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ENGINEERING MATHEMATICS -II

SUBJECT CODE: MA8251

(Regulation 2017)
Common to all branches of B.E

UNIT- IV
COMPLEX INTEGRATION

UNIT –IV

COMPLEX INTEGRATION

4.1 LINE INTEGRAL AND CONTOUR INTEGRAL

If $f(z)$ is a continuous function of the complex variable $z = x + iy$ and C is any continuous curve connecting two points A and B on the z – plane then the complex line integral of $f(z)$ along C from A to B is denoted by $\int_c f(z)dz$

When C is simple closed curve, then the complex integral is also called as a contour integral and is denoted as $\oint_C f(z)dz$. The curve C is always take in the anticlockwise direction.

Note: If the direction of C is reversed (clockwise), the integral changes its sign

$$(ie) \oint_c f(z)dz = -\oint f(z)dz$$

Standard theorems:

1. **Cauchy's Integral theorem (or) Cauchy's Theorem (or) Cauchy's Fundamental Theorem**

   Statement: If $f(z)$ is analytic and its derivative $f'(z)$ is continuous at all points inside and on a simple closed curve C then $\oint_c f(z)\,dz = 0$

2. **Extension of Cauchy's integral theorem (or) Cauchy's theorem for multiply connected Region**

   Statement: If $f(z)$ is analytic at all points inside and on a multiply connected region whose outer boundary is C and inner boundaries are $C_1, C_2, \dots, C_n$ then

$$\int_c f(z)dz = \int_c f(z)dz + \int_{C_2} f(z)dz + \cdots + \int_{c_n} f(z)dz$$

3. **Cauchy's integral formula**

   Statement: If $f(z)$ is analytic inside and on a simple closed curve C of a simply connected region R and if 'a' is any point interior to C, then

$$f(a) = \frac{1}{2\pi i}\int_c \frac{f(z)}{z-a}dz$$

(OR)

$$\int_c \frac{f(x)}{z-a}dz = 2\pi i\, f(a),$$

   the integration around C being taken in the positive direction.

4. **Cauchy's Integral formula for derivatives**

   Statement: If $f(z)$ is analytic inside and on a simple closed curve C of a simply connected Region R and if 'a' is any point interior to C, then

$$\int_c \frac{f(z)}{(z-a)^2}dz = 2\pi i f'(a)$$

$$\int_c \frac{f(z)}{(z-a)^3}dz = 2\pi i f''(a)$$

In general, $\int_c \frac{f(z)}{(z-a)^n}dz = 2\pi i f^{(n-1)}(a)$

Problems based on Cauchy's Integral Theorem

Example: 4.1 Evaluate $\int_0^{3+i} z^2 dz$ along the line joining the points (0, 0) and (3, 1)

Solution:

Given $\int_0^{3+i} z^2 dz$

Let $z = x + iy$

Here $z = 0$ corresponds to (0, 0) and $z = 3 + i$ corresponds to (3, 1)

The equation of the line joining (0, 0) and (3, 1) is

$$y = \frac{x}{3} \Rightarrow x = 3y$$

Now $z^2 dz = (x + iy)^2(dx + idy)$

$$= [x^2 - y^2 + i2xy][dx + idy]$$

$$= [(x^2 - y^2) + i2xy][dx + idy]$$

$$= [(x^2 - y^2)dx - 2xydy] + i[2xydx + (x^2 - y^2)dy]$$

Since $x = 3y \Rightarrow dx = 3dy$

$$\therefore z^2 dz = [8y^2(3dy) - 6y^2dy] + i[18y^2dy + 8y^2dy]$$

$$= 18y^2dy + i26y^2dy$$

$$\therefore \int_0^{3+i} z^2 dz = \int_0^{'}[18y^2 + i26y^2]dy$$

$$= \left[18\,\frac{y^2}{3} + i\,26\,\frac{y^3}{3}\right]_0^{'}$$

$$= 6 + i\frac{26}{3}$$

Example: 4.2 Evaluate $\int_0^{2+i}(x^2 - iy)dz$

Solution:

Let $z = x + iy$

Here $z = 0$ corresponds to (0, 0) and $z = 2 + i$ corresponds to (2, 1)

Now $(x^2 - iy)dz = (x^2 - iy)(dx + idy)$

$$= x^2dx + y\,dy) + i\,(x^2dy - y\,dx)$$

Along the path $y = x^2 \Rightarrow dy = 2xdx$

$$\therefore (x^2 - iy)dz = (x^2dx + 2x^3dx) + i(2x^3dx - x^2dx)$$

$$\int_0^{2+i}(x^2 - iy)dz = \int_0^2(x^2 + 2x^3)dx + i(2x^3 - x^2)dx$$

$$= \left[\frac{x^3}{3} + \frac{2x^4}{4}\right]_0^2 + i\left[\frac{2x^4}{4} = \frac{x^3}{3}\right]_0^2$$

$$= \left(\frac{8}{3} + \frac{16}{2}\right) + i\ \left(\frac{16}{2} - \frac{8}{3}\right)$$

$$= \frac{32}{3} + i\ \frac{16}{3}$$

Example: 4.3 Evaluate $\int_c\ e^{\frac{1}{z}}\, \boldsymbol{dz, where\ C\ is\ |z| = 2}$

Solution:

Let $f(z) = e^{\frac{1}{z}}$ clearly $f(z)$ is analytic inside and on C.

Hence, by Cauchy's integral theorem we get $\int_c\ e^{\frac{1}{z}} dz = 0$

Example: 4.4 Evaluate $\int_c\ \boldsymbol{z^2 e^{\frac{1}{z}} dz, where\ C\ is\ |z| = 1}$

Solution:

Given$\int_c\ z^2\, e^{1/z} dz$

$$= \int_c\ \frac{z^2}{e^{-1/z}} dz$$

$Dr = 0\ \Rightarrow z = 0,$ We get $e^{-\frac{1}{0}} = e^{-\infty} = 0$

$z = 0$ lies inside $|z| = 1$.

Cauchy's Integral formula is

$$\int_c\ z^2\, e^{1/z} dz = 2\pi i f(0) = 0$$

Example: 4.5 Evaluate $\int_c\ \boldsymbol{\frac{1}{2z-3} dz\ \ where\ C\ is\ |z| = 1}$

Solution:

Given $\int_c\ \frac{1}{2z-3} dz$

$Dr = 0\ \Rightarrow 2z - 3 = 0, \Rightarrow z = \frac{3}{2}$

Given $C\ is\ |z| = 1$

$$\Rightarrow |z| = \left|\frac{3}{2}\right|\ = \frac{3}{2} > 1$$

$\therefore z = \frac{3}{2}$ lies outside C

$\therefore$ By Cauchy's Integral theorem, $\int_c\ \frac{1}{2z-3} dz = 0$

Example: 4.6 Evaluate $\int_c\ \boldsymbol{\frac{dz}{z+4}\ \ where\ C\ is\ |z| = 2}$

Solution:

Given $\int_c\ \frac{dz}{z+4}$

$Dr = 0\ \Rightarrow z + 4 = 0\ \Rightarrow z = -4$

Given $C\ is\ |z| = 2$

$$\Rightarrow |z| = |-4| = 4 > 2$$

$\therefore z = -4$ lies outside C.

$\therefore$ By Cauchy's Integral Theorem, $\int_c \frac{dz}{z+4} = 0$

Example: 4.7 Evaluate $\int_c \frac{e^{2z}}{z^2+1} dz, where\ C\ is\ |z| = \frac{1}{2}$

Solution:

Given $\int_c \frac{e^{2z}}{z^2+1} dz$

$Dr = 0 \implies z^2 + 1 = 0 \implies z = \pm i$

Given $C\ is\ |z| = \frac{1}{2}$

$\Rightarrow |z| = |\pm i| = 1 > \frac{1}{2}$

$\therefore$Clearly both the points $z = \pm i$ lies outside C.

$\therefore$ By Cauchy's Integral Theorem, $\int_c \frac{e^{2z}}{z^2+1} dz = 0$

Example: 4.8 Using Cauchy's integral formula Evaluate $\int_c \frac{z+1}{(z-3)(z-1)} dz, where\ C\ is\ |z| = 2$

Solution:

Given $\int_c \frac{z+1}{(z-3)(z-1)} dz$

$Dr = 0 \implies z = 3, 1$

Given $C\ is\ |z| = 2$

$\therefore$Clearly $z = 1$ lies inside C and $z = 3$ lies outside C

$$\int_c \frac{z+1}{(z-3)(z-1)} dz = \int_c \frac{(z+1)/(z-3}{(z-1)} dz$$

$\therefore$ By Cauchy's Integral Theorem

$\int_c \frac{(z+1)/(z-3)}{(z-1)} dz = 2\pi i f(1)$ Where $f(z) = \frac{z+1}{z-3} \Rightarrow f(1) = \frac{2}{-2}$

$$= 2\pi i(-1) = -2\pi i$$

Example: 4.9 Using Cauchy's integral formula, evaluate $\int_c \frac{\sin\pi z^2 + \cos\pi z^2}{(z-2)(z-3)} dz$ **where C is the circle** $|z| = 4$**.**

Solution:

Given $\int_c \frac{\sin\pi z^2 + \cos\pi z^2}{(z-2)(z-3)} dz$

$Dr = 0 \implies z = 2, 3$

Given $C\ is\ |z| = 4$

$\therefore$Clearly $z = 2\ and$ 3 lies inside C.

Consider, $\frac{1}{(z-2)(z-3)} = \frac{A}{z-2} + \frac{B}{z-3}$

$\Rightarrow 1 = A(z-3) + B(Z = 2)$

Put $z = -3 \Rightarrow 1 = B$

Put $z = 2 \Rightarrow -1 = A$

$\therefore \frac{1}{(z-2)(z-3)} = -\frac{1}{z-2} + \frac{1}{z-3}$

$\int_c \frac{\sin \pi z^2 + \cos \pi z^2}{(z-2)(z-3)} dz = -\int \frac{sin\pi z^2 + \cos \pi z^2}{z-2} dz + \int \frac{sin\pi z^2 + \cos \pi z^2}{z-3} dz$

$= -2\pi i f(2) + 2\pi i\, f(3)$ Where $f(z) = \sin(\pi z^2) + \cos \pi z^2$

$= -2\pi i(1) + 2\pi i(-1)$ $f(2) = \sin 4\pi + \cos 4\pi = 1$

$= -4\pi i$ $f(3) = \sin 9\pi + \cos 9\pi - 1 = -1$

Example: 4.10 Evaluate $\int_c \frac{z+4}{z^2+2z+5}$ Where C is the circle $(i)|z+1+i| = 2$ $(ii)|z+1-i| = 2$ (iii) $|z| = 1$

Solution:

Given $\int_c \frac{z+4}{z^2+2z+5} dz$

$Dr = 0 \Rightarrow z^2 + 2z + 5 = 0$

$\Rightarrow z = \frac{-2 \pm \sqrt{4-20}}{2}$

$\Rightarrow z = -1 \pm 2i$

$\therefore \int_c \frac{z+4}{z^2+2z+5} dz = \int_c \frac{(z+4)\, dz}{[z-(-1+2i)[z-(-1-2i)]}$

(i) $|z+1+i| = 2$ $is\ the\ circle$

When $z = -1+2i, |-1+2i+1+i| = |3i| > 2$ lies outside C.

When $z = -1-2i, |-1-2i+1+i| = |-i| < 2$ lies inside C.

$\therefore$ By Cauchy's Integral formula

$\int_c \frac{[(z+1)/(z-(-1+2i)]}{[z-(-1-2i)]} dz = 2\pi i f(-1-2i)$ Where $f(z) = \frac{z+4}{[z-(-1+2i)]}$

$= 2\pi i \left[\frac{3-2i}{-4i}\right]$ $f(-1-2i) = \frac{-1-2i+4}{-1-2i+1-2i} = \frac{3-2i}{-4i}$

$= \frac{\pi}{2}(2i-3)$

(ii) $|z+1-i| = 2$ is the circle

When $z = -1+2i, |-1+2i+1-i| = |i| < 2$ lies inside C

When $z = -1-2i, |-1-2i+1-i| = |-3i| > 2$ lies outside C

$\therefore$ By Cauchy's Integral formula

$\int_c \frac{(z+1)/[z-(-1-2i)]}{[z-(-1+2i)]} dz = 2\pi i\, f(-1+2i)$ Where $f(z) = \frac{z+4}{z-(-1-2i)}$

$= 2\pi i \frac{[3+2i]}{4i}$ $f(-1+2i) = \frac{-1+2i+4}{-1+2i+1+2i} = \frac{3+2i}{4i}$

$= \frac{\pi}{2}(3+2i)$

(iii)$|z| = 1$ is the circle

When $z = -1+2i, 1-1+2i| = \sqrt{5} > 1$ lies outside C

When $z = -1 - 2i, |-1 - 2i| = \sqrt{5} > 1$ lies outside C

$\therefore$ By Cauchy's Integral theorem

$$\int_c \frac{z+4}{z^2+2z+5} dz = 0$$

Example: 4.11 Using Cauchy's integral formula, evaluate $\int_c \frac{z+1}{z^2+2z+4} dz$ where C is the circle $|z+1+i| = 2$

Solution:

Given $\int_c \frac{z+1}{z^2+2z+4} dz$

$Dr = 0 \Rightarrow z^2 + 2z + 4 = 0$

$\Rightarrow z = \frac{-2 \pm \sqrt{4-16}}{2}$

$\Rightarrow z = -1 \pm i\sqrt{3}$

$$\therefore \int_c \frac{z+1}{z^2+2z+4} dz = \int_c \frac{(z+1\ )dz}{[z-(-1+i\sqrt{3})][z-(-1-i\sqrt{3}]}$$

Given $C\ is\ |z+1+i| = 2$

When $z = -1 - i\sqrt{3}, \left|-1 - i\sqrt{3} + 1 + i\right| = |(1-\sqrt{3}i)| < 2$ lies inside C.

When $z = -1 + i\sqrt{3}, \left|-1 + i\sqrt{3} + 1 + i\right| = |i + \sqrt{3}i| > 2$ lies outside C.

$\therefore$ By Cauchy's Integral Formula

$$\int_c \frac{(z+1)/[z-(-1+i\sqrt{3})]}{[z-(-1-i\sqrt{3})]} dz = 2\pi i\, f(-1-i\sqrt{3}) \qquad \text{Where } f(z) = \frac{z+1}{z-(-1+i\sqrt{3})}$$

$$= 2\pi i \left(\frac{1}{2}\right) = \pi i \qquad f(-1-i\sqrt{3}) = \frac{-1-i\sqrt{3}+1}{-1-i\sqrt{3}+1-i\sqrt{3}} = \frac{\sqrt{3}i}{-2i\sqrt{3}} = \frac{1}{2}$$

$$\therefore \int_c \frac{z+1}{z^2+2z+4} dz = \pi i$$

Example: 4.12 Evaluate $\int_c \frac{z^2+1}{z^2-1} dz$ where C is the circle $(i)|z-1| = 1\ (ii)|z+1| = 1 (iii)|z-i| = 1$

Solution:

Given $\int_c \frac{z^2+1}{z^2-1} dz = \int_c \frac{z^2+1}{(z+1)(z-1)} dz$

$Dr = 0 \Longrightarrow z = 1, -1$

(i) $(z-1) = 1$ is the circle

When $z = 1$, $|1-1| = 0 < 1$ lies inside C

When $z = -1, |-1-1| = 2 > 1$ lies outside C

$\therefore$ By Cauchy's Integral formula

$$\int_c \frac{z^2+1}{(z+1)(z-1)} dz = \int_c \frac{(z^2+1)/z+1}{(z-1)} dz$$

$$= 2\pi i f(1) \qquad \text{where } f(z) = \frac{z^2+1}{z+1} \Rightarrow f(1) = 1$$

$$= 2\pi i(1)$$

$$= 2\pi i$$

(ii)$|z+1|=1$ is the circle

When $z=1, |1+1|=2>1$ lies outside C

When $z=-1, |-1+1|=0<1$ lies inside C

$\therefore$ By Cauchy's Integral formula

$$\int_c \frac{(z^2+1)/(z-1)}{z+1}dz = 2\pi i\, f(-1) \qquad \text{where } f(z)=\frac{z^2+1}{z-1} \Rightarrow f(-1)=-1$$

$$= 2\pi i(-1) = -2\pi i$$

(iii) $|z-i|=1$ is the circle

When $z=1, |1-i|=\sqrt{2}>1$ lies outside C

When $z=-1, |-1-i|=\sqrt{2}>1$ lies outside C

$\therefore$ By Cauchy's Integral Formula

$$\int_c \frac{(z^2+1)}{(z+1)(z-1)}dz = 0$$

Problems based on Cauchy's Integral Formula for derivatives

Example: 4.13 If $f(a)=\int_c \frac{3z^2+7z+1}{z-a}dz$ where C is the circle $x^2+y^2=4$ find the values of $f(3), f(1), f'(1-i)$ and $f''(1-i)$

Solution:

Given $f(a)=\int_c \frac{3z^2+7z+1}{z-a}dz$

To find: $f(3)=\int_c \frac{3z^2+7z+1}{z-3}dz$

$Dr=0 \Longrightarrow z=3$

Hence $z=3$ lies outside the circle $x^2+y^2=4$

By Cauchy's Integral theorem

$$\int_c \frac{3x^2+7z+1}{z-3}dz = 0$$

To find: $f(1)=\int_c \frac{3z^2+7z+1}{z-1}dz$

$Dr=0 \Longrightarrow z=1$

Clearly $z=1$ lies inside the circle $x^2+y^2=4$

$\therefore$ By Cauchy's Integral formula

$$\int_c \frac{3z^2+7z+1}{z-1}dz = 2\pi i\, f(1) \qquad \text{Where } f(z)=3z^2+7z+1 \Rightarrow f(1)=11$$

$$= 2\pi i(11)$$

$$= 22\pi i$$

To find: $f'(1-i)=\int_c \frac{3z^2+7z+1}{z-(1-i)}dz$

$Dr=0 \Longrightarrow z=1-i$

and the point $z=1-i$ lies inside the circle $x^2+y^2=4$

∴ By Cauchy's Integral formula

$$f'(1-i) = 2\pi i\varphi'(1-i) \qquad \text{Where } \varphi(z) = 3z^2 + 7z + 1$$

$$= 2\pi i[6(1-i)+7] \qquad \Rightarrow \varphi'(z) = 6z + 7$$

$$= 2\pi i[13-6i] \qquad \Rightarrow \varphi'(1-i) = 6(1-i)+7$$

$$= 2\pi i[13-6i]$$

To find: $f''(1-i) = \int_c \frac{3z^2+7z+1}{z-(1-i)} dz$

Cleary and the point $z = 1 - i$ lies inside the circle $x^2 + y^2 = 4$

∴ By Cauchy's Integral formula

$$f'(1-i) = 2\pi i\varphi''(1-i) \qquad \text{Where } \varphi(z) = 3z^2 + 7z + 1$$

$$= 2\pi i[6] \qquad \varphi''(z) = 6z + 7 \Rightarrow \varphi''(z) = 6$$

$$= 12\pi i$$

Example: 4.14 Using Cauchy's Integral formula evaluate $\int_c \frac{zdz}{(z-1)(z-2)^2}$ where C is the circle $|z-2| = \frac{1}{2}$

Solution:

Given $\int_c \frac{zdz}{(z-1)(z-2)^2}$

$Dr = 0 \Longrightarrow z = 1$ is a pole of order 1, $z = 2$ is a pole of order 2.

