DAM (SEM III)
Module 3

Relations and Functions

UNIT NO :3.2

Relations

Cartesian product

Consider two Non-empty sets X and Y.

The set of all ordered pairs (x,y) where x ∈ X and y ∈ Y is called the **Cartesian product**, of X and Y.

it is denoted by $X \times Y$, which is read "X cross Y."

Definition

$$X \times Y = \{(x, y) \mid x \in X \text{ and } y \in Y\}$$

EXAMPLE

Let $X = \{1, 2\}$ and $Y = \{10, 15, 20\}$. Then write $X \times Y$, $Y \times X$, $X \times X$

$X \times Y$
$= \{(1, 10), (1, 15), (1, 20), (2, 10), (2,15), (2, 20)\}$
$Y \times X$
$= \{(10, 1), (15, 1), (20, 1), (10, 2), (15, 2), (20, 2)\}$

Also, $X \times X = \{(1, 1), (1, 2), (2, 1), (2, 2)\}$

Definition : Relation

A **relation** from a set X to a set Y is any subset of the Cartesian product $X \times Y$**,** those pair (x,y) which are related with each other.

A relation can be stated as a rule (Infinite sets) or can be given as set of ordered pairs (Finite)

EXAMPLE

Let $X = \{1,2\}$ and $Y = \{10,15,20\}$.

And we can define R ={(1,10) (2,20)}

Terminologies

The set of first components in the ordered pairs is called the **domain** of the relation and the set of second components is called the **range** of the relation.

For $\mathrm{X} = \{1, 2\}, \mathrm{Y} = \{10, 15, 20\}$ and

R ={(1,10) (2,20)}

Domain of R ={1,2}

Range of R ={10,20}

Terminologies

Suppose R is a relation from X to Y. Then R is a set of ordered pairs where each first element comes from X and each second element comes from Y. That is, for each pair $x \in X$ and $y \in Y$, exactly one of the following is true:

i. $(x,y) \in R$; we then say "x is R – related to y", written xRy.

ii. $(x,y) \notin R$; we then say "x is not R – related to y", written $x \not R y$

Examples : Relation

1. $A = \{1, 2, 3, 4\}$ Then write R as ordered pairs if relation R "is less than" i.e. $aRb\ if\ a < b$.

R = {(1,2), (1,3), (1,4), (2,3), (2,4), (3,4)}

2. Let A is z^+(set of positive integers) and R defined as "devides" i.e. $aRb\ if\ a\ divides\ b$.

   e.g. 3 R 15, 7 R 35 etc.

3. $A = \{2, 3, 4, 5, 6\}$ relation defined by $aRb\ if$ $|a - b|$ is divisible by 3, write R as set

R = {(2,5), (5,2), (3,6), (6,3)}

(For finite – ordered pair, For infinite – rule)

Definition : Inverse of R

Let R be any relation from a set A to set B.
The **inverse** of R, denoted by R^{-1}, is the relation from B to A which consists of those ordered pairs, when reversed, belong to R.
That is: $R^{-1} = \{(b, a) : (a, b) \in R\}$

REPRESENTATION OF RELATIONS:

Matrix of a Relation (M_R)

Matrices can be easily used to represent relation

EXAMPLE: For A = {1, 2, 3, 4} and B = {a, b, c}

If R = {(1,x), (2,x), (3,y), (3,z)} then matrix of R, M_R is

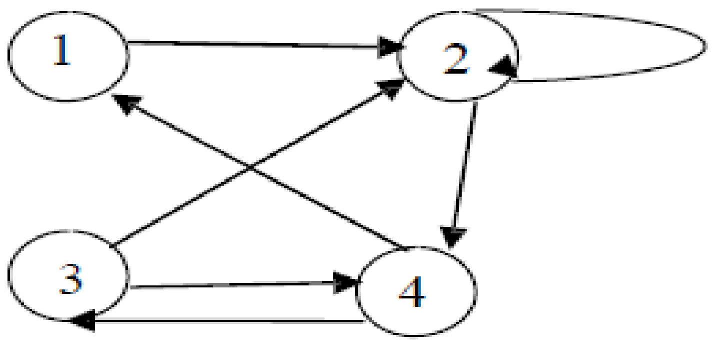
$$
\begin{array}{c|ccc}
 & x & y & z \\
\hline
1 & 1 & 0 & 0 \\
2 & 1 & 0 & 0 \\
3 & 0 & 1 & 1 \\
4 & 0 & 0 & 0
\end{array}
$$

REPRESENTATION OF RELATIONS:

Digraph: Another way of pictorial representation is **diagraph**. i.e. Directed Graph

For A = {1, 2, 3, 4} and

R = {(1, 2), (2, 2), (2, 4), (3, 2), (3, 4), (4, 1), (4, 3)} Then, the diagraph of R is drawn as follows:



The directed graphs are very important data structures that have applications in Computer Science (in the area of networking).

