INDICES

CHAPTER 3 PART 1

Powers, roots and standard form

Powers, roots and standard form

1. Powers and roots
2. Index laws
3. Negative indices and reciprocals
4. Fractional indices
5. Standard form

Square numbers

When we multiply a number by itself we say that we are **squaring** the number.

To square a number we can write a small ^{2} after it.

For example, the number 3 multiplied by itself can be written as

$$3 \times 3 \qquad \text{or} \qquad 3^2$$

The value of three squared is 9.

The result of any *whole* number multiplied by itself is called a **square number**.

Square roots

Finding the **square root** is the inverse of finding the square:

$$\begin{tikzcd}[column sep=huge] \mathbf{8} \arrow[r, bend left, "\textbf{squared}"] & \mathbf{64} \arrow[l, bend left, "\textbf{square rooted}"] \end{tikzcd}$$

We write

$$\sqrt{64} = 8$$

The square root of 64 is 8.

The product of two square numbers

The product of two square numbers is always another square number.

For example,

$$4 \times 25 = 100$$

because

$$2 \times 2 \times 5 \times 5 = 2 \times 5 \times 2 \times 5$$

and

$$(2 \times 5)^2 = 10^2$$

We can use this fact to help us find the square roots of larger square numbers.

Using factors to find square roots

If a number has factors that are square numbers then we can use these factors to find the square root.

For example,

Find $\sqrt{400}$	Find $\sqrt{225}$
$\sqrt{400} = \sqrt{(4 \times 100)}$	$\sqrt{225} = \sqrt{(9 \times 25)}$
$= \sqrt{4} \times \sqrt{100}$	$= \sqrt{9} \times \sqrt{25}$
$= 2 \times 10$	$= 3 \times 5$
$= 20$	$= 15$

Finding square roots of decimals

We can also find the square root of a number can be made be dividing two square numbers.

For example,

Find $\sqrt{0.09}$

Find $\sqrt{0.0144}$

$$\sqrt{0.09} = \sqrt{(9 \div 100)}$$
$$= \sqrt{9} \div \sqrt{100}$$
$$= 3 \div 10$$
$$= 0.3$$

$$\sqrt{0.0144} = \sqrt{(144 \div 10000)}$$
$$= \sqrt{144} \div \sqrt{10000}$$
$$= 12 \div 100$$
$$= 0.12$$

Approximate square roots

If a number cannot be written as a product or quotient of two square numbers then its square root cannot be found exactly.

Use the √ key on your calculator to find out $\sqrt{2}$.

The calculator shows this as 1.414213562

This is an approximation to 9 decimal places.

The number of digits after the decimal point is infinite and non-repeating.

This is an example of an **irrational** number.

Estimating square roots

What is $\sqrt{50}$?

50 is not a square number but lies between 49 and 64.

Therefore,

$$\sqrt{49} < \sqrt{50} < \sqrt{64}$$

So,

$$7 < \sqrt{50} < 8$$

50 is much closer to 49 than to 64, so $\sqrt{50}$ will be about 7.1

Use the √ key on you calculator to work out the answer.

$\mathbf{\sqrt{50} = 7.07}$ (to 2 decimal places.)

Negative square roots

$5 \times 5 = 25$ **and** $-5 \times -5 = 25$

Therefore, the square root of 25 is 5 or –5.

When we use the √ symbol we usually mean the positive square root.

We can also write ±√ to mean both the positive and the negative square root.

