13.5 This question deals with the properties of possible worlds, defined on page 488 assignments to all random variables. We will work with propositions that correspond to exactly one possible world because they pin down the assignments of all the variables. In probability theory, such propositions are called atomic events. For example, with Boolean variables X1, X2, X3, the proposition x1 ∧ ¬x2 ∧ ¬x3 fixes the assignment of the variables; in the language of propositional logic, we would say it has exactly one model.
a. Prove, for the case of n Boolean variables, that any two distinct atomic events are mutually exclusive; that is, their conjunction is equivalent to false.
b. Prove that the disjunction of all possible atomic events is logically equivalent to true.
c. Prove that any proposition is logically equivalent to the disjunction of the atomic events that entail its truth.

a. Prove, for the case of n Boolean variables, that any two distinct atomic events are mutually exclusive; that is, their conjunction is equivalent to false.

A possible world is defined to be an assignment of values to all of the random variables under consideration.

Domain of every boolean variable $X_i = \{true, false\}$
We write the possible values that a boolean variable X_i can take as the literals $x_i \; and \; \neg x_i$

Suppose we have two atomic events a and b with the form $(X_1, X_2, \ ..., \ X_n)$

Example:

$a = (x_1, x_2, \neg x_3, x_4, \ ..., \ x_n)$
$b = (x_1, x_2, \neg x_3, x_4, \ ..., \ x_n)$

If a is different from b then there exists at least one pair of literals at some position i where $X_{ai}! = X_{bi}$.
That means that $(X_{ai} = x_i \ \wedge \ X_{bi} = \neg x_i)$ or that $(X_{ai} = \neg x_i \ \wedge \ X_{bi} = x_i)$.

The intersection of two worlds where there exists a variable $X_i = x_i$ in one world and the same variable $X_i = \neg x_i$ in the other: is False.

Q.E.D.

b. Prove that the disjunction of all possible atomic events is logically equivalent to true.

Using the *inclusion-exclusion principle*:

$P(w_1 \vee w_2) = P(w_1) + P(w_2) - P(w_1 \wedge w_2)$

And knowing that every two distinct atomic events are mutually exclusive:

$P(w_1 \wedge w_2) = 0$

Generalizing and using the same *inclusion-exclusion principle*:

$P(w_1 \vee w_2 \vee ... \vee w_n) = P(w_1) + P(w_2 \vee w_3 \vee ... \vee w_n) - P(w_1 \wedge (w_2 \vee w_3 \vee ... \vee w_n))$
$P(w_1 \vee w_2 \vee ... \vee w_n) = P(w_1) + P(w_2 \vee w_3 \vee ... \vee w_n) - 0$

I) $P(w_1 \vee w_2 \vee ... \vee w_n) = P(w_1) + P(w_2 \vee w_3 \vee ... \vee w_n)$

Now we can go over the second term of the right hand in (I)
$P(w_2 \vee w_3 \vee ... \vee w_n) = P(w_2) + P(w_3 \vee w_4 \vee ... \vee w_n) - P(w_2 \wedge (w_3 \vee w_4 \vee ... \vee w_n))$
$P(w_2 \vee w_3 \vee ... \vee w_n) = P(w_2) + P(w_3 \vee w_4 \vee ... \vee w_n) - 0$
II) $P(w_2 \vee w_3 \vee ... \vee w_n) = P(w_2) + P(w_3 \vee w_4 \vee ... \vee w_n)$

III) Using replacing (II) in (I)
$P(w_1 \vee w_2 \vee ... \vee w_n) = P(w_1) + P(w_2) + P(w_3 \vee ... \vee w_n)$

IV) By recursion we arrive to:
$P(w_1 \vee w_2 \vee ... \vee w_n) = P(w_1) + P(w_2) + P(w_3) + ... + P(w_n)$

V) And we know by Axiom I of Probability Theory that the sum of all atomic events probabilities is 1

$P(w_1) + P(w_2) + P(w_3) + ... + P(w_n) = 1$

VI) Then by (IV) and (V)

$P(w_1 \vee w_2 \vee ... \vee w_n) = 1$

The disjunction of all possible atomic events is True!

Q.E.D.

c. Prove that any proposition is logically equivalent to the disjunction of the atomic events that entail its truth.

The formal definition of entailment is this:

$\alpha \mid = \beta$ if and only if, in every model in which α is true, β is also true
$\alpha \mid = \beta$ if and only if $M(\alpha) \subseteq M(\beta)$.
$(*)$ α *and* β *are propositions and* "$\alpha \mid = \beta$" *must be read as* "α *entails* β"

We use the notation $M(\alpha)$ to represent the set of all atomic events in which α is True:
$M(\alpha) = \{\alpha_1, \alpha_2, ..., \alpha_n\}$

By definition there exists only one model for an atomic event sentence because atomic events are mutually exclusive:

$M(\alpha'_1) = \{\alpha_1\}$
$M(\alpha'_2) = \{\alpha_2\}$
...

$M(\alpha'_n) = \{\alpha_n\}$
We can rewrite $M(\alpha)$ as the union of $M(\alpha'_1),\ M(\alpha'_2), ...,\ and\ M(\alpha'_n)$

Notice the use of the $'$ in α' to make reference to the atomic event sentence directly linked to its atomic event.

Let call β to the disjunction of the atomic events sentences that entails the truth of α:

$$\beta = \alpha'_1 \vee \alpha'_2 \vee ... \vee \alpha'_n$$

Using all of these with the formal definition of entailment we see:

$$\alpha| = (\alpha'_1 \vee \alpha'_2 \vee ... \vee \alpha'_n) \Leftrightarrow \{\alpha_1, \alpha_2, ..., \alpha_n\} \subseteq M(\alpha'_1 \vee \alpha'_2 \vee ... \vee \alpha'_n)$$

We need to find the set of all atomic events in which β is True. Or at least we need to show that every atomic event in $\{\alpha_1, \alpha_2, ..., \alpha_n\}$ is included in $M(\alpha'_1 \vee \alpha'_2 \vee ... \vee \alpha'_n)$.

In any disjunction of atomic events sentences with any atomic event sentence e, we will find at least one model $M(e') = \{e\}$ that belongs to the set of models of the disjunction of the atomic events sentences:

$$\{e\} \subseteq M(... \vee ... \vee e' \vee ... \vee ...)$$

So, we can generalize for all atomic events sentences in the disjunction and say:

$$\{\alpha_1, \alpha_2, ..., \alpha_n\} \subseteq M(\alpha'_1 \vee \alpha'_2 \vee ... \vee \alpha'_n)$$

So, the right hand is true and the exercise affirmation does it too:

$$\alpha| = (\alpha'_1 \vee \alpha'_2 \vee ... \vee \alpha'_n) \Leftrightarrow \{\alpha_1, \alpha_2, ..., \alpha_n\} \subseteq M(\alpha'_1 \vee \alpha'_2 \vee ... \vee \alpha'_n)$$

Q.E.D.