Given C is $|z-2| = \frac{1}{2}$

When $z = 1, |1-2| = 1 > \frac{1}{2}$ lies outside C.

When $z = 2, |2-2| = 0 < \frac{1}{2}$ lies inside C.

∴ By Cauchy's Integral formula

$$\int_c \frac{z/z-1}{(z-2)^2} dz = 2\pi i\, f'(2) \qquad \text{Where } f(z) = \frac{z}{z-1}$$

$$= 2\pi i(-1) \qquad f'(z) = \frac{(z-1)1 - z(1)}{(z-1)^2} \Rightarrow f'(2) = -1$$

$$= -2\pi i$$

Example: 4.15 Evaluate $\int_c \frac{sin^2 z}{\left(z-\frac{\pi}{6}\right)^3} dz$ where C is the circle $|z| = 1$

Solution:

Given $\int_c \frac{sin^2 z}{(z-\frac{\pi}{6})^3} dz$

$Dr = 0 \Rightarrow z = \frac{\pi}{6}$ is a pole of order 3.

Give C is $|z| = 1$.

Clearly $z = \frac{\pi}{6}$ lies inside the circle $|z| = 1$

∴ By Cauchy's Integral formula

$$\int_c \frac{sin^2 z}{(z-\frac{\pi}{6})^3}dz = \frac{2\pi i}{2!}\, f''(\pi/6) \qquad \text{Where } f(z) = sin^2 z$$

$$= \frac{2\pi i}{2!}(1) \qquad f'(z) = 2\sin z\cos z = \sin 2z$$

$$= \pi i \qquad f''(z) = \cos 2z(2) \Rightarrow f''\left(\frac{\pi}{6}\right) = 2\cos\left(\frac{2\pi}{6}\right)$$

$$= 2\cos\frac{\pi}{3} = 2\left(\frac{1}{2}\right) = 1$$

Example: 4.16 Evaluate $\int_c \frac{z}{(z-1)^3}dz$ where C is the circle $|z| = 2$, using Cauchy's Integral formula

Solution:

Given $\int_c \frac{z}{(z-1)^3}dz$

$Dr = 0 \Rightarrow z = 1$ is a pole of order 3.

Given C is $|z| = 2$.

Clearly $z = 1$ lies inside the circle C

∴ By Cauchy's Integral formula

$$\int_c \frac{sin^2 z}{(z-1)^3}dz = \frac{2\pi i}{2!}\, f''(1) \qquad \text{Where } f(z) = z \Rightarrow f'(z) = 1$$

$$= \frac{2\pi i}{2!}(0) \qquad \Rightarrow f''(z) = 0 \Rightarrow f''(1) = 0$$

$$= 0$$

Example: 4.17 Evaluate $\int_c \frac{z^2}{(2z-1)^2}dz$ where C is the circle $|z| = 1$

Solution:

Given $\int_c \frac{z^2}{(2z-1)^2}dz$

$Dr = 0 \Rightarrow 2z = 0 \Rightarrow z = \frac{1}{2}$ is a pole of order 2.

Given C is $|z| = 1$.

Clearly $z = \frac{1}{2}$ lies inside the circle C

∴ By Cauchy's Integral formula

$$\int_c \frac{z^2}{2^2(z-\frac{1}{2})^2}dz = \frac{1}{4}\int_c \frac{z^2}{\left(z-\frac{1}{2}\right)^2}dz \qquad \text{Where } f(z) = z^2 \Rightarrow f'(z) = 2z$$

$$= \frac{1}{4}\left(2\pi i f'\left(\frac{1}{2}\right)\right) \qquad \Rightarrow f'\left(\frac{1}{2}\right) = 1$$

$$= \frac{1}{2}\pi i(1)$$

$$= \frac{\pi i}{2}$$

Exercise: 4.1

Evaluate the following using Cauchy's Integral formula

1. $\int_c \frac{z^2}{z^2+9} dz$ where $C\ is\ |z-1| = \frac{3}{2}$ **Ans:** 0
2. $\int_c \frac{7z-1}{z^2-3z-4}$ where C is the ellipse $x^2 + 4y^2 = 4$ **Ans:** $\frac{16\pi i}{5}$
3. $\int_c \frac{z^3-z}{(z+2)^3} dz$ where $C\ is\ |z| = 3$ **Ans:** $12\pi i$
4. $\int_c \frac{3z-1}{z^2-z} dz$ where $C\ is\ |z| = \frac{1}{2}$ **Ans:** $2\pi i$
5. $\int_c \frac{12z-7}{(2z+3)(z-1)^3} dz$ where C is the circle $x^2 + 4y^2 = 4$ **Ans:** 0
6. $\int_c \frac{e^{2z}}{(z+1)(z-2} dz$ where $C\ is |z| = 3$ **Ans:** $2\pi i(e^4 - e^2)$
7. $\int_c \frac{z+1}{z^4-4z^3+4z^2} dz$ where $C\ is |z-2-i| = 2$ **Ans:** πi
8. $\int_c \frac{z}{z^4-4z^3+4z^2} dz$ where $C\ \ is |z-2\} = \frac{1}{2}$ **Ans:** $4\pi i$
9. $\int_c \frac{z}{(z-2)(z-3)^2} dz$ where $C\ is\ \ |z-3| = \frac{1}{2}$ **Ans:** $-4\pi i$
10. If $f(a) = \int_c \frac{4z^2+z+5}{z-a} dz$ where C is the ellipse$\frac{x^2}{4} + \frac{y^2}{9} = 1$ find the values of $f(1), f(i), f'(1-i)$ and $f''(1+i)$ **Ans:** $20\pi i, 2\pi i(1+i), 2\pi i\ (9-8i), 8\pi i$

4. 2 TAYLORS AND LAURENTS SERIES

In this section, we find a power series for the given analytic function. Taylor's series is a series of positive powers while Laurent's series is a series of both positive and negative powers.

Taylor's Series

If $f\ (z)$ is analytic inside and on a circle C with centre at point 'a' and radius 'R' then at each point Z inside C,

$$f\ (z) = f(a) + (z-a)\frac{f'(a)}{1!} + (z-a)^2\frac{f''(a)}{2!} + \cdots$$

(OR)

$$f(z) = \sum_{n=0}^{\infty} \frac{(z-a)^n}{n!} f^n(a)$$

This is known as Taylor's series of $f(z)$ about $z\ = a.$

Note: 1 Putting $a = 0$ in the Taylor's series we get

$f\ (z) = f(0) + (z-0)\frac{f'(0)}{1!} + (z-0)^2\frac{f''(0)}{1!} + \cdots$ this series is called Maclaurin's Series.

Note: 2 The Maclaurin's for some elementary functions are

1) $(1-z)^{-1} = 1 + z + z^2 + z^3\ + \cdots$, when $|z| < 1$
2) $(1+z)^{-1} = 1 - z + z^2 - z^3\ + \cdots$, when $|z| < 1$
3) $(1-z)^{-2} = 1 + 2z + 3z^2 + 4z^3\ + \cdots$, when $|z| < 1$
4) $(1+z)^{-2} = 1 - 2z + 3z^2 - 4z^3\ + \cdots$, when $|z| < 1$
5) $e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \cdots$ when $|z| < \infty$

6) $e^z = 1 - \frac{z}{1!} + \frac{z^2}{2!} + \cdots$ when $|z| < \infty$

7) $\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots$ when $|z| < \infty$

8) $\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots$ when $|z| < \infty$

LAURENTS SERIES

If c_1 *and* c_2 are two concentric circles with centre at $z = a$ and radii r_1 *and* r_2 $(r_1 < r_2)$ and if $f(z)$ is analytic inside on the circles and within the annulus between c_1 *and* c_2 then for any z in the annulus, we have

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} b_n (z-a)^{-n} \quad \ldots (1)$$

Where $a_n = \frac{1}{2\pi i}\int_{c_1} \frac{f(z)}{(z-a)^{n+1}}\, dz$ and $b_n = \frac{1}{2\pi i}\int_{c_2} \frac{f(z)}{(z-a)^{1-n}}\, dz$ and the integration being taken in positive direction. This series (1) is called Laurent series of $f(z)$ about the point $z = a$

Note:

1) If $f(z)$ is analytic inside c_2, then the Laurent's series reduces to the Taylor series of $f(z)$with centre a,since the negative powers in Laurent's series is Zero.
2) As the Taylor's and Laurent's expansion in the regions are unique, they can find by simpler method such as binomial series.
3) In Laurent's series the part $\sum_{n=0}^{\infty} a_n (z-a)^n$, consisting of positive powers of $(z-a)$ is called the analytic part of Laurent's series, while $\sum_{n=1}^{\infty} b_n (z-a)^{-n}$consisting of negative powers of $(z-a)$ is called the principal part of Laurent's series.
4) The coefficient of $\frac{1}{z-a}$ $(i.e)$ b, in the Laurent's expansion of $f(z)$ about a singularity $z = a$ valid in region $0 < |z-a| < r$ is also called residue.

   $(i.e)$ coeff of $\frac{1}{z-a} = Res\,[f(z), z = a]$

Problems based on Taylor's series

Example: 4.18 Expand $f(z) = cosz$ as a Taylor's series about $z = \frac{\pi}{4}$.

Solution:

Function	Value of function at $z = \frac{\pi}{4}$
$f(z) = cosz$	$f\left(\frac{\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$
$f'(z) = -sinz$	$f'\left(\frac{\pi}{4}\right) = -sin\left(\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}$
$f''(z) = -cosz$	$f''\left(\frac{\pi}{4}\right) = -cos\left(\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}$
$f'''(z) = sinz$	$f''\left(\frac{\pi}{4}\right) = sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$

The Taylor series of $f(z)$ about $z = \frac{\pi}{4}$ is $f\ (z) = f\left(\frac{\pi}{4}\right) + \left(z - \frac{\pi}{4}\right)\frac{f'\left(\frac{\pi}{4}\right)}{1!} + \left(z - \frac{\pi}{4}\right)^2 \frac{f''\left(\frac{\pi}{4}\right)}{2!} + \cdots$

$$cosz = \frac{1}{\sqrt{2}} + \left(z - \frac{\pi}{4}\right)\frac{-\frac{1}{\sqrt{2}}}{1!} + \left(z - \frac{\pi}{4}\right)^2 \frac{-\frac{1}{\sqrt{2}}}{2!} + \cdots$$

Example: 4.19 Expand $f(z) = log\ (1 + z)$ as a Taylor's series about $z = 0$.

Solution:

Function	Value of function at $z = 0$
$f(z) = log\ (1 + z)$	$f(0) = log\ (1 + 0)= 0$
$f'(z) = \dfrac{1}{1 + z}$	$f'(0) = \dfrac{1}{1 + 0} = 1$
$f''(z) = \dfrac{-1}{(1 + z)^2}$	$f''(0) = \dfrac{-1}{(1 + 0)^2} = -1$
$f'''(z) = \dfrac{2}{(1 + z)^3}$	$f'''^{(0)} = \dfrac{2}{(1 + 0)^3} = 2$

The Taylor series of $f(z)$ about $z = 0$ is

$$f\ (z) = f(0) + (z - 0)\frac{f'(0)}{1!} + (z - 0)^2 \frac{f''(0)}{2!} + \cdots$$

$$log\ (1 + z)\ = 0 + (z)\frac{1}{1!} + (z)^2 \frac{-1}{2!} + \cdots$$

$$log\ (1 + z)\ = (z)\frac{1}{1!} - (z)^2 \frac{1}{2!} + \cdots$$

Example: 4.20 Expand $f(z) = \frac{1}{z-2}$as a Taylor's series about $z = 1$.

Solution:

Function	Value of function at $z = 1$
$f(z) = \dfrac{1}{z - 2}$	$f(z) = \frac{1}{1-2} = -1$
$f'(z) = \dfrac{-1}{(z - 2)^2}$	$f'(1) = \dfrac{-1}{(1 - 2)^2} = -1$
$f''(z) = \dfrac{2}{(z - 2)^3}$	$f''(1) = \dfrac{2}{(1 - 2)^3} = -2$
$f'''(z) = \dfrac{-6}{(z - 2)^4}$	$f'''^{(0)} = \dfrac{-6}{(z - 2)^4} = -6$

The Taylor series of $f(z)$ about $z = 1$ is

$$f\ (z) = f(1) + (z - 1)\frac{f'(1)}{1!} + (z - 1)^2 \frac{f''(1)}{2!} + \cdots$$

$$\frac{1}{z-2} = -1 + (z - 1)\frac{-1}{1!} + (z - 1)^2 \frac{-2}{2!} + \cdots$$

Problems based on Laurent's Series

Working rule to expand $f(z)$ as a Laurent's Series

Let $f(z) = \frac{1}{z+a} + \frac{1}{z+b}\ \ with\ a < b$

(i) To expand $f(z)$ in $|z| < a$, rewrite $f(z)$as

$$f(z) = \frac{1}{a(1+{}^{z}/_{a})} + \frac{1}{b(1+{}^{z}/_{b})}$$

$$= \frac{1}{a}(1 + {}^{z}/_{a})^{-1} + \frac{1}{b}\left(1 + {}^{z}/_{b}\right)^{-1}$$

Now use Binomial expansion.

(ii)To expand $f(z)$ in $|z| > a$, rewrite $f(z)$ as

$$f(z) = \frac{1}{z(1+{}^{a}/_{z})} + \frac{1}{z(1+{}^{b}/_{z})}$$

$$= \frac{1}{z}(1 + {}^{a}/_{z})^{-1} + \frac{1}{z}\left(1 + {}^{b}/_{z}\right)^{-1}$$

Now use Binomial expansion.

(iii)To expand $f(z)$ in $a < |z| < b$, rewrite $f(z)$as

$$f(z) = \frac{1}{z(1+{}^{a}/_{z})} + \frac{1}{b(1+{}^{z}/_{b})}$$

$$= \frac{1}{z}(1 + {}^{a}/_{z})^{-1} + \frac{1}{b}\left(1 + {}^{z}/_{b}\right)^{-1}$$

Now use Binomial expansion.

Example: 4.21 Expand $f(z) = \frac{z^2-1}{(z+2)(z+3)}$as a Laurent's series if (i) $|z| < 2$ (ii) $|z| > 3$

(iii) $2 < |z| < 3$

Solution:

Given $f(z) = \frac{z^2-1}{(z+2)(z+3)}$ is an improper fraction. Since degree of numerator and degree of denominator of $f(z)$ are same

$\therefore$ Apply division process

$$z^2 + 5z + 6 \overline{\left)\begin{array}{l} z^2 - 1 \\ z^2 + 5z + 6 \\ \hline -5z - 7 \end{array}\right.}\quad 1$$

$$\therefore \frac{z^2-1}{(z+2)(z+3)} = 1 - \frac{5z+7}{(z+2)(z+3)} \quad \ldots (1)$$

Consider $\frac{5z+7}{(z+2)(z+3)} = \frac{A}{z+2} + \frac{B}{z+3}$

$$\Rightarrow 5z + 7 = A(z + 3) + B(z + 2)$$

Put $z = -2$, we get $-10 + 7 = A\,(1)$

$$\Rightarrow A = -3$$

Put $z = -3$, we get $-15 + 7 = B(-1)$

$$\Rightarrow B = 8$$

$$\therefore \frac{5z+7}{(z+2)(z+3)} = \frac{-3}{z+2} + \frac{8}{z+3}$$

$$\therefore (1) \Rightarrow 1 - \frac{3}{z+2} - \frac{8}{z+3}$$

(i) Given $|z| < 2$

$$\therefore\ f(z) = 1 + \frac{3}{2(1+{}^{z}/_{2})} - \frac{8}{3(1+{}^{z}/_{3})}$$

$$= 1 + \frac{3}{2}\left(1 + {}^{z}/_{2}\right)^{-1} - \frac{8}{3}\left(1 + {}^{z}/_{3}\right)^{-1}$$

$$= 1 + \frac{3}{2}\left[1 - \frac{z}{2} + \left[\frac{z}{2}\right]^{2} + \cdots\right] - \frac{8}{3}\left[1 - \frac{z}{3} + \left[\frac{z}{3}\right]^{2} + \cdots\right]$$

$$= 1 + \frac{3}{2}\sum_{n=0}^{\infty}(-1)^{n}\left[\frac{z}{2}\right]^{n} - \frac{8}{3}\sum_{n=0}^{\infty}(-1)^{n}\left[\frac{z}{3}\right]^{n}$$

(ii) Given $|z| > 3$

$$\therefore\ f(z) = 1 + \frac{3}{z(1+{}^{2}/_{z})} - \frac{8}{z(1+{}^{3}/_{z})}$$

$$= 1 + \frac{3}{z}\left(1 + {}^{2}/_{z}\right)^{-1} - \frac{8}{z}\left(1 + {}^{3}/_{z}\right)^{-1}$$

$$= 1 + \frac{3}{z}\left[1 - \frac{2}{z} + \left[\frac{2}{z}\right]^{2} + \cdots\right] - \frac{8}{z}\left[1 - \frac{3}{z} + \left[\frac{3}{z}\right]^{2} \ldots\right]$$

$$= 1 + \frac{3}{z}\sum_{n=0}^{\infty}(-1)^{n}\left[\frac{2}{z}\right]^{n} - \frac{8}{z}\sum_{n=0}^{\infty}(-1)^{n}\left[\frac{3}{z}\right]^{n}$$

(iii) Given $2 < |z| < 3$

$$\therefore\ f(z) = 1 + \frac{3}{z(1+{}^{2}/_{z})} - \frac{8}{3(1+{}^{z}/_{3})}$$

$$= 1 + \frac{3}{z}\left(1 + {}^{2}/_{z}\right)^{-1} - \frac{8}{3}\left(1 + {}^{z}/_{3}\right)^{-1}$$

$$= 1 + \frac{3}{z}\left[1 - \frac{2}{z} + \left[\frac{2}{z}\right]^{2} + \cdots\right] - \frac{8}{3}\left[1 - \frac{z}{3} + \left[\frac{z}{3}\right]^{2} \ldots\right]$$

$$= 1 + \frac{3}{z}\sum_{n=0}^{\infty}(-1)^{n}\left[\frac{2}{z}\right]^{n} - \frac{8}{3}\sum_{n=0}^{\infty}(-1)^{n}\left[\frac{z}{3}\right]^{n}$$

Example: 4.22 Find the Laurent's series expansion of $\boldsymbol{f(z) = \frac{7z-2}{z(z-2)(z+1)}}$ in $\boldsymbol{1 < |z+1| < 3}$.

