14.4 Consider the Bayesian network in Figure 14.2.
a. If no evidence is observed, are Burglary and Earthquake independent? Prove this from the numerical semantics and from the topological semantics.
b. If we observe Alarm = true, are Burglary and Earthquake independent? Justify your answer by calculating whether the probabilities involved satisfy the definition of conditional independence.

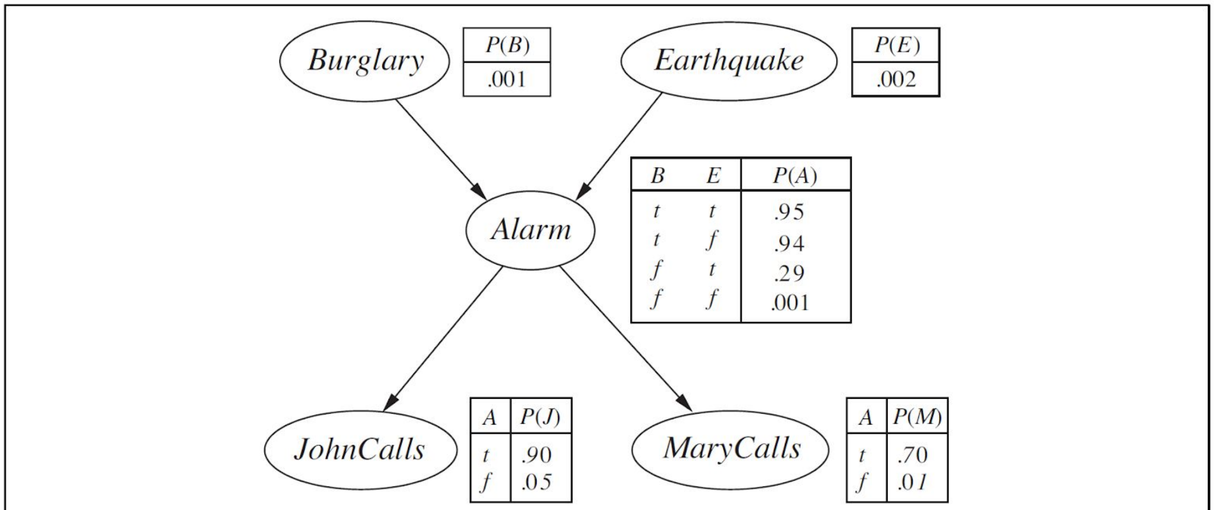

Figure 14.2 A typical Bayesian network, showing both the topology and the conditional probability tables (CPTs). In the CPTs, the letters B, E, A, J, and M stand for $Burglary$, $Earthquake$, $Alarm$, $JohnCalls$, and $MaryCalls$, respectively.

a. If no evidence is observed, are Burglary and Earthquake independent? Prove this from the numerical semantics and from the topological semantics.

Numerical Semantics

I) The numerical semantics of this Bayesian Network says that:

$$P(B,E) = P(B \mid parents(B)) \times P(E \mid parents(E))$$

II) And E and B present no parents, then:

$$P(B,E) = P(B) \times P(E)$$

III) Looking at (II) B and E are independents by definition of independence.

Topological Semantics

The topological semantics says that these variables are independent because they do not share one common ancestor at least. They present no ancestor at all.

b. If we observe Alarm = true, are Burglary and Earthquake independent? Justify your answer by calculating whether the probabilities involved satisfy the definition of conditional independence.

If B and E are dependent given an evidence a then:

$$P(B \wedge E \mid a) = \delta \times P(a \mid B \wedge E) \times P(B \wedge E)$$

If B and E are independent given an evidence a then:

$$P(B \wedge E \mid a) = P(B \mid a) \times P(E \mid a)$$

Then B and E will be independent if we prove that $\delta \times P(a \mid B \wedge E) \times P(B \wedge E) = P(B \mid a) \times P(E \mid a)$

1) First lest calculate the conditional distribution values

$$P(B \wedge E \mid a) = \delta \times P(a \mid B \wedge E) \times P(B \wedge E)$$
$$= \delta \times < P(a \mid b \wedge e) \times P(b \wedge e); P(a \mid b \wedge \neg e) \times P(b \wedge \neg e); P(a \mid \neg b \wedge e) \times P(\neg b \wedge e); P(a \mid \neg b \wedge \neg e) \times P(\neg b \wedge \neg e) >$$

