Nonlinear Dynamics

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strictly following »Nonlinear Dynamics and Chaos« by Steven Strogatz

1 General

- Differential equation → time continuous.
  - 1. Ordinary differential equation (one independent variable, e.g. t): temporal dynamics.
  - 2. Partial differential equation (more independent variable, e.g. t and x): spatio-temporal dynamics.
- Difference equation → discrete in time.
- **Classification**: Linear vs. nonlinear, finite vs. infinite-dimensional, deterministic vs stochastic (characterised by statistics only, noise involved), autonomous (e.g. time-invariant) vs. non-autonomous (build new $n+1$-dimensional state space for solving).

2 1D Problems

2.1 Flows on the line

- Function $\dot{x} = f(x)$. Get FP by: $\dot{x} = f(x^*) = 0$.
  - At a FP the flow vanishes. Stable FP: *Sink*. Unstable FP: *Source*.
  - A FP is globally stable, if all perturbations lend back to it.
  - A FP is locally stable, if small perturbations lend back to it.
- Draw complicated functions separately: $\dot{x} = x \cdot cos(x)$.

2.2 Linear Stability Analysis

- Get quantitative information at FPs (e.g. rate of decay).
- Perturbation at FP: $\eta(t) = x(t) - x^*$
  - $\dot{\eta} = \frac{d}{dt}(x(t) - x^*) = \dot{x} = f(x) = f(x^* + \eta)$
  - Approximation by a Taylor series: $f(x^* + \eta) \approx \eta \cdot f'(x^*) +$ h.o.t.
  - Solution: $\eta(t) \sim e^{f'(x^*) \cdot t}$
    - Decay near FP: $f'(x^*) < 0 \to$ stable
    - Grows near FP: $f'(x^*) > 0 \to$ unstable

2.3 Existence and Uniqueness

- If $f(x)$ is smooth enough ($f(x)$ and $f'(x)$ are continuous on open interval R and $x_0 \in$ R), then a solution exist and is unique.
- **Blow Up:** The system has a solution that reaches infinity in finite time (e.g. $x(t) = \tan(t)$).

2.4 Impossibility of Oscillations $\dot{x} = f(x)$

- All trajectories approach FP or diverge to $\pm\infty$. Trajectories are either forced to increase or decrease monotonically, or remain constant. A solution of a ODE of first order is never complex in any sense. Hence there are no periodic solutions to $\dot{x} = f(x)$.

2.5 Potentials

- Visualize dynamics of first-order system $\dot{x} = f(x)$ based on the physical idea of potential energy. Picture a particle sliding down the walls of a potential well.
- Potential is defined as $f(x) = -\frac{dV}{dx}$.
- $\dot{V} = \frac{dV}{dx}\frac{dx}{dt} = -\dot{x}\dot{x} = -|\dot{x}|^2 < 0$
- $V(x)$ decreases along trajectories/ along $x \to$ particle move to lower potential.
- Local minimum of $V(x)$ = stable FP, local maximum of $V(x)$ = unstable FP.

3 Bifurcations

- **Bifurcation:** *Qualitative* change in dynamics as parameter change (FP created/destroyed, stability changes).
- Parameter values at which bifurcation occurs: bifurcation points.
- Word meaning: splitting into two branches.
- Example: buckling of a beam, fluids: laminar/turbulent flow

3.1 Saddle-Node Bifurcation

- An unstable and a stable FP slide towards each other, annihilate into a half-stable FP, which than vanishes (generation and or loss of a FP).
- Normal form: $\dot{x} = a \cdot (r - r_c) + b \cdot (x - x^*)^2 \to$ half-stable FP at $r = r_c$ and $x = x^*$.

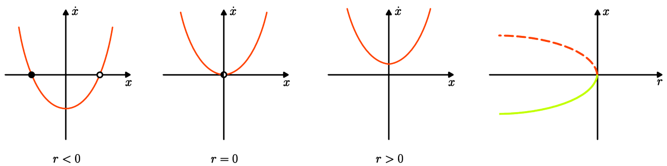
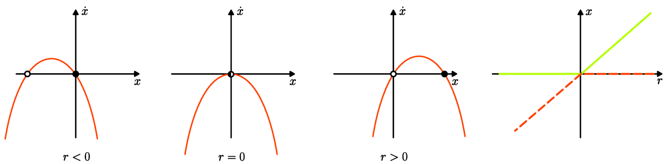


Abbildung 1: Saddle-Node Bifurcation: $\dot{x} = r + x^2$.

3.2 Transcritical Bifurcation

- Normal form: $\dot{x} = rx - x^2$.
- FP is not destroyed, but changes stability.

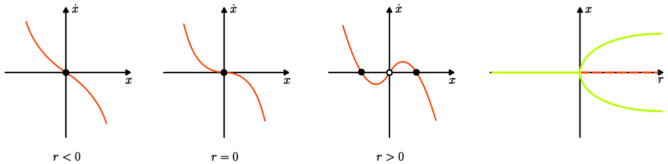


3.3 Pitchfork Bifurcation

- FPs appear/disappear in symmetrical pairs.
- Occurs often in symmetrical problems, e.g. buckling problem.

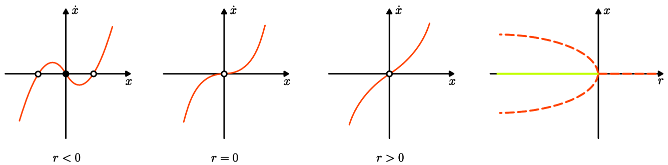
3.3.1 Supercritical Pitchfork Bifurcation

- Normal form: $\dot{x} = rx - x^3$ (invariant: $f(x) \leftrightarrow f(-x)$).
- Stable FP becomes unstable and two new stable FPs appear $x^* = \pm\sqrt{r}$.
- Cubic term is *stabilizing*.
- Terms: forward Bifurcation; smooth/soft, continuous bifurcation; 2nd-order phase transition.



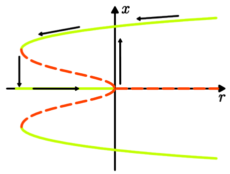
3.3.2 Subcritical Pitchfork Bifurcation

- Normal form: $\dot{x} = rx + x^3$ (invariant: $f(x) \leftrightarrow f(-x)$).
- Unstable FP becomes stable and two new unstable FPs appear $x^* = \pm\sqrt{-r}$ for decreasing r.
- Cubic term is *destabilizing*.
- Terms: inverse Bifurcation; hard/dangerous bifurcation; 1st-order phase transition.



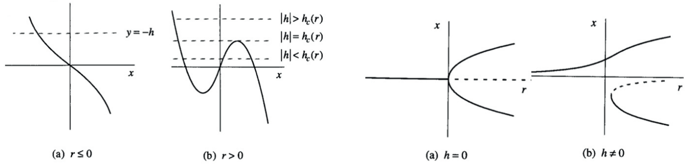
3.4 Jumps and Hysteresis

- Lack of reversibility: hysteresis.



3.5 Imperfect Bifurcation

- Slight imperfections lead to loss of symmetry.
- Example: $\dot{x} = h + rx - x^3$
  - $h = 0$: normal supercritical pitchfork bifurcation.
  - $h \neq 0$: symmetry is broken, h is imperfection parameter.
- $r \leq 0$: one stable FP, $r > 0$: 1, 2 or 3 FP depending on h.



