CHAPTER 3

PART 2

Logarithm

Logarithms

- We've discussed exponential equations of the form

$$y = b^x \qquad (b > 0,\ b \neq 1)$$

- But what about solving the same equation for y?
- You may recall that y is called the logarithm of x to the base b, and is denoted $\log_b x$.
- Logarithm of x to the base b

$$y = \log_b x \quad \text{if and only if} \quad x = b^y \qquad (x > 0)$$

Logarithmic Notation

$\log x = \log_{10} x$ Common logarithm

$\ln x = \log_e x$ Natural logarithm

Logarithmic Function

Logarithmic Function

- The function defined by

$$f(x) = \log_b x \qquad (b > 0,\ b \neq 1)$$

  is called the logarithmic function with base
- The domain of *f* is the set of all positive numbers.

Properties of Logarithmic Functions

- The logarithmic function

$$y = \log_b x \qquad (b > 0, b \neq 1)$$

has the following properties:

1. Its domain is $(0, \infty)$.
2. Its graph passes through the point $(1, 0)$.
3. It is continuous on $(0, \infty)$.
4. It is increasing on $(0, \infty)$ if $b > 1$
   and decreasing on $(0, \infty)$ if $b < 1$.

Logarithmic Functions

If $a > 0$ and $a \neq 1$, the exponential function $f(x) = a^x$ is either increasing or decreasing and so it is one-to-one by the Horizontal Line Test. It therefore has an inverse function f^{-1}, which is called the **logarithmic function with base *a*** and is denoted by $\log_a$.

Logarithmic Functions

If we use the formulation of an inverse function given below

$$f^{-1}(x) = y \quad \Longleftrightarrow \quad f(y) = x$$

then we have

$$\boxed{\log_a x = y \quad \Longleftrightarrow \quad a^y = x}$$

Thus, if $x > 0$, then $\log_a x$ is the exponent to which the base a must be raised to give x.

E.g. 1—Graphing a Logarithmic Function

- Use a graphing calculator to graph

$$f(x) = \log_6 x$$

  - Calculators don't have a key for $\log_{6.}$
  - So, we use the Change of Base Formula to write:

$$f(x) = \log_6 x = \frac{\ln x}{\ln 6}$$

Solving Exponential and Logarithmic Equations

- For some students, logarithms are confusing. The following box gives a list of common student errors when using logarithms.

Common Errors with Logarithms

Each of the following represent common errors when calculating with logarithms:

$$\log_b(m+n) \neq \log_b(m) + \log_b(n)$$

$$\log_b(m-n) \neq \log_b(m) - \log_b(n)$$

$$\log_b\left(\frac{m}{n}\right) \neq \frac{\log_b(m)}{\log_b(n)}$$

$$\log_b(m \cdot n) \neq (\log_b(m))(\log_b(n))$$

$$\log_b(m \cdot n^p) \neq p \log_b(m \cdot n)$$

Laws of Logarithms

Laws of Logarithms

Since logarithms are exponents, the Laws of Exponents give rise to the Laws of Logarithms.

Laws of Logarithms

If m and n are positive numbers, then

1. $\log_b mn = \log_b m + \log_b n$
2. $\log_b \frac{m}{n} = \log_b m - \log_b n$
3. $\log_b m^n = n \log_b m$
4. $\log_b 1 = 0$
5. $\log_b b = 1$

E.g. 1—Using the Laws to Evaluate Expressions

Evaluate each expression.

(a) $\log_4 2 + \log_4 32$

(b) $\log_2 80 - \log_2 5$

(c) $-\frac{1}{3}\log 8$

E.g. 1—Evaluating Expressions Example (a)

$$\log_4 2 + \log_4 32$$

$$= \log_4 (2 \cdot 32)$$ Law 1

$$= \log_4 64$$

$$= 3$$ Because $64 = 4^3$

$$\log_2 80 - \log_2 5$$

$$= \log_2\left(\frac{80}{5}\right)$$ Law 2

$$= \log_2 16$$

$$= 4$$ Because $16 = 2^4$

Expanding and Combining Logarithmic Expressions

Expanding Logarithmic Expressions

The laws of logarithms allow us to write the logarithm of a product or a quotient as the sum or difference of logarithms.

