13.16 It is quite often useful to consider the effect of some specific propositions in the context of some general background evidence that remains fixed, rather than in the complete absence of information. The following questions ask you to prove more general versions of the product rule and Bayes' rule, with respect to some background evidence e:
a. Prove the conditionalized version of the general product rule:
P(X, Y | e) = P(X | Y, e).P(Y | e)
b. Prove the conditionalized version of Bayes' rule in Equation (13.13).
P(Y | X, e) = P(X | Y, e).P(Y | e) / P(X | e)

a. Prove the conditionalized version of the general product rule:
P(X, Y | e) = P(X | Y, e).P(Y | e)

I) Using the conditional probability definition:

$$P((X \wedge Y) \mid e) = P(X \wedge Y \wedge e) / P(e)$$

II) Using the product rule:

$$P(X \wedge (Y \wedge e)) = P(X \mid Y \wedge e) \times P(Y \wedge e)$$

III) Replacing P(X, Y, e) in (I) with (II)

$$P((X \wedge Y) \mid e) = P(X \mid Y \wedge e) \times P(Y \wedge e) / P(e)$$

IV) Using the product rule:

$$P(Y \wedge e) = P(Y \mid e) \times P(e)$$

V) Replacing P(Y, e) in (III) with (IV)

$$P(X \wedge Y \mid e) = P(X \mid Y \wedge e) \times P(Y \mid e) \times P(e) / P(e) = P(X \mid Y \wedge e) \times P(Y \mid e)$$

Q.E.D.

b. Prove the conditionalized version of Bayes' rule in Equation (13.13).
P(Y | X, e) = P(X | Y, e).P(Y | e) / P(X | e)

I) Using the conditional probability definition:

$$P(Y \mid X \wedge e) = P(Y \wedge X \wedge e) / P(X \wedge e)$$

II) Using the product rule:

$$P(Y \wedge X \wedge e) = P(X \wedge Y \wedge e) = P(X \mid Y \wedge e) \times P(Y \wedge e)$$

III) Replacing P(Y, X, e) in (I) with (II)

$$P(Y \mid X \wedge e) = P(X \mid Y \wedge e) \times P(Y \wedge e) / P(X \wedge e)$$

IV) Using the product rule:

$$P(Y \wedge e) = P(Y \mid e) \times P(e)$$

V) Using the product rule:

$$P(X \wedge e) = P(X \mid e) \times P(e)$$

VI) Dividing equalities in (IV) / (V) we keep the equality

$$P(Y \wedge e) / P(X \wedge e) = P(Y \mid e) \times P(e) / (P(X \mid e) \times P(e)) = P(Y \mid e) / P(X \mid e)$$

VII) Replacing “P(Y, e) / P(X,e)” in (III) with (VI)

$$P(Y \mid X \wedge e) = P(X \mid Y \wedge e) \times P(Y \mid e) / P(X \mid e)$$

Q.E.D.