13.20 Let X, Y, Z be Boolean random variables. Label the eight entries in the joint distribution P(X, Y, Z) as a through h. Express the statement that X and Y are conditionally independent given Z, as a set of equations relating a through h. How many non redundant equations are there?

Domain of every Boolean variable:
$X : \{x, \neg x\},\ Y : \{y, \neg y\},\ Z : \{z, \neg z\}$

Joint distribution values for each combination

P(X, Y, Z)	**Joint Distribution (Probabilities)**
$P(x, y, z)$	a
$P(x, y, \neg z)$	b
$P(x, \neg y, z)$	c
$P(x, \neg y, \neg z)$	d
$P(\neg x, y, z)$	e
$P(\neg x, y, \neg z)$	f
$P(\neg x, \neg y, z)$	g
$P(\neg x, \neg y, \neg z)$	h

The sum of all joint distribution values must be 1 by axiom 1 of probability theory:
$a + b + c + d + e + f + g + h = 1$

Then knowing 7 joint distributions values will let us calculate the 8th probability value.
Let's see if we can find some non redundant equations to simplificate prior knowledge requirement.

We can express the statement that X and Y are conditionally independent given Z using the conditional independence equation between two propositions when there are some evidence.

I) By definition of conditional independence (of X and Y given Z):

$$P(X, Y \mid Z) = P(X \mid Z) \times P(Y \mid Z)$$

II) By conditional definition we can rewrite P(X, Y | Z):

$$P((X, Y) \mid Z) = P(X, Y, Z) / P(Z)$$

III) Equation (I) and (II) are equals, so:

$$P(X \mid Z) \times P(Y \mid Z) = P(X, Y, Z) / P(Z)$$

IV) By conditional definition:

$$P(Y \mid Z) = P(Y, Z) / P(Z)$$

V) By conditional definition:

$$P(X \mid Z) = P(X, Z) / P(Z)$$

VI) Replacing (IV) and (V) in (III)

$$P(X, Y, Z) / P(Z) = (P(X, Z) / P(Z)) \times (P(Y, Z) / P(Z))$$
$$P(X, Y, Z) = P(X, Z) \times P(Y, Z) / P(Z)$$

VII) Using marginalization definition over a sample space of 3 boolean variables we can say:

$$P(X, Z) = \sum_{y \in Y} P(X,\ y,\ Z)$$

$$P(Y, Z) = \sum_{x \in X} P(x,\ Y,\ Z)$$

$$P(Z) = \sum_{x \in X, y \in Y} P(x,\ y,\ Z)$$

VIII) Replacing equations of (VII) in (VI) we obtain:

$$P(X, Y, Z) = \sum_{y \in Y} P(X,\ y,\ Z) \times \sum_{x \in X} P(x,\ Y,\ Z) \quad / \sum_{x \in X, y \in Y} P(x,\ y,\ Z)$$

IX) Now we can use equation in (VIII) to resolve the eight probabilistic joints distributions:

P(x, y, z) = a

$$P(x, y, z) = \frac{(P(x,y,z)+P(x,\neg y,\neg z))\times(P(x,y,z)+P(\neg x,y,z))}{P(x,y,z)+P(x,\neg y,z)+P(\neg x,y,z)+P(\neg x,\neg y,z)}$$

$$a = \frac{(a+c) \times (a+e)}{a+c+e+g}$$

$$aa + ae + ac + ce - a \times (a + c + e + g) = 0$$
$$aa + ae + ac + ce - aa - ac - ae - ag = 0$$
$$ce - ag = 0$$
$$ce = ag$$

P(x, y,-z) = b

$$P(x, y, \neg z) = \frac{(P(x,y,\neg z)+P(x,\neg y,\neg z))\times(P(x,y,\neg z)+P(\neg x,y,\neg z))}{P(x,y,\neg z)+P(x,\neg y,\neg z)+P(\neg x,y,\neg z)+P(\neg x,\neg y,\neg z)}$$

$$b = \frac{(b+d) \times (b+f)}{b+d+f+h}$$

$bb + bf + bd + df - b \times (b + d + f + h) = 0$
$bb + bf + bd + df - bb - bd - bf - bh = 0$
$df - bh = 0$
$df = bh$

