Ada and the numbers

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April 10, 2024

Pour Oscar, joyeux Noël.

Chapter 0

This is Ada. Today she decided to go outside and play with her chalks. She wants to draw the digits she has just learnt at school. They have funny shapes and strange names. First Ada draws a line. It starts at the house and goes across the yard to the garden. It will be a good place to put the digits. They mustn't fall off! Ada starts by drawing a circle near her house. This is the 0. It will be the starting point. Then she takes a step towards the garden and follows the line. Ada now draws a 1. It's easy, it looks like a vertical bar. A second step towards the garden and she draws a 2. This one is a bit more complicated: it turns, goes up and down. And don't forget the little horizontal bar underneath. One more step and Ada draws a 3. Two rounded lines and you're done. Next come the 4 (all straight lines), the 5 (a mixture of straight and rounded

lines), the 6 (all rounded lines), the 7 (like an unfinished zigzag), the 8 (like two circles) and the last one: the 9 (be careful not to confuse it with a 6). Ada is now in the middle of the yard. She takes a step back and goes from 9 to 8. She takes another step back and there she is at 7, then 6, 5, 4, 3, 2, 1 and 0, right in front of the house. It's funny, she says to herself, when I take a step towards the garden, I step on the bigger number. And when I step backwards, I step on the smaller number. Let's see what happens when I take three steps forward. Ada stands on the 0 and counts as she walks: 1, 2 and 3! Three more steps: 4, 5 and 6! And one more: 7, 8 and 9!
"And now," says Ada, "in the other direction! Let's see what happens when I go back two steps:

8 and 7 !

6 and 5 !

4 and 3 !

2 and 1 !"

I'm almost there, Ada thinks.

Chapter 1

Addition and subtraction

At that moment, Ada's aunt, Emily, comes out of the house.
"Hello Ada," says Emily, "how are you?"
"Emily!" Ada exclaims.
Ada really likes Emily. She is one of her favorite aunts. Emily is very cool; she has traveled a lot and speaks three different languages. And, her job is to build rockets!
"Look, Emily, I'm playing with numbers!" Ada explains. "I've drawn all the numbers from 0 to 9. And when I take a step forward, I move on to the next one. And when I

take a step backwards, I return to the previous one."
Emily then explained to Ada that when she takes a step forward, she adds one to the number she was on and that you draw $+1$. When she takes a step backwards, she subtracts one from the number she was on and you draw -1. Ada now has two new symbols to use: plus $(+)$ for addition and minus $(-)$ for subtraction. She thinks that these two new symbols are fun. Ada and Emily decide to play together. Starting from 0, Ada takes three steps forward $(+3)$ and then one step back (-1). Now she's at 2. Emily takes out a notebook and a pencil and shows Ada how to draw her path: $0 + 3 - 1$. And as Ada is now on 2, Emily explains that there is a symbol called equal $(=)$, which is used to say that she has arrived at 2: $0 + 3 - 1 = 2$. Ada thinks it sounds like a magic formula. She really wants to try it again.
"Come on, Auntie, you write the numbers and funny symbols in your magic notebook and I'll go along the line."
Emily showed her her notebook where it said: $4 + 3 - 2$. Ada thought about it.
"You have to start at 4, take three steps forward and two back, right?"
"That's it!" replied Emily.
So Ada set off. Starting from 4, she took three steps forward and arrived at 7. Then she turns and takes two steps in the other direction. Ada is now on number 5. Emily

