Example 3

Working with translation invariant spaces

This example demonstrates the functions the MPAW Library provides for shift invariant spaces and introduces the Box spline based de la Vallée poussin means.

```
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Created:        15.08.2013
Last Changed:   15.08.2013
```

License

Loading the Library

The MPAWL is located in the parent directory (see `MPAWL.m`) in order to load the library, we add its path to **`$Path`**.

In[86]:=
```
$Path = Join[$Path, {ParentDirectory[NotebookDirectory[]]}];
SetDirectory[NotebookDirectory[]];(*Set to actual directory*)
Needs["MPAWL`"];
```

defining a function

We use the approach of the Dirichlet kernel, cf. [1], but compute them in the direct approach by setting

In[89]:=
```
mM = {{32, 4}, {-1, 8}}; MatrixForm[mM]
```

Out[89]//MatrixForm=

$$\begin{pmatrix} 32 & 4 \\ -1 & 8 \end{pmatrix}$$

In[90]:=
```
Abs[Det[mM]]
```

Out[90]= 260

In[91]:=
```
max = Table[Max[Ceiling[(1 / 2 Transpose[mM].#) [[j]] &/@
      {{1, 1}, {1, -1}, {-1, 1}, {-1, -1}}]] + 1, {j, 1, 2}]
```

Out[91]= {18, 7}

where in the following table the origin k = 0 is at

In[92]:=
```
origin = max + 1
```

Out[92]= {19, 8}

This table consists of the (not yet normalized or anything) coefficients which the Dirichlet kernel is based on.

In[93]:=
```
ckDM = Table[ If[Max[Abs[Transpose[Inverse[mM]].{k1, k2}]] ≤ 1 / 2, 1, 0],
    {k1, -max[[1]], max[[1]]}, {k2, -max[[2]], max[[2]]}];
MatrixForm[
 ckDM]
```

Out[94]//MatrixForm=

$$\begin{pmatrix}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&1&1&1&1&0&0&0&0&0\\
0&0&1&1&1&1&1&1&1&1&0&0&0&0&0\\
0&0&1&1&1&1&1&1&1&1&0&0&0&0&0\\
0&0&1&1&1&1&1&1&1&1&0&0&0&0&0\\
0&0&1&1&1&1&1&1&1&1&0&0&0&0&0\\
0&0&1&1&1&1&1&1&1&1&0&0&0&0&0\\
0&0&1&1&1&1&1&1&1&1&0&0&0&0&0\\
0&0&1&1&1&1&1&1&1&1&0&0&0&0&0\\
0&0&1&1&1&1&1&1&1&1&1&0&0&0&0\\
0&0&0&1&1&1&1&1&1&1&1&0&0&0&0\\
0&0&0&1&1&1&1&1&1&1&1&0&0&0&0\\
0&0&0&1&1&1&1&1&1&1&1&0&0&0&0\\
0&0&0&1&1&1&1&1&1&1&1&0&0&0&0\\
0&0&0&1&1&1&1&1&1&1&1&0&0&0&0\\
0&0&0&1&1&1&1&1&1&1&1&0&0&0&0\\
0&0&0&1&1&1&1&1&1&1&1&0&0&0&0\\
0&0&0&1&1&1&1&1&1&1&1&1&0&0&0\\
0&0&0&0&1&1&1&1&1&1&1&1&0&0&0\\
0&0&0&0&1&1&1&1&1&1&1&1&0&0&0\\
0&0&0&0&1&1&1&1&1&1&1&1&0&0&0\\
0&0&0&0&1&1&1&1&1&1&1&1&0&0&0\\
0&0&0&0&1&1&1&1&1&1&1&1&0&0&0\\
0&0&0&0&1&1&1&1&1&1&1&1&0&0&0\\
0&0&0&0&1&1&1&1&1&1&1&1&0&0&0\\
0&0&0&0&1&1&1&1&1&1&1&1&1&0&0\\
0&0&0&0&0&1&1&1&1&1&1&1&1&0&0\\
0&0&0&0&0&1&1&1&1&1&1&1&1&0&0\\
0&0&0&0&0&1&1&1&1&1&1&1&1&0&0\\
0&0&0&0&0&1&1&1&1&1&1&1&1&0&0\\
0&0&0&0&0&1&1&1&1&1&1&1&1&0&0\\
0&0&0&0&0&1&1&1&1&1&1&1&1&0&0\\
0&0&0&0&0&1&1&1&1&1&1&1&1&0&0\\
0&0&0&0&0&1&1&1&1&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0
\end{pmatrix}$$

The function is given by

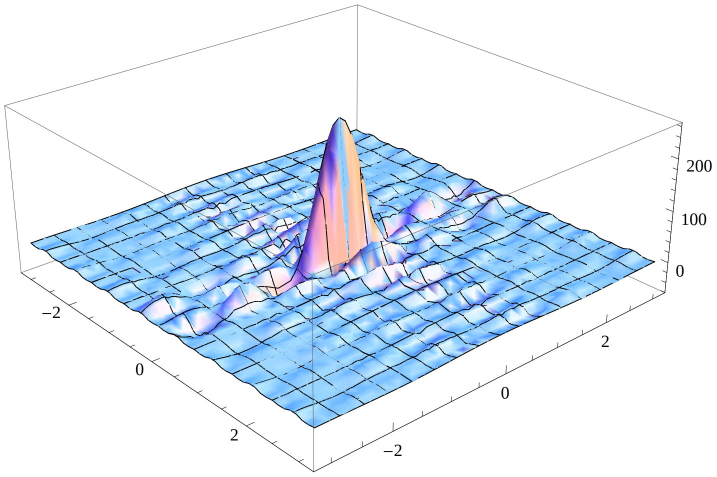
In[95]:=
```
dM[x_] := Sum[ ckDM[[Sequence @@ ({k1, k2} + origin)]] Exp[I {k1, k2}.x] ,
    {k1, -max[[1]], max[[1]]}, {k2, -max[[2]], max[[2]]}];
```

which we simplify for ploting by

In[96]:=
```
dMTerm = dM[{x, y}];
```

In[97]:=
```
Plot3D[dMTerm, {x, -π, π}, {y, -π, π}, PlotRange → All]
```

Out[97]=



If we now look at the corresponding Bracket sums (again adressed with respect to the cycles as mentioned in Example 2)

In[98]:=
```
? computeBracketSums
```

computeBracketSums[data,originIndex,mM]

Compute the sum over the congruence classes h+mM^T*z, where h is from the generating set and z runs through all integers adressing the values in data.