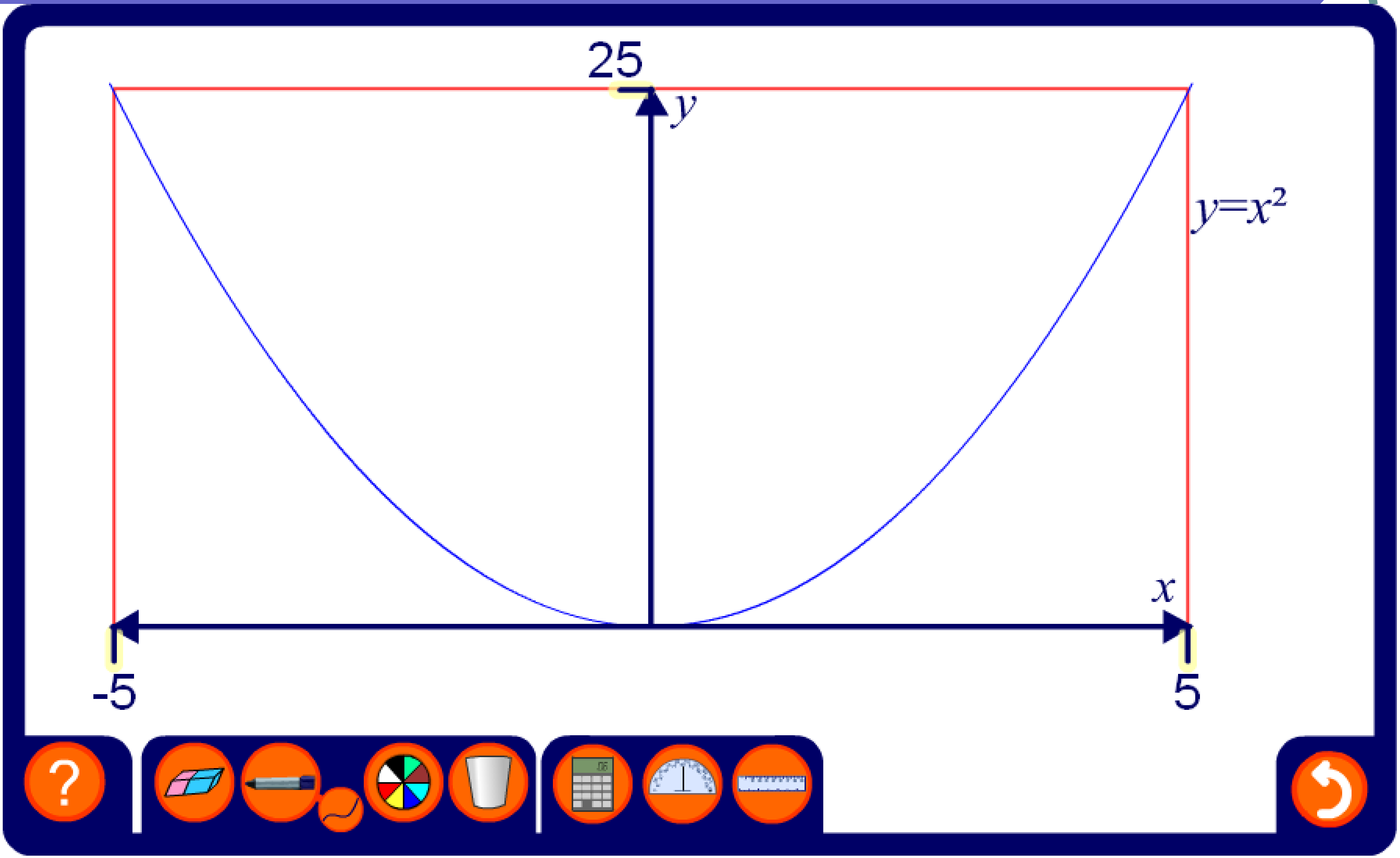
However the equation,

$$x^2 = 25$$

has 2 solutions,

$x = 5$ **or** $x = -5$

Squares and square roots from a graph



Cubes

The numbers 1, 8, 27, 64, and 125 are all:

Cube numbers

$1^3 = 1 \times 1 \times 1 = 1$ ‘1 cubed’ or ‘1 to the power of 3’

$2^3 = 2 \times 2 \times 2 = 8$ ‘2 cubed’ or ‘2 to the power of 3’

$3^3 = 3 \times 3 \times 3 = 27$ ‘3 cubed’ or ‘3 to the power of 3’

$4^3 = 4 \times 4 \times 4 = 64$ ‘4 cubed’ or ‘4 to the power of 3’

$5^3 = 5 \times 5 \times 5 = 125$ ‘5 cubed’ or ‘5 to the power of 3’

Cube roots

Finding the cube root is the inverse of finding the cube:

$$\begin{tikzcd}[column sep=huge] 5 \arrow[r, bend left, "\textbf{cubed}"] & 125 \arrow[l, bend left, "\textbf{cube rooted}"] \end{tikzcd}$$

We write

$$\sqrt[3]{125} = 5$$

The cube root of 125 is 5.

