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ENGINEERING MATHEMATICS -II

SUBJECT CODE: MA8251

(Regulation 2017)
Common to all branches of B.E

UNIT - II

VECTOR CALCULUS

INTRODUCTION

In this chapter we study the basics of vector calculus with the help of a standard vector differential operator. Also we introduce concepts like gradient of a scalar valued function, divergence and curl of a vector valued function, discuss briefly the properties of these concepts and study the applications of the results to the evaluation of line and surface integrals in terms of multiple integrals.

2.1 GRADIENT – DIRECTIONAL DERIVATIVE

Vector differential operator

The vector differential operator ∇ (read as Del) is denoted by $\nabla = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}$ where $\vec{i}, \vec{j}, \vec{k}$ are unit vectors along the three rectangular axes $OX, OY\ and\ OZ$.

It is also called Hamiltonian operator and it is neither a vector nor a scalar, but it behaves like a vector.

The gradient of a scalar function

If $\varphi(x, y, z)$ is a scalar point function continuously differentiable in a given region of space, then the gradient of φ is defined as $\nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$

It is also denoted as Grad φ.

Note

(i) $\nabla\varphi$ is a vector quantity.

(ii) $\nabla\varphi = 0$ if φ is constant.

(iii) $\nabla(\varphi_1\varphi_2) = \varphi_1\nabla\varphi_2 + \varphi_2\nabla\varphi_1$

(iv) $\nabla\left(\frac{\varphi_1}{\varphi_2}\right) = \frac{\varphi_2\nabla\varphi_1 - \varphi_1\nabla\varphi_2}{{\varphi^2}_2}$ if $\varphi_2 \neq 0$

(v) $\nabla(\varphi \pm \chi) = \nabla\varphi \pm \nabla\chi$

Problems based on Gradient

Example: 2.1 Find the gradient of $\boldsymbol{\varphi}$ where $\boldsymbol{\varphi}$ is $\boldsymbol{3x^2y - y^3z^2}$ at $(\boldsymbol{1}, \boldsymbol{-2}, \boldsymbol{1})$.

Solution:

Given $\varphi = 3x^2y - y^3z^2$

Grad $\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$

Now $\frac{\partial\varphi}{\partial x} = 6xy$, $\frac{\partial\varphi}{\partial y} = 3x^2 - 3y^2z^2$, $\frac{\partial\varphi}{\partial z} = -2y^3z$

$\therefore$ grad $\varphi = \vec{i}\,6xy + \vec{j}(3x^2 - 3y^2z^2) - \vec{k}2y^3z$

$\therefore (grad\ \varphi)_{(1,-2,\ 1)} = -12\vec{i} - 9\vec{j} + 16\vec{k}$

Example: 2.2 If $\boldsymbol{\varphi} = \log(\boldsymbol{x^2} + \boldsymbol{y^2} + \boldsymbol{z^2})$ then find $\boldsymbol{\nabla\varphi}$.

Solution:

Given $\varphi = \log(x^2 + y^2 + z^2)$

$$\nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

$$= \vec{i}\left(\frac{2x}{x^2+y^2+z^2}\right) + \vec{j}\left(\frac{2y}{x^2+y^2+z^2}\right) + \vec{k}\left(\frac{2z}{x^2+y^2+z^2}\right)$$

$$= \frac{2}{x^2+y^2+z^2}\left(x\vec{i} + y\vec{j} + z\vec{k}\right) = \frac{2}{r^2}\vec{r}$$

Example: 2.3 Find $\nabla(r), \nabla\left(\frac{1}{r}\right), \nabla(\log r)$ where $r = |\vec{r}|$ and $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$.