The result ist given with respect of the coefficients of th generating set basis, i.e. each h is decomposed with generatingSetBasisDecomp and these coefficients are used to adress the sum of h in the result. Here, originIndex denotes the index in data that corresponds to the origin.

Options

Validate → True | False
whether to perform a check (via isMatrixValid[mM]) on the matrix mM and the check, whether the Origin is in Range.

compute → "Bracket" | "absolute Squares"
despite just summing up the entries, the second option sums up the absolute squares of the data entries.

Index → None
If specified other than None, only this Index is computed and its value returned, provided it is in Range of the data.

In[99]:=
```
dMBS = computeBracketSums[ckDM, origin, mM]
```

Out[99]=
```
{1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1,
 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1,
 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1,
 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1,
 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 2, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1,
 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1,
 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1,
 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1,
 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1}
```

we see that exactely one bracket sum is not equal to 1. To obtain an interpolating basis we divide each Fourier c_k coefficient by its corresponding Bracket sum $[c]_k^M$ multiplied by m = Det[mM]. This can be done by seeing these divisors as coefficients in the space of traslates, i.e. by using

In[100]:=
```
? getFourierFromSpace
```

getFourierFromSpace[coefficients, ckSpace, originIndex, mM]

The coefficients represent the Fourier transform of the weights which applied to the translates --- with respect to mM – of a function φ result in a function f and φ with its Fourier coefficients ckSpace (where originIndex is the index representing the origin), this function reconstructs the Fourier coefficients of f.

Options

Validate → True | False

whether to perform a check (via isMatrixValid[mM]) on the matrix mM and the check, whether the Origin is in Range of the Fourier coefficients.

In[101]:=
```
ckDMIP = getFourierFromSpace[1 / (Abs[Det[mM]] dMBS), ckDM, origin, mM]
```

Out[101]=
```
{{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 1/520, 1/260, 1/260, 1/260, 1/260, 0, 0, 0, 0, 0},
 {0, 0, 1/260, 1/260, 1/260, 1/260, 1/260, 1/260, 1/260, 1/260, 0, 0, 0, 0, 0},
 {0, 0, 1/260, 1/260, 1/260, 1/260, 1/260, 1/260, 1/260, 1/260, 0, 0, 0, 0, 0},
 {0, 0, 1/260, 1/260, 1/260, 1/260, 1/260, 1/260, 1/260, 1/260, 0, 0, 0, 0, 0},
 {0, 0, 1/260, 1/260, 1/260, 1/260, 1/260, 1/260, 1/260, 1/260, 0, 0, 0, 0, 0},
 {0, 0, 1/260, 1/260, 1/260, 1/260, 1/260, 1/260, 1/260, 1/260, 0, 0, 0, 0, 0},
 {0, 0, 1/260, 1/260, 1/260, 1/260, 1/260, 1/260, 1/260, 1/260, 0, 0, 0, 0, 0},
 {0, 0, 1/260, 1/260, 1/260, 1/260, 1/260, 1/260, 1/260, 1/260, 0, 0, 0, 0, 0},
 {0, 0, 1/260, 1/260, 1/260, 1/260, 1/260, 1/260, 1/260, 1/260, 1/260, 0, 0, 0, 0},
 {0, 0, 0, 1/260, 1/260, 1/260, 1/260, 1/260, 1/260, 1/260, 1/260, 0, 0, 0, 0},
 {0, 0, 0, 1/260, 1/260, 1/260, 1/260, 1/260, 1/260, 1/260, 1/260, 0, 0, 0, 0},
 {0, 0, 0, 1/260, 1/260, 1/260, 1/260, 1/260, 1/260, 1/260, 1/260, 0, 0, 0, 0},
 {0, 0, 0, 1/260, 1/260, 1/260, 1/260, 1/260, 1/260, 1/260, 1/260, 0, 0, 0, 0},
 {0, 0, 0, 1/260, 1/260, 1/260, 1/260, 1/260, 1/260, 1/260, 1/260, 0, 0, 0, 0},
```

$\left\{0, 0, 0, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, 0, 0, 0, 0\right\},$

$\left\{0, 0, 0, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, 0, 0, 0, 0\right\},$

$\left\{0, 0, 0, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, 0, 0, 0\right\},$

$\left\{0, 0, 0, 0, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, 0, 0, 0\right\},$

$\left\{0, 0, 0, 0, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, 0, 0, 0\right\},$

$\left\{0, 0, 0, 0, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, 0, 0, 0\right\},$

$\left\{0, 0, 0, 0, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, 0, 0, 0\right\},$

$\left\{0, 0, 0, 0, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, 0, 0, 0\right\},$

$\left\{0, 0, 0, 0, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, 0, 0, 0\right\},$

$\left\{0, 0, 0, 0, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, 0, 0, 0\right\},$

$\left\{0, 0, 0, 0, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, 0, 0\right\},$

$\left\{0, 0, 0, 0, 0, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, 0, 0\right\},$

$\left\{0, 0, 0, 0, 0, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, 0, 0\right\},$

$\left\{0, 0, 0, 0, 0, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, 0, 0\right\},$

$\left\{0, 0, 0, 0, 0, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, 0, 0\right\},$

$\left\{0, 0, 0, 0, 0, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, 0, 0\right\},$

$\left\{0, 0, 0, 0, 0, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, 0, 0\right\},$

$\left\{0, 0, 0, 0, 0, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, 0, 0\right\},$

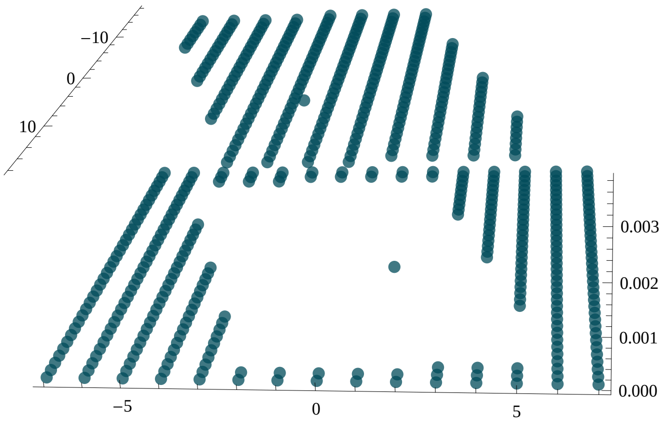
$\left\{0, 0, 0, 0, 0, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{520}, 0, 0, 0, 0, 0\right\},$