- This process—called expanding a logarithmic expression—is illustrated in the next example.

E.g. 1—Expanding Logarithmic Expressions

Use the Laws of Logarithms to expand each expression.

(a) $\log_2(6x)$

(b) $\log_5\left(x^3y^6\right)$

(c) $\ln\left(\dfrac{ab}{\sqrt[3]{c}}\right)$

E.g. 1—Expanding Log Expressions Examples (a) & (b)

$$\log_2(6x) = \log_2 6 + \log_2 x$$ Law 1

$$\log_5\left(x^3y^6\right) = \log_5 x^3 + \log_5 y^6$$ Law 1

$$= 3\log_5 x + 6\log_5 y$$ Law 3

E.g. 1—Expanding Log Expressions

$$\ln\left(\frac{ab}{\sqrt[3]{c}}\right) = \ln(ab) - \ln\sqrt[3]{c} \qquad \text{Law 2}$$

$$= \ln a + \ln b - \ln c^{1/3} \qquad \text{Law 1}$$

$$= \ln a + \ln b - \tfrac{1}{3}\ln c \qquad \text{Law 3}$$

The laws of logarithms also allow us to reverse the process of expanding done in Example 2.

- That is, we can write sums and differences of logarithms as a single logarithm.
- This process—called combining logarithmic expressions—is illustrated in the next example.

E.g. 2—Combining Logarithmic Expressions

Combine $3 \log x + \frac{1}{2} \log(x + 1)$ into a single logarithm.

$$\begin{aligned} &3 \log x + \tfrac{1}{2} \log(x+1) \\ &= \log x^3 + \log(x+1)^{1/2} && \text{(Law 3)} \\ &= \log\left(x^3 (x+1)^{1/2}\right) && \text{(Law 1)} \end{aligned}$$

E.g. 3—Combining Logarithmic Expressions

Combine 3 ln *s* + ½ ln *t* – 4 ln(t^2 + 1) into a single logarithm.

$$3\ln s + \tfrac{1}{2}\ln t - 4\ln(t^2+1)$$

$$= \ln s^3 + \ln t^{1/2} - \ln(t^2+1)^4 \qquad \text{(Law 3)}$$

$$= \ln(s^3 t^{1/2}) - \ln(t^2+1)^4 \qquad \text{(Law 1)}$$

$$= \ln\left(\frac{s^3\sqrt{t}}{(t^2+1)^4}\right) \qquad \text{(Law 2)}$$

Warning

Although the Laws of Logarithms tell us how to compute the logarithm of a **product or a quotient**, there is no corresponding rule for the logarithm of a **sum or a difference**.

Warning

For instance,

$$\log_a (x + y) \cancel{=} \log_a x + \log_a y$$

- In fact, we know that the right side is equal to $\log_a(xy)$.

Also, don’t improperly simplify quotients or powers of logarithms.

- For instance,

$$\frac{\log 6}{\log 2} \neq \log\left(\frac{6}{2}\right)$$

$$(\log_2 x)^3 \neq 3\log_2 x$$

Change of Base

How to Convert Between Different Bases

Calculators and computers generally don't calculate the logarithm to the base 2, but we can use a method to make this easy.

Take for example, the equation $2^x = 32$. We use the change of base formula!! We can change any base to a different base any time we want. The most used bases are obviously base 10 and base *e* because they are the only bases that appear on your calculator!

Change of Base

For some purposes, we find it useful to change from logarithms in one base to logarithms in another base.

- Suppose we are given $\log_a x$ and want to find $\log_b x$.

Change of Base

Let

$$y = \log_b x$$

- We write this in exponential form and take the logarithm, with base *a*, of each side—as follows.

