14.10 The probit distribution defined on page 522 describes the probability distribution for a Boolean child, given a single continuous parent.
a. How might the definition be extended to cover multiple continuous parents?
b. How might it be extended to handle a multivalued child variable? Consider both cases where the child's values are ordered (as in selecting a gear while driving, depending on speed, slope, desired acceleration, etc.) and cases where they are unordered (as in selecting bus, train, or car to get to work). (Hint: Consider ways to divide the possible values into two sets, to mimic a Boolean variable.)

a. How might the definition be extended to cover multiple continuous parents?

The Probit or Probability Unit Function $\Phi(x)$ is defined as the integral of the normal standard distribution:

$$\Phi(x) = \oint_{-\infty}^{x} N(0,1)(x)dx$$

The probability of Buys (boolean) given Cost (continuous) might be:

$$P(buys \mid Cost = c) = \Phi((-c + \mu)/\sigma)$$

How might the above definition be extended to cover multiple continuous variables?

$$P(Boolean = true \mid C_1 = c_1,\ C_2 = c_2, ...,\ C_n = c_n) = \ ?$$

The probability of a boolean variable given multiple continuous parents could be defined using a linear combination of the parent values (that define a new normal distribution) to do the same that we already did in the case where there was present only one continuous parent variable:

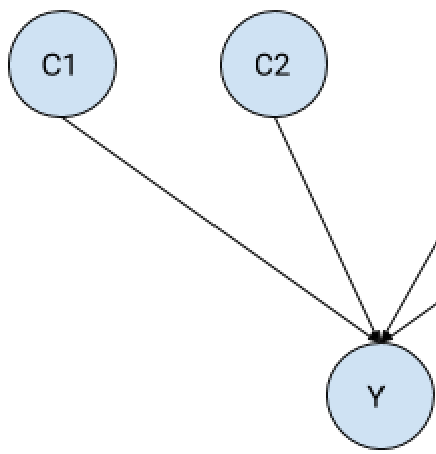
$$P(Boolean = true \mid c_1,\ c_2, ...,\ c_n) = \Phi(\mu - (w_1 c_1 + \ ... \ + w_n c_n))$$

The sum of normal distributed random variables (independent or correlated) is normal distributed.
https://en.wikipedia.org/wiki/Sum_of_normally_distributed_random_variables

b. How might it be extended to handle a multivalued child variable? Consider both cases where the child's values are ordered (as in selecting a gear while driving, depending on speed, slope, desired acceleration, etc.) and cases where they are unordered (as in selecting bus, train, or car to get to work). (Hint: Consider ways to divide the possible values into two sets, to mimic a Boolean variable.)

We need to think a possible way to calculate this probability:
$P(Y = value_i \mid c_1, c_2, ..., c_n) = ?$

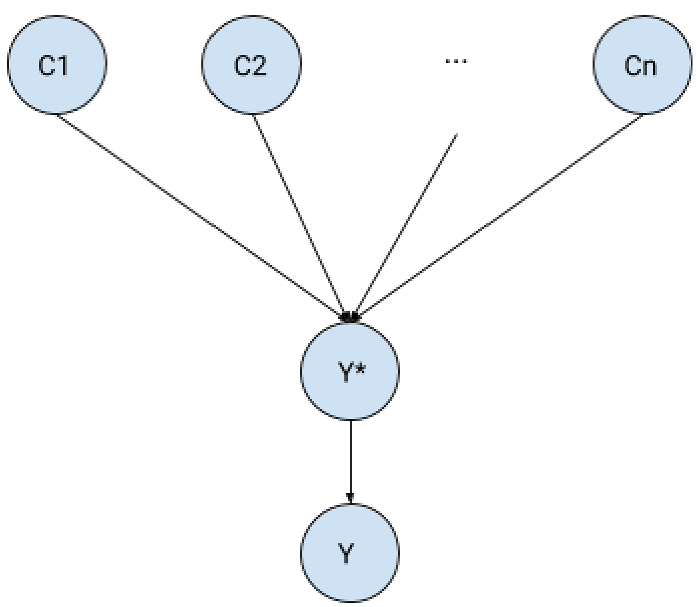
Where:
$Child\ Y\ ordered\ set = \{value_0, value_1, ..., value_k\}$



If we define:

An unobserved dependent variable y^* as a linear combination of parent values, then all possible y^* values are normal distributed (because is the resulted sum of normal distributed variables):