$\{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\},$

$\{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\}\}$

In[118]:=
```
ListPointPlot3D[Table[{k1, k2, ckDMIP[[Sequence @@ ({k1, k2} + origin)]]},
  {k1, -max[[1]], max[[1]]}, {k2, -max[[2]], max[[2]]}], Axes → True]
```

Out[118]=



We see, that all the points on the boundary of (**Transpose[mM]** times) the unit cube are different from the other coefficients and furthermore

In[102]:=
```
computeBracketSums[ckDMIP, origin, mM]
```

Out[102]=

$$
\begin{gathered}
\left\{\frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260},\right. \\
\frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \\
\frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \\
\frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \\
\frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \\
\frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \\
\frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \\
\frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \\
\frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \\
\frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \\
\frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \\
\frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \\
\frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \\
\frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \\
\frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \\
\frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \\
\frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \\
\frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \\
\left.\frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}, \frac{1}{260}\right\}
\end{gathered}
$$

In[25]:= which means, that

In[103]:=
```
dMIP[x_] := Sum[ ckDMIP[[Sequence @@ ({k1, k2} + origin)]] Exp[I {k1, k2}.x] ,
    {k1, -max[[1]], max[[1]]}, {k2, -max[[2]], max[[2]]}];
```

is a Lagrange interpolator, more precisely, from the pattern only y = 0 is 1, all other sampling points are zero : We use as in Example 1

In[104]:=
```
{v} = patternBasis[mM]
```

Out[104]=
$$\left\{\left\{\frac{2}{65}, \frac{1}{260}\right\}\right\}$$

In[105]:=
```
ϵ = IntegerSmithForm[mM, ExtendedForm → False][[2, 2]]
```

Out[105]= 260

to obtain

In[106]:=
```
Table[
 N[dMIP[2 π * modM[k * v, IdentityMatrix[2], Target → "Symmetric"]]], {k, 0, ϵ - 1}]
```

