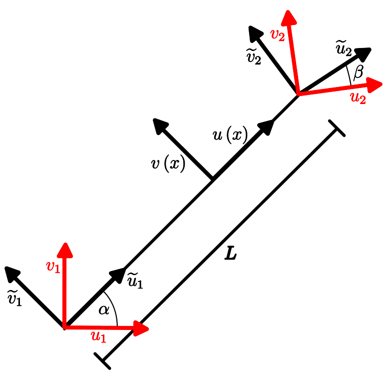
1 Truss Elements

$$K_e = \frac{EA}{L}\begin{bmatrix} (\cos(\alpha))^2 & \sin(\alpha)\cos(\alpha) & -\cos(\alpha)\cos(\beta) & -\cos(\alpha)\sin(\beta) \\ \sin(\alpha)\cos(\alpha) & (\sin(\alpha))^2 & -\sin(\alpha)\cos(\beta) & -\sin(\alpha)\sin(\beta) \\ -\cos(\alpha)\cos(\beta) & -\sin(\alpha)\cos(\beta) & (\cos(\beta))^2 & \sin(\beta)\cos(\beta) \\ -\cos(\alpha)\sin(\beta) & -\sin(\alpha)\sin(\beta) & \sin(\beta)\cos(\beta) & (\sin(\beta))^2 \end{bmatrix}, u_e = \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{bmatrix}$$

$$K_e = \frac{EA}{L}\begin{bmatrix} (\cos(\alpha))^2 & \sin(\alpha)\cos(\alpha) & -\cos(\alpha)\cos(\beta) & -\cos(\alpha)\sin(\beta) \\ & (\sin(\alpha))^2 & -\sin(\alpha)\cos(\beta) & -\sin(\alpha)\sin(\beta) \\ & & (\cos(\beta))^2 & \sin(\beta)\cos(\beta) \\ sym. & & & (\sin(\beta))^2 \end{bmatrix}, u_e = \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{bmatrix}$$



$\boldsymbol{\alpha = 0° = 180°}$, $\boldsymbol{\beta = 0° = 180°}$

$$K = \frac{EA}{L}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\boldsymbol{\alpha = 30° = -150°}$, $\boldsymbol{\beta = 30° = -150°}$

$$K = \frac{EA}{4L}\begin{bmatrix} 3 & \sqrt{3} & -3 & -\sqrt{3} \\ \sqrt{3} & 1 & -\sqrt{3} & -1 \\ -3 & -\sqrt{3} & 3 & \sqrt{3} \\ -\sqrt{3} & -1 & \sqrt{3} & 1 \end{bmatrix}$$

$\boldsymbol{\alpha = 45° = -135°}$, $\boldsymbol{\beta = 45° = -135°}$

$$K = \frac{EA}{2L}\begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \end{bmatrix}$$

$\boldsymbol{\alpha = 60° = -120°}$, $\boldsymbol{\beta = 60° = -120°}$

$$K = \frac{EA}{4L}\begin{bmatrix} 1 & \sqrt{3} & -1 & -\sqrt{3} \\ \sqrt{3} & 3 & -\sqrt{3} & -3 \\ -1 & -\sqrt{3} & 1 & \sqrt{3} \\ -\sqrt{3} & -3 & \sqrt{3} & 3 \end{bmatrix}$$

$\boldsymbol{\alpha = 90° = -90°}$, $\boldsymbol{\beta = 90° = -90°}$

$$K = \frac{EA}{L}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

$\boldsymbol{\alpha = 120° = -60°}$, $\boldsymbol{\beta = 120° = -60°}$

$$K = \frac{EA}{4L}\begin{bmatrix} 1 & -\sqrt{3} & -1 & \sqrt{3} \\ -\sqrt{3} & 3 & \sqrt{3} & -3 \\ -1 & \sqrt{3} & 1 & -\sqrt{3} \\ \sqrt{3} & -3 & -\sqrt{3} & 3 \end{bmatrix}$$

$\boldsymbol{\alpha = 135° = -45°}$, $\boldsymbol{\beta = 135° = -45°}$

$$K = \frac{EA}{2L}\begin{bmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

$\boldsymbol{\alpha = 150° = -30°}$, $\boldsymbol{\beta = 150° = -30°}$

$$K = \frac{EA}{4L}\begin{bmatrix} 3 & -\sqrt{3} & -3 & \sqrt{3} \\ -\sqrt{3} & 1 & \sqrt{3} & -1 \\ -3 & \sqrt{3} & 3 & -\sqrt{3} \\ \sqrt{3} & -1 & -\sqrt{3} & 1 \end{bmatrix}$$

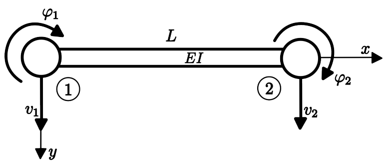
by Keno Moenck and Nils Albrecht

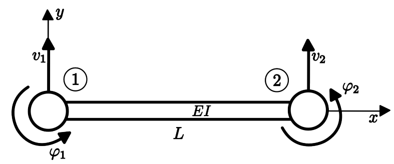
2 Hermite Beam Elements

2.1 Steifigkeitsmatrix

$$K_e = \frac{EI}{L^3}\begin{bmatrix} 12 & 6\,L & -12 & 6\,L \\ 6\,L & 4\,L^2 & -6\,L & 2\,L^2 \\ -12 & -6\,L & 12 & -6\,L \\ 6\,L & 2\,L^2 & -6\,L & 4\,L^2 \end{bmatrix} = \frac{EI}{L^3}\begin{bmatrix} 12 & 6\,L & -12 & 6\,L \\ & 4\,L^2 & -6\,L & 2\,L^2 \\ & & 12 & -6\,L \\ sym. & & & 4\,L^2 \end{bmatrix}, u_e = \begin{bmatrix} w_1 \\ \varphi_1 \\ w_2 \\ \varphi_2 \end{bmatrix}$$

2.2 Definition



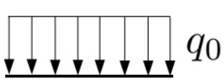


2.3 Einheiten

$$\left[\frac{E\cdot I}{L^3}\cdot \boldsymbol{K_e}\cdot \boldsymbol{U_e}\right] = \underbrace{\frac{\frac{N}{m^2}\cdot m^4}{m^3}}_{\frac{N}{m}}\begin{bmatrix} & m & & m \\ m & m^2 & m & m^2 \\ & m & & m \\ m & m^2 & m & m^2 \end{bmatrix}\cdot\begin{bmatrix} m \\ \circ \\ m \\ \circ \end{bmatrix} = \begin{bmatrix} N \\ Nm \\ N \\ Nm \end{bmatrix} = [\boldsymbol{R}]$$

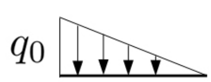
2.4 Kraftinterpolation und Flächenlast

$$H^T = \begin{bmatrix} 1 - 3\,\frac{x^2}{L^2} + 2\,\frac{x^3}{L^3} \\ x - 2\,\frac{x^2}{L} + \frac{x^3}{L^2} \\ 3\,\frac{x^2}{L^2} - 2\,\frac{x^3}{L^3} \\ -\frac{x^2}{L} + \frac{x^3}{L^2} \end{bmatrix} \qquad \int dx = \begin{bmatrix} x - \frac{x^3}{L^2} + \frac{1}{2}\,\frac{x^4}{L^3} \\ \frac{1}{2}\,x^2 - \frac{2}{3}\,\frac{x^3}{L} + \frac{1}{4}\,\frac{x^4}{L^2} \\ \frac{x^3}{L^2} - \frac{1}{2}\,\frac{x^4}{L^3} \\ -\frac{1}{3}\,\frac{x^3}{L} + \frac{1}{4}\,\frac{x^4}{L^2} \end{bmatrix} \qquad R = \int_0^L H^T\cdot q(x)dx$$



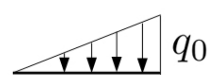
$$q(x) = q_0$$

$$R = \begin{bmatrix} \frac{1}{2}\,Lq_0 \\ \frac{1}{12}\,L^2q_0 \\ \frac{1}{2}\,Lq_0 \\ -\frac{1}{12}\,L^2q_0 \end{bmatrix}$$



$$q(x) = (1-\tfrac{x}{L})\cdot q_0$$

$$R = \begin{bmatrix} \frac{7}{20}\,Lq_0 \\ \frac{1}{20}\,L^2q_0 \\ \frac{3}{20}\,Lq_0 \\ -\frac{1}{30}\,L^2q_0 \end{bmatrix}$$



$$q(x) = \tfrac{x}{L}\cdot q_0$$

$$R = \begin{bmatrix} \frac{3}{20}\,Lq_0 \\ \frac{1}{30}\,L^2q_0 \\ \frac{7}{20}\,Lq_0 \\ -\frac{1}{20}\,L^2q_0 \end{bmatrix}$$

