Prof. Sushma Kadge
16/09/21, 11:30am
Thursday, September 16, 2021 11:12 AM

Fourier series (FS)

Introduction :-

- Journey into frequency begins with Fourier series → only periodic s/g
- Periodic s/g representation unfolds → frequency content / spectrum

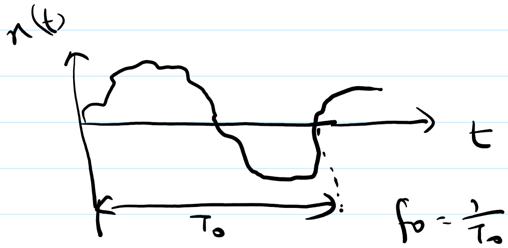
→ Synthesis → Combine 2 or more s/g → another periodic s/g
Analysis → Separation of periodic s/g into its periodic components.

FS :- describes a periodic s/g $x_p(t)$ as sum of harmonics at fundamental freq (f_0) of $x_p(t)$ & its multiples nf_0

[note :- Analysis of a s/g is more convenient in freq. domain]

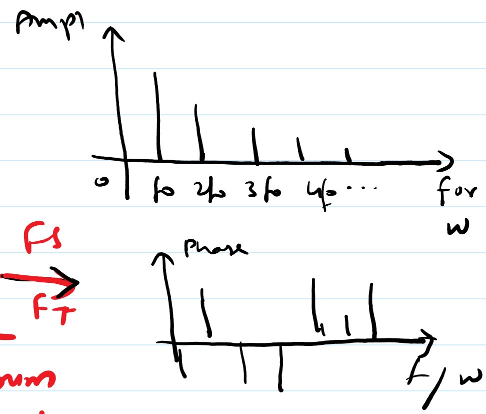
3 important class — FS → Fourier series → periodic s/g
FT " Transform → Both (nonperiodic)
LT Laplace " → Both

In frequency domain representation → x axis → freq (w or f)
↓
Line spectrum graph — Amplitude Spectrum (mag.)
Phase Spectrum.



$f_0 = \frac{1}{T_0}$

freq. compo ? present → FS → FT
↓
line spectrum all info.



Information
→ Type of s/g (Per / Aper)
→ T_0
→ f_0
→ Shape of s/g

Line & phase Spectrum :-

$x(t) = A\cos(\omega_0 t + \phi)$

$\omega_0 = 2\pi f_0$; $f_0 \rightarrow$ fundamental freq.

$T_0 = \frac{1}{f_0}$

$f_0 = \frac{2\pi}{\omega_0}$

One sided spectrum
Double sided spectrum. [mag. $\rightarrow$ even symmetry
phase $\rightarrow$ odd symmetry.

eg: Plot the line spectrum of sig

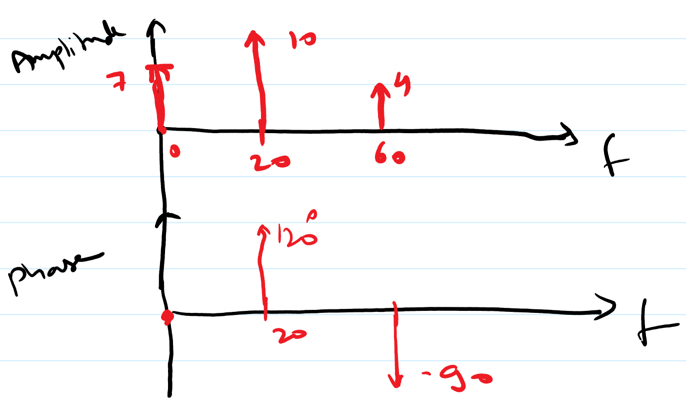
$x(t) = 7 - 10\cos(40\pi t - 60^\circ) + 4\sin(120\pi t)$ $\quad \omega t \rightarrow 2\pi f t$

$\Rightarrow x(t) = 7\cos(2\pi 0 t) + 10\cos(2\times 20\pi t + 120^\circ) + 4\cos(2\times 60\pi t - 90^\circ)$ $\quad 2\times\pi\times 20\times t$, $40\pi t$

$f_0 =$ zero freq, Ampl $\rightarrow 7$, $\phi_0 = 0^\circ$ — dc component

$f_1 = 20$, $\phi_1 = -60^\circ, -10$ or $120^\circ, +10$ — 1st harmonic

$f_2 = 60$ Hz, $\phi_2 = -90^\circ$ — 2nd harmonic



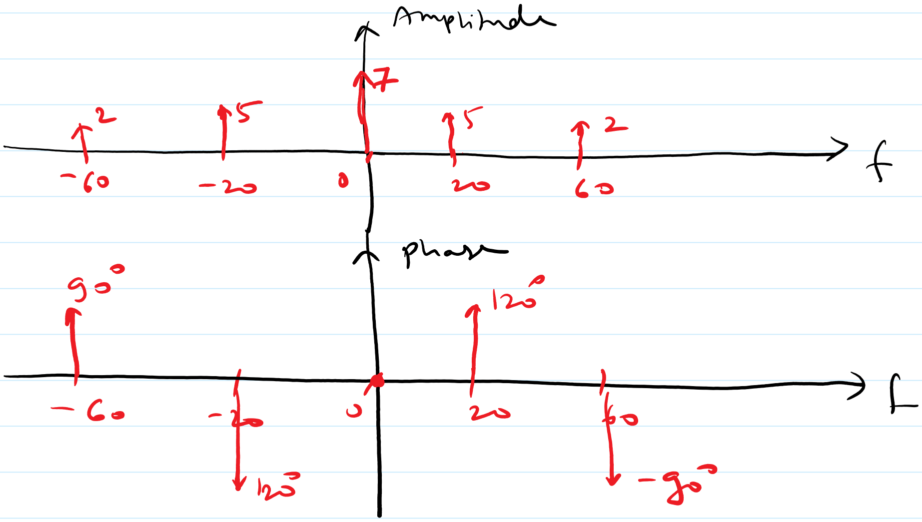
one sided mag & phase spectrum

Draw Double sided spectrum for the above sig

↑ Amplitude

Ampl. $\rightarrow$ half.

Draw double sided spectrum for the above sig



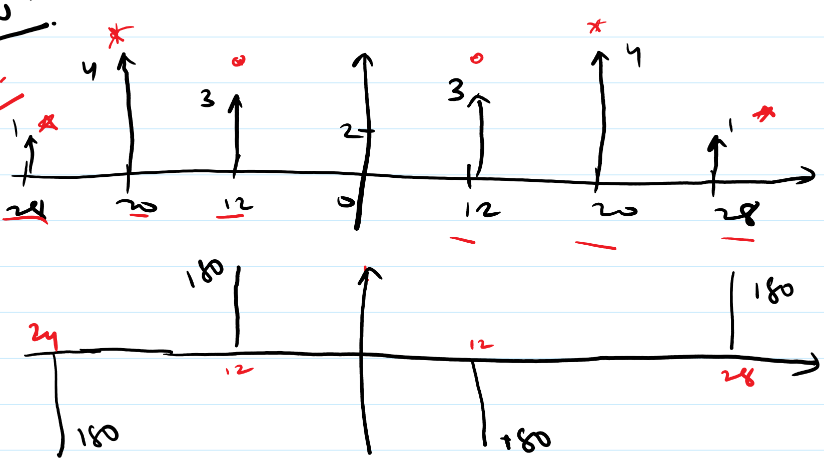
Ampl. → half.

Even symm.

Odd Symmetry

H.W.:

Eg 2:-



$x(t) = ?$

fundamental freq = ?

→ fundamental freq → GCD(12, 20, 28)

= 4 Hz

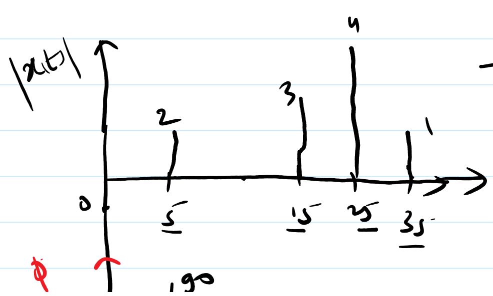
$\omega = 2\pi f$

$x(t) = 2\cos(\omega t) + 6\cos(\omega t + \phi) + 8\cos(\omega t) + 2\cos(\omega t)$

$f = 0 \quad f = 12 \quad f = 20 \quad f = 28$

$$\boxed{x(t) = 2 + 6\cos(24\pi t - 180^\circ) + 8\cos(40\pi t) + 2\cos(56\pi t + 180^\circ)}$$

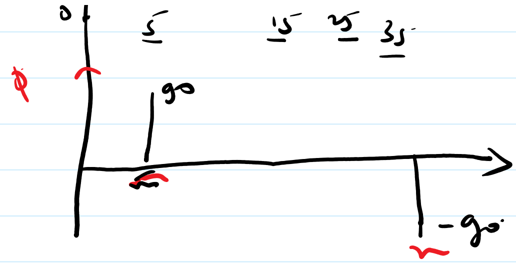
Eg:-



→ single sided spectrum.

$f_0 = GCD(\underset{\omega_0}{5}, \underset{\omega_1}{15}, \underset{\omega_2}{25}, \underset{\omega_3}{35})$

$f_0 = 5\ Hz$



$f_0 = 5\,Hz$

$x(t) = A\cos(\omega t)$ $\quad \omega = 2\pi f$ $\quad f = 5, 15, 25, 35$

$$x(t) = 2\cos(10\pi t + 90^\circ) + 3\cos(30\pi t + 0^\circ) + 4\cos(50\pi t) + 1\cos(70\pi t - 90^\circ)$$

Orthogonality :-

Two functions $f(t)$ & $g(t)$

Two functions are said to be orthogonal if area of their product $f(t)\cdot g^*(t)$ over the interval $\alpha \le t \le \beta$ equal to zero

$$\int_\alpha^\beta f(t)\cdot g^*(t)\,dt = 0$$

* Conjugation is req^ed only complex signals

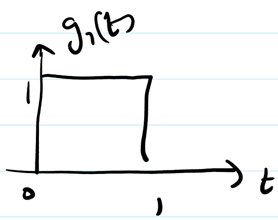
$$\int_\alpha^\beta f(t)\cdot g(t)\,dt = 0$$

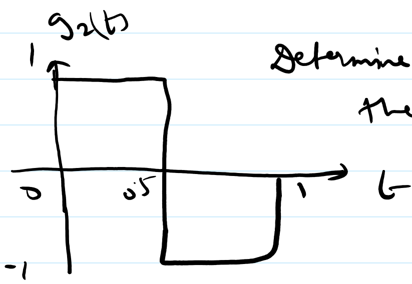
If the functions are same

$$\int_\alpha^\beta x_1(t)\,x_2(t) \neq 0 \quad \underline{\text{const}}\ ?$$

Kn.

Eg:-





Determine whether they are orthogonal

⇒

$g_1(t) = 1 \qquad 0 \le t < 1$

$g_2(t) = 1 \qquad 0 \le t < 0.5$

$\quad\;\; = -1 \qquad 0.5 \le t < 1$

$$g(t) = \int_{\alpha}^{\beta} g_1(t) \cdot g_2(t)$$

$$= \int_{0}^{0.5} 1 \cdot 1 \, dt + \int_{0.5}^{1} 1 \cdot (-1) \, dt$$

$$g(t) = 0.5 - 0.5 = \underline{\underline{0}}$$

Eg:- S.T following s/g are orthogonal over an interval $[0, 1]$

$f(t) = 1 \qquad g(t) = \sqrt{3}(1-2t)$

$$\Rightarrow \int_{\alpha}^{\beta} f(t) \, g(t) = 0$$

$$\int_{0}^{1} 1 \cdot \sqrt{3}(1-2t) \, dt$$

$$\int_{0}^{1} \sqrt{3} \, dt - \sqrt{3}\int_{0}^{1} 2t \, dt$$

$$\sqrt{3}\left[t - \frac{2t^2}{2}\right]_0^1$$

$$\underline{\underline{0}}$$

Eg:- S.T set of s/g $\{\sin n\omega_0 t\}$ forms an orthogonal set over the interval $\left\{0, \frac{2\pi}{\omega_0}\right\}$, Also find k_n.

$$\Rightarrow \int_{0}^{\frac{2\pi}{\omega_0}} \sin n\omega_0 t \cdot \sin m\omega_0 t \, dt = 0$$

$$\frac{1}{2}\int_{0}^{\frac{2\pi}{\omega_0}} 2 \sin n\omega_0 t \cdot \sin m\omega_0 t \, dt$$

$2\sin A \cdot \sin B$

$\cos(A-B) - \cos(A+B)$

$$\frac{1}{2}\int \cos(m-n)\omega_0 t - \cos(m+n)\omega_0 t \; dt$$

$$\frac{1}{2}\left[\frac{\sin (m-n)\omega_0 t}{(m-n)\omega_0} - \frac{\sin (m+n)\omega_0 t}{(m+n)\omega_0}\right]_0^{2\pi/\omega_0}$$

$\sin p(2\pi) = 0$

$= 0$ Hence verified.

$$K_n = \int_0^{2\pi/\omega_0} \sin^2 n\omega_0 t \; dt$$

$$= \int_0^{2\pi/\omega_0} \frac{1-\cos 2n\omega_0 t}{2}$$

$$= \int_0^{2\pi/\omega_0} \frac{1}{2}\, dt - \frac{1}{2}\int_0^{2\pi/\omega_0} \cos 2n\omega_0 t \; dt$$

sin. $= 0$

$$= \frac{1}{2}\,\frac{2\pi}{\omega_0}.$$

$$\boxed{K_n = \frac{\pi}{\omega_0}}$$

H.W. (1) $\int_0^{\pi} \cos mt \cdot \cos nt \; dt \Rightarrow$ S.T. they are orthogonal.