Solution:

Given $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$$\Rightarrow |\vec{r}| = r = \sqrt{x^2 + y^2 + z^2}$$

$$\Rightarrow r^2 = x^2 + y^2 + z^2$$

$$2r\frac{\partial r}{\partial x} = 2x, \quad 2r\frac{\partial r}{\partial y} = 2y, \quad 2r\frac{\partial r}{\partial z} = 2z$$

$$\Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}$$

(i) $\nabla(r) = \vec{i}\frac{\partial r}{\partial x} + \vec{j}\frac{\partial r}{\partial y} + \vec{k}\frac{\partial r}{\partial z}$

$$= \vec{i}\frac{x}{r} + \vec{j}\frac{y}{r} + \vec{k}\frac{z}{r}$$

$$= \frac{1}{r}\left(x\vec{i} + y\vec{j} + z\vec{k}\right) = \frac{1}{r}\vec{r}$$

(ii) $\nabla\left(\frac{1}{r}\right) = \vec{i}\frac{\partial\left(\frac{1}{r}\right)}{\partial x} + \vec{j}\frac{\partial\left(\frac{1}{r}\right)}{\partial y} + \vec{k}\frac{\partial\left(\frac{1}{r}\right)}{\partial z}$

$$= \vec{i}\left(\frac{-1}{r^2}\right)\frac{\partial r}{\partial x} + \vec{j}\left(\frac{-1}{r^2}\right)\frac{\partial r}{\partial y} + \vec{k}\left(\frac{-1}{r^2}\right)\frac{\partial r}{\partial z}$$

$$= \left(-\frac{1}{r^2}\right)\left[\vec{i}\frac{x}{r} + \vec{j}\frac{y}{r} + \vec{k}\frac{z}{r}\right]$$

$$= -\frac{1}{r^3}\left(x\vec{i} + y\vec{j} + z\vec{k}\right) = -\frac{1}{r^3}\vec{r}$$

(iii) $\nabla(\log r) = \sum\vec{i}\,\frac{\partial(\log r)}{\partial x}$

$$= \sum\vec{i}\,\frac{1}{r}\,\frac{\partial r}{\partial x}$$

$$= \sum\vec{i}\,\frac{1}{r}\,\frac{x}{r}$$

$$= \sum\vec{i}\,\frac{x}{r^2}$$

$$= \frac{1}{r^2}\left(x\vec{i} + y\vec{j} + z\vec{k}\right) = \frac{1}{r^2}\vec{r}$$

Example: 2.4 Prove that $\nabla(r^n) = nr^{n-2}\ \vec{r}$

Solution:

Given $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$$\nabla(r^n) = \vec{i}\frac{\partial r^n}{\partial x} + \vec{j}\frac{\partial r^n}{\partial y} + \vec{k}\frac{\partial r^n}{\partial z}$$

$$= \vec{i}\, nr^{n-1}\frac{\partial r}{\partial x} + \vec{j}\, nr^{n-1}\frac{\partial r}{\partial y} + \vec{k}\, nr^{n-1}\frac{\partial r}{\partial z}$$

$$= nr^{n-1}\left[\vec{i}\left(\frac{x}{r}\right) + \vec{j}\left(\frac{y}{r}\right) + \vec{k}\left(\frac{z}{r}\right)\right]$$

$$= \frac{nr^{n-1}}{r}\left(x\vec{i} + y\vec{j} + z\vec{k}\right) = nr^{n-2}\vec{r}$$

Example: 2.5 Find $|\nabla\varphi|$ if $\varphi = 2xz^4 - x^2y$ at $(2,-2,-1)$

Solution:

Given $\varphi = 2xz^4 - x^2y$

$$\nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

Now $\frac{\partial\varphi}{\partial x} = 2z^4 - 2xy,\ \ \frac{\partial\varphi}{\partial y} = -x^2,\ \ \frac{\partial\varphi}{\partial z} = 8xz^3$

$$\therefore \nabla\varphi = \vec{i}\,(2z^4 - 2xy) + \vec{j}(-x^2) + \vec{k}(8xz^3)$$

$$\therefore (\nabla\varphi)_{(2,-2,-1)} = 10\vec{i} - 4\vec{j} - 16\vec{k}$$

$$|\nabla\varphi| = \sqrt{100 + 16 + 256} = \sqrt{372}$$

Directional Derivative (D.D) of a scalar point function

The derivative of a point function (scalar or vector) in a particular direction is called its directional derivative along the direction.

The directional derivative of a scalar function φ in a given direction $\vec{a}$ is the rate of change of φ in that direction. It is given by the component of $\nabla\varphi$ in the direction of $\vec{a}$.

