13.12 Show that the three forms of independence in Equation (13.11) are equivalent.

Equation (13.11):

$P(a \mid b) = P(a)$ *or* $P(b \mid a) = P(b)$ *or* $P(a \wedge b) = P(a) \times P(b)$

Bayes Rule says:

$P(a \mid b) = P(b \mid a) \times P(a) / P(b)$

If P $P(a \mid b) = P(a)$ then we can use the Bayes Rule formula and replace $P(a \mid b)$ with $P(a)$

$P(a) = P(b \mid a) \times P(a) / P(b)$

Dividing the whole formula by $P(a)$:

$1 = P(b \mid a) / P(b)$
$P(b \mid a) = P(b)$

So, if $P(a \mid b) = P(a)$ then $P(b \mid a) = P(b)$.

Using the same idea we arrive to the conclusion that if $P(b \mid a) = P(b)$ then $P(a \mid b) = P(a)$
So saying $P(b \mid a) = P(b)$ is equivalent to say that $P(a \mid b) = P(a)$

To show that last equation is equivalent to the other ones I need to use the Product Rule that says:

$P(a \wedge b) = P(a \mid b) \times P(b)$

And when $P(a \wedge b) = P(a) \times P(b)$ we can replace and say that:

$P(a \wedge b) = P(a) \times P(b) = P(a \mid b) \times P(b)$
$P(a) \times P(b) = P(a \mid b) \times P(b)$
$P(a) = P(a \mid b)$

So, if $P(a \wedge b) = P(a) \times P(b)$ then $P(a) = P(a \mid b)$

We know the intersection operator is commutative so:

$P(a \wedge b) = P(b \wedge a) = P(b \mid a) \times P(a)$ by the product rule

And when $P(a \wedge b) = P(a) \times P(b)$ we can replace and say that:

$P(b \mid a) \times P(a) = P(b) \times P(a)$
$P(b \mid a) = P(b)$

So, if $P(a \wedge b) = P(a) \times P(b)$ then $P(b) = P(b \mid a)$

Then the three equations are equivalent when a and b are independents. Q.E.D.