Index notation

We use index notation to show repeated multiplication by the same number.

For example

we can use index notation to write **$2 \times 2 \times 2 \times 2 \times 2$** as

Index or power

$$2^5$$

base

This number is read as ‘two to the power of five’.

$$2^5 = \quad 2 \times 2 \times 2 \times 2 \times 2 = \qquad 32$$

Index notation

Evaluate the following:

$6^2 = \quad 6 \times 6 = \quad 36$

$3^4 = \quad 3 \times 3 \times 3 \times 3 = \quad 81$

When we raise a negative number to an odd power the answer is negative.

$(-5)^3 = \quad -5 \times -5 \times -5 = \quad -125$

$2^7 = \quad 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 = \quad 128$

When we raise a negative number to an even power the answer is positive.

$(-1)^5 = \quad -1 \times -1 \times -1 \times -1 \times -1 = \quad -1$

$(-4)^4 = \quad -4 \times -4 \times -4 \times -4 = \quad 256$

INDICES

What are Indices?

- Indices provide a way of writing numbers in a more convenient form
- Indices is the plural of Index
- An Index is often referred to as a power

For example

$5 \times 5 \times 5 = 5^3$

$2 \times 2 \times 2 \times 2 = 2^4$

$7 \times 7 \times 7 \times 7 \times 7 = 7^5$

7 is the BASE NUMBER

5 is the INDEX

7^5 & 2^4 are numbers in INDEX FORM

Combining numbers

$$5 \times 5 \times 5 \times 2 \times 2 \times 2 \times 2$$

$$= 5^3 \times 2^4$$

We can not write this any more simply

Can ONLY do that if BASE NUMBERS are the same

INDEX LAW

IF ONLY **BASE NUMBERS** ARE **SAME**

Rule 1 : Multiplication

$2^6 \times 2^4 = 2^{10}$

$2^4 \times 2^2 = 2^6$

$3^5 \times 3^7 = 3^{12}$

General Rule

$a^m \times a^n = a^{m+n}$

What Goes In The Box ? 1

Simplify the expressions below :

(1) $6^4 \times 6^3$ $= 6^7$

(2) $9^7 \times 9^2$ $= 9^9$

(3) $11^6 \times 11$ $= 11^7$

(4) $14^9 \times 14^{12}$ $= 14^{21}$

(5) $27^{25} \times 27^{30}$ $= 27^{55}$

(6) $2^2 \times 2^3 \times 2^5$ $= 2^{10}$

(7) $8^7 \times 8^{10} \times 8$ $= 8^{18}$

(8) $5^{20} \times 5^{30} \times 5^{50}$ $= 5^{100}$

Rule 2 : Division

$2^6 \div 2^4 \quad = 3^2$

$2^5 \div 2^2 \quad = 2^3$

$3^5 \div 3^7 \quad = 3^{-2}$

General Rule

$a^m \div a^n = a^{m-n}$

What Goes In The Box ? 2

Simplify the expressions below :

(1) $5^{9} \div 5^{2}$ $=5^{7}$

(2) $7^{12} \div 7^{5}$ $= 7^{7}$

(3) $19^{6} \div 19$ $= 19^{5}$

(4) $36^{15} \div 36^{10}$ $= 36^{5}$

(5) $18^{40} \div 18^{20}$ $= 18^{20}$

(6) $2^{32} \div 2^{27}$ $= 2^{5}$

(7) $8^{70} \div 8^{39}$ $= 8^{31}$

(8) $5^{200} \div 5^{180}$ $= 5^{20}$

Rule 3 : Brackets

$(2^6)^2 = 2^6 \times 2^6 = 2^{12}$

$(3^5)^3 = 3^5 \times 3^5 \times 3^5 = 3^{15}$

General Rule

$(a^m)^n = a^{m \times n}$

Rule 4 : Negative Index Numbers

Simplify the expression below:

$5^{3} \div 5^{7}$ $\boxed{= 5^{-4}}$

To understand this result fully consider the following:

Write the original expression again as a quotient:

$$\frac{5^3}{5^7}$$

Expand the numerator and the denominator:

$$= \frac{5\times5\times5}{5\times5\times5\times5\times5\times5\times5}$$

Cancel out as many fives as possible:

$$= \frac{1}{5\times5\times5\times5}$$

Write as a power of five:

$$\boxed{= \frac{1}{5^4}}$$

Now compare the two results:

The result on the previous slide allows us to see the following results:

Turn the following powers into fractions:

(1) 2^{-3}

$$= \frac{1}{2^3}$$

$$= \frac{1}{8}$$

(2) 3^{-4}

$$= \frac{1}{3^4}$$

$$= \frac{1}{81}$$

(3) 10^{-6}

$$= \frac{1}{10^6}$$

$$= \frac{1}{1000000}$$

We can now write down our fourth rule of index numbers:

Rule 4 : For negative indices:.

$$a^{-m} = \frac{1}{a^m}$$

Reciprocals

A number raised to the power of –1 gives us the **reciprocal** of that number.

The reciprocal of a number is what we multiply the number by to get 1.

The reciprocal of a is $\dfrac{1}{a}$

The reciprocal of $\dfrac{a}{b}$ is $\dfrac{b}{a}$

Finding the reciprocals

Find the reciprocals of the following:

1) 6 — The reciprocal of 6 = $\frac{1}{6}$ or $6^{-1} = \frac{1}{6}$

2) $\frac{3}{7}$ — The reciprocal of $\frac{3}{7} = \frac{7}{3}$ or $\left(\frac{3}{7}\right)^{-1} = \frac{7}{3}$

3) 0.8 — $0.8 = \frac{4}{5}$ The reciprocal of $\frac{4}{5} = \frac{5}{4} = 1.25$

or $0.8^{-1} = 1.25$

Index laws for negative indices

Here is a summery of the index laws for negative indices.

$$x^{-1} = \frac{1}{x}$$ → The reciprocal of x is $\frac{1}{x}$

$$x^{-n} = \frac{1}{x^n}$$ → The reciprocal of x^n is $\frac{1}{x^n}$

More On Negative Indices

Simplify the expressions below leaving your answer as a positive index number each time:

(1) $$\frac{3^{-6}\times 3^{9}}{3^{-5}}$$

$$= 3^{-6+9-(-5)}$$

$$= 3^{-6+9+5}$$

$$= 3^{8}$$

(2) $$\frac{7^{-4}\times 7^{3}}{7^{8}\times 7^{-2}}$$

$$= \frac{7^{-4+3}}{7^{8+(-2)}}$$

$$= \frac{7^{-1}}{7^{6}}$$

$$= 7^{-1-6}$$

$$= 7^{-7}$$

$$= \frac{1}{7^{7}}$$

What Goes In The Box ? 3

Change the expressions below to fractions:

(1) 2^{-5}

$$= \frac{1}{32}$$

(2) 3^{-3}

$$= \frac{1}{27}$$

(3) $\dfrac{4^{-2}}{2^{-3}}$

$$= \frac{1}{2}$$

(4) $\dfrac{6^{-2}}{3^{-3}}$

$$= \frac{3}{4}$$

Simplify the expressions below leaving your answer with a positive index number at all times:

(5) $\dfrac{4^{-5} \times 4^{6}}{4^{-3}}$

$$= 4^4$$

(6) $\dfrac{7^{7} \times 7^{-6}}{7^{10} \times 7^{-11}}$

$$= 7^2$$

(7) $\dfrac{3^{2} \times 3^{-4} \times 3^{3}}{3^{6} \times 3^{-4} \times 3^{2}}$

$$= \frac{1}{3^3}$$

What Goes In The Box ? 4

Simplify the expressions below leaving your answer as a positive index number.

(1) $(7^4)^5$

$= 7^{20}$

(2) $(5^3)^{-6}$

$= \dfrac{1}{5^{18}}$

(3) $(10^{-7})^{-3}$

$= 10^{21}$

(4) $(8^2 \times 8^4)^3$

$= 8^{18}$

(5) $(7^{-3} \times 7^2)^{-5}$

$= 7^5$

(6) $(11^{-6} \times 11^{-5})^{-10}$

$= 11^{110}$

Rule 5 : Index of 1

Find the value of the following using your calculator:

6^1 47^1 0.9^1 -5^1 0^1

Any number raised to the power of 1 is equal to the number itself. In general, $x^1 = x$

Because of this we don't usually write the power when a number is raised to the power of 1.

