Complex Integration , Taylor's and Laurent's series

TYPE-I   LINE INTEGRAL

1. If O is the origin, L is the point $z = 3$, M is the point $z = 3 + i$, evaluate $\int z^2 dz$ along

(i) the path OM, **(ii)** the path OLM, **(iii)** the path OLMO

2. Evaluate $\int_0^{3+i} z^2 dz$

(i) along the real axis from 0 to 3 and then vertically to $3 + i$.

(ii) along the imaginary axis from 0 to i and then horizontally to $3 + i$

(iii) along the parabola $x = 3y^2$

3. Integrate the function $f(z) = x^2 + ixy$ from $A(1,1)\ to\ B\ (2,4)$ along the curve $x = t, y = t^2$

4. Evaluate $\int_0^{1+2i} z^2 dz$ along the curve $2x^2 = y$

5. Evaluate $\int_0^{1+i} z^2 dz$, along **(i)** the line $y = x$, **(ii)** the parabola $x = y^2$ Is the line integral independent of the path? Explain?

6. Evaluate $\int_0^{1+i} (x^2 + iy) dz$, along the path **(i)** $y = x$, **(ii)** $y = x^2$

Is the line integral independent of the path?

7. Evaluate $\int f(z) dz$ along the parabola $y = 2x^2$ from $z = 0$ to $z = 3 + 18i$ where $f(z) = x^2 - 2iy$

8. Evaluate $\int f(z) dz$, along the parabola $y = 2x^2$ from $z = 0\ to\ z = 3 + 18i$ where $f(z) = x^2 - 2i\, xy$.

9. Evaluate $\int_c z^2\, dz$, where C is the circle $x = r cos\theta, y = r \sin\theta$, from $\theta = 0\ to\ \theta = \pi/3$

10. Show that $\int_c \log z\, dz = 2\pi i$, where C is the unit circle in the z – plane.

11. Evaluate $\int_c (z^2 - 2\bar{z} + 1) dz$, where C is the circle $x^2 + y^2 = 2$

12. Evaluate $\int_c (z^2 + 3z^{-4}) dz$, where C is upper half of the unit circle from $(1,0) to\ (-1, 0)$

13. Find $\int Im(z)\, dz$ along **(i)** the unit circle described once in positive direction from $z = 1, to\ z = 1$

(ii) the straight line from $P(z_1)\ to\ Q(z_2)$

14. Evaluate $\int_c (3z^2 + 2z + 1) dz$, where C is arc of the cycloid $x = a(\theta + sin\theta), y = a(1 - cos\theta)$

Between $\theta = 0\ to\ \theta = 2\pi$

15. Evaluate $\int f(z) dz$ along the square whose vertices are $(1,1), (2,1), (2,2), (1,2)$ in anti – clockwise direction where $f(z) = x - 2iy$

TYPE-II CAUCHY'S INTEGRAL THEOREM, CAUCHY'S INTEGRAL FORMULA

1. Evaluate $\int_c \cot z . dz$ where C is $\left|z + \frac{1}{2}\right| = \frac{1}{3}$

2. Evaluate $\oint_c \frac{e^{3z}}{z - \pi i} dz$ where C is the curve $|z - 2| + |z + 2| = 6$

3. Evaluate $\int_c \frac{1}{z} . \cos z\, dz$ where C is the ellipse $9x^2 + 4y^2 = 1$

4. Evaluate $\int_c \frac{e^{3z}}{z - i} dz$ where C is the curve $|z - 2| + |z + 2| = 6$

5. Evaluate $\int_c \frac{3z^2+z}{z^2-1} dz$, where C is the circle $|z| = 2$

6. Evaluate $\int_c \frac{e^{2z}}{(z-1)(z-2)} dz$, where C is the circle $|z| = 3$

7. If C is the circle $|z| = 1$, using the integral $\int_c \frac{e^{kz}}{z} dz$ where k is real, show that $\int_0^{\pi} e^{\mathrm{k}\cos\theta} \cos(k\sin\theta) d\theta = \pi$