Change of Base

$b^y = x$ Exponential form

$\log_a \left(b^y \right) = \log_a x$ Taking $\log_a$ of each side

$y \log_a b = \log_a x$ Law 3

$$y = \frac{\log_a x}{\log_a b}$$

- This proves the following formula.

Change of Base Formula

$$\log_b x = \frac{\log_a x}{\log_a b}$$

- In particular, if we put $x = a$, then $\log_a a = 1$, and this formula becomes:

$$\log_b a = \frac{1}{\log_a b}$$

We can now evaluate a logarithm to any base by:

1. Using the Change of Base Formula to express the logarithm in terms of common logarithms or natural logarithms.
2. Using a calculator.

Change-of-Base Formula

Let a, b, and x be positive real numbers such that $a \neq 1$ and $b \neq 1$. Then $\log_a x$ can be converted to a different base using any of the following formulas.

Base b	*Base 10*	*Base e*
$\log_a x = \dfrac{\log_b x}{\log_b a}$	$\log_a x = \dfrac{\log_{10} x}{\log_{10} a}$	$\log_a x = \dfrac{\ln x}{\ln a}$

E.g. 1—Evaluating Logarithms

Use the Change of Base Formula and common or natural logarithms to evaluate each logarithm, correct to five decimal places.

(a) $\log_8 5$

(b) $\log_9 20$

E.g. 1—Evaluating Logarithms Example (a)

We use the Change of Base Formula with *b* = 8 and *a* = 10:

$$\log_8 5 = \frac{\log_{10} 5}{\log_{10} 8}$$

$$\approx 0.77398$$

E.g. 1—Evaluating Logarithms Example (b)

We use the Change of Base Formula with $b = 9$ and $a = e$:

$$\log_9 20 = \frac{\ln 20}{\ln 9}$$

$$\approx 1.36342$$

Common Logarithms

A common logarithm has a base of 10.

If there is no base given explicitly, it is common.

You can easily find common logs of powers of ten.

You can use your calculator to evaluate common logs.

Properties of Common Logarithms

General Properties

1) $\log_b 1 = 0$

2) $\log_b b = 1$

3) $\log_b b^x = x$

4) $10^{\log_b x} = x$

3 and 4 are inverse properties

Common Logarithms

1) log 1 = 0

2) log 10 = 1

3) $\log 10^x = x$

4) $10^{\log x} = x$

EXAMPLES

Examples

- **Solve** $\log_3 x = 4$ **for** x**:**

<u>Solution</u>

- **By definition,** $\log_3 x = 4$ **implies** $x = 3^4 = 81$**.**

Examples

- **Solve** $\log_{16}4 = x$ **for** x**:**

Solution

- $\log_{16}4 = x$ **is equivalent to** $4 = 16^x = (4^2)^x = 4^{2x}$**, or** $4^1 = 4^{2x}$**, from which we deduce that**

$$2x = 1$$

$$x = \frac{1}{2}$$

Examples

- Solve $\log_x 8 = 3$ for x:

Solution

- By definition, we see that $\log_x 8 = 3$ is equivalent to

$$8 = 2^3 = x^3$$

$$x = 2$$

Examples

- **Given that** $\log 2 \approx 0.3010$, $\log 3 \approx 0.4771$, **and** $\log 5 \approx 0.6990$, **use the laws of logarithms to find**

$$\begin{aligned}\log 15 &= \log 3 \cdot 5 \\ &= \log 3 + \log 5 \\ &\approx 0.4771 + 0.6990 \\ &= 1.1761\end{aligned}$$

Examples

- **Given that** $\log 2 \approx 0.3010$, $\log 3 \approx 0.4771$, **and** $\log 5 \approx 0.6990$, **use the laws of logarithms to find**

$$
\begin{aligned}
\log 7.5 &= \log(15/2) \\
&= \log(3 \cdot 5/2) \\
&= \log 3 + \log 5 - \log 2 \\
&\approx 0.4771 + 0.6990 - 0.3010 \\
&= 0.8751
\end{aligned}
$$

