Matrix

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Chapter 1

Matrix

1.1 Introduction

A matrix is an $m \times n$ array of numbers arranged in m rows and n columns. The matrix is then described as being of order $m \times m$. Eq.1.1 illustrates a matrix with m rows and n columns.

$$\underline{a} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & a_{24} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & a_{44} & \dots & a_{3n} \\ a_{41} & a_{42} & a_{43} & a_{44} & \dots & a_{4n} \\ . & . & . & . & \dots & . \\ a_{m1} & a_{m2} & a_{m3} & a_{m4} & \dots & a_{mn} \end{pmatrix} \tag{1.1}$$

If $m \neq n$ in matrix Eq.1.1 , the matrix is called rectangular. If $m = 1$ and $n > 1$, the elements of Eq.1.1 form a single row called a row matrix. If $m > 1$, and $n = 1$ the elements form single columns called a columns matrix. If $m = n$, the array is called a square matrix. Matrix is usually denoted by [a] or d. Row matrices and rectangular matrices are denoted by using brackets (j, and column matrices are denoted by using braces {}. For simplicity, matrices (row, column, or rectangular) are often denoted by using a line under a variable instead of surrounding it with brackets or braces. The order of the matrix should then be apparent from the context of its use. The force and displacement matrices used in structural analysis are column matrices, whereas the stiffness matrix is a square matrix.

We represent the elements by a_{ij}, where the subcripts i and j indicate the row number and the column number.

A rectangular matix a is given by :

$$\underline{a} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 4 \\ 6 & 2 & 3 \end{pmatrix} \tag{1.2}$$

where a has three rows and three columns.

A square matrix b is given by:

$$\underline{b} = \begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix} \tag{1.3}$$

where b has two rows and two columns. A row matrix c is given by:

$$\underline{c} = \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix} \tag{1.4}$$

where c has one rows and three columns. A columns matrix b is given by:

$$\underline{d} = \begin{pmatrix} 3 & 4 & 1 & 2 \end{pmatrix} \tag{1.5}$$

where d has four rows and one columns. Matrices and matrix notation are often used to express algebraic equations in compact form and are frequently used in the finite element formulation of equations.

1.2 Matrix Operations

We will now present some common matrix operations that will be used in this text.

1.2.1 Multiplication of a Matrix by a Scalar

If we have a scalar k and a matrix c then the product $a = k \times \underline{c}$ is given by

$$d_{ij} = Ka_{ij} \tag{1.6}$$

e.g

$$\underline{a} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 4 \\ 6 & 2 & 3 \end{pmatrix} k = 4$$

The product $d_{ij} = Ka_{ij}$ is

$$\underline{d} = \begin{pmatrix} 4 & 8 & 12 \\ 4 & 8 & 16 \\ 24 & 8 & 12 \end{pmatrix}$$

Note that if d is of order $m \times n$,then a is also of order $m \times n$

1.2.2 Addition of a Matrix

Matrices of the same order can be added together by summing corresponding elements of the matrices. Subtractions is performed in similar manner. Matrices of unlike order cannot be added or subtracted.

$$\underline{c} = \underline{a} + \underline{b} = \underline{b} + \underline{a} \tag{1.7}$$

$$\underline{a} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 4 \\ 6 & 2 & 3 \end{pmatrix} \underline{b} = \begin{pmatrix} 7 & 4 & 4 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix}$$

$$\underline{c} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 4 \\ 6 & 2 & 3 \end{pmatrix} + \begin{pmatrix} 7 & 4 & 4 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix} = \begin{pmatrix} 8 & 6 & 7 \\ 3 & 6 & 9 \\ 9 & 7 & 9 \end{pmatrix}$$

Again,remember that the matrix $\underline{\text{a}}$,$\underline{\text{b}}$ and $\underline{\text{c}}$ must all be same. For e.g., a 2×2 marix cannot be added to a 3×3 matrix.