(2) $\int_0^{\pi} \sin nt \cdot \sin mt \; dt$

(3) P.T $\{\sin nt\}$ $\{0, 2\pi\}$ they are orthogonal

Also find $K_n = \pi$

(4) P.T $\{\cos nt\}$ $\{(0, 2\pi)\}$ orthogonality

$K_n = \pi$

Fourier series :- Sum of sinusoids at fundamental frequency f_0 & its multiples of nf_0.

Any periodic signal $f(t)$ can be represented by

$$f(t) = a_0 \cos(\underbrace{0}_{\text{zero}}\omega t) + a_1 \cos(\omega_0 t) + a_2 \cos(2\omega_0 t) + \cdots$$

$$\cdots + a_n \cos(n\omega_0 t) + b_0 \sin(0\omega_0 t) + b_1 \sin(\omega_0 t) + b_2 \sin(2\omega_0 t) + \cdots + b_n \sin(n\omega_0 t)$$

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t) + \sum_{n=1}^{\infty} b_n \sin(n\omega_0 t)$$

a_0, a_n & b_n = Coefficient of trigonometric F.S

a_0 : dc component or offset.

(1) Trignometric F.S
(2) Polar F.S
(3) Exponential F.S

(1) Trignometric F.S.

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega t) + \sum_{n=1}^{\infty} b_n \sin(n\omega t)$$

$$\omega = 2\pi f_0 = \frac{2\pi}{T_0} \quad (1)$$

$$a_0 = \frac{1}{T_0}\int^{T_0} x(t)\,dt \quad \rightarrow (2)$$

$$a_n = \frac{2}{T_0}\int_0^{T_0} x(t)\cos(n\omega t)\,dt \rightarrow (3)$$

$$b_n = \frac{2}{T_0}\int_0^{T_0} x(t)\sin(n\omega t)\,dt \rightarrow (4)$$

(2) Polar series :- $(D_0 \text{ \& } D_n)$

$$D_0 = a_0$$

$$D_n = \sqrt{a_n^2 + b_n^2}$$

Spectrum $\left[\begin{array}{l}\text{mag}\\ \text{phase}\end{array}\right.$

$$\phi_n = -\tan^{-1}\frac{b_n}{a_n}$$

$$x(t) = D_0 + \sum_{n=1}^{\infty} D_n \cos(n\omega t + \phi_n)$$

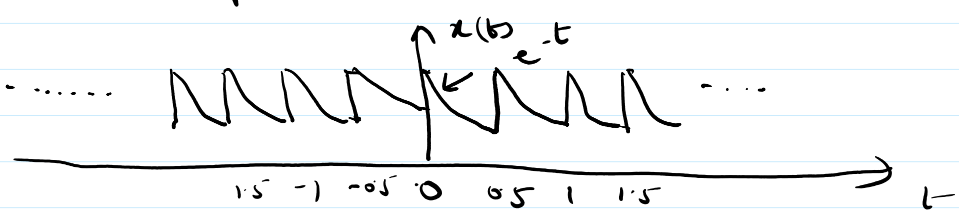
(3) Exponential F.S

$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega t}$$

$$C_n = \frac{1}{T_0}\int x(t)\, e^{-jn\omega t}\,dt.$$

eg:-

Find trigonometric F.S for the periodic signal



$\Rightarrow$ $T_0 = 0.5$

$f_0 = 2$

$a_0,\ a_n,\ b_n = ?$

$$a_0 = \frac{1}{T_0}\int_0^{T_0} x(t)\,dt$$

$$= \frac{1}{0.5}\int_0^{0.5} e^{-t}\,dt$$

$$= \frac{1}{0.5}\left[-e^{-t}\right]_0^{0.5}$$

$$= \frac{1}{0.5}\left[1 - e^{-0.5}\right] = 0.7869$$

$$\boxed{a_0 = 0.7869}$$

$a_n = ?$

$$a_n = \frac{2}{T_0}\int_0^{T_0} x(t)\cos(n\omega t)\,dt$$

$\omega = 2\pi f_0$
$= 2\pi \times 2$
$= 4\pi$

$$= \frac{2}{0.5}\int_0^{0.5} e^{-t}\cos(n4\pi t)\,dt$$

$$\boxed{\int e^{ax}\cdot\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}\left(a\cos bx + b\sin bx\right)}$$

$a = -1 \qquad b = 4\pi n$

$$a_n = 4\left\{\frac{e^{-t}}{1+(4\pi n)^2}\left(-1\cos 4\pi n t + 4\pi n \sin 4\pi n t\right)\right]_0^{0.5}$$

$$a_n = 4\left\{\frac{e^{-0.5}}{1+(4\pi n)^2}\left[-\cos 4\pi n(0.5) + 4\pi n \sin 4\pi n(0.5)\right)\right.$$

$$- e^{0} \quad (-1\cos 0 + 4\pi n \sin 0)\}$$

$$- \frac{e^0}{1+(4\pi n)^2}\left(-1\cos 0 + 4\pi n \sin 0\right)\Big\}$$

$$= \frac{4}{1+(4\pi n)^2}\Big\{0.606\left[-\cos(2\pi n) + 4\pi n \sin(2\pi n)\right]$$

$$- \left[-\cos 0 + 4\pi n \sin 0\right]\Big\}$$

$$= \frac{4}{1+(4\pi n)^2}\left\{-0.606 + 0 + 1 + 0\right\}$$

$$\boxed{a_n = \frac{1.576}{1+(4\pi n)^2}}$$

$$b_n = \frac{2}{T_0}\int_0^{T_0} x(t)\sin(n\omega t)\,dt$$

$$= \frac{2}{0.5}\int_0^{0.5} e^{-t}\sin(n4\pi t)\,dt$$

$$\boxed{\int e^{-ax}\sin bx\,dx = \frac{e^{-ax}}{a^2+b^2}\left[-a\sin bx + b\cos bx\right]}$$

$$b_n = \frac{e^{-t}}{1+(4\pi n)^2}\Big[1\sin 4\pi n t + 4\pi n \sin 4\pi n t\Big]_0^{0.5}$$

$$\vdots$$

$$\boxed{b_n = \frac{6.32\pi n}{1+(4\pi n)^2}}$$

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n\cos(n\omega t) + \sum_{n=1}^{\infty} b_n\sin(n\omega t)$$

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega t) + \sum_{n=1}^{\infty} b_n \sin(n\omega t)$$

$$\boxed{x(t) = 0.7869 + \sum_{n=1}^{\infty} \frac{1.576}{1+(4\pi n)^2} \cos(4\pi n t) + \sum_{n=1}^{\infty} \frac{6.32\pi n}{1+(4\pi n)^2} \sin 4\pi n}$$

Eg 2: Find Polar series of the above eg 1. & plot its magnitude & phase spectrum.

$\Rightarrow$

D_0 & D_n = ?

$$x(t) = D_0 + \sum_{n=1}^{\infty} D_n \cos(n\omega t + \phi_n)$$

$$\boxed{D_0 = a_0 = 0.7869}$$

$$D_n = \sqrt{a_n^2 + b_n^2}$$

$$= \sqrt{\frac{(1.576)^2}{[1+(4\pi n)^2]^2} + \frac{(6.32\pi n)^2}{[1+(4\pi n)^2]^2}}$$

$$= \sqrt{\frac{(1.576)^2 [1+(4\pi n)^2]}{1+(4\pi n)^2}}$$

$$\boxed{D_n = \frac{1.576}{\sqrt{1+(4\pi n)^2}}}$$

$$\phi_n = -\tan^{-1} \frac{b_n}{a_n}$$

$$= -\tan^{-1} \frac{6.32\pi n / 1+(4\pi n)^2}{1.576 / 1+(4\pi n)^2}$$

$1.576 / 1 + (4\pi n)^2$

$$\boxed{\phi_n = -\tan^{-1}(4\pi n)}$$

$$\boxed{x(t) = 0.786 + \sum_{n=1}^{\infty} \frac{1.576}{\sqrt{1+(4\pi n)^2}} \cos\left[4\pi n - \tan^{-1}(4\pi n)\right]}$$

Magnitude & Phase spectrum :-

we know that $T_0 = 0.5$ $f_0 = 2$.

$$\omega_0 = 2\pi f_0 = 4\pi$$

Magnitude spectrum is given by

$$|D_n| = \frac{1.576}{\sqrt{1+(4\pi n)^2}}$$

Plot $|D_n|$ vs ω.

$$\omega = n\omega_0$$

$n = 0, 1, 2, 3 \ldots\ldots$

✓ $\omega = 4\pi n$.

$= 0, 4\pi, 8\pi, 12\pi \ldots\ldots\ldots$

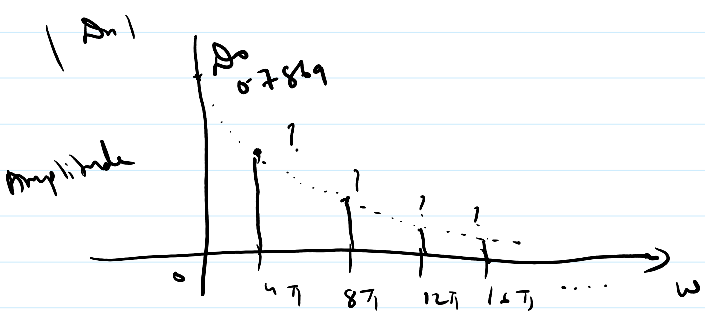
$$|D_n| = \frac{1.576}{\sqrt{1+\omega^2}}$$

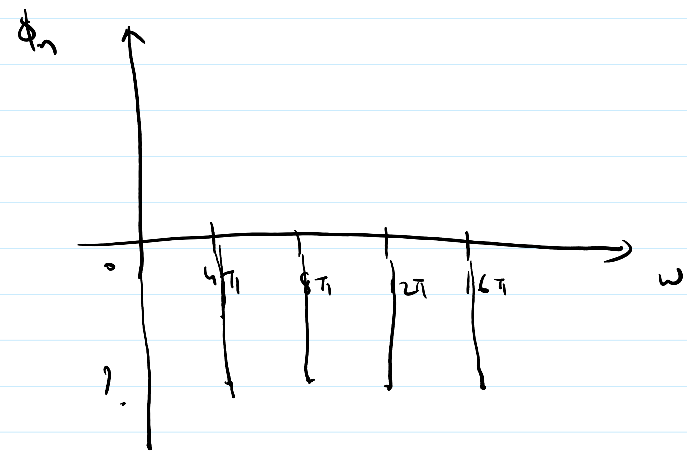
$$\phi_n = -\tan^{-1}(4\pi n)$$

$$= -\tan(\omega)$$

$\omega = n\omega_0 = 4\pi n$	$\lvert D_n\rvert = \frac{1.576}{\sqrt{1+\omega^2}}$	$\phi_n = -\tan^{-1}\omega$
n=1 4π	$\frac{1.576}{\sqrt{1+(4\pi)^2}} =$	$\phi_n = -\tan^{-1}(4\pi) =$

n=1	4π	$\frac{1.576}{\sqrt{1+(4\pi)^2}} =$	$\phi_n = -\tan^{-1}(4\pi) =$
n=2	8π	$\frac{1.576}{\sqrt{1+(8\pi)^2}} =$	$\phi_n = -\tan^{-1}(8\pi)$
n=3	12π	$\frac{1.576}{\sqrt{1+(12\pi)^2}} =$	$\phi_n = -\tan^{-1}(12\pi)$
n=4	16π	$\frac{1.576}{\sqrt{1+(16\pi)^2}} =$	$\phi_n = -\tan^{-1}(16\pi)$





