SOMAIYA VIDYAVIHAR UNIVERSITY

Knowledge Alone Liberates · Somaiya Vidyavihar

Batch: A-2 Roll No.: 16014022050
Name: Ketaki Mahajan
Course: EXCP
~~Experiment / assignment~~ / tutorial No. 12
Grade: Signature of the Faculty with date

TUTORIAL 12 : Complex Integration

TYPE I : Line Integral

Q.1. $\int_0^{3+i} z^2 \, dz$

(i) Along real axis from 0 to 3 and then vertically to 3+i.

Here C consists of lines $OA = C_1$ & $AP = C_2$.

$$I = \int_C (x+iy)^2 (dx+idy)$$

$$= \int_{C_1} (x+iy)^2 (dx+idy) + \int_{C_2} (x+iy)^2 (dx+idy)$$

C_1 : $y=0$, $dy=0$; $x : 0 \to 3$

C_2 : $x=3$, $dx=0$; $y : \cancel{x}^{0} \to 1$

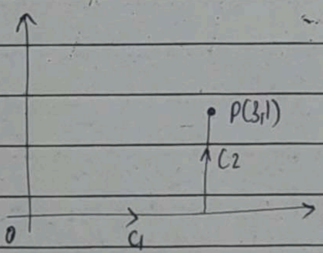
$$\int_{C_2} (x+iy)^2 (dx+idy) = \int_{y=0}^{1} \cancel{(3+iy)^2}^{x^2} \, \cancel{idy}^{dx} = \left[\frac{x^3}{3}\right] = 9 \quad \text{— (i)}$$

$$\int_{C_2} (x+iy)^2 (dx+idy) = \int_{y=0}^{1} (3+iy)^2 \, idy$$

$$= \int_0^1 (9+6iy-y^2) \, idy$$

$$= \int_0^1 (9i-6y-iy^2) \, dy = 9iy - 3y^2 - \frac{i}{3}y^3 \Big|_0^1$$

$$= 9i - 3 - \frac{i}{3} \quad \text{— (ii)}$$

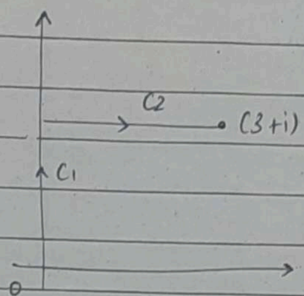


From (i) & (ii), $I = 9 + 9i - 3 - i/3 = 6 + \frac{26i}{3}$

(ii) Along the imaginary axis from 0 to i & then horizontally to 3+i.

Here, C consists of line $OB = C$ & $BP = C_2$.

Q.1. (ii). $I = \int_C (x+iy)^2 (dx+idy) = \int_{C1} (x+iy)^2 (dx+idy) + \int_{C2} (x+iy) (dx + idy)$



$C_1 : x=0, dx=0 \; ; \; y: 0 \longrightarrow 1$

$C_2 : y=1, dy=0 \; ; \; x: 0 \longrightarrow 3.$

$\int_{C1} (x+iy)^2 (xd+idy) = \int_{y=0}^{1} (iy)^2\, i dy = \int_0^1 i^3 y\, dy = -i\left[\frac{y^2}{3}\right]_0^1$

$= \frac{-i}{3}$ —(i).

$\int_{C2} (x+iy)^2 (dx+idy) = \int_{x=0}^{3} (x+i)^2 (dx) = \int_0^3 (x^2+2ix-1)\, dx$

$= \left[\frac{x^3}{3} + ix^2 - x\right]_0^3 = 9+9i-3 = 6+9i$ —(ii)

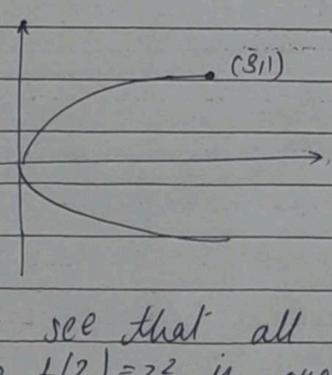
From (i) & (ii), $I = \frac{-i}{3} + 6+9i = 6+\frac{26i}{3}$

(iii). Along parabola $x = 3y^2$.