notes $4 + 3 - 2 = 5$.
"Now it's my turn," Ada shouts.
Emily gives her her notebook and pencil and Ada writes: $1+7-3$. Emily looks at the paper, gets up and walks over to number 5. "Voilà!", she said, and write $1 + 7 - 3 = 5$ in the notebook.
"What?" asks Ada. "You're cheating! You have to start from 1, take seven steps forward and three back."
And Ada showed her. She stood on the 1, took seven steps forward (she was now on the 8) and then took three steps back to arrive exactly where Emily was, on the number 5. Emily then explained to Ada that, thanks to the notebook and pencil, you don't always have to go back and forth. You can calculate the result of operations (that's what Emily calls the magic the magic formulae with the digits, the $+$ and the $-$) and find out the result (what comes after the $=$) without even moving. Ada is sceptical. She asks Emily for another example. Emily stands on the 0 and writes down $2 + 4 - 5 + 2 - 3$ in her notebook. Ada stands on the 2, takes four steps forward, then five in the other direction, turns around and takes two steps forward and three steps back. Phew! She almost felt dizzy. Emily didn't move. Ada looked down at her feet and saw the 0. Emily smiled at her and wrote $2 + 4 - 5 + 2 - 3 = 0$.
"You see," she says, "I knew we'd both end up here on 0, even before you started."

"It's true," replies Ada, "it's handy for those really long formulas that make you dizzy. But I think it's more fun to travel for real!"

Chapter 2

Running out of digits

Ada continued to play with additions and subtractions. Emily left her notebook and pencil and went to pick flowers in the garden. After a few rounds, Ada has a problem. She ran out of digits! She wrote $4 + 5 + 2 - 3$ in the notebook, stood on the 4, took five steps forward and now she is stuck. She is now on 9, the last digit in the line. Ada has to take two steps forward and then three steps back. But after 9, Ada hasn't drawn anything. She could walk forward and then backward along the line without any digits to put her feet on. But how does she know if she's in the right place? She knows that there are things after 9. She has heard of numbers like twenty-one, sixty-three, thirty-six and fifteen.

But Ada doesn't know how to draw them, or even in what order. Which comes first? twenty-one or fifteen? sixty-three or thirty-six? She was quite worried. What should she do? Ask her aunt Emily? No, Emily was probably very busy, and Ada thought she could do it on her own. So she thought about it. She thought she could come up with some new digits. All you have to do is draw new symbols and find new names for them. E asier said than done. After four new digits, "ga", "bu", "zo" and "meu"[1], Ada has run out of ideas. It's not that easy. You have to think of new names and new symbols to draw (and who invented the digits 0 to 9? Ada wonders). And it's hard to remember things like that: she was already having trouble remembering the names of the digits 0 to 9, so imagine if she had to remember a new name and symbol for each new digit. So, no, Ada decided that inventing new digits might not be such a good idea after all. But what could she do? Still standing on the 9, Ada had to take two more steps forward and three backwards. She's thinking again. Maybe she could use the digits again? After 9, you can write 0, 1, 2, etc. again. That way you don't have to learn new symbols. That's good, thinks Ada, but the problem

[1] Author's note: these numbers were not invented by Ada, but come from a cartoon series called The Shadoks, see `https://en.wikipedia.org/wiki/Les_Shadoks`.

is that you can get lost. How will I know how far away I am from the house? If I'm on a 5, am I on the first 5 - the one next to the house, the second? the third? How can I find my way around, Ada wondered. She thought again. What if she used colours? One colour for the first series of digits from 0 to 9, say green. Then another colour, say yellow, for the second series of digits, then red, then blue. Hmm, is blue bigger than red? Or would it be better to use blue for the third series and red for the fourth? And what colour for the fifth? Ada says to herself that this solution causes more problems than it solves. The colours may not be the right solution, but it gives her another idea. Ada is going to write down the number of times she uses all the digits. She starts at home: 0, 1, 2, 3, 4, 5, 6, 7, 8 and 9. After 9, Ada starts the sequence again. She writes down 0. And so that she doesn't forget that she just used the whole series the first time, she writes 1 to the left. Ada then gets 1 0 ('one-zero'). She goes on to write 1 1 ('one-one'), 1 2 ('one-two'), 1 3 ('one-three'), 1 4, 1 5, 1 6, 1 7, 1 8 and 1 9. At this point, Ada extends the line along the path to the garden. Since she has used the whole series of numbers a second time, she will now write 2 0 ('two-zero') and continue: 2 1 ('two-one'), 2 2 ('two-two'), 2 3, 2 4, 2 5. Ada pauses for a moment to look at her solution. She likes it. First of all, you never get lost, she thinks (you always know whether you're near or far from home), and