$U = [w_1,\, \varphi_1,\, w_2,\, \varphi_2]^T$: Nodal Displacement

$w(x)$ an der Stelle x:
$w\,(x) = H\,(x)\cdot U$

φ an der Stelle x:
$\varphi\,(x) = w'\,(x) = H'\,(x)\cdot U$

$$\frac{\mathrm{d}}{\mathrm{d}x}H^T = \begin{bmatrix} -6\,\frac{x}{L^2} + 6\,\frac{x^2}{L^3} \\ 1 - 4\,\frac{x}{L} + 3\,\frac{x^2}{L^2} \\ 6\,\frac{x}{L^2} - 6\,\frac{x^2}{L^3} \\ -2\,\frac{x}{L} + 3\,\frac{x^2}{L^2} \end{bmatrix}$$

3 Isoparametric Plate Elements

3.1 Definitions

- Determine h_i-functions from table.
- Calculate $x(r,\, s)$ and $y(r,\, s)$:

$$x(r,\, s) = \sum x_i\, h_i$$
$$y(r,\, s) = \sum y_i\, h_i$$

  x_i and y_i are the corresponding $(x,\, y)$-coordinates to the node i.

- Set up the Jacobian and determine the determinant:

$$J = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \end{bmatrix},\ det(J)$$

- Calculate the inverse of the Jacobian:

$$J^{-1} = \frac{adj(J)}{det(J)}$$

- Determine ∇H for each node:

$$\nabla H = \begin{bmatrix} \frac{\partial h_i}{\partial x} \\ \frac{\partial h_i}{\partial y} \end{bmatrix} = J^{-1} \begin{bmatrix} \frac{\partial h_i}{\partial r} \\ \frac{\partial h_i}{\partial s} \end{bmatrix}$$

- Calculate Strain-Displacement Matrix B:

$$B = \begin{bmatrix} \frac{\partial h_i}{\partial x} & 0 & \\ 0 & \frac{\partial h_i}{\partial y} & \ldots \\ \frac{\partial h_i}{\partial y} & \frac{\partial h_i}{\partial x} & \end{bmatrix}$$

- Find Material Matrix C from table and Poission number ν.

3.2 Stiffness Matrix

- General case:

$$K = \int_V B^T\, C\, B\, dV$$

- For a **Quadrilateral** Element:

$$K = t \int_{-1}^{1} \int_{-1}^{1} B^T\, C\, B\ \cdot det(J)\ dr\, ds$$

- For a **Triangular** Element:

$$K = t \int_{0}^{1} \int_{0}^{1-r} B^T\, C\, B\ \cdot det(J)\ ds\, dr$$

3.3 Displacement

$$\begin{bmatrix} u(r,\, s) \\ v(r,\, s) \end{bmatrix} = H \cdot \hat{U} \tag{1}$$

$$\begin{bmatrix} u(r,\, s) \\ v(r,\, s) \end{bmatrix} = \begin{bmatrix} h_1 & 0 & h_2 & 0 & \\ 0 & h_1 & 0 & h_2 & \ldots \end{bmatrix} \cdot \begin{bmatrix} \hat{u}_1 \\ \hat{v}_1 \\ \hat{u}_2 \\ \hat{v}_2 \\ \ldots \end{bmatrix} \tag{2}$$

3.4 Strain

$\boldsymbol{\epsilon}$ is a inner deformation quantity.

$$\boldsymbol{\epsilon} = \frac{\Delta L}{L} \tag{3}$$

$$\boldsymbol{\epsilon}(r,\, s) = \begin{bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \gamma_{xy} \end{bmatrix} = B(r,\, s) \cdot \begin{bmatrix} \hat{u}_1 \\ \hat{v}_1 \\ \hat{u}_2 \\ \hat{v}_2 \\ \ldots \end{bmatrix} \tag{4}$$

3.5 Stress

$\boldsymbol{\sigma}$ or sometimes $\boldsymbol{\tau}$ (same, but different notation!) is a inner stress quantity.

$$\sigma = E\, \epsilon \tag{5}$$
$$\boldsymbol{\sigma} = C\, \boldsymbol{\epsilon} \tag{6}$$
$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{bmatrix} = C\, B(r,\, s)\, \hat{U} \tag{7}$$

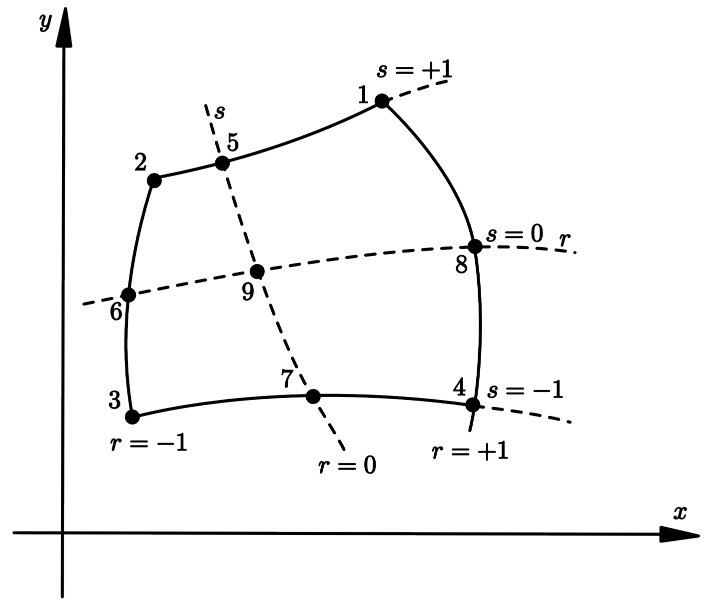
mit den Normalspannungen (σ_{xx} und σ_{yy}) und der Schubspannung τ_{xy}.

3.6 Load Vector

- Determine $q(r, s)$ in global coordinates and depending on $(r,\, s)$.
- Integrate over the Area A (triangular or quadrilateral element)

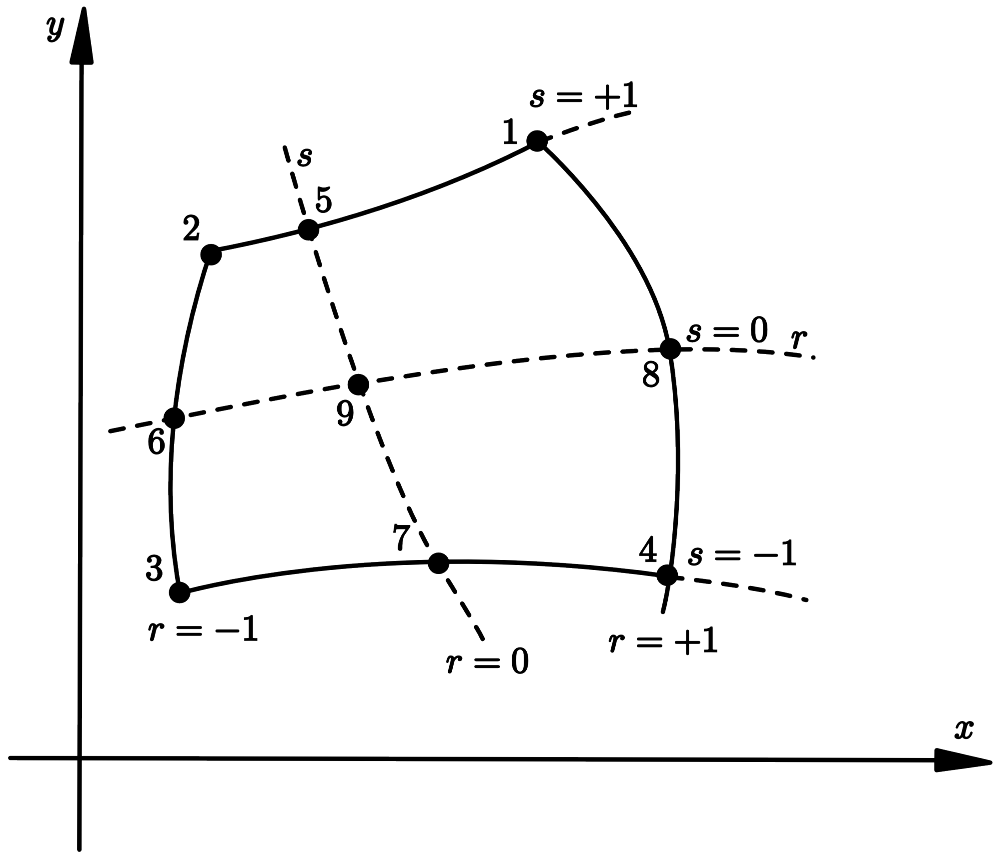
$$R = t \int \int_A H^T\, q(r, s)\ \cdot det(J)\ dr\, ds$$