1.2.3 Multiplications of Matrix

For two matrices $\underline{\text{a}}$ and $\underline{\text{b}}$ to be multiplied in the order shown in Eq.1.8 , the number of columns in $\underline{\text{a}}$ must equal the number of rows in b. For

$$\underline{c} = \underline{a} \times \underline{b} \tag{1.8}$$

If $\underline{\text{a}}$ is an M x N matrix, then $\underline{\text{b}}$ must have n rows. Using subscript notation, we can write the product of matrices $\underline{\text{a}}$ and $\underline{\text{b}}$ as

$$[c_{ij}] = \sum_{e=1}^{n} a_{ie} b_{ej} \tag{1.9}$$

where n is the total number of columns in $\underline{\text{a}}$ or of rows in $\underline{\text{b}}$. For matrix $\underline{\text{a}}$ of order 2x2 and matrix $\underline{\text{b}}$ of order 2x2, after multiplying the two matrices, we have

$$\underline{c} = \begin{pmatrix} a_{11} \times b_{11} + a_{12} \times b_{21} & a_{11} \times b_{12} + a_{12} \times b_{22} \\ a_{21} \times b_{11} + a_{22} \times b_{21} & a_{21} \times b_{12} + a_{22} \times b_{22} \end{pmatrix}. \tag{1.10}$$

for example, let:

$$\underline{a} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 4 \\ 6 & 2 & 3 \end{pmatrix} \underline{b} = \begin{pmatrix} 7 & 4 & 4 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix}$$

$$\underline{c} = \underline{a} \times \underline{b} = \begin{pmatrix} 20 & 27 & 32 \\ 23 & 32 & 38 \\ 55 & 47 & 52 \end{pmatrix} \tag{1.11}$$

In general, matrix multiplication is not commutative; that is,

$$\underline{ab} \neq \underline{ba} \tag{1.12}$$

The validity of the products of two matrices $\underline{a}$ and $\underline{b}$ is commanly illustrated by

$$\underline{a} \quad \underline{b} = \underline{c}(i \times e)(e \times j)(i \times j)$$

where the product matrix $\underline{c}$ will be of order $i \times j$; that is, it will have the same number of rows as matrix $\underline{a}$ and the same number of columns as matrix $\underline{b}$.

1.2.4 Transpose of Matrix

Any matrix, whether a row, column, or rectangular matrix, can be transposed. This operation is frequently used in finite element equation formulations. The transpose of a matrix $\underline{a}$ is commonly denoted by $\underline{a}^T$. The superscript T is used to denote the transpose of matrix throughout this text. The transpose of matrix is obtained by interchanging rows and columns, that is, the first row becomes the first column, the second row becomes the second column, and so on.. For the transpose of matrix $\underline{a}$

$$[a_{ij}] = [a_{ji}]^T \tag{1.13}$$

For example, if we let

$$\underline{a} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 4 \\ 6 & 2 & 3 \end{pmatrix} \ \underline{a}^T = \begin{pmatrix} 1 & 1 & 6 \\ 2 & 2 & 2 \\ 3 & 4 & 3 \end{pmatrix}$$

Where we have interchanged the rows and columns of $\underline{a}$ to obtain its transpose. Another important relationship that involves the transpose is

$$(\underline{ab})^T = \underline{b}^T \underline{a}^T \tag{1.14}$$

That is, the transpose of the product of matrices $\underline{a}$ and b is equal to the transpose of the latter matrix [b multiplied by the transpose of matrix $\underline{a}$ in that order, provided the order of the initial matrices continues to satisfy the rule for matrix multiplication, Eq.1.13 . In general, this property holds for any number of matrices; that is:

$$(\underline{abc}...\underline{k})^T = \underline{k}^T ... \underline{c}^T \underline{b}^T \underline{a}^T \tag{1.15}$$