4 Flows on the Circle

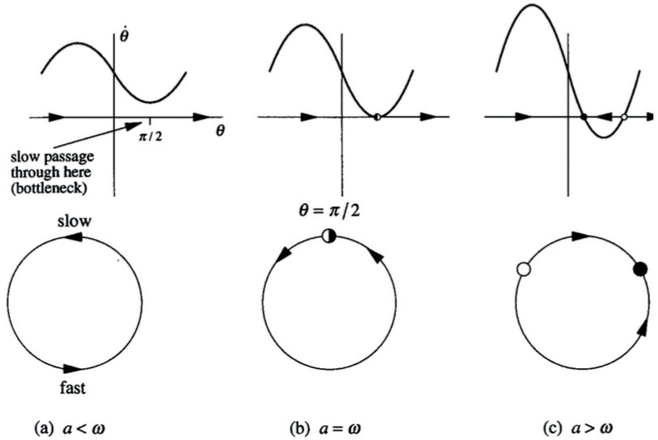
- **Definition:** A rule that assigns a unique velocity vector to every point on the circle.
- Vector field on the circle: $\dot{\theta} = f(\theta)$.
  - $\theta \in [0; 2\pi[$: point on the circle.
  - $f(\theta)$: real-valued-2π-periodic function.
- → Periodic solutions. Oscillations are possible.

4.1 Uniform Oscillator

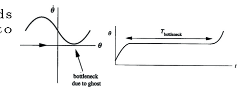
- Normal form: $\dot{\theta} = \omega$ with $\omega = const.$ and $\theta(t) = \omega t + \theta_0$ ($T = \frac{2\pi}{\omega}$).
- Beat phenomenon: Two non-interacting oscillators with different ω will periodically go in and out of phase.

4.2 Nonuniform Oscillator

- $\dot{\theta} = \omega - a\sin(\theta)$ (here: $a \geq 0$, $\omega \geq 0$).
  - $a < \omega$: Phase takes long through bottleneck $\frac{\pi}{2}$, speeds up afterwards.
    - Increasing a leads to longer period $T = \frac{2\pi}{\sqrt{\omega^2 - a^2}}$.
  - $a = \omega$: System stops oscillating at new half-stable FP $\theta = \frac{\pi}{2}$.
  - $a > \omega$: Half-stable FP splits into a stable and unstable FP.



- **Bottleneck**: very slow passage.
- **Ghost**: a former saddle-node (a slightly $< \omega$) that leads to a bottleneck. The trajectory needs most of its time to go through this point.



4.3 Overdamped Pendulum

- System: $mL^2\ddot{\theta} + b\dot{\theta} + mgL\sin(\theta) = \Gamma$ (m: mass, L: length of the pendulum, b: viscous damping, g: gravity constant, Γ: constant applied torque).
- With very large $b \to b\dot{\theta} + mgL\sin(\theta) = \Gamma$ (acceleration $\to 0$).
- Non-dimensionalize $\frac{b}{mgL}\dot{\theta} = \frac{\Gamma}{mgL} - \sin(\theta)$ with $\tau = \frac{mgL}{b}t$ and $\gamma = \frac{\Gamma}{mgL}$, then $\theta' = \gamma - \sin(\theta)$ with $\theta' = \frac{d\theta}{d\tau}$.
  - $\gamma > 1$: pendulum is rotating.
  - $\gamma = 1$: stable FP appear at $\frac{\pi}{2}$.
  - $\gamma < 1$: stable FP → two FPs wandering.
  - $\gamma = 0$: $\Gamma = 0$ and two equilibria (top, bottom).

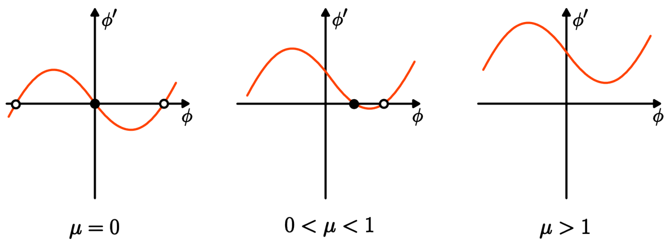
4.4 Synchronization (Fireflies)

4.4.1 Model

- $\theta(t)$: phase of the firefly's flashing rhythm ($\theta = 0$ instant of the flash).
- $\dot{\theta} = \omega$: eigenfrequency of a single fly, $\dot{\Theta} = \Omega$ eigenfrequency of the swarm.
- $\dot{\theta} = \omega + A\sin(\Theta - \theta)$ with A as the resetting strength.
- For example, if Θ is ahead of θ the firefly speeds up ($\dot{\theta} > \omega$).

4.4.2 Analysis

- Phase difference: $\phi = \Theta - \theta$, Velocity difference: $\dot{\phi} = \dot{\Theta} - \dot{\theta} = \Omega - \omega - A\sin(\phi)$
- Nondimensionalize
  - $\tau = At$, $\mu = \frac{\Omega - \omega}{A}$
  - $\frac{d\phi}{d\tau} = \phi' = \mu - \sin(\phi)$
- Different cases for : $\mu \geq 0$
  - $\mu = 0$: Synchronization without phase difference (stable FP $\phi^* = 0$).
  - $0 < \mu < 1$: synchronization with phase difference (phase lock, stable FP $\phi^* > 0$).
  - $\mu = 1$: saddle-node bifurcation
  - $\mu > 1$: no synchronization (phase drift, ϕ increases continuously).
- Entrainment for $\omega - A \leq \Omega \leq \omega + A$.
- $|\mu| > 1 \to$ beating



5 2D Linear Systems

- Form $\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x}$ with $\boldsymbol{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ and $\boldsymbol{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.
- Linear System: if $\boldsymbol{x_1}$ and $\boldsymbol{x_2}$ is a solution, then $\boldsymbol{x_3} = c_1\boldsymbol{x_1} + c_2\boldsymbol{x_2}$ is solution.
- $\boldsymbol{x^*} = 0$ is always a solution for any choice of $\boldsymbol{A}$.
- Mainly used for classification of non-linear systems.

5.1 Eigenvalues and eigenvectors

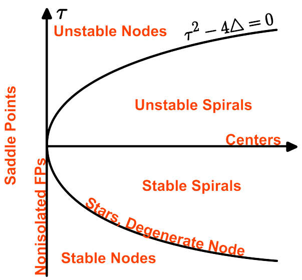
- 1. $\boldsymbol{A}\boldsymbol{v} = \lambda\boldsymbol{v} \to (\boldsymbol{A} - \lambda\boldsymbol{I})\,\boldsymbol{v} = 0$.
- 2. λ is a eigenvalue of $\boldsymbol{A}$ iff $\boldsymbol{A} - \lambda\boldsymbol{I}$ is singular: $det(\boldsymbol{A} - \lambda\boldsymbol{I}) = 0$.
- 3. The eigenvector $\boldsymbol{v_i}$ is determined by $(\boldsymbol{A} - \lambda_i\boldsymbol{I})\,\boldsymbol{v_i} = 0$.

