ENGINEERING MATHEMATICS 1

LECTURE 6 (PART 1)

MATRICES

Matrices

- Definitions ,order , types of matrices
- Operation transpose
- inverse of a square matrix
- solution of sets of linear equation
- Gaussian elimination method
- Eigen values and eigenvectors

1.0 INTRODUCTION

There are many information in real life displayed in table form. For example production, manufacturing, dietary table, flight schedule etc. Let see the following example of a dietary table of three kind of foods

	Units per ounce		
	Food A	Food B	Food C
Calcium	30	10	20
Iron	10	10	20
Vitamin A	10	30	20

If we remove the labels from the table, the result would be a matrix

$$\begin{bmatrix} 30 & 10 & 20 \\ 10 & 10 & 20 \\ 10 & 30 & 20 \end{bmatrix}$$

Hence, we can define a matrix as follows, Definition

A matrix is a set of number(s) arrange in rectangular array form denoted by

$$A = \left[a_{ij}\right] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix};$$

Where a_{ij} is called an entry /element, the i-th row of A is $\left[a_{i1} \quad a_{i2} \quad \cdots \quad a_{in}\right]$, $1 \le i \le m$

, and the j-th column of A is $\begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix}$, $1 \le j \le n$

If a matrix A has m rows and n columns, then A is a matrix of order m by n. (written as $m \times n$)

If $m = n$, then A is called a square matrix of order n, and that the elements $a_{11}, a_{22}, \ldots, a_{nn}$ are on the main diagonal of A.

Equality of Matrices

Let A and B be matrices of order $m \times n$, then A and B are equal if and only if $a_{ij} = b_{ij}$ for all $i = 1,2,\ldots,m$ and $j = 1,2,\ldots,n$.

Example 1

The matrices $A = \begin{bmatrix} 1 & 2 & -3 \\ 2 & -1 & 4 \\ 5 & -4 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & w \\ 2 & x & 4 \\ 5 & -4 & z \end{bmatrix}$ are equal $(A = B)$, if and only if $w = -3$, $x = -1$, and $z = 0$.

1.1 ARITHMETIC OPERATIONS AND PROPERTIES

There are three types of arithmetic operations that can be carried with matrices, which are addition, subtraction, and multiplication. However, for multiplication, there are two types of matrix multiplication which are scalar multiplication and matrix multiplication.

Matrix Addition and Subtraction

If A and B be matrices of order $m \times n$, then $A + B = \lfloor a_{ij} + b_{ij} \rfloor$ and $A - B = \lfloor a_{ij} - b_{ij} \rfloor$ for all $i = 1,2,\ldots,m$ and $j = 1,2,\ldots,n$.

Properties of Matrix Addition:

If A, B and C are matrices of order $m \times n$, then

a) $A + B = B + A$

b) $A + (B + C) = (A + B) + C$

c) There is a unique $m \times n$ matrix $O_{m \times n}$ such that

$$O + A = A + O = A$$

for any $m \times n$ matrix A. The matrix $O_{m \times n}$ is called the $m \times n$ zero matrix.

The $m \times n$ zero matrix is called the additive identity of A.

d) For each $m \times n$ matrix A, there is a unique $m \times n$ matrix D such that $A + D = O$, where $D = -A$. The matrix $-A$ is called the additive inverse of A.

Example 2

a) Let $A = \begin{bmatrix} 1 & 3 \\ 5 & 7 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

b) Let $A = \begin{bmatrix} 1 & -2 & 3 \\ 8 & -1 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} -3 & 2 & 1 \\ 1 & 0 & -1 \end{bmatrix}$.

c) Let $A = \begin{bmatrix} 1 & -7 & 4 \\ 5 & -9 & 6 \\ 2 & 0 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 & -3 \\ 0 & -9 & 0 \\ 1 & -8 & 4 \end{bmatrix}$.

d) For each $m \times n$ matrix A, there is a unique $m \times n$ matrix D such that $A + D = O$, where $D = -A$. The matrix $-A$ is called the additive inverse of A.