Note that the transpose of a column matrix is a row matrix. As a numerical example of the use of Eq.1.14 , let:

$$\underline{a} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 4 \\ 6 & 2 & 3 \end{pmatrix} \underline{c} = \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix}$$

First

$$\underline{ab} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 4 \\ 6 & 2 & 3 \end{pmatrix} \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 14 \\ 15 \\ 29 \end{pmatrix}$$

Then

$$(\underline{ab})^T = \begin{pmatrix} 14 & 15 & 29 \end{pmatrix} \tag{1.16}$$

Because $\underline{b}^T$ and $\underline{a}^T$ can be multiplied according to the rule for matrix multiplication, We have

$$\underline{b}^T \underline{a}^T = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 4 \\ 6 & 2 & 3 \end{pmatrix} \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 14 & 15 & 29 \end{pmatrix} \tag{1.17}$$

Hence, on comparing Eq.1.16 and Eq.1.17 , we have shown (for this case) the validity of Eq.1.14 . A simple proof of the general validity of Eq.1.14 is left to your discretion.

1.2.5 Symmetric Matrix

If a square matrix is equal to its transpose, it is called a symmetric matrix, that is,

$$\underline{a} = \underline{a}^T \tag{1.18}$$

if then $\underline{\text{a}}$ is a symmetric matrix. As an example,

$$\underline{a} = \begin{pmatrix} 3 & 1 & 2 \\ 1 & 4 & 0 \\ 2 & 0 & 3 \end{pmatrix} \tag{1.19}$$

is a symmetric matrix because each element a_{ij} equals a_{ji} for $i \neq j$. In Eq.1.19 , note that the main diagonal running from the upper left corner to the lower right corner is the line of symmetry of the symmetric matrix $\underline{\text{a}}$. Remember that the only a square matrix can be symmetric.

1.2.6 Unit Matrix

The unit (or identity) matrix I is such that

$$\underline{aI} = \underline{Ia} = \underline{a} \tag{1.20}$$

The unit matrix acts in the same way that the number one acts in conventional multiplication. The unit matrix is always a square matrix of any possible order with each element of the main diagonal equal to one and all other elements equals to zero. For example, the 3×3 unit matrix is given by

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{1.21}$$

1.2.7 Inverse of a Matrix

The inverse of a matrix is a matrix such that

$$\underline{a}^{-1}\underline{a} = \underline{aa}^{-1} = \underline{I} \tag{1.22}$$

where the superscript, -1, denotes the inverse of a as $\underline{a}^{-1}$. Section 1.4 provides more information regarding the properties of the inverse of a matrix and gives a method for determining it.

1.2.8 Orthogonal Matrix

A matrix T is an orthogonal matrix if

$$\underline{T}^T\underline{T} = \underline{TT}^T = 1 \tag{1.23}$$

Hence, for an orthogonal matrix, we have

$$\underline{T}^{-1} = \underline{T}^T \tag{1.24}$$

An orthogonal matrix frequently used is the transformation or rotation matrix T. In two dimensional space, the transformation matrix relates components of a vector in one coordinate system to components in another system. For instance, the displacement (and force as well) vector components of d expressed in the x-y system are related to those in the x-y system Figure 1.1 and 1.4 by

$$\hat{d} = \underline{Td} \tag{1.25}$$

or

$$\begin{pmatrix} \hat{d}_x \\ \hat{d}_y \end{pmatrix} = \begin{pmatrix} cos\theta & sin\theta \\ -sin\theta & cos\theta \end{pmatrix} \begin{pmatrix} d_x \\ d_y \end{pmatrix} \tag{1.26}$$

where T is the square matrix on the right side of Eq.1.26 .

Another use of an orthogonal matrix is to change from the local stiffness matrix to a global stiffness matrix for an element. That is, given a local stiffness matrix $\underline{\hat{h}}$ for an element, if the element is arbitrarily oriented in the x-y plane, then:

$$\underline{k} = \underline{T}^T\underline{\hat{k}T} = \underline{T}^{-1}\underline{\hat{k}T} \tag{1.27}$$

Eq.1.27 is used throughout this text to express the stiffness matrix k in the x-y plane.