4 Isoparametric quadrilateral element



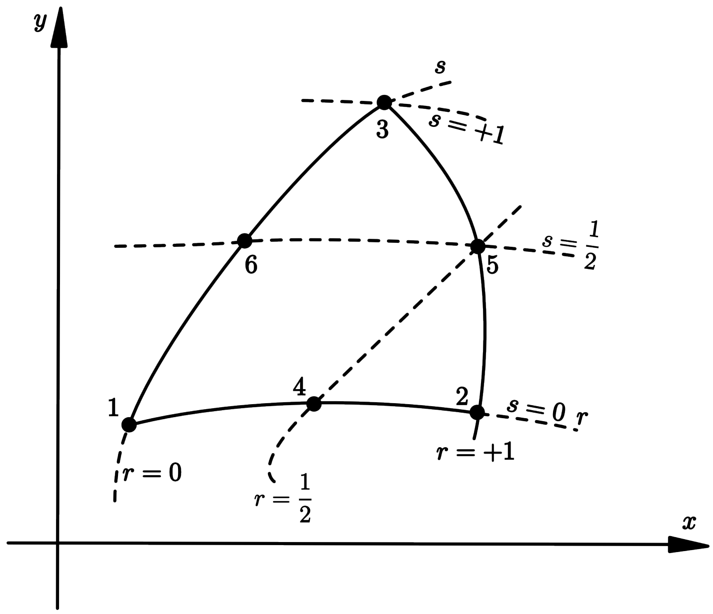
h	Node		$i=5$	$i=6$	$i=7$	$i=8$	$i=9$	X	Y
	1	$\frac{1}{4}(1+r)(1+s)$ $\frac{1}{4}(1+r+s+rs)$	$\frac{-1}{4}(1-r^2)(1+s)$ $\frac{-1}{4}(1+s-r^2-r^2s)$			$\frac{-1}{4}(1-s^2)(1+r)$ $\frac{-1}{4}(1+r-s^2-rs^2)$	$\frac{-1}{4}(1-r^2)(1-s^2)$ $\frac{-1}{4}(1-r^2-s^2+r^2s^2)$		
	2	$\frac{1}{4}(1-r)(1+s)$ $\frac{1}{4}(1-r+s-rs)$	$\frac{-1}{4}(1-r^2)(1+s)$ $\frac{-1}{4}(1+s-r^2-r^2s)$	$\frac{-1}{4}(1-s^2)(1-r)$ $\frac{-1}{4}(1-r-s^2+rs^2)$			$\frac{-1}{4}(1-r^2)(1-s^2)$ $\frac{-1}{4}(1-r^2-s^2+r^2s^2)$		
	3	$\frac{1}{4}(1-r)(1-s)$ $\frac{1}{4}(1-r-s+rs)$		$\frac{-1}{4}(1-s^2)(1-r)$ $\frac{-1}{4}(1-r-s^2+rs^2)$	$\frac{-1}{4}(1-r^2)(1-s)$ $\frac{-1}{4}(1-s-r^2+r^2s)$		$\frac{-1}{4}(1-r^2)(1-s^2)$ $\frac{-1}{4}(1-r^2-s^2+r^2s^2)$		
	4	$\frac{1}{4}(1+r)(1-s)$ $\frac{1}{4}(1+r-s-rs)$			$\frac{-1}{4}(1-r^2)(1-s)$ $\frac{-1}{4}(1-s-r^2+r^2s)$	$\frac{-1}{4}(1-s^2)(1+r)$ $\frac{-1}{4}(1+r-s^2-rs^2)$	$\frac{-1}{4}(1-r^2)(1-s^2)$ $\frac{-1}{4}(1-r^2-s^2+r^2s^2)$		
	5	$\frac{1}{2}(1-r^2)(1+s)$ $\frac{1}{2}(1+s-r^2-r^2s)$					$\frac{-1}{2}(1-r^2)(1-s^2)$ $\frac{-1}{2}(1-r^2-s^2+r^2s^2)$		
	6	$\frac{1}{2}(1-s^2)(1-r)$ $\frac{1}{2}(1-r-s^2+rs^2)$					$\frac{-1}{2}(1-r^2)(1-s^2)$ $\frac{-1}{2}(1-r^2-s^2+r^2s^2)$		
	7	$\frac{1}{2}(1-r^2)(1-s)$ $\frac{1}{2}(1-s-r^2+r^2s)$					$\frac{-1}{2}(1-r^2)(1-s^2)$ $\frac{-1}{2}(1-r^2-s^2+r^2s^2)$		
	8	$\frac{1}{2}(1-s^2)(1+r)$ $\frac{1}{2}(1+r-s^2-rs^2)$					$\frac{-1}{2}(1-r^2)(1-s^2)$ $\frac{-1}{2}(1-r^2-s^2+r^2s^2)$		
	9	$(1-r^2)(1-s^2)$ $1-r^2-s^2+r^2s^2)$							

5 Isoparametric quadrilateral element shape function derivates



h	Node		$i=5$	$i=6$	$i=7$	$i=8$	$i=9$
	$\frac{\delta h_1}{\delta r}=$ $\frac{\delta h_1}{\delta s}=$	$\frac{1}{4}(1+s)$ $\frac{1}{4}(1+r)$	$\frac{1}{2}r(1+s)$ $\frac{1}{4}(r^2-1)$			$\frac{-1}{4}(1-s^2)$ $\frac{1}{2}s(1+r)$	$\frac{-1}{2}r(s^2-1)$ $\frac{-1}{2}(r^2-1)s$
	$\frac{\delta h_2}{\delta r}=$ $\frac{\delta h_2}{\delta s}=$	$\frac{-1}{4}(1+s)$ $\frac{1}{4}(1-r)$	$\frac{1}{2}r(1+s)$ $\frac{1}{4}(r^2-1)$	$\frac{-1}{4}(s^2-1)$ $\frac{-1}{2}(r-1)s$			$\frac{-1}{2}r(s^2-1)$ $\frac{-1}{2}(r^2-1)s$
	$\frac{\delta h_3}{\delta r}=$ $\frac{\delta h_3}{\delta s}=$	$\frac{-1}{4}(1-s)$ $\frac{-1}{4}(1-r)$		$\frac{-1}{4}(s^2-1)$ $\frac{-1}{2}(r-1)s$	$\frac{-1}{2}r(s-1)$ $\frac{1}{4}(1-r^2)$		$\frac{-1}{2}r(s^2-1)$ $\frac{-1}{2}(r^2-1)s$
	$\frac{\delta h_4}{\delta r}=$ $\frac{\delta h_4}{\delta s}=$	$\frac{1}{4}(1-s)$ $\frac{-1}{4}(1+r)$			$\frac{-1}{2}r(s-1)$ $\frac{1}{4}(1-r^2)$	$\frac{-1}{4}(1-s^2)$ $\frac{1}{2}s(1+r)$	$\frac{-1}{2}r(s^2-1)$ $\frac{-1}{2}(r^2-1)s$
	$\frac{\delta h_5}{\delta r}=$ $\frac{\delta h_5}{\delta s}=$	$-r(1+s)$ $\frac{1}{2}(1-r^2)$					$r(s^2-1)$ $-(r^2-1)s$
	$\frac{\delta h_6}{\delta r}=$ $\frac{\delta h_6}{\delta s}=$	$\frac{1}{2}(s^2-1)$ $s(r-1)$					$r(s^2-1)$ $-(r^2-1)s$
	$\frac{\delta h_7}{\delta r}=$ $\frac{\delta h_7}{\delta s}=$	$r(s-1)$ $\frac{1}{2}(r^2-1)$					$r(s^2-1)$ $-(r^2-1)s$
	$\frac{\delta h_8}{\delta r}=$ $\frac{\delta h_8}{\delta s}=$	$\frac{1}{2}(1-s^2)$ $-s(1+r)$					$r(s^2-1)$ $-(r^2-1)s$
	$\frac{\delta h_9}{\delta r}=$ $\frac{\delta h_9}{\delta s}=$	$2r(s^2-1)$ $2s(r^2-1)$					