85 to 90°

③ Find the exponential f.s for eq ①

$$\Rightarrow \quad x(t) = \sum_{n=-\infty}^{\infty} C_n e^{j\omega n t}$$

$\omega = 2\pi f$

$= \frac{2\pi}{T_0}$

$$C_n = \frac{1}{T_0}\int_0^{T_0} x(t)\, e^{-j\omega n t}\, dt$$

$\omega = 4\pi$

$T_0 = 0.5 \qquad f_0 = 2$

$$C_n = \frac{1}{\ } \int^{0.5} e^{-t}\; e^{-j4\pi n t}\; dt$$

$T_0 = 0.5 \qquad \omega_0 = 2$

$$C_n = \frac{1}{0.5}\int_0^{0.5} e^{-t}\, e^{-j4\pi n t}\, dt$$

$$= \frac{1}{0.5}\int_0^{0.5} e^{-(1+j4\pi n)t}\, dt$$

$$= 2\left[\frac{e^{-(1+j4\pi n)t}}{-(1+j4\pi n)}\right]_0^{0.5}$$

$$= \frac{2}{1+j4\pi n}\left[\underset{②}{-e^{-(1+j4\pi n)0.5}} + \underset{①}{e^{0}}\right]$$

$$= \frac{2}{1+j4\pi n}\left[1 - e^{-0.5}\cdot e^{-j2\pi n}\right]$$

$= 1$

$\rightarrow \cos 2\pi n$

$-j\sin 2\pi n$

0

$$= \frac{2}{1+j4\pi n}\left[1 - 0.606\, \underset{1}{e^{-j2\pi n}}\right]$$

$$\boxed{C_n = \frac{0.789}{1+j4\pi n}}$$

$$x(t) = \sum_{n=-\infty}^{\infty} C_n\, e^{j\omega_0 n t}$$

$$\boxed{x(t) = \sum_{n=-\infty}^{\infty} \frac{0.789}{1+j4\pi n}\, e^{j\omega_0 n t}}$$

Magnitude & phase spectrum of C_n.

$$C_n = \frac{0.789}{1+j4\pi n} \times \frac{1-j4\pi n}{1-j4\pi n}$$

$$= \frac{0.789 - j\,0.789\times 4\pi n}{1-(j4\pi n)^2}$$

$$\overline{\qquad\qquad\qquad}$$
$$1-(j4\pi n)^2$$

$$\boxed{j^2=-1}$$

$$= \underbrace{\frac{0.789}{1+(4\pi n)^2}}_{\text{real}} - j\,\underbrace{\frac{0.789 \times 4\pi n}{1+(4\pi n)^2}}_{\text{imag.}}$$

magnitude of c_n

$$|c_n| = \sqrt{\left(\frac{0.789}{1+(4\pi n)^2}\right)^2 + \left(\frac{0.789 \times 4\pi n}{1+(4\pi n)^2}\right)^2}$$

$$= \sqrt{\frac{(0.789)^2 + 0.789^2(4\pi n)^2}{\left[1+(4\pi n)^2\right]^2}}$$

$$= 0.789\sqrt{\frac{1+(4\pi n)^2}{\left[(1+4\pi n)^2\right]^2}}$$

$$|c_n| = \frac{0.789}{\sqrt{1+(4\pi n)^2}}$$

Plot c_n vs ω

$$\omega_0 = 2\pi f_0 = 4\pi$$

$$\omega = n\omega_0 = 4\pi n \qquad n = \pm 1, \pm 2, 3 \ldots$$

$$|c_n| = \frac{0.7869}{\sqrt{1+\omega^2}} \qquad \omega = 0, \pm 4\pi, \pm 8\pi, \pm 12\pi \ldots$$

$$\phi_n = -\tan^{-1}\frac{\text{Imag}}{\text{Real}}$$

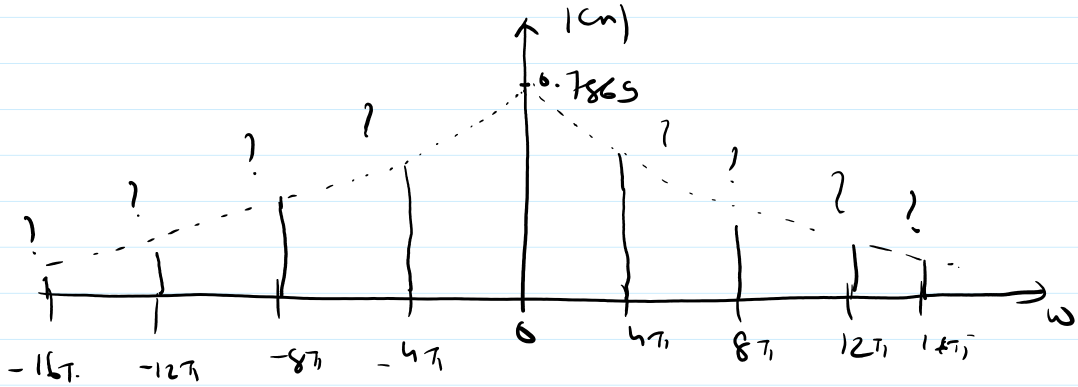
$$= -\tan^{-1}\frac{\cancel{0.7869} \times 4\pi n / \cancel{\left[1+(4\pi n)^2\right]}}{\qquad}$$

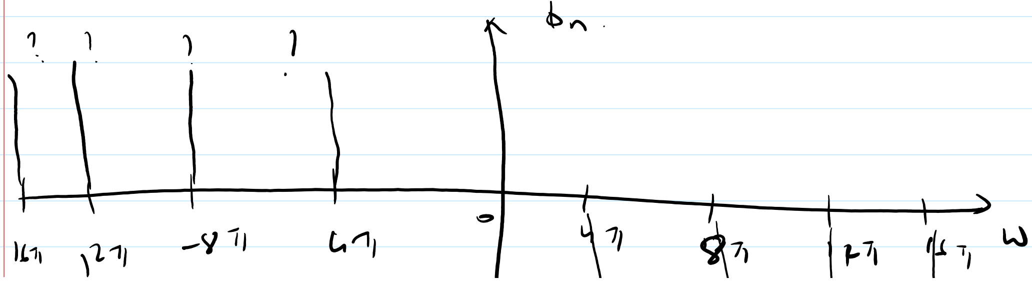
$$= -\tan^{-1} \frac{\cancel{0.7869} \times 4\pi n / \cancel{[1+(4\pi n)^2]}}{\cancel{0.7869}\ \cancel{[1+(4\pi n)^2]}}$$

$$= -\tan^{-1} 4\pi n$$

$$\phi_n = -\tan^{-1}(\omega)$$

	$\omega = n\omega_0$ $= 4\pi n$	$\lvert c_n \rvert = \frac{0.7869}{\sqrt{1+\omega^2}}$	$\phi_n = -\tan^{-1}(\omega)$
0	0	$c_0 = 0.7869 = a_0$	$-\tan^{-1}(0) =$
± 1	$\pm 4\pi$	$= \frac{0.7869}{\sqrt{1+(4\pi)^2}} = ?$	$-\tan^{-1}(4\pi) =$ $-\tan^{-1}(-4\pi) =$
± 2	$\pm 8\pi$	$\frac{0.7869}{\sqrt{1+(8\pi)^2}} = ?$	$-\tan^{-1}(8\pi)$ $-\tan^{-1}(-8\pi)$
± 3	$\pm 12\pi$	$\frac{0.7869}{\sqrt{1+(12\pi)^2}} = ?$	$-\tan^{-1}(12\pi)$ $-\tan^{-1}(-12\pi)$





on comparison of both (polar & expo) spectrum

① $D_0 = C_0 = 0.7869$

② $D_1 = 2C_1 =$

$D_2 = 2C_2$

$D_3 = 2C_3$

$\boxed{\text{ie } |D_n| = 2|C_n|}$

③ magnitude spectrum has even symmetry

④ ϕ_n is same

⑤ phase spectrum has odd symmetry.

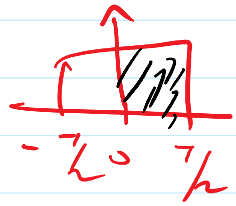
Symmetry of the signals

Even function * Even Function = Even Function

Odd function * odd Function = Even Function

Even function * Odd Function = Odd Function

Odd function * Even Function = Odd Function



$$\int_{-T/2}^{T/2} x_e(t)\, dt = 2\int_0^{T/2} x_e(t)\, dt$$

$$\int_{-T/2}^{T/2} x_o(t)\, dt = 0$$

* Even periodic function

$$a_0 = \frac{1 \cdot 2}{T_0}\int_0^{T_0/2} x(t)\, dt$$

$$a_n = \frac{2 \cdot 2}{T_0}\int_0^{T_0/2} \underbrace{x(t)}_{\text{even}} \underbrace{\cos(n\omega_0 t)}_{\text{even}}\, dt.$$

$$b_n = \frac{2}{T_0}\int \underbrace{x(t)}_{\text{even}} \underbrace{\sin(n\omega_0 t)}_{\ast \text{ odd}}\, dt = 0$$

only find a_0 & a_n

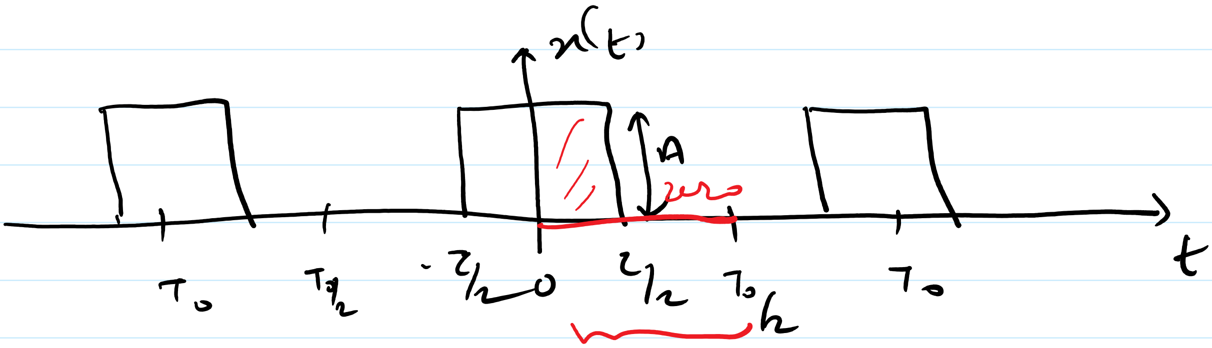
* odd periodic function

$$a_n = \frac{2}{T_0}\int \underbrace{x(t)}_{\text{odd}} \cdot \underbrace{\cos n\omega_0 t}_{\text{even}}\, dt = 0$$

$$\boxed{a_n = 0}$$

$$b_n = \frac{2\cdot 2}{T_0}\int_0^{T_0/2} x(t)\cdot \sin n\omega_0 t\, dt$$

*



⇒ we can the symmetry → even
$x_e(t) = x_e(-t)$

$a_0, a_n = ?$ $\qquad b_n = 0$

$$a_0 = \frac{2}{T_0}\int_0^{T_0/2} x(t)\, dt$$

$$= \frac{2}{T_0}\int_0^{\tau/2} x(t)\, dt + \underbrace{\int_{\tau/2}^{T_0} x(t)\, dt}_{\text{zero}}$$

$$= \frac{2}{\ } \int^{\tau/2} A\, dt = 2A \cdot t \Big|^{\tau/2}$$

$$= \frac{2}{T_0} \int_0^{\tau/2} A\, dt \qquad = \frac{2A}{T_0} \cdot t \Big|_0^{\tau/2}$$

$$\boxed{a_0 = \frac{A\tau}{T_0}}$$

$$a_n = \frac{4}{T_0} \int_0^{T_0/2} x(t) \cos n\omega_0 t\, dt$$

$$= \frac{4}{T_0} \int_0^{\tau/2} A \cdot \cos n\omega_0 t\, dt + \int_{\tau/2}^{T_0/2} 0 \cdot \cos n\omega_0\, dt$$

$$= \frac{4A}{T_0} \left[\frac{\sin n\omega_0 t}{n\omega_0} \right]_0^{\tau/2} + 0$$

$$= \frac{4A}{nT_0\omega_0} \left[\sin n\omega_0 \frac{\tau}{2} - \sin 0 \right]$$

$$\omega_0 = \frac{2\pi}{T_0}$$

$$= \frac{\cancel{4}^{2} A}{n \cancel{T_0} \frac{\cancel{2}\pi}{\cancel{T_0}}} \left[\sin n \frac{\cancel{2}\pi}{T_0} \cdot \frac{\tau}{\cancel{2}} \right]$$

$$a_n = \frac{2A}{\ldots} \left[\sin n\pi\tau \ldots \right]$$

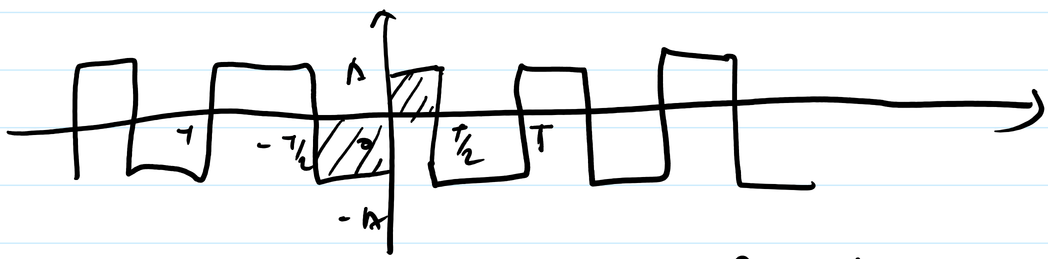
$$\boxed{a_n = \frac{2A}{n\pi}\left[\sin\frac{n\pi\tau}{T_0}\right]}$$

