13.21 (Adapted from Pearl (1988).) Suppose you are a witness to a nighttime hit-and-run accident involving a taxi in Athens. All taxis in Athens are blue or green. You swear, under oath, that the taxi was blue. Extensive testing shows that, under the dim lighting conditions, discrimination between blue and green is 75% reliable.
a. Is it possible to calculate the most likely color for the taxi? (Hint: distinguish carefully between the proposition that the taxi is blue and the proposition that it appears blue.)
b. What if you know that 9 out of 10 Athenian taxis are green?

a. Is it possible to calculate the most likely color for the taxi? (Hint: distinguish carefully between the proposition that the taxi is blue and the proposition that it appears blue.)

$L = \text{"}the\ taxi\ Looks\ blue\text{"}$
$B = \text{"}the\ taxi\ is\ Blue\text{"}$

We suppose a population of taxis where its color can be blue or green (boolean variable). If color is not blue, then it is green ($\neg B = G$) and if taxi does not look blue then it looks green ($\neg L = \text{"}the\ taxi\ Looks\ Green\text{"}$).

The exercise says that discrimination between taxi color blue and green is 75% reliable. Which says that the probability of seeing a taxi blue given that it is blue is 0.75, and that the probability that of seeing a taxi green given that it is green is 0.75. We express these with conditional symbols as:

$P(L \mid B) = 0.75$
$P(-L \mid -B) = 0.75$
$P(-L \mid B) = 1 - P(L \mid B) = 0.25$
$P(L \mid -B) = 1 - P(L \mid -B) = 0.25$

We look for the probability that the taxi was blue when it looks blue. Using Bayes Rule:

$P(B \mid L) = P(L \mid B) \times P(B) / P(L)$
$P(B \mid L) = 0.75 \times P(B) / P(L)$

$P(-B \mid L) = P(L \mid -B) \times P(-B) / P(L)$
$P(-B \mid L) = 0.25 \times P(-B) / P(L)$

$If\ P(B \mid L) / P(-B \mid L) > 1:\ most\ likely\ to\ be\ Blue$

$P(B \mid L) / P(\neg B \mid L) = (0.75 \times P(B) / P(L)) / (0.25 \times P(\neg B) / P(L))$
$P(B \mid L) / P(\neg B \mid L) = 0.75 \times P(B) / 0.25 \times P(\neg B)$
$P(B \mid L) / P(\neg B \mid L) = 3 \times P(B) / P(\neg B)$
$P(B \mid L) / P(\neg B \mid L) = 3 \times (1 - P(\neg B)) / P(\neg B)$
$P(B \mid L) / P(\neg B \mid L) = (3 - 3 \times P(\neg B)) / P(\neg B)$
$P(B \mid L) / P(\neg B \mid L) = 3/P(\neg B) - 3$

$P(B \mid L) / P(\neg B \mid L) > 1$
$3/P(\neg B) - 3 > 1$
$3/P(\neg B) > 4$

$3 > 4 \times P(\neg B)$
$3/4 > P(\neg B)$
$0.75 > P(\neg B)$
$0.75 > 1 - P(B)$
$P(B) > 0.25$

We can not make a choice because we need some more data (prior information). But we can say that if the probability of finding a blue taxi is greater than 0.25, then if a taxi looks blue it will be most likely that the taxi is blue.

b. What if you know that 9 out of 10 Athenian taxis are green?

$P(Green) = P(\neg B) = 0.9$

From previous point we saw that if P(-B) < 0.75 then it will be more likely that the taxi is blue when it looks blue. We can reverse this and say that if P(-B) > 0.75 the taxi will be more likely to be green. And here P(-B) is greater than 0.75, then the taxi will be more likely to be green.

But we can go into more details:

$P(B \mid L) = 0.75 \times P(B) / P(L) = 0.75 \times 0.1 / P(L)$
$P(-B \mid L) = 0.25 \times P(\neg B) / P(L) = 0.25 \times 0.9 / P(L)$

$P(B \mid L) + P(\neg B \mid L) = 1$
$0.75 \times 0.1 + 0.25 \times 0.9 = P(L)$
$0.075 + 0.225 = P(L)$
$P(L) = 0.3$

$P(B \mid L) = 0.75 \times 0.1 / 0.3 = 0.075 / 0.3 = 0.25$
$P(\neg B \mid L) = 1 - 0.25 = 0.75$

$P(L \mid B) = P(B \mid L) \times P(L) / P(B)$
$P(L \mid B) = 0.25 \times 0.3 / 0.1 = 0.75$
$P(\neg L \mid B) = 0.25$

$P(B \mid \neg L) = P(\neg L \mid B) \times P(B) / P(\neg L)$
$P(B \mid \neg L) = 0.25 \times 0.1 / 0.7 = 0.025 / 0.7 = 1 / 28$
$P(\neg B \mid \neg L) = 1 - 1/28 = 27/28$

$P(L \mid \neg B) = P(\neg B \mid L) \times P(L) / P(\neg B)$
$P(L \mid \neg B) = 0.75 \times 0.3 / 0.9 = 0.25$
$P(\neg L \mid \neg B) = 1 - 0.25 = 0.75$

$P(L) = Probability\ that\ a\ taxi\ looks\ blue : 0.3$
$P(\neg L) = Probability\ that\ a\ taxi\ looks\ green : 0.7$
$P(B \mid L) = Probability\ that\ the\ taxi\ was\ blue\ when\ it\ looked\ blue : 0.25$

$P(\neg B \mid L) \;=\; Probability\ that\ the\ taxi\ was\ green\ when\ it\ looked\ blue :\; 0.75$
$P(B \mid \neg L) \;=\; Probability\ that\ the\ taxi\ was\ blue\ when\ it\ looked\ green :\; 1/28$
$P(\neg B \mid \neg L) \;=\; Probability\ that\ the\ taxi\ was\ green\ when\ it\ looked\ green :\; 27/28$