Rule 6 : Index of 0

How could you get an answer of 3^0?

$$3^5 \div 3^5 = 3^{5-5} = 3^0$$

$$3^0 = 1$$

General Rule

$$a^0 = 1$$

Putting them together?

$$\frac{2^6 \times 2^4}{2^3} = \frac{2^{10}}{2^3} = 2^7$$

$$\frac{3^5 \times 3^7}{3^4} = \frac{3^{12}}{3^4} = 3^8$$

$$\frac{2^5 \times 2^3}{2^4 \times 2^2} = \frac{2^8}{2^6} = 2^2$$

Works with algebra too!

$$a^6 \times a^4 = a^{10}$$

$$b^5 \times b^7 = b^{12}$$

$$\frac{c^5 \times c^3}{c^4} = \frac{c^8}{c^4} = c^4$$

$$\frac{a^5 \times a^3}{a^4 \times a^6} = \frac{a^8}{a^{10}} = a^{-2}$$

..and a mixture…

$2a^3 \times 3a^4 = 2 \times 3 \times a^3 \times a^4 = 6a^7$

$8a^6 \div 4a^4 = (8 \div 4) \times (a^6 \div a^4) = 2a^2$

$$\frac{\cancel{8}^{2}a^{\cancel{6}^{2}}}{\cancel{4}a^{\cancel{4}}}$$

Rule 7 : Fractional indices

Indices can also be fractional. Suppose we have $9^{\frac{1}{2}}$.

$$9^{\frac{1}{2}} \times 9^{\frac{1}{2}} = 9^{\frac{1}{2}+\frac{1}{2}} = 9^1 = 9$$

But, $\sqrt{9} \times \sqrt{9} = 9$

Because 3 × 3 = 9

In general, $\boxed{x^{\frac{1}{2}} = \sqrt{x}}$

Similarly, $8^{\frac{1}{3}} \times 8^{\frac{1}{3}} \times 8^{\frac{1}{3}} = 8^{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}} = 8^1 = 8$

But, $\sqrt[3]{8} \times \sqrt[3]{8} \times \sqrt[3]{8} = 8$

Because
2 × 2 × 2 = 8

In general, $\boxed{x^{\frac{1}{3}} = \sqrt[3]{x}}$

Fractional indices

What is the value of $25^{\frac{3}{2}}$?

We can think of $25^{\frac{3}{2}}$ as $25^{\frac{1}{2} \times 3}$.

Using the rule that $(x^a)^b = x^{ab}$ we can write

$$25^{\frac{1}{2} \times 3} = (\sqrt{25})^3$$

$$= (5)^3$$

$$= 125$$

In general,

$$\boxed{x^{\frac{m}{n}} = (\sqrt[n]{x})^m}$$

Evaluate the following

1) $49^{\frac{1}{2}}$ $\qquad$ $49^{\frac{1}{2}} = \sqrt{49} = 7$

2) $1000^{\frac{2}{3}}$ $\qquad$ $1000^{\frac{2}{3}} = (\sqrt[3]{1000})^2 = 10^2 = 100$

3) $8^{-\frac{1}{3}}$ $\qquad$ $8^{-\frac{1}{3}} = \dfrac{1}{8^{\frac{1}{3}}} = \dfrac{1}{\sqrt[3]{8}} = \dfrac{1}{2}$

4) $64^{-\frac{2}{3}}$ $\qquad$ $64^{-\frac{2}{3}} = \dfrac{1}{64^{\frac{2}{3}}} = \dfrac{1}{(\sqrt[3]{64})^2} = \dfrac{1}{4^2} = \dfrac{1}{16}$

5) $4^{\frac{5}{2}}$ $\qquad$ $4^{\frac{5}{2}} = (\sqrt{4})^5 = 2^5 = 32$

Index laws for fractional indices

Here is a summary of the index laws for fractional indices.

$$\boxed{x^{\frac{1}{2}} = \sqrt{x}}$$

$$\boxed{x^{\frac{1}{n}} = \sqrt[n]{x}}$$

$$\boxed{x^{\frac{m}{n}} = \sqrt[n]{x^m} \quad \text{or} \quad (\sqrt[n]{x})^m}$$

POWER OF TEN

Powers of ten

Our decimal number system is based on **powers of ten**.