$y^* = w_1c_1 + w_2c_2 + ... + w_nc_n$



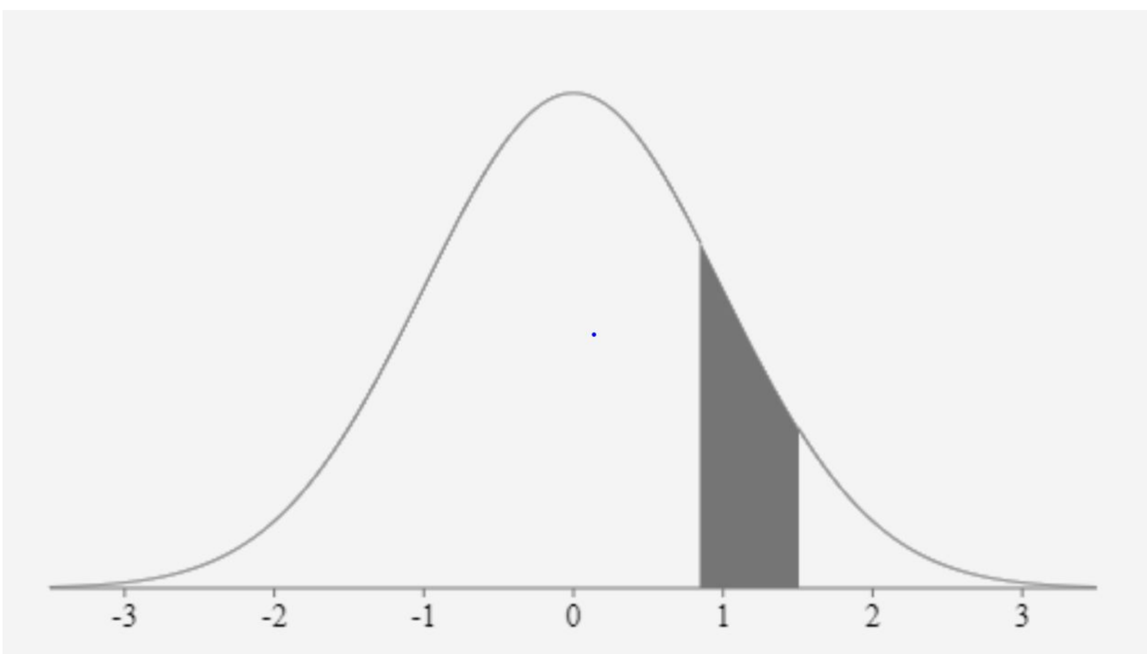
We also define (and find) some ordered coefficients μ as separators of the distribution y^* where:

$Y = value_0 \ if -\infty = \mu_{-1} \leq y^* \leq \mu_0$
$Y = value_1 \ if \ \mu_0 \leq y^* \leq \mu_1$
$Y = value_2 \ if \ \mu_1 \leq y^* \leq \mu_2$
...
$Y = value_i \ if \ \mu_{i-1} \leq y^* \leq \mu_i$
...
$Y = value_k \ if \ \mu_{k-1} \leq y^* \leq \mu_k = +\infty$

Then:

$P(Y = value_0 \mid c_1, c_2, ..., c_n) = P(-\infty = \mu_{-1} \leq y^* \leq \mu_0) = P(y^* \leq \mu_0) = \Phi(\mu_0 - y^*)$
$P(Y = value_1 \mid c_1, c_2, ..., c_n) = P(\mu_0 \leq y^* \leq \mu_1) = \Phi(\mu_1 - y^*) - \Phi(\mu_0 - y^*)$
$P(Y = value_2 \mid c_1, c_2, ..., c_n) = P(\mu_1 \leq y^* \leq \mu_2) = \Phi(\mu_2 - y^*) - \Phi(\mu_1 - y^*)$
...
$P(Y = value_i \mid c_1, c_2, ..., c_n) = P(\mu_{i-1} \leq y^* \leq \mu_i) = \Phi(\mu_i - y^*) - \Phi(\mu_{i-1} - y^*)$
...
$P(Y = value_k \mid c_1, c_2, ..., c_n) = P(\mu_{k-1} \leq y^* \leq \mu_k) = \Phi(\mu_k - y^*) - \Phi(\mu_{k-1} - y^*) = 1 - \Phi(\mu_{k-1} - y^*)$

Example: a possible Y^* distribution highlighting region between coefficients $\mu_{i-1} = 0.85$ and $\mu_i = 1.50$.
$P(Y = value_i \mid c_1, c_2, ..., c_n) = P(Y = value_i \mid Y^* = y^*) = P(\mu_{i-1} \leq y^* \leq \mu_i) = \Phi(\mu_i - y^*) - \Phi(\mu_{i-1} - y^*)$



Reference for unordered probit model
https://www.bristol.ac.uk/media-library/sites/cmm/migrated/documents/unordered-multi-r-models.pdf