then you can draw in any colour you like! Emily, who had just returned from the garden, said to her:
"What are you doing on twenty-five?"
Ada asks, "Is 'two-five' actually called twenty-five?"
"Yes," answers Emily, "all numbers have names."
"Numbers?" asks Ada.
"Yes," says Emily, "that's what we call them. The digits are the symbols: 0, 1, 2, 3, 4, 5, 6, 7, 8 and 9. There are only ten of them. You can write numbers with digits. For example, twenty-five with a two and a five: 25."
"And what's the one with a one and a four?" asks Ada.
"It's called fourteen."
"And what about this one? The funny one with a one and a seven?"
"It's seventeen."
Emily then taught Ada the names of the numbers: "ten" (10), "eleven" (11), "twelve" (12), etc. up to 25, where Ada stopped. Ada wonders who decides the names of the numbers. Emily explains that the digits have travelled a long way: China, India, the Middle East, Central Asia, North Africa and Europe. Numbers as we know them are used all over the world. Ada finds this very practical. She imagines a world where numbers change from one country to another. How complicated!

Chapter 3

Infinity

Ada picks up where she left off. She takes the notebook and reads $4 + 5 + 2 - 3$. Ada places herself on the 4 and takes five steps forward. She is now on 9. She takes two steps forward. She is now on 11. Ada turns around and takes three steps in the opposite direction, stopping on 8. At last! Ada can now write $4 + 5 + 2 - 3 = 8$ in her notebook. She turns to Emily and asks her:
"Say, Auntie, what's the biggest number?
Emily looked at her and smiled.
"The biggest number you've drawn on your line so far is twenty-five."
"No!" replied Ada, "you know, the biggest of ALL num-

bers. After 25 I can write 2 6 (Ada then pronounces “two-six” and Emily tells her its name: “twenty-six”), and 2 7 (“two-seven”, “twenty-seven”), 28, 29. Then I write 30 to say that I’ve used all the digits from 0 to 9 three times. And I continue: I write 31, 32, 33 and so on. You see, if the line continues through the garden, to the neighbour’s house, across the road, I should be able to write 99. If there’s still room on the line (and if Mum lets me go to the neighbours, cross the road and continue on the line), I should be able to go one step further and add 1, right?"
"That’s right," answers Emily, "you can always go one step further, or add 1. After 99 comes 100, after 999 comes 1000, after 9999 comes $10,000$."
"So," asks Ada, "which is the bigger number? Because if you can always add 1, that means there’s always a bigger number, and another bigger number than the biggest of the biggest. It never ends!"
Ada was almost dizzy.
"It’s true," explains Emily, "you’re right. It never ends. We say there are infinite numbers and we write ∞."
"Like an eight lying down?"
"Yes, like the digit eight. But be careful, infinity is not a number. As you said, you can always find a greater number, and another greater number, and another greater number than the greatest of the greatest, without ever stopping. That’s what we mean when we write ∞."

Ada watches this newcomer. When she looks at him, she sees a path that never ends. She thinks she could travel very far in the land of numbers.

Chapter 4

Negative numbers

Ada feels proud: she has figured out how to write numbers bigger than 9, and she has also discovered an infinity of numbers. But Ada is also a little sad, because in her playground she can only play with a few numbers. And what's more, they're small: most of them only have two digits! Seeing this, Emily asks her if she still wants to play with additions and subtractions. But Ada is no longer amused. She feels there is nothing left to discover. Emily suggests going for a snack. But before leaving, she writes down a new formula in the notebook: $3 - 5 =$ and hands it to Ada.

