TYPE-I ANALYTIC FUNCTIONS

1. Prove that an analytic function with its derivative zero is constant.

2. If $f(z) = u + iv$ is an analytic function and v = constant then $f(z)$ is constant.

3. If $f(z)$ is an analytic function with constant modulus then, prove that $f(z)$is constant.

4. Every analytic function $w = u + iv$ can be expressed as a function of z only.

5. Show that the function $f(z) = e^{2z}$ is analytic and find its derivatives.

6. Determine whether the following functions are analytic and if so find their derivatives

(i) $\cos h\, z$ **(ii)** $\cos z$ **(iii)** $\frac{1}{z}$ **(iv)** $z^2 + z$

(v) $z^2 - \bar{z}$ **(vi)** $e^x(\cos y - isin\, y)$ **(vii)** $\frac{1}{2}log(x^2 + y^2) + itan^{-1}\frac{y}{x}$

(viii) $e^{-x}(\cos y - i\sin y)$ **(ix)** $x^2 - y^2 + 2i\, xy$

(x) $(x^3 - 3xy^2 + 3x) + i(3x^2y - y^3 + 3y)$ **(xi)** $z\, e^{2z}$

7. Verify that the real and Imaginary parts of $f(z) = ze^{2z}$ satisfy the Cauchy – Riemann equations.

8. Show that the functions **(i)** $f(z) = \bar{z}$ **(ii)** $f(z) = z|z|$ are not analytic

9. Prove that $f(z) = (x^3 - 3xy^2 + 2xy) + i(3x^2y - x^2 + y^2 - y^3)$ is analytic and find $f'(z)$ *and* $f(z)$ in terms of z.

10. Find the constants a, b, c, d, e if

(i) $f(z) = (ax^3 + bxy^2 + 3x^2 + cy^2 + x) + i(dx^2y - 2y^3 + exy + y)$ are analytic

(ii) $f(z) = (ax^4 + bx^2y^2 + cy^4 + dx^2 - 2y^2) + i(4x^3 - exy^3 + 4xy)$ are analytic

11. Find the values of z for which the following function is not analytic.

$z = \sin u \cos hv + i \cos u \sin hv$

12. Find the value of k if $f(z) = r^3 \cos k\,\theta + \; i\, r^{\,k} \sin k\,\theta$ is analytic.

13. Is $f(z) = \frac{z}{\bar{z}}$ analytic?

14. If $w = log\, z$ determine whether w is analytic and find $\frac{dw}{dz}$.

15. If $f(z) = u + iv$ is analytic in R show that **(i)** $\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = |f'(z)|^2$

(ii) $\left[\frac{\partial |f(z)|}{\partial x}\right]^2 + \left[\frac{\partial |f(z)|}{\partial y}\right]^2 = |f'(z)|^2$

Type-II Harmonic Function & Milne Thomson Method

Construct an analytic function whose real part is

1. $\frac{\sin 2x}{\cos h\, 2y + \cos 2x}$

2. $x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$

3. $log\sqrt{x^2 + y^2}$

4. $sinx \cos h\, y$

5. $e^x(xcos\, y - y \sin y)$

6. $\frac{x}{2}log(x^2 + y^2) - ytan^{-1}\left(\frac{y}{x}\right) + \sin xcos\, hy$

Find an analytic function whose imaginary part is

7. $log(x^2 + y^2) + x - 2y$

8. $\frac{x-y}{x^2+y^2}$

9. $\sin h\, x \cos y$

10. $e^{-x}(x \sin y - y\cos\, y)$

11. $\frac{\sin h\, 2y}{\cos 2x + \cos h\, 2y}$

12. $e^{-x}[2xy \cos y + (y^2 - x^2) \sin y]$

13. $\frac{x}{x^2+y^2} + \cos hx \cos y$

14. $\frac{y}{x^2+y^2}$

15. Show that there does not exist an analytic function whose real part is

(i) $3x^2 + \sin x + y^2 + 5y + 4$ **(ii)** $3x^2 - 2x^2y + y^2$

16. Show that the following functions are harmonic

(i) $e^x \cos y + x^3 - 3xy^2$ **(ii)** $e^{2x}(x \cos 2y - y \sin 2y)$

(iii) $\log\sqrt{x^2 + y^2}$

17. Show that the following functions are harmonic and find the corresponding analytic function $f(z) = u + iv$

(i) $v = e^{-x}(x \cos y + y \sin y)$ **(ii)** $v = e^{2x}(y \cos 2y + x \sin 2y)$

(iii) $u = e^x \cos y - x^2 + y^2$ **(iv)** $u = x^2 - y^2$

(v) $u = x^3 - 3xy^2 + 3x^2y - y^3 + 1$ **(vi)** $u = (x-1)^3 - 3xy^2 + 3y^2$

18. Show that the following functions are harmonic. Also find the corresponding harmonic conjugate function and analytic function.

(i) $u = e^x \cos y$ **(ii)** $u = x^2 - y^2 - 2xy - 2x + 3y$

(iii) $u = 3x^2y + 2x^2 - y^3 - 2y^2$ **(iv)** $u = e^x(x \cos y - y \sin y)$

(v) $u = e^{-2xy} sin(x^2 - y^2)$ **(vi)** $u = \frac{\sin 2x}{\cos h\, 2\, y + \cos 2x}$

19. Prove that $u(x,y) = x^2 - y^2$ and $v(x,y) = -y/(x^2 + y^2)$ are both harmonic functions but $u + iv$ is not analytic.

20. Find the analytic function $f(z)$ whose real part is

(i) $r^2 \cos 2\theta - r \sin \theta$ **(ii)** $r^n \cos n\theta$

21. Show that $u = r - \frac{a^2}{r} \sin\theta$ cannot be the real part of an analytic function $f(re^{i\theta}) = u(r,\theta) + iv(r,\theta)$

22. If $f(z)$ is an analytic function, prove that $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(u^2 + v^2) = 4\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right]$

Or $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4|f'(z)|^2$

23. If $f(z)$ is an analytic function, prove that $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) log|f'(z)| = 0$

24. Find the orthogonally trajectories of the family of the curves

1. $3x^2y - y^3 = c$

2. $x^2 - y^2 + x = c$

3. $e^{-x}(xsin\, y - ycos\, y) = c$

4. $x^2 - y^2 - 2xy + 2x - 3y = c$

5. $e^x \cos y - xy = c$

6. $3x^2y + 2x^2 - y^3 - 2y^2 = c$