$b_n = 0$ [even symmetry]

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t$$

$$\boxed{x(t) = \frac{A\tau}{T_0} + \sum_{n=1}^{\infty} \frac{2A}{n\pi} \sin\left(\frac{\pi n\tau}{T_0}\right) \cdot \cos(n\omega_0 t)}$$

*



Period T

→ Identify whether Symmetry exists?

odd or even

$x(t) = -x(-t)$ ✓

$a_n = 0$ odd symmetry.

$$b_n = \frac{4}{T_0}\int_0^{T_0/2} x(t) \cdot \sin(n\omega_0 t)\, dt$$

$$b_n = \frac{4}{\ } \int^{T/2}$$

$$b_n = \frac{4}{T}\int_0^{T/2} A \cdot \sin(n\omega_0 t)\,dt$$

$$= \frac{4A}{T}\left[-\frac{\cos n\omega_0 t}{n\omega_0}\right]_0^{T/2}$$

$$\omega_0 = 2\pi/T$$

$$= \frac{\cancel{4}^{2}A}{\cancel{T}\,n\,\frac{\cancel{2}\pi}{\cancel{T}}}\left[-\cos n\,\frac{\cancel{2}\pi}{\cancel{T}}\cdot\frac{\cancel{T}}{\cancel{2}} + \cos 0\right]$$

$$= \frac{2A}{n\pi}\left[-\cos n\pi + 1\right]$$

$$\boxed{b_n = \frac{2A}{n\pi}\left[1 - \cos n\pi\right]}$$

Dirchlet Conditions:

1. The function $x(t)$ should be absolutely integrable i.e $\int^{T} x(t)\,dt < \infty$
2. The function should have finite number of maxima & minima in the interval T_0
3. The function $x(t)$ should have at the most finite number of discontinuities in the interval T_0

(4) The function $x(t)$ should be single valued within the interval T_0.

Parserval power theorem: -

It states that the total average power of the periodic s/g $x(t)$, is equal to sum of the average power of its individual components.

F.S $\rightarrow$ expands $x(t)$ $\Rightarrow$ Sum of phasor $C_n e^{j2\pi n f_0 t}$

The avg. power of each Phasor component

$$P = \left| C_0 n^2 + C_1 n^2 + \ldots\ldots \right|$$

$$\left\langle \left| C_n e^{j2\pi n f_0} \right| \right\rangle = |C_n|^2$$

$$\boxed{\text{Total avg. power } P = \sum_{n=-\infty}^{\infty} |C_n|^2}$$

$$P = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} |x(t)|^2 \, dt$$

$$|x(t)|^2 = x(t) \cdot x^*(t)$$

$$P = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t) \cdot x^*(t) \, dt$$

Expo. series

$$x(t) = \sum_{n=-\infty}^{\infty} c_n \, e^{j2\pi n f_0 t}$$

$$x^*(t) = \left[\sum_{n=-\infty}^{\infty} c_n \, e^{j2\pi n f_0 t}\right]^*$$

$$x^*(t) = \sum_{n=-\infty}^{\infty} c_n^* \, e^{-j2\pi n f_0 t}$$

$$P = \frac{1}{T_0}\int x(t)\underline{\left[\sum_{n=-\infty}^{\infty} c_n^* \, e^{-j2\pi n f_0 t}\right]} dt.$$

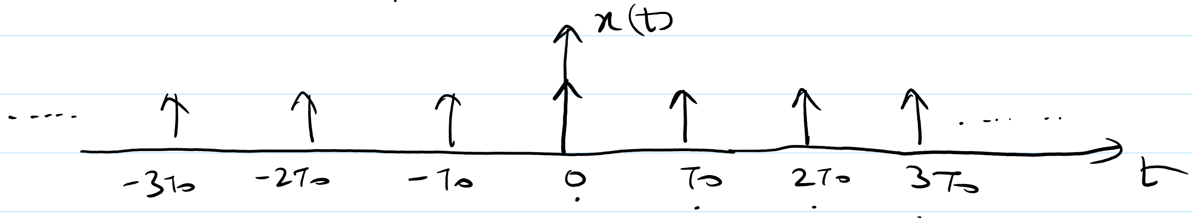
$$P = \sum_{n=-\infty}^{\infty} \underbrace{\left[\frac{1}{T_0}\int x(t) \, e^{-j2\pi n f_0 t} \, dt\right]}_{c_n} c_n^*$$

$$P = \sum_{n=-\infty}^{\infty} c_n \cdot c_n^*$$

$$\boxed{P = \sum_{n=-\infty}^{\infty} |c_n|^2}$$

Parseval's power theorem

Eg :- find the total normalized avg. power of the unit impulse train



$\Rightarrow$

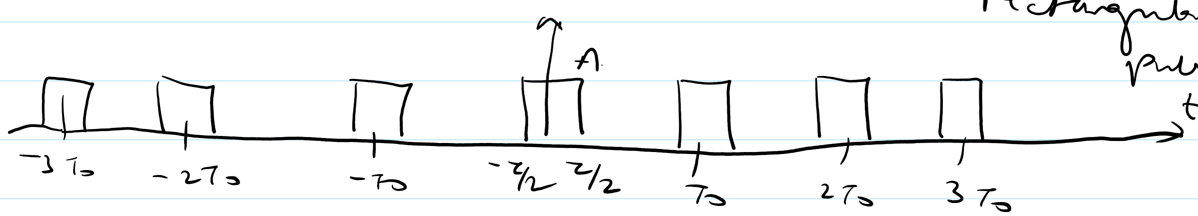
$$P = \sum_{n=-\infty}^{\infty} |c_n|^2 \qquad \text{step 1}$$

$c_n = ?$

$c_n \rightarrow$ Expo. F.S coefficient

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{j2\pi n f_0 t}$$

$$c_n = \frac{1}{T_0} \int x(t)\, e^{-j\frac{2\pi n}{T_0} \cdot t}\, dt$$



Rectangular pulse train

width $\rightarrow \tau$

Ampl $\rightarrow A$

period $\rightarrow T_0$

$$c_n = \frac{1}{T_0} \int_{-\tau/2}^{\tau/2} A \cdot e^{-j\frac{2\pi n t}{T_0}}\, dt$$

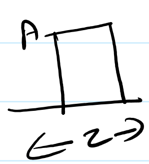
$$= \frac{A}{\cancel{T_0}} \left[\frac{e^{-j2\pi n t/T_0}}{\frac{-j2\pi n}{\cancel{T_0}}} \right]_{-\tau/2}^{\tau/2}$$

$$= \frac{A}{\pi n} \left[\frac{-e^{-j\frac{2\pi n}{T_0}\frac{\tau}{2}} + e^{+j\frac{2\pi n}{T_0}\cdot\frac{\tau}{2}}}{2j} \right]$$

$$= \frac{A}{\pi n} \left[\sin\left(\frac{\pi n \tau}{T_0} \right) \right]$$

$$= \frac{A}{\cancel{\pi n}} \cdot \frac{\cancel{\pi n}\tau}{T_0} \; \frac{\sin\left(\frac{\pi n \tau}{T_0} \right)}{\frac{\pi n \tau}{T_0}}$$

$$C_n = \frac{A\tau}{T_0} \operatorname{sinc}\left(\frac{\pi n \tau}{T_0} \right)$$



$\rightarrow$

$\tau = 0$

$A\tau = 1$

$$C_n = \lim_{\substack{A\tau \to 1 \\ \tau \to 0}} \frac{\overset{1}{A\tau}}{T_0} \operatorname{sinc}\left(\frac{n\pi\tau}{T_0} \right)$$

$$\boxed{\operatorname{sinc} 0 = 1}$$

$$\boxed{C_n = \frac{1}{T_0}}$$

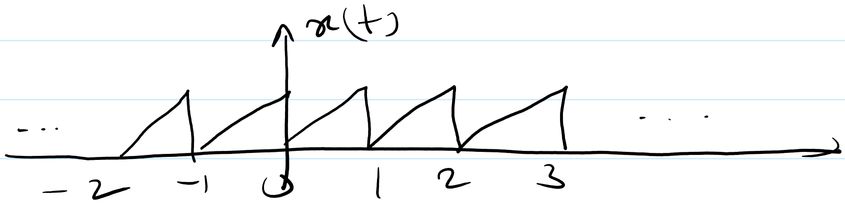
$$\operatorname{sinc} 0 = 1$$

By Parseval's theorem, total avg. power

$$P = \sum_{n=-\infty}^{\infty} |C_n|^2 = \sum_{n=-\infty}^{\infty} \frac{1}{T_0^2}$$

$$\boxed{P = \frac{1}{T_0^2}}$$

Eg H.W



Ans $|C_n| = \frac{1}{2T_1 n}$

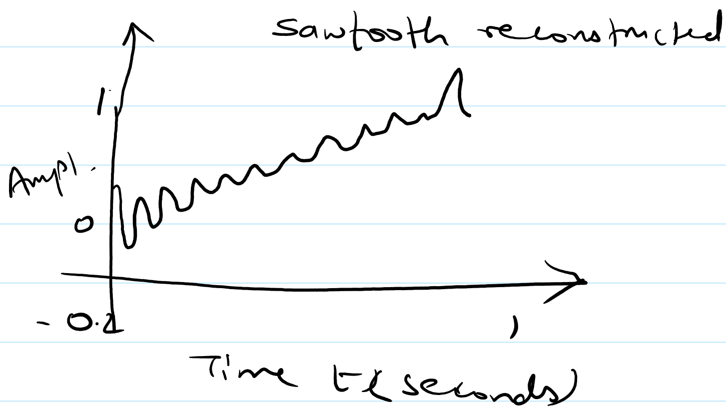
$$\boxed{P = \sum_{n=-\infty}^{\infty} \frac{1}{4T_1^2 n^2}}$$

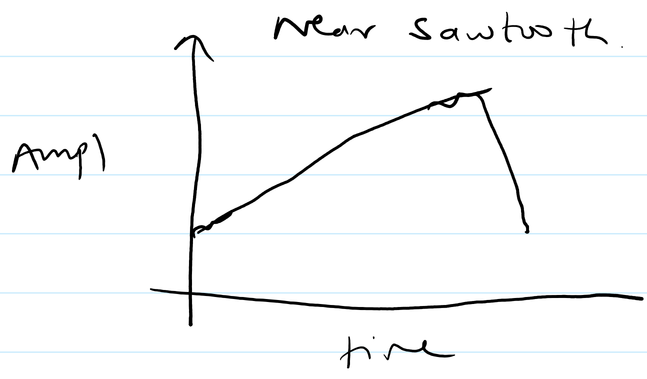
Signal reconstruction and the Gibbs effect

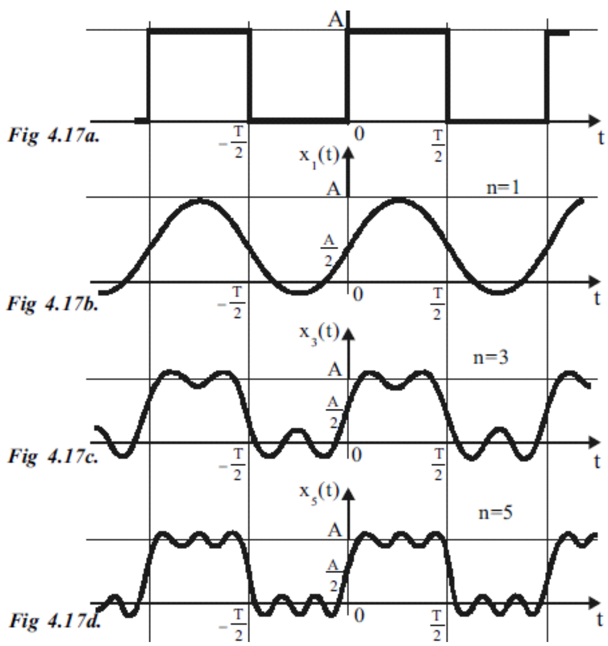
* Peaks in the harmonics used in S/g reconstruction occur at different times & lead to oscillations ie maxima & minima or ripples in the reconstruction.

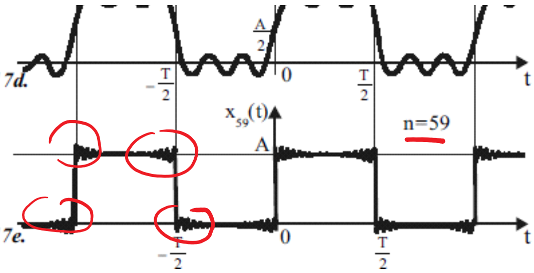
* No. of peaks in the ripples → indication of non-zero harmonics N → reconstruction.