We can write powers of ten using **index notation**.

$10 = 10^1$

$100 = 10 \times 10 = 10^2$

$1000 = 10 \times 10 \times 10 = 10^3$

$10\,000 = 10 \times 10 \times 10 \times 10 = 10^4$

$100\,000 = 10 \times 10 \times 10 \times 10 \times 10 = 10^5$

$1\,000\,000 = 10 \times 10 \times 10 \times 10 \times 10 \times 10 = 10^6$ …

Negative powers of ten

Any number raised to the power of 0 is 1, so

$$1 = 10^0$$

Decimals can be written using negative powers of ten

$$0.1 = \frac{1}{10} = \frac{1}{10^1} = 10^{-1}$$

$$0.01 = \frac{1}{100} = \frac{1}{10^2} = 10^{-2}$$

$$0.001 = \frac{1}{1000} = \frac{1}{10^3} = 10^{-3}$$

$$0.0001 = \frac{1}{10000} = \frac{1}{10^4} = 10^{-4}$$

$$0.00001 = \frac{1}{100000} = \frac{1}{10^5} = 10^{-5}$$

$$0.000001 = \frac{1}{1000000} = \frac{1}{10^6} = 10^{-6} \ldots$$

Very large numbers

Use you calculator to work out the answer to
40 000 000 × 50 000 000.

Your calculator may display the answer as:

$2\ \times10^{15}$	,	$2\ E^{15}$	or	$2\ ^{15}$

What does the 15 mean?

The 15 means that the answer is 2 followed by 15 zeros or:

$\boxed{2 \times 10^{15}}$ = 2 000 000 000 000 000

Very small numbers

Use you calculator to work out the answer to
0.0002 ÷ 30 000 000.

Your calculator may display the answer as:

1.5 ×10 –12 , 1.5 E –12 or 1.5 –12

What does the –12 mean?

The –12 means that the 15 is **divided** by 1 followed by 12 zeros.

1.5×10^{-12} = 0.000000000002

STANDARD FORM

Standard form

2×10^{15} and 1.5×10^{-12} are examples of a number written in **standard form**.

Numbers written in standard form have two parts:

A number between 1 and 10	×	A power of 10

This way of writing a number is also called **standard index form** or **scientific notation**.

Any number can be written using standard form, however it is usually used to write very large or very small numbers.

Standard form – writing large numbers

For example, the mass of the planet earth is about
5 970 000 000 000 000 000 000 000 kg.

We can write this in standard form
as a number between 1 and 10
multiplied by a power of 10.

$$\boxed{5.97 \times 10^{24} \text{ kg}}$$

A number between 1 and 10

A power of ten

Standard form – writing large numbers

How can we write these numbers in standard form?

80 000 000 = 8×10^7

230 000 000 = 2.3×10^8

724 000 = 7.24×10^5

6 003 000 000 = 6.003×10^9

371.45 = 3.7145×10^2

Standard form – writing large numbers

These numbers are written in standard form. How can they be written as ordinary numbers?

$5 \times 10^{10} =$ **50 000 000 000**

$7.1 \times 10^{6} =$ **7 100 000**

$4.208 \times 10^{11} =$ **420 800 000 000**

$2.168 \times 10^{7} =$ **21 680 000**

$6.7645 \times 10^{3} =$ **6764.5**

Standard form – writing large numbers

We can write very small numbers using negative powers of ten.

For example, the width of this shelled amoeba is 0.00013 m.

We write this in standard form as:

$$\boxed{1.3 \times 10^{-4} \text{ m.}}$$

A number between 1 and 10

A negative power of 10

Standard form – writing small numbers

How can we write these numbers in standard form?

0.0006 = 6×10^{-4}

0.00000072 = 7.2×10^{-7}

0.0000502 = 5.02×10^{-5}

0.0000000329 = 3.29×10^{-8}

0.001008 = 1.008×10^{-3}

Standard form – writing small numbers

These numbers are written in standard form. How can they be written as ordinary numbers?