$dx = 6y\, dy$

$I = \int_{y=0}^{1} (3y^2+iy)^2 (6y dy + idy) = \int_0^1 (9y^4 + i6y^3 - y^2)(6y+i)\, dy$

$= \int_0^1 (54y^5 + i36y^4 - 6y^3 - 19y^4 - 6y^3 - iy^2)\, dy$



$= 9[y^6]_0^1 - 3[y^4]_0^1 + i9[y^5]_0^1 - i/3\,[y^3]_0^1$

$= 9 - 3 + i9 - i/3 = 6 + i26/3$

We can see that all integrals along different path is same. This is because $f(z) = z^2$ is analytic function. Integral of analytic function is path independent.

②

SOMAIYA VIDYAVIHAR UNIVERSITY

Batch: A-2 Roll No.: 16014022050

Name: Ketaki Mahajan

Course: CASOT

Experiment / assignment / tutorial No. 12

Grade: Signature of the Faculty with date

Q2. Show $\int_C \log z \, dz = 2\pi i$, C is unit circle in z-plane

Equation of circle with centre at origin & radius = 1,

Assuming C is around anticlockwise,

$\theta : 0 \to 2\pi$.

$$I = \int_0^{2\pi} \log e^{i\theta} \cdot ie^{i\theta} d\theta$$

$$= \int_0^{2\pi} i\theta \cdot ie^{i\theta} d\theta$$

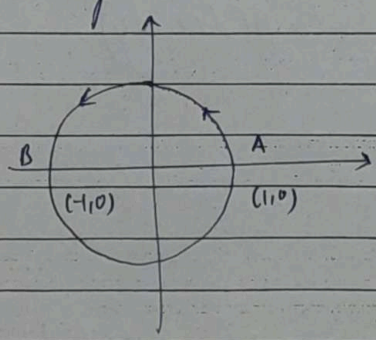
$$= -\int_0^{2\pi} \theta \cdot e^{i\theta} d\theta$$

$$= -\left[\theta \frac{e^{i\theta}}{i} - \frac{e^{i\theta}}{(-1)}\right]_0^{2\pi} = -\left[-i2\pi e^{2\pi i} + e^{2\pi i} - i\right]$$

$$= 2\pi i$$

Hence proved that $\int \log z \, dz = 2\pi i$ where C is unit circle in z plane.

Q.3. Evaluate $\int_C (z^2 + 3z^{-4}) \, dz$, where C is upper half of unit circle from (1,0) to (-1,0).



C is circle $|z| = 1$ from $A(1,0)$ to $B(-1,0)$.

The equation of circle is $z = e^{i\theta}$.

$\therefore dz = ie^{i\theta} d\theta$, $0 \le \theta \le \pi$

$$I = \int_C z^2 + 3z^{-4} \, dz$$

$$I = \int_0^{\pi} (e^{2i\theta} + 3e^{-4i\theta})\, ie^{i\theta}\, d\theta = i\int_0^{\pi} (e^{3i\theta} + 3e^{-3i\theta})\, d\theta$$

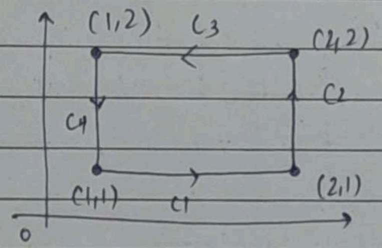
$$= i\left[\frac{e^{3i\theta}}{3i} + \frac{3e^{-3i\theta}}{-3i} \right]_0^{\pi}$$

$$= \left[\frac{e^{3i\theta}}{3} - e^{-3i\theta} \right]_0^{\pi} = \frac{e^{3\pi i}}{3} - e^{3\pi i} - \frac{1}{3}$$

$$= \left[\frac{-1}{3} - \left(\cancel{\frac{1}{3}}\right) - \cancel{\frac{1}{3}}\right) \quad \ldots\ldots \quad \left\{ - \left[e^{3\pi i} - e^{-3\pi i} = -1 \right] \right\}$$

$$= -\frac{1}{3}$$

Q.4. $\int f(z)\, dz$ along the square whose vertices are $(1,1)$, $(2,1)$, $(2,2)$, $(1,2)$ anticlockwise where $f(z) = x - 2iy$.