<u>Example 2</u>

a) Let $A = \begin{bmatrix} 1 & 3 \\ 5 & 7 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

Then, $A + B = \begin{bmatrix} 1 & 4 \\ 6 & 7 \end{bmatrix}$ and $A - B = \begin{bmatrix} 1 & 2 \\ 4 & 7 \end{bmatrix}$.

b) Let $A = \begin{bmatrix} 1 & -2 & 3 \\ 8 & -1 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} -3 & 2 & 1 \\ 1 & 0 & -1 \end{bmatrix}$.

Then, $A + B = \begin{bmatrix} -2 & 0 & 4 \\ 9 & -1 & 3 \end{bmatrix}$ and $A - B = \begin{bmatrix} 4 & -4 & 2 \\ 7 & -1 & 5 \end{bmatrix}$.

c) Let $A = \begin{bmatrix} 1 & -7 & 4 \\ 5 & -9 & 6 \\ 2 & 0 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 & -3 \\ 0 & -9 & 0 \\ 1 & -8 & 4 \end{bmatrix}$.

Then, $A + B = \begin{bmatrix} 3 & -7 & 1 \\ 5 & -18 & 6 \\ 3 & -8 & 7 \end{bmatrix}$ and $A - B = \begin{bmatrix} -1 & -7 & 7 \\ 5 & 0 & 6 \\ 1 & 8 & 3 \end{bmatrix}$.

Scalar Multiplication

Let $A = \lfloor a_{ij} \rfloor$ be a matrix of order $m \times n$ and c is a real number, then the scalar multiple of A by c, $cA = \lfloor ca_{ij} \rfloor$ for all $i = 1,2,\ldots,m$ and $j = 1,2,\ldots n$.

Properties of Scalar Multiplication:

If c and d are real numbers and A and B are matrices, then

a) $c(dA) = (cd)A$

b) $1A = A$

c) $c(A + B) = cA + cB$

d) $(c + d)A = cA + dA$

Example 3

Let $A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & -1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & 3 & 0 \\ 4 & -5 & 0 \end{bmatrix}$,

Find

a) $-2(A+B)$.

b) $-3A+4B$.

Solution:

a) $A+B = \begin{bmatrix} -1 & 1 & 3 \\ 6 & -6 & 0 \end{bmatrix}$

Example 3

Let $A=\begin{bmatrix}1 & -2 & 3\\ 2 & -1 & 0\end{bmatrix}$ and $B=\begin{bmatrix}-2 & 3 & 0\\ 4 & -5 & 0\end{bmatrix}$,

Find

a) $-2(A+B)$.

b) $-3A+4B$.

Solution:

a) $A+B=\begin{bmatrix}-1 & 1 & 3\\ 6 & -6 & 0\end{bmatrix}$

$-2(A+B)=\begin{bmatrix}2 & -2 & -6\\ -12 & 12 & 0\end{bmatrix}$

b) $-3A=\begin{bmatrix}-3 & 6 & -9\\ -6 & 3 & 0\end{bmatrix}$ and $4B=\begin{bmatrix}-8 & 12 & 0\\ 16 & -20 & 0\end{bmatrix}$

Then, $-3A+4B=\begin{bmatrix}-11 & 18 & -9\\ 10 & -17 & 0\end{bmatrix}$

Matrix Multiplication

Let A be a matrix of order $m \times n$ and B be a matrix of order $n \times p$, then the product of A and B, is a matrix C of order $m \times p$, $AB = C_{m \times p}$, where all the elements of C define by,

$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} = a_{i1} b_{1j} + a_{i2} b_{2j} + \cdots + a_{in} b_{nj}$, for all $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, p$.

Note:

- AB is defined only when the number of columns of A is the same as the number of rows of B, i.e., $A_{m \times n} B_{n \times p} = C_{m \times p}$.
- In general $AB \neq BA$.