5.2 Stability language

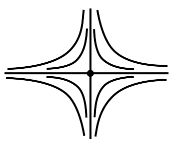
- Stable manifold: all initial conditions $\boldsymbol{x_0}$, such that $\boldsymbol{x}(t) \to \boldsymbol{x}^*$ as $t \to \infty$.
- Unstable manifold: all initial conditions $\boldsymbol{x_0}$, such that $\boldsymbol{x}(t) \to \boldsymbol{x}^*$ as $t \to -\infty$.
- FP is locally attractive, if $\boldsymbol{x}(t) \to \boldsymbol{x}^*$, as $t \to \infty$ for $\boldsymbol{x_0}$ close to $\boldsymbol{x}^*$.
- FP is globally attractive, if $\boldsymbol{x}(t) \to \boldsymbol{x}^*$, as $t \to \infty$ for all $\boldsymbol{x_0}$.
- FP is Lyapunov stable, if all trajectories that start sufficiently close $\boldsymbol{x}^*$ remain close to it for all time: if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that, if $\|\boldsymbol{x}(0) - \boldsymbol{x}^*\| < \delta$, then for every $t \geq 0$ we have $\|\boldsymbol{x}(t) - \boldsymbol{x}^*\| < \varepsilon$.
- FP is neutrally stable, if it is Lyapunov stable but not attractive.
- FP is asymptotically stable, if it is Lyapunov stable and attractive.
- FP is unstable, if it is neither attracting nor Lyapunov stable.

5.3 Classification of Linear Systems

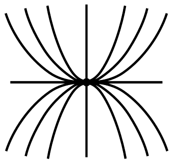
- Use eigenanalysis to create picture of phase portrait.
- Eigenvectors and eigenvalues $\to$ qualitative phase portrait.
- Find eigensolutions of the form: $\boldsymbol{x}(t) = \boldsymbol{v}e^{\lambda t} \to \boldsymbol{x}(t) = c_1\boldsymbol{v_1}e^{\lambda_1 t} + c_2\boldsymbol{v_2}e^{\lambda_2 t}$.
- $\lambda_{1,2} = \frac{\tau \pm \sqrt{\tau^2 - 4\Delta}}{2}$
  - $\tau = \lambda_1 + \lambda_2$
  - $\Delta = \lambda_1 \lambda_2$
- $ax^2 + bx + c = 0 \to x_{1,2} = \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$



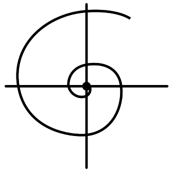
- $\triangle < 0$: Isolated FP: Saddle Point
- $\triangle = 0$: Nonisolated FP (λ_1 or $\lambda_2 = 0$)
  - $\tau < 0$: Line of Lyapunov stable FPs.
  - $\tau = 0$: Plane of FPs ($\lambda_1 = \lambda_2 = 0$).
  - $\tau > 0$: Line of unstable FPs.
- $\triangle > 0$: Isolated FP
  - $\tau < -\sqrt{4\triangle}$: Stable node ($\lambda_1$ and λ_2 real and negative).
  - $\tau = -\sqrt{4\triangle}$:
    - If there are no uniquely determined eigenvectors, no eigendirection: Stable star.
    - If there is one uniquely determined eigenvector, one eigendirection: Stable degenerate node.
  - $-\sqrt{4\triangle} < \tau < 0$: Stable spiral ($\lambda_1$ and λ_2 are complex conjugate).
  - $\tau = 0$: Stable center (λ_1 and λ_2 are purely imaginary and complex conjugate).
  - $0 < \tau < \sqrt{4\triangle}$: Unstable spiral ($\lambda_1$ and λ_2 are complex conjugate).
  - $\tau = \sqrt{4\triangle}$:
    - If there are no uniquely determined eigenvectors, no eigendirection: Unstable star.
    - If there is one uniquely determined eigenvector, one eigendirection: Unstable degenerate node.
  - $\sqrt{4\triangle} < \tau$: Unstable node (λ_1 and λ_2 real and positive).
- **General:** $\lambda_1 < \lambda_2 < 0$: Fixed point is stable, trajectories approach fixed point tangent to slow eigendirection λ_2, for $t \to -\infty$ trajectories become parallel to fast eigendirection λ_1.
- Centers, stars, degenerate nodes, and non-isolated fixed points are **borderline cases** that occur along curves in the (Δ, τ) plane.
- Centers can occur in nonlinear frictionless mechanical systems where energy is conserved.
- **Hartman-Grobman Theorem**: The local phase portrait near a hyperbolic FP ($Re(\lambda_i) \neq 0$) is topologically equivalent to the phase portrait of the linearization.
- **Basin of Attraction**: Region in phase plane in which every particle moves to the same FP.



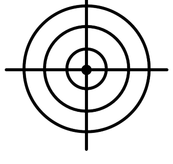
Saddle Point



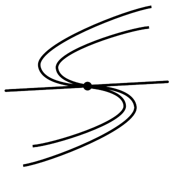
Node



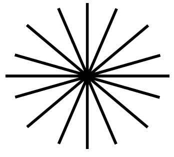
Spiral



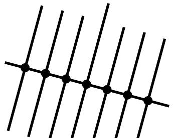
Center



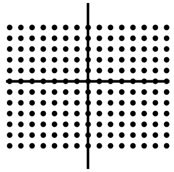
Degenerate Node



Star



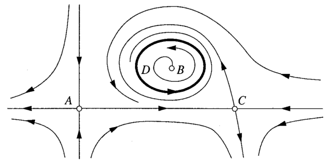
Line of Lyapunov stable FPs



Plane of FPs

6 2D Nonlinear Systems

- General form of a vector field in phase plane: $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$. By flowing along the vector field, a phase point traces out a solution $\boldsymbol{x}(t)$, corresponding to a trajectory.
- Features of a phase portrait:
  - 1. FPs: $\boldsymbol{f}(\boldsymbol{x}^*) = \boldsymbol{0}$ (like A, B, C).
  - 2. Closed Orbits: Periodic solutions $\boldsymbol{x}(t + T) = \boldsymbol{x}(t)\,\forall t$ and $T > 0$ (like D)
  - 3. Flow close to FPs an close orbits.
  - 4. Stability and instability of FPs and closed orbits. (stable: A, B, D and unstable: C).



- A phase portrait is structurally stable, if its topology is robust against perturbations (e.g. saddles are structurally stable, centers are not).
- Homoclinic Orbit: Trajectory that starts and ends at the same FP.
- Heteroclinic Orbit: Trajectory that starts and ends at two different FPs.

6.1 Existance and Uniqueness Theorem

- Existance $\leftrightarrow$ Smoothness.
- Uniqueness $\leftrightarrow$ Trajectories do not cross/intersect. Otherwise at the intersection point two solutions would start (not unique).
  - Consider $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$, $\boldsymbol{x}(0) = \boldsymbol{x_0}$, suppose that $\boldsymbol{f}$ and its derivatives are smooth. Then for $\boldsymbol{x_0}$ the initial value problem has a unique solution $\boldsymbol{x}(t)$.

6.2 Calculus of Mechanical Equations

- The system of the form $\ddot{x} + f(x, \dot{x})\dot{x} + h(x, \dot{x})x + c = 0$ can be expressed as:
  - $\dot{x} = y$ and
  - $\dot{y} = -f(x, y)y - h(x, y)x - c$.