Out[106]= $\{1., 5.55112\times10^{-17}+0.\,i, 0.+0.\,i, 5.55112\times10^{-17}+0.\,i, 0.+0.\,i, 0.+0.\,i, 0.+0.\,i,$
$5.55112\times10^{-17}+0.\,i, 0.+0.\,i, 5.55112\times10^{-17}+0.\,i, 0.+0.\,i, 5.55112\times10^{-17}+0.\,i,$
$0.+0.\,i, 0.+0.\,i, 0.+0.\,i, 0.+0.\,i, 0.+0.\,i, 5.55112\times10^{-17}+0.\,i, 0.+0.\,i,$
$5.55112\times10^{-17}+0.\,i, 5.55112\times10^{-17}+0.\,i, 5.55112\times10^{-17}+0.\,i, 0.+0.\,i,$
$5.55112\times10^{-17}+0.\,i, 0.+0.\,i, 0.+0.\,i, 2.77556\times10^{-17}+0.\,i, 5.55112\times10^{-17}+0.\,i,$
$0.+0.\,i, 5.55112\times10^{-17}+0.\,i, 0.+0.\,i, 5.55112\times10^{-17}+0.\,i, 0.+0.\,i,$
$5.55112\times10^{-17}+0.\,i, 0.+0.\,i, 0.+0.\,i, 0.+0.\,i, 5.55112\times10^{-17}+0.\,i, 0.+0.\,i,$
$0.+0.\,i, 5.55112\times10^{-17}+0.\,i, 5.55112\times10^{-17}+0.\,i, 0.+0.\,i, 5.55112\times10^{-17}+0.\,i,$
$0.+0.\,i, 0.+0.\,i, 0.+0.\,i, 5.55112\times10^{-17}+0.\,i, 0.+0.\,i, 5.55112\times10^{-17}+0.\,i,$
$0.+0.\,i, 5.55112\times10^{-17}+0.\,i, 0.+0.\,i, 5.55112\times10^{-17}+0.\,i, 0.+0.\,i,$
$0.+0.\,i, 0.+0.\,i, 5.55112\times10^{-17}+0.\,i, 0.+0.\,i, 5.55112\times10^{-17}+0.\,i,$
$5.55112\times10^{-17}+0.\,i, 5.55112\times10^{-17}+0.\,i, 0.+0.\,i, 5.55112\times10^{-17}+0.\,i,$
$0.+0.\,i, 0., 0.+0.\,i, 5.55112\times10^{-17}+0.\,i, 0.+0.\,i, 5.55112\times10^{-17}+0.\,i,$
$0.+0.\,i, 5.55112\times10^{-17}+0.\,i, 0.+0.\,i, 5.55112\times10^{-17}+0.\,i, 0.+0.\,i, 0.+0.\,i,$
$0.+0.\,i, 5.55112\times10^{-17}+0.\,i, 2.77556\times10^{-17}+0.\,i, 5.55112\times10^{-17}+0.\,i,$
$5.55112\times10^{-17}+0.\,i, 5.55112\times10^{-17}+0.\,i, 0.+0.\,i, 5.55112\times10^{-17}+0.\,i,$
$0.+0.\,i, 0.+0.\,i, 0.+0.\,i, 5.55112\times10^{-17}+0.\,i, 0.+0.\,i, 5.55112\times10^{-17}+0.\,i,$
$0.+0.\,i, 0.+0.\,i, 0.+0.\,i, 5.55112\times10^{-17}+0.\,i, 0.+0.\,i, 0.+0.\,i, 0.+0.\,i,$
$5.55112\times10^{-17}+0.\,i, 0.+0.\,i, 5.55112\times10^{-17}+0.\,i, 5.55112\times10^{-17}+0.\,i,$
$5.55112\times10^{-17}+0.\,i, 0.+0.\,i, 5.55112\times10^{-17}+0.\,i, 0.+0.\,i, 0.+0.\,i, 0.+0.\,i,$
$5.55112\times10^{-17}+0.\,i, 0.+0.\,i, 5.55112\times10^{-17}+0.\,i, 0.+0.\,i, 5.55112\times10^{-17}+0.\,i,$
$0.+0.\,i, 5.55112\times10^{-17}+0.\,i, 0.+0.\,i, 0.+0.\,i, 0.+0.\,i, 0.+0.\,i, 0.+0.\,i,$
$5.55112\times10^{-17}+0.\,i, 5.55112\times10^{-17}+0.\,i, 5.55112\times10^{-17}+0.\,i, 0.+0.\,i,$
$5.55112\times10^{-17}+0.\,i, 0.+0.\,i, 0.+0.\,i, 0.+0.\,i, 5.55112\times10^{-17}+0.\,i, 0.+0.\,i,$
$5.55112\times10^{-17}+0.\,i, 0., 5.55112\times10^{-17}+0.\,i, 0.+0.\,i, 5.55112\times10^{-17}+0.\,i,$
$0.+0.\,i, 0.+0.\,i, 0.+0.\,i, 5.55112\times10^{-17}+0.\,i, 0.+0.\,i, 5.55112\times10^{-17}+0.\,i,$
$5.55112\times10^{-17}+0.\,i, 5.55112\times10^{-17}+0.\,i, 0.+0.\,i, 0.+0.\,i, 0.+0.\,i, 0.+0.\,i,$
$0.+0.\,i, 5.55112\times10^{-17}+0.\,i, 0.+0.\,i, 5.55112\times10^{-17}+0.\,i, 0.+0.\,i,$
$5.55112\times10^{-17}+0.\,i, 0.+0.\,i, 5.55112\times10^{-17}+0.\,i, 0.+0.\,i, 0.+0.\,i, 0.+0.\,i,$
$5.55112\times10^{-17}+0.\,i, 0.+0.\,i, 5.55112\times10^{-17}+0.\,i, 5.55112\times10^{-17}+0.\,i,$
$5.55112\times10^{-17}+0.\,i, 0.+0.\,i, 5.55112\times10^{-17}+0.\,i, 0.+0.\,i, 0.+0.\,i, 0.+0.\,i,$
$5.55112\times10^{-17}+0.\,i, 0.+0.\,i, 0.+0.\,i, 0.+0.\,i, 5.55112\times10^{-17}+0.\,i, 0.+0.\,i,$
$5.55112\times10^{-17}+0.\,i, 0.+0.\,i, 0.+0.\,i, 0.+0.\,i, 5.55112\times10^{-17}+0.\,i, 0.+0.\,i,$
$5.55112\times10^{-17}+0.\,i, 5.55112\times10^{-17}+0.\,i, 5.55112\times10^{-17}+0.\,i, 2.77556\times10^{-17}+0.\,i,$
$5.55112\times10^{-17}+0.\,i, 0.+0.\,i, 0.+0.\,i, 0.+0.\,i, 5.55112\times10^{-17}+0.\,i, 0.+0.\,i,$
$5.55112\times10^{-17}+0.\,i, 0.+0.\,i, 5.55112\times10^{-17}+0.\,i, 0.+0.\,i, 5.55112\times10^{-17}+0.\,i,$
$0.+0.\,i, 0., 0.+0.\,i, 5.55112\times10^{-17}+0.\,i, 0.+0.\,i, 5.55112\times10^{-17}+0.\,i,$
$5.55112\times10^{-17}+0.\,i, 5.55112\times10^{-17}+0.\,i, 0.+0.\,i, 5.55112\times10^{-17}+0.\,i, 0.+0.\,i,$
$0.+0.\,i, 0.+0.\,i, 5.55112\times10^{-17}+0.\,i, 0.+0.\,i, 5.55112\times10^{-17}+0.\,i, 0.+0.\,i,$
$5.55112\times10^{-17}+0.\,i, 0.+0.\,i, 5.55112\times10^{-17}+0.\,i, 0.+0.\,i, 0.+0.\,i, 0.+0.\,i,$
$5.55112\times10^{-17}+0.\,i, 0.+0.\,i, 5.55112\times10^{-17}+0.\,i, 5.55112\times10^{-17}+0.\,i, 0.+0.\,i,$
$0.+0.\,i, 5.55112\times10^{-17}+0.\,i, 0.+0.\,i, 0.+0.\,i, 0.+0.\,i, 5.55112\times10^{-17}+0.\,i,$
$0.+0.\,i, 5.55112\times10^{-17}+0.\,i, 0.+0.\,i, 5.55112\times10^{-17}+0.\,i, 0.+0.\,i,$
$5.55112\times10^{-17}+0.\,i, 2.77556\times10^{-17}+0.\,i, 0.+0.\,i, 0.+0.\,i, 5.55112\times10^{-17}+0.\,i,$
$0.+0.\,i, 5.55112\times10^{-17}+0.\,i, 5.55112\times10^{-17}+0.\,i, 5.55112\times10^{-17}+0.\,i,$
$0.+0.\,i, 5.55112\times10^{-17}+0.\,i, 0.+0.\,i, 0.+0.\,i, 0.+0.\,i, 0.+0.\,i, 0.+0.\,i,$
$5.55112\times10^{-17}+0.\,i, 0.+0.\,i, 5.55112\times10^{-17}+0.\,i, 0.+0.\,i, 5.55112\times10^{-17}+0.\,i,$
$0.+0.\,i, 0.+0.\,i, 0.+0.\,i, 5.55112\times10^{-17}+0.\,i, 0.+0.\,i, 5.55112\times10^{-17}+0.\,i\}$

In[21]:= where we ignore rounding errors to obtain

In[107]:= **Chop[%]**

Out[107]= {1., 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}

In[72]:= We can also use the method of the Bracket sums to see

In[108]:= **dMBSq = computeBracketSums[ckDM, origin, mM, Compute → "absolute Squares"]**

Out[108]= {1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1,
1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1,
1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1,
1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1,
1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 2, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1,
1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1,
1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1,
1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1,
1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1}

Which is in this case of course the same. In order to orthonormalize the translates (cf. Cor. 3.6 in [1]), we can compute similar to the last case

In[109]:= **ckDMon = getFourierFromSpace[1 / (Sqrt[Abs[Det[mM]] dMBS]), ckDM, origin, mM]**