6 Isoparametric triangular element shape function derivates



h	Node		$i=4$	$i=5$	$i=6$	X	Y
	1	$1-r-s$	$-2r(1-r-s)$ $-2(r-r^2-rs)$		$-2s(1-r-s)$ $-2(s-rs-s^2)$		
	2	r	$-2r(1-r-s)$ $-2(r-r^2-rs)$	$-2rs$			
	3	s		$-2rs$	$-2s(1-r-s)$ $-2(s-rs-s^2)$		
	4	$4r(1-r-s)$ $4(r-r^2-rs)$					
	5	$4rs$					
	4	$4s(1-r-s)$ $4(s-rs-s^2)$					

h	Node		$i=4$	$i=5$	$i=6$	X	Y
	$\frac{\delta h_1}{\delta r}=$ $\frac{\delta h_1}{\delta s}=$	-1 -1	$2(2r+s-1)$ $2r$		$2s$ $2(r+2s-1)$		
	$\frac{\delta h_2}{\delta r}=$ $\frac{\delta h_2}{\delta s}=$	1 0	$2(2r+s-1)$ $2r$	$-2s$ $-2r$			
	$\frac{\delta h_3}{\delta r}=$ $\frac{\delta h_3}{\delta s}=$	0 1		$-2s$ $-2r$	$2s$ $2(r+2s-1)$		
	$\frac{\delta h_4}{\delta r}=$ $\frac{\delta h_4}{\delta s}=$	$-4(2r+s-1)$ $-4r$					
	$\frac{\delta h_5}{\delta r}=$ $\frac{\delta h_5}{\delta s}=$	$4s$ $4r$					
	$\frac{\delta h_6}{\delta r}=$ $\frac{\delta h_6}{\delta s}=$	$-4s$ $-4(r+2s-1)$					

7 Plain Stress

$\nu = 0.1$

$$C = \frac{100\,E}{99}\begin{bmatrix} 1 & \frac{1}{10} & 0 \\ \frac{1}{10} & 1 & 0 \\ 0 & 0 & \frac{9}{20} \end{bmatrix}$$

$\nu = 0.2$

$$C = \frac{25\,E}{24}\begin{bmatrix} 1 & \frac{1}{5} & 0 \\ \frac{1}{5} & 1 & 0 \\ 0 & 0 & \frac{2}{5} \end{bmatrix}$$

$\nu = 0.25$

$$C = \frac{16\,E}{15}\begin{bmatrix} 1 & \frac{1}{4} & 0 \\ \frac{1}{4} & 1 & 0 \\ 0 & 0 & \frac{3}{8} \end{bmatrix}$$

$\nu = \frac{1}{3}$

$$C = \frac{9\,E}{8}\begin{bmatrix} 1 & \frac{1}{3} & 0 \\ \frac{1}{3} & 1 & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$$

$\nu = 0.4$

$$C = \frac{25\,E}{21}\begin{bmatrix} 1 & \frac{2}{5} & 0 \\ \frac{2}{5} & 1 & 0 \\ 0 & 0 & \frac{3}{10} \end{bmatrix}$$

$\nu = 0.5$

$$C = \frac{4\,E}{3}\begin{bmatrix} 1 & \frac{1}{2} & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix}$$

$\nu = 0.6$

$$C = \frac{25\,E}{16}\begin{bmatrix} 1 & \frac{3}{5} & 0 \\ \frac{3}{5} & 1 & 0 \\ 0 & 0 & \frac{1}{5} \end{bmatrix}$$

$\nu = 0.7$

$$C = \frac{100\,E}{51}\begin{bmatrix} 1 & \frac{7}{10} & 0 \\ \frac{7}{10} & 1 & 0 \\ 0 & 0 & \frac{3}{20} \end{bmatrix}$$

$\nu = 0.75$

$$C = \frac{16\,E}{7}\begin{bmatrix} 1 & \frac{3}{4} & 0 \\ \frac{3}{4} & 1 & 0 \\ 0 & 0 & \frac{1}{8} \end{bmatrix}$$

$\nu = 0.8$

$$C = \frac{25\,E}{9}\begin{bmatrix} 1 & \frac{4}{5} & 0 \\ \frac{4}{5} & 1 & 0 \\ 0 & 0 & \frac{1}{10} \end{bmatrix}$$

$\nu = 0.9$

$$C = \frac{100\,E}{19}\begin{bmatrix} 1 & \frac{9}{10} & 0 \\ \frac{9}{10} & 1 & 0 \\ 0 & 0 & \frac{1}{20} \end{bmatrix}$$

$\nu = \frac{1}{6}$

$$C = \frac{36\,E}{35}\begin{bmatrix} 1 & \frac{1}{6} & 0 \\ \frac{1}{6} & 1 & 0 \\ 0 & 0 & \frac{5}{12} \end{bmatrix}$$

8 Plain Strain

$\nu = 0.1$

$$C = \frac{45\,E}{44}\begin{bmatrix} 1 & \frac{1}{9} & 0 \\ \frac{1}{9} & 1 & 0 \\ 0 & 0 & \frac{4}{9} \end{bmatrix}$$

$\nu = 0.2$

$$C = \frac{10\,E}{9}\begin{bmatrix} 1 & \frac{1}{4} & 0 \\ \frac{1}{4} & 1 & 0 \\ 0 & 0 & \frac{3}{8} \end{bmatrix}$$

$\nu = 0.25$

$$C = 6/5\,E\begin{bmatrix} 1 & \frac{1}{3} & 0 \\ \frac{1}{3} & 1 & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$$

$\nu = 0.3$

$$C = \frac{35\,E}{26}\begin{bmatrix} 1 & \frac{3}{7} & 0 \\ \frac{3}{7} & 1 & 0 \\ 0 & 0 & \frac{2}{7} \end{bmatrix}$$

$\nu = 0.4$

$$C = \frac{15\,E}{7}\begin{bmatrix} 1 & \frac{2}{3} & 0 \\ \frac{2}{3} & 1 & 0 \\ 0 & 0 & \frac{1}{6} \end{bmatrix}$$

$\nu = 0.6$

$$C = -\frac{5}{4}\,E\begin{bmatrix} 1 & \frac{3}{2} & 0 \\ \frac{3}{2} & 1 & 0 \\ 0 & 0 & -\frac{1}{4} \end{bmatrix}$$

$\nu = 0.7$

$$C = -\frac{15\,E}{34}\begin{bmatrix} 1 & \frac{7}{3} & 0 \\ \frac{7}{3} & 1 & 0 \\ 0 & 0 & -\frac{2}{3} \end{bmatrix}$$

$\nu = \frac{1}{3}$

$$C = \frac{3}{2}\,E\begin{bmatrix} 1 & \frac{1}{2} & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix}$$

$\nu = 0.75$

$$C = -\frac{2}{7}\,E\begin{bmatrix} 1 & 3 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$\nu = 0.8$

$$C = -\frac{5\,E}{27}\begin{bmatrix} 1 & 4 & 0 \\ 4 & 1 & 0 \\ 0 & 0 & -\frac{3}{2} \end{bmatrix}$$

$\nu = 0.9$

$$C = -\frac{5\,E}{76}\begin{bmatrix} 1 & 9 & 0 \\ 9 & 1 & 0 \\ 0 & 0 & -4 \end{bmatrix}$$

$\nu = \frac{1}{6}$

$$C = \frac{15\,E}{14}\begin{bmatrix} 1 & \frac{1}{5} & 0 \\ \frac{1}{5} & 1 & 0 \\ 0 & 0 & \frac{2}{5} \end{bmatrix}$$

9 Gaussian integration

Minimum Gaussian order for exact result: $n_i \geq \frac{p_i+1}{2}$
For multi dimensional integrals: $n_i \ \times \ n_j$
Order for exact integration of non polynomials: $n = \inf$

9.1 Transformation

Gaussian integration is only defined for integrals $\int_{-1}^{1}$, otherwise transformation necessary:
For an integral:

$$\int_a^b F(x)dx$$

holds:

$$x(r) = \tfrac{b-a}{2}r + \tfrac{b+a}{2} \qquad det(J) = \tfrac{b-a}{2}$$

$$\int_a^b F(x)dx = \int_{-1}^{1} F(x(r))det(J)dr = \int_{-1}^{1} F(\frac{b-a}{2}r + \frac{b+a}{2})\frac{b-a}{2}dr$$

9.2 Integration

For single dimension integrals:	For multi dimension integrals:
$I(x) = \int_{-1}^{1} f(x)dx$	$I(x,y) = \iint_{-1}^{1} f(x,y)\, dxdy$
$I(x) = \sum_i \alpha_i f(r_i)$	$I(x,y) = \sum_{i,j} \alpha_i \alpha_j f(r_i, rj)$

Use all possible i, j combinations and add them. In total $n_i \times n_j$ summands.