Also find the residue of $\boldsymbol{f(a)}$ at $\boldsymbol{z = -1}$

Solution:

Given $f(z) = \frac{7z-2}{z(z-2)(z+1)}$

$$\frac{7z-2}{z(z-2)(z+1)} = \frac{A}{z} + \frac{B}{z-2} + \frac{C}{z+1}$$

$$7z - 2 = A(z-2)(z+1) + Bz(z+1) + Cz(z-2)$$

Put $z = 2$, we get $14 - 2 = B\,(2)\,(2+1)$

$$\Rightarrow 12 = 6B$$

$$\Rightarrow B = 2$$

Put $z = -1$, we get $-7 - 2 = C(-1)(-1-2)$

$$\Rightarrow -9 = 3C$$

$$\Rightarrow C = -3$$

Put $z = 0$ we get $-2 = A(-2)$

$$\Rightarrow A = 1$$

$$\therefore f(z) = \frac{1}{z} + \frac{2}{z-2} - \frac{3}{z+1}$$

Given region is $1 < |z+1| < 3$

Let $u = z+1 \Rightarrow z = u-1$

$(i.e)\ 1 < |u| < 3$

Now $f(z) = \frac{1}{u-1} + \frac{2}{u-3} - \frac{3}{u}$

$$= \frac{1}{u(1-{}^{1}/_{u})} + \frac{2}{-3(1-{}^{u}/_{3})} - \frac{3}{u}$$

$$= \frac{1}{u}\left(1 - {}^{1}/_{u}\right)^{-1} - \frac{2}{3}\left(1 - {}^{u}/_{3}\right)^{-1} - \frac{3}{u}$$

$$= \frac{1}{u}\left[1 + \frac{1}{u} + \left[\frac{1}{u}\right]^2 + \cdots\right] - \frac{2}{3}\left[1 + \frac{u}{3} + \left[\frac{u}{3}\right]^2 + \cdots\right] - \frac{3}{u}$$

$$= \frac{1}{z+1}\left[1 + \frac{1}{z+1} + \left[\frac{1}{z+1}\right]^2 + \cdots\right] - \frac{2}{3}\left[1 + \frac{z+1}{3} + \left[\frac{z+1}{3}\right]^2 + \cdots\right] - \frac{3}{z+1}$$

$$= \frac{1}{z+1}\sum_{n=0}^{\infty}\left[\frac{1}{z+1}\right]^n - \frac{2}{3}\sum_{n=0}^{\infty}\left[\frac{1}{\frac{z+1}{3}}\right]^n - \frac{3}{z+1}$$

Also $Res[f(z), z = -1] =$coefficient of $\frac{1}{z+1} = -2$

Example: 4.23 Expand $f(z) = \frac{1}{(z-1)(z-2)}$ in a Laurent's series valid in the region

(i)$|z-1| > 1$(ii) $0 < |z-2| < 1$ (iii) $|z| > 2$ (iv) $0 < |z-1| < 1$

Solution:

Given $f(z) = \frac{1}{(z-1)(z-2)}$

Consider $\frac{1}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2}$

$$\Rightarrow 1 = A(z-2) + B(z-1)$$

Put $z = 2$, we get $1 = B(1)$

$$\Rightarrow B = 1$$

Put $z = 1$we get $1 = A(1-2)$

$$\Rightarrow A = -1$$

$$\therefore f(z) = \frac{-1}{z-1} + \frac{1}{z-2}$$

(i) Given region is $|z-1| > 1$

Let $u = z-1 \Rightarrow z = u+1$

$(i.e)\ |u| > 1$

Now $f(z) = -\frac{1}{u} + \frac{1}{u-1}$

$$= \frac{-1}{u} + \frac{1}{u(1-{}^{1}/_{u})}$$

$$= \frac{-1}{u} + \frac{1}{u}\left(1 - {}^{1}/_{u}\right)^{-1}$$

$$= \frac{-1}{u} + \frac{1}{u}\left[1 + \frac{1}{u} + \left[\frac{1}{u}\right]^2 + \cdots\right]$$

$$= \frac{-1}{z+1} + \frac{1}{z+1}\left[1 + \frac{1}{z+1} + \left[\frac{1}{z+1}\right]^2 + \cdots\right]$$

$$= \frac{-1}{z+1} + \frac{1}{z+1}\sum_{n=0}^{\infty}\left[\frac{1}{z+1}\right]^n$$

(ii) Given $0 < |z-2| < 1$

Let $u = z - 2 \Rightarrow z = u + 2$

$(i.e)\ 0 < |u| < 1$

Now $f(z) = -\frac{1}{u+1} + \frac{1}{u}$

$$= -(1+u)^{-1} + \frac{1}{u}$$

$$= -[1 - u + [u]^2 + \cdots] + \frac{1}{u}$$

$$= -[1 - (z-2) + [z-2]^2 + \cdots] + \frac{1}{z-2}$$

$$= -\sum_{n=0}^{\infty}[-1]^n[z-2]^n + \frac{1}{z-2}$$

(iii) Given $|z| > 2$

Now $f(z) = -\frac{1}{z\left(1-\frac{1}{z}\right)} + \frac{1}{z\left(1-\frac{2}{z}\right)}$

$$= -\frac{1}{z}\left(1 - \frac{1}{z}\right)^{-1} + \frac{1}{z}\left(1 - \frac{2}{z}\right)^{-1}$$

$$= -\frac{1}{z}\left[1 + \frac{1}{z} + \left[\frac{1}{z}\right]^2 + \cdots\right] + \frac{1}{z}\left[1 + \frac{2}{z} + \left[\frac{2}{z}\right]^2 + \cdots\right]$$

$$= -\frac{1}{z}\sum_{n=0}^{\infty}\left[\frac{1}{z}\right]^n + \frac{1}{z}\sum_{n=0}^{\to\infty}\left[\frac{2}{z}\right]^n$$

(iv) Given $0 < |z-1| < 1$

Let $u = z - 1 \Rightarrow z = u + 1$

$(i.e)\ 0 < |u| < 1$

Now $f(z) = -\frac{1}{u} + \frac{1}{u-1}$

$$= -\frac{1}{u} + \frac{1}{-1\,[1-u]}$$

$$= -\frac{1}{u} - (1-u)^{-1}$$

$$= -\frac{1}{u} - [1 + u + [u]^2 + \cdots]$$

$$= -\frac{1}{z-1} - [1 + z - 1 + [z-1]^2 + \cdots]$$

$$= -\frac{1}{z-1} - \sum_{n=0}^{\infty}[z-1]^n$$

Example: 4.24 Expand $f(z) = \frac{z}{(z+1)(z-2)}$ in a Laurent's series about (i) $z = -1$ (ii) $z = 2$

Solution:

Consider $\frac{z}{(z+1)(z-2)} = \frac{A}{z+1} + \frac{B}{z-2}$

$$\Rightarrow\ z = A(z-2) + B(z+1)$$

Put $z =\ 2$, we get $2 = B(3)$

$$\Rightarrow B = \frac{2}{3}$$

Put $z = -\ 1$we get $-1 = A(-3)$

$$\Rightarrow A = \frac{1}{3}$$

$$\therefore f(z) = \frac{1}{3(z+1)} + \frac{2}{3(z-2)}$$

(i)To expand $f(z)$ about $z = -1$

(or) $|z-1| < 1$

Put $z + 1 = u \Rightarrow z = u - 1$

$$\Rightarrow |z-1| < 1 \Rightarrow |u| < 1$$

Now $f(z) = \frac{1}{3u} + \frac{2}{3(u-3)}$

$$= \frac{1}{3u} + \frac{2}{3\left((-3)(1-{}^{u}/_{3})\right)}$$

$$= \frac{1}{3u} - \frac{2}{9}\left(1 - {}^{u}/_{3}\right)^{-1}$$

$$= \frac{1}{3u} - \frac{2}{9}\left[1 + \frac{u}{3} + \left[\frac{u}{3}\right]^2 + \cdots\right]$$

$$= \frac{1}{3(z+1)} - \frac{2}{9}\left[1 + \frac{(z+1)}{3} + \left[\frac{(z+1)}{3}\right]^2 + \cdots\right]$$

$$= \frac{1}{3(z+1)} - \frac{2}{9}\sum_{n=0}^{\infty}\left[\frac{(z+1)}{3}\right]^n$$

(ii) To expand $f(z)$ about $z = 2$

(or) $|z-2| < 1$

Put $z - 2 = u \Rightarrow z = u + 2$

$$\Rightarrow |z-2| < 1 \Rightarrow |u| < 1$$

Now $f(z) = \frac{1}{3(u+3)} + \frac{2}{3(u)}$

$$= \frac{1}{3\left((3)(1+{}^{u}/_{3})\right)} + \frac{2}{3(u)}$$

$$= \frac{1}{9}\left(1 + {}^{u}/_{3}\right)^{-1} + \frac{2}{3(u)}$$

$$= \frac{1}{9}\left[1 - \frac{u}{3} + \left[\frac{u}{3}\right]^2 + \cdots\right] + \frac{2}{3(u)}$$

$$= \frac{1}{9}\left[1 - \frac{(z-2)}{3} + \left[\frac{(z-2)}{3}\right]^2 + \cdots\right] + \frac{2}{3(z-2)}$$

$$= \frac{1}{9}\sum_{n=0}^{\infty}(-1)^n\left[\frac{(z-2)}{3}\right]^n + \frac{2}{3(z-2)}$$

Example: 4.25 Expand the Laurent's series about for $f(z) = \frac{6z+5}{z(z-2)(z+1)}$in the region $1 < |z+1| < 3$

Solution:

Consider $\frac{6z+5}{z(z-2)(z+1)} = \frac{A}{z} + \frac{B}{z-2} + \frac{c}{z+1}$

$$\Rightarrow 6z + 5 = A(z-2)(z+1) + Bz(z+1) + Cz(z-2)$$

Put $z = 0$, we get $5 = A(-2)(1)$

$$\Rightarrow A = \frac{-5}{2}$$

Put $z = -1$we get $-11 = C(-1)(-3)$

$$\Rightarrow C = -\frac{11}{3}$$

Put $z = 2$we get $17 = B(2)(3)$

$$\Rightarrow B = \frac{17}{6}$$

$$\therefore f(z) = \frac{-5}{2z} + \frac{17}{6(z-2)} - \frac{11}{3(z+1)}$$

Given region $1 < |z+1| < 3$

Put $z + 1 = u \Rightarrow z = u - 1$

$(i.e)\ 1 < |u| < 3$

Now $f(z) = \frac{-5}{2(u-1)} + \frac{17}{6(u-3)} - \frac{11}{3u}$

$$= \frac{-5}{2u(1-\frac{1}{u})} + \frac{17}{6(-3)(1-\frac{u}{3})} - \frac{11}{3u}$$

$$= \frac{-5}{2u}\left[1 - \frac{1}{u}\right]^{-1} - \frac{17}{18}\left[1 - \frac{u}{3}\right]^{-1} - \frac{11}{3u}$$

$$= \frac{-5}{2u}\left[1 + \frac{1}{u} + \left[\frac{1}{u}\right]^2 + \cdots\right] - \frac{17}{18}\left[1 + \frac{u}{3} + \left[\frac{u}{3}\right]^2 + \cdots\right] - \frac{11}{3u}$$

$$= \frac{-5}{2(z+1)}\left[1 + \frac{1}{(z+1)} + \left[\frac{1}{(z+1)}\right]^2 + \cdots\right] - \frac{17}{18}\left[1 + \frac{(z+1)}{3} + \left[\frac{(z+1)}{3}\right]^2 + \cdots\right] - \frac{11}{3(z+1)}$$

$$= \frac{-5}{2(z+1)}\sum_{n=0}^{\infty}\left[\frac{1}{(z+1)}\right]^n - \frac{17}{18}\sum_{n=0}^{\infty}\left[\frac{(z+1)}{3}\right]^n - \frac{11}{3(z+1)}$$

Example: 4.26 Find the Laurent's series which represents the function $\frac{z}{(z+1)(z+2)}$ in (i) $|z| < 1$ (ii) 1< $|z| < 2$ (iii) $|z| > 2$

Solution:

Consider $\frac{z}{(z+1)(z+2)} = \frac{A}{z+1} + \frac{B}{z+2}$

$$\Rightarrow z = A(z+2)(z+1)$$

Put $z = -2$we get $-2 = B(-1)$

$$\Rightarrow B = 2$$

Put $z = -1$ we get $-1 = A(1)$

$$\Rightarrow A = -1$$

$$\therefore f(z) = \frac{-1}{z+1} + \frac{2}{z+2}$$

(i) Given region $|z| < 1$

$$f(z) = \frac{-1}{z+1} + \frac{2}{2(1+z/2)}$$

$$= -(1+z)^{-1} + \left(1 + z/2\right)^{-1}$$

$$= -[1 - z + z^2 - \cdots] + \left[1 - \frac{z}{2} + \left[\frac{z}{2}\right]^2 - \cdots\right]$$

$$= (-1)\sum_{n=0}^{\infty}(-1)^n z^n + \sum_{n=0}^{\infty}(-1)^n \left[\frac{z}{2}\right]^n$$

(ii) Given region $1 < |z| < 2$

$$f(z) = \frac{-1}{z(1+1/z)} + \frac{2}{2(1+z/2)}$$

$$= -1/z\left(1 + 1/z\right)^{-1} + \left(1 + z/2\right)^{-1}$$

$$= -1/z\,[1 - 1/z + (1/z)^2 - \cdots] + \left[1 - \frac{z}{2} + \left[\frac{z}{2}\right]^2 - \cdots\right]$$

$$= (-1/z)\sum_{n=0}^{\infty}(-1)^n (1/z)^n + \sum_{n=0}^{\infty}(-1)^n \left[\frac{z}{2}\right]^n$$

(iii) Given region $|z| > 2$

$$f(z) = \frac{-1}{z(1+1/z)} + \frac{2}{z(1+2/z)}$$

$$= -1/z\left(1 + 1/z\right)^{-1} + \frac{2}{z}\left(1 + 2/z\right)^{-1}$$

$$= -1/z\,[1 - 1/z + (1/z)^2 - \cdots] + \frac{2}{z}\left[1 - \frac{2}{z} + \left[\frac{2}{z}\right]^2 - \cdots\right]$$

$$= (-1/z)\sum_{n=0}^{\infty}(-1)^n (1/z)^n + \frac{2}{z}\sum_{n=0}^{\infty}(-1)^n \left[\frac{2}{z}\right]^n$$

(iv) Given region $|z+1| < 1$

Put $z + 1 = u \Rightarrow z = u - 1$

$\therefore\ |z+1| < 1 \Rightarrow |u| < 1$

$$f(z) = \frac{-1}{u} + \frac{2}{u+1}$$

$$= \frac{-1}{u} + 2(1+u)^{-1}$$

$$= \frac{-1}{u} + 2[1 - u + u^2 - \cdots]$$

$$= \frac{-1}{z+1} + 2[1 - (z+1) + ((z+1)^2) - \cdots]$$

$$= \frac{-1}{z+1} + 2\sum_{n=0}^{\infty}(-1)^n (z+1)^n$$

EXERCISE: 4.2

(1) Expand $f(z) = e^z$ in a Taylor's series about $z = 0$. **Ans:** $f(z) = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \cdots$

(2) Expand $f(z) = sinz$ in a Taylor's series about $z = \frac{\pi}{4}$. **Ans:** $f(z) = \frac{1}{\sqrt{2}}\left[1 + \frac{\left[z-\frac{\pi}{4}\right]}{1!} - \frac{\left[\left[z-\frac{\pi}{4}\right]\right]^2}{2!} + \cdots\right]$

(3) Find the Laurent's series expansion of $f(z) = \frac{z-1}{(z+2)(z+3)}$ valid in the region $2 < |z| < 3$

Ans: $f(z) = \frac{-3}{2}\sum_{n=0}^{\infty}(-1)^n\left[\frac{2}{z}\right]^n + \frac{4}{3}\sum_{n=0}^{\infty}(-1)^n\left[\frac{z}{3}\right]^n$

(4) Expand the Laurent's series the function $f(z) = \frac{3z-7}{(z-1)(z-2)}$ in the region $1 < |z-1| < 2$

Ans: $f(z) = \frac{-2}{z-1}\sum_{n=0}^{\infty}\left(\frac{1}{z-1}\right)^n - 2\sum_{n=0}^{\infty}\left[\frac{z-1}{2}\right]^n$

(5) Find the Laurent's series the function $f(z) = \frac{1}{z(1-z)}$ valid in the region (i) $|z+1| < 1$

(ii) $1 < |z+1| < 2$ (iii) $|z+1| > 2$

Ans: (i) $\sum_{n=0}^{\infty}\left(\frac{1}{2^{n+1}} - 1\right)(z+1)^n$ (ii) $\sum_{n=0}^{\infty}\left(\frac{1}{z+1}\right)^{n+1} + \sum_{n=0}^{\infty}\frac{(z+1)^n}{2^{n+1}}$

(iii) $\sum_{n=0}^{\infty}\frac{1-2^n}{(z+1)^{n+1}}$

(6) Find the Laurent's series expansion $f(z) = \frac{1}{z^2+4z+3}$ valid in the region (i) $|z| < 1$ and

(ii) $0 < |z+1| < 2$ **Ans:** (i) $f(z) = \frac{1}{2(z+1)} - \frac{1}{4}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z+1}{2}\right)^n$

(ii) $f(z) = \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n[z]^n - \frac{1}{6}\sum_{n=0}^{\infty}(-1)^n\left[\frac{z}{3}\right]^n$

(7) Expand $f(z) = \frac{z+3}{(z)(z^2-z-2)}$ the Laurent's series for the region

(i) $|z| < 1$ (ii) $1 < |z| < 2$ (iii) $|z| < 2$

Ans: (i) $f(z) = \frac{-3}{2z} + \frac{2}{3}\sum_{n=0}^{\infty}(-1)^n(z)^n - \frac{5}{12}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{2}\right)^n$

(ii) $f(z) = \frac{-3}{2z} + \frac{2}{3z}\sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{z}\right)^n - \frac{5}{12}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n$

(iii) $f(z) = \frac{-3}{2z} + \frac{2}{3z}\left[\sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{z}\right)^n\right] - \frac{5}{6z}\sum_{n=0}^{\infty}\left(\frac{2}{z}\right)^n$

(8) Expand $\frac{z^2+6z-1}{(z-1)(z-3)(z+2)}$ in $3 < |z+2| < 5$ as a Laurent's Series.

Ans: $f(z) = \frac{-1}{z+1} + \frac{8}{5}\sum_{n=0}^{\infty}\left(\frac{3}{z+2}\right)^n - \frac{13}{25}\sum_{n=0}^{\infty}\left(\frac{z+2}{5}\right)^n$

(9) Obtain the Laurent's Series expansion of the function $\frac{e^z}{(z-1)^2}$ in the neighbourhood of its singular point.

Hence find the residue at that pole. **Ans:** The residue is 'e'.

(10) Find the residues of $f(z) = \frac{z^2}{(z-1)^2(z-2)}$ at its isolated singularities using Laurent's Series expansions.

Ans: Residue at $z = -1$ is – 3, Residue at $z = 2$ is 4

4.3 SINGULARITIES – RESIDUES – RESIDUE THEOREM

Zeros of an analytic function

If a function $f(z)$ is analytic in a region R, is zero at a point $z = z_0$ in R, then z_0 is called a zero of $f(z)$.