Out[109]=
$$\Big\{\{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\},$$
$$\{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\},$$
$$\Big\{0, 0, 0, 0, 0, \frac{1}{2\sqrt{130}}, \frac{1}{2\sqrt{65}}, \frac{1}{2\sqrt{65}}, \frac{1}{2\sqrt{65}}, \frac{1}{2\sqrt{65}}, 0, 0, 0, 0, 0\Big\},$$
$$\Big\{0, 0, \frac{1}{2\sqrt{65}}, \frac{1}{2\sqrt{65}}, \frac{1}{2\sqrt{65}}, \frac{1}{2\sqrt{65}}, \frac{1}{2\sqrt{65}}, \frac{1}{2\sqrt{65}}, \frac{1}{2\sqrt{65}},$$
$$\frac{1}{2\sqrt{65}}, 0, 0, 0, 0, 0\Big\}, \Big\{0, 0, \frac{1}{2\sqrt{65}}, \frac{1}{2\sqrt{65}}, \frac{1}{2\sqrt{65}}, \frac{1}{2\sqrt{65}},$$
$$\frac{1}{2\sqrt{65}}, \frac{1}{2\sqrt{65}}, \frac{1}{2\sqrt{65}}, \frac{1}{2\sqrt{65}}, 0, 0, 0, 0, 0\Big\}, \Big\{0, 0, \frac{1}{2\sqrt{65}}, \frac{1}{2\sqrt{65}},$$
$$\frac{1}{2\sqrt{65}}, \frac{1}{2\sqrt{65}}, \frac{1}{2\sqrt{65}}, \frac{1}{2\sqrt{65}}, \frac{1}{2\sqrt{65}}, \frac{1}{2\sqrt{65}}, 0, 0, 0, 0, 0\Big\},$$
$$\Big\{0, 0, \frac{1}{2\sqrt{65}}, \frac{1}{2\sqrt{65}}, \frac{1}{2\sqrt{65}}, \frac{1}{2\sqrt{65}}, \frac{1}{2\sqrt{65}}, \frac{1}{2\sqrt{65}}, \frac{1}{2\sqrt{65}}, \frac{1}{2\sqrt{65}},$$
$$0, 0, 0, 0, 0\Big\}, \Big\{0, 0, \frac{1}{2\sqrt{65}}, \frac{1}{2\sqrt{65}}, \frac{1}{2\sqrt{65}}, \frac{1}{2\sqrt{65}}, \frac{1}{2\sqrt{65}},$$
$$\frac{1}{2\sqrt{65}}, \frac{1}{2\sqrt{65}}, \frac{1}{2\sqrt{65}}, 0, 0, 0, 0, 0\Big\}, \Big\{0, 0, \frac{1}{2\sqrt{65}}, \frac{1}{2\sqrt{65}},$$
$$\frac{1}{2\sqrt{65}}, \frac{1}{2\sqrt{65}}, \frac{1}{2\sqrt{65}}, \frac{1}{2\sqrt{65}}, \frac{1}{2\sqrt{65}}, \frac{1}{2\sqrt{65}}, 0, 0, 0, 0, 0\Big\},$$

$$\{0,\ 0,\ \frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},$$

$$\frac{1}{2\sqrt{65}},\ 0,\ 0,\ 0,\ 0,\ 0\},\ \{0,\ 0,\ \frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},$$

$$\frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},\ 0,\ 0,\ 0,\ 0\},$$

$$\{0,\ 0,\ 0,\ \frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},$$

$$\frac{1}{2\sqrt{65}},\ 0,\ 0,\ 0,\ 0\},\ \{0,\ 0,\ 0,\ \frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},$$

$$\frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},\ 0,\ 0,\ 0,\ 0\},\ \{0,\ 0,\ 0,\ \frac{1}{2\sqrt{65}},$$

$$\frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},\ 0,\ 0,\ 0,\ 0\},$$

$$\{0,\ 0,\ 0,\ \frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},$$

$$\frac{1}{2\sqrt{65}},\ 0,\ 0,\ 0,\ 0\},\ \{0,\ 0,\ 0,\ \frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},$$

$$\frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},\ 0,\ 0,\ 0,\ 0\},\ \{0,\ 0,\ 0,\ \frac{1}{2\sqrt{65}},$$

$$\frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},\ 0,\ 0,\ 0,\ 0\},$$

$$\{0,\ 0,\ 0,\ \frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},$$

$$\frac{1}{2\sqrt{65}},\ 0,\ 0,\ 0,\ 0\},\ \{0,\ 0,\ 0,\ \frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},$$

$$\frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},\ 0,\ 0,\ 0\},\ \{0,\ 0,\ 0,\ 0,\ \frac{1}{2\sqrt{65}},$$

$$\frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},\ 0,\ 0,\ 0\},$$

$$\{0,\ 0,\ 0,\ 0,\ \frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},$$

$$\frac{1}{2\sqrt{65}},\ 0,\ 0,\ 0\},\ \{0,\ 0,\ 0,\ 0,\ \frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},$$

$$\frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},\ 0,\ 0,\ 0\},\ \{0,\ 0,\ 0,\ 0,\ \frac{1}{2\sqrt{65}},$$

$$\frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},\ 0,\ 0,\ 0\},$$

$$\{0,\ 0,\ 0,\ 0,\ \frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},$$

$$\frac{1}{2\sqrt{65}},\ 0,\ 0,\ 0\},\ \{0,\ 0,\ 0,\ 0,\ \frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},$$

$$\frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},\ 0,\ 0,\ 0\Big\},\ \Big\{0,\ 0,\ 0,\ 0,\ \frac{1}{2\sqrt{65}},$$

$$\frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},\ 0,\ 0,\ 0\Big\},$$

$$\Big\{0,\ 0,\ 0,\ 0,\ \frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},$$

$$\frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},\ 0,\ 0\Big\},\ \Big\{0,\ 0,\ 0,\ 0,\ 0,\ \frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},$$

$$\frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},\ 0,\ 0\Big\},$$

$$\Big\{0,\ 0,\ 0,\ 0,\ 0,\ \frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},$$

$$\frac{1}{2\sqrt{65}},\ 0,\ 0\Big\},\ \Big\{0,\ 0,\ 0,\ 0,\ 0,\ \frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},$$

$$\frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},\ 0,\ 0\Big\},\ \Big\{0,\ 0,\ 0,\ 0,\ 0,\ \frac{1}{2\sqrt{65}},$$

$$\frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},\ 0,\ 0\Big\},$$

$$\Big\{0,\ 0,\ 0,\ 0,\ 0,\ \frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},$$

$$\frac{1}{2\sqrt{65}},\ 0,\ 0\Big\},\ \Big\{0,\ 0,\ 0,\ 0,\ 0,\ \frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},$$

$$\frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},\ 0,\ 0\Big\},\ \Big\{0,\ 0,\ 0,\ 0,\ 0,\ \frac{1}{2\sqrt{65}},$$

$$\frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},\ 0,\ 0\Big\},$$

$$\Big\{0,\ 0,\ 0,\ 0,\ 0,\ \frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{65}},\ \frac{1}{2\sqrt{130}},\ 0,\ 0,\ 0,\ 0,\ 0\Big\},$$

$$\{0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0\},$$

$$\{0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0\}\Big\}$$

In[110]:=
```
dMon[x_] := Sum[ckDMon[[Sequence @@ ({k1, k2} + origin)]] Exp[I {k1, k2}.x] ,
   {k1, -max[[1]], max[[1]]}, {k2, -max[[2]], max[[2]]}];
```