Composite Relation

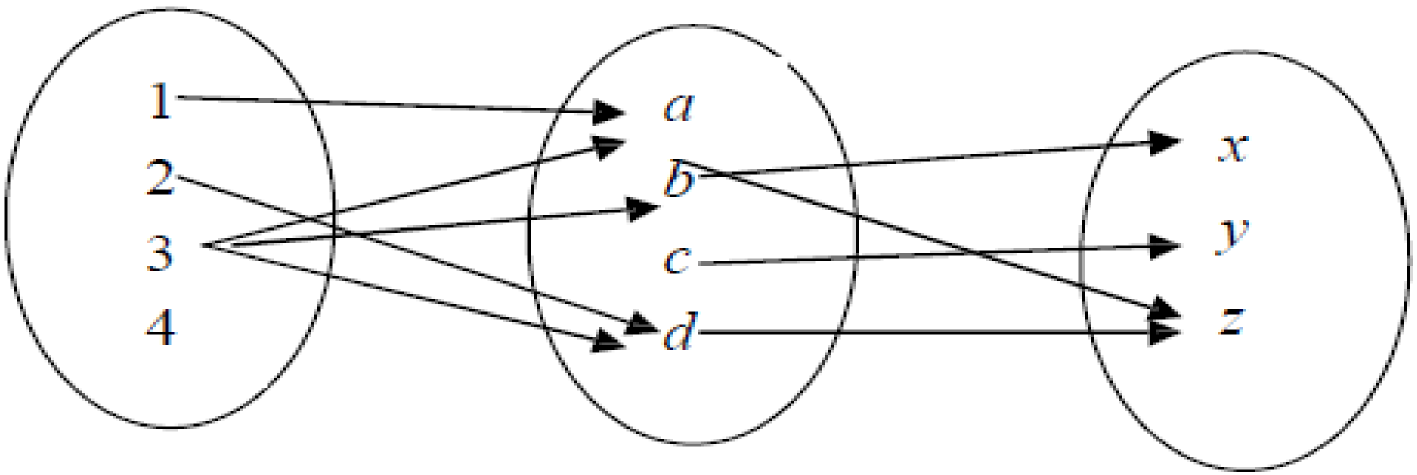
Let A, B and C be three sets.

Let R be a relation from A to B and S be a relation from B to C.

Then, composite relation **SoR** is a relation from A to C defined by,

a(**SoR** c, if there is some b ∈ B, such that $a\ R\ b$ and $b\ S\ c$.

Example : Let $A = \{1, 2, 3, 4\}$, $B = \{a, b, c, d\}$, $C = \{x, y, z\}$ and let $R = \{(1, a), (2, d), (3, a), (3, b), (3, d)\}$ and $S = \{(b, x), (b, z), (c, y), (d, z)\}$. Write SOR



SoR will be given as below.

SoR $= \{(2, z), (3, x), (3, z)\}$.

Properties of Relation
Reflexive Relation

Let A be a nonempty set, a relation R on A is said to be reflexive if for each a ∈ A, (a, a) ∈ R.

Example

Let A = {a, b, c, d} and R be defined as follows:

$R = \{(a, a), (a, c), (b, a), (b, b), (c, c), (d, c), (d, d)\}$.

Is R a reflexive relation ?

YES

Properties of Relation

Symmetric Relation

Let A be a nonempty set, a relation R on A is said to be symmetric if for each pair of elements a, b ∈ A,

(a, b) ∈ R implies (b, a) ∈ R.

Example

Let *A* = {1, 2, 3, 4} and R be defined as:

R = {(1, 2), (2, 3), (2, 1), (3, 2), (3, 3)},

Is *R* a symmetric relation ?

YES

Observations

If we draw a diagraph of a reflexive relation, then all the vertices will have a loop.

Also if we represent reflexive relation using a matrix, then all its diagonal entries will be 1.

