13.18 Suppose you are given a bag containing n unbiased coins. You are told that n – 1 of these coins are normal, with heads on one side and tails on the other, whereas one coin is a fake, with heads on both sides.
a. Suppose you reach into the bag, pick out a coin at random, flip it, and get a head. What is the (conditional) probability that the coin you chose is the fake coin?
b. Suppose you continue flipping the coin for a total of k times after picking it and see k heads. Now what is the conditional probability that you picked the fake coin?
c. Suppose you wanted to decide whether the chosen coin was fake by flipping it k times. The decision procedure returns fake if all k flips come up heads; otherwise it returns normal. What is the (unconditional) probability that this procedure makes an error?

a. Suppose you reach into the bag, pick out a coin at random, flip it, and get a head. What is the (conditional) probability that the coin you chose is the fake coin?

Using Bayes Rule:
$P(fake \mid head) = P(head \mid fake) \times P(fake) \,/\, P(head)$

$P(head \mid fake) \;=\; 1$
$P(fake) \;=\; 1/n$

A number of n coins presents a total of 2n faces where n+1 faces are heads and n-1 faces are tails.
$P(head) \;=\; (n+1) \,/\, 2n$

With the previous data we can calculate P(fake | head)
$P(fake \mid head) \;=\; 1 \times 1/n \,/\, ((n+1) \,/\, 2n) = 2/(n+1)$

b. Suppose you continue flipping the coin for a total of k times after picking it and see k heads. Now what is the conditional probability that you picked the fake coin?

Using conditional probabilities:

$P(Fake \mid heads_k) = \delta \times < P(fake \mid heads_k); P(\neg fake \mid heads_k) >$
$P(Fake \mid heads_k) = \delta \times < P(heads_k \mid fake) \times P(fake);\; P(heads_k \mid \neg fake) \times P(\neg fake) >$
$P(Fake \mid heads_k) = \delta \times < P(heads_k \mid fake) \times 1/n;\; P(heads_k \mid \neg fake) \times (n-1)/n >$
$P(Fake \mid heads_k) = \delta \times < 1 \times 1/n;\; 1/2^k \times (n-1)/n >$
$P(Fake \mid heads_k) = \delta \times < 1/n; (n-1)/(n \times 2^k) >$
$\delta \times (1/n \;+\; (n-1)/(n \times 2^k)) = 1$
$\delta = (n \times 2^k) \,/\, (2^k \;+ n - 1)$
$P(Fake \mid heads_k) = \;\; < (n \times 2^k) \,/\, (2^k \;+ n - 1) \;\times (1/n);\; (n \times 2^k) \,/\, (2^k \;+ n - 1) \times ((n-1)/(n \times 2^k)) >$
$P(Fake \mid heads_k) = \;\; < 2^k \,/\, (2^k + n - 1);\; (n-1) \,/\, (2^k + n - 1) >$

Using probability and combinatorial theories:

Number of atomic events for 1 coin: 2^k
Number of atomic events for n coins: $n \times 2^k$

Number of atomic events with the fake coin: 2^k
Number of atomic events with all heads: atomic events with fake coin + atomic events with normal coins
Number of atomic events with all heads: $2^k + (n-1)$

$P(fake \mid heads_k) = atomic\ events\ with\ fake\ coin\ /\ atomic\ events\ where\ we\ see\ all\ heads$
$P(fake \mid heads_k) = 2^k / (2^k + n - 1)$

$P(\neg fake \mid heads_k) = atomic\ events\ with\ normal\ coin\ /\ atomic\ events\ where\ we\ see\ all\ heads$
$P(\neg fake \mid heads_k) = (n-1) / (2^k + n - 1)$

c. Suppose you wanted to decide whether the chosen coin was fake by flipping it k times. The decision procedure returns fake if all k flips come up heads; otherwise it returns normal. What is the (unconditional) probability that this procedure makes an error?

Using the product rule:

$P(\neg fake,\ heads_k) = P(heads_k | \neg fake) \times P(\neg fake)$
$P(\neg fake,\ heads_k) = 1/2^k \times (n-1)/n\ = (n-1) / (n \times 2^k)$

Using probabilities:

Number of atomic events where we see k heads and the coin is not fake: $n-1$
Sample space size: $n \times 2^k$

$P(\neg fake,\ heads_k) = (n-1) / (n \times 2^k)$