In[111]:=
```
dMonTerm = Simplify[dMon[{x1, x2}]];
```

Then the usual scalar product on $L^2_{2\pi}(\mathbb{T}^2)$ yields

In[112]:=
```
1 / (2 π) ^2 NIntegrate[Abs[dMonTerm] ^2 , {x1, -π, π}, {x2, -π, π}]
```

Out[112]= 1.

and as an example of the orthogonality of the translates using one translate by **v** and the Parseval's equation on $L^2_{2\pi}(\mathbb{T}^2)$

In[113]:=
```
Sum[ckDMon[[Sequence @@ ({k1, k2} + origin)]] *
  Conjugate[Exp[-2 π I {k1, k2}.v] * ckDMon[[Sequence @@ ({k1, k2} + origin)]]],
 {k1, -max[[1]], max[[1]]}, {k2, -max[[2]], max[[2]]}]
```

Out[113]=

$$\frac{1}{260} e^{-\frac{i\pi}{130}} + \frac{1}{260} e^{\frac{i\pi}{130}} + \frac{1}{260} e^{-\frac{i\pi}{65}} + \frac{1}{260} e^{\frac{i\pi}{65}} + \frac{1}{260} e^{-\frac{3i\pi}{130}} + \frac{1}{260} e^{\frac{3i\pi}{130}} + \frac{1}{260} e^{-\frac{2i\pi}{65}} + \frac{1}{260} e^{\frac{2i\pi}{65}} +$$

$$\frac{1}{260} e^{-\frac{i\pi}{26}} + \frac{1}{260} e^{\frac{i\pi}{26}} + \frac{1}{260} e^{-\frac{3i\pi}{65}} + \frac{1}{260} e^{\frac{3i\pi}{65}} + \frac{1}{260} e^{-\frac{7i\pi}{130}} + \frac{1}{260} e^{\frac{7i\pi}{130}} + \frac{1}{260} e^{-\frac{4i\pi}{65}} + \frac{1}{260} e^{\frac{4i\pi}{65}} +$$

$$\frac{1}{260} e^{-\frac{9i\pi}{130}} + \frac{1}{260} e^{\frac{9i\pi}{130}} + \frac{1}{260} e^{-\frac{i\pi}{13}} + \frac{1}{260} e^{\frac{i\pi}{13}} + \frac{1}{260} e^{-\frac{11i\pi}{130}} + \frac{1}{260} e^{\frac{11i\pi}{130}} + \frac{1}{260} e^{-\frac{6i\pi}{65}} + \frac{1}{260} e^{\frac{6i\pi}{65}} +$$

$$\frac{1}{260} e^{-\frac{i\pi}{10}} + \frac{1}{260} e^{\frac{i\pi}{10}} + \frac{1}{260} e^{-\frac{7i\pi}{65}} + \frac{1}{260} e^{\frac{7i\pi}{65}} + \frac{1}{260} e^{-\frac{3i\pi}{26}} + \frac{1}{260} e^{\frac{3i\pi}{26}} + \frac{1}{260} e^{-\frac{8i\pi}{65}} + \frac{1}{260} e^{\frac{8i\pi}{65}} +$$

$$\frac{1}{260} e^{-\frac{17i\pi}{130}} + \frac{1}{260} e^{\frac{17i\pi}{130}} + \frac{1}{260} e^{-\frac{9i\pi}{65}} + \frac{1}{260} e^{\frac{9i\pi}{65}} + \frac{1}{260} e^{-\frac{19i\pi}{130}} + \frac{1}{260} e^{\frac{19i\pi}{130}} + \frac{1}{260} e^{-\frac{2i\pi}{13}} +$$

$$\frac{1}{260} e^{\frac{2i\pi}{13}} + \frac{1}{260} e^{-\frac{21i\pi}{130}} + \frac{1}{260} e^{\frac{21i\pi}{130}} + \frac{1}{260} e^{-\frac{11i\pi}{65}} + \frac{1}{260} e^{\frac{11i\pi}{65}} + \frac{1}{260} e^{-\frac{23i\pi}{130}} + \frac{1}{260} e^{\frac{23i\pi}{130}} +$$

$$\frac{1}{260} e^{-\frac{12i\pi}{65}} + \frac{1}{260} e^{\frac{12i\pi}{65}} + \frac{1}{260} e^{-\frac{5i\pi}{26}} + \frac{1}{260} e^{\frac{5i\pi}{26}} + \frac{1}{260} e^{-\frac{i\pi}{5}} + \frac{1}{260} e^{\frac{i\pi}{5}} + \frac{1}{260} e^{-\frac{27i\pi}{130}} +$$

$$\frac{1}{260} e^{\frac{27i\pi}{130}} + \frac{1}{260} e^{-\frac{14i\pi}{65}} + \frac{1}{260} e^{\frac{14i\pi}{65}} + \frac{1}{260} e^{-\frac{29i\pi}{130}} + \frac{1}{260} e^{\frac{29i\pi}{130}} + \frac{1}{260} e^{-\frac{3i\pi}{13}} + \frac{1}{260} e^{\frac{3i\pi}{13}} +$$

$$\frac{1}{260} e^{-\frac{31i\pi}{130}} + \frac{1}{260} e^{\frac{31i\pi}{130}} + \frac{1}{260} e^{-\frac{16i\pi}{65}} + \frac{1}{260} e^{\frac{16i\pi}{65}} + \frac{1}{260} e^{-\frac{33i\pi}{130}} + \frac{1}{260} e^{\frac{33i\pi}{130}} + \frac{1}{260} e^{-\frac{17i\pi}{65}} +$$

$$\frac{1}{260} e^{\frac{17i\pi}{65}} + \frac{1}{260} e^{-\frac{7i\pi}{26}} + \frac{1}{260} e^{\frac{7i\pi}{26}} + \frac{1}{260} e^{-\frac{18i\pi}{65}} + \frac{1}{260} e^{\frac{18i\pi}{65}} + \frac{1}{260} e^{-\frac{37i\pi}{130}} + \frac{1}{260} e^{\frac{37i\pi}{130}} +$$

$$\frac{1}{260} e^{-\frac{19i\pi}{65}} + \frac{1}{260} e^{\frac{19i\pi}{65}} + \frac{1}{260} e^{-\frac{3i\pi}{10}} + \frac{1}{260} e^{\frac{3i\pi}{10}} + \frac{1}{260} e^{-\frac{4i\pi}{13}} + \frac{1}{260} e^{\frac{4i\pi}{13}} + \frac{1}{260} e^{-\frac{41i\pi}{130}} +$$