Also if we represent symmetric relation using a matrix then the matrix will be symmetric matrix

Properties of Relation

Anti-Symmetric Relation

Let A be a nonempty set,

A relation R on A is said to be anti-symmetric,

if $a\ R\ b$ and $b\ R\ a$, then $a = b$, for every $a, b \in A$

Thus, R is not anti-symmetric if there exists $a, b \in A$ such that $a\ R\ b$ and $b\ R\ a$ but $a \neq b$.

If R is not symmetric or Anti-symmetric then it is called asymmetric

Example

Example1: Let $A = \{a, b, c, d\}$

R be defined as: $R = \{(a, b), (b, a), (a, c), (c, d), (d, b)\}$.

Check whether R is symmetric or anti-symmetric ?

R is not symmetric, as $a\ R\ c$ but c R a .

R is not anti-symmetric, because $a\ R\ b$ and $b\ R\ a$, but $a \neq b$,
Hence R is asymmetric

Example2: The relation "less than or equal to (≤)", on set of real is an anti- symmetric relation

Because If $a \le b\ and\ b \le a\ then\ a = b$

Check relation "is subset of (⊆)", on set of all subsets of A is an anti- symmetric relation

Properties of Relation

Transitive Relation

Let A be a nonempty set, a relation R on A is said to be transitive if for each triplet of element

$a, b, c \in A$, If $(a, b), (b, c) \in R \Longrightarrow (a,c) \in R$.

Example

Relation "*a* divides *b*", on the set of integers, is a transitive relation.

If a|b and b|c then a|c

The relation "less than or equal to (≤) Or (≥)", on set of real numbers is a transitive relation.

If $a \leq b \; and \; b \leq c \; then \; a \leq c$

Properties of Relation

Partial order Relation

A relation *R* on the set *A* is said to be ***partial order relation***, if it is reflexive, anti-symmetric and transitive.

Example : Let $A = \{a, b, c, d, e\}$.

Relation R, represented using following matrix

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Is R partial order relation ?

ANS :Yes

Example : Let A be a set of natural numbers and relation R be "less than or equal to relation ($\leq$)". Then R is a partial order relation on A.

Answer :

For any $m, n, k \in N$,

$n \leq n$ (reflexive);

if $m \leq n$ and $n \leq$ m, then $m = n$ (anti-symmetric);

lastly, if $m \leq n$ and $n \leq k$, then $m \leq k$ (transitive)

Properties of Relation

Equivalence Relation

Let A be a nonempty set.

A relation R on set A is said to be equivalence relation if R is reflexive, symmetric and transitive

Example : Consider the set *L* of lines in the Euclidean plane. Two lines in the plane are said to be related, if they are parallel to each other.

Is this relation an equivalence relation?

Yes .

R = $\{(L_1, L_2)$: L_1 is parallel to $L_2\}$

R is reflexive as any line L_1 is parallel to itself i.e., $(L_1, L_1) \in R$.

Now,

Let $(L_1, L_2) \in R$.

$\Rightarrow L_1$ is parallel to L_2.

$\Rightarrow L_2$ is parallel to L_1.

$\Rightarrow (L_2, L_1) \in R$

$\therefore$ R is symmetric.

Now,

Let $(L_1, L_2), (L_2, L_3) \in R$.

$\Rightarrow L_1$ is parallel to L_2. Also, L_2 is parallel to L_3.

$\Rightarrow L_1$ is parallel to L_3.

$\therefore$R is transitive.

Hence, R is an equivalence relation.

Example: Determine whether the relation R on a set A is reflexive, symmetric, antisymmetric, transitive, equivalence or partial order.

A = set of all positive integers, a R b iff $|a-b| \leq 2$

R is reflexive because $|a-a| = 0 < 2, \forall\, a \in A$

R is symmetric because $|a-b| \leq 2 \Rightarrow |b-a| \leq 2 \;\; \therefore a\,R\,b \Rightarrow b\,R\,a$

R is not antisymmetric because $1\,R\,2$ & $2\,R\,1$ $1\,R\,2 \Rightarrow |1-2| \leq 2$ &

$2\,R\,1 \Rightarrow |2-1| \leq 2$. But $1 \neq 2$

R is not transitive because 5 R 4, 4 R 2 but 5 $\not{R}$ 2

Since it is Not transitive it can not be Partial order or equivalence relation

Terminologies

Congruence

Let *m* be a fixed positive integer.

Two integers, *a*, *b* are said to be congruent modulo *m*, if *m* divides $a – b$ (i.e. a-b = km where k is an integer) written as: $a \equiv b \pmod{m}$.

The congruence relation is an equivalence relation (Check!!)