By further examination of T, we see that the trigonometric terms in T can be interpreted as the direction cosines of line O$\hat{x}$ or dx, we have from Eq.1.26

$$< t11 \quad t12 >=< cos\theta \quad sin\theta > \tag{1.28}$$

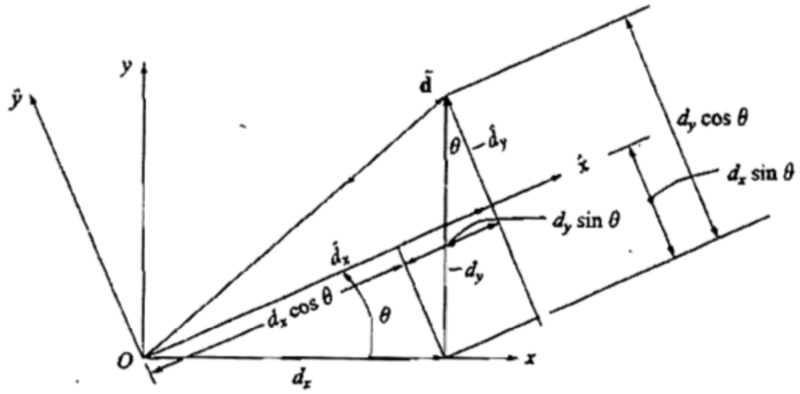

Figure 1.1: Components of vector in x-y and $\hat{x} - \hat{y}$ coordinates

and for Oy or dy, we have

$$< t21 \quad t22 >=< -sin\theta \quad cos\theta > \tag{1.29}$$

or unit vectors $\bar{i}$ and $\bar{j}$ can be represented in terms of unit vectors $\bar{\hat{i}}$ and $\bar{\hat{i}}$.

$$\bar{\hat{i}} = icos\theta + jsin\theta \tag{1.30}$$

$$\bar{\hat{j}} = -isin\theta + jcos\theta \tag{1.31}$$

and hence

$$t_{11}^2 + t_{12}^2 = 1 t_{21}^2 + t_{22}^2 = 1 \tag{1.32}$$

and since these vectors ($\bar{\hat{i}}$ and $\bar{\hat{i}}$) are orthogonal, by the dot product, we have

$$< t_{11}\bar{i} + t_{12}\bar{j} > . < t_{21}\bar{i} + t_{22}\bar{j} > \tag{1.33}$$

or

$$t_{11}t_{21} + t_{12}t_{22} = 0 \tag{1.34}$$

or we say $\underline{\text{T}}$ is orthogonal and therefore $\underline{T}^T\underline{T} = \underline{TT}^T = I$ and that the transpose is its inverse. That is,

$$\underline{T}^T = \underline{T}^{-1} \tag{1.35}$$

1.2.9 Differentiating a Matrix

A matrix is differentiated by differentiating every element in matrix in the conventional manner. For example, if

$$\underline{a} = \begin{pmatrix} x^3 & 2\,x^2 & 3\,x \\ 2\,x^2 & x^4 & x \\ 3\,x & x & x^5 \end{pmatrix} \tag{1.36}$$

the derivative $\frac{\mathrm{d}\underline{a}}{\mathrm{d}x}$ is given by

$$\frac{\mathrm{d}\underline{a}}{\mathrm{d}x} = \begin{pmatrix} 3\,x^2 & 4\,x & 3 \\ 4\,x & 4\,x^3 & 1 \\ 3 & 1 & 5\,x^4 \end{pmatrix} \tag{1.37}$$