6.3 Fixed Points and Linearization

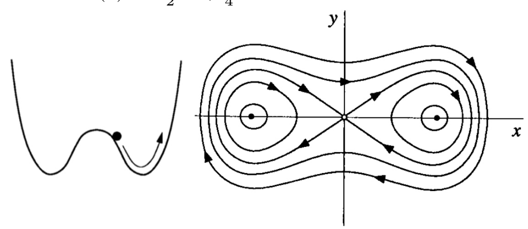
- Approximation of the phase portrait near a FP by a corresponding linear system.
- Nonlinear System:
  - $\dot{x} = f(x, y)$ and $\dot{y} = g(x, y)$
  - with FP at (x^*, y^*).
- A small disturbances near the FP: $u = x - x^*$ and $v = y - y^*$.
- Approximation by a Taylor series leads to:
  - $\begin{bmatrix}\dot{u}\\ \dot{v}\end{bmatrix} = \begin{bmatrix}\frac{\delta f}{\delta x} & \frac{\delta f}{\delta y}\\ \frac{\delta g}{\delta x} & \frac{\delta g}{\delta y}\end{bmatrix}\begin{bmatrix}u\\ v\end{bmatrix} +$ h.o.t
  - Dropping the h.o.t, one obtain a **linearized system**.
  - Neglecting h.o.t. gives qualitatively correct pictures for saddle, nodes and spirals, but not for borderline cases.

6.4 Nullclinics

- Graph lines: $\dot{x} = f(x, y) = 0$, $\dot{y} = g(x, y) = 0$
- Intersection points are FP that move by changing μ.
- The vector directions on the nullclinics are either vertical or horizontal.

6.5 Conservative Systems

- There exists a conserved quantity $E(\boldsymbol{x})$, i.e. a quantity that is const. along trajectories $\frac{dE}{dt} = 0$.
- $m\ddot{x} = F(x)$ and $(F \neq F(\dot{x}, t)) \to$ No damping or time-dependent driving forces.
- Conservative systems have **no attractive or repulsive FPs**.
- In conservative systems there are **only saddles or centers**.
- **Example**: $\ddot{x} = x - x^3 = F(x) \to \dot{x} = y$ and $\dot{y} = x - x^3$ (2D nonlinear system).
  - Energy $E = \frac{1}{2}y^2 - \frac{1}{2}x^2 + \frac{1}{4}x^4 = const.$
  - Potenial $V(x) = -\frac{1}{2}x^2 + \frac{1}{4}x^4$.



- $\to$ Two trajectories can be described by heteroclinic orbits. All other solutions are periodic.
- General idea: Define $V(x)$ as potential energy and set $F(x) = -\frac{dV}{dx} \to m\ddot{x} + \frac{dV}{dx} = 0 \Leftrightarrow m\dot{x}\ddot{x} + \frac{dV}{dx}\dot{x} = 0 \Leftrightarrow \frac{d}{dt}\left(\frac{1}{2}m\dot{x}^2 + V(x)\right) = 0$
- Preventing trivial solution $E(x) \neq 0$, then $E(x) = const.$ as a function of time.

6.5.1 Nonlinear Centers Theorem

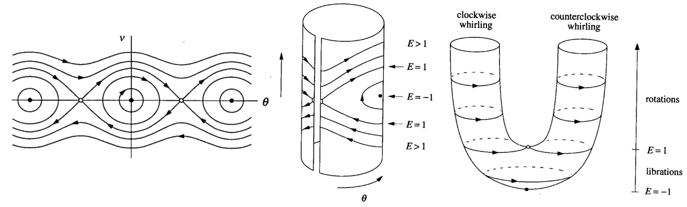
- Consider a conservative system $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$, $\boldsymbol{x} \in \mathbb{R}^2$ and $\boldsymbol{f}$ is smooth. Suppose there exists a conserved quantity $E(\boldsymbol{x})$ and suppose that $\boldsymbol{x}^*$ is an isolated FP. If $\boldsymbol{x}^*$ is a local minimum of E, then all trajectories sufficiently close to $\boldsymbol{x}^*$ are closed.

6.6 Reversible System

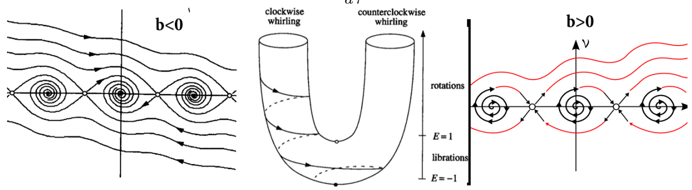
- Short: Dynamics are the same, whether the time runs forward or backward.
- Definition: Any 2-nd order system that is invariant under time $t \to -t$.
- Reversible systems have many of the same properties as conservative systems, e.g. centers are robust.

6.7 Pendulum

- **Equation of motion**: $\ddot{\theta} + \frac{g}{L}sin(\theta) = 0$.
- **Nondimensionalize**: Set $\omega = \sqrt{\frac{g}{L}}$ and $\tau = \omega t$, then $\ddot{\theta} + sin(\theta) = 0$ with overdot denotes differentiation with respect to τ.
- **System**: $\dot{\theta} = \nu$ and $\dot{\nu} = -sin(\theta)$.
- FPs: $(\theta^*, \nu^*) = (k\pi, 0)$ (saddle) and $(\theta^*, \nu^*) = (0, 0)$ (center). FPs are 2π-periodic.
- Energy: $E = \frac{1}{2}\dot{\theta}^2 - cos(\theta)$. Physical interpretation:
  - $E = -1$: stable equilibrium, pendulum hangs down.
  - $E < 1$: Small oscillations about equilibrium.
  - $E = 1$: Critial case, heteroclinic trajectories: Motion in which the pendulum slows to a halt precisely as it approaches the inverted position (saddle node).
  - $E > 1$: Pendulum whirls repeatedly over the top $\to$ periodic solution.
- A cylinder is the natural phase space of the pendulum (periodic whirling motions look periodic. They are the closed orbits that encircle the cylinder for $E > 1$).



- **Adding damping**: $\ddot{\theta} + b\dot{\theta} + sin(\theta) = 0$ and $b > 0$.
- The change in energy along a trajectory: $\frac{dE}{d\tau} = -b\dot{\theta}^2 \leq 0$.

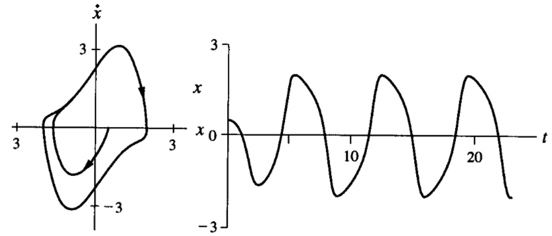


7 Limit Cycles

- Definition: A limit cycle is an isolated closed trajectory. Isolated means that neighboring trajectories are not closed. They spiral either toward or away from the LC.
- Types: stable, unstable, half-stable (trajectories from one side towards the LC, from the other side away).
- System has a standard oscillation and if perturbed returns to standard cycle (e.g. heartbeat, body temperature, feedback controlled system).
- LC can not occur in linear systems:
  - Linear: If $\boldsymbol{x}(t)$ is a solution, $c\cdot\boldsymbol{x}(t)$ is a solution. Thus, the solution is not isolated.
  - Nonlinear: LC oscillations are determined by the structure of the system itself. The amplitude of a oscillation is not set entirely by its initial conditions and disturbances will not persist forever.