C_1: $y = 1$, $dy = 0$; $x: 1 \to 2$
C_2: $x = 2$, $dx = 0$; $y: 1 \to 2$
C_3: $y = 2$, $dy = 0$; $x: 2 \to 1$
C_4: $x = 1$, $dx = 0$; $y: 2 \to 1$.

$$\int_C f(z)\, dz = \int_C (x - 2iy)(dx + i\,dy)$$

$$= \int_{C_1} (x-2iy)(dx+i\,dy) + \int_{C_2} (x-2iy)(dx+i\,dy) + \int_{C_3} (x-2i)(dx+i\,dy) + \int_{C_4} (x-2iy)(dx+i\,dy)$$

$$\int_{C_1} (x-2iy)(dx+i\,dy) = \int_1^2 (x-2i)\, dx$$

$$= \left[\frac{x^2}{2} - 2ix \right]_1^2 = 2 - 4i - \left(\frac{1}{2} - 2i\right) = \frac{3}{2} - 2i$$

③

SOMAIYA VIDYAVIHAR UNIVERSITY

Batch: A-2 Roll No.: 16014022050
Name: Ketaki Mahajan
Course: CASOT
Experiment / ~~assignment~~ / tutorial No. 12
Grade: Signature of the Faculty with date

Q.4. $\int_{C_2} (x-2iy)(dx+idy) = \int_1^2 (2i-2y)\,dy = i\,[2y-iy^2]_1^2$

$= i\,[4-4i+i-2]$

$= i[2-3i] = 3+2i$

$\int_{C_3} (x-2iy)(dx+idy) = \int_1^2 (x-4i)\,dx = \left[\frac{x^2}{2}-4ix\right]_2^1$

$= \frac{1}{2}-4i-(2-8i)$

$= \frac{-3}{2}+4i$

$\int_{C_4} (x-2iy)(dx+idy) = \int_2^1 (1-2iy)\,i\,dy = \int_2^1 (i+2y)\,dy$

$= [iy]_2^1 + [y^2]_2^1$

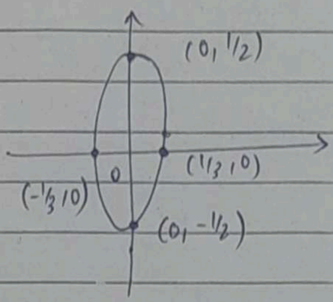
$= (i-2i)-3 = -3-i$

$I = \frac{3}{2} - 2i + 3 + 2i + 4i - 3 - i = 3i$

$\therefore \int f(z)\,dz = 3i$

TYPE 2: CAUCHY'S LINE INTEGRAL THEOREM, CAUCHY'S INTEGRAL FORMULA.

Q.5. $\int_c \frac{1}{z} \cos z \, dz$, C is ellipse $9x^2 + 4y^2 = 1$



$\left(\frac{x}{1/3}\right)^2 + \left(\frac{y}{1/2}\right)^2 = 1$

$z=0$ is singular point & it lies inside C.

$f(z) = \cos z$, $z_0 = 0$.

By Cauchy Integral formula,

$\oint \frac{f(z)}{z-z_0} dz = 2\pi i f(z_0)_{z=0}$

$= 2\pi i (\cos 0) \Rightarrow \therefore \int_c \frac{\cos z}{z} dz = 2\pi i$ //

Q.6. $\int_c \frac{e^{3z}}{z-i} dz$ where C is curve $|z-2| + |z+2| = 6$.

Now $|z-2| + |z+2| = 6$ is an ellipse with foci $= (2,0)$ & $(-2,0)$

major axis $= 6$

Semi major axis $b^2 = \left(\frac{k}{2}\right)^2 - c^2 = \left(\frac{6}{2}\right)^2 - (2^2) = 5$

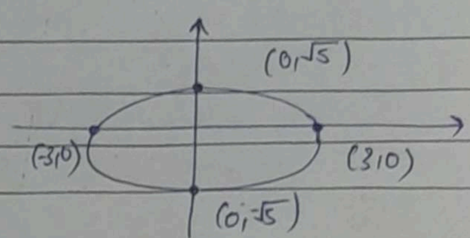
$\therefore b = \sqrt{5}$.