9.3 Coefficients

n	Points r_i	Approximately r_i	Weights α_i	Approximately α_i
1	0	0	2	2
2	$\pm\frac{1}{\sqrt{3}}$	± 0.57735	1	1
3	0	0	$\frac{8}{9}$	0.888889
	$\pm\sqrt{\frac{3}{5}}$	± 0.774597	$\frac{5}{9}$	0.555556
4	$\pm\sqrt{\frac{3}{7} - \frac{2}{7}\sqrt{\frac{6}{5}}}$	± 0.339981	$\frac{18+\sqrt{30}}{36}$	0.652145
	$\pm\sqrt{\frac{3}{7} + \frac{2}{7}\sqrt{\frac{6}{5}}}$	± 0.861136	$\frac{18-\sqrt{30}}{36}$	0.347855
5	0	0	$\frac{128}{225}$	0.568889
	$\pm\frac{1}{3}\sqrt{5 - 2\sqrt{\frac{10}{7}}}$	± 0.538469	$\frac{322+13\sqrt{70}}{900}$	0.478629
	$\pm\frac{1}{3}\sqrt{5 + 2\sqrt{\frac{10}{7}}}$	± 0.90618	$\frac{322-13\sqrt{70}}{900}$	0.236927
n	-	-	$\sum_{i=1}^{n} \alpha_i \ = 2$	-

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n_x	α_x	n_y	α_y	n_z	α_z	$\alpha_x\alpha_y\alpha_z$	r_x	r_y	r_z	***F(x, y, z)***	

10 Newton-Cotes Integration

Minimum Newton-Cotes order for exact result: $n_i \geq p_i$.
For multi dimensional integrals: $n_x \times n_y$.
Order for exact integration of non polynomials: $n = \inf$.
$n_i = n_{samplingpoints} - 1$: Number of intervals.
$n_i + 1$: polynomials up to order n can be integrated exactly.

10.1 Integration

For single dimension integrals:	For multi dimension integrals:
$I(x) = \int_a^b F(x)dx$	$I(x,y) = \int_{a_y}^{b_y} \int_{a_x}^{b_x} F(x,y)\,dxdy$
$I(x) = (b-a) \sum \alpha_c F(s_c)$	$I(x,y) = (b_x - a_x) \cdot (b_y - a_y) \sum_{i=0}^{n_x} \sum_{j=0}^{n_y} \alpha_{c_i} \alpha_{c_j} F(s_{x_{ci}}, s_{y_{cj}})$

- α_c aus Tabelle ablesen.
- s_c berechnen.
- Use all possible i, j combinations and add them. In total $n_i \times n_j$ summands.

10.2 Coefficients

$$s_{ci} = \frac{b-a}{n} i + a$$

For all i from 0....n:

n	points s_c for [0..1]	Weights α_c
1	0 1	$\frac{1}{2}$ $\frac{1}{2}$
2	0 $\frac{1}{2}$ 1	$\frac{1}{6}$ $\frac{4}{6}$ $\frac{1}{6}$
3	0 $\frac{1}{3}$ $\frac{2}{3}$ 1	$\frac{1}{8}$ $\frac{3}{8}$ $\frac{3}{8}$ $\frac{1}{8}$
4	0 $\frac{1}{4}$ $\frac{2}{4}$ $\frac{3}{4}$ 1	$\frac{7}{90}$ $\frac{32}{90}$ $\frac{12}{90}$ $\frac{32}{90}$ $\frac{7}{90}$

11 Newmark Integration Method

Stability is guaranted for $\delta = 0,5$, $\alpha = 0,25$, respectively $2\alpha \geq \delta \geq \frac{1}{2}$ independent from the timestep.

Needed Variables:

- Stiffness Matrix K
- Mass Matrix M
- Inital Values ${}^0U$, ${}^0\dot{U}$, ${}^0\ddot{U}$
- Time Step Δt

	For $\Delta t = 0.1$, $\alpha = 0.25$ and $\delta = 0.5$:
$\alpha_0 = \frac{1}{\alpha \Delta t^2}$	$= 400 \frac{1}{s^2}$
$\alpha_1 = \frac{\delta}{\alpha \Delta t}$	$= 20 \frac{1}{s}$
$\alpha_2 = \frac{1}{\alpha \Delta t}$	$= 40 \frac{1}{s}$
$\alpha_3 = \frac{1}{2\alpha} - 1$	$= 1$
$\alpha_4 = \frac{\delta}{\alpha} - 1$	$= 1$
$\alpha_5 = \frac{\Delta t}{2}(\frac{\delta}{\alpha} - 2)$	$= 0s$
$\alpha_6 = \Delta t(1 - \delta)$	$= 0.05s$
$\alpha_7 = \delta \Delta t$	$= 0.05s$

Guideline: $\Delta t_{Newmark} = \frac{T_n}{10} = \frac{2\pi}{10\,\omega}$

The Newmark parameters α and δ determine stability and damping of the method. Often: $\alpha = 0.25$ and $\delta = 0.5$

11.1 Initial Calculation

- Effektive stiffness Matrix:
  $\hat{K} = K + \alpha_0\, M + \alpha_1\, C$

  The effective mass matrix must be Triangular, or Triangularized.

11.2 Repeat per Step

- Calculate the effective load vector at time $t + \Delta t$ ::

  General case:
  $$^{t+\,\Delta t}\hat{R} =^{t+\,\Delta t} R + M\left(\alpha_0\,{}^tU + \alpha_2\,{}^t\dot{U} + \alpha_3\,{}^t\ddot{U}\right) + C\left(\alpha_1\,{}^tU + \alpha_4\,{}^t\dot{U} + \alpha_5\,{}^t\ddot{U}\right)$$
  Without Damping:
  $$^{t+\,\Delta t}\hat{R} =^{t+\,\Delta t} R + M\left(\alpha_0\,{}^tU + \alpha_2\,{}^t\dot{U} + \alpha_3\,{}^t\ddot{U}\right)$$

- Get displacement:
  $$^{t+\Delta t}U = \hat{K}^{-1}\,{}^{t+\Delta t}\hat{R}$$

- Calculate central differences if needed:
  $$^{t+\Delta t}\ddot{U} = \alpha_0\left[{}^{t+\Delta t}U -^{t} U\right] - \alpha_2\,{}^t\dot{U} - \alpha_3\,{}^t\ddot{U}$$
  $$^{t+\Delta t}\dot{U} = {}^t\dot{U} + \alpha_6\,{}^t\ddot{U} + \alpha_7\,{}^{t+\Delta t}\ddot{U}$$

Newmark Coefficients

	$\alpha =$	0,25	$\delta =$	0,5													
$\Delta t =$	0,1	0,2	0,3	0,4	0,5	0,6	0,7	0,8	0,9	1	1,2	1,4	1,6	1,8	2		
$\alpha_0 = \frac{1}{\alpha \Delta t^2}$	400	100	44,444	25	16	11,111	8,163	6,25	4,938	4	2,778	2,041	1,563	1,235	1	$\frac{1}{s^2}$	*a0*
$\alpha_1 = \frac{\delta}{\alpha \Delta t}$	20	10	6,667	5	4	3,333	2,857	2,5	2,222	2	1,667	1,429	1,25	1,111	1	$\frac{1}{s}$	*a1*
$\alpha_2 = \frac{1}{\alpha \Delta t}$	40	20	13,333	10	8	6,667	5,714	5	4,444	4	3,333	2,857	2,5	2,222	2	$\frac{1}{s}$	*a2*
$\alpha_3 = \frac{1}{2\alpha} - 1$	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1		*a3*
$\alpha_4 = \frac{\delta}{\alpha} - 1$	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1		*a4*
$\alpha_5 = \frac{\Delta t}{2}(\frac{\delta}{\alpha} - 2)$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	s	*a5*
$\alpha_6 = \Delta t(1 - \delta)$	0,05	0,1	0,15	0,2	0,25	0,3	0,35	0,4	0,45	0,5	0,6	0,7	0,8	0,9	1	s	*a6*
$\alpha_7 = \delta \Delta t$	0,05	0,1	0,15	0,2	0,25	0,3	0,35	0,4	0,45	0,5	0,6	0,7	0,8	0,9	1	s	*a7*