The directional derivative of a scalar point function in the direction of $\vec{a}$ is given by

$$\textbf{D.D} = \frac{\nabla\varphi\cdot\vec{a}}{|\vec{a}|}$$

The maximum directional derivative is $|\nabla\varphi|$ or $|\text{grad}\ \varphi|$.

Problems based on Directional Derivative

Example: 2.6 Find the directional derivative of $\varphi = 4xz^2 + x^2yz$ at $(1,-2,1)$ in the direction of $2\vec{i} - \vec{j} - 2\vec{k}$.

Solution:

Given $\varphi = 4xz^2 + x^2yz$

$$\nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

$$= \vec{i}\,(2xyz + 4z^2) + \vec{j}\,(x^2z) + \vec{k}\,(x^2y + 8xz)$$

$$\therefore (\nabla\varphi)_{(1,-2,-1)} = 8\vec{i} - \vec{j} - 10\vec{k}$$

Given $\vec{a} = 2\vec{i} - \vec{j} - 2\vec{k}$

$$|\vec{a}| = \sqrt{4+1+4} = 3$$

$$\text{D. D} = \frac{\nabla\varphi\cdot\vec{a}}{|\vec{a}|}$$

$$= \left(8\vec{i} - \vec{j} - 10\vec{k}\right)\cdot\frac{(2\vec{i} - \vec{j} - 2\vec{k})}{3}$$

$$= \frac{1}{3}\ (16 + 1 + 20) = \frac{37}{3}$$

Example: 2.7 Find the directional derivative of $\boldsymbol{\varphi(x, y, z) = xy^2 + yz^3}$ at the point $\mathbf{P(2, -1, 1)}$ in the direction of PQ where Q is the point $\mathbf{(3, 1, 3)}$

Solution:

Given $\varphi = xy^2 + yz^3$

$$\nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

$$= \vec{i}\,(y^2) + \vec{j}\,(2xy + z^3) + \vec{k}\,(3yz^2)$$

$$\therefore (\nabla\,\varphi)_{(2,-1,\ 1)} = \vec{i} - 3\vec{j} - 3\vec{k}$$

Given $\vec{a} = \overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$

$$= \left(3\vec{i} + \vec{j} + 3\vec{k}\right) - \left(2i - \vec{j} + \vec{k}\right)$$

$$= \vec{i} + 2\vec{j} + 2\vec{k}$$

$$|\vec{a}| = \sqrt{1 + 4 + 4} = 3$$

$$\text{D. D} = \frac{\nabla\varphi \cdot \vec{a}}{|\vec{a}|}$$

$$= \frac{\left(\vec{i} - 3\vec{j} - 3\vec{k}\right) \cdot \left(\vec{i} + 2\vec{j} + 2\vec{k}\right)}{3}$$

$$= \frac{1}{3}\ (1 - 6 - 6) = -\frac{11}{3}$$

Example: 2.8 In what direction from $\mathbf{(-1, 1, 2)}$ is the directional derivative of $\boldsymbol{\varphi = xy^2\, z^3}$ a maximum? Find also the magnitude of this maximum.

Solution:

Given $\varphi = xy^2\, z^3$

$$\nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

$$= \vec{i}\,(y^2z^3) + \vec{j}\,(2xy\,z^3) + \vec{k}\,(3xy^2z^2)$$

$$\therefore (\nabla\,\varphi)_{(-1,\ 1,\ 2)} = 8\vec{i} - 16\vec{j} - 12\vec{k}$$

The maximum directional derivative occurs in the direction of $\nabla\,\varphi = 8\vec{i} - 16\vec{j} - 12\vec{k}$.

$\therefore$ The magnitude of this maximum directional derivative

$$|\nabla\varphi| = \sqrt{64 + 256 + 144} = \sqrt{464}$$

Example: 2.9 Find the directional derivative of the scalar function $\boldsymbol{\varphi = xyz}$ in the direction of the outer normal to the surface $\boldsymbol{z = xy}$ at the point$\mathbf{(3, 1, 3)}$.