* Can you count 20 peaks in the reconstruction









Gibbs phenomenon :-

The persistence of overshoot & imperfect reconstruction for signals with jumps → describes the Gibbs phenomenon / Gibbs effect.

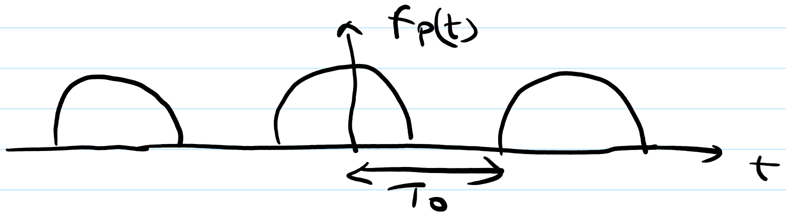
$N \to \infty$ 9% upward or downward.

Prof. Sushma Kadge

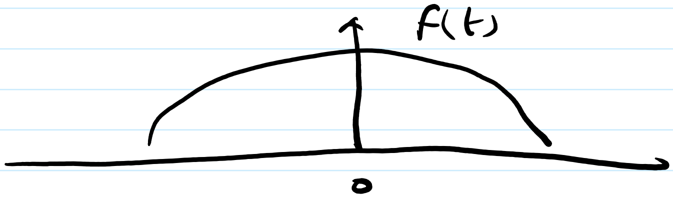
Tuesday, October 12, 2021 10:30 AM

* Fourier Transform * (FT)

* Scope & objective
  - freq. domain description of time domain signals
  - Extension of FS
  - Aperiodic signal.
- transforming a periodic s/g to an aperiodic s/g by stretching the period without limit.



periodic s/g



Non periodic s/g

FT → transformation technique, which transforms the signals from CT domain to corresponding freq domain & vice versa, → applied to both periodic & aperiodic (study) signals.

CTFT → (FT) ✓

✓ DTFT } higher sem.
✓ DFT

→ FT → extremely useful mathematical tool (LTI)

Application :-

— Signal Analysis (ECG, image, video)

— Signal Processing

— Photography

— astronomy

— Biomedical.

— Radar

— Communication.

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}$$

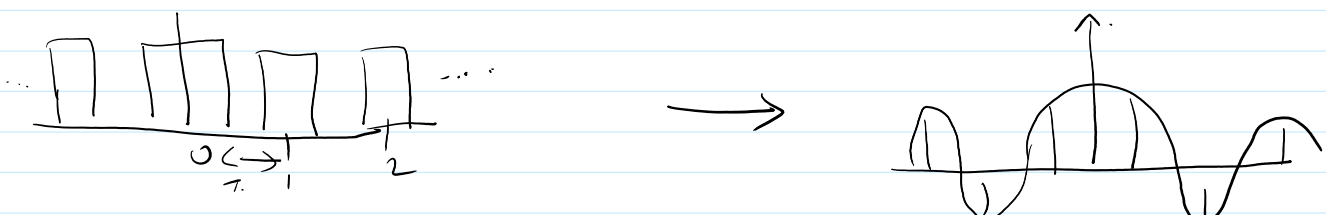
$$c_n = \frac{1}{T} \int x(t)\, e^{-jn\omega_0 t}\, dt$$

F.S → F.T?

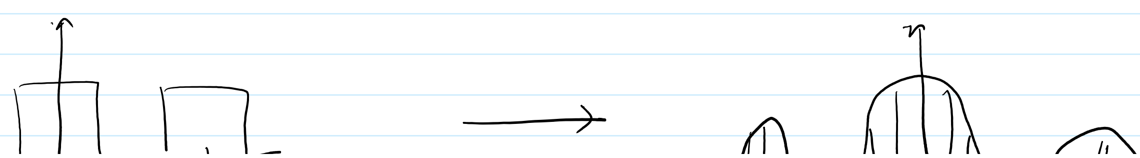
$$x(t) = \sum_{n=-\infty}^{\infty} \left[\frac{1}{T} x(t) e^{-jn\omega_0 t} dt \right] e^{jn\omega_0 t}$$

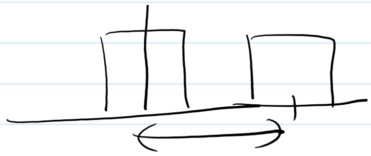
$$\omega_0 = 2\pi f = \frac{2\pi}{T} \Rightarrow T = \frac{2\pi}{\omega_0}$$

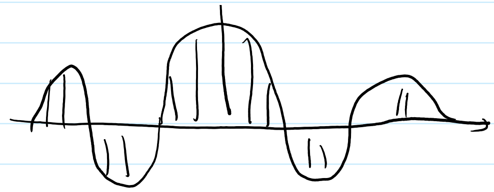
$$x(t) = \sum_{n=-\infty}^{\infty} \left[\frac{1}{2\pi} x(t) e^{-jn\omega_0 t} dt \right] e^{jn\omega_0 t}\, \omega_0$$



$T_0 = 1$ $F_0 = 1$ $\omega_0 = 2\pi$

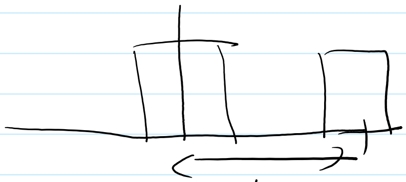


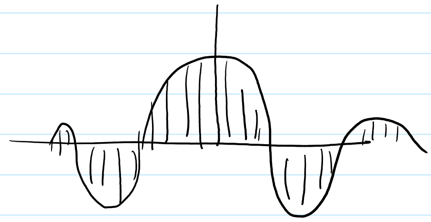




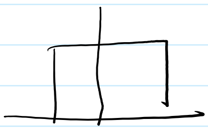
2

$T_0 = 2 \qquad \omega_0 = \pi$





$T_0 = 4 \qquad \omega_0 = \pi/2$



$T_0 \to \infty \qquad \omega_0 \to 0$ (infinitesimal small)

$$\omega_0 = \Delta\omega_0 \to d\omega \qquad \Sigma \to \int$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \underbrace{\left[\int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt \right]}_{F.T.} e^{j\omega t}\, d\omega$$

F.T. $$\boxed{X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt} \qquad (1)$$

IFT $$\boxed{x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega} \qquad (2)$$

$$F\{x(t)\} \xrightarrow[F.T]{} X(\omega)$$

$$F^{-1}\{X(\omega)\} \xrightarrow[IFT]{} x(t)$$

$$\boxed{x(t) \longleftrightarrow X(\omega)}$$

Condition for existence F.T

(1) The fn $x(t)$ should be absolutely integrable fn

That means $\int_{-\infty}^{\infty} x(t)\,dt < \infty$

eg. vol $\rightarrow$ Resistor. energy dissipiated.

(2) The function $x(t)$ should be single value in any finite time interval T

(3) finite number of discontinuities in any finite interval T

(4) " " " maxima & minima " " "

* Amplitude & phase spectrums.

* " " " are continuous in nature

$$X(\omega) = X_R(\omega) + j X_I(\omega)$$

$$|X(\omega)| = \sqrt{X_R(\omega^2) + X_I(\omega)^2}$$

$$\angle X(\omega) = -\tan^{-1} \frac{X_I(\omega)}{X_R(\omega)}$$

Plot $\rightarrow$ $|X(\omega)|$ Vs ω $\rightarrow$ Amplitude spectrum } Freq Spectrum

$\angle X(\omega)$ Vs ω $\rightarrow$ Phase "

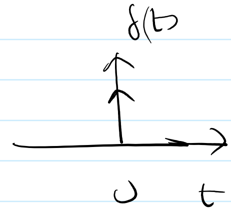
Fourier Transform of Standard Signals:-

→ Impulse function $\delta(t)$ $x(t) = \delta(t)$ → given

$$\delta(t) = \begin{cases} 1 & t = 0 \\ 0 & t \neq 0 \end{cases}$$

$X(\omega) = ?$

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$



$$= \int_{-\infty}^{\infty} \delta(t)\, e^{-j\omega t}\, dt$$

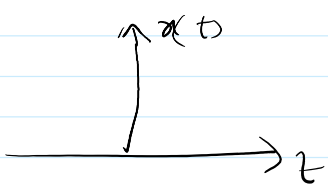
$$= 1 \cdot e^{-j\omega t}\Big|_{t=0}$$

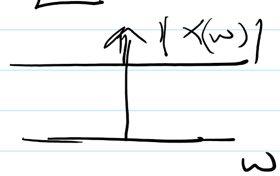
$$\boxed{X(\omega) = 1}$$

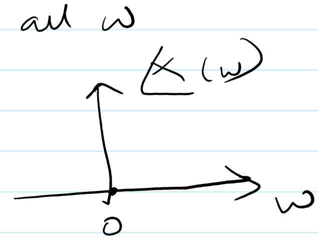
$$\boxed{\delta(t) \xleftrightarrow{F.T} 1}$$

Spectrum :- $|X(\omega)| = 1$ for all ω

$\angle X(\omega) = 0$ for all ω







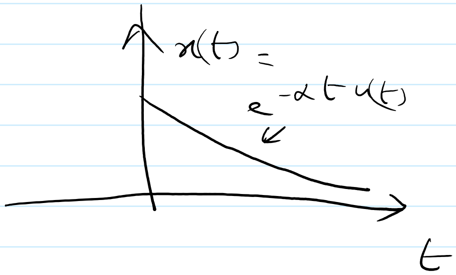
Impulse fn mag. spectrum phase spectrum.

$$\boxed{F\,\delta(t - t_0) \longrightarrow e^{-j\omega t_0}}$$

② Single sided exponential function $e^{-\alpha t} u(t)$

$\Rightarrow$

$$x(t) = e^{-\alpha t} \quad t \geq 0$$
$$= 0 \quad t < 0$$



$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$= \int_{0}^{\infty} e^{-\alpha t}\, e^{-j\omega t}\, dt$$

$$= \int_{0}^{\infty} e^{-(\alpha + j\omega) t}\, dt$$

$$= \left. \frac{e^{-(\alpha + j\omega) t}}{-(\alpha + j\omega)} \right|_{0}^{\infty}$$

$$= \frac{-1}{\alpha + j\omega} \left[e^{-(\alpha + j\omega)\infty} - e^{0} \right]$$

$$\boxed{X(\omega) = \frac{1}{\alpha + j\omega}}$$

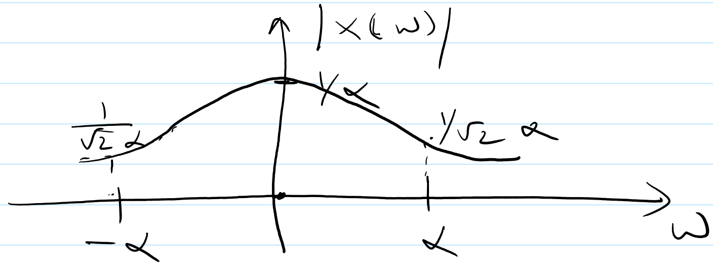
$$\boxed{e^{-\alpha t} u(t) \Longleftrightarrow \frac{1}{\alpha + j\omega}}$$

Spectrum :-

$$\boxed{|X(\omega)| = \frac{1}{\sqrt{\alpha^2 + \omega^2}}}$$ mag.

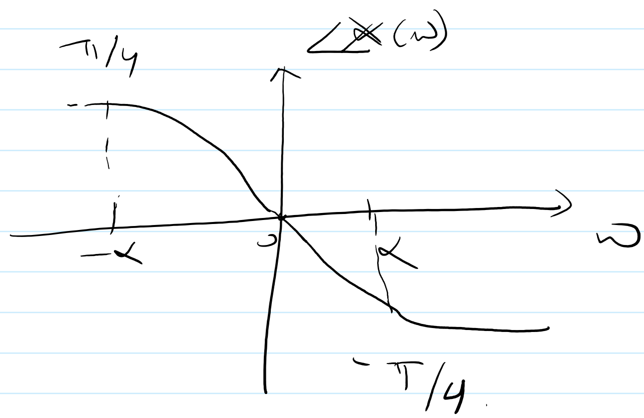
$$\boxed{\angle X(\omega) = -\tan^{-1}\frac{\omega}{\alpha}}$$ Phase

ω	$\lvert X(\omega)\rvert$
0	$1/\alpha$
$\pm\alpha$	$\frac{1}{\sqrt{2}\,\alpha}$

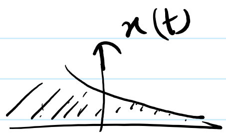


Phase Spectrum

ω	$\angle X(\omega) = -\tan^{-1}\frac{\omega}{\alpha}$
0	0
$+\alpha$	$-\pi/4$
$-\alpha$	$\pi/4$
∞	$-\pi/2$



13/10/21

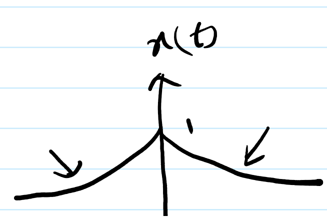
Note : 

$e^{-\alpha t}$ → area under curve is not finite → hence application of FT formula give absurd result.