8. Evaluate $\int_c \frac{e^{2z}}{z-1} dz$ where C is the circle, find **(i)** $|z| = 2,$ **(ii)** $|z| = 1/2$

9. Evaluate $\int_c \frac{dz}{\sin hz}$ where C is the circle $x^2 + y^2 = 16$

10. Evaluate $\int_c \frac{\sin 3z}{z+(\pi/2)} dz$ where C is the circle $|z| = 5$

11. If $f(z) = z^3 + iz^2 - 4z - 4i$ evaluate $\int_c \frac{f'(z)}{f(z)} dz$

where C is a simple closed curve enclosing zeros of $f(z)$

12. Evaluate $\int_c \frac{z+3}{z^2+2z+5}$ where C is the circle **(i)** $|z| = 1$ **(ii)** $|z + 1 - i| = 2$

13. Evaluate $\int_c \frac{4z-1}{z^2-3z-4}$ where C is the ellipse $x^2 + 4y^2 = 4.$

14. Evaluate $\int_c \frac{z^2}{z^4-1}$ where C is the circle **(i)** $|z| = 1/2$ **(ii)** $|z - 1| = 1$ **(iii)** $|z + i| = 1$

15 Evaluate $\int_c \frac{\sin\pi z^2 + \cos\pi z^2}{z^2+3z+2}$ where C is **(i)** $|z| = 1$ **(ii)** $|z| = 2$

16. Evaluate $\int_c \frac{z\cos\pi z}{z^2-z-2} dz$ where C is $|z - i| = 2$

17. Evaluate $\int_c \frac{z+1}{z^3-2z} dz$ where C is the circle $|z| = 1$

18. Evaluate $\int_c \frac{z^2+4}{(z-2)(z+3i)} dz$ where C is **(i)** $|z + 1| = 2$ **(ii)** $|z - 2| = 2$

19. Find **(a)** $\int_c \frac{\boldsymbol{dz}}{\boldsymbol{z-z_0}}$ **(b)** $\int \frac{dz}{(z-z_0)^n}$, $n \neq 1$ where C is a simple closed curve and $z = z_0$ is a point

(a) outside C, **(b)** inside C.

20. Evaluate $\int_c \frac{\sin^6 z}{(z-\pi/6)^3} dz$ where C is $|z| = 1$

21. If $f(z)$ is analytic in and on a simple closed curve C, prove that $f'''(a) = \frac{3!}{2\pi i}\int_c \frac{f(z)}{(z-a)^4} dz$

Hence, evaluate $\int_c \frac{e^{iz}}{z^4} dz$ where C is the circle $|z| = 2.$

22. Evaluate $\int_c \frac{e^{2z}}{(z-1)^4} dz$ where C is the circle $|z| = 2$

23. Evaluate $\int_c \frac{z-1}{(z+1)^2(z-2)} dz$ where C is $|z - i| = 2$

24. Evaluate $\int_c \frac{ze^{2z}}{(z-1)^3} dz$ where C is $|z + i| = 2$

25. Evaluate $\int_c \frac{\sin^6 z}{(z-\pi/6)^n} dz$ where C is the circle $|z| = 1$ for $n = 1, n = 3$

26. Evaluate $\int_c \frac{z+1}{z^3-2z^2} dz$ where C is **(a)** the circle $|z| = 1,$ **(b)** the circle $|z - 2 - i| = 2,$