$8 \times 10^{-4} =$ **0.0008**

$2.6 \times 10^{-6} =$ **0.0000026**

$9.108 \times 10^{-8} =$ **0.00000009108**

$7.329 \times 10^{-5} =$ **0.00007329**

$8.4542 \times 10^{-2} =$ **0.084542**

Ordering numbers in standard form

Write these numbers in order from smallest to largest:

5.3×10^{-4}, 6.8×10^{-5}, 4.7×10^{-3}, 1.5×10^{-4}.

To order numbers that are written in standard form start by comparing the powers of 10.

Remember, 10^{-5} is smaller than 10^{-4}. That means that 6.8×10^{-5} is the smallest number in the list.

When two or more numbers have the same power of ten we can compare the number parts. 5.3×10^{-4} is larger than 1.5×10^{-4} so the correct order is:

6.8×10^{-5}, 1.5×10^{-4}, 5.3×10^{-4}, 4.7×10^{-3}

Ordering planet sizes

Diameters in kilometres

- Jupiter 1.4×10^5
- Earth 1.3×10^4
- Mars 6.8×10^3
- Pluto 2.4×10^3
- Saturn 1.2×10^5
- Venus 1.2×10^4
- Mercury 4.9×10^3
- Uranus 5.2×10^4
- Neptune 4.9×10^4

Calculations involving standard form

What is 2×10^5 multiplied by 7.2×10^3 ?

To multiply these numbers together we can multiply the number parts together and then the powers of ten together.

$$2 \times 10^5 \times 7.2 \times 10^3 = (2 \times 7.2) \times (10^5 \times 10^3)$$

$$= 14.4 \times 10^8$$

This answer is **not** in standard form and must be converted!

$$14.4 \times 10^8 = 1.44 \times 10 \times 10^8$$

$$= 1.44 \times 10^9$$

Calculations involving standard form

What is 1.2×10^{-6} divided by 4.8×10^{7} ?

To divide these numbers we can divide the number parts and then divide the powers of ten.

$$(1.2 \times 10^{-6}) \div (4.8 \times 10^{7}) = (1.2 \div 4.8) \times (10^{-6} \div 10^{7})$$

$$= 0.25 \times 10^{-13}$$

This answer is **not** in standard form and must be converted.

$$0.25 \times 10^{-13} = 2.5 \times 10^{-1} \times 10^{-13}$$

$$= 2.5 \times 10^{-14}$$

Calculations involving standard form

How long would it take a space ship travelling at an average speed of 2.6×10^3 km/h to reach Mars 8.32×10^7 km away?

Rearrange $\text{speed} = \dfrac{\text{distance}}{\text{time}}$ to give $\text{time} = \dfrac{\text{distance}}{\text{speed}}$

$$\text{Time to reach Mars} = \frac{8.32 \times 10^7}{2.6 \times 10^3}$$

$$= 3.2 \times 10^4 \text{ hours}$$

This is 8.32 ÷ 2.6

This is $10^7 \div 10^3$

Calculations involving standard form

Use your calculator to work out how long 3.2×10^4 hours is in years.

You can enter 3.2×10^4 into your calculator using the EXP key:

3 **.** **2** **EXP** **4**

Divide by 24 to give the equivalent number of days.

Divide by 365 to give the equivalent number of years.

3.2×10^4 hours is over 3½ years.

CONCLUSIONS

The Rules Of Indices

Rule 1 : Multiplication of Indices.

$$a^{n} \times a^{m} = \ldots\ldots\ldots$$

Rule 2 : Division of Indices.

$$a^{n} \div a^{m} = \ldots\ldots\ldots$$

Rule 4 : For Powers Of Index Numbers.

$(a^{m})^{n} = \ldots\ldots$

Rule 3 : For negative indices

$$a^{-m} = \ldots\ldots\ldots$$

$x^0 = 1$ (for $x \neq 0$)

$$x^{\frac{1}{2}} = \sqrt{x}$$

$$x^{\frac{1}{n}} = \sqrt[n]{x}$$

$$x^{\frac{m}{n}} = \sqrt[n]{x^m} \quad \text{or} \quad (\sqrt[n]{x})^m$$

The Rules Of Standard Form

A number between 1 and 10	×	A power of 10