Calculate the four components of the distribution:

$$P(a \mid b \wedge e) \times P(b \wedge e) = 0.95 \times (0.001 \times 0.002) = 0.95 \times 0.000002 = 0.0000019$$
$$P(a \mid b \wedge \neg e) \times P(b \wedge \neg e) = 0.94 \times (0.001 \times 0.998) = 0.94 \times 0.000998 = 0.00093812$$
$$P(a \mid \neg b \wedge e) \times P(\neg b \wedge e) = 0.29 \times (0.999 \times 0.002) = 0.29 \times 0.001998 = 0.00057942$$
$$P(a \mid \neg b \wedge \neg e) \times P(\neg b \wedge \neg e) = 0.001 \times (0.999 \times 0.998) = 0.001 \times 0.997002 = 0.000997002$$

$$P(B \wedge E \mid a) = \delta \times < 0.0000019; 0.00093812; 0.00057942; 0.000997002 >$$
$$P(B \wedge E \mid a) = (\delta/1000000000) \times < 19000;\ 938120;\ 579420;\ 997002 >$$
$$P(B \wedge E \mid a) = \delta' \times < 1900;\ 938120;\ 579420;\ 997002 >$$

$$1/\delta' = 1900 + 938120 + 579420 + 997002$$
$$1/\delta' = 2516442$$
$$\delta' = 1/2516442$$

$$P(B \wedge E \mid a) = (1/2516442) \times < 1900;\ 938120;\ 579420;\ 997002 >$$

The conditional distribution for B and E given evidence A=True is:

$$P(B \wedge E \mid a) \simeq\ < 0.000755;\ 0.372796;\ 0,230254;\ 0,396195 >$$

2) Now, lets calculate the distribution with the independent formula

$P(B \wedge E \mid a) = P(B \mid a) \times P(E \mid a)$

But before we will require to calculate some previous probabilities:

Lets calculate $P(a \mid b)$ **and** $P(\neg a \mid b)$**:**
$P(A \mid b) = \delta \times < P(a \mid b); P(\neg a \mid b) >$

Using Marginalization:
$P(A \mid b) = \delta \times < P(a \mid b, e) + P(a \mid b, \neg e); P(\neg a \mid b, e) + P(\neg a \mid b, \neg e) >$
$P(A \mid b) = \delta \times < 0.94 + 0.95; 0.06 + 0.05 > \; = \delta \times < 1.89;\; 0.11 >$
$P(A \mid b) = \; < 0.945;\; 0.055 >$

Lets calculate $P(a \mid \neg b)$ **and** $P(\neg a \mid \neg b)$**:**
$P(A \mid \neg b) = \delta \times < P(a \mid \neg b); P(\neg a \mid \neg b) >$
Using Marginalization:
$P(A \mid \neg b) = \delta \times < P(a \mid \neg b, e) + P(a \mid \neg b, \neg e); P(\neg a \mid \neg b, e) + P(\neg a \mid \neg b, \neg e) >$
$P(A \mid \neg b) = \delta \times < 0.29 + 0.001; 0.71 + 0.999 > \; = \delta \times < 0.291; 1.700 > \; = \; < 0.291/1.991; 1.700/1.991 >$
$P(A \mid \neg b) \simeq \; < 0.146;\; 0.854 >$

Lets calculate $P(a \mid e)$ **and** $P(\neg a \mid e)$**:**
$P(A \mid e) = \delta \times < P(a \mid e); P(\neg a \mid e) >$
Using Marginalization:
$P(A \mid e) = \delta \times < P(a \mid b, e) + P(a \mid \neg b, e); P(\neg a \mid b, e) + P(\neg a \mid \neg b, e) >$
$P(A \mid e) = \delta \times < 0.95 + 0.29;\; 0.05 + 0.71 > \; = \delta \times < 1.24;\; 0.75 > \; = \; < 1.24/1.99; 0.75/1.99 >$
$P(A \mid e) \simeq \; < 0.623; 0.377 >$

Lets calculate $P(a \mid \neg e)$ **and** $P(\neg a \mid \neg e)$**:**
$P(A \mid \neg e) = \delta \times < P(a \mid \neg e); P(\neg a \mid \neg e) >$
Using Marginalization:
$P(A \mid \neg e) = \delta \times < P(a \mid b, \neg e) + P(a \mid \neg b, \neg e); P(\neg a \mid b, \neg e) + P(\neg a \mid \neg b, \neg e) >$
$P(A \mid \neg e) = \delta \times < 0.94 + 0.001;\; 0.06 + 0.999 > \; = \delta \times < 0.941; 1.059 > \; = \; < 0.941/2;\; 1.059/2 >$
$P(A \mid \neg e) = \; < 0.4705; 0.5295 >$

Now, we go back to the independence formula and resolve it:

$P(B \wedge E \mid a) = P(B \mid a) \times P(E \mid a)$
$= < P(b \wedge e \mid a);\; P(b \wedge \neg e \mid a); P(\neg b \wedge e \mid a); P(\neg b \wedge \neg e \mid a) >$
$= < P(b \mid a) \times P(e \mid a);\; P(b \mid a) \times P(\neg e \mid a);\; P(\neg b \mid a) \times P(e \mid a);\; P(\neg b \mid a) \times P(\neg e \mid a) >$

Calculate the four components of the distribution:

$P(b \mid a) \times P(e \mid a) = [P(a \mid b) \times P(b) / P(a)] \times [P(a \mid e) \times P(e) / P(a)]$
$P(b \mid a) \times P(\neg e \mid a) = [P(a \mid b) \times P(b) / P(a)] \times [P(a \mid \neg e) \times P(\neg e) / P(a)]$
$P(\neg b \mid a) \times P(e \mid a) = [P(a \mid \neg b) \times P(\neg b) / P(a)] \times [P(a \mid e) \times P(e) / P(a)]$

$P(\neg b \mid a) \times P(\neg e \mid a) = [P(a \mid \neg b) \times P(\neg b) / P(a)] \times [P(a \mid \neg e) \times P(\neg e) / P(a)]$

$P(b \mid a) \times P(e \mid a) = (1/P(a)^2) \times [P(a \mid b) \times P(b)] \times [P(a \mid e) \times P(e)]$
$P(b \mid a) \times P(\neg e \mid a) = (1/P(a)^2) \times [P(a \mid b) \times P(b)] \times [P(a \mid \neg e) \times P(\neg e)]$
$P(\neg b \mid a) \times P(e \mid a) = (1/P(a)^2) \times [P(a \mid \neg b) \times P(\neg b)] \times [P(a \mid e) \times P(e)]$
$P(\neg b \mid a) \times P(\neg e \mid a) = (1/P(a)^2) \times [P(a \mid \neg b) \times P(\neg b)] \times [P(a \mid \neg e) \times P(\neg e)]$

$P(b \mid a) \times P(e \mid a) = (1/P(a)^2) \times [P(a \mid b) \times 0.001] \times [P(a \mid e) \times 0.002]$
$P(b \mid a) \times P(\neg e \mid a) = (1/P(a)^2) \times [P(a \mid b) \times 0.001] \times [P(a \mid \neg e) \times 0.998]$
$P(\neg b \mid a) \times P(e \mid a) = (1/P(a)^2) \times [P(a \mid \neg b) \times 0.999] \times [P(a \mid e) \times 0.002]$
$P(\neg b \mid a) \times P(\neg e \mid a) = (1/P(a)^2) \times [P(a \mid \neg b) \times 0.999] \times [P(a \mid \neg e) \times 0.998]$

$P(b \mid a) \times P(e \mid a) = (1/P(a)^2) \times P(a \mid b) \times P(a \mid e) \times 0.000002$
$P(b \mid a) \times P(\neg e \mid a) = (1/P(a)^2) \times P(a \mid b) \times P(a \mid \neg e) \times 0.000998$
$P(\neg b \mid a) \times P(e \mid a) = (1/P(a)^2) \times P(a \mid \neg b) \times P(a \mid e) \times 0.001998$
$P(\neg b \mid a) \times P(\neg e \mid a) = (1/P(a)^2) \times P(a \mid \neg b) \times P(a \mid \neg e) \times 0.997002$

We had already calculated the values of:

- $P(a \mid b) = 0.945$
- $P(a \mid \neg b) \simeq 0.146$
- $P(a \mid e) \simeq 0.623$
- $P(a \mid \neg e) = 0.4705$

So we can use them to calculate:

$P(b \mid a) \times P(e \mid a) \simeq (1/P(a)^2) \times 0.945 \times 0.623 \times 0.000002 = (1/P(a)^2) \times 0.00000117747$
$P(b \mid a) \times P(\neg e \mid a) \simeq (1/P(a)^2) \times 0.945 \times 0.4705 \times 0.000998 = (1/P(a)^2) \times 0.000443733255$
$P(\neg b \mid a) \times P(e \mid a) \simeq (1/P(a)^2) \times 0.146 \times 0.623 \times 0.001998 = (1/P(a)^2) \times 0.000181734084$
$P(\neg b \mid a) \times P(\neg e \mid a) \simeq (1/P(a)^2) \times 0.146 \times 0.4705 \times 0.997002 = (1/P(a)^2) \times 0.068487058386$

And finally we are in conditions to resolve our distribution:

$P(B \wedge E \mid a) = P(B \mid a) \times P(E \mid a)$
$\simeq (1/P(a)^2) \times < 0.00000117747;\ 0.000443733255;\ 0.000181734084; 0.068487058386 >$
$\simeq (1/0.069113703195) \times < 0.00000117747;\ 0.000443733255;\ 0.000181734084; 0.068487058386 >$
$\simeq < 0.0000170367; 0.006420337;\ 0.002629494; 0.990933132 >$

Then we can check that the distribution found in 1) and 2) are differents.
$< 0.000755;\ 0.372796;\ 0,230254;\ 0,396195 > \neq < 0.0000170367; 0.006420337;\ 0.002629494; 0.990933132 >$

Conclusion: then we can conclude that variables B and E are not independent given evidence A=True