14.1 We have a bag of three biased coins a, b, and c with probabilities of coming up heads of 20%, 60%, and 80%, respectively. One coin is drawn randomly from the bag (with equal likelihood of drawing each of the three coins), and then the coin is flipped three times to generate the outcomes X1, X2, and X3.
a. Draw the Bayesian network corresponding to this setup and define the necessary CPTs.
b. Calculate which coin was most likely to have been drawn from the bag if the observed flips come out heads twice and tails once.

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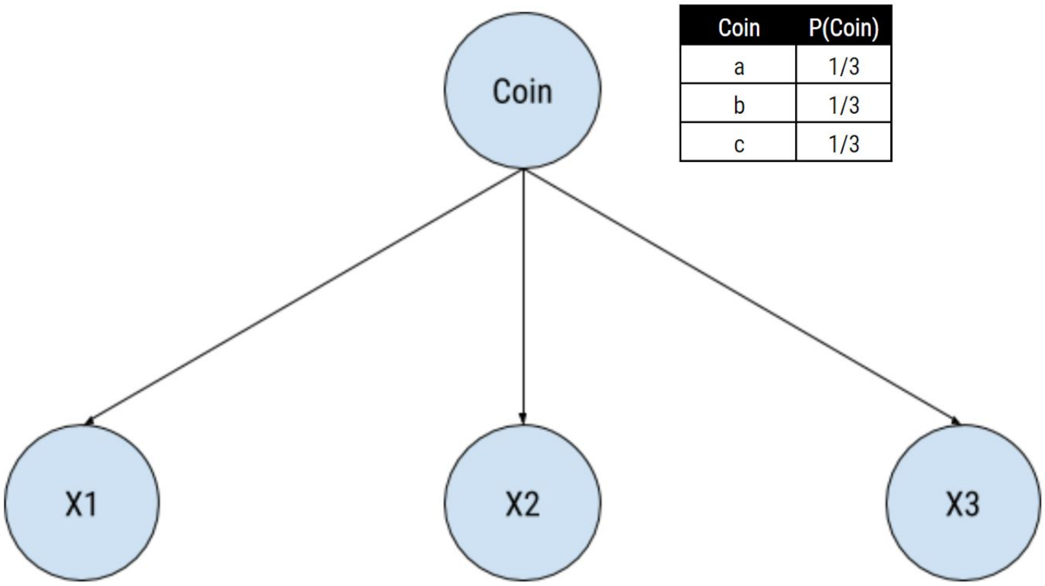

Coin	P(Coin)
a	1/3
b	1/3
c	1/3

Coin	X1	P(X1 | Coin)
a	H	0.2
b	H	0.6
c	H	0.8

Coin	X2	P(X2 | Coin)
a	H	0.2
b	H	0.6
c	H	0.8

Coin	X3	P(X3 | Coin)
a	H	0.2
b	H	0.6
c	H	0.8

b. Calculate which coin was most likely to have been drawn from the bag if the observed flips come out heads twice and tails once.

So, we need to discover which coin presents the highest probability value given that (X_1, X_2, X_3) was $HHT,\ HTH\ or\ THH$ (these are all the possible cases where there are present two heads and one tail).

If we pay attention to the Bayesian network above, we see that $P(X_i \mid Coin)$ value is the same for the same coin for all X_i. So, finding one case value will be used for all cases:

$$P(Coin \mid X_1 = H,\ X_2 = H,\ X_3 = T) = P(Coin \mid X_1 = H,\ X_2 = T,\ X_3 = H) = P(Coin \mid X_1 = T,\ X_2 = H,\ X_3 = T)$$
$$P(Coin \mid Two\ Heads\ and\ One\ Tail) = 3 \times P(Coin \mid X_1 = H,\ X_2 = H,\ X_3 = T)$$

Then, we only need to calculate $P(Coin \mid X_1 = H,\ X_2 = H,\ X_3 = T)$ and check which coin present the highest value.

$P(Coin \mid X_1 = H,\ X_2 = H,\ X_3 = T) = P(X_1 = H,\ X_2 = H,\ X_3 = T \mid Coin) \times P(Coin) / P(X_1 = H,\ X_2 = H,\ X_3 = T)$

$P(Coin)$ and $P(X_1 = H,\ X_2 = H,\ X_3 = T)$ are the same for all possible coins. Then:

$P(Coin \mid X_1 = H,\ X_2 = H,\ X_3 = T) = \alpha \times P(X_1 = H,\ X_2 = H, X_3 = T \mid Coin)$

Then we only need to calculate $P(X_1 = H,\ X_2 = H,\ X_3 = T \mid Coin)$ and check which coin presents the highest value. We see in the Bayesian Network that all X_i are independent of each other.

$P(X_1 = H, X_2 = H, X_3 = T \mid Coin) = P(X_1 = H \mid Coin) \times P(X_2 = H \mid Coin) \times P(X_3 = T \mid Coin)$

$P(X_1 = H \mid Coin = a) \times P(X_2 = H \mid Coin = a) \times P(X_3 = T \mid Coin = a) = 0.2 \times 0.2 \times 0.8 = 0.032$
$P(X_1 = H \mid Coin = b) \times P(X_2 = H \mid Coin = b) \times P(X_3 = T \mid Coin = b) = 0.6 \times 0.6 \times 0.4 = 0.144$
$P(X_1 = H \mid Coin = c) \times P(X_2 = H \mid Coin = c) \times P(X_3 = T \mid Coin = c) = 0.8 \times 0.8 \times 0.2 = 0.128$

$P(X_1 = H, X_2 = H, X_3 = T \mid Coin) = \ < 0.032;\ 0.144;\ 0.128 >$

The coin b was the coin most likely to have been drawn from the bag. And we can go further to estimate the distribution probabilities (normalizing):

$P(Coin \mid Two\ Heads\ and\ One\ Tail) = \beta \times < 0.032;\ 0.144;\ 0.128 >$

$0.032 + 0.144 + 0.128 = 0.304$

$P(Coin \mid Two\ Heads\ and\ One\ Tail) = \ (1/0.304) \times < 0.032;\ 0.144;\ 0.128 >$
$P(Coin \mid Two\ Heads\ and\ One\ Tail) \simeq \ < 0.105;\ 0.474;\ 0.421 >$