7.1 Van-der-Pol Oscillator

- System: $\ddot{x} + \mu\left(x^2 - 1\right)\dot{x} + x = 0$ with $\mu \geq 0$ (one unique stable LC for $\mu > 0$).
- Harmonic oscillator with nonlinear damping term $\left(x^2 - 1\right)$. Damps oscillations for $|x| > 1$ and pumps up for $|x| < 1$.
- System settles into a self-sustained oscillation, where the dissipated energy balances the energy pumped in.



7.2 Ruling Out Closed Orbits

7.2.1 Gradient System

- Gradient system with potential function $V(\boldsymbol{x})$: $\dot{\boldsymbol{x}} = -\bigtriangledown V$.
- Closed orbits are impossible in gradient systems.
- Example: $\dot{x} = \sin(y)$ and $\dot{y} = x\cos(y) \to V(x, y) = -x\sin(y)$.

- $-\frac{\delta V}{\delta x} = \sin(y)$ and $-\frac{\delta V}{\delta y} = x\cos(y)$.
- No LC.

- **Proof**: Assume there is a limit cycle. Going one time around would mean $\bigtriangledown V = 0$. But on the other hand: $\triangle V = \int_0^T \frac{dV}{dt}\,dt = \int_0^T (\bigtriangledown V \cdot \dot{\boldsymbol{x}})\,dt = -\int_0^T \|\dot{\boldsymbol{x}}\|^2\,dt < 0$ (unless $\dot{\boldsymbol{x}} = \mathbf{0}$, trajectory is a fixed point).

7.2.2 Lyapunov Function

- **Idea**: Find a Lyapunov function that decays monotonically along arbitrary trajectories $\to$ no LC.
- **Theorem**: If for a given system $\dot{\boldsymbol{x}} = f(\boldsymbol{x})$ with FP $\boldsymbol{x}^*$ a Lyapunov function exists, then $\boldsymbol{x}^*$ is global asymptotically stable for all initial conditions and closed orbits are not possible.
- Lyapunov function:
  - $V(\boldsymbol{x}) > 0$ for all $\boldsymbol{x} \neq \boldsymbol{x}^*$ and $V(\boldsymbol{x}^*) = 0$ (V is positive definite).
  - $\dot{V} < 0$ for all $\boldsymbol{x} \neq \boldsymbol{x}^*$ (all trajectories flow downhill toward $\boldsymbol{x}^*$).

7.3 Proof Existence of Closed Orbit

7.3.1 Poincaré-Bendixson Theorem

- Suppose that:
  - R is a closed, bounded subset of the plane.
  - $\boldsymbol{x} = f(\boldsymbol{x})$ is a continuously differentiable vector field on an open set containing R.
  - R does not contain any fixed points.
  - There exists a trajectory C that is confined in R, in the sense that it starts in R and stays in R for all future time.
- Then either C is a closed orbit, or it spirals toward a closed orbit as $t \to \infty$. In either case, R contains a closed orbit.
- In 2D-systems, you can not have something more complicated then a FP or a closed orbit, chaos is not possible in 2D.

7.4 Nonlinear Oscillator

- System: $\ddot{x} + f(x, \dot{x})\dot{x} + g(x) = 0$
  - Nonlinear damping: $f(x, \dot{x})\dot{x}$.
  - Restoring forces: $g(x)$.
- Distinction:
  - Weakly: response is nearly harmonic.
  - Strongly: shocks, jumps, discontinuous dynamical systems

7.4.1 Weakly Nonlinear Oscillator

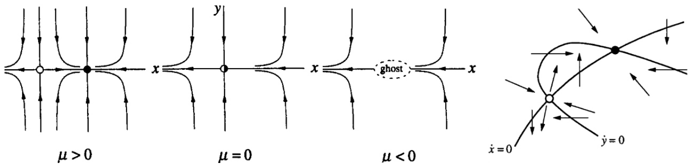
- System: $\ddot{\boldsymbol{x}} + \epsilon h(\boldsymbol{x}, \dot{\boldsymbol{x}}) + \boldsymbol{x} = 0$ with $0 < \epsilon < 1$ and $h(\boldsymbol{x}, \dot{\boldsymbol{x}})$ is smooth.
  - Van-der-pol oscillator: $\ddot{\boldsymbol{x}} + \epsilon\left(\boldsymbol{x}^2 - 1\right)\dot{\boldsymbol{x}} + \boldsymbol{x} = 0$.
  - Duffing oscillator: $\ddot{\boldsymbol{x}} + \epsilon \boldsymbol{x}^3 + \boldsymbol{x} = 0$.
- Trajectories are nearly elliptical.

8 2D Bifurcations

- The phase portrait changes its topological structure as a parameter is varied, e.g. change in number and stability of FPs, closed orbits or saddle connections.
- **Classification:**
  - 1. Bifurcations of FP:
    - $\lambda = 0$ Bifurcation (Saddle-Node, Transcritical, Pitchfork).
    - $\lambda = \pm i\omega$ Hopf Bifurcation $\to$ creation of closed orbits.
  - 2. Bifurcation of Closed Orbits
    - Coalescence of Cycles (Saddle-Node Bifurcation of Cycles).
    - Sniper, Snic (Saddle-Node-Infinite-Period Bifurcation).
    - Homoclinic Bifurcation

8.1 Saddle-Node Bifurcation

- Example: $\dot{x} = \mu - x^2$ and $\dot{y} = -y$.
  - Two FP at $(\pm\sqrt{\mu}, 0)$ and eigenvalues $\lambda_1 = -2x^*$, $\lambda_2 = -1$.
- As μ decreases, the saddle and node approach and collide at $\mu = 0$ ($\lambda_1 = 0$) and disappear. It remains a ghost that slows down incoming trajectories.



8.2 Transcritical and Pitchfork Bifurcations

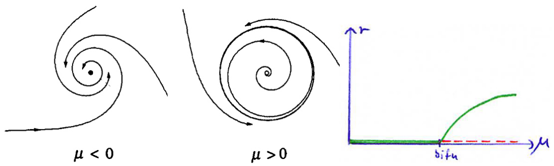
- Transcritical: $\dot{x} = \mu x - x^2$, $\dot{y} = -y$ (FP changes stability.)
- Supercritical Pitchfork: $\dot{x} = \mu x - x^3$, $\dot{y} = -y$ (Two new stable FPs)
- Subcritical Pitchfork: $\dot{x} = \mu x + x^3$, $\dot{y} = -y$ (Two new unstable FPs)

8.3 Hopf Bifurcation

- FP changes stability, both complex conjugate eigenvalues cross the imaginary axis into the right half plane. Dimensions ≥ 2.
- Linearization does not provide a distinction between sub- or supercritical bifurcations.

8.3.1 Supercritical Hopf Bifurcation (soft, continuous, safe)

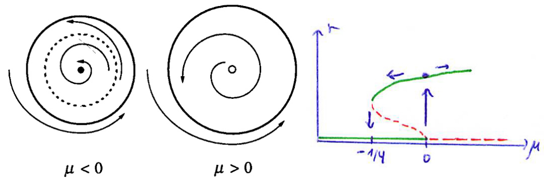
- Stable spiral changes into an unstable spiral surrounded by a small, nearly elliptical limit cycle when control-parameter μ is changed (or vice versa).
- Example: $\dot{r} = \mu r - r^3$, $\dot{\theta} = \omega + br^2$



- Rules of thumb: 1. Size of the LC grows continuously from zero with $\sqrt{\mu}$. 2. The frequency of the limit cycle is approximately $\omega = \mathrm{Im}(\lambda)$, evaluated at $\mu = 0$.