Solution:

Given $\varphi = xyz$

$$\nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

$$= \vec{i}\,(yz) + \vec{j}\,(xz) + \vec{k}\,(xy)$$

$\therefore (\nabla \varphi)_{(3,\ 1,\ 3)} = 3\vec{i} + 9\vec{j} + 3\vec{k}$

Given surface is $z = xy \Rightarrow z - xy = 0$

$\nabla\chi = \vec{i}\frac{\partial\chi}{\partial x} + \vec{j}\frac{\partial\chi}{\partial y} + \vec{k}\frac{\partial\chi}{\partial z}$

$= \vec{i}\,(-y) + \vec{j}\,(-x) + \vec{k}\,(1)$

Let $\vec{a} = \nabla\chi_{(3,1,3)} = -\vec{i} - 3\vec{j} + \vec{k}$

$\Rightarrow |\vec{a}| = \sqrt{1+9+1} = \sqrt{11}$

D. D $= \frac{\nabla\varphi \cdot \vec{a}}{|\vec{a}|}$

$= \frac{(3\vec{i}+9\vec{j}+3\vec{k})\cdot(-\vec{i}-3\vec{j}+\vec{k})}{\sqrt{11}}$

$= \frac{1}{\sqrt{11}}(-3-27+3) = -\frac{27}{\sqrt{11}}$

Example: 2.10 Find the directional derivative of $\varphi = xy + yz + zx$ at $(1, 2, 0)$ in the direction of $\vec{i} + 2\vec{j} + 2\vec{k}$. Find also its maximum value.

Solution:

Given $\varphi = xy + yz + zx$

$\nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$

$= \vec{i}\,(y+z) + \vec{j}\,(x+z) + \vec{k}\,(y+x)$

$\therefore (\nabla \varphi)_{(1,\ 2,\ 0)} = 2\vec{i} + \vec{j} + 3\vec{k}$

Given $\vec{a} = \vec{i} + 2\vec{j} + 2\vec{k}$

$|\vec{a}| = \sqrt{1+4+4} = 3$

D. D $= \frac{\nabla\varphi \cdot \vec{a}}{|\vec{a}|}$

$= \frac{(2\vec{i}+\vec{j}+3\vec{k})\cdot(\vec{i}+2\vec{j}+2\vec{k})}{3}$

$= \frac{1}{3}(2+2+6) = \frac{10}{3}$

Maximum value is $|\nabla\varphi| = \sqrt{4+1+9} = \sqrt{14}$

Unit normal vector to the surface

If $\varphi(x, y, z)$ be a scalar function, then $\varphi(x, y, z) = c$ represents a surface and the unit normal vector to the surface φ is given by $\hat{n} = \frac{\nabla\varphi}{|\nabla\varphi|}$

Normal Derivative $= |\nabla\varphi|$

Problems based on unit normal vector

Example: 2.11 Find the unit normal to the surface $x^2 + y^2 = z$ at the point $(1, -2, 5)$.

Solution:

Given $\varphi = x^2 + y^2 - z$

$$\nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

$$= \vec{i}\,(2x) + \vec{j}\,(2y) + \vec{k}\,(-1)$$

$$\therefore (\nabla\varphi)_{(1,-2,\ 5)} = 2\vec{i} - 4\vec{j} - \vec{k}$$

$$|\nabla\varphi| = \sqrt{4+16+1} = \sqrt{21}$$

Unit normal $\hat{n} = \frac{\nabla\varphi}{|\nabla\varphi|} = \frac{2\vec{i}-4\vec{j}-\vec{k}}{\sqrt{21}}$

Example: 2.12 Find the unit normal to the surface $\boldsymbol{x^2 + xy + y^2 + xyz}$ at the point $\boldsymbol{(1, -2, 1)}$.