"Such a small formula," thinks Ada, "can't be too com-

plicated. I'll solve it (as Emily says) before Émily comes back."
Ada stands on the 3 and starts to move backwards: one step, two steps, three steps and... Oh no, here it comes again! Ada is now on the 0, in front of the house door, at the beginning of the line. She still has two steps to go to finish her formula, but the line stops. So Ada picks up her chalk and continues the line behind the house to the garage. She can see that she needs to go back two steps to finish her formula. There is enough room for at least twenty more steps, Ada thinks. The problem is, there is no number on this side of the line.

Ada thinks. She could reuse the numbers she already knows. Starting from in front of the house (0), she takes one step back and notes 1, then two steps back and notes 2. The problem, Ada says to herself, is that we can't tell the difference between the 1 behind the house and the 1 in front, in the yard. She thought she could use some color. Green for the front and orange for the back. Color is fine, but Ada finds another problem. When she goes backwards, i.e. when she removes one, the numbers get bigger and bigger. And when she moves forward, i.e. adds one, the numbers get smaller and smaller. It is the opposite of before! You can't have rules for one side of the house and other rules for the other side.
"We'll end up mixing everything up," says Ada.

She thinks again, looks at her notebook with all its formulas and thinks: when it says -1, that means I take one off and go back one step. When it says -2, it means I take off two and go back two steps. When it says -3, it means I'm taking off three and going back three steps. Ada stands in front of the house again, on the 0. She takes a step back and writes down -1 (I've taken a step back from zero). She takes another step back and notes -2 (I've gone back two steps from zero). She continues to list the steps, writing -3, -4, -5, etc. all the way to the garage, where Ada writes down -21. Just then, Emily returns with snacks.
"Ada, you've discovered negative numbers!" exclaims her aunt.
Ada is proud of herself. Emily explains to her that negative numbers have also traveled a long way and have long been considered strange numbers. Ada says to herself that there is nothing weird about them and now, thanks to them, she can finish Emily's formula. Ada places herself on the 3 and takes five steps back. Now she stands on -2.
"There," says Ada, "$3 - 5 = 2$. And now, can we get a snack?"

Chapter 5

Multiplication

Ada's little brother Charles joins his sister and Emily for a snack. After eating a few cakes, Ada asked Emily:
"Say, Auntie, can you jump with both feet? Look at that!"
Ada stands on 0, makes her first jump and lands on 2. A second jump and she's on 4. Another and Ada lands on 6.
"Look, Emily, every time I jump, I go two steps further! Now it's your turn!"
Emily also places herself on the 0, makes her first jump and lands on the 3. One more and she's on 6 too.
"My turn!" shouts Charles.
And he too starts jumping. From the 0, he lands on the 1. Then onto the 2 and the 3, and so on, until, after six jumps,

he arrives with Emily and Ada on the 6. Ada then wonders where she'd land if she did 10 jumps in a row. She starts again from the beginning (0). After three jumps, she's back on 6. She makes a fourth jump and lands on 8. After a fifth, she's on 10. She continues counting the number of jumps and arrives, after ten, on 20.
It's your turn now," says Ada to Emily.
Emily stands on 0 and makes one jump (landing on 3), then a second (landing on 6) and a third (on 9), and so on. After the tenth jump, she's much further away than Ada, at number 30. Ada is amazed; she and Emily have each made ten jumps. But since Emily's jumps are bigger than Ada's, Emily has gone much further than Ada. And now it's your turn, Charles," says Ada to her brother."
And Charles starts jumping ten times in a row. Starting at 0, he finally reaches 10. Ada wants to remember all these numbers. She takes out her notebook and writes:

$$2+2+2+2+2+2+2+2+2+2 = 20 \text{ for her,}$$

$$3+3+3+3+3+3+3+3+3+3 = 30 \text{ for Emily and}$$

$$1+1+1+1+1+1+1+1+1+1 = 10 \text{ for Charles.}$$

How long and complicated, Ada thinks. Imagine, if we had to make 50 jumps or 100, there wouldn't be room to write it all down! We'd need a faster way of writing that we'd

done the same thing 10 times. We'd only need two numbers: for example, the number of jumps (10) and the size of each jump (2). It would be a lot more practical. Ada shows Emily her notebook and asks if there's a simpler way of writing than $2+2+2+2+2+2+2+2+2 = 20$. Emily then introduces him to a new symbol: a little cross called "fois" that goes like this: $\times$. Emily shows her that $2+2+2+2+2+2+2+2+2 = 20$ can be written $10 \times 2 = 20$ and that we say ten times two. Ada looks at this newcomer and then writes $10 \times 3 = 30$ in her notebook. It's still more practical than writing $3+3+3+3+3+3+3+3+3 = 30$," she says, "and that's exactly what I needed. And for you, Charles, we write $10 \times 1 = 10$."
But something is troubling Ada.
"Say, Emily," she asks, "you told me that when you write $+$, you're adding. When we write $-$, we subtract. What do we do when we write $\times$?"
"It's called multiplication," answers Emily, "we say we multiply two numbers together."
"So if I did 50 jumps, what number would I arrive at?"
"A hundred, because $50 times 2 = 100$."
"And if I do 100?"
"Two hundred, because 100×2 =200."
Ada imagines she'd be very tired from jumping so much.

Chapter 6

Division

Ada is still a little hungry. There's no more cake, but fortunately Emily has brought some raspberries from the garden. There are twelve of them. Ada wonders how to share them. Emily and Charles also like raspberries. Ada thinks. I'll start by giving one to Charles, one to Emily and one to me. There are (Ada counts the raspberries aloud) nine left. Here's a second one for Charles, a second one for Emily and a second one for me. Only six left. So here's a third for Charles, Emily and me. There are now only three left. Finally, Ada, give each of you a fourth raspberry. And that's the twelve raspberries shared.
"Who's still hungry? Who wants raspberries?" asks Ada.

"Look, we all have four."
"Bravo," says Emily, "you know the division too!"
"The what?" asks Ada.
"Division," explains Emily. "It is what you just did to share the raspberries. Division is a friend of multiplication, subtraction and multiplication."
"Ada replies, her mouth full of fruit. And how do you draw it?"
"We often use a line to show that we're dividing (cutting) one number by another."
"We cut the numbers!" exclaims Ada. "
Ada imagines numbers cut in half. An 8 having lost its upper half (ơ), a footless 4 (⊿)! Strange!.
"And it doesn't hurt?"
"No," reassures Emily. "Here, take a look."
Emily takes the notebook and writes: $12/3 = 4$.
"Twelve raspberries divided into three equals four raspberries per person. You can also write like this."
Emily hands Ada the notebook and shows her:

$$\frac{12}{3} = 4$$

Ada thinks the division is rather pretty. She wonders if it works for all numbers and not just raspberries. She'd like to know what 7 divided by 3 is. How do you do it? For the raspberries, it was easy: all you had to do was hand them

out. One for Ada, one for Charles, one for Emily, two for Ada, two for Charles, two for Emily, and so on, until there were no raspberries left. But Ada's run out of raspberries to try. Ada thinks. She has an idea. She's going to use her coloured chalks and some pebbles. Ada gets up and takes three pieces of chalk: one green, one orange and one purple. Then she draws three circles on the ground: one green, one orange and one purple. Then she picks up seven pebbles. First she puts one in the green circle, one in the orange circle and one in the purple circle. It's the same as with the raspberries, except that Ada distributes the pebbles. Ada continues. She has four pebbles left. She puts one in the green circle, one in the orange circle and one in the purple circle. Each circle now has two pebbles. Ada looks in her hand. There is only one pebble left to distribute. Not enough to put one in each coloured circle! It's not going to work.
"Emily? Your division isn't working.""
"Really?" asks Emily.
"Yes," resumes Ada, "I wanted to divide 7 by 3. I have three baskets: one green, one orange and one purple. I already put two pebbles in each basket. But I've still got one more to go, but I can't divide it into three!"
"It's normal," explains Emily, "not all numbers can be divided completely. Sometimes there's a remainder."
"A remainder?"