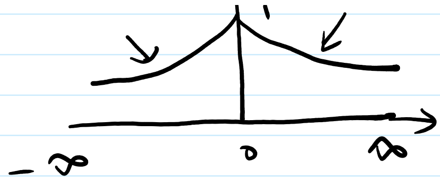
Integral doesnot converge. ∴ does not FT

③ Double sided exponential signal

F.T of $e^{-\alpha|t|}$



⇒



$$\Rightarrow \quad X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{0} e^{\alpha t}\, e^{-j\omega t}\, dt + \int_{0}^{\infty} e^{-\alpha t} \cdot e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{0} e^{(\alpha - j\omega)t}\, dt + \int_{0}^{\infty} e^{-(\alpha + j\omega)t}\, dt$$

$$= \left. \frac{e^{(\alpha - j\omega)t}}{\alpha - j\omega} \right|_{-\infty}^{0} + \left. \frac{e^{-(\alpha + j\omega)t}}{-(\alpha + j\omega)} \right|_{0}^{\infty}$$

$$= \frac{1}{\alpha - j\omega} + \frac{1}{\alpha + j\omega}$$

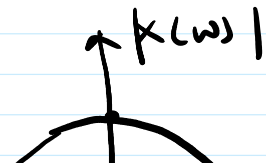
$$= \frac{\alpha + \cancel{j\omega} + \alpha - \cancel{j\omega}}{\alpha^2 + \omega^2} = \frac{2\alpha}{\alpha^2 + \omega^2}$$

$$\boxed{X(\omega) = \frac{2\alpha}{\alpha^2 + \omega^2}}$$

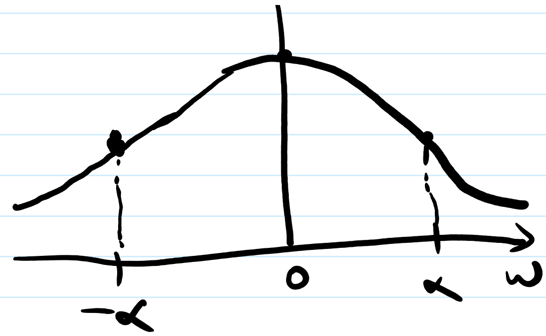
Spectrum :-

ω	$\lvert X(\omega) \rvert$

Amplitude Spectrum.



0	$\rightarrow 2/\alpha$
$\mp\alpha$	$\mp\frac{1}{\alpha}$



phase Spectrum :-

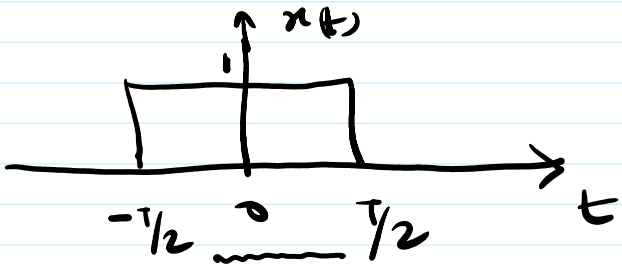
phase is zero at all values of w

$$\angle X(\omega) = -\tan^{-1}\frac{\text{Imag}}{\text{Real}} \rightarrow$$

(4) F.T of gate function / rectangular pulse

$\Rightarrow$ Function is expressed as

$$G_T(t) = 1 \qquad T/2 \le t < T/2$$
$$0 \qquad \text{otherwise}$$



$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$= \int^{T/2} 1 \cdot e^{-j\omega t}\, dt$$

$$= \int_{-T/2} 1 \cdot e^{-j\omega t} \, dt$$

$$= \left. \frac{e^{-j\omega t}}{-j\omega} \right|_{-T/2}^{T/2}$$

$$= \frac{1}{j\omega} \left[-e^{-j\omega T/2} \underbrace{-(}_{+} -e^{+j\omega T/2}) \right]$$

$$X(\omega) = \frac{1}{j\omega} \left[e^{+j\omega T/2} - e^{-j\omega T/2} \right] \times \frac{2}{2}$$

$$= \frac{2}{\omega} \left[\frac{e^{+j\omega T/2} - e^{-j\omega T/2}}{2j} \right]$$

$$\boxed{X(\omega) = \frac{2}{\omega} \sin \frac{\omega T}{2}} \quad *$$

This fn is written using sinc(x) or Sa(x) → sampling fn. & it is

$$\text{Sinc } x = \frac{\sin x}{x} \quad *$$

$$X(\omega) = \cancel{2} \cdot \frac{T}{\cancel{2}} \frac{\sin \omega T/2}{\omega \, T/2}$$

form $\Rightarrow \frac{\sin x}{x}$

$$X(\omega) = T \cdot \frac{\sin \omega T/2}{\omega T/2}$$

$\omega T/2$

$$\boxed{X(\omega) = T \cdot \operatorname{sinc}(\omega T/2)}$$

$\operatorname{sinc}(x) = \frac{\sin x}{x}$ $\quad x = \frac{\omega T}{2}$

$\operatorname{sinc}(0) = 1$

$\operatorname{sinc}(\pi) = 0$

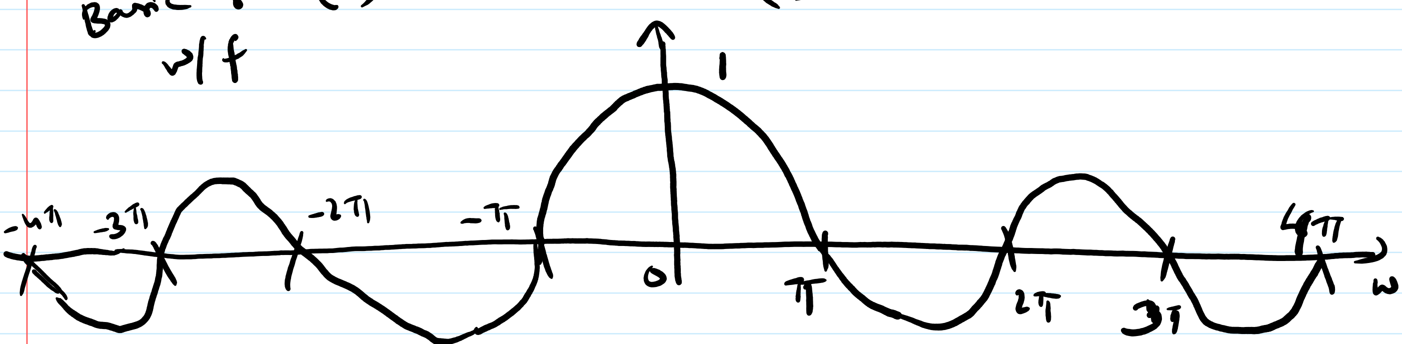
$x = \pm p\pi$ where 'p' is any integer

$$\frac{\omega T}{2} = \pm p\pi$$

$$\omega = \pm \frac{2p\pi}{T}$$

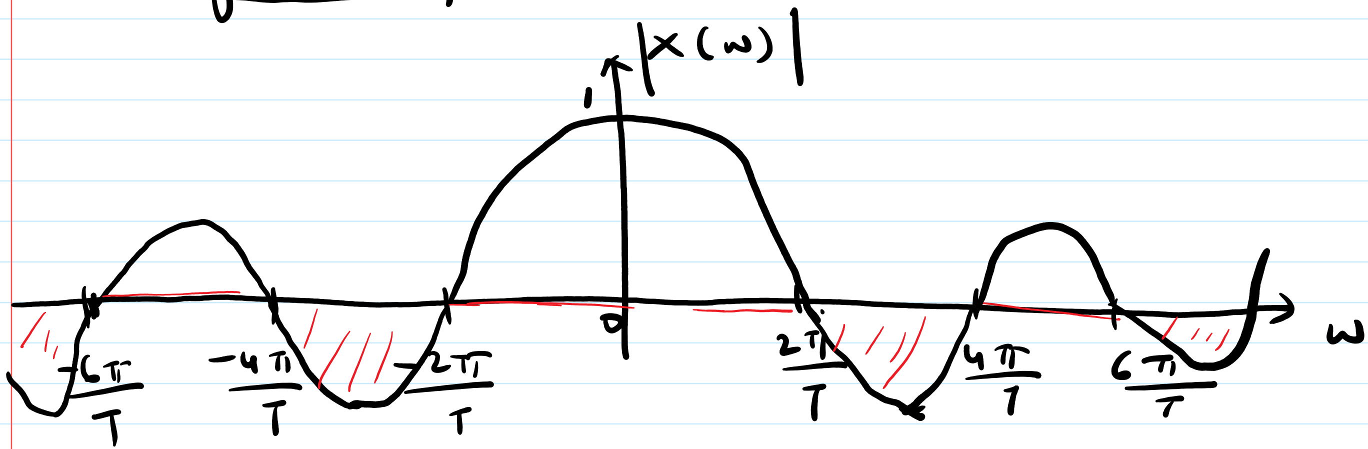
$$= \pm \frac{2\pi}{T}, \frac{4\pi}{T}, \frac{6\pi}{T} \dots\dots\dots$$

Basic sinc(x) w/f



<u>Rectangular pulse/gate fn.</u>

magnitude spectrum :-

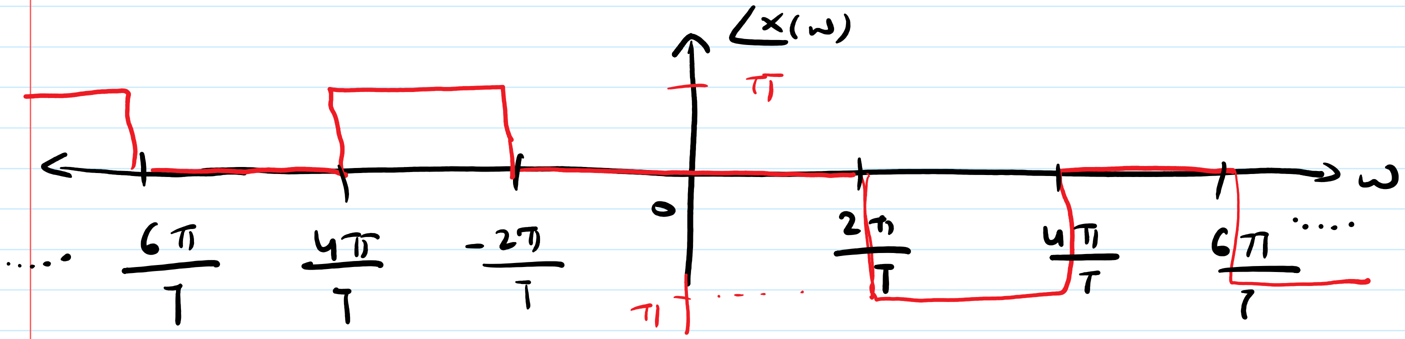


phase spectrum :-

$\angle X(\omega) \rightarrow$ Considering the angle $\pm\pi$

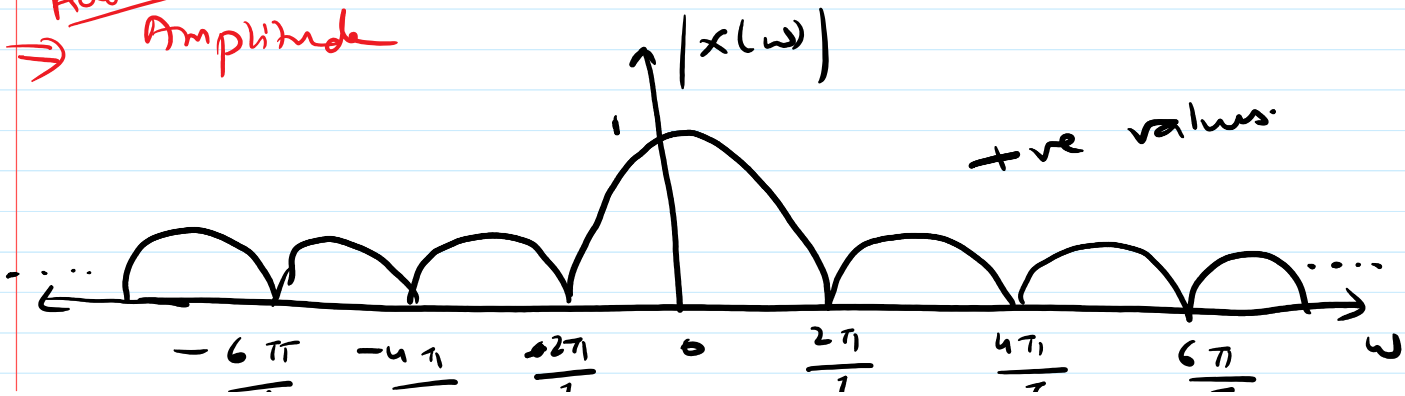
Angle is zero $\rightarrow |X(\omega)| = +ve$