8.3.2 Subcritical Hopf Bifurcation (hard, discontinuous, dangerous)

- Example: $\dot{r} = \mu r + r^3 - r^5$, $\dot{\theta} = \omega + br^2$
- Unstable cycle surrounds a stable spiral. Cycle shrinks and at $\mu = 0$ FP becomes unstable. $\mu > 0$ large LC is suddenly the only attractor $\to$ small oscillations now grow to large oscillations.
- Hysteresis: Large oscillations can not turned off by decreasing μ.

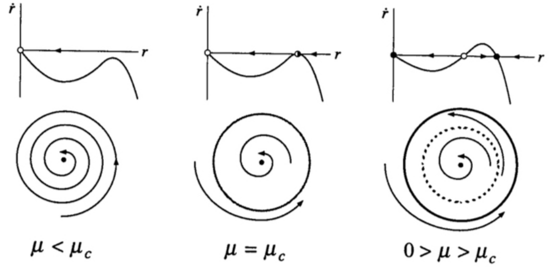


8.4 Global Bifurcation

- In 2D there are four ways to generate a LC: 1. Hopf bifurcation (local), 2. Saddle-Node bifurcation (global), 3. Infinte period bifurcation (global), 4. Homoclinic bifurcation (global).
- $\to$ Analysis of the whole state space is required.

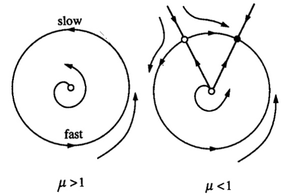
8.4.1 Saddle-Node Bifurcation of Cycles (Coalesce/Annihilation of LCs)

- Phenomenon: Two LC merges and disappear.
- Example: $\dot{r} = \mu r + r^3 - r^5$, $\dot{\theta} = \omega + br^2$.
- At $\mu_c = \frac{1}{4}$ a new half-stable cycle is born and splits into a pair of LCs (one stable, one unstable) as μ increases.
- Special: The origin remains stable throughout, it does not participate in this bifurcation.



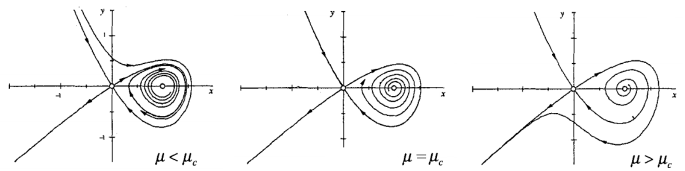
8.4.2 Infinite-Period Bifurcation

- Phenomenon: Saddle-node bifurcation in phase angle. A LC is destroyed via infinite period.
- Example: $\dot{r} = r\left(1 - r^2\right)$, $\dot{\theta} = \mu - \sin(\theta)$, $\mu \geq 0$.
- $\mu > 1$: Angular motion is everywhere clockwise. $\mu \to 1^+$: Oscillation period lengthens and becomes infinite at $\mu_c = 1$, when a FP appears on the circle at $\frac{\pi}{2}$. $\mu < 1$: FPs split into a saddle and a node.



8.4.3 Homoclinic Bifurcation

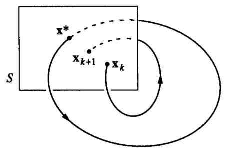
- Phenomenon: LC detaches from homoclinic orbit as μ decreases.
- Example: $\dot{x} = y$, $\dot{y} = \mu y + x - x^2 + xy$.
- $\mu < \mu_c$: Stable LC close to a saddle point at the origin. As μ increases the LC gets closer to the homoclinic orbit. $\mu = \mu_c$: The LC becomes a homoclinic orbit. $\mu > \mu_c$: The saddle point connection breaks and the loop is destroyed.



8.5 Coupled Oscillators and Quasiperiodicity

- 2D-Phase-Space: Torus: $[0, 2\pi] \times [0, 2\pi]$.
- Natural phase space for systems: $\dot{\theta_1} = f_1(\theta_1, \theta_2)$ and $\dot{\theta_2} = f_2(\theta_1, \theta_2)$, where f_1 and f_2 are periodic functions.
- Example: Coupled Oscillators: $\dot{\theta_1} = \omega_1 + K_1 \sin(\theta_2 - \theta_1)$ and $\dot{\theta_2} = \omega_2 + K_2 \sin(\theta_1 - \theta_2)$.
- Visualize on torus or on a square with periodic boundary conditions.
- **Uncoupled System:**
  - $K_1 = K_2 = 0$, $\dot{\theta_1} = \omega_1$, $\dot{\theta_2} = \omega_2$.
  - Trajectories on square are straight lines with slope $\frac{\omega_2}{\omega_1}$.
  - Case 1: Slope is rational $\to$ all trajectories are closed orbits on torus.
  - Case 2: Slope is irrational $\to$ flow is quasiperiodic, every trajectory winds around endlessly on torus, never intersecting, never closing.
- **Coupled System:**
  - Consider phase-difference: $\phi = \theta_1 - \theta_2 \to \dot{\phi} = \omega_1 - \omega_2 - (K_1 + K_2)\sin(\phi)$.

8.6 Poincaré Maps



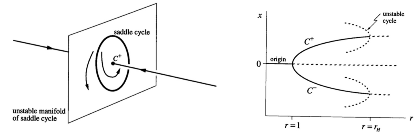
- A Poincaré map or Poincaré return map is the intersection of a periodic orbit in the state space of a continuous dynamical system with a certain lower-dimensional subspace.
- Let $\dot{\boldsymbol{x}} = f(\boldsymbol{x})$ be an n-dimensional system. Let S be an $n-1$-dimensional surface intersecting the flow transversally (<u>Surface of section</u>).
- Poincaré map P: mapping from S to itself by following trajectories from one intersection with S to the next: $\boldsymbol{x_{k+1}} = P(\boldsymbol{x_k})$.
- FPs are solutions to $x_{k+1} - x_k = 0$.
- If $\boldsymbol{x}^*$ is a FP: $P(\boldsymbol{x}^*) = \boldsymbol{x}^* \to \boldsymbol{x}^*$ returns to $\boldsymbol{x}^* \to$ closed orbit.
- Looking at behaviour of P near fixed point, one can determine stability of closed orbits. Poincaré map converts problems about closed orbits to problems about FPs of a mapping.

8.6.1 Linear Stability of Periodic Orbits

- Determine the stability of a closed orbit $\to$ determine the stability of the corresponding FP $\boldsymbol{x}^*$ on the Poincaré map.
- $\boldsymbol{x}^* + \boldsymbol{v_1} = P(\boldsymbol{x}^* + \boldsymbol{v_0}) = P(\boldsymbol{x}^*) + [DP(\boldsymbol{x}^*)]\,\boldsymbol{v_0} + O\left(\|\boldsymbol{v_0}\|^2\right)$ with $DP(\boldsymbol{x}^*)$ is the linearized Poincaré map at $\boldsymbol{x}^*$. Then $\boldsymbol{v_1} = [DP(\boldsymbol{x}^*)]\,\boldsymbol{v_0}$.
- Closed orbit is linearly stable iff eigenvalues $\left|\lambda_j\right| < 1$ of $DP(\boldsymbol{x}^*)$ (also called the characteristic or Floquet multiplier).