Solution:

Given $\varphi = x^2 + xy + y^2 + xyz$

$$\nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

$$= \vec{i}\,(2x + y + yz) + \vec{j}\,(x + 2y + xz) + \vec{k}\,(xy)$$

$$\therefore (\nabla\varphi)_{(1,-2,\ 1)} = -2\vec{i} - 2\vec{j} - 2\vec{k}$$

$$|\nabla\varphi| = \sqrt{4+4+4} = \sqrt{12} = 2\sqrt{3}$$

Unit normal $\hat{n} = \frac{\nabla\varphi}{|\nabla\varphi|} = \frac{-2\vec{i}-2\vec{j}-2\vec{k}}{2\sqrt{3}}$

$$= \frac{-1}{\sqrt{3}}\left(\vec{i} + \vec{j} + \vec{k}\right)$$

Example: 2.13 Find the normal derivative to the surface $\boldsymbol{x^2y + xz^2}$ at the point $\boldsymbol{(-1, 1, 1)}$.

Solution:

Given $\varphi = x^2y + xz^2$

$$\nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

$$= \vec{i}\,(2xy + z^2) + \vec{j}\,(x^2) + \vec{k}\,(2xz)$$

$$\therefore (\nabla\varphi)_{(-1,\ 1,\ 1)} = -\vec{i} + \vec{j} - 2\vec{k}$$

Normal derivative $|\nabla\varphi| = \sqrt{1+1+4} = \sqrt{6}$

Example: 2.14 What is the greatest rate of increase of $\boldsymbol{\varphi = xyz^2}$ at the point $\boldsymbol{(1, 0, 3)}$.

Solution:

Given $\varphi = xyz^2$

$$\nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

$$= \vec{i}\,(y\,z^2) + \vec{j}\,(xz^2) + \vec{k}\,(2xyz)$$

$$\therefore (\nabla\varphi)_{(1,\ 0,\ 3)} = 0\vec{i} + 9\vec{j} + 0\vec{k}$$

$\therefore$ Greatest rate of increase $|\nabla\varphi| = \sqrt{9^2} = 9$

Angle between the surfaces

$$\cos\theta = \frac{\nabla\varphi_1 \cdot \nabla\varphi_2}{|\nabla\varphi_1|\,|\nabla\varphi_2|}$$

$\Rightarrow \theta = cos^{-1}\left[\frac{\nabla \varphi_1 \cdot \nabla \varphi_2}{|\nabla\varphi_1||\nabla\varphi_2|}\right]$

Problems based on angle between two surfaces

Example: 2.15 Find the angle between the surfaces $z = x^2 + y^2 - 3$ and $x^2 + y^2 + z^2 = 9$ at the point $(2, -1, 2)$.

Solution:

Given $\varphi = x^2 + y^2 - z - 3$

$$\nabla\varphi_1 = \vec{i}\frac{\partial\varphi_1}{\partial x} + \vec{j}\frac{\partial\varphi_1}{\partial y} + \vec{k}\frac{\partial\varphi_1}{\partial z}$$

$$= \vec{i}\,(2x) + \vec{j}\,(2y) + \vec{k}\,(-1)$$

$$\therefore (\nabla\varphi_1)_{(2,-1,\ 2)} = 4\vec{i} - 2\vec{j} - \vec{k}$$

$$|\nabla\varphi_1| = \sqrt{16+4+1} = \sqrt{21}$$

$$\nabla\varphi_2 = \vec{i}\frac{\partial\varphi_2}{\partial x} + \vec{j}\frac{\partial\varphi_2}{\partial y} + \vec{k}\frac{\partial\varphi_2}{\partial z}$$

$$= \vec{i}\,(2x) + \vec{j}\,(2y) + \vec{k}\,(2z)$$

$$\therefore (\nabla\varphi_2)_{(2,-1,\ 2)} = 4\vec{i} - 2\vec{j} + 4\vec{k}$$

$$|\nabla\varphi_2| = \sqrt{16+4+16} = \sqrt{36} = 6$$

The angle between the surfaces is $\cos\theta = \frac{\nabla \varphi_1 \cdot \nabla \varphi_2}{|\nabla\varphi_1||\nabla\varphi_2|}$

$$= \frac{(4\vec{i}-2\vec{j}-\vec{k})\,(4\vec{i}-2\vec{j}+4\vec{k})}{\sqrt{21}(6)}$$

$$= \frac{16+4-4}{\sqrt{21}(6)}$$

$$= \frac{16}{\sqrt{21}(6)} = \frac{8}{3\sqrt{21}}$$

$$\Rightarrow \theta = cos^{-1}\left[\frac{8}{3\sqrt{21}}\right]$$

Example: 2.16 Find the angle between the normals to the surfaces $x^2 = yz$ at the point $(1, 1, 1)$ *and* $(2, 4, 1)$.