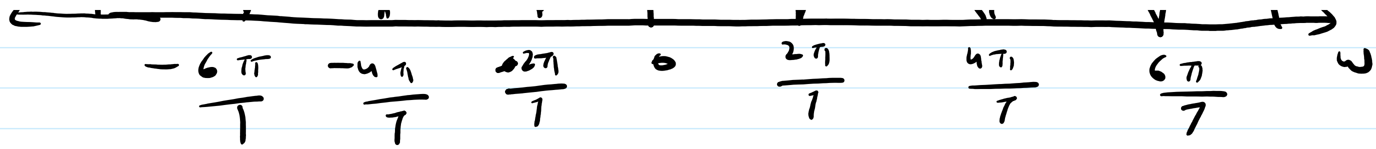
Angle is $\pm\pi$ $\quad |X(\omega)| = -ve$



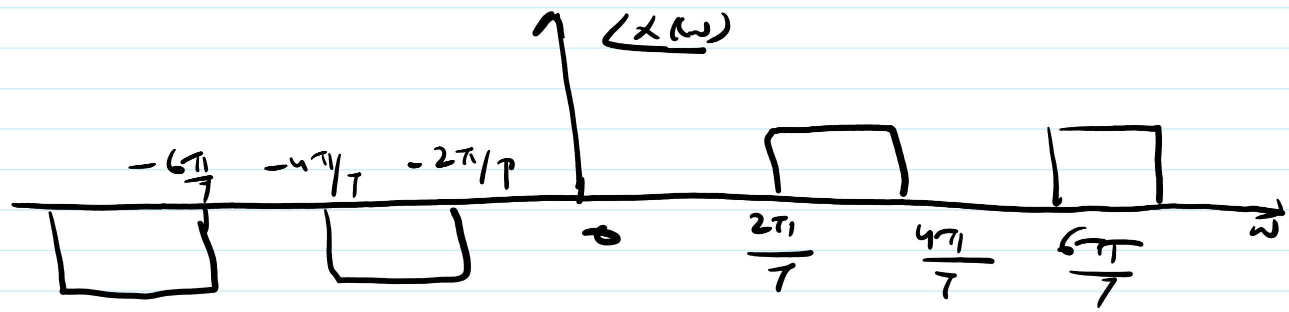
Additional

$\Rightarrow$ Amplitude





−ve values of amplitude → +ve by phase shift of ± 180 in the phase spectrum

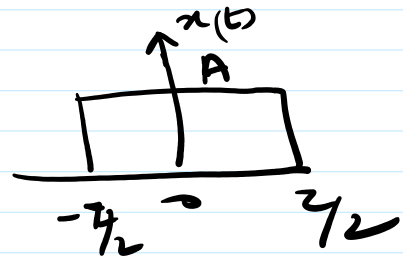


(5) Shifted Rectangular pulse → determine F.T

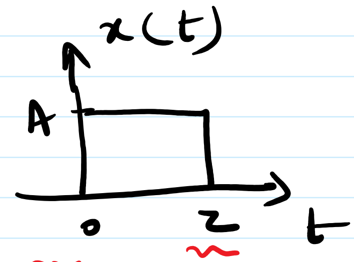
↓

delay by $\tau/2$ or $T/2$

⇒



Shifted ⇒ pulse



⇓

F.T = ?

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$= \int_{0}^{\tau} A \cdot e^{-j\omega t}\, dt$$

$$= A \cdot \left[\frac{e^{-j\omega t}}{-j\omega}\right]_{0}^{\tau}$$

Shifted pulse

↓

$$= \frac{A}{j\omega}\left[1 - e^{-j\omega\tau}\right]$$

$\downarrow$

F.T. sinc

$\frac{\sin x}{x}$

$$= \frac{A}{j\omega}\, e^{-j\omega\tau/2}\left[e^{j\omega\tau/2} - e^{-j\omega\tau/2}\right]$$

multiply & divide by 2

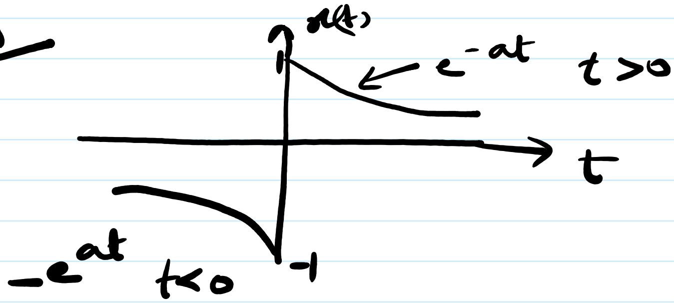
$$X(\omega) = \frac{2A}{\omega}\, \underbrace{e^{-j\omega\tau/2}}_{\text{shift}} \cdot \sin\omega\tau/2$$

$$X(\omega) = \cancel{2}A \cdot \tau/\cancel{2}\; e^{-j\omega\tau/2}\; \frac{\sin\omega\tau/2}{\omega\tau/2}$$

$$\boxed{X(\omega) = A\tau\, \underbrace{e^{-j\omega\tau/2}}_{\text{shift by } \tau/2}\, \mathrm{sin}c(\omega\tau/2)}$$

H.W spectrum.

14/10/21



$$\Rightarrow \quad X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

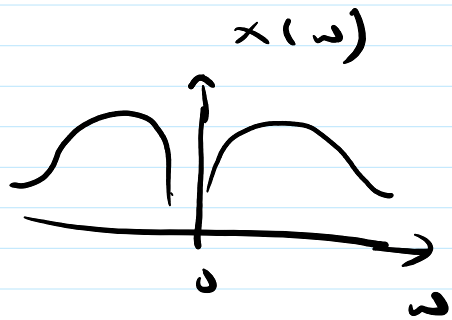
$$= \int_{-\infty}^{0} -1 \cdot e^{at}\, e^{-j\omega t}\, dt + \int_{0}^{\infty} e^{-at}\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{0} -e^{(a-j\omega)t}\, dt + \int_{0}^{\infty} e^{-(a+j\omega)t}\, dt$$

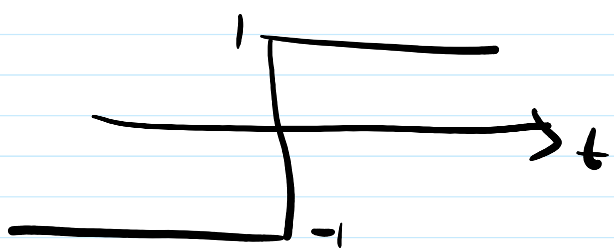
$$= -\frac{1}{\alpha - j\omega} + \frac{1}{\alpha + j\omega}$$

$$X(\omega) = \frac{-\cancel{\alpha} - j\omega + \cancel{\alpha} - j\omega}{\alpha^2 + \omega^2} = \frac{-2j\omega}{\alpha^2 + \omega^2}$$

$$\boxed{X(\omega) = \frac{-2j\omega}{\alpha^2 + \omega^2}}$$



*.



$\Rightarrow$ signum function $\operatorname{sgn}(t) = \begin{cases} 1 & t > 0 \\ -1 & t < 0 \end{cases}$

function is not absolutely integrable.

so cannot directly find its F.T

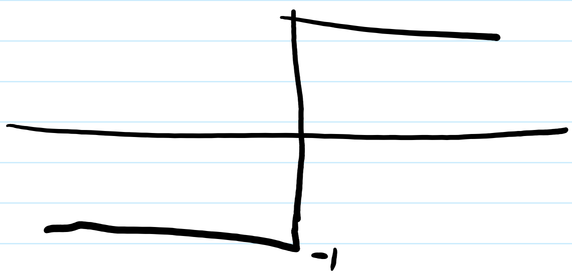
how = ?

multiplying fn by expo.

step

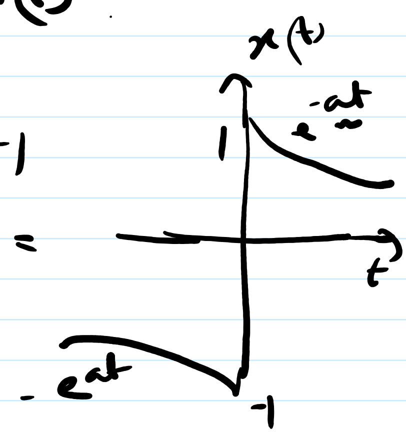
① $x(t) = e^{-a|t|}\,\mathrm{sgn}(t)$.

② $a \to 0 \to$ obtain $\mathrm{sgn}(t)$



$\times\; e^{-a|t|}$

=



$$X(\omega) = \lim_{a\to 0}\left[\mathrm{sgn}(t)\,e^{-a|t|}\right]$$

from previous problem.

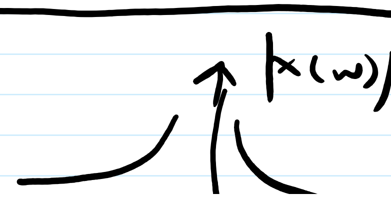
$$= \lim_{a\to 0} \frac{-2j\omega}{a^2+\omega^2}$$

$$= \frac{-2j\omega}{\omega^2} = \frac{-2j}{\omega} = \frac{2}{j\omega}$$

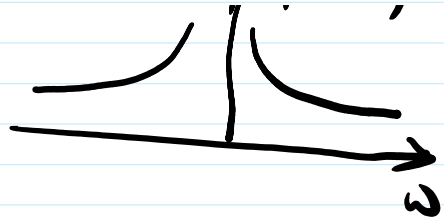
$$\boxed{X(\omega) = \frac{2}{j\omega}} \quad \text{or} \quad \boxed{\mathrm{sgn}(t) \longleftrightarrow \frac{2}{j\omega}}$$

magnitude spectrum

$$|X(\omega)| = \frac{2}{\omega}$$



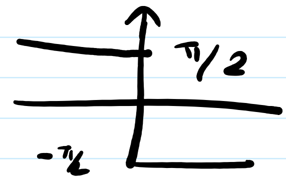
$$|X(\omega)| = \frac{2}{\omega}$$



phase spectrum

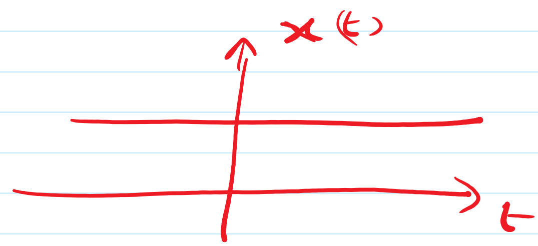
$$\angle X(\omega) = \frac{-\pi}{2} \quad \text{for } \omega > 0$$

$$= +\frac{\pi}{2} \quad \text{for } \omega < 0$$



* F.T of a constant

⇒ Let $f(t) = 1$

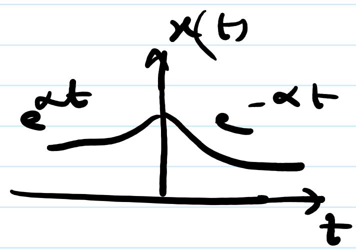


— Not absolutely integrable

— area under $|x(t)|$ infinite → attempt to evaluate F.T integral fails.

Step

① So → consider for $e^{-\alpha|t|}$



② $\alpha \to 0$

$$x(t) = 1 = \lim_{\alpha \to 0} F\left[e^{-\alpha|t|}\right]$$

$$F\left(e^{-\alpha|t|}\right) \Longleftrightarrow \frac{2\alpha}{\alpha^2 + \omega^2}$$

$$X(\omega) = \lim_{\alpha \to 0} \frac{2\alpha}{\alpha^2 + \omega^2}$$

$= 0$ except at $\omega = 0$

[Case 1:- $\alpha \to 0$ $\omega =$ some value (10)

$$\lim_{\alpha \to 0} \frac{2}{\alpha + \frac{\omega^2}{\alpha}} = \underline{\underline{0}}$$

Case 2 $\omega = 0$

$$\lim_{\alpha \to 0} \frac{2}{\alpha + \frac{\omega^2}{\alpha}} =$$ some const value. depending α.

$$F\{x(t) = 1\} = A\,\delta(\omega)$$

$$\int_{-\infty}^{\infty} A\,\delta(t)\,dt = A$$

$$A = \int_{-\infty}^{\infty} \frac{2\alpha}{\alpha^2 + \omega^2}\,d\omega = 2 \cdot \tan^{-1}\frac{\omega}{\alpha}\Big|_{-\infty}^{\infty}$$