Similarly,the partial derivative of a matrix is

$$\frac{\delta \underline{a}}{\delta x} = \frac{\delta \begin{pmatrix} x^2 & xy & xz \\ xy & y^2 & yz \\ xz & yz & z^2 \end{pmatrix}}{\delta x} = \begin{pmatrix} 2\,x & y & z \\ y & 0 & 0 \\ z & 0 & 0 \end{pmatrix} \tag{1.38}$$

In structural analysis theory, we sometimes differentiate an expression of the form

$$U = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{1.39}$$

where U might represent the strain energy in a bar. Eq.1.39 is known as a quadratic form. By matrix multiplication of Eq.1.39 , we obtain.

$$U = \frac{1}{2}(a_{11}x^2 + 2a_{12}xy + a_{22}y^2) \tag{1.40}$$

Differentiating U now yields

$$\frac{\delta U}{\delta x} = a_{11}x + a_{12}y\frac{\delta U}{\delta y} = a_{12}x + a_{22}y \tag{1.41}$$

Eq.1.41 in matrix form becomes

$$\begin{pmatrix} \frac{\delta U}{\delta x} \\ \frac{\delta U}{\delta y} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{1.42}$$

A general form of Eq.1.39 is

$$U = \frac{1}{2}\{X\}^T \underline{a}\{X\} \tag{1.43}$$

Then by comparing Eq.1.39 and Eq.1.42 , we obtain

$$\frac{\delta U}{\delta x_i} = \underline{a}\{X\} \tag{1.44}$$

where x_i denotes x and y. Here Eq.1.44 depends on matrix $\underline{\text{a}}$ in Eq.1.43 being summetric.

1.2.10 Integrating a Matrix

Just as in matrix differentiation, to integrate a matrix, we must integrate every element in the matrix in the conventional manner. For example, if

$$\underline{a} = \begin{pmatrix} x^3 & 2\,x^2 & 3\,x \\ 2\,x^2 & x^4 & x \\ 3\,x & x & x^5 \end{pmatrix}$$

we obtain the integration of $\underline{a}$ as

$$\int \underline{a}dx = \begin{pmatrix} \frac{1}{4}\,x^4 & \frac{2}{3}\,x^3 & \frac{3}{2}\,x^2 \\ \frac{2}{3}\,x^3 & \frac{1}{5}\,x^5 & \frac{1}{2}\,x^2 \\ \frac{3}{2}\,x^2 & \frac{1}{2}\,x^2 & \frac{1}{6}\,x^6 \end{pmatrix}$$

In our finite element formulation of equations, we often integrate an expression of the form

$$\int \underline{X}^T \underline{a}\underline{X} dxdy \tag{1.45}$$

The triple product in Eq.1.45 will be symmetric if $\underline{a}$ is symmetric. The form $\underline{X}^T \underline{a}\underline{X}$ is also called a quadratic form. For example, letting

$$\underline{a} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 4 \\ 6 & 2 & 3 \end{pmatrix} \underline{X} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

we obtain

$$\begin{aligned} X^T \underline{a} X &= \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 4 \\ 6 & 2 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \\ &= \left((x_1 + x_2 + 6\,x_3)x_1 + 2\,(x_1 + x_2 + x_3)x_2 + (3\,x_1 + 4\,x_2 + 3\,x_3)x_3 \right) \end{aligned} \tag{1.46}$$

which is in quadratic form.

1.3 Cofactor or Adjoint Method to Determine the inverse of a Matrix

We will now introduce a method for finding the inverse of matrix. This method is useful for longhand determination of the inverse of smaller order square matrices (preferably of order 4×4 or less). A matrix $\underline{a}$ must be square for us to determine its inverse.

We must first define the determinant of a matrix. This concept is necessary in determining the inverse of a matrix by the cofactor method. A determinant is a ısquare array of elements expressed by

$$|\underline{a}| = |\underline{a_{ij}}| \tag{1.47}$$

where the straight vertical bar, ||, on each side of the array denote the determinant. The resulting determinant of an array will be a single numerical value when the array is evaluated.