$$\frac{1}{260} e^{\frac{41i\pi}{130}} + \frac{1}{260} e^{-\frac{21i\pi}{65}} + \frac{1}{260} e^{\frac{21i\pi}{65}} + \frac{1}{260} e^{-\frac{43i\pi}{130}} + \frac{1}{260} e^{\frac{43i\pi}{130}} + \frac{1}{260} e^{-\frac{22i\pi}{65}} + \frac{1}{260} e^{\frac{22i\pi}{65}} +$$

$$\frac{1}{260} e^{-\frac{9i\pi}{26}} + \frac{1}{260} e^{\frac{9i\pi}{26}} + \frac{1}{260} e^{-\frac{23i\pi}{65}} + \frac{1}{260} e^{\frac{23i\pi}{65}} + \frac{1}{260} e^{-\frac{47i\pi}{130}} + \frac{1}{260} e^{\frac{47i\pi}{130}} + \frac{1}{260} e^{-\frac{24i\pi}{65}} +$$

$$\frac{1}{260} e^{\frac{24i\pi}{65}} + \frac{1}{260} e^{-\frac{49i\pi}{130}} + \frac{1}{260} e^{\frac{49i\pi}{130}} + \frac{1}{260} e^{-\frac{5i\pi}{13}} + \frac{1}{260} e^{\frac{5i\pi}{13}} + \frac{1}{260} e^{-\frac{51i\pi}{130}} + \frac{1}{260} e^{\frac{51i\pi}{130}} +$$

$$\frac{1}{260} e^{-\frac{2i\pi}{5}} + \frac{1}{260} e^{\frac{2i\pi}{5}} + \frac{1}{260} e^{-\frac{53i\pi}{130}} + \frac{1}{260} e^{\frac{53i\pi}{130}} + \frac{1}{260} e^{-\frac{27i\pi}{65}} + \frac{1}{260} e^{\frac{27i\pi}{65}} + \frac{1}{260} e^{-\frac{11i\pi}{26}} +$$

$$\frac{1}{260} e^{\frac{11i\pi}{26}} + \frac{1}{260} e^{-\frac{28i\pi}{65}} + \frac{1}{260} e^{\frac{28i\pi}{65}} + \frac{1}{260} e^{-\frac{57i\pi}{130}} + \frac{1}{260} e^{\frac{57i\pi}{130}} + \frac{1}{260} e^{-\frac{29i\pi}{65}} + \frac{1}{260} e^{\frac{29i\pi}{65}} +$$

$$\frac{1}{260} e^{-\frac{59i\pi}{130}} + \frac{1}{260} e^{\frac{59i\pi}{130}} + \frac{1}{260} e^{-\frac{6i\pi}{13}} + \frac{1}{260} e^{\frac{6i\pi}{13}} + \frac{1}{260} e^{-\frac{61i\pi}{130}} + \frac{1}{260} e^{\frac{61i\pi}{130}} + \frac{1}{260} e^{-\frac{31i\pi}{65}} +$$

$$\frac{1}{260} e^{\frac{31i\pi}{65}} + \frac{1}{260} e^{-\frac{63i\pi}{130}} + \frac{1}{260} e^{\frac{63i\pi}{130}} + \frac{1}{260} e^{-\frac{32i\pi}{65}} + \frac{1}{260} e^{\frac{32i\pi}{65}} + \frac{1}{260} e^{-\frac{33i\pi}{65}} + \frac{1}{260} e^{\frac{33i\pi}{65}} +$$

$$\frac{1}{260} e^{-\frac{67i\pi}{130}} + \frac{1}{260} e^{\frac{67i\pi}{130}} + \frac{1}{260} e^{-\frac{34i\pi}{65}} + \frac{1}{260} e^{\frac{34i\pi}{65}} + \frac{1}{260} e^{-\frac{69i\pi}{130}} + \frac{1}{260} e^{\frac{69i\pi}{130}} + \frac{1}{260} e^{-\frac{7i\pi}{13}} +$$

$$\frac{1}{260} e^{\frac{7i\pi}{13}} + \frac{1}{260} e^{-\frac{71i\pi}{130}} + \frac{1}{260} e^{\frac{71i\pi}{130}} + \frac{1}{260} e^{-\frac{36i\pi}{65}} + \frac{1}{260} e^{\frac{36i\pi}{65}} + \frac{1}{260} e^{-\frac{73i\pi}{130}} + \frac{1}{260} e^{\frac{73i\pi}{130}} +$$

$$\frac{1}{260} e^{-\frac{37i\pi}{65}} + \frac{1}{260} e^{\frac{37i\pi}{65}} + \frac{1}{260} e^{-\frac{15i\pi}{26}} + \frac{1}{260} e^{\frac{15i\pi}{26}} + \frac{1}{260} e^{-\frac{38i\pi}{65}} + \frac{1}{260} e^{\frac{38i\pi}{65}} + \frac{1}{260} e^{-\frac{77i\pi}{130}} +$$

$$\frac{1}{260} e^{\frac{77i\pi}{130}} + \frac{1}{260} e^{-\frac{3i\pi}{5}} + \frac{1}{260} e^{\frac{3i\pi}{5}} + \frac{1}{260} e^{-\frac{79i\pi}{130}} + \frac{1}{260} e^{\frac{79i\pi}{130}} + \frac{1}{260} e^{-\frac{8i\pi}{13}} + \frac{1}{260} e^{\frac{8i\pi}{13}} +$$

$$\frac{1}{260} e^{-\frac{81i\pi}{130}} + \frac{1}{260} e^{\frac{81i\pi}{130}} + \frac{1}{260} e^{-\frac{41i\pi}{65}} + \frac{1}{260} e^{\frac{41i\pi}{65}} + \frac{1}{260} e^{-\frac{83i\pi}{130}} + \frac{1}{260} e^{\frac{83i\pi}{130}} + \frac{1}{260} e^{-\frac{42i\pi}{65}} +$$

$$\frac{1}{260} e^{\frac{42i\pi}{65}} + \frac{1}{260} e^{-\frac{17i\pi}{26}} + \frac{1}{260} e^{\frac{17i\pi}{26}} + \frac{1}{260} e^{-\frac{43i\pi}{65}} + \frac{1}{260} e^{\frac{43i\pi}{65}} + \frac{1}{260} e^{-\frac{87i\pi}{130}} + \frac{1}{260} e^{\frac{87i\pi}{130}} +$$