12 Central Difference Method

Für die numerische Berechnung muss der kritische Zeitschritt Δt_{crit} berücksichtigt werden (Stabilität):

$\Delta t_{crit} = \frac{2}{\omega_{max}} \geq \Delta t$

Needed Variables:

- Stiffness Matrix K
- Mass Matrix M
- Inital Values ${}^{0}U$, ${}^{0}\dot{U}$, ${}^{0}\ddot{U}$
- Time Step Δt

	For $\Delta t = 0.1$:
$\alpha_0 = \frac{1}{\Delta t^2}$	$= 100 \frac{1}{s^2}$
$\alpha_1 = \frac{1}{2\,\Delta t}$	$= 5 \frac{1}{s}$
$\alpha_2 = 2\,\alpha_0$	$= 200 \frac{1}{s^2}$
$\alpha_3 = \frac{1}{\alpha_2}$	$= 0.005 s^2$

12.1 Initial Calculation

- Calculate past timestep:

$$ {}^{-\Delta t}U =^{0} U - \Delta t \, {}^{0}\dot{U} + \alpha_3 \cdot \, {}^{0}\ddot{U} $$

- Effektive Mass Matrix

$$ \hat{M} = \alpha_0 \, M + \alpha_1 \, C $$

The effective mass matrix must be Triangular, or Triangularized.

12.2 Repeat per Step

- Calculate the effective load vector at time T:

General case:

$$ {}^{t}\hat{R} = \, {}^{t}R - (K - \alpha_2 M) \, {}^{t}U - (\alpha_0 M - \alpha_1 C) \, {}^{t-\Delta t}U $$

Without Damping:

$$ {}^{t}\hat{R} = \, {}^{t}R - (K - \alpha_2 M) \, {}^{t}U - \alpha_0 M \, {}^{t-\Delta t}U $$

- Get displacement:

$$ {}^{t+\Delta t}U = \hat{M}^{-1} \cdot \, {}^{t}\hat{R} $$

- Calculate central differences if needed:

$$ {}^{t}\ddot{U} = \alpha_0 \left[{}^{t-\Delta t}U - 2 \, {}^{t}U + \, {}^{t+\,\Delta t}U \right] $$

$$ {}^{t}\dot{U} = \alpha_1 \left[- \, {}^{t-\Delta t}U + \, {}^{t+\,\Delta t}U \right] $$

Central Difference Coefficients

	$\Delta t =$	0,1	0,2	0,3	0,4	0,5	0,6	0,7	0,8	0,9	1	1,2	1,4	1,6	1,8	2		
$\alpha_0 = \frac{1}{\Delta t^2}$		100	25	11,111	6,25	4	2,778	2,041	1,563	1,235	1	0,694	0,51	0,391	0,309	0,25	$\frac{1}{s^2}$	*a0*
$\alpha_1 = \frac{1}{2\,\Delta t}$		5	2,5	1,667	1,25	1	0,833	0,714	0,625	0,556	0,5	0,417	0,357	0,313	0,278	0,25	$\frac{1}{s}$	*a1*
$\alpha_2 = 2\,\alpha_0$		200	50	22,222	12,5	8	5,556	4,082	3,126	2,47	2	1,388	1,02	0,782	0,618	0,5	$\frac{1}{s^2}$	*a2*
$\alpha_3 = \frac{1}{\alpha_2}$		0,005	0,02	0,045	0,08	0,125	0,18	0,245	0,32	0,405	0,5	0,72	0,98	1,279	1,618	2	s^2	*a3*

13 Method of Steepest-Descend

Die Methode kann verwendet werden um Gleichungssysteme $Ax = b$ iterativ zu lösen. Im Gegensatz zu direkten Solvern kann bei einer gewünschten Genauigkeit abgebrochen werden und eine exakte Lösung muss nicht berechnet werden. Dadurch ergeben sich Geschwindigkeitsvorteile wenn eine approximierte Lösung genügt.

Der Solver startet an einem Startpunkt und iteriert in Richtung des Gradienten der quadrierten Funktion.

$$f(x) = \frac{1}{2}x^T Ax - x^T b$$

$$\nabla f(x) = Ax - b = -r_0$$

Den Gradienten der quadratischen Gleichung gegen 0 zu bringen ist äquivalent zur Lösung des ursprünglichen LGS.

- Den Startwert x_0 festlegen bzw. raten.
- Den Ersten Gradienten r_0 berechnen:

$$r_0 = b - A\,x_0$$

- Bis zur gewünschten Genauigkeit iterieren:

$$\alpha = \frac{r_i^T\ r_i}{r_i^T\ A\ r_i}$$

$$x_{i+1} = x_i + \alpha\, r_i$$

$$r_{i+1} = b - A\,x_{i+1}$$

- Fehler je Schritt:

$$e = \sqrt{r_{i+1}^T\ r_{i+1}}$$

If A is positive definit, there is one minimum and one solution of the method of Steepest-Descend.

14 Transient Problems

14.1 Equation of Motion

$$M\ddot{u} + C\dot{u} + Ku = R$$

$$M = \int_V \rho H^T H dV$$

$$C = \int_V \kappa H^T H dV$$

14.2 Eigenfrequencies

Für $n = 1$:	Für $n > 1$:
$\omega = \sqrt{\frac{K}{M}}$	$det(K - \omega^2 M) = 0$

15 Modalraum Transformation

15.1 Generalized EV-Problem

$$\underline{K}\underline{\varphi} = \omega^2 \underline{M}\underline{\varphi}$$

n eigensolutions $(\omega_1, \varphi_1), ... (\omega_n, \varphi_n)$

$\underline{\varphi_i}$: eigenvector of the i-th mode (eigenmode)

ω_i: respective eigenfrequency [rad/s]

$0 \leq \omega_1^2 \leq \omega_2^2 \leq ... \leq \omega_n^2$

$\underline{\Phi} = \left[\underline{\varphi}_1\ \underline{\varphi}_2\ ..\underline{\varphi}_n\right]$

$\underline{\Omega}^2 = \begin{bmatrix} \omega_1^2 & 0 & 0 \\ 0 & .. & 0 \\ 0 & 0 & \omega_n^2 \end{bmatrix}$

15.2 Transformation

If the damping is neglected, the transformes equation* of motion is reduced to:

$$\underline{\ddot{X}}(t) + \underline{\Omega}^2 \underline{X}(t) = \underline{\Phi}^T \underline{R}(t)$$

Solution:

$$\underline{U}(t) = \underline{\Phi}\ \underline{X}(t)$$

$$\underline{U}(t) = \sum_{i=1}^{n} \underline{\varphi}_i X_i(t)$$

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$\boldsymbol{\alpha = 0^\circ = 180^\circ}$

$\boldsymbol{\beta = 0^\circ = 180^\circ}$

$$K = \frac{EA}{L}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\boldsymbol{\alpha = 30^\circ = -150^\circ}$

$\boldsymbol{\beta = 30^\circ = -150^\circ}$

$$K = \frac{EA}{4L}\begin{bmatrix} 3 & \sqrt{3} & -3 & -\sqrt{3} \\ \sqrt{3} & 1 & -\sqrt{3} & -1 \\ -3 & -\sqrt{3} & 3 & \sqrt{3} \\ -\sqrt{3} & -1 & \sqrt{3} & 1 \end{bmatrix}$$

$\boldsymbol{\alpha = 45^\circ = -135^\circ}$

$\boldsymbol{\beta = 45^\circ = -135^\circ}$

$$K = \frac{EA}{2L}\begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \end{bmatrix}$$

$\boldsymbol{\alpha = 60^\circ = -120^\circ}$

$\boldsymbol{\beta = 60^\circ = -120^\circ}$

$$K = \frac{EA}{4L}\begin{bmatrix} 1 & \sqrt{3} & -1 & -\sqrt{3} \\ \sqrt{3} & 3 & -\sqrt{3} & -3 \\ -1 & -\sqrt{3} & 1 & \sqrt{3} \\ -\sqrt{3} & -3 & \sqrt{3} & 3 \end{bmatrix}$$