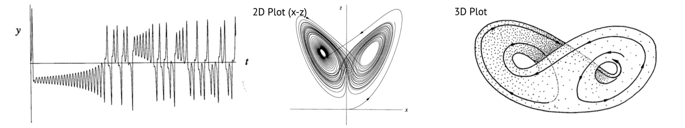
9 Lorenz Equation

- $\dot{x} = \sigma(y - x)$, $\dot{y} = rx - y - xz$, $\dot{z} = xy - bz$ with $\sigma, r, b > 0$.
- <u>Observations</u>: Determinisitc system, that exhibits strange dynamics, aperiodic behaviour, never exactly repeating, but always in a bounded region of the phase space (strange attractor).
- Dissipative nature: Phase space volume shrinks. The system is <u>globally contractive</u> $\Leftrightarrow$ any volume of initial conditions shrinks to zero.
  - 1. No quasiperiodic solutions (they would live on a torus).
  - 2. No repellent FP or repelling LC $\to$ all FP are either sinks or saddles.
  - 3. Closed orbits are stable or saddle-like.
- Symmetry: $(x, y, z) = (-x, -y, z)$.
- Analysis for $\left(\sigma = 10, b = \frac{8}{3}\right)$:
  - Global stable FP at origin exists always and if $r < 1$.
  - $r = 1$: Supercritical pitchfork bifurcation.
  - $1 < r < r_h$: Origin turns unstable, two new linearly stable FP at $x^* = y^* = \pm\sqrt{b(r-1)}$ and $z^* = r - 1$. Stable FP are surrounded by a sadle cycle.
  - $r = r_H$: Subcritical Hopf bifurcation, cycle shrinks and is absorbed by FP, that changes into a saddle point.
  - $r > r_H$: No attractors in the neighborhood remain.



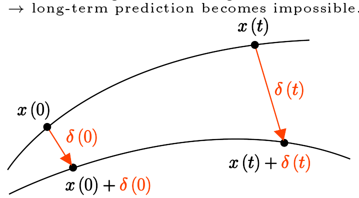
9.1 Chaos on a Strange Attractor

- Plots with $\sigma = 10$, $b = \frac{8}{3}$ and $r = 28 > r_H$.
- $y(t)$-plot: Solution settles into an irregular, non-repeating (aperiodic) motion.
- 2D: Trajectory spirals on both sides. Number of circuits are unpredictable.
- 3D: Infinite sets of surfaces, extremely close to each other. This set with zero volume but infinite surface area is called fractal.



9.1.1 Exponential Divergence of Nearby Trajectories

- Motion in the attractor exhibits. Sensitive dependence on initial conditions. Two trajectories starting very close together will exponentially fast diverge $\|\delta(t)\| \approx \|\delta(0)\|\,e^{\lambda t}$ $\to$ long-term prediction becomes impossible.



- **Lyapunov exponent** is a quantity that characterizes the rate of separation of infinitesimal close trajectories. The separation can be different in direction, thus there are actually n different Lyapunov exponents for an n-dimensional system. λ is chosen as the largest Lyapunov exponent (dominant stretching direction of ellipsoid of initial condition).

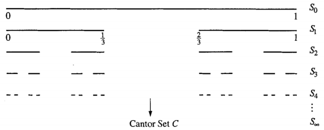
- **Horizon of Predictability**: For systems with a positive Lyapunov exponent there is a time horizon, beyond which prediction breaks down. Initial conditions are known only approximately. The true dynamics separates from the assumed one.

10 Fractals

- Motivation: Geometry of strange attractors are fractals.
- Fractals are complex geometric shapes with fine structure at arbitrarily small scales. Usually they have some degree of self-similarity. (Magnify a tiny part of the fractal and you see features reminiscent of the whole).
- Their dimension is not an integer.
- Distance between two points on the fractal $\rightarrow \infty$.

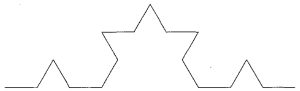
10.1 Cantor Set

- The limiting set $C = S_\infty$ is the Cantor set.
  - C has structure at arbitrarily small scales. If we enlarge part of C repeatedly, we continue to see a complex pattern of points separated by gaps of various sizes.
  - C is self-similar. It contains smaller copies of itself at all scales.
  - The dimension of C is not an integer: $d = \frac{ln2}{ln3} = 0.63$.
  - C has measure zero and it consists of uncountably many points.

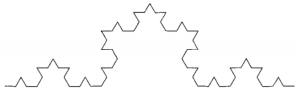


10.2 Cantor Set

- $d = \frac{ln4}{ln3} = 1.26$



S_2



S_3

10.3 Dimensions

- **Box Counting**: Consider covering the object with boxes of size ε. Then:
  - $d = \lim_{\varepsilon \to 0} \frac{lnN(\varepsilon)}{lnN\left(\frac{1}{\varepsilon}\right)}$
- **Pointwise and Correlation Dimension**: Fix a point $\boldsymbol{x}$ on attractor A. Let $N_x(\varepsilon)$ denote number of points on A in a ball of radius ε about $\boldsymbol{x}$. Vary ε over many $\boldsymbol{x}$ and determine pointwise dimension d at $\boldsymbol{x}$: $N_x(\varepsilon) \propto \varepsilon^d$. Average $N_x(\varepsilon)$ over many $\boldsymbol{x}$ to get correlation dimension $C(\varepsilon) \propto \varepsilon^d$.

11 Strange Attractor

11.1 Stretching and Folding Property of Dynamic System

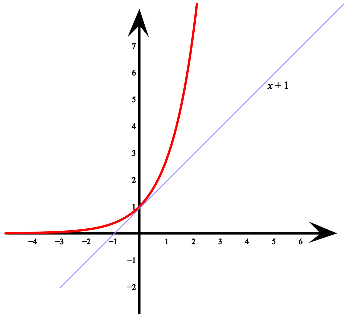
- How can trajectories diverge endlessly and stay bounded by stretching and folding:
  - 1. Consider a region of initial conditions in phase space.
  - 2. A strange attractor arises when flow contracts the region in some direction and stretches in others.
  - 3. Region must be folded back on itself to remain in bounded region.

11.2 Attractor Reconstruction

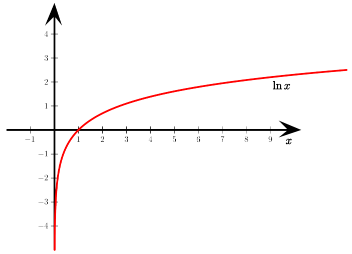
- Idea: Dynamics in the full phase space can be reconstructed from a single time serie $B(t)$
  - 1. Define n-dimensional vector $\boldsymbol{x}(t) = (B(t), B(t+\tau), \ldots, B(t+n\tau))$.
  - 2. From a certain dimension n the trajectories do not intersect anymore.
  - 3. The topological structure of the strange attractor is preserved in $\boldsymbol{x}(t)$.
  - 4. E.g. obtain Poincaré section, compute intersection of the orbits $\boldsymbol{x}(t)$ with a fixed plane.