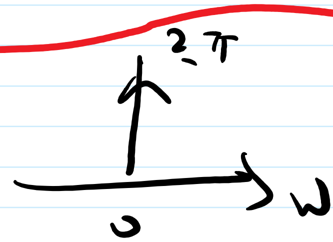
$$= 2\left[\tan^{-1}(\infty) - \tan^{-1}(-\infty)\right]$$

$$= 2\left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right)$$

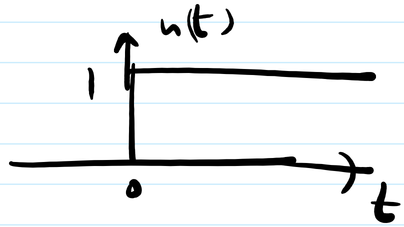
$$= 2\pi$$

$$\therefore \boxed{F\{x(t) = 1\} = 2\pi\,\delta(\omega)}$$

$$\therefore \boxed{F\{x(t)=1\} = 2\pi\,\delta(\omega)}$$

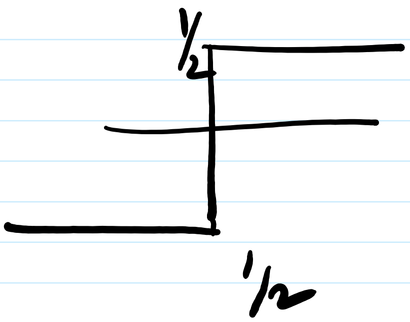


* F.T of unit step u(t)

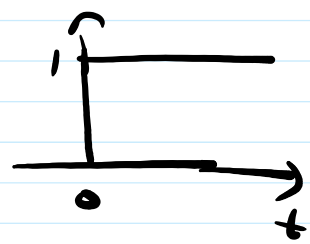


$\Rightarrow$ – Not absolutely integrable

– express u(t) in terms of sgn(t)



$$+ \frac{1}{2} =$$



$$\frac{1}{2}\,\mathrm{sgn}(t) + \frac{1}{2} = u(t)$$

$$X(\omega) = F\left[\frac{1}{2}\left(1 + \mathrm{sgn}(t)\right)\right]$$

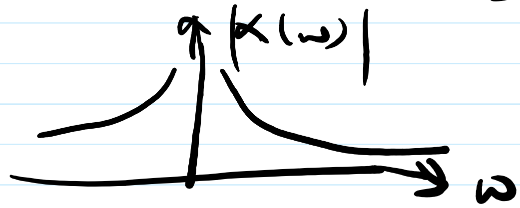
$$= \frac{1}{2}\left[2\pi\,\delta(\omega) + \frac{2}{j\omega}\right]$$

$$\boxed{X(\omega) = \pi\delta(\omega) + \frac{1}{j\omega}}$$

$$\boxed{u(t) \longleftrightarrow \pi\delta(\omega) + \frac{1}{j\omega}}$$

$|X(\omega)| = \infty$ at $\omega = 0$ & zero at $\omega = \pm\infty$

Spectrum.



Let, $\mathcal{F}\{x(t)\} = X(j\Omega)$; $\mathcal{F}\{x_1(t)\} = X_1(j\Omega)$; $\mathcal{F}\{x_2(t)\} = X_2(j\Omega)$

$j\Omega \; (\omega)$

Property	Time domain signal	Frequency domain signal
Linearity	$a_1 x_1(t) + a_2 x_2(t)$	$a_1 X_1(j\Omega) + a_2 X_2(j\Omega)$
Time shifting	$x(t - t_0)$	$e^{-j\Omega t_0} X(j\Omega)$
Time scaling	$x(at)$	$\frac{1}{\lvert a\rvert} X\left(\frac{j\Omega}{a}\right)$
Time reversal	$x(-t)$	$X(-j\Omega)$
Conjugation	$x^*(t)$	$X^*(-j\Omega)$
Frequency shifting	$e^{j\Omega_0 t} x(t)$	$X(j(\Omega - \Omega_0))$
Time differentiation	$\frac{d}{dt} x(t)$	$j\Omega\, X(j\Omega)$ → $(j\omega\, X(\omega))$
Time integration	$\int_{-\infty}^{t} x(\tau)\, d\tau$	$\frac{X(j\Omega)}{j\Omega} = \pi X(0)\, \delta(\Omega)$
Frequency differentiation	$t\, x(t)$	$j \frac{d}{d\Omega} X(j\Omega)$
imp Time convolution	$x_1(t) * x_2(t) = \int_{-\infty}^{+\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau$	$X_1(j\Omega) \cdot X_2(j\Omega)$
Frequency convolution (or Multiplication)	$x_1(t)\, x_2(t)$	$\frac{1}{2\pi} \int_{\lambda=-\infty}^{\lambda=+\infty} X_1(j\lambda)\, X_2(j(\Omega - \lambda))\, d\lambda$
Symmetry of real signals	$x(t)$ is real	$X(j\Omega) = X^*(j\Omega)$ $\lvert X(j\Omega)\rvert = \lvert X(-j\Omega)\rvert$; $\angle X(j\Omega) = -\angle X(-j\Omega)$ $\mathrm{Re}\{X(j\Omega)\} = \mathrm{Re}\{X(-j\Omega)\}$ $\mathrm{Im}\{X(j\Omega)\} = -\mathrm{Im}\{X(-j\Omega)\}$
Real and even	$x(t)$ is real and even	$X(j\Omega)$ are real and even
Real and odd	$x(t)$ is real and odd	$X(j\Omega)$ are imaginary and odd
Duality	If $x_2(t) \equiv X_1(j\Omega)$ [i.e., $x_2(t)$ and $X_1(j\Omega)$ are similar functions] then $X_2(j\Omega) \equiv 2\pi x_1(-j\Omega)$ [i.e., $X_2(j\Omega)$ and $2\pi x_1(-j\Omega)$ are similar functions]	
Area under a frequency domain signal	$\int_{-\infty}^{+\infty} X(j\Omega)\, d\Omega = 2\pi\, x(0)$	
Area under a time domain signal	$\int_{-\infty}^{+\infty} x(t)\, dt = X(0)$	
Parseval's relation	Energy in time domain is, $E = \int_{-\infty}^{+\infty} \lvert x(t)\rvert^2\, dt$	Energy in frequency domain is, $E = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \lvert X(j\Omega)\rvert^2\, d\Omega$
	$\int_{-\infty}^{+\infty} \lvert x(t)\rvert^2\, dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \lvert X(j\Omega)\rvert^2\, d\Omega$	

$(j\omega)^2 X(\omega)$

$(j\omega)^3 X(\omega)$

Examples

① If $m(t - t_0) + m(t + t_0) = p(t)$.

F.T of $P(t)$ using time shifting property.

→ Time shifting.

$$x(t \mp t_0) \longleftrightarrow e^{\mp j\omega t_0} X(\omega)$$

$$m(t - t_0) \xleftrightarrow{f} e^{-j\omega t_0} m(\omega) \quad \text{(1)}$$

$$m(t + t_0) \xleftrightarrow{f} e^{+j\omega t_0} M(\omega) \rightarrow \text{(2)}$$

$$p(t) \longleftrightarrow \underline{P(\omega)} = ?$$

$$P(\omega) = e^{-j\omega t_0} m(\omega) + e^{+j\omega t_0} m(\omega)$$

$$= 2 \underbrace{\left[\frac{e^{+j\omega t_0} + e^{-j\omega t_0}}{2} \right]} m(\omega)$$

$$\boxed{P(\omega) = 2 \cos(\omega t_0)\, m(\omega)}$$

21/10/21
11:30 am

(2) LTI system, initially at rest is described by differential eqn.

$$\underbrace{\frac{d^2 y(t)}{dt^2}}_{2^{nd}} + 3 \underbrace{\frac{d\, y(t)}{dt}}_{1^{st}} + 2y(t) = \frac{d\, x(t)}{dt} + 3x(t)$$

Determine transfer function & impulse response of the system.

→ $H(\omega) = \frac{Y(\omega)}{X(\omega)}$ → $h(t)$

⇒ Same problem by F.T / L.T

$x(t)$ i/p → $\boxed{\text{System } h(t)}$ → o/p $y(t)$ ⇒ conv.

$X(\omega)$. —— . $H(\omega)$

$Y(\omega)$ = multiplica

$X(\omega)$

$H(\omega)$

$Y(\omega) \rightarrow$

$H(\omega) = \dfrac{Y(\omega)}{X(\omega)}$?

$Y(\omega) = X(\omega) \cdot H(\omega)$

$y(t) \rightarrow$ o/p of system

$$\frac{d^2}{dt^2} = (j\omega)^2$$

$$\frac{d}{dt} = j\omega$$

$$y(t) \overset{F.T}{\longleftrightarrow} Y(\omega)$$

$$x(t) \longleftrightarrow X(\omega)$$

By time differentiation property.

$$(j\omega)^2 Y(\omega) + 3j\omega Y(\omega) + 2Y(\omega) = j\omega X(\omega) + 3X(\omega)$$

$$\boxed{H(\omega) = ?}$$

$$\boxed{H(\omega) = \frac{Y(\omega)}{X(\omega)}}$$

$$\boxed{H(\omega) = \frac{j\omega + 3}{[(j\omega)^2 + 3j\omega + 2]}}$$

① $h(t) = ?$ root of denominator

$$\text{roots} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= -1, -2$$

$$H(\omega) = \frac{3 + j\omega}{(j\omega+1)(j\omega+2)}$$

$$H(\omega) \xleftrightarrow{\text{IFT}} h(t)$$

PFE $$H(\omega) = \frac{A}{j\omega+1} + \frac{B}{j\omega+2}$$

2 Coeff:
$$A = (j\omega+1) \cdot H(\omega) \Big|_{j\omega=-1} = 2$$

$$B = (j\omega+2)\, H(\omega) \Big|_{j\omega=-2} = -1$$

$$H(\omega) = \frac{2}{j\omega+1} - \frac{1}{j\omega+2}$$

3 IFT $$e^{-at}\,u(t) \longleftrightarrow \frac{1}{a+j\omega}$$

$$\boxed{h(t) = 2 \cdot e^{-t}\,u(t) - 1 \cdot e^{-2t}\,u(t)}$$

This is impulse response of system.

Eg
$$\frac{d^2 y(t)}{dt^2} + \frac{3\,d\,y(t)}{dt} + 2y(t) = x(t)$$

Calculate o/p $\rightarrow$ $y(t)$ if input $x(t) = e^{-3t}u(t)$

$\Rightarrow$ 1 Take F.T on B.S

$$(j\omega)^2 Y(\omega) + 3(j\omega) Y(\omega) + 2Y(\omega) = X(\omega)$$

$$\frac{Y(\omega)}{X(\omega)} = \frac{1}{(j\omega)^2 + 3j\omega + 2}$$

2 $x(t) = e^{-3t} u(t)$

$$X(\omega) = \frac{1}{3 + j\omega}$$

$$Y(\omega) = \frac{1}{(j\omega)^2 + 3j\omega + 2)} \cdot \frac{1}{3 + j\omega}$$

3 $y(t) = ?$

$$Y(\omega) \xleftrightarrow{IFT} y(t)$$

a roots, PFE

$$Y(\omega) = \frac{1}{(1 + j\omega)(2 + j\omega)} \cdot \frac{1}{3 + j\omega}$$

$$Y(\omega) = \frac{A}{1 + j\omega} + \frac{B}{2 + j\omega} + \frac{C}{3 + j\omega}$$

Calculate A, B, C.

$A = 1/2 \quad B = -1, \quad C = 1/2$

$$Y(\omega) = \frac{1}{2} \frac{1}{1 + j\omega} - 1 \cdot \frac{1}{2 + j\omega} + 1/2 \cdot \frac{1}{3 + j\omega}$$

IFT

$$\boxed{y(t) = \frac{1}{2} e^{-t} u(t) - e^{-2t} u(t) + \frac{1}{2} e^{-3t} u(t)}$$

IFT

$$\boxed{y(t) = \frac{1}{2} e^{-t} u(t) - e^{-2t} u(t) + \frac{1}{2} e^{-3t} u(t)}$$

This is the o/p of system

Determine convolution of $x_1(t) = e^{-2t} u(t)$

$x_2(t) = e^{-6t} u(t)$.

$\Rightarrow$

$$X_1(\omega) = \frac{1}{2 + j\omega} \qquad X_2(\omega) = \frac{1}{6 + j\omega}.$$

convolution property. $\rightarrow$ conv $\longrightarrow$ multiplic^n freq.

$$X(\omega) = X_1(\omega) \cdot X_2(\omega)$$

$$X(\omega) = \frac{1}{2 + j\omega} \cdot \frac{1}{6 + j\omega}$$

PFE $\quad X(\omega) = \frac{A}{2 + j\omega} + \frac{B}{6 + j\omega}$

$$X(\omega) = \frac{0.25}{2 + j\omega} - \frac{0.25}{6 + j\omega}.$$

$$\boxed{x(t) = 0.25 e^{-2t} u(t) - 0.25 e^{-6t} u(t)}$$

Eg: use Time shifting property. find F.T of

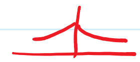
$$x(t) = e^{-a|t - t_0|}$$

$\Rightarrow \quad X(\omega) = ?$

$$x(t - t_0) = e^{-j\omega t_0} X(\omega)$$

$$x(t - t_0) = e^{-j\omega t_0} X(\omega)$$

$$e^{-a|t|} = \frac{2a}{a^2+\omega^2}$$



$$\boxed{e^{-a|t-t_0|} \xleftrightarrow[\text{property}]{\text{Time}} e^{-j\omega t_0} \cdot \frac{2a}{a^2+\omega^2}}$$