Properties of Matrix Multiplication:

Let A, B and C are the matrices with the appropriate sizes and c is a scalar, then,

a) $A(BC)=(AB)C$

b) $(A+B)C=AC+BC$

c) $C(A+B)=CA+CB$

d) $c(AB)=(cA)B$

e) There a square matrix I where all the elements are equal 0 except the diagonal equal to 1 such that

$IA=AI=A$

The matrix I is called the identity matrix. Below is the different size of identity matrices.

$$\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}_{2\times 2} \quad \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}_{3\times 3} \quad \begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & 1\end{bmatrix}_{4\times 4} \quad \cdots$$

Example 4

1. $\begin{bmatrix} a & b & c \end{bmatrix}_{1\times 3} \begin{bmatrix} p \\ q \\ r \end{bmatrix}_{3\times 1} = \begin{bmatrix} ap + bq + cr \end{bmatrix}_{1\times 1}$

2. $\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}_{2\times 3} \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}_{3\times 1} = \begin{bmatrix} 2a + 3b + 4c \\ 2d + 3e + 4f \end{bmatrix}_{2\times 1}$

3. $A = \begin{bmatrix} 2 & 0 \\ -1 & 2 \end{bmatrix}_{2\times 2}$, and $B = \begin{bmatrix} 3 & -1 & 2 \\ 1 & 0 & 5 \end{bmatrix}_{2\times 3}$, then

Example 4

1. $\begin{bmatrix} a & b & c \end{bmatrix}_{1\times 3} \begin{bmatrix} p \\ q \\ r \end{bmatrix}_{3\times 1} = \begin{bmatrix} ap+bq+cr \end{bmatrix}_{1\times 1}$

2. $\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}_{2\times 3} \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}_{3\times 1} = \begin{bmatrix} 2a+3b+4c \\ 2d+3e+4f \end{bmatrix}_{2\times 1}$

3. $A = \begin{bmatrix} 2 & 0 \\ -1 & 2 \end{bmatrix}_{2\times 2}$, and $B = \begin{bmatrix} 3 & -1 & 2 \\ 1 & 0 & 5 \end{bmatrix}_{2\times 3}$, then

$$AB = \begin{bmatrix} 2\times 3+0\times 1 & 2\times(-1)+0\times 0 & 2\times 2+0\times 5 \\ -1\times 3+2\times 1 & -1\times(-1)+2\times 0 & -1\times 2+2\times 5 \end{bmatrix} = \begin{bmatrix} 6 & -2 & 4 \\ -1 & 1 & 8 \end{bmatrix}_{2\times 3}$$

However, BA is not defined.

Transpose of a Matrix

Let A be a matrix of order $m \times n$, then the transpose of A, denoted by A^T, is the $n \times m$ matrix defined by $A^T = [a_{ij}^T] = [a_{ji}]$. Thus, the A^T is obtained from A by interchanging the rows and columns of A.

Properties of Transpose:

If A and B are matrices, and c is a scalar, then

a) $(A^T)^T = A$

b) $(A+B)^T = A^T + B^T$

c) $(AB)^T = B^T A^T$

d) $(cA)^T = cA^T$

Example 5

If $A = \begin{bmatrix} 1 & -3 \\ 0 & 2 \\ -1 & 7 \end{bmatrix}_{3\times 2}$, then $A^T = \begin{bmatrix} 1 & 0 & -1 \\ -3 & 2 & 7 \end{bmatrix}_{2\times 3}$.

1.2 DETERMINANT OF A SQUARE MATRIX

Determinant is a mathematical process whereby from a given matrix (as an input) become a real number (as an output). Determinant can only be done on a square matrix (a matrix that has the same number of rows and column). If the matrix is not a square matrix, it will not be possible to find the determinant of such matrix.

input		output
A	$\rightarrow$	$\lvert A \rvert$
square matrix		real number (= determinant of A)

Determinant of a 1×1 matrix (Order 1 matrix)

If $A = [a]$ is a square matrix of order 1, then $|A| = |a| = a$.

Example 1

$$A = \begin{bmatrix} -5 \end{bmatrix}, \text{ then } |A| = |-5| = -5$$

Note: Determinant of order 1 matrix is the element number of its matrix.

Determinant of a 2×2 matrix (Order 2 matrix)

If $A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$ is a square matrix of order 2, then $|A| = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1$.

Example 2

1. $\begin{vmatrix} 2 & 1 \\ 3 & -4 \end{vmatrix} = (2)(-4) - (1)(3) = -8 - 3 = -11$

2. $\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = (1)(1) - (0)(0) = 1$

3. $\begin{vmatrix} x & 0 \\ y & 1 \end{vmatrix} = (x)(1) - (0)(y) = x$