Solution:

Given $\varphi = x^2 - yz$

$$\nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

$$= \vec{i}\,(2x) + \vec{j}\,(-z) + \vec{k}\,(-y)$$

$$\therefore (\nabla\varphi_1)_{(1,\ 1,\ 1)} = 2\vec{i} - \vec{j} - \vec{k}$$

$$|\nabla\varphi_1| = \sqrt{4+1+1} = \sqrt{6}$$

$$\therefore (\nabla\varphi_2)_{(2,\ 4,\ 1)} = 4\vec{i} - \vec{j} - 4\vec{k}$$

$$|\nabla\varphi_2| = \sqrt{16+1+16} = \sqrt{33}$$

The angle between the surfaces is $\cos\theta = \frac{\nabla \varphi_1 \cdot \nabla \varphi_2}{|\nabla\varphi_1||\nabla\varphi_2|}$

$$= \frac{(2\vec{\imath}-\vec{\jmath}-\vec{k})\ (4\vec{\imath}-\vec{\jmath}-4\vec{k})}{\sqrt{6}\sqrt{33}}$$

$$= \frac{8+1+4}{\sqrt{6}\sqrt{33}}$$

$$= \frac{13}{\sqrt{2(3)}\sqrt{11(3)}} = \frac{13}{3\sqrt{22}}$$

$$\Rightarrow \theta = cos^{-1}\left[\frac{13}{3\sqrt{22}}\right]$$

Example: 2.17 Find the angle between the surfaces $\boldsymbol{x\ log\ z = y^2 - 1}$ and $\boldsymbol{x^2 y = 2 - z}$ at the point $\boldsymbol{(1, 1, 1)}$.

Solution:

Given $\varphi_1 = y^2 - x\ log\ z - 1$

$$\nabla\varphi_1 = \vec{\imath}\frac{\partial\varphi_1}{\partial x} + \vec{\jmath}\frac{\partial\varphi_1}{\partial y} + \vec{k}\frac{\partial\varphi_1}{\partial z}$$

$$= \vec{\imath}\,(-\log z) + \vec{\jmath}\,(2y) + \vec{k}\left(-\frac{x}{z}\right)$$

$$\therefore (\nabla\varphi_1)_{(1,\ 1,\ 1)} = 0\vec{\imath} + 2\vec{\jmath} - \vec{k}$$

$$|\nabla\varphi_1| = \sqrt{0+4+1} = \sqrt{5}$$

$$\nabla\varphi_2 = \vec{\imath}\frac{\partial\varphi_2}{\partial x} + \vec{\jmath}\frac{\partial\varphi_2}{\partial y} + \vec{k}\frac{\partial\varphi_2}{\partial z}$$

$$= \vec{\imath}\,(2xy) + \vec{\jmath}\,(x^2) + \vec{k}\,(1)$$

$$\therefore (\nabla\varphi_2)_{(1,\ 1,\ 1)} = 2\vec{\imath} + \vec{\jmath} + \vec{k}$$

$$|\nabla\varphi_2| = \sqrt{4+1+1} = \sqrt{6}$$

The angle between the surfaces is $\cos\theta = \frac{\nabla\varphi_1 \cdot \nabla\varphi_2}{|\nabla\varphi_1|\ |\nabla\varphi_2|}$

$$= \frac{(0\vec{\imath}+2\vec{\jmath}-\vec{k})\cdot(2\vec{\imath}+\vec{\jmath}+\vec{k})}{\sqrt{5}\sqrt{6}}$$

$$= \frac{0+2-1}{\sqrt{30}}$$

$$= \frac{1}{\sqrt{30}}$$

$$\Rightarrow \theta = cos^{-1}\left[\frac{1}{\sqrt{30}}\right]$$

Problems based on orthogonal surfaces

Two surfaces are orthogonal if $\nabla\varphi_1 \cdot \nabla\varphi_2 = 0$

Example: 2.18 Find $\boldsymbol{a}$ and $\boldsymbol{b}$ such that the surfaces $\boldsymbol{ax^2 - byz = (a+2)x}$ and $\boldsymbol{4x^2 y + z^3 = 4}$ cut orthogonally at $\boldsymbol{(1, -1, 2)}$.

