14.13 Consider the network shown in Figure 14.22(ii), and assume that the two telescopes work identically. N ∈ {1, 2, 3} and M1, M2 ∈ {0, 1, 2, 3, 4}, with the symbolic CPTs as described in Exercise 14.12. Using the enumeration algorithm (Figure 14.9 on page 525), calculate the probability distribution P(N | M1=2, M2=2).

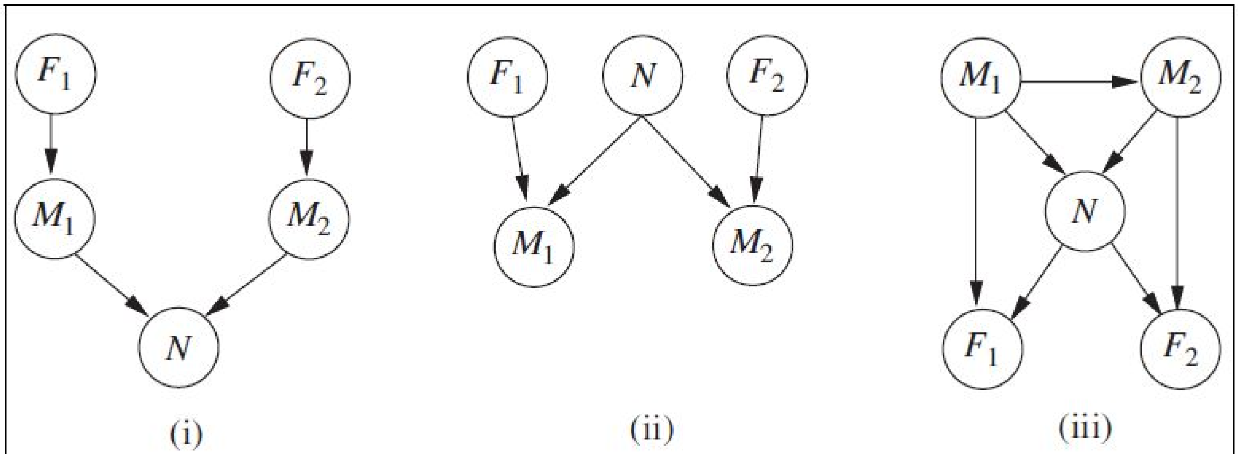

Figure 14.22 Three possible networks for the telescope problem.

function ENUMERATION-ASK(X, **e**, bn) **returns** a distribution over X
  inputs: X, the query variable
    e, observed values for variables **E**
    bn, a Bayes net with variables $\{X\} \cup \mathbf{E} \cup \mathbf{Y}$ /* **Y** = *hidden variables* */

  $\mathbf{Q}(X) \leftarrow$ a distribution over X, initially empty
  for each value x_i of X **do**
    $\mathbf{Q}(x_i) \leftarrow$ ENUMERATE-ALL(bn.VARS, $\mathbf{e}_{x_i}$)
      where $\mathbf{e}_{x_i}$ is **e** extended with $X = x_i$
  return NORMALIZE($\mathbf{Q}(X)$)

---

function ENUMERATE-ALL($vars$, **e**) **returns** a real number
  if EMPTY?($vars$) **then return** 1.0
  $Y \leftarrow$ FIRST($vars$)
  if Y has value y in **e**
    then return $P(y \mid parents(Y)) \times$ ENUMERATE-ALL(REST($vars$), **e**)
    else return $\sum_y P(y \mid parents(Y)) \times$ ENUMERATE-ALL(REST($vars$), $\mathbf{e}_y$)
      where $\mathbf{e}_y$ is **e** extended with $Y = y$

Figure 14.9 The enumeration algorithm for answering queries on Bayesian networks.