12 Math

- $ax^2 + bx + c = 0 \rightarrow x_{1,2} = \frac{-b}{2a} \pm \frac{\sqrt{b^2-4ac}}{2a}$
- $\bigtriangledown = \begin{bmatrix} \frac{\delta}{\delta x_1} \\ \vdots \\ \frac{\delta}{\delta x_n} \end{bmatrix}$
- Taylor series about point a: $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + h.o.t.$
- $\sin(x) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!} = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} \mp \cdots$
- $\cos(x) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!} = \frac{x^0}{0!} - \frac{x^2}{2!} + \frac{x^4}{4!} \mp \cdots$
- $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$



$f(x) = e^x$



$f(x) = \ln(x)$

13 Questions

13.1 General

- **What is dynamical system?**
  In mathematics, a dynamical system is a system in which a function describes the time dependence of a point in a geometrical space. We consider two types of dynamical systems: flows (continuous time) and maps (discrete time steps).
- **Which kind of system depends explicitly on the time?**
  Non-Autonomous time-dependent system.
- **What is an autonomous system?**
  System of ODEs which does not explicitly depend on the independent variable. Is closely related to dynamical systems. Any autonomous system can be transformed into a dynamical system and, using very weak assumptions, a dynamical system can be transformed into an autonomous system.
- **What is a fixed point?**
  Equilibrium point of the system.
- **Give and describe two different concepts for stability.**
  A non-wandering set (FP, LC, quasiperiodic orbits, chaotic orbits) may be stable or unstable. There are two types of stability, a weaker and a stronger one:
  - Lyapunov stability: Every orbit starting in a neighborhood of the non-wandering set remains in a neighborhood.
  - Asymptotic stability: In addition to the Lyapunov stability, every orbit in a neighborhood approaches the non-wandering set asymptotically.
- **What is linear stability analysis?**
  A dynamical system is called linearly stable, if the linearization of the system has the form $\dot{\boldsymbol{x}} = A\boldsymbol{x}$ and all eigenvalues of A have negative parts.
- **What is the basin of attraction?**
  The basin of attraction is the set of all initial states approaching the attractor in the long time limit.
- **What is a ghost?**
  Just after fixed points collide, there is a saddle-node remnant or ghost that leads to slow passage through a bottleneck.
- **What are trajectories?**
  Trajectories are solutions in the phase space. The solution is defined by its initial conditions.
- **Are there examples, which does not fulfill the existence and uniqueness theorem?**
  Very unstable FP is not unique, e.g. infinite slope at FP.
- **What is quasiperiodicity?**
  Irregular periodic system.
- **What is a separatrix?**
  Boundary in the phase plane separating two modes of different behaviours, e.g. heteroclinic orbit in phase space of undampled pendulumn.

13.2 Bifurcation

- **Why is it called saddle-node?**
  In general: FPs are created or destroyed. In 2D it looks like horse-saddle.
- **What is saddle-node bifurcation?**
  A saddle-node bifurcation, tangential bifurcation or fold bifurcation is a local bifurcation in which two fixed points (or equilibria) of a dynamical system collide and annihilate each other. If the phase space is one-dimensional, one of the equilibrium points is unstable (the saddle), while the other is stable (the node).
- **How to distinguish local and global bifurcations?**
  - A **local** bifurcation occurs when a parameter change causes the stability of an equilibrium (or fixed point) to change. In continuous systems, this corresponds to the real part of an eigenvalue of an equilibrium passing through zero. Changes by the bifurcation are confined to a finite neighbourhood. They can be detected by using linear stability analysis.
  - **Global** bifurcations occur when larger invariant sets, such as periodic orbits, collide with equilibria. This causes changes in the topology of the trajectories in the phase space which cannot be confined to a small neighbourhood. Analysis of the whole state space is required. They can not be detected by linear stability analysis.
- **Which bifurcation is more dangerous (subcritical or supercritical)?**
  Subcritical: jumps, hysteresis.
- **Definition of a global bifurcation?**
  Analysis of the whole state space is required. Global bifurcations occur when 'larger' invariant sets, such as periodic orbits, collide with equilibria. This causes changes in the topology of the trajectories in the phase space which cannot be confined to a small neighbourhood, as is the case with local bifurcations.
- **Which bifurcation occurs using stable higher order terms in a system with subcritical Hopf bifurcation?**
  Saddle-Node bifurcation.

13.3 Limit Cycles

- **How to determine the existence of LC?**
  1. Gradient Systems does not have LC. 2. If a Lyapunov function exists that decays monotonically along arbitrary trajectories, then there are no LCs. 3. One can proof the existence of a LC using the Poincaré-Bendixson theorem.
- **How to determine the existence of LC in very high dimensional systems?**
  Using Poincaré return maps.
- **What is a Poincaré map?**
  It is the intersection of a periodic orbit in the state space of a continuous dynamical system with a certain lower-dimensional subspace. A Poincaré map turns a continuous dynamical system into a discrete one. Poincaré maps are invertible maps because one gets x_n from x_{n+1} by following the orbit backwards.
- **How to determine the stability of LCs using Poincaré maps?**
  Determine the stability of the FPs.

13.4 Chaos

- **What is chaos?**
  Chaos is aperiodic long-term behavior in a deterministic system that exhibits sensitive dependence on initial conditions.
  - Aperiodic long-term behavior means that there are trajectories which do not settle down to fixed points, periodic orbits, or quasiperiodic orbits as t goes to infinity.
  - Deterministic means that the system has no random or noisy inputs or parameters. The irregular behavior arises from the system's nonlinearity, rather than from noisy driving forces.
  - Sensitive dependence on initial conditions means that nearby trajectories separate exponentially fast, i.e., the system has a positive Lyapunov exponent.
- **What is a Lyapunov exponent?**
  The Lyapunov exponent is a measure of how fast neighboring trajectories diverge or converge exponentially in a chaotic system.
- **In terms of Lyapunov exponents, when is a system chaotic?**
  A positive MLE (largest Lyapunov exponent) is usually taken as an indication that the system is chaotic.
- **How many Lyapunov exponents exist in a system?**
  As many as dimensions.
- **What means a Lyapunov exponent of zero?**
  Trajectories stay periodic.
- **What if the largest Lyapunov exponent is zero?**
  In most cases the solution is periodic, nevertheless, it will not converge to a single point.
- **What is an attractor?**
  An attractor is a closed set A with following properties:
  - 1. Invariant set: Any trajectory $\boldsymbol{x}(t)$, that starts in A stays in A for all time.
  - 2. A attracts an open set of initial conditions: A attracts all trajectories that start sufficiently close to it.
  - 3. A is minimal: There is no proper subset of A that satisfies cond. 1 and 2.
- **What is a strange attractor?**
  Attractor + Sensitive dependency on initial conditions.
- **Which are strange attractor?**
  Lorenz, Rössler, Hénon.
- **What means volume contraction?**
  If the system is dissipative, volumes in phase space contract under the flow.
- **What are characteristics of a strange attractor?**
  Attractive behavior in the phase space, sensitive to initial conditions, fractal structure, deterministic chaos.
- **How to determine the box-dimension?**
  Scaling and Box-Counting.

14 References

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