$\boldsymbol{\alpha = 90^\circ = -90^\circ}$

$\boldsymbol{\beta = 90^\circ = -90^\circ}$

$$K = \frac{EA}{L}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

$\boldsymbol{\alpha = 120^\circ = -60^\circ}$

$\boldsymbol{\beta = 120^\circ = -60^\circ}$

$$K = \frac{EA}{4L}\begin{bmatrix} 1 & -\sqrt{3} & -1 & \sqrt{3} \\ -\sqrt{3} & 3 & \sqrt{3} & -3 \\ -1 & \sqrt{3} & 1 & -\sqrt{3} \\ \sqrt{3} & -3 & -\sqrt{3} & 3 \end{bmatrix}$$

$\boldsymbol{\alpha = 135^\circ = -45^\circ}$

$\boldsymbol{\beta = 135^\circ = -45^\circ}$

$$K = \frac{EA}{2L}\begin{bmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

$\boldsymbol{\alpha = 150^\circ = -30^\circ}$

$\boldsymbol{\beta = 150^\circ = -30^\circ}$

$$K = \frac{EA}{4L}\begin{bmatrix} 3 & -\sqrt{3} & -3 & \sqrt{3} \\ -\sqrt{3} & 1 & \sqrt{3} & -1 \\ -3 & \sqrt{3} & 3 & -\sqrt{3} \\ \sqrt{3} & -1 & -\sqrt{3} & 1 \end{bmatrix}$$

$\boldsymbol{\alpha = 0°}$

$\beta = 30°$

$$K = -\frac{EA}{4L}\begin{bmatrix} -4 & 0 & 2\sqrt{3} & 2 \\ 0 & 0 & 0 & 0 \\ 2\sqrt{3} & 0 & -3 & -\sqrt{3} \\ 2 & 0 & -\sqrt{3} & -1 \end{bmatrix}$$

$\beta = -30°$

$$K = -\frac{EA}{4L}\begin{bmatrix} -4 & 0 & 2\sqrt{3} & -2 \\ 0 & 0 & 0 & 0 \\ 2\sqrt{3} & 0 & -3 & \sqrt{3} \\ -2 & 0 & \sqrt{3} & -1 \end{bmatrix}$$

$\beta = 45°$

$$K = -\frac{EA}{2L}\begin{bmatrix} -2 & 0 & \sqrt{2} & \sqrt{2} \\ 0 & 0 & 0 & 0 \\ \sqrt{2} & 0 & -1 & -1 \\ \sqrt{2} & 0 & -1 & -1 \end{bmatrix}$$

$\beta = -45°$

$$K = -\frac{EA}{2L}\begin{bmatrix} -2 & 0 & \sqrt{2} & -\sqrt{2} \\ 0 & 0 & 0 & 0 \\ \sqrt{2} & 0 & -1 & 1 \\ -\sqrt{2} & 0 & 1 & -1 \end{bmatrix}$$

$\boldsymbol{\alpha = 30°}$

$\beta = 0°$

$$K = \frac{EA}{4L}\begin{bmatrix} 3 & \sqrt{3} & -2\sqrt{3} & 0 \\ \sqrt{3} & 1 & -2 & 0 \\ -2\sqrt{3} & -2 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\beta = 15°$

$$\frac{EA}{L}\begin{bmatrix} 0.75 & 0.43 & -0.84 & -0.22 \\ 0.43 & 0.25 & -0.48 & -0.13 \\ -0.84 & -0.48 & 0.93 & 0.25 \\ -0.22 & -0.13 & 0.25 & 0.067 \end{bmatrix}$$

$\beta = -15°$

$$\frac{EA}{L}\begin{bmatrix} 0.75 & 0.43 & -0.84 & 0.22 \\ 0.43 & 0.25 & -0.48 & 0.13 \\ -0.84 & -0.48 & 0.93 & -0.25 \\ 0.22 & 0.13 & -0.25 & 0.067 \end{bmatrix}$$

$\boldsymbol{\alpha = 45°}$

$\beta = 0°$

$$K = -\frac{EA}{2L}\begin{bmatrix} -1 & -1 & \sqrt{2} & 0 \\ -1 & -1 & \sqrt{2} & 0 \\ \sqrt{2} & \sqrt{2} & -2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\beta = 15°$

$$\frac{EA}{L}\begin{bmatrix} 0.50 & 0.50 & -0.68 & -0.18 \\ 0.50 & 0.50 & -0.68 & -0.18 \\ -0.68 & -0.68 & 0.93 & 0.25 \\ -0.18 & -0.18 & 0.25 & 0.067 \end{bmatrix}$$

$\beta = -15°$

$$\frac{EA}{L}\begin{bmatrix} 0.50 & 0.50 & -0.68 & 0.18 \\ 0.50 & 0.50 & -0.68 & 0.18 \\ -0.68 & -0.68 & 0.93 & -0.25 \\ 0.18 & 0.18 & -0.25 & 0.067 \end{bmatrix}$$

$\boldsymbol{\alpha = 60^\circ}$

$\boldsymbol{\beta = 0^\circ}$

$$K = \frac{EA}{4L}\begin{bmatrix} 1 & \sqrt{3} & -2 & 0 \\ \sqrt{3} & 3 & -2\sqrt{3} & 0 \\ -2 & -2\sqrt{3} & 4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\boldsymbol{\beta = 15^\circ}$

$$\frac{EA}{L}\begin{bmatrix} 0.25 & 0.43 & -0.48 & -0.13 \\ 0.43 & 0.75 & -0.84 & -0.22 \\ -0.48 & -0.84 & 0.93 & 0.25 \\ -0.13 & -0.22 & 0.25 & 0.067 \end{bmatrix}$$

$\boldsymbol{\beta = -15^\circ}$

$$\frac{EA}{L}\begin{bmatrix} 0.25 & 0.43 & -0.48 & 0.13 \\ 0.43 & 0.75 & -0.84 & 0.22 \\ -0.48 & -0.84 & 0.93 & -0.25 \\ 0.13 & 0.22 & -0.25 & 0.067 \end{bmatrix}$$

$\boldsymbol{\alpha = 90^\circ}$

$\boldsymbol{\beta = 0^\circ}$

$$K = \frac{EA}{L}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\boldsymbol{\beta = 15^\circ}$

$$\frac{EA}{L}\begin{bmatrix} 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 1.0 & -0.97 & -0.26 \\ 0.0 & -0.97 & 0.93 & 0.25 \\ 0.0 & -0.26 & 0.25 & 0.067 \end{bmatrix}$$

$\boldsymbol{\beta = -15^\circ}$

$$\frac{EA}{L}\begin{bmatrix} 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 1.0 & -0.97 & 0.26 \\ 0.0 & -0.97 & 0.93 & -0.25 \\ 0.0 & 0.26 & -0.25 & 0.067 \end{bmatrix}$$

$\boldsymbol{\alpha = 120^\circ}$

$\boldsymbol{\beta = 0^\circ}$

$$K = -\frac{EA}{4L}\begin{bmatrix} -1 & \sqrt{3} & -2 & 0 \\ \sqrt{3} & -3 & 2\sqrt{3} & 0 \\ -2 & 2\sqrt{3} & -4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\boldsymbol{\beta = 15^\circ}$

$$\frac{EA}{L}\begin{bmatrix} 0.25 & -0.43 & 0.48 & 0.13 \\ -0.43 & 0.75 & -0.84 & -0.22 \\ 0.48 & -0.84 & 0.93 & 0.25 \\ 0.13 & -0.22 & 0.25 & 0.067 \end{bmatrix}$$

$\boldsymbol{\beta = -15^\circ}$

$$\frac{EA}{L}\begin{bmatrix} 0.25 & -0.43 & 0.48 & -0.13 \\ -0.43 & 0.75 & -0.84 & 0.22 \\ 0.48 & -0.84 & 0.93 & -0.25 \\ -0.13 & 0.22 & -0.25 & 0.067 \end{bmatrix}$$

$\boldsymbol{\alpha = 135^\circ}$

$\boldsymbol{\beta = 0^\circ}$

$$K = \frac{EA}{2L}\begin{bmatrix} 1 & -1 & \sqrt{2} & 0 \\ -1 & 1 & -\sqrt{2} & 0 \\ \sqrt{2} & -\sqrt{2} & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\boldsymbol{\beta = 15^\circ}$

$$\frac{EA}{L}\begin{bmatrix} 0.50 & -0.50 & 0.68 & 0.18 \\ -0.50 & 0.50 & -0.68 & -0.18 \\ 0.68 & -0.68 & 0.93 & 0.25 \\ 0.18 & -0.18 & 0.25 & 0.067 \end{bmatrix}$$

