13.2 Using the axioms of probability, prove that any probability distribution on a discrete random variable must sum to 1.

Axiom I
Being Ω the sample space (conformed by all the possible worlds):

$$0 \leq P(\omega) \leq 1 \; for\ every\ \omega\ and \sum_{\omega \in \Omega} P(\omega) = 1$$

Axiom II
For any proposition φ:

$$P(\varphi) = \sum_{\omega \in \varphi} P(\omega)$$

Axiom III

$$P(\neg a) = 1 - P(a)$$

Inclusion-Exclusion Principle

$$P(a \vee b) = P(a) + P(b) - P(a \wedge b)$$

Proof

A discrete variable is a variable which can only take a countable number of values. We will call this variable X in the proof.

Domain of variable $X = \{a_1, a_2, a_3, a_4, ..., a_n\}$ (This could be interpreted as the sample space Ω)

I) Using the *inclusion-exclusion principle* we can say that:

$$P(X = a_1 \vee X = a_2) = P(X = a_1) + P(X = a_2) - P(X = a_1 \wedge X = a_2)$$

The possible worlds are mutually exclusive and exhaustive: two possible worlds cannot both be the case, and one possible world must be the case. So, the possibility of defining X as two different worlds at the same time is imposible:

II) $P(X = a_1 \wedge X = a_2) = 0$

III) Replacing with (II) in (I):

$$P(X = a_1 \vee X = a_2) = P(X = a_1) + P(X = a_2) - 0$$
$$P(X = a_1 \vee X = a_2) = P(X = a_1) + P(X = a_2)$$

IV) We can try to generalize:

$$P((X = a_1 \vee X = a_2) \vee X = a_3) = P(X = a_1 \vee X = a_2) + P(X = a_3) - P((X = a_1 \wedge X = a_2) \wedge X = a_3)$$

As we have already seen, the random variable X can not take more than one possible value at the same time. So the probability of this situation is zero too.

V) Probability that X presents 3 differents values is zero:

$P(X = a_1 \wedge X = a_2 \wedge X = a_3)$ = 0

VI) Then we can use (V) in (IV):

$$P(X = a_1 \vee X = a_2 \vee X = a_3) = P(X = a_1 \vee X = a_2) + P(X = a_3) - 0$$

VII) We can use (III) in (VII):

$$P(X = a_1 \vee X = a_2 \vee X = a_3) = P(X = a_1) + P(X = a_2) + P(X = a_3)$$

VIII) And then by recursion we can really generalize:

$$P(X = a_1 \vee X = a_2 \vee X = a_3 \vee ... \vee X = a_n) = P(X = a_1) + P(X = a_2) + P(X = a_3) + ... + P(X = a_n)$$

IX) We know that the probability that X takes the value of one value between all its possible values (worlds) is 1. In other case X would have to take a value outside its own domain, and this is not possible.

$P(X = a_1 \vee X = a_2 \vee X = a_3 \vee ... \vee X = a_n) = 1$

X) Tough the sum of all probabilities of the different possible worlds/values of X is 1 (by (VIII) and (IX))

$$1 = P(X = a_1) + P(X = a_2) + P(X = a_3) + ... + P(X = a_n)$$

That can be simplified to:

$$\sum_{i=1}^{n} P(X = a_i) = 1$$

Q.E.D.