To evaluated the determinant of $\underline{a}$, we must first determinant the cofactors of $\underline{a}_{ij}$. The cofactors of $\underline{a}_{ij}$ are given by

$$\underline{c}_{ij} = (-1)^{i+j}|\underline{d}| \tag{1.48}$$

where the matrix $\underline{d}$, called the first minor of $\underline{a}_{ij}$, is matrix $\underline{a}$ with row i and column j deleted. The inverse of matrix $\underline{a}$ is then given by

$$\underline{a}^{-1} = \frac{C^T}{|\underline{a}|} \tag{1.49}$$

where $\underline{\mathrm{C}}$ is the cofactor matrix and $|\underline{a}|$ is the determine the inverse of a matrix $\underline{\mathrm{a}}$ given by

$$\underline{a} = \begin{pmatrix} -1 & 3 & -2 \\ 2 & -4 & 2 \\ 0 & 4 & 1 \end{pmatrix} \tag{1.50}$$

Using eq ,we find that the cofactors of matrix $\underline{\mathrm{a}}$ are

$$\begin{aligned}
C_{11} &= (-1)^{1+1}\begin{vmatrix} -4 & 2 \\ 4 & 1 \end{vmatrix} = -12 \\
C_{12} &= (-1)^{1+2}\begin{vmatrix} 2 & 2 \\ 0 & 1 \end{vmatrix} = -2 \\
C_{13} &= (-1)^{1+3}\begin{vmatrix} 2 & -4 \\ 0 & 4 \end{vmatrix} = 8 \\
C_{21} &= (-1)^{2+1}\begin{vmatrix} 3 & -2 \\ 4 & 1 \end{vmatrix} = -11 \\
C_{22} &= (-1)^{2+2}\begin{vmatrix} -1 & -2 \\ 0 & 1 \end{vmatrix} = -1 \\
C_{23} &= (-1)^{2+3}\begin{vmatrix} -1 & 3 \\ 0 & 4 \end{vmatrix} = 4
\end{aligned} \tag{1.51}$$

similarly,

$$\begin{aligned}
c_{31} &= -2 \\
c_{32} &= -2 \\
c_{33} &= -2
\end{aligned} \tag{1.52}$$

Therefore, from Eq.1.51 and Eq.1.52 ,we have

$$\underline{c} = \begin{pmatrix} -12 & -11 & -2 \\ -2 & -1 & -2 \\ 8 & 4 & -2 \end{pmatrix} \tag{1.53}$$

the determinant of $\underline{a}$ is then

$$|\underline{a}| = \sum_{j=1}^{n} a_{ij}C_{ij} \text{ with i any row number} (1 < i < n) \tag{1.54}$$

or

$$|\underline{a}| = \sum_{j=1}^{n} a_{ji}C_{ji} \text{ with i any column number} (1 < i < n) \tag{1.55}$$

For instance, if we choose the first of $\underline{a}$ and $\underline{c}$, then i=1 in Eq.1.54 , and j is summed from 1 to 3 such that

$$\begin{aligned} |\underline{a}| = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} \\ = -10 \end{aligned} \tag{1.56}$$

Using the definition of the inverse given by Eq.1.49 ,we have

$$\underline{a^{-1}} = \frac{\underline{C}^T}{|\underline{a}|} = \begin{pmatrix} \frac{6}{5} & \frac{11}{10} & \frac{1}{5} \\ \frac{1}{5} & \frac{1}{10} & \frac{1}{5} \\ -\frac{4}{5} & -\frac{2}{5} & \frac{1}{5} \end{pmatrix} \tag{1.57}$$