$z=i$ is singular point & lies inside C.

$z = e^{3z} = e^{3x+i3y}$

$= e^{3x} e^{i3y}$

$= e^{3x} (\cos 3y + i \sin 3y)$



(4)

SOMAIYA VIDYAVIHAR UNIVERSITY

Batch: A-2 Roll No.: 16014022050

Name: Ketaki Mahajan

Course: CASOT

Experiment / assignment / tutorial No. 11

Grade: Signature of the Faculty with date

Q.6. $\because u = e^{3x}\cos 3y$ & $V = e^{3x}\sin 3y$.

$u_x = \dfrac{e^{3y}\cos 3y}{3}$, $v_x = \dfrac{e^{3x}\sin 3y}{3}$

$u_y = -\dfrac{e^{3x}\sin 3y}{3}$, $v_y = \dfrac{e^{3x}\cos 3y}{3}$

$V_x = V_y$ & $-V_x = u_y$

$\therefore$ CR equations are satisfied.

$\therefore$ z' is an analytic function. By cauchy Integral formula,

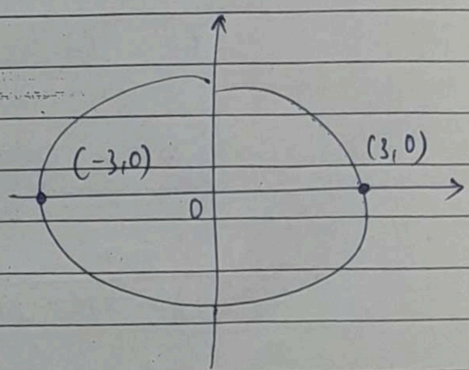
$$\int \frac{e^{3z}}{z-1}\,dz = 2\pi i f(z_0) = 2\pi i\,[e^{3z}]_{z=i}$$

$$= 2\pi i e^{3i}$$

Q.7. $\displaystyle\int_c \frac{e^{2z}}{(z-1)(z-2)}\,dz$, c is the circle $|z| = 3$.

$z=1$ & $z=2$ are singular points & both lies inside the c.

$$\frac{1}{(z-1)(z-2)} = \frac{A}{(z-1)} + \frac{B}{(z-2)}$$



$A(z-2) + B(z-1) = 1$

Putting $z = 2 \implies \therefore B = 1$

$z = 1 \implies \therefore A = -1$

Q.7. $\therefore \quad \dfrac{1}{(z-1)(z-2)} = \dfrac{1}{z-2} - \dfrac{1}{z-1}$

$$\int_c \frac{e^{2z}}{(z-1)(z-2)}\, dz = \int_c \frac{e^{2x}}{z-2}\, dx - \int_c \frac{e^{2x}}{z-1}\, dz$$

$\therefore$ By cauchy integral formulla;

$$= 2\pi i\, f(z_0) - 2\pi i\, f(z_0) = 2\pi i\left[e^{2z}\right]_{z_0=2} - 2\pi i\left[e^{2z}\right]_{z_0=1}$$

$$= 2\pi i e^{4} - 2\pi i e^{2}$$
$$= 2\pi i e^{2}(e^{2}-1)$$

Q.8. If $f(z) = z^3 + iz^2 - 4z - 4i$, evaluate $\int_c \dfrac{f'(z)}{f(z)}\, dz$, where, c is a simple closed curve enclosing zeros of $f(z)$.

$f(z) = z^3 + iz^2 - 4z - 4i$

$f'(z) = 3z^2 - 2iz - 4$

$$I = \int_c \frac{3z^2 + 2iz - 4}{z^3 + iz^2 - 4z - 4i}\, dz$$

$$= \int_c \frac{3z^2 + iz - 4}{(z+i)(z+2)(z-2)}\, dz$$

$z = i$, $z = 2$, $z = -2$ are singular points all lie inside c.

