13.3 For each of the following statements, either prove it is true or give a counterexample.
a. If P(a | b, c) = P(b | a, c), then P(a | c) = P(b | c)
b. If P(a | b, c) = P(a), then P(b | c) = P(b)
c. If P(a | b) = P(a), then P(a | b, c) = P(a | c)

a. If P(a | b, c) = P(b | a, c), then P(a | c) = P(b | c)

I) By product rule:

$$P(a|b,c) = \frac{P(a,b,c)}{P(b,c)}$$

$$* P(b,c) > 0 \text{ by hypothesis}$$

II) By product rule definition:

$$P(b|a,c) = \frac{P(b,a,c)}{P(a,c)}$$

$$* P(a,c) > 0 \text{ by hypothesis}$$

III) By hypotheses (I) = (II):

$$\frac{P(a,b,c)}{P(b,c)} = \frac{P(b,a,c)}{P(a,c)}$$

IV) By commutative law of intersection operation:

$$P(a,b,c) = P(b,a,c)$$

V) By (IV) and (III):

$$P(b,c) = P(a,c)$$

VI) Using product rule definition in left and right hands of (V)

$$P(b|c) \times P(c) = P(a|c) \times P(c)$$

VII) Then:

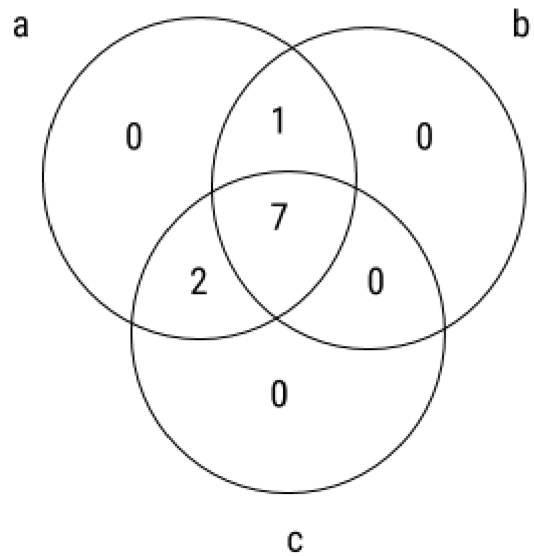
$$P(b|c) = P(a|c)$$

Q.E.D.

Conclusion: item (a) statement is True.

b. If P(a | b, c) = P(a), then P(b | c) = P(b)

I can imagine a world with 10 differents elements that can presents 3 possible categories. This categories are named a, b and c. Next below I draw a Venn's diagram example with 10 elements that will help to calculate the probabilities required for an element e to fall into one or more categories.



$P(a) = (1+2+7)/(1+2+7) = 1$
$P(a \mid b, c) = 7/7 = 1$

Here the exercise first condition is True: $P(a \mid b, c) = P(a)$

$P(b) = (1+7)/(1+2+7) = 8/10 = 4/5$
$P(b \mid c) = 7/(7+2) = 7/9$

Here the exercise second condition is False: $P(b \mid c) = P(b)$

We found a case where Modus Ponens is False ($p \Rightarrow q$)
$p = P(a \mid b, c) = P(a)\ and\ q = P(b \mid c) = P(b)$

We found a case where $p = True\ and\ q = False$ that makes the hypothesis to fail.

Conclusion: the exercise item (b) affirmation is False

c. If P(a | b) = P(a), then P(a | b, c) = P(a | c)

Let experiment with the rolling two identified dices.

Meaning of prepositions a, b and c will be:

- a : "$dice_1$ *is* 1"
- b : "$dice_2$ *is* 3"
- c : "$dice_1$ *and* $dice_2$ *sum* 4"

Some useful probabilities to keep in mind:

$P(a) =$ 1/6 $\{(11), (1,2), (1,3), (1,4), (1,5), (1,6)\}$
$P(b) =$ 1/6 $\{(1,3), (2,3), (3,3), (4,3), (5,3), (6,3)\}$
$P(c) =$ 3/36 = 1/12 $\{(1,3), (2,2), (3,1)\}$
$P(a,b) =$ 1/36 $\{(1,3)\}$
$P(a,c) =$ 1/36 $\{(1,3)\}$
$P(b,c) =$ 1/36 $\{(1,3)\}$
$P(a,b,c) =$ 1/36 $\{(1,3)\}$

We can use the above probabilities to find new ones.

I) Verify that left hand of hypothesis is True with my prepositions:

$$P(a \mid b) = P(a,b) / P(b) = (1/36) / (1/6) = 1/6 = P(a)$$

II) By product rule:

$$P(a \mid b,c) = P(a,b,c) / P(b,c) = (1/36) / (1/36) = 1$$

III) By product rule:

$$P(a \mid c) = P(a,c) / P(c) = (1/36) / (3/36) = 1/3$$

IV) By comparing (II) and (III) we see that:

$$1/36 = P(a \mid b,c) \neq P(a \mid c) = 1/3$$

Then the hypothesis in exercise item (c) is False.