We can then check that

$$\underline{a}a^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{1.58}$$

The transpose of the cofactor matrix is often defined as the adjoint matrix; that is,

$$adj\underline{a} = \underline{C}^T \tag{1.59}$$

Therefore, an alternative equation for the inverse of $\underline{a}$ is

$$\underline{a}^{-1} = \frac{adj\underline{a}}{|\underline{a}|} \tag{1.60}$$

An important property associated with the determinant of a matrix is that if the determinant of a matrix is zero ; that is, $|\underline{a}| = 0$, then the matrix is said to be singular. A singular matrix does not have an inverse. The stiffness matrices used in the finite element method are singular until sufficient boundary conditions (support conditions) are applied. This characteristics of the stiffness matrix is further discussed in the text. The inverse of a nonsingular square matrix $\underline{a}$ can be found by the method of row reduction (sometimes called the Gauss-Jordan method) by performing identical simultaneous operations on the matrix $\underline{a}$ becomes an identity matrix and the original identity matrix becomes the inverse of $\underline{a}$.

1.4 Inverse of a Matrix by Row Reduction

A numerical example will best illustrate the procedure. We begin by converting matrix $\underline{a}$ to an upper triangular form by setting all elements below the main diagonal equal to zero, starting with the first column and continuing with succeeding columns. We then proceed from the last column to the first, setting all elements above the main diagonal equal to zero.

We will invert the following matrix by row reduction.

$$\underline{a} = \begin{pmatrix} 2 & 2 & 1 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \tag{1.61}$$

To find $\underline{a}^{-1}$, we need to find $\underline{x}$ such that $\underline{ax} = I$, where

$$\underline{x} = \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \\ a_{41} & a_{42} & a_{43} \end{pmatrix} \tag{1.62}$$

That is,solve

$$\begin{pmatrix} 2 & 2 & 1 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \underline{x} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{1.63}$$

We begin by writing $\underline{a}$ and $\underline{I}$ side by side as

$$\begin{pmatrix} 2 & 2 & 1 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{pmatrix} \tag{1.64}$$

where the vertical dashed line separates $\underline{a}$ and $\underline{I}$. 1. Divide the first row of Eq.1.64 by 2.

$$\begin{pmatrix} 1 & 1 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{pmatrix} \tag{1.65}$$

2.Multiply the first row of Eq.1.65 by -2 and add the result to the second row.

$$\begin{pmatrix} 1 & 1 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & -1 & -1 & -1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{pmatrix} \tag{1.66}$$

3. Subtract the first row of Eq.1.66 from the third row.

$$\begin{pmatrix} 1 & 1 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & -1 & -1 & -1 & 1 & 0 \\ 0 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 & 1 \end{pmatrix} \tag{1.67}$$

4. Multiply the second row of Eq.1.67 by -1 and the third row by 2.

$$\begin{pmatrix} 1 & 1 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 2 \end{pmatrix} \tag{1.68}$$

5. Subtract the third row of Eq.1.68 from the second row.

$$\begin{pmatrix} 1 & 1 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & 2 & -1 & -2 \\ 0 & 0 & 1 & -1 & 0 & 2 \end{pmatrix} \tag{1.69}$$

6. Multiply the third row of Eq.1.69 by $-\frac{1}{2}$ and add the result to the first row.

$$\begin{pmatrix} 1 & 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 2 & -1 & -2 \\ 0 & 0 & 1 & -1 & 0 & 2 \end{pmatrix} \tag{1.70}$$

7. Subtract the second row of Eq.1.70 from the first row.

$$\begin{pmatrix} 1 & 0 & 0 & -1 & 1 & 1 \\ 0 & 1 & 0 & 2 & -1 & -2 \\ 0 & 0 & 1 & -1 & 0 & 2 \end{pmatrix} \tag{1.71}$$

The replacement of $\underline{a}$ by the inverse matrix is now complete. The inverse of $\underline{a}$ is then the right side of Eq.1.71 ; that is,

$$\underline{a}^{-1} = \begin{pmatrix} -1 & 1 & 1 \\ 2 & -1 & -2 \\ -1 & 0 & 2 \end{pmatrix} \tag{1.72}$$