13.9 In his letter of August 24, 1654, Pascal was trying to show how a pot of money should be allocated when a gambling game must end prematurely. Imagine a game where each turn consists of the roll of a die, player E gets a point when the die is even, and player O gets a point when the die is odd. The first player to get 7 points wins the pot. Suppose the game is interrupted with E leading 4–2. How should the money be fairly split in this case? What is the general formula? (Fermat and Pascal made several errors before solving the problem, but you should be able to get it right the first time.)

I assume a fair game where $P(even) = P(odd) = 1/2$

The situations to be considered are the ones where the match ends. So, the possible ending situations where players E or O win:

- Player E wins: 7-2, 7-3, 7-4, 7-5, 7-6
- Player O wins: 4-7, 5-7, 6-7

Another way to see that is noticing that player E requires just 3 more even numbers to win before player O does obtain 5 more odds numbers. Using 0-0 as the initial state instead of 4-2, the possible ending situations where each one of the players win are:

- Player E wins: 3-0, 3-1, 3-2, 3-3, 3-4
- Player O wins: 0-5, 1-5, 2-5

I will use combinatorial notation now. Remember:

$$C_k^n = \binom{n}{k} = \frac{n!}{k!\times(n-k)!}$$

Player E wins 3-0:

Cases: 1 (*eee*)

There is only one way to arrive to a situation of 3 even numbers and 0 odd numbers .
Calculation: $C_0^3 = 1$

P(3-0) = P(*eee*) = $(1/2)^3$

Player E wins 3-1:

Combinations of 3 elements of one type and 1 element of other type presents 4 alternatives:
$C_1^4 = 4!/1!/3! = 4$

The cases are: 4 (*oeee, eoee, eeoe, eeeo*)

We observe that the case *"eeeo"* was previously evaluated in cases where player E wins 3-0.
So, to calculate the number of cases where player E wins 3-1 we need to subtract the number of cases where the match ends previously.

Cases: 3 (*oeee*, *eoee*, *eeoe*)

Calculation: $C_1^4 - C_0^3 = \binom{4}{1} - \binom{3}{0} = 4 - 1 = 3$

$$P(3-1) = P(oeee) + P(eoee) + P(eeoe) = P(oeee) \times 3 = [(1/2)^3 \times (1/2)^1] \times 3 = (1/2)^4 \times 3$$

Player E wins 3-2:

Cases: 6 (*eeooe*, *eoeoe*, *eooee*, *oeeoe*, *oeoee*, *ooeee*)

Calculation: $C_2^5 - C_1^4 = \binom{5}{2} - \binom{4}{1} = 10 - 4 = 6$

$$P(3-2) = P(ooeee) \times 6 = [(1/2)^3 \times (1/2)^2] \times 6 = (1/2)^5 \times 6$$

Player E wins 3-3:

Cases: 10 (*oooeee*, *ooeoee*, *oeooee*, *eoooee*, *ooeeoe*, *oeoeoe*, *eooeoe*, *oeeooe*, *eoeooe*, *eeoooe*)

Calculation: $C_3^6 - C_2^5 = \binom{6}{3} - \binom{5}{2} = 20 - 10 = 10$

$$P(3-3) = P(oooeee) \times 10 = [(1/2)^3 \times (1/2)^3] \times 10 = (1/2)^6 \times 10$$

Player E wins 3-4:

Cases: 15 (*ooooeee*, *oooeoee*, *ooeooee*, *oeoooee*, *eooooee*, *oooeeoe*, *ooeoeoe*, *oeooeoe*, *eoooeoe*, *ooeeooe*, *oeoeooe*, *eooeooe*, *oeeoooe*, *eoeoooe*, *eeoooe*)

Calculation: $C_4^7 - C_3^6 = \binom{7}{4} - \binom{6}{3} = 35 - 20 = 15$

$$P(3-4) = P(ooooeee) \times 15 = [(1/2)^3 \times (1/2)^4] \times 15 = (1/2)^7 \times 15$$