(c) the circle $|z - 1 - 2i| = 2$

27. Evaluate $\int_c \frac{(z+4)^2}{z^4+5z^3+6z^2} dz$ where C is $|z| = 1$

28. If $f(\zeta) = \int_c \frac{4z^2+z+4}{z-\zeta} dz$ where C is the ellipse $4x^2 + 9y^2 = 36$ find the values of

(i) $f(4)$ **(ii)** $f(1)$ **(iii)** $f(i)$ **(iv)** $f'(-1)$ **(v)** $f''(-i)$

29. If $\Phi(\alpha) = \int_c \frac{ze^z}{z-\alpha} dz$, where C is $|z - 2i| = 3$, find the values of **(i)** $\Phi(1)$ **(ii)** $\Phi'(2)$ **(iii)** $\Phi(3)$

30. If $f(\zeta) = \int_c \frac{3z^2+7z+1}{z-\zeta} dz$ where C is the circle $|z| = 2$ find the values of

(i) $f(-3)$ **(ii)** $f(i)$ **(iii)** $f'(1-i)$ **(iv)** $f''(1-i)$

TYPE-III TAYLOR'S & LAURENT'S SERIES

1. Obtain Taylor's expansion of $f(z) = \frac{z-1}{z+1}$ indicating the region of convergence

2. Find the Taylor's series expansion of $f(z) = \frac{1}{(z-1)(z-3)}$ about the point $z = 4$. Find the region of convergence.

3. Obtain Laurent's series for $f(z) = \frac{1}{z(z+2)(z+1)} about\ z = -2$

4. Find the Laurent's series for $f(z) = z^3 e^{1/z}$ about $z = 0$.

5. Obtain the expansion of $f(z) = \frac{z+1}{(z-3)(z-4)} about\ z = 2$

6. Expand the function $f(z) = \frac{\sin z}{z-\pi} about\ z = \pi$

7. Expand $f(z) = \frac{1}{z^3-3z^2+2z}$ as Laurent's series about $z = 0$ for

(i) $|z| < 1$ **(ii)** $1 < |z| < 2$ **(iii)** $|z| > 2$

8. Obtain Taylor's and Laurent's expansions of $f(z) = \frac{z-1}{z^2-2z-3}$ indicating regions of convergence.

9. Obtain the Laurent's series valid in the indicated region.

(i) $\frac{1}{z^2(z-2)}$; $0 < |z| < 2$ **(ii)** $\frac{z-1}{z^2}$; $|z-1| > 1$

(iii) $\frac{(z-2)(z+2)}{(z+1)(z+4)}$ **(a)** $1 < |z| < 4$ **(b)** $|z| > 4$

10. Find all possible Laurent's expansions of the function $f(z) = \frac{2-z^2}{z(1-z)(2-z)}$ about $z = 0$ indicating the region of convergence in each case.

11. Find the Laurent's series of $f(z) = \frac{4z+3}{z(z-3)(z+2)}\ valid\ for\ 2 < |z| < 3$

12. Obtain two distinct Laurent's series for $f(z) = \frac{2z-3}{z^2-4z-3}$ in powers of $(z-4)$ indicating the regions of convergence.

13. Find series which represents the function $f(z) = \frac{2}{(z-1)(z-2)} when$

(i) $|z| < 1,$ **(ii)** $1 < |z| < 2$ **(iii)** $|z| > 2$

14. Expand $f(z) = \frac{1}{(z-1)(z-2)}$ in the regions

(i) $1 < |z-1| < 2.$ **(ii)** $1 < |z-3| < 2$ **(iii)** $|z| < 1$

15. Expand $f(z) = \frac{1}{z^2(z-1)(z+2)}$ $about\ z = 0\ for$

(i) $|z| < 1$ **(ii)** $1 < |z| < 2$ **(iii)** $|z| > 2$

16. Expand $f(z) = \frac{z^2-1}{z^2+5z+6}$ $around\ z = 1$

17. Obtain two distinct Laurent's series expansion of $f(z) = \frac{1}{z^2(2-z)}$

18. Expand $f(z) = \frac{3z-3}{(2z-1)(z-2)}$ in a Laurent's series about $z = 1$. Convergent in $\frac{1}{2} < |z-1| < 1$