$$\frac{1}{260} e^{-\frac{44 i \pi}{65}} + \frac{1}{260} e^{\frac{44 i \pi}{65}} + \frac{1}{260} e^{-\frac{89 i \pi}{130}} + \frac{1}{260} e^{\frac{89 i \pi}{130}} + \frac{1}{260} e^{-\frac{9 i \pi}{13}} + \frac{1}{260} e^{\frac{9 i \pi}{13}} + \frac{1}{260} e^{-\frac{7 i \pi}{10}} +$$
$$\frac{1}{260} e^{\frac{7 i \pi}{10}} + \frac{1}{260} e^{-\frac{46 i \pi}{65}} + \frac{1}{260} e^{\frac{46 i \pi}{65}} + \frac{1}{260} e^{-\frac{93 i \pi}{130}} + \frac{1}{260} e^{\frac{93 i \pi}{130}} + \frac{1}{260} e^{-\frac{47 i \pi}{65}} + \frac{1}{260} e^{\frac{47 i \pi}{65}} +$$
$$\frac{1}{260} e^{-\frac{19 i \pi}{26}} + \frac{1}{260} e^{\frac{19 i \pi}{26}} + \frac{1}{260} e^{-\frac{48 i \pi}{65}} + \frac{1}{260} e^{\frac{48 i \pi}{65}} + \frac{1}{260} e^{-\frac{97 i \pi}{130}} + \frac{1}{260} e^{\frac{97 i \pi}{130}} + \frac{1}{260} e^{-\frac{49 i \pi}{65}} +$$
$$\frac{1}{260} e^{\frac{49 i \pi}{65}} + \frac{1}{260} e^{-\frac{99 i \pi}{130}} + \frac{1}{260} e^{\frac{99 i \pi}{130}} + \frac{1}{260} e^{-\frac{10 i \pi}{13}} + \frac{1}{260} e^{\frac{10 i \pi}{13}} + \frac{1}{260} e^{-\frac{101 i \pi}{130}} + \frac{1}{260} e^{\frac{101 i \pi}{130}} +$$
$$\frac{1}{260} e^{-\frac{51 i \pi}{65}} + \frac{1}{260} e^{\frac{51 i \pi}{65}} + \frac{1}{260} e^{-\frac{103 i \pi}{130}} + \frac{1}{260} e^{\frac{103 i \pi}{130}} + \frac{1}{260} e^{-\frac{4 i \pi}{5}} + \frac{1}{260} e^{\frac{4 i \pi}{5}} + \frac{1}{260} e^{-\frac{21 i \pi}{26}} +$$
$$\frac{1}{260} e^{\frac{21 i \pi}{26}} + \frac{1}{260} e^{-\frac{53 i \pi}{65}} + \frac{1}{260} e^{\frac{53 i \pi}{65}} + \frac{1}{260} e^{-\frac{107 i \pi}{130}} + \frac{1}{260} e^{\frac{107 i \pi}{130}} + \frac{1}{260} e^{-\frac{54 i \pi}{65}} + \frac{1}{260} e^{\frac{54 i \pi}{65}} +$$
$$\frac{1}{260} e^{-\frac{109 i \pi}{130}} + \frac{1}{260} e^{\frac{109 i \pi}{130}} + \frac{1}{260} e^{-\frac{11 i \pi}{13}} + \frac{1}{260} e^{\frac{11 i \pi}{13}} + \frac{1}{260} e^{-\frac{111 i \pi}{130}} + \frac{1}{260} e^{\frac{111 i \pi}{130}} + \frac{1}{260} e^{-\frac{56 i \pi}{65}} +$$
$$\frac{1}{260} e^{\frac{56 i \pi}{65}} + \frac{1}{260} e^{-\frac{113 i \pi}{130}} + \frac{1}{260} e^{\frac{113 i \pi}{130}} + \frac{1}{260} e^{-\frac{57 i \pi}{65}} + \frac{1}{260} e^{\frac{57 i \pi}{65}} + \frac{1}{260} e^{-\frac{23 i \pi}{26}} + \frac{1}{260} e^{\frac{23 i \pi}{26}} +$$
$$\frac{1}{260} e^{-\frac{58 i \pi}{65}} + \frac{1}{260} e^{\frac{58 i \pi}{65}} + \frac{1}{260} e^{-\frac{9 i \pi}{10}} + \frac{1}{260} e^{\frac{9 i \pi}{10}} + \frac{1}{260} e^{-\frac{59 i \pi}{65}} + \frac{1}{260} e^{\frac{59 i \pi}{65}} + \frac{1}{260} e^{-\frac{119 i \pi}{130}} +$$
$$\frac{1}{260} e^{\frac{119 i \pi}{130}} + \frac{1}{260} e^{-\frac{12 i \pi}{13}} + \frac{1}{260} e^{\frac{12 i \pi}{13}} + \frac{1}{260} e^{-\frac{121 i \pi}{130}} + \frac{1}{260} e^{\frac{121 i \pi}{130}} + \frac{1}{260} e^{-\frac{61 i \pi}{65}} + \frac{1}{260} e^{\frac{61 i \pi}{65}} +$$
$$\frac{1}{260} e^{-\frac{123 i \pi}{130}} + \frac{1}{260} e^{\frac{123 i \pi}{130}} + \frac{1}{260} e^{-\frac{62 i \pi}{65}} + \frac{1}{260} e^{\frac{62 i \pi}{65}} + \frac{1}{260} e^{-\frac{25 i \pi}{26}} + \frac{1}{260} e^{\frac{25 i \pi}{26}} + \frac{1}{260} e^{-\frac{63 i \pi}{65}} +$$
$$\frac{1}{260} e^{\frac{63 i \pi}{65}} + \frac{1}{260} e^{-\frac{127 i \pi}{130}} + \frac{1}{260} e^{\frac{127 i \pi}{130}} + \frac{1}{260} e^{-\frac{64 i \pi}{65}} + \frac{1}{260} e^{\frac{64 i \pi}{65}} + \frac{1}{260} e^{-\frac{129 i \pi}{130}} + \frac{1}{260} e^{\frac{129 i \pi}{130}}$$

In[114]:=
```
Simplify[%]
```

Out[114]= 0

and hence orthogonality.

Literature

[1] D.Langemann, J.Prestin, *Multivariate periodic wavelet analysis*, Appl. Comput. Harmon. Anal.28 (2010) 46–66, doi: 10.1016/j.acha.2009.07.001