To discover the distribution of P(N | M1=2, M2=2) we will call to **Enumeration-ASK(X={N}, e={M1=2, M2=2})**
This method will first calculate the N distribution given the evidence using each possible value of N:

$$Q[N=1] = \sum_{f_1 \in F_1} P(F_1 = f_1) \times \sum_{f_2 \in F_2} P(F_2 = f_2) \times P(N=1) \times P(M_1 = 2 \mid F_1 = f_1, N=1) \times P(M_2 = 2 \mid F_2 = f2, N=1)$$

$$Q[N=2] = \sum_{f_1 \in F_1} P(F_1 = f_1) \times \sum_{f_2 \in F_2} P(F_2 = f_2) \times P(N=2) \times P(M_1 = 2 \mid F_1 = f_1, N=2) \times P(M_2 = 2 \mid F_2 = f2, N=2)$$

$$Q[N=3] = \sum_{f_1 \in F_1} P(F_1 = f_1) \times \sum_{f_2 \in F_2} P(F_2 = f_2) \times P(N=3) \times P(M_1 = 2 \mid F_1 = f_1, N=3) \times P(M_2 = 2 \mid F_2 = f2, N=3)$$

For N in range 1 to 3 if any telescope is out of focus, then the measure will be forced to be 0. Then, because the evidence says that M1 and M2 are greater than zero, we can be sure that the telescopes are both focused. Then we can simplify the above formulas to:

$$Q[N=1] = P(F_1 = f_1) \times P(F_2 = f_2) \times P(N=1) \times P(M_1 = 2 \mid F_1 = f_1, N=1) \times P(M_2 = 2 \mid F_2 = f2, N=1)$$
$$Q[N=2] = P(F_1 = f_1) \times P(F_2 = f_2) \times P(N=2) \times P(M_1 = 2 \mid F_1 = f_1, N=2) \times P(M_2 = 2 \mid F_2 = f2, N=2)$$
$$Q[N=3] = P(F_1 = f_1) \times P(F_2 = f_2) \times P(N=3) \times P(M_1 = 2 \mid F_1 = f_1, N=3) \times P(M_2 = 2 \mid F_2 = f2, N=3)$$

If we define $\alpha = P(F_1 = f_1) \times P(F_2 = f_2)$

$$Q[N=1] = \alpha \times P(N=1) \times P(M_1 = 2 \mid F_1 = f_1, N=1) \times P(M_2 = 2 \mid F_2 = f2, N=1)$$
$$Q[N=2] = \alpha \times P(N=2) \times P(M_1 = 2 \mid F_1 = f_1, N=2) \times P(M_2 = 2 \mid F_2 = f2, N=2)$$
$$Q[N=3] = \alpha \times P(N=3) \times P(M_1 = 2 \mid F_1 = f_1, N=3) \times P(M_2 = 2 \mid F_2 = f2, N=3)$$

Both telescopes work the same by hypothesis, then:

$$Q[N=1] = \alpha \times P(N=1) \times P(M_1 = 2 \mid F_1 = f_1, N=1)^2$$
$$Q[N=2] = \alpha \times P(N=2) \times P(M_1 = 2 \mid F_1 = f_1, N=2)^2$$
$$Q[N=3] = \alpha \times P(N=3) \times P(M_1 = 2 \mid F_1 = f_1, N=3)^2$$

And by hypothesis these probabilities are:

$$Q[N=1] = \alpha \times P(N=1) \times e^2$$
$$Q[N=2] = \alpha \times P(N=2) \times (1-2e)^2$$
$$Q[N=3] = \alpha \times P(N=3) \times e^2$$

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$$\alpha \times P(N=1) \times e^2 + \alpha \times P(N=2) \times (1-2e)^2 + \alpha \times P(N=3) \times e^2 = 1$$
$$P(N=1) \times e^2 + P(N=2) \times (1-2e)^2 + P(N=3) \times e^2 = 1/\alpha$$
$$\alpha = (P(N=1) \times e^2 + P(N=2) \times (1-2e)^2 + P(N=3) \times e^2)^{-1}$$

And finally we can write the distribution as:

$$P(N \mid M1 = 2,\ M2 = 2) = \alpha \times < P(N=1) \times e^2;\ P(N=2) \times (1-2e)^2;\ P(N=3) \times e^2 >$$