Divides

a is said to be divisible by b (or b divides a)

if a= b.k where k is an integer

divides is a Partial order relation (Check!!)

Let *R* be a relation defined on a set of integers as *a R b* if *a* ≡ *b* (mod *5*) prove that R is an equivalence relation

1.Reflexive: for every integer x, $x - x = 0$ is divisible by 5 so $x \equiv x$(mod 5).

2. Symmetric: if $x \equiv y$ (mod 5) then x-y is divisible by 5

$\Rightarrow x - y$ =5k where k is an integer

$\Rightarrow y - x = -5k$

$\Rightarrow y - x$ is also divisible by 5

hence $y \equiv x \pmod 5$.

3. Transitive: assume $x \equiv y \pmod 5$ and $y \equiv z \pmod 5$.

Then $x - y = 5$m and $y - z$ =5n where m ,n are integers

From here, $x - z = (x - y) + (y - z)$ =5m+5n=5(m+n)

$\Rightarrow$x-z is divisible by 5

$\Rightarrow x \equiv z \pmod 5$.

Example: Check whether relation R on a set of real numbers is reflexive, symmetric, or transitive. a R b if $a \le b^2$

R= $\{(a, b) / a \le b^2\}$

Since $(1/2) > (1/2)^2$

$\Rightarrow (1/2, 1/2) \notin R$

$\therefore$R is not reflexive.

Now, $(1, 4) \in R$ as $1 < 4^2$

But, 4 is not less than 1^2.

$\therefore (4, 1) \notin R$

$\therefore$R is not symmetric.

$(3, 2), (2, 1.5) \in R$

(as $3 < 2^2 = 4$ and $2 < (1.5)^2 = 2.25$)

But, $3 > (1.5)^2 = 2.25$

$\therefore (3, 1.5) \notin R$

$\therefore$ R is not transitive.

Partition

A partition of a set A is a collection of non-empty subsets $A_1, A_2, A_3, \ldots$ of A which are pairwise disjoint and whose union equals A

$$1.\ A_i \cap A_j = \Phi \qquad \text{for } i \neq j$$

$$2.\ \cup_n A_n = A$$

Example: Is P= {{1,2},{3,5},{4,5,6}} partition of A={1,2,3,4,5,6}?

Let $A = \{1, 2, 3, 4, 5, 6\}$.

$A_1 = \{1, 2\}; A_2 = \{3, 5\}; A_3 = \{4, 5, 6\}$.

$A = A_1 \cup A_2 \cup A_3$ but $A_2 \cap A_3 \neq \phi$..

P is not partition of A

Example:

Is P= {{1,2}{3,5}{4}} partition of A={1,2,3,4,5} ?

$A_1 = \{1, 2\}; A_2 = \{3, 5\}; A_3 = \{4\}.$

$A_1 \cap A_2 = \phi, A_1 \cap A_3 = \phi, \text{ and } A_2 \cap A_3 = \phi.$

$A = A_1 \cup A_2 \cup A_3$

P is partition of A

Equivalence Class

Let R be an equivalence relation on a set A

Let $\mathrm{x} \in \mathrm{A}$

the set of elements of A related to x is called the equivalence class of x, represented by $[\mathrm{x}]$

$[\mathrm{x}] = \{\mathrm{y} \in \mathrm{A} \mid \mathrm{yRx}\}$.

The collection of equivalence classes, represented $\mathrm{A/R}$

$\mathrm{A/R} = \{[\mathrm{x}] \mid \mathrm{x} \in \mathrm{A}\}$, is called Quotient set of A by R

If *R* is an equivalence relation on *A*, then collection of sets [a] or *R*(*a*) is called as equivalence classes of *R*.

Theorem

Let R be an equivalence relation on a set A.

Then A|R is a partition of A.

Specifically:

(i) For each a in A, we have a $\in$ [a]. (So every element is covered, nothing is left)

(ii) [a] = [b] if and only if (a, b) $\in$ R or b $\in$ [a]

(iii) If $[a] \neq [b]$ *or* $b \notin [a]$, then [a] and [b] are disjoint.

Example

Let *A* = {1, 2, 3, 4} and *R* = {(1, 1), (1, 2), (2, 1), (2, 2), (3,4), (4, 3), (3, 3), (4, 4)}. Show that *R* is an equivalence relation on *A* hence find partition of A induced by R

Consider $M_R = \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

It is Reflexive since all diagonal elements are 1.