Simple zero

If $f(z_0) = 0$ and $f'(z_0) \neq 0$, then $z = z_0$ is called a simple zero of $f(z)$ or a zero of the first order.

Zero of order n

If $f(z_0) = f'(z_0) = \cdots = f^{n-1}(z_0) = 0$ and $f^n(z_0) \neq 0$, then z_0 is called zero of order.

Problems based on zeros

Example: 4.27 Find the zeros of $f(z) = \frac{z^2+1}{1-z^2}$

Solution:

The zeros of $f(z)$are given by $f(z) = 0$

$$(i.e.) f(z) = \frac{z^2+1}{1-z^2} = \frac{(z+i)(z-i)}{1-z^2} = 0$$

$$\Rightarrow (z+i)(z-i) = 0$$

$\Rightarrow z = i$ and $-i$ are simple zero.

Example: 4.28 Find the zeros of $f(z) = \sin\frac{1}{z-a}$

Solution:

The zeros are given by $f(z) = 0$

$$(i.e.) \sin\frac{1}{z-a} = 0$$

$$\Rightarrow \frac{1}{z-a} = n\pi, n = \pm 1, \pm 2, \ldots$$

$$\Rightarrow (z-a)n\pi = 1$$

$\therefore$ The zeros are $z = a + \frac{1}{n\pi}, n = \pm 1, \pm 2, ,,$

Example: 4.29 Find the zeros of $f(z) = \frac{\sin z - z}{z^3}$

Solution:

The zeros are given by $f(z) = 0$

$$(i.e.) \frac{\sin z - z}{z^3} = 0$$

$$\Rightarrow \frac{\left[z - \frac{z^3}{3!} + \frac{z^5}{5!} \ldots.\right]}{z^3} - z = 0$$

$$\Rightarrow \frac{-\frac{z^3}{3!} + \frac{z^5}{5!}}{z^3} \ldots. = 0$$

$$\Rightarrow -\frac{1}{3!} + \frac{z^2}{5!} \ldots = 0$$

But $\lim\limits_{z\to 0} \frac{\sin z - z}{z^3} = -\frac{1}{3!} + 0$

$\therefore f(z)$ has no zeros.

Example: 4.30 Find the zeros of $f(z) = \frac{1-e^{2z}}{z^4}$

Solution:

The zeros are given by $f(z) = 0$

$$(i.e.) \frac{1-e^{2z}}{z^4} = 0$$

$$\Rightarrow 1 - e^{2z} = 0$$

$$\Rightarrow e^{2z} = e^{2in\pi}$$

$(i.e.)2z = 2in\pi$

$\Rightarrow z = in\pi, n = 0, \pm 1;\ \pm 2 \ldots$

Singular points

A point $z = z_0$ at which a function $f(z)$ fails to be analytic is called a singular point or singularity of $f(z)$.

Example: Consider $f(z) = \frac{1}{z-5}$

Here, $z = 5$, is a singular point of $f(z)$

Types of singularity

A point $z = z_0$ is said to be isolated singularity of $f(z)$ if

(i) $f(z)$ is not analytic at $z = z_0$

(ii) There exists a neighbourhood of $z = z_0$ containing no other singularity

Example: $f(z) = \frac{z}{z^2-1}$

This function is analytic everywhere except at $z = 1, -1$

$\therefore z = 1, -1$ are two isolated singular points.

When $z = z_0$ is an isolated singular point of $f(z)$, it can expand $f(z)$ as a Laurent's series about $z = z_0$

Thus

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=0}^{\infty} b_n (z - z_0)^{-n}$$

Note: If $z = z_0$ is an isolated singular point of a function $f(z)$, then the singularity is called

(i) a removable singularity (or)

(ii) a pole (or)

(iii) an essential singularity

According as the Laurent's series about $z = z_0$ of $f(z)$ has

(i) no negative powers (or)

(ii) a finite number of negative powers (or)

(iii) an infinite number of negative powers

Removable singularity

If the principal part of $f(z)$ in Laurent's series expansion contains no term $(i.e.)b_n = 0$ for all n, then the singularity $z = z_0$ is known as the removable singularity of $f(z)$

$$\therefore f(z) = \sum_{n=0}^{\infty} a_n (z - z_o)^n$$

(OR)

A singular point $z = z_0$ is called a removable singularity of $f(z)$, if $\lim_{z \to z_0} f(z)$ ezists finitely

Example: $f(z) = \frac{\sin z}{z}$

$$\frac{\sin z}{z} = \frac{1}{z}\left[z - \frac{z^3}{3!} + \frac{z^5}{5!} \ldots\right]$$

$$= 1 - \frac{z^2}{3!} + \frac{z^4}{5!}$$

There is no negative powers of z.

$\therefore z = 0$ is a removable singularity of $f(z)$.

Poles

If we can find the positive integer n such that $\lim_{z \to z_0} (z - z_0)^n f(z) \neq 0$, then $z = z_0$ is called a pole of order n for $f(z)$.

(or)

If $\lim_{z \to z_0} f(z) = \infty, then\, z = z_0$ is a pole of $f(z)$

Simple pole

A pole of order one is called a simple pole.

Example: $f(z) = \frac{1}{(z-1)^2(z+2)}$

Here $z = 1$ is a pole of order 2

$z = 2$ is a pole of order 1.

Essential singularity

If the principal part of $f(z)$ in Laurent's series expansion contains an infinite number of non zero terms, then $z = z_0$ is known as an essential singularity.

Example: $f(z) = e^{1/z} = 1 + \frac{\frac{1}{z}}{1!} + \frac{\left(\frac{1}{z}\right)^2}{2!} + \cdots$ has $z = 0$ as an essential singularity since, $f(z)$ is an infinite series of negative powers of z.

$f(z) = e^{\frac{1}{z}-4}$ has $z = 4$ an essential singularity

Note: The removable singularity and the poles are isolated singularities. But, the essential singularity is either an isolated or non-isolated singularity.

Entire function (or) Integral function

A function $f(z)$ which is analytic everywhere in the finite plane (except at infinity) is called an entire function or an integral function.

Example: $e^z, \sin z, \cos z$ are all entire functions.

Problems Based on Singularities

Example: 4.31 What is the nature of the singularity $z = 0$ of the function $f(z) = \frac{sinz - z}{z^3}$

Solution:

Given $f(z) = \frac{sinz - z}{z^3}$

The function $f(z)$is not defined at $z = 0$

By L' Hospital's rule.

$$\lim_{z\to 0}\frac{sinz-z}{z^3}=\lim_{z\to 0}\frac{cosz-1}{3z^2}$$

$$=\lim_{z\to 0}\frac{-sinz}{6z}$$

$$=\lim_{z\to 0}-\frac{cosz}{6z}=\frac{-1}{6}$$

Since, the limit exists and is finite, the singularity at $z=0$ is a removable singularity.

Example: 4.32 Classify the singularities for the function$\boldsymbol{f(z)=\frac{z-sinz}{z}}$

Solution:

Given $f(z)=\frac{\boldsymbol{z-sinz}}{\boldsymbol{z}}$

The function $f(z)$is not defined at $z=0$

But by L' Hospital's rule.

$$\lim_{z\to 0}\frac{z-sinz}{z}=\lim_{z\to 0}1-cosz\quad =1-1=0$$

Since, the limit exists and is finite, the singularity at $z=0$ is a removable singularity.

Example: 4.33 Find the singularity of$\boldsymbol{f(z)=\frac{e^{1/z}}{(z-a)^2}}$

Solution:

Given $f(z)=\frac{e^{1/z}}{(z-a)^2}$

Poles of $f(z)$ are obtained by equating the denominator to zero.

$$(i.e.)(z-a)^2=0$$

$\Rightarrow z=a$ is a pole of order 2.

Now, Zeros of $f(z)$

$$\lim_{z\to 0}\frac{e^{1/z}}{(z-a)^2}=\frac{\infty}{a^2}=\infty\neq 0$$

$\Rightarrow$ $z=0$is a removable singularity.

$\therefore$ $f(z)$ has no zeros.

Example: 4.34 Find the kind of singularity of the function $\boldsymbol{f(z)=\frac{cot\pi z}{(z-a)^2}}$

Solution:

Given $f(z)=\frac{cot\pi z}{(z-a)^2}$

$$=\frac{\cos\pi z}{sin\pi z(z-a)^2}$$

Singular points are poles, are given by

$$\Rightarrow sin\pi z(z-a)^2=0$$

$$(i.e.)sin\pi z=0,(z-a)^2=0$$

$\pi z=n\pi$, where $n=0,\pm1,\pm2,...$

$$(i.e.)z=n$$

$z = a$ is a pole of order 2

Since $z = n, n = 0, \pm 1, \pm 2, \ldots$

$z = \infty$ is a limit of these poles.

$\therefore z = \infty$ is non- isolated singularity.

Example: 4.35 Find the singular point of the function $f(z) = sinz\frac{1}{z-a}$. State nature of singularity.

Solution:

Given $f(z) = sinz\frac{1}{z-a}$

$z = a$ is the only singular point in the finite plane.

$$sinz\frac{1}{z-a} = \frac{1}{z-a} - \frac{1}{3!(z-a)^3} + \frac{1}{5!(z-a)^5} - \cdots$$

$z = a$ is an essential singularity

It is an isolated singularity.

Example: 4.36 Identify the type of singularity of the function $f(z) = sin\left(\frac{1}{1-z}\right)$.

Solution:

$z = 1$ is the only singular point in the finite plane.

$z = 1$ is an essential singularity

It is an isolated singularity.

Example: 4.37 Find the singular points of the function $f(z) = \left(\frac{1}{sin\frac{1}{z-a}}\right)$, state their nature.

Solution:

$f(z)$has an infinite number of poles which are given by

$$\frac{1}{z-a} = n\pi, n = \pm 1, \pm 2, \ldots$$

$(i.e.) z - a = \frac{1}{n\pi}\ ; z = a + \frac{1}{n\pi}$

But $z = a$ is also a singular point.

It is an essential singularity.

It is a limit point of the poles.

So, It is an non - isolated singularity.

Example: 4.38 Classify the singularity of $f(z) = \frac{tanz}{z}$.

Solution:

Given $f(z) = \frac{tanz}{z}$

$$= \frac{z + \frac{z^3}{3} + \frac{2z^5}{15} + \ldots}{z}$$

$$= 1 + \frac{z^2}{3} + \frac{2z^4}{15} + \ldots$$

$$\lim_{z\to 0}\frac{tanz}{z} = 1 \neq 0$$

$\Rightarrow\ z = 0$is a removable singularity of $f(z)$.

Example: 4.39 Find the residue of $\frac{1-e^z}{z^4}$ at $z = 0$

Solution:

$$\text{Given } f(z) = \frac{1-e^z}{z^4} = \frac{1-\left[1+\frac{2z}{1!}+\frac{(2z)^2}{2!}+\frac{(2z)^3}{3!}+\frac{(2z)^4}{4!}+\ldots\right]}{z^4}$$

$$= \frac{-\left[\frac{2}{1!}+\frac{4z}{2!}+\frac{8z^2}{3!}+\frac{16z^3}{4!}+\ldots\right]}{z^4}$$

Here, $z = 0$ is a pole of order 3

$$[Res\, f(z), z = 0] = \frac{1}{2!}\lim_{z\to 0}\frac{d^2}{dz^2}[(z)^3 f(z)]$$

$$= \frac{1}{2}\lim_{z\to 0}\frac{d^2}{dz^2}\left[-\left[\frac{2}{1!} + 2z + \frac{4z^2}{3} + \frac{2z^3}{3} + \ldots\right]\right]$$

$$= \frac{1}{2}\lim_{z\to 0}\frac{d}{dz}\left[-\left[2 + \frac{8}{3}z + \frac{6z^2}{3} + \ldots\right]\right]$$

$$= \frac{1}{2}\lim_{z\to 0}\left[-\left(\frac{8}{3} + \frac{12}{3}z + \cdots\right)\right]$$

$$= \frac{1}{2}\left(\frac{-8}{3}\right) = \frac{-4}{3}$$

Example: 4.40 Find the residue of $f\,(z) = tanz$ at $z = \frac{\pi}{2}$

Solution:

$$\left[\text{Res f(z)}, z = \frac{\pi}{2}\right] = \lim_{z\to\frac{\pi}{2}}\left(z - \frac{\pi}{2}\right)\ \tan z$$

$$= \lim_{z\to\frac{\pi}{2}}\frac{z-\frac{\pi}{2}}{\cot z} \quad \left[\frac{0}{0}\right] \text{ form}$$

$$= \lim_{z\to\frac{\pi}{2}}\frac{1}{-\operatorname{cosec}^2 z} = -1[\text{By L'Hospital rule}]$$

Residue

The residue of $f(z)$ at $z = z_0$ is the coefficient of $\frac{1}{z-z_0}$ in the Laurent series of $f(z)$ about $z = z_0$

Evaluation of Residues

(i) If $z = z_0$ is a pole of order one (simple pole) for $f(z)$, then

$$[Res\, f(z), z = z_0] = \lim_{z\to z_0}(z - z_0)\ f(z).$$

(ii) If $z = z_0$ is a pole of order n for $f(z)$, then

$$[Res\, f(z), z = z_0] = \lim_{z\to z_0}\frac{1}{(n-1)!}\frac{d^{\,n-1}}{dz^{n-1}}(z - z_0)^n\ f(z)$$

Problems based on Residues

Example: 4.41 Calculate the residue of $f(z) = \frac{e^{2z}}{(z+1)^2}$ at its pole.

Solution:

Given $f(z) = \frac{e^{2z}}{(z+1)^2}$ Here, $z = -1$ is a pole of order 2.

We know that,

$$[Res\, f(z), z = z_0] = \lim_{z \to z_0} \frac{1}{(m-1)!} \frac{d^{\,m-1}}{dz^{m-1}} (z - z_0)^m\ f(z)$$

Here, $m = 2$

$$[Res\, f(z), z = -1] = \lim_{z \to -1} \frac{1}{1!} \frac{d}{dz} (z+1)^2 \frac{e^{2z}}{(z+1)^2}$$

$$= \lim_{z \to -1} \frac{d}{dz} [e^{2z}] \quad = \lim_{z \to -1} 2[e^{2z}] = 2\, e^{-2}$$

Example: 4.42 Find the residues at $z = 0$ of the function (i) $f(z) = e^{1/z}$ (ii) $f(z) = \frac{sinz}{z^4}$

(iii) $f(z) = zcos\frac{1}{z}$

Solution:

The residues are the coefficients of $\frac{1}{z}$ in the Laurent's expansions of $f(z)$ about $z = 0$

(i) $e^{1/z} = 1 + \frac{\left(\frac{1}{z}\right)}{1!} + \frac{\left(\frac{1}{z}\right)^2}{2!} + \cdots$

$$= 1 + \frac{1}{1!}\left(\frac{1}{z}\right) + \frac{1}{2!}\left(\frac{1}{z}\right)^2 + \frac{1}{3!}\left(\frac{1}{z}\right)^3 + \cdots$$

$[Res\, f(z), 0\,]$ = coefficient of $\frac{1}{z}$ in Laurent's expansion.

$[Res\, f(z), 0\,] = \frac{1}{1!} = 1$by definition of residue.

(ii) $f(z) = \frac{sinz}{z^4} = \frac{1}{z^4}\left[z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right] = \frac{1}{z^3} - \frac{1}{3!}\frac{1}{z} + \frac{z^5}{5!} -$ ...

$[Res\, f(z), 0\,]$ = coefficient of $\frac{1}{z}$ in Laurent's expansion.

$[Res\, f(z), 0\,] = -\frac{1}{3!} = -\frac{1}{6}$ by definition of residue.

(iii) $f(z) = zcos\frac{1}{z} = z\left[1 - \frac{1}{2!}\frac{1}{z^2} + \frac{1}{4!}\frac{1}{z^4} - \cdots\right]$

$$= z - \frac{1}{2!}\frac{1}{z} + \frac{1}{4!}\frac{1}{z^3} - \cdots$$

$[Res\, f(z), 0\,]$ = coefficient of $\frac{1}{z}$ in Laurent's expansion.

$[Res\, f(z), 0\,] = -\frac{1}{2!} = -\frac{1}{2}$

Example: 4.43 Find the residue of $z^2 sin\left(\frac{1}{z}\right)$ at $z = 0$

Solution:

Let $f(z) = z^2 sin\left(\frac{1}{z}\right) = z^2\left[\frac{\left(\frac{1}{z}\right)}{1!} - \frac{\left(\frac{1}{z}\right)^3}{3!} + \cdots\right] = \frac{z}{1!} - \frac{1}{6z} + \cdots$

$[Res\, f(z), 0\,]$ = coefficient of $\frac{1}{z}$ in Laurent's expansion.

$$= -\frac{1}{6}$$

Example: 4.44 Find the residue of the function $f(z) = \frac{4}{z^3(z-2)}$ at a simple pole.

Solution:

Here, $z = 2$ is a simple pole.

$$[Res\, f(z), z = 2] = \lim_{z \to 2}(z-2)\frac{4}{z^3(z-2)}$$

$$= \lim_{z \to 2}\frac{4}{z^3} = \frac{4}{8} = \frac{1}{2}$$

Example: 4.45 Find the residue of $\frac{1-e^{-z}}{z^3}$ at $z = 0$

Solution:

Given $f(z) = \frac{1-e^{-z}}{z^3} = \frac{1-\left[1-\frac{z}{1!}+\frac{(z)^2}{2!}-\frac{(z)^3}{3!}+\frac{(z)^4}{4!}-\ \dots\right]}{z^3}$

$$= \frac{\left[1-\frac{z}{2!}+\frac{z^2}{3!}-\frac{z^3}{4!}+\dots\right]}{z^2}$$

Here, $z = 0$ is a pole of order 2.

$$[Res\, f(z), z = 0] = \frac{1}{1!}\lim_{z \to 0}\frac{d}{dz}[(z)^2 f(z)]$$

$$= \lim_{z \to 0}\frac{d}{dz}\left[\left[1-\frac{z}{2!}+\frac{z^2}{3!}-\frac{z^3}{4!}+\dots\right]\right]$$

$$= \lim_{z \to 0}\left[\frac{-1}{2!}+\frac{2z}{3!}-\frac{3z^2}{4!}+\dots\right]$$

$$= \frac{-1}{2!} = -\frac{1}{2}$$

4.4 CAUCHY RESIDUE THEOREM

Statement:

If $f(z)$ is analytic inside and on a simple closed curve C, except at a finite number of singular points $a_1, a_2, \dots a_n$ inside C, then

$$\int_c f(z)dz = 2\pi i\,[sum\ of\ residues\ of\ f(z) at\ a_1, a_2, \dots a_n]$$

Note: Formulae for evaluation of residues

(i) If $z = a$ is a simple pole of $f(z)$ then

$$[Res\, f(z), z = a] = \lim_{z \to a}(z-a)\,f(z)$$

(ii) If $z = a$ is a pole of order n of $f(z)$, then

$$\big[[Res\, f(z)], z = a\big] = \frac{1}{(n-1)!}\lim_{z \to a}\left\{\frac{d^{n-1}}{dz^{n-1}}[(z-a)^n f(z)]\right\}$$

Problems based on Cauchy Residue theorem

Example: 4.46 Find the residue of $f(z) = \frac{z+2}{(z-2)(z+1)^2}$ about each singularity.