Solution:

Given $ax^2 - byz = (a+2)x$

Let $\varphi_1 = ax^2 - byz - (a+2)x$

$$\nabla\varphi_1 = \vec{\imath}\frac{\partial\varphi_1}{\partial x} + \vec{\jmath}\frac{\partial\varphi_1}{\partial y} + \vec{k}\frac{\partial\varphi_1}{\partial z}$$

$= \vec{i}\left(2ax - (a+2)\right) + \vec{j}\,(-bz) + \vec{k}\,(-by)$

$\therefore (\nabla\varphi_1)_{(1,-1,\ 2)} = \vec{i}(a-2) + \vec{j}\,(-2b) + \vec{k}(b)$

Let $\varphi_2 = 4x^2y + z^3 - 4$

$\nabla\varphi_2 = \vec{i}\frac{\partial\varphi_2}{\partial x} + \vec{j}\frac{\partial\varphi_2}{\partial y} + \vec{k}\frac{\partial\varphi_2}{\partial z}$

$= \vec{i}\,(8xy) + \vec{j}\,(4x^2) + \vec{k}\,(3z^2)$

$\therefore (\nabla\varphi_2)_{(1,-1,\ 2)} = -8\vec{i} + 4\vec{j} + 12\vec{k}$

Since the two surfaces are orthogonal if $\nabla\varphi_1 \cdot \nabla\varphi_2 = 0$

$\Rightarrow \left(\vec{i}(a-2) + \vec{j}\,(-2b) + \vec{k}(b)\right) \cdot \left(-8\vec{i} + 4\vec{j} + 12\vec{k}\right) = 0$

$\Rightarrow -8\,(a-2) - 8b + 12b = 0$

$\Rightarrow -8a + 16 - 8b + 12b = 0$

$\Rightarrow -8a + 16 + 4b = 0$

$\div$ by 4 $\Rightarrow -2a + 4 + b = 0$

$\Rightarrow 2a - b - 4 = 0 \;\; \ldots (1)$

To find a and b we need another equation in a and b.

The point $(1,-1,2)$ lies in $ax^2 - byz - (a+2)x = 0$

$\therefore a - b\,(-1)\,(2) - (a+2)\,(1) = 0$

$\Rightarrow a + 2b - a - 2 = 0$

$\Rightarrow 2b - 2 = 0$

$\Rightarrow b = 1$

Substitute $b = 1$ in (1) we get

$\Rightarrow 2a - 1 - 4 = 0$

$\Rightarrow 2a - 5 = 0$

$\Rightarrow a = \frac{5}{2}$

Example: 2.19 Find the values of a and b so that the surfaces $ax^3 - by^2z = (a+3)x^2$ and $4x^2y - z^3 = 11$ may cut orthogonally at $(2,-1,-3)$.

Solution:

Given $ax^3 - by^2z = (a+3)x^2$

Let $\varphi_1 = ax^3 - by^2z - (a+3)x^2$

$\nabla\varphi_1 = \vec{i}\frac{\partial\varphi_1}{\partial x} + \vec{j}\frac{\partial\varphi_1}{\partial y} + \vec{k}\frac{\partial\varphi_1}{\partial z}$

$= \vec{i}\left(3ax^2 - 2x(a+3)\right) + \vec{j}\,(-2byz) + \vec{k}\,(-by^2)$

$\therefore (\nabla\varphi_1)_{(2,-1,-3)} = \vec{i}(8a-12) + \vec{j}\,(-6b) + \vec{k}(-b)$

Let $\varphi_2 = 4x^2y - z^3 - 11$

$\nabla\varphi_2 = \vec{i}\frac{\partial\varphi_2}{\partial x} + \vec{j}\frac{\partial\varphi_2}{\partial y} + \vec{k}\frac{\partial\varphi_2}{\partial z}$

