14.19 This exercise explores the stationary distribution for Gibbs sampling methods.
a. The convex composition [α, q1; 1 - α, q2] of q1 and q2 is a transition probability distribution that first chooses one of q1 and q2 with probabilities α and 1 - α, respectively, and then applies whichever is chosen. Prove that if q1 and q2 are in detailed balance with π, then their convex composition is also in detailed balance with π. (Note: this result justifies a variant of GIBBS-ASK in which variables are chosen at random rather than sampled in a fixed sequence.)
b. Prove that if each of q1 and q2 has π as its stationary distribution, then the sequential composition q = q1 ∘ q2 also has π as its stationary distribution.

a. The convex composition [α, q1; 1 - α, q2] of q1 and q2 is a transition probability distribution that first chooses one of q1 and q2 with probabilities α and 1 - α, respectively, and then applies whichever is chosen. Prove that if q1 and q2 are in detailed balance with π, then their convex composition is also in detailed balance with π. (Note: this result justifies a variant of GIBBS-ASK in which variables are chosen at random rather than sampled in a fixed sequence.)

Being $\pi_t(x)$ the probability that the system is in state x at time t, then the STATIONARY DISTRIBUTION is reached when $\pi_t = \pi_{t+1}$. We call the stationary distribution: π

$$\pi(x') = \sum_x \pi(x) \times q(x \to x') \, for \, all \, x'$$

We say that the transition $q(x \to x')$ is in DETAILED BALANCE with $\pi(x)$ when:
$\pi(x).q(x \to x') = \pi(x').q(x' \to x) \, for \, all \, x, x'$

If q_1 and q_2 are in detailed balance, by definition we can say that:
$\pi(x).q_1(x \to x') = \pi(x').q_1(x' \to x) \;\; for \, all \, x, \; x'$
$\pi(x).q_2(x \to x') = \pi(x').q_2(x' \to x) \;\; for \, all \, x, \; x'$

Let call the transition probability distribution $q' = \alpha.q_1 + (1-\alpha).q_2$ and then verify if we can prove that:
$\pi(x).q'(x \to x') = \pi(x').q'(x' \to x)$

Using our transition probability distribution q':
$\pi(x).q'(x \to x') = \pi(x).(\alpha.q_1(x \to x') + (1-\alpha).q_2(x \to x'))$
$\pi(x).q'(x \to x') = \alpha.\pi(x).q_1(x \to x') + (1-\alpha).\pi(x).q_2(x \to x')$
$\pi(x).q'(x \to x') = \alpha.(\pi(x).q_1(x \to x')) + (1-\alpha).(\pi(x).q_2(x \to x'))$
Replacing $\pi(x).q_1(x \to x')$ and $\pi(x).q_2(x \to x')$ because they are detailed balanced:
$\pi(x).q'(x \to x') = \alpha.(\pi(x').q_1(x' \to x)) + (1-\alpha).(\pi(x').q_2(x' \to x))$
$\pi(x).q'(x \to x') = \pi(x').(\alpha.q_1(x' \to x) + (1-\alpha).q_2(x' \to x))$
$\pi(x).q'(x \to x') = \pi(x').q'(x' \to x)$
Q.E.D.

b. Prove that if each of q1 and q2 has π as its stationary distribution, then the sequential composition q = q1 ∘ q2 also has π as its stationary distribution.

Sequential composition q of q1 and q2 is:

I) $q(x \to x') = q_1 \circ q_2(x \to x') = \sum_{x''} q_1(x \to x'') \times q_2(x'' \to x')$

By hypothesis π is the stationary distribution for q1 and q2. Then:

II) $\pi(x'') = \sum_{x} \pi(x) \times q_1(x \rightarrow x'')\, for\ all\ x''$

III) $\pi(x') = \sum_{x''} \pi(x'') \times q_2(x'' \rightarrow x')\, for\ all\ x'$

Let start by (II)

$$\pi(x') \ = \ \sum_{x''} \pi(x'') \times q_2(x'' \rightarrow x')$$

$$\pi(x') \ = \ \sum_{x''} q_2(x'' \rightarrow x') \times \pi(x'')$$

Replace $\pi(x'')$ with (III):

$$\pi(x') \ = \ \sum_{x''} q_2(x'' \rightarrow x') \times \sum_{x} \pi(x) \times q_1(x \rightarrow x'')$$

Reorder of factors:

$$\pi(x') \ = \ \sum_{x} \pi(x) \times \sum_{x''} q_1(x \rightarrow x'') \times q_2(x'' \rightarrow x')$$

Replace $\sum_{x''} q_1(x \rightarrow x'') \times q_2(x'' \rightarrow x')$ with (I)

$$\pi(x') \ = \ \sum_{x} \pi(x) \times (q_1 \circ q_2)(x \rightarrow x')$$

$$\pi(x') \ = \ \sum_{x} \pi(x) \times q(x \rightarrow x')$$

And this prove that π is the stationary distribution q (the sequential composition of q1 and q2).

Q.E.D.