Solution:

Given $f(z) = \frac{z+2}{(z-2)((z+1)^2)}$

The poles are given by $(z-2)(z+1)^2$

$$\Rightarrow z-2=0, z+1=0$$

$$\Rightarrow z=2 \text{ and } z=-1$$

$\therefore$ The Poles of $f(z)$are $z=2$ is a simple pole and $z=-1$ is a pole of order 2

$$[Res\, f(z)]_{z=2} = \lim_{z\to 2}(z-2)\, f(z)$$

$$[Res\, f(z)]_{z=2} = \lim_{z\to 2}(z-2)\, \frac{\boldsymbol{z+2}}{(\boldsymbol{z-2})(z+1)^2}$$

$$= \lim_{z\to 2}\frac{\boldsymbol{z+2}}{(z+1)^2} = \frac{4}{9}$$

$$[Res\, f(z)]_{z=1} = \lim_{z\to -1}\frac{d}{dz}[(z+1)^2 f(z)]$$

$$= \lim_{z\to -1}\frac{d}{dz}\left[(z+1)^2\,\frac{\boldsymbol{z+2}}{(\boldsymbol{z-2})(z+1)^2}\right]$$

$$= \lim_{z\to -1}\frac{d}{dz}\left(\frac{z+2}{z-2}\right)$$

$$= \lim_{z\to -1}\left[\frac{(z-2)(1)-(z+2)(1)}{(z-2)^2}\right] = -\frac{4}{9}$$

Example: 4.47 Evaluate using Cauchy's residue theorem, $\int_c \frac{\sin\pi z^2+\cos\pi z^2}{(z-1)(z-2)}\,dz$, where C is $|z|=3$

Solution:

Let $f(z) = \frac{\boldsymbol{\sin\pi z^2+\cos\pi z^2}}{(\boldsymbol{z-1})(\boldsymbol{z-2})}$

The poles are given by $(z-1)(z-2)=0$

$\Rightarrow z=1,2$ are poles of order 1.

Given C is$|z|=3$

$\therefore$ Clearly $z=1$and $z=2$ lies inside $|z|=3$

To find the residues:

(i) When $z=1$

$$[Res\, f(z)]_{z=1} = \lim_{z\to 1}(z-1)f(z)$$

$$= \lim_{z\to 1}(z-1)\,\frac{\cos\pi z^2+\sin\pi z^2}{(z-1)(z-2)}$$

$$= \lim_{z\to 1}\frac{\cos\pi z^2+\sin\pi z^2}{(z-2)}$$

$$= \frac{\cos\pi+\sin\pi}{-1}$$

$$= \frac{-1+0}{-1} = 1$$

(ii) When $z=2$

$$[Res\, f(z)]_{z=2} = \lim_{z\to 2}(z-2)f(z)$$

$$= \lim_{z\to 2}(z-2)\,\frac{\cos\pi z^2+\sin\pi z^2}{(z-1)(z-2)}$$

$$= \lim_{z\to 2}\frac{\cos\pi z^2+\sin\pi z^2}{(z-1)}$$

$$= \frac{\cos 4\pi + \sin 4\pi}{1}$$

$$= \frac{1+0}{1} = 1$$

∴ By Cauchy's Residue theorem

$$\int_c f(z)dz = 2\pi i \ (sum\ of\ residues)$$

$$= 2\pi i\ (1+1) \ = 4\pi i$$

Example: 4.48 Evaluate $\int_c \frac{z^2}{z^2+1} dz$ where C is $|z| = 2$ using Cauchy's residue theorem.

Solution:

Let $f(z) = \frac{z^2}{z^2+1}$

The poles are given by $z^2 + 1 = 0$

$\Rightarrow z = \pm i$ are poles of order 1.

Given C is $|z| = 2$

∴ Clearly $z = i, -i$ lies inside $|z| = 2$

To find the residue:

(i) When $z = i$

$$[Res\ f(z)]_{z=i} = \lim_{z\to i}(z-i)\frac{z^2}{(z+i)(z-i)}$$

$$= \lim_{z\to i}\frac{z^2}{(z+i)} = \frac{-1}{2i}$$

(ii) When $z = -i$

$$[Res\ f(z)]_{z=-i} = \lim_{z\to -i}(z+i)\frac{z^2}{(z+i)(z-i)}$$

$$= \lim_{z\to -i}\frac{z^2}{(z-i)}$$

$$= \frac{-1}{-2i} = \frac{1}{2i}$$

∴ By Cauchy's Residue theorem

$$\int_c f(z)dz = 2\pi i \ (sum\ of\ residues)$$

$$= 2\pi i \left(\frac{-1}{2i} + \frac{1}{2i}\right) = 0$$

$$\therefore \int_c \frac{z^2}{z^2+1} dz = 0$$

Example: 4.49 Evaluate $\int_c \frac{(z-1)}{(z+1)^2(z-2)} dz$ where C is the circle $|z - i| = 2$ using Cauchy's residue theorem.

Solution:

Let $f(z) = \frac{(z-1)}{(z+1)^2(z-2)}$

The poles are given by $(z+1)^2(z-2) = 0$

$$\Rightarrow z + 1 = 0; z - 2 = 0$$

$$\Rightarrow z = -1 \; is \; a \; pole \; of \; order \; 2 \; and$$

$$\Rightarrow z = 2 \text{ is a pole of order 1.}$$

Given C is $|z - i| = 2$

When $z = -1, |z - i| = |-1 - i| = \sqrt{2} < 2$

$\therefore z = -1$ lies inside C

When $z = 2, |z - i| = |2 - i| = \sqrt{5} > 2$

$\therefore z = -1$ lies inside C

To find the residue for the inside pole:

$$[Res\, f(z)]_{z=-1} = \lim_{z \to -1} \frac{d}{dz}[(z+1)^2 f(z)]$$

$$= \lim_{z \to -1} \frac{d}{dz}\left[(z+1)^2 \frac{(\boldsymbol{z-1})}{(\boldsymbol{z+1})^2(\boldsymbol{z-2})}\right]$$

$$= \lim_{z \to -1} \frac{d}{dz}\left(\frac{z-1}{z-2}\right)$$

$$= \lim_{z \to -1} \left[\frac{(z-2)(1)-(z-1)(1)}{(z-2)}\right] = -\frac{1}{9}$$

$\therefore$ By Cauchy's Residue theorem

$$\int_c f(z)dz = 2\pi i \; (sum \; of \; residues)$$

$$= 2\pi i \left(-\frac{1}{9}\right)$$

$$\therefore \int_c \frac{(z-1)}{(z+1)^2(z-2)} \, dz = -\, 2\pi i \left(\frac{1}{9}\right)$$

Example: 4.50 Evaluate $\int_c \frac{\boldsymbol{dz}}{(\boldsymbol{z^2+4})^2}$ where C is the circle $|\boldsymbol{z} - \boldsymbol{i}| = \boldsymbol{2}$ using Cauchy's residue theorem.

Solution:

Let $f(z) = \frac{1}{(z^2+4)^2}$

The poles are given by $(z^2 + 4)^2$

$$\Rightarrow z^2 + 4 = 0$$

$$\Rightarrow z = \pm 2i \text{ are poles of order 2}$$

Given C is $|z - i| = 2$

When $z = 2i, |z - i| = |2i - i| = 1 < 2$

$\therefore z = 2i$ lies inside C

When $z = -2i, |z - i| = |-2i - i| = 3 > 2$

$\therefore z = -2i$ lies outside C

To find the residue for the inside pole

$$[Res\, f(z)]_{z=2i} = \lim_{z \to 2i} \frac{d}{dz}[(z-2i)^2 f(z)]$$

$$= \lim_{z \to 2i} \frac{d}{dz}\left[(z-2i)^2 \frac{1}{(z-2i)^2((z+2i)^2)}\right]$$

$$= \lim_{z\to 2i} \frac{d}{dz}\left(\frac{1}{z+2i}\right)^2$$

$$= \lim_{z\to 2i} \left[\frac{-2)}{(z+2i)^3)}\right]$$

$$= -\frac{2}{(4i)^3} = -\frac{2}{-64i} = \frac{1}{32i}$$

∴ By Cauchy's Residue theorem

$$\int_c f(z)dz = 2\pi i\,(sum\ of\ residues)$$

$$= 2\pi i\left(\frac{1}{32i}\right)$$

$$\therefore \frac{dz}{(z^2+4)^2} = \frac{\pi}{16}$$

Example: 4.51 Evaluate $\int_c \frac{e^z dz}{(z^2+\pi^2)^2}$ where C is the circle $|z| = 4$ using Cauchy's residue theorem.

Solution:

Let $f(z) = \frac{e^z}{(z^2+\pi^2)^2}$

The poles are given by $(z^2 + \pi^2)^2 = 0$

$$\Rightarrow z^2 + \pi^2 = 0$$

$\Rightarrow z = \pm\pi i$ are poles of order 2

Given C is $|z| = 4$

Clearly $z = +\pi i\,, z = \ \pi i$ lies inside $|z| = 4$

To find the residue

(i) When $z = +\pi i$

$$[Res\, f(z)]_{z=\pi i} = \lim_{z\to\pi i} \frac{d}{dz}[(z-\pi i)^2 f(z)]$$

$$= \lim_{z\to\pi i} \frac{d}{dz}\left[(z-\pi i)^2 \frac{e^z}{(z-\pi i)^2(z+\pi i)^2}\right]$$

$$= \lim_{z\to\pi i} \frac{d}{dz}\left(\frac{e^z}{(z+\pi i)^2}\right)$$

$$= \lim_{z\to\pi i} \left[\frac{(z+\pi i)^2 e^z - 2(z+\pi i)e^z)}{(z+\pi i)^4)}\right]$$

$$= \lim_{z\to\pi i} \left[\frac{(z+\pi i)e^z[z+\pi i-2]}{(z+\pi i)^4}\right]$$

$$= \frac{e^{\pi i}(2\pi i-2)}{(2\pi i)^3}$$

$$= \frac{e^{\pi i}(\pi i-1)}{-4\pi^3 i}$$

$$= \frac{(cos\pi + isin\pi)(1-\pi i)}{4\pi^3 i}$$

$$= \frac{(-1+0)(1-\pi i)}{4\pi^3 i}$$

$$= \frac{(\pi i-1)}{4\pi^3 i}$$

(ii) When $z = -\pi i$

$$[Res\, f(z)]_{z=-\pi i} = \lim_{z\to -\pi i} \frac{d}{dz}[(z+\pi i)^2 f(z)]$$

$$= \lim_{z\to -\pi i} \frac{d}{dz}\left[(z+\pi i)^2 \frac{e^z}{(z-\pi i)^2(z+\pi i)^2}\right]$$

$$= \lim_{z\to -\pi i} \frac{d}{dz}\left(\frac{e^z}{(z-\pi i)^2}\right)$$

$$= \lim_{z\to -\pi i}\left[\frac{(z-\pi i)^2 e^z - 2(z-\pi i)e^z)}{(z-\pi i)^4}\right]$$

$$= \lim_{z\to -\pi i}\left[\frac{(z-\pi i)e^z[z-\pi i-2]}{(z-\pi i)^4}\right]$$

$$= \frac{e^{-\pi i}(-2\pi i-2)}{(-2\pi i)^3}$$

$$= \frac{(-2)(cos\pi - isin\pi)(\pi i+1)}{8\pi^3 i}$$

$$= \frac{-(-1-0)(\pi i+1)}{4\pi^3 i}$$

$$= \frac{(1+\pi i)}{4\pi^3 i}$$

∴ By Cauchy's Residue theorem

$$\int_c f(z)dz = 2\pi i\,(sum\ of\ residues)$$

$$= 2\pi i\left[\frac{(\pi i-1)}{4\pi^3 i} + \frac{(\pi i+1)}{4\pi^3 i}\right]$$

$$= \frac{2\pi i}{4\pi^3 i}[2\pi i\,] = \frac{i}{\pi}$$

$$\therefore \int_c \frac{e^z dz}{(z^2+\pi^2)^2} = \frac{i}{\pi}$$

Example: 4.52 Evaluate $\int_c \frac{dz}{zsinz}$ where C is the circle $|z| = 1$ using Cauchy's residue theorem.

Solution:

Let $f(z) = \frac{1}{zsinz}$

The poles are given by $zsinz = 0$

$$\Rightarrow z\left[z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right] = 0$$

$$\Rightarrow z^2\left[1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots\right] = 0$$

$\Rightarrow z = 0$ is a pole of order 2

Given C is $|z| = 1$

$\therefore z = 0$ lies inside C

To find the residue for the inside pole

$$[Res\, f(z)]_{z=0} = \lim_{z\to 0}\frac{d}{dz}[(z-0)^2 f(z)]$$

$$= \lim_{z\to 0}\frac{d}{dz}\left[(z)^2 \frac{1}{zsinz}\right]$$

$$= \lim_{z\to 0}\frac{d}{dz}\left(\frac{z}{sinz}\right)$$

$$= \lim_{z\to 0}\left[\frac{sinz(1)-z(cosz)}{(sinz)^2}\right]$$

$$= \frac{0-0}{0} = \left[\frac{0}{0}\right]\text{form}$$

$$= \lim_{z\to 0}\frac{cosz-[z\,(-sinz)+cosz(1)]}{2sinzcosz}\text{(by L' Hospital rule)}$$

$$= \lim_{z\to 0}\frac{cosz+zsinz-cosz}{2sinzcosz}$$

$$= \lim_{z\to 0}\frac{zsinz}{2sinzcosz}$$

$$= \lim_{z\to 0}\frac{z}{2cosz}$$

$$= \frac{0}{2} = 0$$

∴ By Cauchy's Residue theorem

$$\int_c f(z)dz = 2\pi i\ (sum\ of\ residues)$$

$$= 2\pi i\ [0]$$

$$\therefore \int_c \frac{dz}{zsinz} = 0$$

Example: 4.53 Evaluate $\int_c \frac{\tan^{z}/_2}{(z-a)^2}$, where $-2 < a < 2$ and C is the boundary of the square whose sides lie along $x = \pm 2$ and $y = \pm 2$

Solution:

Let $f(z) = \frac{\tan^{z}/_2}{(z-a)^2}$

The poles are given by $(z-a)^2 = 0$

$\Rightarrow z = a$ is a pole of order 2

C is the square with vertices $(-2,2), (2,-2)(2,2) and\ (-2,-2)$

Clearly $z = a$ lies inside C

To find the residue for the inside pole

$$[Res\ f(z)]_{z=a} = \lim_{z\to a}\frac{d}{dz}[(z-a)^2 f(z)]$$

$$= \lim_{z\to a}\frac{d}{dz}\left[(z-a)^2\frac{\tan^{z}/_2}{(z-a)^2}\right]$$

$$= \lim_{z\to a}\frac{d}{dz}\left(\tan{}^{z}/_2\right)$$

$$= \lim_{z\to a}\left[sec^2\frac{z}{2}\left({}^{1}/_2\right)\right]$$

$$= \frac{1}{2}sec^2\left(\frac{a}{2}\right)$$

∴ By Cauchy's Residue theorem

$$\int_c f(z)dz = 2\pi i\ (sum\ of\ residues)$$

$$= 2\pi i\left[\frac{1}{2}sec^2\left(\frac{a}{2}\right)\right]$$

$$\int_c \frac{\tan^z/_2}{(z-a)^2} = \pi i \left[sec^2\left(\frac{a}{2}\right)\right]$$

Example: 4.54 Evaluate $\int_c \frac{dz}{z^2 sinhz}$ where C is the circle $|z-1| = 2$ using Cauchy's residue theorem.

Solution:

Let $f(z) = \frac{1}{z^2 sinhz}$

The poles are given by $z^2 sinhz = 0$

$\Rightarrow z^2 = 0\ (or) sinhz = 0$

$\Rightarrow z = 0\ or)\ z = \ sinh^{-1}(0) = 0$is a pole of order 1.

Given C is $|z-1| = 2$

∴Clearly $z = 0$ lies inside C.

To find residue for the inside pole at $z = 0$

$$f(z) = \frac{1}{z^2 sinhz}$$

$$= \frac{1}{z^2\left[z+\frac{z^3}{3!}+\frac{z^5}{5!}+\cdots\right]}$$

$$= \frac{1}{z^3\left[1+\frac{z^2}{6}+\frac{z^4}{120}+\cdots\right]}$$

$$= \frac{1}{z^3}[1+u]^{-1} \qquad \text{where} \quad u = 1+\frac{z^2}{6}+\cdots$$

$$= \frac{1}{z^3}[1-u+u^2-u^3 \ldots]$$

$$= \frac{1}{z^3}\left[1-\left(\frac{z^2}{6}+\frac{z^4}{120}+\cdots\right)+\left(\frac{z^2}{6}+\frac{z^4}{120}+\cdots\right)^2 \ldots\right]$$

$$= \frac{1}{z^3} - \frac{1}{6z} - \frac{z}{120} + \cdots$$

$[Res\ f(z)]_{z=0}$ = Coefficient of $\frac{1}{z}$ in the Laurent's expansion of $f(z)$

$$\therefore [Res\ f(z)]_{z=0} = -\frac{1}{6}$$

∴ By Cauchy's Residue theorem

$$\int_c f(z)dz = 2\pi i\ (sum\ of\ residues)$$

$$= 2\pi i\left[-\frac{1}{6}\right]$$

$$\therefore \int_c \frac{dz}{z^2 sinhz} = \frac{-\pi i}{3}$$

Example: 4.55 Evaluate $\int_c \frac{z}{cosz}\,dz$ where C is the circle $\left|z-\frac{\pi}{2}\right| = \frac{\pi}{2}$

Solution:

Let $f(z) = \frac{z}{cosz}$

The poles are given by $cosz = 0$

$\Rightarrow z = \ (2n+1)\frac{\pi}{2}$, $n = 0, \pm 1, \pm 2, \ldots$ are poles of order 1

Given C is $\left|z-\frac{\pi}{2}\right|=\frac{\pi}{2}$

Here $z=\frac{\pi}{2}$ lies inside the circle and others lies outside.

$$[Res\, f(z)]_{z=\frac{\pi}{2}} = \lim_{z\to\frac{\pi}{2}}\left(z-\frac{\pi}{2}\right)f(z)$$

$$[Res\, f(z)]_{z=\frac{\pi}{2}} = \lim_{z\to\frac{\pi}{2}}\left(z-\frac{\pi}{2}\right)\frac{z}{cosz}$$

$$=\frac{0}{0}\text{ (form)}$$

Using L ‘ Hospital’s rule

$$= \lim_{z\to\frac{\pi}{2}}\frac{\left(z-\frac{\pi}{2}\right)(1)+z(1)}{-sinz}$$

$$= \lim_{z\to\frac{\pi}{2}}\frac{\left(z-\frac{\pi}{2}\right)+z}{-sinz}$$

$$=-\frac{\pi}{2}$$

∴ By Cauchy’s Residue theorem

$$\int_c f(z)dz = 2\pi i\,(sum\ of\ residues)$$

$$= 2\pi i\left[-\frac{\pi}{2}\right]$$

$$\therefore \int_c \frac{z}{cosz}\,dz = -\pi^2 i$$

Example: 4.56 Evaluate $\int_c z^2 e^{1/z}\,dz$ where C is the unit circle using Cauchy’s residue theorem.