Examples

- **Given that** $\log 2 \approx 0.3010$**,** $\log 3 \approx 0.4771$**, and** $\log 5 \approx 0.6990$**, use the laws of logarithms to find**

$$\begin{aligned} \log 81 &= \log 3^4 \\ &= 4 \log 3 \\ &\approx 4(0.4771) \\ &= 1.9084 \end{aligned}$$

Examples

- **Given that** $\log 2 \approx 0.3010$**,** $\log 3 \approx 0.4771$**, and** $\log 5 \approx 0.6990$**, use the laws of logarithms to find**

$$\begin{aligned} \log 50 &= \log 5 \cdot 10 \\ &= \log 5 + \log 10 \\ &\approx 0.6990 + 1 \\ &= 1.6990 \end{aligned}$$

Examples

- **Expand** **and** **simplify** **the expression:**

$$\log_3 x^2 y^3 = \log_3 x^2 + \log_3 y^3$$
$$= 2\log_3 x + 3\log_3 y$$

Examples

- **Expand** **and** **simplify** **the expression:**

$$\log_2 \frac{x^2+1}{2^x} = \log_2\left(x^2+1\right) - \log_2 2^x$$

$$= \log_2\left(x^2+1\right) - x\log_2 2$$

$$= \log_2\left(x^2+1\right) - x$$

Examples

- **Expand** and **simplify** the expression:

$$\ln\frac{x^2\sqrt{x^2-1}}{e^x} = \ln\frac{x^2(x^2-1)^{1/2}}{e^x}$$

$$= \ln x^2 + \ln(x^2-1)^{1/2} - \ln e^x$$

$$= 2\ln x + \frac{1}{2}\ln(x^2-1) - x\ln e$$

$$= 2\ln x + \frac{1}{2}\ln(x^2-1) - x$$

Examples

- **Use the properties of logarithms to solve the equation for *x*:**

$$\log_3(x+1) - \log_3(x-1) = 1$$

$$\log_3 \frac{x+1}{x-1} = 1$$

Law 2

$$\frac{x+1}{x-1} = 3^1 = 3$$

Definition of logarithms

$$x+1 = 3(x-1)$$

$$x+1 = 3x-3$$

$$4 = 2x$$

$$x = 2$$

Examples

- **Use the properties of logarithms to solve the equation for x:**

$$\log x + \log(2x-1) = \log 6$$

$$\log x + \log(2x-1) - \log 6 = 0$$

$$\log \frac{x(2x-1)}{6} = 0$$

Laws 1 and 2

$$\frac{x(2x-1)}{6} = 10^0 = 1$$

Definition of logarithms

$$x(2x-1) = 6$$

$$2x^2 - x - 6 = 0$$

$$(2x+3)(x-2) = 0$$

$$x = 2$$

$x = -\frac{3}{2}$ **is out of the domain of** $\log x$**, so it is discarded.**

Examples

- **Solve the equation** $2e^{x+2} = 5$.

Solution

- **Divide both sides of the equation by** 2 **to obtain:**

$$e^{x+2} = \frac{5}{2} = 2.5$$

- **Take the natural logarithm of each side of the equation and solve:**

$$\ln e^{x+2} = \ln 2.5$$

$$(x+2)\ln e = \ln 2.5$$

$$x+2 = \ln 2.5$$

$$x = -2 + \ln 2.5$$

$$x \approx -1.08$$

Examples

- **Solve the equation** $5 \ln x + 3 = 0$.

Solution

- **Add** -3 **to both sides of the equation and then divide both sides of the equation by** 5 **to obtain:**

$$5 \ln x = -3$$

$$\ln x = -\frac{3}{5} = -0.6$$

and so:

$$e^{\ln x} = e^{-0.6}$$

$$x = e^{-0.6}$$

$$x \approx 0.55$$