

13.17 Show that the statement of conditional independence

$P(X, Y | Z) = P(X | Z) \cdot P(Y | Z)$

is equivalent to each of the statements

$P(X | Y, Z) = P(X | Z)$ and $P(Y | X, Z) = P(Y | Z)$

Comment: the exercise should say $P(Y | X, Z)$ instead of $P(B | X, Z)$

I) By statement of conditional independence:

$$P(X \wedge Y | Z) = P(X | Z) \times P(Y | Z)$$

II) Using the conditional probability definition:

$$P(X \wedge Y | Z) = P(X \wedge Y \wedge Z) / P(Z)$$

III) Using the conditional probability definition:

$$P(Y | Z) = P(Y \wedge Z) / P(Z)$$

IV) Replacing (II) and (III) in (I):

$$P(X \wedge Y \wedge Z) / P(Z) = P(X | Z) \times P(Y \wedge Z) / P(Z)$$

$$P(X \wedge Y \wedge Z) = P(X | Z) \times P(Y \wedge Z)$$

V) Using product rule:

$$P(X \wedge Y \wedge Z) = P(X | Y \wedge Z) \times P(Y \wedge Z)$$

VI) Dividing (V) by (IV)

$$P(X \wedge Y \wedge Z) / P(X \wedge Y \wedge Z) = P(X | Y \wedge Z) \times P(Y \wedge Z) / (P(X | Z) \times P(Y \wedge Z))$$

$$P(X | Y \wedge Z) / P(X | Z) = 1$$

$$P(X | Y \wedge Z) = P(X | Z)$$

Then, saying (I) is equivalent to say that $P(X | Y, Z) = P(X | Z)$ **Q.E.D. (Part 1/2)**

VII) Using the conditional probability definition:

$$P(X | Z) = P(X \wedge Z) / P(Z)$$

VIII) Replacing (II) and (III) in (I):

$$P(X \wedge Y \wedge Z) / P(Z) = P(Y | Z) \times P(X \wedge Z) / P(Z)$$

$$P(X \wedge Y \wedge Z) = P(Y | Z) \times P(X \wedge Z)$$

IX) Using product rule:

$$P(X \wedge Y \wedge Z) = P(Y | X \wedge Z) \times P(X \wedge Z)$$

X) Dividing (IX) by (VIII):

$$P(X \wedge Y \wedge Z) / P(X \wedge Y \wedge Z) = P(Y | X \wedge Z) \times P(X \wedge Z) / (P(Y | Z) \times P(X \wedge Z))$$

$$P(Y | X \wedge Z) / P(Y | Z) = 1$$

$$P(Y | X \wedge Z) = P(Y | Z)$$

Then, saying (I) is equivalent to say that $P(Y | X, Z) = P(Y | Z)$

Q.E.D. (Part 2/2)