Solution:

Let $f(z)=z^2e^{1/z}$

Here $z=0$ is the only singular point.

Given C is $|z|=1$

∴Clearly $z=0$ lies inside C.

To find residue of $f(z)$ at $z=0$

We find the Laurent’s series of $f(z)$ $about\ \ z=0$

$$\Rightarrow f(z)=z^2e^{1/z}$$

$$\Rightarrow z^2\left[1+\frac{1}{z}+\frac{1}{2!z^2}+\cdots\right]$$

$[Res\, f(z)]_{z=0}$ = C0efficient of $\frac{1}{z}$ in the Laurent’s expansion of $f(z)$

$$\therefore [Res\, f(z)]_{z=0} = \frac{1}{6}$$

∴ By Cauchy’s Residue theorem

$$\int_c f(z)dz = 2\pi i\,(sum\ of\ residues)$$

$$= 2\pi i\left[\frac{1}{6}\right]$$

$$\therefore \int_c z^2 e^{1/z}\, dz = \frac{\pi i}{3}$$

Exercise: 4.3

(1) Using Cauchy's Resi (i) 0 (ii) 0due, evaluate $\int_c \frac{zdz}{(z-1)^2(z+1)}$ where C is the circle (i) $|z| = \frac{1}{2}$, (ii) $|z| = 2$

Ans: (i) 0 (ii) 2

(2) Obtain the residue of the function $f(z) = \frac{z-3}{(z+1)(z+2)}$ at its poles.

Ans: For pole $-1\ res = -4,$ For pole $-2\ res = 5$

(3) Using Cauchy's Residue theorem, evaluate $\int_c \frac{4-3z}{z(z-1)(z-2)}\, dz$ where C is the circle $|z| = \frac{3}{2}$

Ans: $2\pi i$

(4) Evaluate $\int_c \frac{3z^2+z-1}{(z^2-1)(z-3)}\, dz$using the residue theorem where C is the circle $|z| = 2$. **Ans:** $\frac{-5\pi i}{4}$

(5) Evaluate $\int_c \frac{z^2+1}{(z^2-1)}\, dz$ where C is the circle $|z-i| = 1$using the Cauchy residue theorem. **Ans:** 0

(6) Evaluate $\int_c \frac{z-3}{z^2+2z+5}\, dz$ where C is the circle $|z+1+i| = 2$using Cauchy Residue theorem.

Ans: $\pi(i+2)$

(7) Evaluate $\int_c \frac{z^2}{(z-1)^2(z+2)}\, dz$ where C is the circle $|z| = 3$ using Cauchy Residue theorem. **Ans:** $2\pi i$

(8) Evaluate $\int_c \frac{z^3+z-1}{(z^2-1)(z-3)}\, dz$ around the circle $|z| = 2$using Cauchy Residue theorem. **Ans:** $\frac{-5\pi i}{4}$

(9) Evaluate $\int_c \frac{z^3}{2z+1}\, dz$ where C is the unit circle. **Ans:** $\frac{-\pi i}{8}$

(10) Evaluate $\int_c \frac{(2z-1)}{z(z+2)(2z+1)}\, dz$where C is the circle $|z| = 1$using Cauchy Residue theorem. **Ans:** $\frac{5\pi i}{3}$

4.5 Contour Integration

Evaluation of Real Integrals

The evaluation of certain types of real definite integrals of complex functions over suitable closed paths or contours and applying Cauchy's Residue theorem is known as Contour Integration.

Type 1: Integration round the unit circle

Integrals of the form $\int_0^{2\pi} f(\cos\theta, \sin\theta)d\theta$ where f is a rational function in $cos\theta$ and $\sin\theta$

To evaluate this type of integrals

We take the unit circle $|z| = 1$ as the contour C.

On $|z| = 1, let\ z = e^{i\theta}$

$$\Longrightarrow \frac{dz}{d\theta} = ie^{i\theta} = iz$$

$$\therefore d\theta = \frac{dz}{iz}$$

Also, $\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{z+\frac{1}{z}}{2} = \frac{z^2+1}{2z}$

and, $\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i} = \frac{z^2-1}{2iz}$

$|z| = 1 \Rightarrow \theta$ varies from $0\ to\ 2\pi$

$$\therefore \int_0^{2\pi} f\,(\cos\theta, \sin\theta)d\theta = \int_c\ f\ \left(\frac{z^2+1}{2z}, \frac{z^2-1}{2iz}\right)\frac{dz}{iz}$$

Now applying Cauchy's Residue theorem, we can evaluate the integral on the right side.

Problems based on Contour Integration

Example: 4.57 Evaluate $\int_o^{2\pi} \frac{d\theta}{5+4\sin\theta}$ using Contour integration.

Solution:

Replacement Let $z = e^{i\theta}$

$$\Rightarrow d\theta = \frac{dz}{iz}\ and \sin\theta = \frac{z^2-1}{2iz}$$

$$\therefore \int_o^{2\pi} \frac{d\theta}{5+4\sin\theta} = \int_c \frac{\frac{dz}{iz}}{5+4\left(\frac{z^2-1}{2iz}\right)}\ where\ c\ is\ |z| = 1$$

$$= \int_c \frac{\frac{dz}{iz}}{\frac{5iz+2z^2-2}{iz}}$$

$$= \int_c \frac{dz}{2z^2+5iz-2}$$

$$= \int_c f(z)\,dz \quad ...(1)$$

Where, $f(z) = \frac{1}{2z^2+5iz-2}$

To Evaluate, $\int_c f(z)\,dz$

To find poles of $f(z)$, put $2z^2 + 5iz - 2 = 0$

$$z = \frac{-5i \pm \sqrt{-25+16}}{4} = \frac{-5i \pm 3i}{4}$$

$$z = -\frac{i}{2}, -2i \text{ are poles of order one}$$

Given $C\ is\ |z| = 1$

Consider $z = -\frac{i}{2}$

$$\Rightarrow |z| = \left|\frac{-i}{2}\right| = \frac{1}{2} < 1$$

$\therefore z = -\frac{i}{2}$ lies inside C

Consider $z = -2i$

$$\Rightarrow |z| = |-2i| = 2 > 1$$

$\therefore z = -2i$ lies outside C.

Find the residue for inside pole $z = -\frac{i}{2}$

$$[Res\ f(z)]_{z=-\frac{i}{2}} = \lim_{z\to-\frac{i}{2}} \left(z+\frac{i}{2}\right)f(z)$$

$$= \lim_{z \to -\frac{i}{2}} \left(z + \frac{i}{2}\right) \frac{1}{2\left(Z+\frac{i}{2}\right)(Z+2i)}$$

$$= \frac{1}{2\left(-\frac{i}{2}+2i\right)} = \frac{1}{3i}$$

∴ By Cauchy's residue theorem

$$\int_c f(z)dz = 2\pi i \quad [Sum\ of\ residues]$$

$$= 2\pi i\left(\frac{1}{3i}\right) = \frac{2\pi}{3}$$

$$(1) \Longrightarrow \int_0^{2\pi} \frac{d\theta}{5+4\sin\theta} = \frac{2\pi}{3}$$

Example: 4.58 Evaluate $\int_0^{2\pi} \frac{d\theta}{13+5\sin\theta}$ using Contour Integration.

Solution:

Replacement Let $z = e^{i\theta}$

$\Rightarrow d\theta = \frac{dz}{iz}$ and $\sin\theta = \frac{z^2-1}{2iz}$

$$\therefore \int_0^{2\pi} \frac{d\theta}{13+5\sin\theta} = \int_c \frac{dz/iz}{13+5\left(\frac{z^2-1}{2iz}\right)} \ where\ C is\ |z| = 1$$

$$= \int_c \frac{dz/iz}{\frac{26iz+5z^2-5}{2iz}}$$

$$= 2\int_c \frac{dz}{5z^2+26iz-5}$$

$$= 2\int_c f(z)dz \qquad \ldots(1)$$

Where, $f(z) = \frac{1}{5z^2+26iz-5}$

To evaluate $\int_c f(z)dz$

To find poles of $f(z)$, put $5z^2 + 26iz - 5 = 0$

$$z = \frac{-26i \pm \sqrt{-676+100}}{10} = \frac{-26i \pm 24i}{10}$$

$\Rightarrow z = -\frac{i}{5}, -5i$ are poles of order one.

Given C is $|z| = 1$

Consider $z = -\frac{i}{5}$

$$\Rightarrow |z| = \left|-\frac{i}{5}\right| = \frac{1}{5} < 1$$

$\therefore z = -\frac{i}{5}$ lies inside C

Consider $z = -5i$

$\Rightarrow |z| = |-5i| = 5 > 1$

$\therefore z = -5i$ lies outside C.

Find the residue for inside pole $z = -\frac{i}{5}$

$$[Res\ f(z)]_{z=-\frac{i}{5}} = \lim_{z \to -\frac{i}{5}} \left(z + \frac{i}{5}\right) f(z)$$

$$= \lim_{z \to -\frac{i}{5}} \left(z + \frac{i}{5}\right) \frac{1}{5z^2 + 26iz - 5}$$

$$= \lim_{z \to -\frac{i}{5}} \left(z + \frac{i}{5}\right) \frac{1}{(5z+i)(z+5i)}$$

$$= \lim_{z \to -\frac{i}{5}} \left(z + \frac{i}{5}\right) \frac{1}{(5(z+\frac{i}{5})(z+5i)}$$

$$= \frac{1}{5\left(-\frac{i}{5}+5i\right)} = \frac{1}{24i}$$

∴ By Cauchy's residue theorem

$$\int_c f(z)dz = 2\pi i[Sum\ of\ residues]$$

$$= 2\pi i\left(\frac{1}{24i}\right) = \frac{\pi}{12}$$

$$\Rightarrow \int_0^{2\pi} \frac{d\theta}{13+5\sin\theta} = 2\left(\frac{\pi}{12}\right) = \frac{\pi}{6}$$

Example: 4.59 Evaluate $\int_0^{2\pi} \frac{d\theta}{a+b\cos\theta}, a > b > 0$ by using contour integration.

Solution:

Replacement Let $z = e^{i\theta}$

$$\Rightarrow d\theta = \frac{dz}{iz}\ and \cos\theta = \frac{z^2+1}{2z}$$

$$\therefore \int_o^{2\pi} \frac{d\theta}{a+b\cos\theta} = \int_c \frac{dz/iz}{a+b\left(\frac{z^2+1}{2z}\right)}\ where\ c\ is\ |z| = 1$$

$$= \int_c \frac{dz/iz}{\frac{2az+bz^2+b}{2z}}$$

$$= \frac{2}{i}\int_c \frac{dz}{bz^2+2az+b}$$

$$= \frac{2}{i}\int_c f(z)dz \qquad \dots (1)$$

Where, $f(z) = \frac{1}{bz^2+2az+b}$

To evaluate $\int_c f(z)dz$

To find poles of $f(z), put\ bz^2 + 2az + b$

$$z = \frac{-2a \pm \sqrt{4(a^2-b^2)}}{2b} = \frac{-a \pm \sqrt{a^2-b^2}}{b}$$

$z = \frac{-a+\sqrt{a^2-b^2}}{b}, \frac{-a+\sqrt{a^2-b^2}}{b}$ are poles of order one.

Clearly, $z = \frac{-a+\sqrt{a^2-b^2}}{b} = \propto$ lies inside c

$and\ z = \frac{-a-\sqrt{a^2-b^2}}{b} = \beta$ lies outside c

Since a > b, we can write $bz^2 + 2az + b = b(z-\alpha)(z-\beta)$

Find the residue for inside pole $z = \alpha$

$$Res\ f(z)]_{z=\alpha} = \lim_{z\to\alpha}(z-\alpha)\frac{1}{b(z-\alpha)(z-\beta)}$$

$$= \frac{1}{b(\alpha-\beta)}$$

$$= \frac{1}{2\sqrt{a^2-b^2}}$$

∴ By Cauchy residue theorem

$$\int_c f(z)dz = 2\pi i\ [sum\ of\ residues]$$

$$= 2\pi i\left[\frac{1}{2\sqrt{a^2-b^2}}\right]$$

$$= \frac{\pi i}{\sqrt{a^2-b^2}}$$

$$(1) \Longrightarrow \int_o^{2\pi} \frac{d\theta}{a+b\cos\theta} = \frac{2}{i}\left[\frac{\pi i}{\sqrt{a^2-b^2}}\right]$$

$$= \frac{2\pi}{\sqrt{a^2-b^2}}$$

Example: 4.60 Show that $\int_o^{2\pi} \frac{d\theta}{1-2p\cos\theta+p^2} = \frac{2\pi}{1-p^2}\ if\ |p| < 1$

Solution:

Replacement Let $z = e^{i\theta}$

$$\Rightarrow d\theta = \frac{dz}{iz}\ and \cos\theta = \frac{z^2+1}{2z}$$

$$\therefore \int_o^{2\pi} \frac{d\theta}{1-2p\cos\theta+p^2} = \int_c \frac{dz/iz}{\left[1-2p\left(\frac{z^2+1}{2z}\right)+P^2\right]}$$

Where, C is $|z| = 1$

$$= \int_c \frac{dz}{iz\left[\frac{z-P(z^2+1)+P^2z}{z}\right]}$$

$$= -\frac{1}{i}\int_c \frac{dz}{P(z^2+1)-P^2z-z}$$

$$= -\frac{1}{i}\int_c \frac{dz}{P(z^2-(1+P^2)z+P}$$

$$= -\frac{1}{i}\int_c f(z)dz \qquad \text{... (1)}$$

Where, $f(z) = \frac{1}{Pz^2-(1+P^2)z+P}$

To evaluate $\int_c \boldsymbol{f(z)dz}$

To find poles of $f(z), put\ pz^2 - (1+p^2)z + p = 0$

$$z = \frac{(P^2+1)\pm\sqrt{(P^2+1)^2-4P^2}}{2p}$$

$$= \frac{(P^2+1)\pm(P^2-1)}{2p}$$

$\Rightarrow z = p, \frac{1}{p}$ are poles of order one.

Since $|p| < 1$, the pole $z = p$ lies inside C and the pole $z = \frac{1}{p}$ lies outside C.

Find the residue for the inside pole $z = p$

$$[Res\ f(z)]_{z=p} = \lim_{z\to p}(z-p)\ f(z)$$

$$= \lim_{z\to p}\left[(z-p)\frac{1}{p(z-p)\left(z-\frac{1}{p}\right)}\right]$$

$$= \frac{1}{P\left(P-\frac{1}{P}\right)} = \frac{1}{P^2-1}$$

$\therefore$ By Cauchy Residue theorem

$$\int_c f(z)dz = 2\pi i\,[sum\ of\ residues]$$

$$= 2\pi i\left[\frac{1}{P^2-1}\right]$$

$$(1) \Rightarrow \int_O^{2\pi}\frac{d\theta}{1-2p\cos\theta+P^2} = -\frac{1}{i}\left[\frac{2\pi i}{P^2-1}\right]$$

$$= \frac{2\pi}{1-P^2}$$

Example: 4.61 Evaluate $\int_O^{2\pi}\frac{\cos 3\theta\, d\theta}{5-4\cos\theta}$ using contour integration.

Solution:

Replacement Let $z = e^{i\theta}$

$$\Rightarrow d\theta = \frac{dz}{iz}\ and \cos\theta = \frac{z^2+1}{2z}$$

$$\cos 3\theta = Real\ part\ of\ e^{i3\theta} = R.P\ (z^3)$$

$$\therefore \int_O^{2\pi}\frac{\cos 3\theta\, d\theta}{5-4\cos\theta} = \int_O^{2\pi}\frac{R.P\frac{(Z^3)dz}{iz}}{5-4\left(\frac{z^2+1}{2z}\right)}\ where\ C\ is\ |z| = 1$$

$$= R.P\int_c \frac{z^3dz/iz}{\frac{5z-(2z^2+2)}{z}}$$

$$= R.P\ \left(-\frac{1}{i}\right)\int_c f(z)\,dz \qquad ...(1)$$

Where, $f(z) = \frac{z^3}{2z^2-5z+2}$

To evaluate $\int_c \boldsymbol{f(z)dz}$

To find poles of $f(z), put\ 2z^2 - 5z + 2 = 0$

$$z = \frac{5+\sqrt{25-16}}{4} = \frac{5\pm3}{4}$$

$\Rightarrow z = 2, \frac{1}{2}$ are poles of order one.

Given $C\ is\ |z| = 1$

Consider $z = \frac{1}{2}$

$$\Rightarrow |z| = \left|\frac{1}{2}\right| = \frac{1}{2} < 1$$

$\therefore z = \frac{1}{2}$ lies inside C

Consider $z = 2$

$$\Rightarrow |z| = |2| = 2 > 1$$

$\therefore z = 2$ lies outside C

Find the residue for inside pole $z = \frac{1}{2}$

$$[Res\, f(z)]_{z=\frac{1}{2}} = \lim_{z\to\frac{1}{2}} \left(z-\frac{1}{2}\right) f(z)$$

$$= \lim_{z\to\frac{1}{2}} \left(z-\frac{1}{2}\right) f(z)$$

$$= \lim_{z\to\frac{1}{2}} \left(z-\frac{1}{2}\right) \frac{z^3}{(2z-1)(z-2)}$$

$$= \lim_{z\to\frac{1}{2}} \left(z-\frac{1}{2}\right) \frac{z^3}{2\left(z-\frac{1}{2}\right)(z-2)}$$

$$= \frac{\left(\frac{1}{2}\right)^3}{2\left(\frac{1}{2}-2\right)} = -\frac{1}{24}$$

∴ By Cauchy's Residue theorem

$$\int_c f(z)dz = 2\pi i\, [sum\ of\ residues]$$

$$= 2\pi i\left(-\frac{1}{24}\right) = -\frac{\pi i}{12}$$

$$(1) \Rightarrow \int_O^{2\pi} \frac{\cos 3\theta}{5-4\cos\theta}\, d\theta = R.P\ \left(-\frac{1}{i}\right)\left(-\frac{\pi i}{12}\right) = \frac{\pi}{12}$$

Example: 4.62 Evaluate $\int_O^{2\pi} \frac{\sin^2\theta d\theta}{5-3\cos\theta} = \int_O^{2\pi} \frac{\cos 2\theta d\theta}{10-6\cos\theta}$

Solution:

Replacement Let $z = e^{i\theta}$

$$\Rightarrow d\theta = \frac{dz}{iz}\ and \cos\theta = \frac{z^2+1}{2z}$$

$$\cos 2\theta = Real\ part\ of\ e^{i2\theta} = R.P\ (Z^2)$$

$$\therefore \int_O^{2\pi} \frac{\sin^2\theta}{5-3\cos\theta}\, d\theta = \int_O \frac{1-R.P\frac{(z^2)dz}{iz}}{10-6\left(\frac{z^2+1}{2z}\right)}$$

$where\ \ C\ is\ |z| = 1$

$$= R.P\ \int_C \frac{(1-z^2)dz/iz}{\frac{10z-3z^2-3}{z}}$$

$$= R.P\ \left(-\frac{1}{i}\right)\int_C \frac{(1-z^2)dz}{3z^2-10Z+3}$$

$$= R.P\ \left(-\frac{1}{i}\right)\int_C\ f(z)dz \qquad \dots (1)$$

Where, $f(z) = \frac{1-z^2}{3z^2-10z+3}$

To evaluate $\int_c\ \boldsymbol{f(z)dz}$

To find poles of $f(z), put\ 3z^2 - 10z + 3 = 0$

$$z = \frac{10\pm\sqrt{100-36}}{6} = \frac{10\pm 8}{6}$$

$\therefore z = 3, \frac{1}{3}$ are poles of order one.