P(x,-y, z) = c

$$P(x, \neg y, z) = \frac{(P(x,y,z)+P(x,\neg y,z))\times(P(x,\neg y,z)+P(\neg x,\neg y,z))}{P(x,y,z)+P(x,\neg y,z)+P(\neg x,y,z)+P(\neg x,\neg y,z)}$$

$$c = \frac{(a + c) \times (c + g)}{a + c + e + g}$$

$ac + ag + cc + cg - c \times (a + c + e + g) = 0$
$ac + ag + cc + cg - ca - cc - ce - cg = 0$
$ag - ce = 0$
$ce = ag$

P(x,-y,-z) = d

$$P(x, \neg y, \neg z) = \frac{(P(x,y,\neg z)+P(x,\neg y,\neg z))\times(P(x,\neg y,\neg z)+P(\neg x,\neg y,\neg z))}{P(x,y,\neg z)+P(x,\neg y,\neg z)+P(\neg x,y,\neg z)+P(\neg x,\neg y,\neg z)}$$

$$d = \frac{(b + d) \times (d + h)}{b + d + f + h}$$

$bd + bh + dd + dh - d \times (b + d + f + h) = 0$
$bb + bf + bd + df - db - dd - df - dh = 0$
$bh - df = 0$
$df = bh$

P(-x, y, z) = e

$$P(\neg x, y, z) = \frac{(P(\neg x,y,z)+P(\neg x,\neg y,z))\times(P(x,y,z)+P(\neg x,y,z))}{P(x,y,z)+P(x,\neg y,z)+P(\neg x,y,z)+P(\neg x,\neg y,z)}$$

$$e = \frac{(e + g) \times (a + e)}{a + c + e + g}$$

$ae + ee + ag + eg - e \times (a + c + e + g) = 0$
$ae + ee + ag + eg - ae - ce - ee - ge = 0$
$ag - ce = 0$
$ce = ag$

P(-x, y,-z) = f

$$P(\neg x, y, \neg z) = \frac{(P(\neg x,y,\neg z)+P(\neg x,\neg y,\neg z))\times(P(x,y,\neg z)+P(\neg x,y,\neg z))}{P(x,y,\neg z)+P(x,\neg y,\neg z)+P(\neg x,y,\neg z)+P(\neg x,\neg y,\neg z)}$$

$$f = \frac{(f + h) \times (b + f)}{b + d + f + h}$$

$$bf + ff + bh + fh - f \times (b + d + f + h) = 0$$
$$bf + ff + bh + fh - bf - df - ff - fh = 0$$
$$bh - df = 0$$
$$df = bh$$

P(-x, -y, z) = g

$$P(\neg x, \neg y, z) = \frac{(P(\neg x,y,z)+P(\neg x,\neg y,z))\times(P(x,\neg y,z)+P(\neg x,\neg y,z))}{P(x,y,z)+P(x,\neg y,z)+P(\neg x,y,z)+P(\neg x,\neg y,z)}$$

$$g = \frac{(e + g) \times (c + g)}{a + c + e + g}$$

$$ce + eg + cg + gg - g \times (a + c + e + g) = 0$$
$$ce + eg + cg + gg - ag - cg - eg - gg = 0$$
$$ag - ce = 0$$
$$ce = ag$$

P(-x,-y,-z) = h

$$P(\neg x, \neg y, \neg z) = \frac{(P(\neg x,y,\neg z)+P(\neg x,\neg y,\neg z))\times(P(x,\neg y,\neg z)+P(\neg x,\neg y,\neg z))}{P(x,y,\neg z)+P(x,\neg y,\neg z)+P(\neg x,y,\neg z)+P(\neg x,\neg y,\neg z)}$$

$$h = \frac{(f + h) \times (d + h)}{b + d + f + h}$$

$$df + fh + dh + hh - h \times (b + d + f + h) = 0$$
$$df + fh + dh + hh - bh - dh - fh - hh = 0$$
$$bh - df = 0$$
$$df = bh$$

So, after all there are two non redundant equations more with conditional independence detected:

$$ce = ag$$
$$df = bh$$

Then we can define a and b for example:

$$a = ce/g$$
$$b = df/h$$

And as we already said we have one more equation thanks to axiom 1 of probability theory: the sum of all joint distribution values must be 1 that will help to define c for example :

$$a + b + c + d + e + f + g + h = 1$$
$$(ce/g) + (df/h) + c + d + e + f + g + h = 1$$
$$c = 1 - (ce/g) - (df/h) - d - e - f - g - h$$

Exercise's answer: there are a total of 3 non redundant equations when there is conditional independence between X and Y given Z. So by knowing just 5 joint distributions we can derive the other 3 ones.