$\boldsymbol{\beta = -15^\circ}$

$$\frac{EA}{L}\begin{bmatrix} 0.50 & -0.50 & 0.68 & -0.18 \\ -0.50 & 0.50 & -0.68 & 0.18 \\ 0.68 & -0.68 & 0.93 & -0.25 \\ -0.18 & 0.18 & -0.25 & 0.067 \end{bmatrix}$$

$\boldsymbol{\alpha = 150^\circ}$

$\boldsymbol{\beta = 0^\circ}$

$$K = -\frac{EA}{4L}\begin{bmatrix} -3 & \sqrt{3} & -2\sqrt{3} & 0 \\ \sqrt{3} & -1 & 2 & 0 \\ -2\sqrt{3} & 2 & -4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\boldsymbol{\beta = 15^\circ}$

$$\frac{EA}{L}\begin{bmatrix} 0.75 & -0.43 & 0.84 & 0.22 \\ -0.43 & 0.25 & -0.48 & -0.13 \\ 0.84 & -0.48 & 0.93 & 0.25 \\ 0.22 & -0.13 & 0.25 & 0.067 \end{bmatrix}$$

$\boldsymbol{\beta = -15^\circ}$

$$\frac{EA}{L}\begin{bmatrix} 0.75 & -0.43 & 0.84 & -0.22 \\ -0.43 & 0.25 & -0.48 & 0.13 \\ 0.84 & -0.48 & 0.93 & -0.25 \\ -0.22 & 0.13 & -0.25 & 0.067 \end{bmatrix}$$

$\boldsymbol{\alpha = -30^\circ}$

$\boldsymbol{\beta = 0^\circ}$

$$K = -\frac{EA}{4L}\begin{bmatrix} -3 & \sqrt{3} & 2\sqrt{3} & 0 \\ \sqrt{3} & -1 & -2 & 0 \\ 2\sqrt{3} & -2 & -4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\boldsymbol{\beta = 15^\circ}$

$$\frac{EA}{L}\begin{bmatrix} 0.75 & -0.43 & -0.84 & -0.22 \\ -0.43 & 0.25 & 0.48 & 0.13 \\ -0.84 & 0.48 & 0.93 & 0.25 \\ -0.22 & 0.13 & 0.25 & 0.067 \end{bmatrix}$$

$\boldsymbol{\beta = -15^\circ}$

$$\frac{EA}{L}\begin{bmatrix} 0.75 & -0.43 & -0.84 & 0.22 \\ -0.43 & 0.25 & 0.48 & -0.13 \\ -0.84 & 0.48 & 0.93 & -0.25 \\ 0.22 & -0.13 & -0.25 & 0.067 \end{bmatrix}$$

$\boldsymbol{\alpha = -45^\circ}$

$\beta = 0^\circ$

$$K = -\frac{EA}{2L}\begin{bmatrix} -1 & 1 & \sqrt{2} & 0 \\ 1 & -1 & -\sqrt{2} & 0 \\ \sqrt{2} & -\sqrt{2} & -2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\beta = 15^\circ$

$$\frac{EA}{L}\begin{bmatrix} 0.50 & -0.50 & -0.68 & -0.18 \\ -0.50 & 0.50 & 0.68 & 0.18 \\ -0.68 & 0.68 & 0.93 & 0.25 \\ -0.18 & 0.18 & 0.25 & 0.067 \end{bmatrix}$$

$\beta = -15^\circ$

$$\frac{EA}{L}\begin{bmatrix} 0.50 & -0.50 & -0.68 & 0.18 \\ -0.50 & 0.50 & 0.68 & -0.18 \\ -0.68 & 0.68 & 0.93 & -0.25 \\ 0.18 & -0.18 & -0.25 & 0.067 \end{bmatrix}$$

$\boldsymbol{\alpha = -60^\circ}$

$\beta = 0^\circ$

$$K = -\frac{EA}{4L}\begin{bmatrix} -1 & \sqrt{3} & 2 & 0 \\ \sqrt{3} & -3 & -2\sqrt{3} & 0 \\ 2 & -2\sqrt{3} & -4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\beta = 15^\circ$

$$\frac{EA}{L}\begin{bmatrix} 0.25 & -0.43 & -0.48 & -0.13 \\ -0.43 & 0.75 & 0.84 & 0.22 \\ -0.48 & 0.84 & 0.93 & 0.25 \\ -0.13 & 0.22 & 0.25 & 0.067 \end{bmatrix}$$

$\beta = -15^\circ$

$$\frac{EA}{L}\begin{bmatrix} 0.25 & -0.43 & -0.48 & 0.13 \\ -0.43 & 0.75 & 0.84 & -0.22 \\ -0.48 & 0.84 & 0.93 & -0.25 \\ 0.13 & -0.22 & -0.25 & 0.067 \end{bmatrix}$$

$\boldsymbol{\alpha = -90^\circ}$

$\beta = 0^\circ$

$$K = \frac{EA}{L}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\beta = 15^\circ$

$$\frac{EA}{L}\begin{bmatrix} 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 1.0 & 0.97 & 0.26 \\ 0.0 & 0.97 & 0.93 & 0.25 \\ 0.0 & 0.26 & 0.25 & 0.067 \end{bmatrix}$$

$\beta = -15^\circ$

$$\frac{EA}{L}\begin{bmatrix} 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 1.0 & 0.97 & -0.26 \\ 0.0 & 0.97 & 0.93 & -0.25 \\ 0.0 & -0.26 & -0.25 & 0.067 \end{bmatrix}$$

	$\boldsymbol{\alpha = -120^\circ}$	$\boldsymbol{\alpha = -135^\circ}$	$\boldsymbol{\alpha = -150^\circ}$
$\boldsymbol{\beta = 0^\circ}$	$K = \frac{EA}{4L}\begin{bmatrix} 1 & \sqrt{3} & 2 & 0 \\ \sqrt{3} & 3 & 2\sqrt{3} & 0 \\ 2 & 2\sqrt{3} & 4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$	$K = \frac{EA}{2L}\begin{bmatrix} 1 & 1 & \sqrt{2} & 0 \\ 1 & 1 & \sqrt{2} & 0 \\ \sqrt{2} & \sqrt{2} & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$	$K = \frac{EA}{4L}\begin{bmatrix} 3 & \sqrt{3} & 2\sqrt{3} & 0 \\ \sqrt{3} & 1 & 2 & 0 \\ 2\sqrt{3} & 2 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$
$\boldsymbol{\beta = 15^\circ}$	$\frac{EA}{L}\begin{bmatrix} 0.25 & 0.43 & 0.48 & 0.13 \\ 0.43 & 0.75 & 0.84 & 0.22 \\ 0.48 & 0.84 & 0.93 & 0.25 \\ 0.13 & 0.22 & 0.25 & 0.067 \end{bmatrix}$	$\frac{EA}{L}\begin{bmatrix} 0.50 & 0.50 & 0.68 & 0.18 \\ 0.50 & 0.50 & 0.68 & 0.18 \\ 0.68 & 0.68 & 0.93 & 0.25 \\ 0.18 & 0.18 & 0.25 & 0.067 \end{bmatrix}$	$\frac{EA}{L}\begin{bmatrix} 0.75 & 0.43 & 0.84 & 0.22 \\ 0.43 & 0.25 & 0.48 & 0.13 \\ 0.84 & 0.48 & 0.93 & 0.25 \\ 0.22 & 0.13 & 0.25 & 0.067 \end{bmatrix}$
$\boldsymbol{\beta = -15^\circ}$	$\frac{EA}{L}\begin{bmatrix} 0.25 & 0.43 & 0.48 & -0.13 \\ 0.43 & 0.75 & 0.84 & -0.22 \\ 0.48 & 0.84 & 0.93 & -0.25 \\ -0.13 & -0.22 & -0.25 & 0.067 \end{bmatrix}$	$\frac{EA}{L}\begin{bmatrix} 0.50 & 0.50 & 0.68 & -0.18 \\ 0.50 & 0.50 & 0.68 & -0.18 \\ 0.68 & 0.68 & 0.93 & -0.25 \\ -0.18 & -0.18 & -0.25 & 0.067 \end{bmatrix}$	$\frac{EA}{L}\begin{bmatrix} 0.75 & 0.43 & 0.84 & -0.22 \\ 0.43 & 0.25 & 0.48 & -0.13 \\ 0.84 & 0.48 & 0.93 & -0.25 \\ -0.22 & -0.13 & -0.25 & 0.067 \end{bmatrix}$