"Yes, a remainder, that's what it's called. In your case, 7 divided by 3 equals 2, leaving 1. You could also say that 7 equals 3 packets of 2 plus 1. All the same. And we can write it down like this: $7 = 3 \times 2 + 1$."
"Huh?" exclaims Ada. "What happened to my division? You're cheating again, Auntie!"
"Of course not. Multiplication and division are such good friends that you can change one into the other. We say that division is the inverse of multiplication."
Ada looks at her stunned. Then after a while she says:
"Okay, so please, can you get us six candies? Two each? Because 6 = 3 =2 and 6/3 =2, right?"

Chapter 7

Fractions

Emily returned to the house. Ada sits on a low wall and thinks. She thinks it's strange that some numbers can't be completely divided. Meanwhile, Charles is having fun on the number line.
"Seven! two! nine! zero! eight! six! three! five! one! three! Look, Charles the number tamer!
"Charles, stop messing around. You're not even walking properly on the line!" complained Ada."
Unable to concentrate, Ada watches Charles play on the number line. She realizes that it takes him two steps to go from 0 to 1, then two more to go from 1 to 2, and so on. Ada realizes that Charles is taking half-steps. It's

true, Ada says to herself, there are lots of things that are 1 but can be cut in half: for example, an apple or a deck of cards. And then there are things that are 1 and can't be cut in half: a pebble or a pencil, for example. Ada notices a sparrow hopping around in the garden. Its steps are even smaller than Charles'. Maybe it takes ten steps to take one of Ada's. On the low wall where she's sitting, Ada notices some ants. They must take at least a hundred steps to take the equivalent of one of Ada's. This gives her an idea. If Charles has to take two steps when Ada takes one, then one step from Charles is equal to one step from Ada cut in half: 1/2. One sparrow step is equal to one Ada step cut in ten: 1/10. And one step of ants equals one step of Ada cut into a hundred! 1/100. Ada takes out her notebook and writes $2 \times$ Charles $= 1 \times$ Ada. Ada imagines two Charles. Two little brothers! Oh no, one little brother is already a lot, so two is a bit much. Ada writes $1 \times$ Charles = Ada/2. Ada imagines herself cut in two. She's not happy about the idea either. What's more, she wonders if she's allowed to mix numbers with letters. Just then, Charles calls out to her. "Ada! I've got all the numbers! I want some new ones!"

Ada gets up, grabs her chalk and heads for the number line.

"Look, Charles, I'm going to draw your numbers for you."

Ada draws 1/2 between the 0 and the 1. Then Ada counts.

To get to 1, Charles has to take two steps. Ada's two half-steps: $2/2 = 1$. No need to draw this one, so Ada continues. To get between 1 and 2, Charles has to take three steps. Ada writes 3/2. To get to the two, Charles has to take four steps: $4/2 = 2$. One more, which Ada doesn't need to draw. Between the 2 and the 3, Ada writes 5/2 and so on. Between 3 and 4, 7/2. Between the 4 and the 5, 9/2. Ada continues to 10. Charles is delighted.

"Numbers just for me, thanks Ada! And what are their names? "

Ada thinks. Charles takes half-steps.

"The first, 1/2, is called one half. Next, 3/2, is three halves. 5/2, five halves, and so on."