It is symmetric since Matrix is symmetric.

Due to block structure of Matrix it is transitive. (Check)

Hence R is Equivalence relation on the set A.

We observe that *R*(1) = $[1] = \{1, 2\} = [2] =$ *R*(2)

and *R*(3) = $[3] = \{3, 4\} = [4] =$ *R*(4)

hence A/R = P = { {1, 2}, {3, 4} } which is partition of set A.

Example let A = {1, 2, …., 8}. Let R be the relation defined by $x \equiv y \bmod(4)$. Write R as a set of ordered pairs, Check that it is equivalence relation. Find the partition of A induced by R .

$$\mathrm{R} = \{(1,1), (1,5), (2,2), (2,6), (3,3), (3,7), (4,4), (4,8),$$
$$(5,1), (5,5), (6,2), (6,6), (7,3), (7,7), (8,4), (8,8)\}$$

(Prove equivalence same as previous examples)

Then Equivalence classes of R are

[1] = {1, 5} = [5], [2] = {2, 6} = [6],

$[3] = \{3,7\} = [7]$ and [4] = {4,8} = [8]

So, the partition of A induced by R is

$A|R = \{[1], [2], [3], [4]\}$ $\boldsymbol{or}$ $\{[1], [2], [7], [8]\}$ etc.

Example : Consider the set L of lines in the Euclidean plane. Two lines in the plane are said to be related, if they are parallel to each other. Show that R is an equivalence relation on A hence find an equivalence class of y=2x+4

We have already proved the First Part. Now

The set of all lines related to the line $y = 2x + 4$ is the set of all lines that are parallel to the line $y = 2x + 4$.

Slope of line $y = 2x + 4$ is $m = 2$

It is known that parallel lines have the same slopes.

The line parallel to the given line is of the form $y = 2x + c$, where $c \in \mathbf{R}$.

Hence, the set of all lines related to the given line is given by $y = 2x + c$, where $c \in \mathbf{R}$.

Construction of Z_5

Let A = Z (set of integers) and define R as

$R = \{(a, b) \in A \times A : a \equiv b \pmod 5\}$. Then, we have,

$R(1) = \{......, -14, -9, -4, 1, 6, 11, ... \}$

$R(2) = \{......, -13, -8, -3, 2, 7, 12, ... \}$

$R(3) = \{......, -12, -7, -2, 3, 8, 13, ... \}$

$R(4) = \{......, -11, -6, -1, 4, 9, 14, ... \}$

$R(5) = \{......, -10, -5, 0, 5, 10, 15, ... \}$.

$R(1)$, $R(2)$, $R(3)$, $R(4)$ and $R(5)$ form partition on Z with respect to given equivalence relation.

Z|R= {R(1),R(2),R(3),R(4),R(5)}

$Z_{5=} \{\bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}\}$

Similarly we can construct any set Z_n using equivalence classes of modulo n

Example 7.47 : *Let R and S are equivalence relation on $A = \{1, 2, 3, 4\}$ given by*

$R = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 3), (3, 3), (4, 4)\}$

$S = \{(1, 1), (2, 2), (3, 3), (4, 4), (1, 3), (3, 1)\}$

Determine partition of A induced by

(i) R^{-1} *(ii)* $R \cap S$

Solution : (i)

$$R = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 3), (3, 3), (4, 4)\}$$
$$R^{-1} = \{(1, 1), (2, 1), (1, 2), (2, 2), (4, 3), (3, 4), (3, 3), (4, 4)\}$$

$\therefore$
$$[1]_{R^{-1}} = \{1, 2\}$$
$$[2]_{R^{-1}} = \{1, 2\}$$
$$[3]_{R^{-1}} = \{3, 4\}$$
$$[4]_{R^{-1}} = \{3, 4\}$$

Here, $[1]_{R^{-1}} = [2]_{R^{-1}}$ and $[3]_{R^{-1}} = [4]_{R^{-1}}$

$\therefore$ Partition of A induced by $R^{-1} = [\{1, 2\}, \{3, 4\}]$

(ii)
$$R \cap S = \{(1, 1), (2, 2), (3, 3), (4, 4)\}$$
$$[1]_{R \cap S} = \{1\}$$
$$[2]_{R \cap S} = \{2\}$$
$$[3]_{R \cap S} = \{3\}$$
$$[4]_{R \cap S} = \{4\}$$

$\therefore$ Partition of A induced by $R \cap S = [\{1\}, \{2\}, \{3\}, \{4\}]$