$$\frac{1}{(z+i)(z+2)(z-2)} = \frac{A}{(z+i)} + \frac{B}{(z+2)} + \frac{C}{(z-2)}$$

$$1 = A(z+2)(z-2) + B(z+i)(z-2) + C(z+i)(z+2)$$

⑥

SOMAIYA VIDYAVIHAR UNIVERSITY

Batch: A2 Roll No.: 16014022050

Name: Ketaki Mahajan

Course: CASOT

~~Experiment / assignment~~ / tutorial No. 12

Grade: Signature of the Faculty with date

Q8. Putting $z = 2$, $C = \frac{1}{4(2+i)}$

$z = -2$, $B = \frac{1}{4(2-i)}$

$z = -i$, $A = \frac{-1}{5}$

$$I = \frac{-1}{5}\int \frac{3z^2+iz-4}{z+i}\,dz + \frac{1}{4(2-i)}\int \frac{3z^2+iz-4}{z+2}\,dz$$

$$+ \frac{1}{4(2+i)}\int \frac{3z^2+iz-4}{z-2}\,dz$$

$\therefore$ By cauchy's integral formula,

$$I = \frac{-2\pi i}{5}\left[3z^2+iz-4\right]_{z=-i} + \frac{2\pi i}{4(2-i)}\left[3z^2+iz-4\right]_{z=-2}$$

$$+ \frac{2\pi i}{4(2+i)}\left[3z^2+iz-4\right]_{z=2}$$

$$= \frac{-2\pi i}{5}(-9) + \frac{2\pi i}{\cancel{4(2-i)}}\cancel{(8-4i)} + \frac{2\pi i}{\cancel{4(2+i)}}\cancel{4(2+i)}$$

$$= \frac{18\pi i}{5} + 2\pi i + 2\pi i = \frac{38\pi i}{5}$$

$$\therefore \int \frac{f'(z)}{f(z)} = \frac{38\pi i}{5}$$

Q9. $\int_c \frac{\sin^6 z}{(z-\pi/6)^3}\,dz$ where c is $|z|=1$.

Singular point is $z = \pi/6$.

Q.9. $z=\frac{\pi}{6}$ is inside the circle.

$f(z)=\sin^6 z,\ z_0=\frac{\pi}{6},\ n=3.$

By Cauchy's Integral Theorem,

$$\int \frac{\sin^6 z}{(z-\pi/6)^3}\,dz = \frac{2\pi i}{2!}\cdot f''(z_0) = \pi i f''(z_0)$$

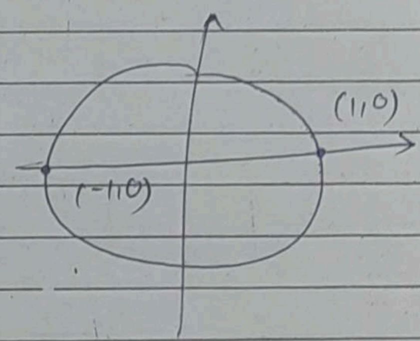
$f(z)=\sin^6 z,\quad f'(z)=6\sin^5 z\cos z.$

$$f''(z)=6[\sin^5 z(-\sin z)+\cos z\,5\sin^4 z\cos z]$$

$$=6[5\sin z\cos^2 z-\sin^6 z]$$

$$f''(z_0)=6[5\sin 30\cos^2 30-\sin^6 30] = 6\left[5\times\frac{1}{2}\times\left(\frac{\sqrt{3}}{2}\right)^2-\left(\frac{1}{2}\right)^6\right]$$

$$=\frac{6}{8}\left[15-\frac{1}{8}\right] = \frac{357}{32}$$

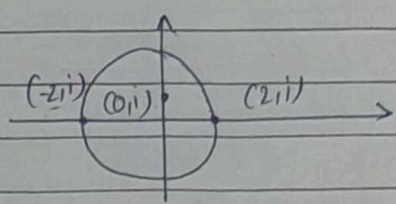


$$\therefore \int \frac{\sin^6 z}{(z-\pi/6)^3}\,dz = \pi i\times\frac{714}{64}$$

$$=\frac{357\pi i}{32}$$

Q10. $\int_c \frac{z-1}{(z+1)^2(z-2)}\,dz$ where c is $|z-i|=2$.

$z=-1,\ 1$ are singular points.



$z=2$, lies outside C.

$z=-1$, lies inside of C.