Probability than player E wins:

$$P(E\ wins) = P(3-0) + P(3-1) + P(3-2) + P(3-3) + P(3-4)$$
$$P(E\ wins) = (1/2)^3 + (1/2)^4 \times 3 + (1/2)^5 \times 6 + (1/2)^6 \times 10 + (1/2)^7 \times 15$$
$$P(E\ wins) = (1/2)^3 \times (1 + (1/2)^1 \times 3 + (1/2)^2 \times 6 + (1/2)^3 \times 10 + (1/2)^4 \times 15)$$
$$P(E\ wins) = 0.125 \times (1 + 1.5 + 1.5 + 1.25 + 0.9375)$$
$$P(E\ wins) = 0.125 \times (6.1875) = 0.7734375$$

We can use the same procedure for player O

Player O wins 0-5:

Cases: 1 (*ooooo*)

Calculation: $C_0^5 = \binom{5}{0} = 1$

$P(0-5) = P(ooooo) = (1/2)^5$

Player O wins 1-5:

Cases: 5 (*eooooo, oeoooo, ooeooo, oooeoo, ooooeo*)

Calculation: $C_1^6 - C_0^5 = \binom{6}{1} - \binom{5}{0} = 6 - 1 = 5$

$P(1-5) = P(eooooo) + P(oeoooo) + P(ooeooo) + P(oooeoo) + P(ooooeo)$

$P(1-5) = P(eooooo) \times 5 = [(1/2)^1 x\ (1/2)^5]\ x\ 5 = (1/2)^6\ x\ 5$

Player O wins 2-5:

Cases: 15
Calculation: $C_2^7 - C_1^6 = \binom{7}{2} - \binom{6}{1} = 21 - 6 = 15$

$P(2-5) = P(eeooooo) \times 15 = [(1/2)^2 x\ (1/2)^5]\ x\ 15 = (1/2)^7\ x\ 15$

Probability than player O wins

Now, we can use all of this probabilities to estimate the probability that player O wins:

$P(O\ wins) = P(0-5) + P(1-5) + P(2-5)$
$P(O\ wins) = (1/2)^5 + (1/2)^6 \times 5 + (1/2)^7 \times 15$
$P(O\ wins) = (1/2)^5 \times (1 + (1/2)^1 \times 5 + (1/2)^2 \times 15)$
$P(O\ wins) = (1/2)^5 \times (1 + (1/2)^1 \times 5 + (1/2)^2 \times 15)$
$P(O\ wins) = 0.03125\ \times (1 + 2.5 + 3.75)$
$P(O\ wins) = 0.03125\ \times (7.25) = 0.2265625$

And it is consistent with $P(O\ wins) + P(E\ wins) = 1$

Fairly splitting of the money

So, the money should be fairly splitted with player E receiving the 77.34375% of the pot against player O receiving 22.65625% of the pot.

General Formula

n = the number of points required to win the game
e = the number of points obtained by player E
o = the number of points obtained by player O
p = probability of obtaining an even number $P(even)$

$1-p$ = probability of obtaining an odd number $P(odd)$

$$P(E\ wins) = p^{n-e} \times \sum_{i=0}^{n-o-1} \{(1-p)^i \times [C_i^{n+i-o} - C_{i-1}^{n+i-o-1}]\}$$

Observations about the number of combinations for one case minus the number of combinations for the previous case:

$$C_m^n - C_{m-1}^{n-1} = n!/(m!(n-m)!) \ - (n-1)!/((m-1)!(n-m)!)$$
$$C_m^n - C_{m-1}^{n-1} = n!/(m(m-1)!(n-m)!) \ - (n-1)!/((m-1)!(n-m)!)$$
$$C_m^n - C_{m-1}^{n-1} = n! - (n-1)!m/(m!(n-m)!)$$
$$C_m^n - C_{m-1}^{n-1} = ((n-1)!(n-m)/(m!(n-m)!)$$
$$C_m^n - C_{m-1}^{n-1} = ((n-1)!/(m!(n-1\ -m)!)$$
$$C_m^n - C_{m-1}^{n-1} = C_m^{n-1}$$

So, we can rewrite a simplified version for the general formula:

$$P(E\ wins) = p^{n-e} \times \sum_{i=0}^{n-o-1} \{(1-p)^i \times C_i^{n+i-1}\}$$

Or another one based on the opponent probabilities:

$$P(E\ wins) = 1 - P(O\ wins)$$

$$P(E\ wins) = 1\ -\ (\,p^{n-o} \times \sum_{i=0}^{n-e-1} \{(1-p)^i \times C_i^{n+i-1}\})$$