Given $C\ is\ |z| = 1$

Consider $z = \frac{1}{3}$

$$\Rightarrow |z| = \left|\frac{1}{3}\right| = \frac{1}{3} < 1$$

$\therefore z = \frac{1}{3}$ lies inside C

Consider $z = 3$

$$\Rightarrow |z| = |3| = 3 < 1$$

$\therefore z = 3$ lies outside C

Find the residue for inside pole $z = \frac{1}{3}$

$$[Res\, f(z)]_{z=\frac{1}{3}} = \lim_{z\to\frac{1}{3}} \left(z - \frac{1}{3}\right) f(z)$$

$$= \lim_{z\to\frac{1}{3}} \left(z - \frac{1}{3}\right) \frac{(1-Z^2)}{3\left(Z-\frac{1}{3}\right)(Z-3)}$$

$$= \frac{1-\left(\frac{1}{3}\right)^2}{3\left(\frac{1}{3}-3\right)} = -\frac{1}{9}$$

$\therefore$ By Cauchy's Residue theorem

$$\int_C f(z)dz = 2\pi i\, [sum\ of\ residues]$$

$$= 2\pi i \left(-\frac{1}{9}\right)$$

$$(1) \Rightarrow \int_0^{2\pi} \frac{sin^2\theta\, d\theta}{5-3\cos\theta} = R.P\left(-\frac{1}{i}\right)\left(-\frac{2\pi i}{9}\right) = \frac{2\pi}{9}$$

Example: 4.63 Using Contour Integration, evaluate the real integral $\int_0^{\pi} \frac{1+2\cos\theta}{5+4\cos\theta} d\theta$

Solution:

Replacement Let $z = e^{i\theta}$

$$\Rightarrow d\theta = \frac{dz}{iz}\ and \cos\theta = \frac{Z^2+1}{2Z}$$

Now, $\int_0^{\pi} \frac{1+2\cos\theta}{5+4\cos\theta} d\theta = \frac{1}{2}\int_0^{2\pi} \frac{1+2\cos\theta}{5+4\cos\theta} d\theta$

$$\left[\because \int_0^{2a} f(x)dx = 2\int_0^{a} f(x)dx, if\ f(2a-x) = f(x)\right]$$

$$\therefore \frac{1}{2}\int_0^{2\pi} \frac{1+2\cos\theta}{5+4\cos\theta} d\theta = \frac{1}{2}\int_C \frac{1+\left[1+2\left(\frac{z^2+1)}{2z}\right)\right]\frac{dz}{iz}}{5+4\left(\frac{z^2+1)}{2z}\right)}$$

$$= \frac{1}{2i}\int_C \frac{(z^2+z+1)}{z(2z^2+5z+2}$$

$$= \frac{1}{2i}\int_C f(z)dz \qquad \ldots (1)$$

Where, $f(z) = \frac{z^2+z+1}{Z(2z^2+5z+2)}$

To evaluate $\int_C f(z)dz$

To find poles of $f(z), put\ z(2z^2 + 5z + 2) = 0$

$$\Rightarrow z = 0;\ 2z^2 + 5z + 2 = 0$$

$\Rightarrow z = 0; z = -2, z = -\frac{1}{2}$ are poles of order one.

Given $C\ is\ |z| = 1$

Consider $z = 0$

$$\Rightarrow |z| = |0| = 0 < 1$$

$\therefore z = 0$ lies inside C

Consider $z = -2$

$$\Rightarrow |z| = |-2| = 2 > 1$$

$\therefore z = -2$ lies outside C

Consider $z = -\frac{1}{2}$

$$\Rightarrow |z| = \left|-\frac{1}{2}\right| = \frac{1}{2} < 1$$

$\therefore z = -\frac{1}{2}$ lies inside C

Find the residue for the inside pole

(i) When $z = 0$

$$[Res\ f(z)]_{z=0} = \underset{z\to O}{\text{Lt}}(z-0)f(z)$$

$$\underset{z\to O}{\text{Lt}}\, z\ \frac{(z^2+z+1)}{Z(2Z^2+5z+2)} = \frac{1}{2}$$

(ii) When $z = -\frac{1}{2}$

$$[Res\ f(z)_{z=-\frac{1}{2}} = \lim_{z\to-\frac{1}{2}}\left(z+\frac{1}{2}\right)f(z)$$

$$= \lim_{z\to-\frac{1}{2}}\left(z+\frac{1}{2}\right)\frac{z^2+z+1}{Z(2z+1)(z+2)}$$

$$= \lim_{z\to-\frac{1}{2}}\left(z+\frac{1}{2}\right)\frac{z^2+z+1}{Z\ 2\left(z+\frac{1}{2}\right)(z+2)}$$

$$= \frac{\frac{1}{4}-\frac{1}{2}+1}{2\left(-\frac{1}{2}\right)\left(-\frac{1}{2}+2\right)}$$

$$= \frac{\frac{3}{4}}{-\frac{3}{2}} = -\frac{1}{2}$$

$\therefore$ By Cauchy's Residue Theorem

$$\int_C\ f(z)dz = 2\pi i\ [Sum\ of\ residues]$$

$$= 2\pi i\left[\frac{1}{2}-\frac{1}{2}\right] = 0$$

$$(1) \Rightarrow \frac{1}{2}\int_O^{2\pi}\frac{1+2\cos\theta}{5+4\cos\theta}d\theta = \frac{1}{2i}[0] = 0$$

Example: 4.64 Evaluate $\int_0^{2\pi}\frac{d\theta}{2+\cos\theta}$ using Contour Integration.

Solution:

Replacement Let $z = e^{i\theta}$

$$\Rightarrow d\theta = \frac{dz}{iz}\ and\ \cos\theta = \frac{z^2+1}{2z}$$

$$\therefore \int_0^{2\pi} \frac{d\theta}{2+\cos\theta} = \int_C^{2\pi} \frac{dz/iz}{2+\left(\frac{z^2+1}{2z}\right)} \quad where\ C\ is\ |z| = 1$$

$$= \int_C \frac{dz/iz}{\frac{z^2+4z+1}{2z}}$$

$$= \frac{2}{i}\int_C \frac{dz}{z^2+4z+1}$$

$$= \frac{2}{i}\int_C f(z)dz \qquad \ldots (1)$$

Where, $f(z) = \frac{1}{z^2+4z+1}$

To evaluate $\int_C f(z)dz$

To find poles of $f(z), put\ z^2 + 4z + 1 = 0$

$$z = \frac{-4\pm\sqrt{16\pm4}}{2} = \frac{-4\pm2\sqrt{3}}{2}$$

$\therefore z = -2 + \sqrt{3}, -2 - \sqrt{3}$ are the poles of order one.

Given $C\ is\ |z| = 1$

Consider $z = -2 + \sqrt{3}$

$$\Rightarrow |z| = \left|-2 + \sqrt{3}\right| < 1$$

$\therefore z = -2 + \sqrt{3}$ lies inside C

Consider $z = -2 - \sqrt{3}$

$$\Rightarrow |z| = \left|-2 - \sqrt{3}\right| < 1$$

$\therefore z = -2 - \sqrt{3}$ lies outside C

Find the residue for the inside pole $z = -2 + \sqrt{3}$

$$[Res\ f(z)]_{z=-2+\sqrt{3}} = \lim_{z\to-2+\sqrt{3}} \left(z - (-2 + \sqrt{3}\right) f(z)$$

$$\lim_{z\to-2+\sqrt{3}} \left[z - \left(-2 + \sqrt{3}\right)\right] \frac{1}{[Z-(-2+\sqrt{3})][Z-2-\sqrt{3})}$$

$$= \frac{1}{-2+\sqrt{3}+2+\sqrt{3}} = \frac{1}{2\sqrt{3}}$$

$\therefore$ By Cauchy's Residue Theorem

$$\int_C f(z)dz = 2\pi i\ [Sum\ of\ residues]$$

$$= 2\pi i\left[\frac{1}{2\sqrt{3}}\right] = \frac{\pi i}{\sqrt{3}}$$

$$(1) \Longrightarrow \int_0^{2\pi} \frac{d\theta}{2+\cos\theta} = \frac{2}{i}\left[\frac{\pi i}{\sqrt{3}}\right] = \frac{2\pi}{\sqrt{3}}$$

Exercise: 4.4

Evaluate the following Integrals

1) $\int_0^{2\pi} \frac{d\theta}{1+2\cos\theta}$ **Ans:** $\frac{2\pi}{\sqrt{3}}$

2) $\int_0^{2\pi} \frac{cos2\theta}{5+4\cos\theta} d\theta$ **Ans:** $\frac{\pi}{6}$

3) $\int_0^{2\pi} \frac{sin^2\theta}{5+4\cos\theta} d\theta$ **Ans:** $\frac{\pi}{4}$

4) $\int_0^{2\pi} \frac{d\theta}{5-4\ \cos\theta}$ **Ans:** $\frac{2\pi}{3}$

5) $\int_0^{2\pi} \frac{cos2\theta}{1-2a\cos\theta+a^2}; |a| < 1$ **Ans:** $\frac{2\pi a^3}{1-a^2}$

6) $\int_0^{2\pi} \frac{d\theta}{1-2x\sin\theta+x^2}; [x < n < 1]$ **Ans:** $\frac{2\pi}{1-x^2}$

7) $\int_0^{2\pi} \frac{d\theta}{17-8\cos\theta}$ **Ans:** $\frac{2\pi}{15}$

8) $\int_0^{2\pi} \frac{d\theta}{13+12\cos\theta}$ **Ans:** $\frac{2\pi}{5}$

9) $\int_0^{2\pi} \frac{d\theta}{13+12\cos\theta}$ **Ans:** $\frac{2\pi}{b^2}\left[a - \sqrt{a^2\sqrt{-b^2}}\right]$

10) $\int_0^{2\pi} \frac{cos^2 3\theta}{5-4\cos\theta} d\theta$ **Ans:** $\frac{3\pi}{8}$

Type II: Integration around semi – circular contour

Integrals of the form $\int_{-\infty}^{\infty} \frac{f(x)}{g(x)} dx$,

where $f(x)$ and $g(x)$ are polynomials in x, such that the degree of $f(x)$ is less than that of $g(x)$ atleast by two and $g(x)$ does not vanish for any value of x.

Let C be a closed contour of real axis from $-R$ to R and semicircle $'S'$ of radius R above real axis.

Thus,

$$\int_C \frac{f(z)}{g(z)} = \int_{-R}^{R} \frac{f(x)}{g(x)} dx + \int_S \frac{f(z)}{g(z)} dz$$

As $R \to \infty, \int_C \frac{f(z)}{g(z)} dz \to 0$ by Cauchy's lemma

$$= \int_C \frac{f(z)}{g(z)} dz = \int_{-\infty}^{\infty} \frac{f(x)}{g(x)} dx$$

Now applying Cauchy's Residue theorem, we can evaluate the integral on the left side.

Note: Cauchy's lemma: If $f(z)$ is continuous function such that $|zf(z) \to 0|$ uniformly as $|z| \to \infty$ on S, then $\int_c f(z)dz \to 0$ as $R \to \infty$, where 'S' is semicircle of radius $'R'$ above the real axis.

Problems based on Contour Integration

Example: 4.66 Evaluate $\int_{-\infty}^{\infty} \frac{x^2 dx}{(x^2+a^2)(x^2+b^2)}$ where $a > b > 0$

Solution:

Replacement $put\ x = z \Rightarrow dx = dz$

$$\therefore \int_{-\infty}^{\infty} \frac{x^2 dx}{(x^2+a^2)(x^2+b^2)} = \int_C \frac{z^2 dz)}{(z^2+a^2)(z^2+b^2)} where$$

Where C is the upper semi circle

$$= \int_C f(z)dz \quad \dots(1)$$

Where, $f(z) = \frac{z^2}{(z^2+a^2)(z^2+b^2)}$

To find the poles, put $(z^2 + a^2)(z^2 + b^2) = 0$

$\Rightarrow z = \pm ai, z = \pm bi$, are poles of order one.

Here $z = ai, bi$ lies in upper, half of the z −plane.

Find the residue for the inside pole

(i) When $z = ai$

$$[Res\, f(z)]_{z\to ai} = \lim_{z\to ai}(z - ai)\frac{z^2}{(z+ai)(z-ai)(z^2+b^2)}$$

$$= \frac{-a^2}{2ai\,(b^2-a^2)}$$

$$= \frac{a}{2i\,(a^2-b^2)}$$

(ii) When $z = bi$

$$[Res\, f(z)]_{z=bi} = \lim_{z\to bi}(z - bi)f(z)$$

$$= \lim_{z\to bi}(z - bi)\frac{z^2}{(z^2+a^2)(z+bi)(z-bi)}$$

$$= -\frac{b^2}{(a^2-b^2)2bi}$$

$$= -\frac{b}{2i\,(a^2-b^2)}$$

∴ By Cauchy's Residue theorem

$$\int_c f(z)dz = 2\pi i\,[sum\ of\ residues]$$

$$= 2\pi i\left[\frac{a}{2i(a^2-b^2)} - \frac{b}{2i\,(a^2-b^2)}\right]$$

$$= \frac{2\pi i}{2i}\left[\frac{a-b}{(a-b)(a+b)}\right]$$

$$= \frac{\pi}{a+b}$$

$$(1) \Rightarrow \int_{-\infty}^{\infty}\frac{x^2dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{a+b}$$

Example: 4.67 Evaluate $\boldsymbol{\int_0^\infty \frac{dx}{(x^2+a^2)(x^2+b^2)}, a > 0, b > 0}$

Solution:

$$\int_0^\infty \frac{dx}{(x^2+a^2)(x^2+b^2)} = \frac{1}{2}\int_{-\infty}^{\infty}\frac{dx}{(z^2+a^2)(x^2+b^2)}$$

Replacement put $x = z$

$$\Rightarrow dx = dz$$

$$\therefore \frac{1}{2}\int_{-\infty}^{\infty}\frac{dx}{(x^2+a^2)(x^2+b^2)} = \frac{1}{2}\int_{-\infty}^{\infty}\frac{dx}{(x^2+a^2)(x^2+b^2)}$$

Where C is the upper semi circle

$$= \frac{1}{2}\int_c f(z)dz \ldots (1)$$

Where, $f(z) = \frac{1}{(z^2+a^2)(z^2+b^2)}$

To find the poles, put $(z^2 + a^2)(z^2 + b^2) = 0$

$\Rightarrow z = \pm ai, \pm bi$ are poles of order one.

Here $z = ai, bi$ lies in the upper half of the z- plane.

Find the residue for the inside pole

(i) When $z = ai$

$$[Res\, f(z)]_{z=ai} = \lim_{z\to ai}(z-ai)f(z)$$

$$= \lim_{z\to ai}(z-ai)\frac{1}{(z+ai)(z-ai)(z^2+b^2)}$$

$$= \frac{1}{2ai(b^2-a^2)} = -\frac{1}{2ai\,(a^2-b^2)}$$

(ii) When $z = bi$

$$[Res\, f(z)]_{z=bi} = \lim_{z\to ai}(z-bi)f(z)$$

$$= \lim_{z\to bi}(z-bi) = \frac{1}{(z^2+a^2)(z+bi)(z-bi)}$$

$$= \frac{1}{(a^2-b^2)2bi}$$

∴ By Cauchy's Residue theorem

$$\int_c f(z)dz = 2\pi i\,[-\frac{1}{2ai\,(a^2-b^2)} + \frac{1}{2bi\,(a^2-b^2)}]$$

$$= \frac{2\pi i}{2i(a^2-b^2)}\left[-\frac{1}{a}+\frac{1}{b}\right]$$

$$= \frac{\pi}{(a+b)(a-b)}\left(\frac{a-b}{ab}\right)$$

$$= \frac{\pi}{ab\,(a+b)}$$

$$(1) \implies \frac{1}{2}\int_{-\infty}^{\infty}\frac{dx}{(x^2+a^2)(x^2+b^2)} = \frac{1}{2}\frac{\pi}{ab\,(a+b)}$$

$$= \frac{\pi}{2ab\,(a+b)}$$

Example: 4.68 Evaluate $\int_{-\infty}^{\infty}\frac{dx}{x^2+a^2}$

Solution:

Replacement $put\ x = z$

$$\Rightarrow dx = dz$$

$\therefore \int_{-\infty}^{\infty}\frac{dx}{x^2+a^2} = \int_C \frac{dz}{z^2+a^2}$ where C is the upper semi circle

$$= \int_c f(z)dz \quad \dots (1)$$

Where, $f(z) = \frac{1}{z^2+a^2}$

To find the poles put $z^2 + a^2 = 0$

$\Rightarrow z \pm ai$ are poles of order one.

Here $z = ai$ lies in the upper half of z plane.

Find the residue for the inside pole.

(i) When $z = ai$

$$[Res\, f(z)]_{z=ai} = \lim_{z\to ai}(z-ai)f(z)$$

$$= \lim_{z\to ai}(z-ai)\frac{1}{(z+ai)(z-ai)}$$

$$= \frac{1}{2ai}$$

∴ By Cauchy's Residue Theorem

$$\int_C f(z)dz = 2\pi i\,[sum\,of\,residues]$$

$$= 2\pi i\left[\frac{1}{2ai}\right] = \frac{\pi}{a}$$

$$(1)\quad \Longrightarrow \int_{-\infty}^{\infty} \frac{dx}{x^2+a^2} = \frac{\pi}{a}$$

Example: 4.69 Evaluate $\int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)^2}$

Solution:

Replacement $put\ x = z \Rightarrow dx = dz$

Now, $\int_{o}^{\infty} \frac{dx}{(x^2+a^2)^2} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)}$

$= \frac{1}{2}\int_c \frac{dz}{(z^2+a^2)^2}$ where C is the upper semi circle

$= \frac{1}{2}\int_c f(z)dz \qquad \ldots(1)$

Where, $f(z) = \frac{1}{(z^2+a^2)^2}$

To find the poles, put $(z^2 + a^2)^2 = 0$

$\Rightarrow z = \pm ai$ are poles of order 2 here $z = ai$ lies in the upper half of z – plane. Find the residue of the inside pole.