SOMAIYA VIDYAVIHAR UNIVERSITY

Batch: A-2 Roll No.: 16014022050

Name: Ketaki Mahajan

Course: CASOT

Experiment / ~~assignment~~ / tutorial No. 12

Grade: Signature of the Faculty with date

Q.10. $\displaystyle\int \frac{(z-1)(z-2)}{(z+1)^2}\,dz$

here, $f(z) = \dfrac{z-1}{z-2}$, $z_0 = -1$, $n = 2$.

By cauchy integral formula,

$$\oint \frac{(z-1)(z-2)}{(z+1)^2} = \frac{2\pi i}{2!} f'(z_0) = 2\pi i f'(z_0)$$

$$f'(z) = \frac{(z-2)(1)-(z-1)(1)}{(z-2)^2} = \frac{(\cancel{z}-2)-(\cancel{z}+1)}{(z-2)^2} = \frac{-1}{(z-2)^2}$$

$$f'(z_0) = \frac{(-1)}{(-1-2)^2} = \frac{-1}{9}$$

$$\therefore \oint \frac{(z-1)(z-2)}{(z+1)^2} = \frac{-2\pi i}{9}$$

Q.11. $\displaystyle\int_C \frac{z e^{2z}}{(z-1)^3}\,dz$ where c is $|z+i| = 2$.

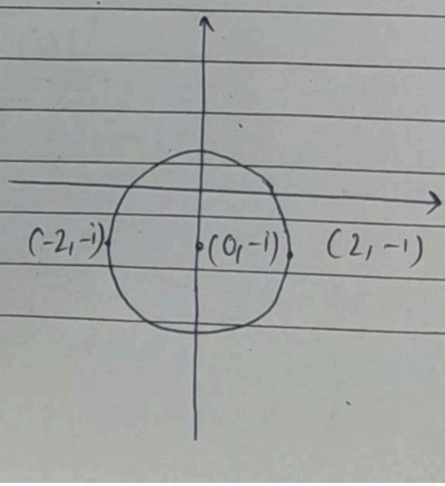
$z = 1$ is singular point & it lies inside c.

$z_0 = 1$, $f(z) = z e^{2z}$.

$f'(z) = e^{2z} + 2e^{2z}$

$f''(z) = 2e^{2} + 2e^{2z} + 22z e^{2z}$

$= 4e^{2z} + 4z e^{2z}$



Q.11. By Cauchy's Integral Theorem,

$$\oint \frac{ze^{2z}}{(z-1)^3} = \frac{2\pi i}{2!} \cdot f''(z_0)$$

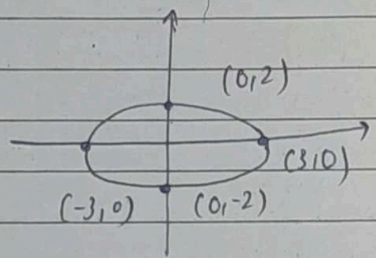
$$= \pi i \left[4e^{2z} + 4ze^{2z} \right]_{z_0=1} = \cancel{4}\, \pi i \left[e^2 \cdot 4 + 4e^2 \right]$$

$$= 8e^2 \pi i \;//$$

Q.12. $f(z) = \oint_c \frac{4z^2+z+4}{z-7}$ where C is ellipse $4x^2+9y^2=36$.

$$4x^2+9y^2 = 36$$

$$\frac{4x^2+9y^2}{36} = 1 \longrightarrow \frac{x^2}{9} + \frac{y^2}{4} = 1$$



CASE I : If z is outside C, then, $\frac{4z^2+z+4}{z-z}$ is analytic inside & on C.

$\therefore f(z) = 0 \;//$.

NOW, let $\phi(z) = 4z^2+z+4$.

CASE II : If z is inside c, then, $f(z) = 2\pi i \,\phi(z)$.

(a). $z=4$ is outside C

$\therefore f(z) = 0 \;/\!/$

(b). $z = i$ is inside C

$\therefore f(i) = 2\pi i (4(i)^2 + i + 4)$

$= 2\pi i (i) = -2\pi \;//$

(d). $z=-1$ is inside c.

$\therefore f'(z) = 2\pi i\,(8z+1)$

$f'(-1) = 2\pi i\,(8(-1)+1)$

$= -14\pi i \;/\!/$

(e). $z=-1$ is inside C

$f''(z) = 2\pi i\,(8)$

$\therefore f''(-i) = 16\pi i \;//$