Charles now takes a step on each of his numbers. Ada wonders what 1 cut into three (1/3) or 1 cut into four (1/4) is called. What about the others? 1/5, 1/6, 1/7, 1/8 and 1/9? For Ada, 1/10 will now be called a sparrow. And 1/100, an ant. These new numbers are strange, Ada thinks. When you cut 1 by a larger number, the result becomes smaller. 1/100 is smaller than 1/10. 1/1000 is smaller than 1/100. 1/100000 is even smaller! Ada imagines herself on the hunt for the smallest number and becoming tiny, smaller than a sparrow, smaller than an ant, infinitely small.

Chapter 8

Equations

Émile is now back. Unfortunately without any candy. It's a shame, Ada thinks, but it doesn't matter, Ada has to show Emily things first. We'll deal with the candy later.
"Look, Emily," says Ada, "Charles takes half-steps. When I take one step, he has to take two."
"Is that right? And how much do Emily's steps equal Ada's steps," asked her aunt.
"I don't know, but let's give it a try," replies Ada. "I've got an idea. We stand next to each other, walk together and see how many steps we have to take to get to the same place."
Ada and Emily both sit in front of the house, on the 0.

Ada takes a step and arrives on the 1. Emily also takes a step, but her steps are bigger and she's stuck between the 1 and the 2. Ada takes a second step and passes Emily. Emily takes a second step and arrives on the 3. Ada has to take a third step to reach Emily. Ada writes 3 Ada = 2 Emily in her notebook. Three steps from Ada are worth two steps from Emily. Sounds complicated, says Ada. I'd rather know how much one of my steps is worth in Émile steps. If three of my steps equal two of Emily's steps, then by cutting into three, I get one of Ada's steps equal to two of Emily's steps cut into three. Ada notes 1 Ada = 2/3 Emily. Emily looks at the notebook and exclaims
"Oh, the beautiful equations!"
"Equations," says Ada, astonished. "Equations," resumes Emily. "They're the formulas you just wrote down. That's what we call them. They're like recipes that tell us how to transform things."
I knew it," says Ada, "you can mix numbers and letters."
"Yes, as long as it adds up!"
"I wonder how many steps you'd have to take if I got to number 12," said Ada.
"Well, we could try," replies Emily. I stand on the 0. I take one or two steps forward and I'm on the 3. After three and four steps, I'm on 6. Five and six steps to 9. Seven and eight steps. And that's it! I'm on 12. Eight steps, that's your answer Ada."

"Hey, don't leave me alone!" suddenly shouts Charles from the house.
"You're not alone," replies Ada, "and what's more, you're not even far away."
"If I'm very far, at least a hundred and fifty-eight!"
"Nonsense," says Ada, "we're on 12 and you're on 0. Since you're doing half-steps, you must be doing twice as many as me. That's two packs of 12 and that makes...," Ada calculates in her head, "twenty-four, right Emily?"
"Yes, that's right. Come on, Charles, it's not that far."
Charles runs to join them.
"And now," says Emily, "I have to go get some salad from the garden. Who's coming with me?"
"Me!" exclaimed Charles.
"Are you joining, Ada?"
"I don't know, the salads are at the bottom of the garden and that seems a long way away. What would be nice," continues Ada, "is to have a method of knowing how many steps I have to take to reach you. The best thing would be to have the answer right away, just by looking, without calculating."
"I can draw you the equations," suggests Emily.
"But I've already written them in my notebook!"
"No, look."
Emily draws two lines in the notebook: a horizontal one for her steps and a vertical one for Ada's steps. Emily then

notes 0 where the lines cross, and adds numbers from 1 to 9 on each. She shows Ada how to draw the equation. When Emily takes two steps, Ada must take three. Emily places the pen on the 2 number on the horizontal line and moves up to the 3 number on the vertical line. There, she scores her first point. When Emily takes four steps, Ada must take six. In the same way, Emily scores a second point. When Emily takes six steps, Ada must take nine. Finally, Emily marks a third point on the sheet. Then she draws a line connecting all the dots.

"You see," says Emily, "this line I've just drawn represents my equation. If you want to know how many steps you have to take to reach me, just follow this line."