(i) When $z = ai$

$$[Res\,f(z)]_{z=ai} = \lim_{z\to ai} \frac{d}{dz}(z-ai)^2 f(z)$$

$$= \lim_{z\to ai} \frac{d}{dz}\left[(z-ai)^2 \frac{1}{(z-ai)^2(z+ai)^2}\right]$$

$$= \lim_{z\to ai} \frac{d}{dz}\left[\frac{1}{(z+ai)^2}\right]$$

$$= \lim_{z\to ai}\left[\frac{-2}{(z+ai)^3}\right]$$

$$= -\frac{2}{(2ai)^3} = -\frac{2}{-8a^3 i} = \frac{1}{4ia^3}$$

∴By Cauchy's Residue theorem,

$$\int_C f(z)dz = 2\pi i\,[sum\,of\,residues]$$

$$= 2\pi i\left(\frac{1}{4ia^3}\right)$$

$$= \frac{\pi}{2a^3}$$

Example: 4.70 Evaluate $\int_{-\infty}^{\infty} \frac{x^2-x+2}{(x^4+10x^2+9)}dx$

Solution:

Replacement Put $x = z \Rightarrow dx = dz$

$$\therefore \int_{-\infty}^{\infty} \frac{x^2-x+2}{(x^4+10x^2+9)}\,dx = \int_C \frac{z^2-z+2}{(z^4+10z^2+9)}\,dz,$$ where C is the upper semi circle.

$$= \int_c f(z)dz \qquad \ldots(1)$$

Where, $f(z) = \frac{z^2-z+2}{(z^4+10z^2+9)}$

To find the poles, put $z^4 + 10z^2 + 9 = 0$

$$\Rightarrow (z^2+1)\,(z^2+9) = 0$$

$\Rightarrow z = \pm i, \pm 3i$ are poles of order one.

Here $z = i, 3i$ lies in the inside pole

Find the residue of the inside pole.

(i) When $z = i$

$$[Res\, f(z)]_{z=i} = \lim_{z\to i}(z-i)\, f(z)$$

$$= \lim_{z\to i}\left[(z-i)\frac{(z^2-z+2)}{(z+i)(z-i)(z^2+9)}\right]$$

$$= \lim_{z\to i}\left[\frac{(z^2-z+2)}{(z+i)(z^2+9)}\right]$$

$$= \frac{-1-i+2}{(2i)(8)} = \frac{1-i}{16i}$$

(ii) When $z = 3i$

$$[Res\, f(z)]_{z=3i} = \lim_{z\to 3i}\frac{d}{dz}(z-3i)\, f(z)$$

$$= \lim_{z\to 3i}\left[(z-3i)\,\frac{(z^2-z+2)}{(z^2+1)(z+3i)(z-3i)}\right]$$

$$= \lim_{z\to 3i}\frac{(z^2-z+2)}{(z^2+1)(z+3i)}$$

$$= \frac{-9-3i+2}{(-8)(6i)} = \frac{-7-3i}{-48i}$$

$$= \frac{7+3i}{48i}$$

$\therefore$By Cauchy's Residue theorem,

$$\int_C f(z)dz = 2\pi i\,[sum\ of\ residues]$$

$$= 2\pi i\left(\frac{1-i}{16i} + \frac{7+3i}{48i}\right)$$

$$= 2\pi i\left(\frac{3-3i+7+3i}{48i}\right)$$

$$= 2\pi i\left(\frac{10}{48i}\right) = \frac{5\pi}{12}$$

$$(1) \implies \int_{-\infty}^{\infty}\frac{x^2-x+2}{(x^4+10x^2+9)}\,dx = \frac{5\pi}{12}$$

Example: 4.71 Evaluate $\int_0^{\infty}\frac{dx}{x^4+a^4}$

Solution:

$$\int_0^{\infty}\frac{dx}{x^4+a^4} = \frac{1}{2}\int_{-\infty}^{\infty}\frac{dx}{x^4+a^4}$$

Replacement Put $x = z \Rightarrow dx = dz$

$$\therefore \frac{1}{2}\int_{-\infty}^{\infty} \frac{dx}{x^4+a^4} = \frac{1}{2}\int_C \frac{dz}{z^4+a^4} \text{ where C is the upper semi circle.}$$

$$= \frac{1}{2}\int_c f(z)dz \qquad \dots (1)$$

Where, $f(z) = \frac{1}{z^4+a^4}$

To find the poles, put $z^4 + a^4 = 0$

$$\Rightarrow z^4 = -a^4$$

$$\Rightarrow z = (-a^4)^{\frac{1}{4}}$$

$$\Rightarrow z = (-1)^{1/4} a$$

$$= (\cos\pi + i\sin\pi)^{\frac{1}{4}} a$$

$$= [\cos(\pi + 2k\pi) + i\sin(\pi + 2k\pi)]^{\frac{1}{4}} a$$

$$= \left[\cos\left(\frac{\pi+2k\pi}{4}\right) + i\sin\left(\frac{\pi+2k\pi}{4}\right)\right] a$$

$$= ae^{+i\left(\frac{\pi+2k\pi}{4}\right)} ; k = 0, 1, 2, 3 \dots \dots$$

When $k = 0, z = ae^{\frac{i\pi}{4}}$

When $k = 1, z = ae^{\frac{3i\pi}{4}}$

When $k = 2, z = ae^{\frac{5i\pi}{4}}$

When $k = 3, z = ae^{\frac{7i\pi}{4}}$ are all poles of order one.

Here $z = ae^{\frac{i\pi}{4}}$ and $, z = ae^{\frac{3i\pi}{4}}$ lies in the upper half of the z plane.

Find the residue for the inside pole

(i)When $z = ae^{\frac{i\pi}{4}}$

$$[Res\, f(z)]_{z\to ae^{\frac{i\pi}{4}}} \left(z - ae^{\frac{i\pi}{4}}\right) f(z)$$

$$= \lim_{z\to ae^{\frac{i\pi}{4}}} \left[\left(z - ae^{\frac{i\pi}{4}}\right)\frac{1}{(z^4+a^4)}\right]$$

$$= \frac{0}{0} \; [Apply\ L'Hospital\ rule]$$

$$= \lim_{z\to ae^{\frac{i\pi}{4}}} \frac{1}{4z^3}$$

$$= \frac{1}{4a^3 e^{\frac{3i\pi}{4}}}$$

(ii) When $z = ae^{\frac{3i\pi}{4}}$

$$[Res\, f(z)]_{z=ae^{\frac{3i\pi}{4}}} = \lim_{z\to ae^{\frac{3i\pi}{4}}} \left(z - ae^{\frac{3i\pi}{4}}\right) f(z)$$

$$= \lim_{z \to ae^{\frac{3i\pi}{4}}} \left(z - ae^{\frac{3i\pi}{4}}\right) \frac{1}{z^4+a^4}$$

$$= \frac{0}{0} \ [Apply\ L'Hospital\ rule]$$

$$= \lim_{z \to ae^{\frac{3i\pi}{4}}} \frac{1}{4z^3}$$

$$= \frac{1}{4a^3 e^{\frac{9i\pi}{4}}}$$

∴ By Cauchy's Residue theorem,

$$\int_C f(z)dz = 2\pi i\ [sum\ of\ residues]$$

$$= 2\pi i \left(\frac{1}{4a^3 e^{\frac{3i\pi}{4}}} + \frac{1}{4a^3 e^{\frac{9i\pi}{4}}} \right)$$

$$= \frac{2\pi i}{4a^3} \left(e^{\frac{-i3\pi}{4}} + e^{\frac{-i9\pi}{4}} \right)$$

$$= \frac{\pi i}{2a^3} \left(e^{-\pi i} e^{\frac{i\pi}{4}} + e^{-i2\pi} e^{-\frac{i\pi}{4}} \right) \qquad [\because\ e^{-\pi i} = -1]$$

$$= \frac{\pi i}{2a^3} \left((-1) e^{\frac{i\pi}{4}} + (-1) e^{-\frac{i\pi}{4}} \right) \qquad [\because e^{-2\pi i} = -1]$$

$$= \frac{-\pi i}{a^3} \left(\frac{e^{\frac{i\pi}{4}} - e^{-\frac{i\pi}{4}}}{2} \right) \qquad \left[\because \frac{e^{ix} - e^{-ix}}{2} = i \sin x\right]$$

$$= \frac{-\pi i}{a^3} \left(i \sin \frac{\pi}{4} \right)$$

$$= \frac{\pi}{a^3} \left(\frac{1}{\sqrt{2}} \right)$$

$$(1) \ \Rightarrow \int_o^\infty \frac{dx}{(x^4+a^4)} = \frac{1}{2} \left(\frac{\pi}{\sqrt{2} a^3} \right)$$

Example: 4.72 Evaluate $\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)^3}$

Solution:

Replacement Put $x = z \Rightarrow dx = dz$

$\therefore \int_{-\infty}^{\infty} \frac{dx}{(x^4+1)^3} = \int_C \frac{dz}{(z^4+1)^3}$ where C is the upper semi circle.

$$= \int_c f(z)dz \qquad \ldots (1)$$

Where, $f(z) = \frac{1}{(z^4+1)^3}$

To find the poles, put $(z^2 + 1)^3 = 0$

$$\Rightarrow z^2 + 1 = 0$$

$\Rightarrow z = \pm i$ are poles of order 3.

Here $z = i$ lies in the upper half of z – plane.

Find the residue for the inside pole $z = i$

$$[Res\ f(z)]_{z=i} = \frac{1}{2} \operatorname{Lim}_{z \to i} \frac{d^2}{dz^2} (z - i)^3 f(z)$$

$$= \frac{1}{2!} \operatorname{Lim}_{z \to i} \frac{d^2}{dz^2} \left[(z - i)^3 \frac{1}{(z+i)^3 (z-i)^3} \right]$$

$$= \frac{1}{2!} \operatorname*{Lim}_{z \to i} \frac{d^2}{dz^2} \left[\frac{1}{(z+i)^3}\right]$$

$$= \frac{1}{2} \lim_{z \to i} \frac{d}{dz} \left[\frac{-3}{(z+i)^4}\right]$$

$$= \frac{1}{2} \frac{12}{(2i)^5} = \frac{6}{32i} = \frac{3}{16i}$$

∴By Cauchy's Residue theorem,

$$\int_C f(z)dz = 2\pi i\,[sum\ of\ residues]$$

$$= 2\pi i \left(\frac{3}{16i}\right) = \frac{3\pi}{8}$$

$$(1) \Rightarrow \int_{-\infty}^{\infty} \frac{dx}{(x^4+1)^3} = \frac{3\pi}{8}$$

Type III

Integrals of the form

$$\int_{-\infty}^{\infty} \frac{f(x)}{g(z)} \sin(nx)dx\ (or) \int_{-\infty}^{\infty} \frac{f(x)}{g(x)} \cos(nx)dx$$

To evaluate this integral, write $\sin(nx)$ and $\cos(nx)$ in terms of e^{inx} thus,

$$\int_C \frac{f(z)}{g(z)} e^{inz} dz = \int_{-\infty}^{\infty} \frac{d(z)}{g(x)} e^{inx} dx$$

Where C is the closed curve as in type II and finally equate imaginary part or real part accordingly to get the required integral.

Problems based on Contour Integration

Example: 4.73 Evaluate $\int_o^{\infty} \frac{\cos mx}{x^2+a^2} dx, a > 0, m > 0$

Solution:

Replacement put $x = z \Rightarrow dx = dz$ and $\cos mn = R.P\ e^{imn}$

Now, $\int_o^{\infty} \frac{\cos mx}{x^2+a^2} dx = \frac{1}{2} \int_{-\infty}^{\infty} \frac{RP\ e^{imx}}{x^2+a^2} dx$

$= \frac{1}{2} \int_C \frac{RP\ e^{imz}}{z^2+a^2} dz$ where C is the upper semi circle.

$= \frac{R.P}{2} \int_c f(z)dz \qquad \ldots (1)$

Where $f(z) = \frac{e^{imz}}{z^2+a^2}$

To find the poles, put $z^2 + a^2 = 0$

$\Rightarrow z = \pm ai$ are poles of order one.

Here $z = ai$ lies in the upper half of z – plane.

Find the residue for the inside pole $z = ai$

$$[Res\, f(z)]_{z=ai} = \lim_{z \to ai} (z - ai)\, f(z)$$

$$= \lim_{z \to ai} (z - ai) \frac{e^{imz}}{(z+ai)(z-ai)}$$

$$= \lim_{z \to ai} \frac{e^{imz}}{(z+ai)}$$

$$= \frac{e^{-ma}}{2ai}$$

∴By Cauchy's Residue theorem,

$$\int_C f(z)dz = 2\pi i\ [sum\ of\ residues]$$

$$= 2\pi i\ \left(\frac{e^{-ma}}{2ai}\right)$$

$$= \frac{\pi\ e^{-ma}}{a}$$

$$(1) \implies \int_o^{\infty} \frac{\cos mx}{x^2+a^2} dx = \frac{R.P}{2}\left(\pi\ \frac{e^{-ma}}{a}\right) = \frac{\pi}{2a} e^{-ma}$$

Example: 4.74 Evaluate $\int_o^{\infty} \frac{x \sin mx}{x^2+a^2} dx$ where $a > 0, m > 0$

Solution:

Replacement $put\ x = z\ \implies dx = dz\ and \sin(mx) = IP\ e^{imx}$

Now, $\int_O^{\infty} \frac{x \sin mx}{x^2+a^2} dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{xIP\ e^{imx}}{x^2+a^2} dx$

$= \frac{I.P}{2}\int_c \frac{z\ e^{imz}}{z^2+a^2} dz$ where C is the upper semi circle.

$= \frac{I.P}{2}\int_c f(z)dz$ ... (1)

Where, $f(z) = \frac{z\ e^{imz}}{z^2+a^2}$

To find the poles, put $f(z)$, put $z^2 + a^2 = 0$

$\Rightarrow z = \pm ai$ are poles of order one.

Here $z = ai$ lies in the upper half of z – plane.

Find the residue for the inside pole $z = ai$

$$[Res\ f(z)]_{z=ai} = \lim_{z\to ai}(z - ai)\ f(z)$$

$$= \lim_{z\to ai}(z - ai)\frac{ze^{imz}}{(z+ai)(z-ai)}$$

$$= \frac{(ai)e^{-ma}}{2ai} = \frac{e^{-ma}}{2}$$

∴By Cauchy's Residue theorem,

$$\int_C f(z)dz = 2\pi i\ [sum\ of\ residues]$$

$$= 2\pi i\ \left(\frac{e^{-ma}}{2}\right) = \pi\ e^{-ma}$$

$$(1) \implies \int_o^{\infty} \frac{x \sin mx}{x^2+a^2} dx = \frac{I.P}{2}(\pi i\ e^{-ma}) = \frac{\pi}{2} e^{-ma}$$

Example: 4.75 Evaluate $\int_{-\infty}^{\infty} \frac{\cos x\ \ dx}{(x^2+a^2)(x^2+b^2)}, a > b > 0$

Solution:

Replacement $put\ n = z\ \Rightarrow dz = dz \cos x = R.P\ \ e^{ix}$

Now, $\int_{-\infty}^{\infty} \frac{\cos x}{x^2+a^2}\frac{dx}{(x^2+b^2)} = \int_c \frac{RP\ e^{iz}\ \ dz}{(z^2+a^2)(z^2+b^2)}$

where C is the upper semi circle.

$$= \frac{R.P}{2}\int_c \ f(z)dz$$

Where, $f(z) = \frac{e^{iz}}{(z^2+a^2)(z^2+b^2)}$

To find the poles, put $f(z)$, put $(z^2 + a^2)(z^2 + b^2) = 0$

$\Rightarrow z = \pm ai, \pm bi$ are poles of order one here $z = ai, bi$ lies in the upper half of z – plane.

To find the residue for the inside pole

(i) when $z = ai$

$$[Res\ f(z)]_{z=ai} = \lim_{z\to ai}(z - ai)\ f(z)$$

$$= \lim_{z\to ai}(z - ai)\frac{e^{iz}}{(z+ai)(z-ai)(z^2+b^2)}$$

$$= \lim_{z\to ai}\frac{e^{iz}}{(z+ai)(z^2+b^2)}$$

$$= \frac{e^{-a}}{(2ai)(b^2-a^2)} = \frac{-e^{-a}}{(2ai)(a^2-b^2)}$$

(ii) when $z = bi$

$$[Res\ f(z)]_{z=bi} = \lim_{z\to bi}(z - bi)\ f(z)$$

$$= \lim_{z\to bi}(z - bi)\frac{e^{iz}}{(z^2+a^2)(z+bi)(z-bi)}$$

$$= \lim_{z\to bi}\frac{e^{iz}}{(z^2+a^2)(z+bi)}$$

$$= \frac{e^{-b}}{2bi(a^2-b^2)}$$

∴By Cauchy's Residue theorem,

$$\int_C \ f(z)dz = 2\pi i\ [sum\ of\ residues]$$

$$= 2\pi i\left(\frac{e^{-a}}{(2ai)(a^2-b^2)} + \frac{e^{-b}}{(2bi)(a^2-b^2)}\right)$$

$$= \frac{2\pi i}{(2i)(a^2-b^2)}\left[\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right]$$

$$= \frac{\pi}{(a^2-b^2)}\left[\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right]$$

$$(1) \implies \int_{-\infty}^{\infty}\frac{\cos x\ dx}{(x^2+a^2)(x^2+b^2)} R.P.\frac{\pi}{a^2-b^2}\left(\frac{ae^{-b}-be^{-a}}{ab}\right) = \frac{\pi}{ab\ (a^2-b^2)}(ae^{-b} - be^{-a})$$

Exercise: 4.4

Evaluate the following integrals

1) $\int_{-\infty}^{\infty}\frac{x^2}{(x^2+1)(x^2+4)}$ **Ans:** $\frac{\pi}{3}$

2) $\int_0^{\infty}\frac{x^2\ dx}{(x^2+9)(x^2+4)}$ **Ans:** $\frac{\pi}{10}$

3) $\int_0^{\infty}\frac{dx}{(x^2+1)^2}$ **Ans:** $\frac{\pi}{4}$

4) $\int_0^{\infty}\frac{\cos 3x\ dx}{(x^2+1)(x^2+4)}$ **Ans:** $\frac{\pi}{6}\left[\frac{1}{e^3} - \frac{e}{2e^6}\right]$

5) $\int_0^\infty \frac{\cos 3x \ dx}{(x^2+1)(x^2+4)}$ **Ans:** $\frac{\pi}{6}\left[\frac{1}{e^3} - \frac{e}{2e^6}\right]$

6) $\int_0^\infty \frac{x^2}{(x^2+1)} dx$ **Ans:** $\frac{\pi}{4}$

7) $\int_0^\infty \frac{dx}{(x^2+a^2)^3}$ **Ans:** $\frac{3\pi}{16a^5}$

8) $\int_0^\infty \frac{dx}{x^4+10x^2+9}$ **Ans:** $\frac{\pi}{24}$

9) $\int_{-\infty}^\infty \frac{\cos 3x}{(x^2+1)} dx$ **Ans:** $\frac{2\pi}{e^3}$

10) $\int_0^\infty \frac{x \sin x}{x^2+1} dx$ **Ans:** $\frac{\pi}{2e}$

11) $\int_{-\infty}^\infty \frac{\sin x}{x^4+4x+5} \ dx$ **Ans:** $\frac{-\pi \sin 2}{e}$

12) $\int_0^\infty \frac{2x^2-1}{x^4+5x^2+4} dx$ **Ans:** $\frac{\pi}{4}$