"Follow the line?"

"Yes, look. If I take eight steps, the line tells me you have to take twelve."

"I already knew that," replies Ada.

"If I take ten steps, you have to take fifteen."

"And if you only do five?" asks Ada, "what does the line tell us?"

"It tells us you have to do seven and a half."

Ada looks at the drawing in her notebook. After magic spells and recipes, here's a treasure map.

Chapter 9

Multiplying negative numbers

While Emily picks salad, Ada and Charles play "Ada says" together. The aim is simple: Ada gives Charles orders, and Charles must carry them out only if Ada says "Ada said".
"Ada said, take three steps back!"
Charles does indeed take three steps back.
"Bravo! Ada said, hop on one foot!"
Charles stands on one foot and begins to hop.
"Now run forward five steps."
Not holding back, Charles starts running toward Ada.
"Lost! I didn't say "Ada said"."

"Oh, no, you didn't! Come on, it's my turn! Charles said $3 \times (-2)$."
"$3 \times (-2)$!? Wait a minute..." Ada thinks about it and says: "2 means you take two steps back, so $3 \times (-2)$ means you take three two-step steps back."
Ada takes six steps back. "Charles said $(-2) \times 3$!"
Ada takes six steps back again.
"Hey, you're cheating!" exclaims Charles.
"How am I cheating?" replies Ada.
"Yes, you are! Before it was $3 \times (-2)$ and now it's $(-2) \times 3$, so it's not the same."
"$(-2) \times 3$ is to take two steps back three times, it's like taking two steps back three times. $3 \times (-2) = (-2) \times 3$."
Just then, Emily arrives with a salad in each hand.
"Ada is right. Multiplication is commutative."
"Commuta-what?" ask the siblings in chorus.
"Commutative. This means that you can change the place of the numbers around the $\times$ sign: $5 \times 4 = 4 \times 5$, $7 \times 3 = 3 \times 7$, or even $3 \times (-2) = (-2) \times 3$. It also works for addition: $1 + 3 = 3+1$, $5 + 2 = 2 + 5$."
What about subtraction and division?" asks Ada.
"They're not commutative. You can't move the numbers around as you like: $3-2$ doesn't equal $2-3$ and $6/3$ doesn't equal $3/6$."
Ada looks at Charles.
"That means I've won!"

"No, one more. Charles said $(-4) \times (-3)$!"
Ada thinks. She's never multiplied two negative numbers before.
"When you take two steps forward, you write 2 and when you take two steps back, you write (-2). If we had $4 \times (-3)$, four packets of (-3), we'd go back four times by three steps."
Ada calculates.
"We'd go back 12 steps. And since we have $(-4) \times (-3)$, we do the same thing in the other direction. We'd take 12 steps forward."
Ada asks Emily.
"Say, Auntie, is $(-4) \times (-3)$ equal to 12?
"Bravo Ada," replies Emily, "when you do multiplication with positive and negative numbers, you have to juggle with the signs."
"Juggling with signs? Sounds difficult."
"No, you just did it. Multiplying two positive numbers together or two negative numbers together always gives a positive result. Multiplying a positive number with a negative number gives a negative result."
Suddenly, a voice from the house calls out to the children:
"Ada, Charles! It's time to go home!"
"We're coming!" the siblings answer in unison.

Chapter 10

Epilogue

It's evening. After Charles has gone to bed, Ada asks Emily to tell her a story. Emily tells the story of numbers that travel, that move backwards and forwards, that jump and add up. The story of a little girl who wonders, thinks and plays with symbols. And finally, the story of numbers that are written with letters, drawn and juggled with their signs like a big circus. And under its big tent there are many other shows: a number i that turns heads, several infinities, each greater than the last, and many other things with strange names: logarithms, cosines and cube roots... But that's another story, because now, shhh! Ada has fallen asleep. Good night, Ada!

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