$= \vec{i}\,(8xy) + \vec{j}\,(4x^2) + \vec{k}\,(-3z^2)$

$\therefore (\nabla\varphi_2)_{(2,-1,-3)} = -16\vec{i} + 16\vec{j} - 27\vec{k}$

Given the two surfaces cut orthogonally if $\nabla\varphi_1 \cdot \nabla\varphi_2 = 0$

$\Rightarrow \left(\vec{i}(8a-12) + \vec{j}\,(-6b) - \vec{k}(b)\right) \cdot \left(-16\vec{i} + 16\vec{j} - 27\vec{k}\right) = 0$

$\Rightarrow -16\,(8a-12) - 16(6b) + 27b = 0$

$\Rightarrow -128a + 192 - 69b = 0$

$\Rightarrow 128a + 69b - 192 = 0 \;\; \ldots (1)$

To find a and b we need another equation in a and b.

The point $(2,-1,-3)$ lies in $ax^3 - by^2z - (a+3)x^2 = 0$

$\therefore 8a - b\,(1)\,(-3) - (a+3)\,(4) = 0$

$\Rightarrow 4a + 3b - 12 = 0 \ldots (2)$

Solving (1) and (2) we get, $a = -\frac{7}{3}$ & $b = \frac{64}{9}$

Equation of the tangent plane and normal to the surface

Equation of the tangent plane is $(\vec{r} - \vec{a}) \cdot \nabla\varphi = 0$

Equation of the normal line is $(\vec{r} - \vec{a}) \times \nabla\varphi = \vec{0}$

Problems based on tangent plane

Example: 2.20 Find the equation of the tangent plane and normal line to the surface $xyz = 4$ at the point $\vec{i} + 2\vec{j} + 2\vec{k}$.

Solution:

Given $\varphi = xyz - 4$

$\nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$

$= \vec{i}\,(yz) + \vec{j}\,(xz) + \vec{k}\,(xy)$

$\therefore (\nabla\varphi)_{(1,\ 2,\ 2)} = 4\vec{i} + 2\vec{j} + 2\vec{k}$

Equation of the tangent plane at the point $\vec{a} = \vec{i} + 2\vec{j} + 2\vec{k}$ is $(\vec{r} - \vec{a}) \cdot \nabla\varphi = 0$

$\Rightarrow \left[\left(x\vec{i} + y\vec{j} + z\vec{k}\right) - \vec{i} + 2\vec{j} + 2\vec{k}\right] \cdot \left(4\vec{i} + 2\vec{j} + 2\vec{k}\right) = 0$

$\Rightarrow \left[(x-1)\,\vec{i} + (y-2)\,\vec{j} + (z-2)\,\vec{k}\right] \cdot \left(4\vec{i} + 2\vec{j} + 2\vec{k}\right) = 0$

$\Rightarrow 4(x-1) + 2(y-2) + 2(z-2) = 0$

$\Rightarrow 4x - 4 + 2y - 4 + 2z - 4 = 0$

$\Rightarrow 4x + 2y + 2z = 12$

$\Rightarrow 2x + y + z = 6$

Equation of the normal line $(\vec{r} - \vec{a}) \times \nabla\varphi = \vec{0}$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x-1 & y-2 & z-2 \\ 4 & 2 & 2 \end{vmatrix} = \vec{0}$$

$$\Rightarrow \vec{i}\,[2\,(y-2) - 2\,(z-2)] - \vec{j}\,[2\,(x-1) - 4\,(z-2)] + \vec{k}\,[2\,(x-1) - 4\,(y-2)]$$

$$= 0\,\vec{i} + 0\,\vec{j} + 0\vec{k}$$

Equating the coefficients of $\vec{i}, \vec{j}, \vec{k}$ we get

$$\Rightarrow 2\,(y-2) - 2\,(z-2) = 0$$

$$\Rightarrow (y-2) = (z-2) \quad \dots (1)$$

$$\Rightarrow 2\,(x-1) - 4\,(z-2) = 0$$

$$\Rightarrow (x-1) = 2\,(z-2)$$

$$\Rightarrow \frac{x-1}{2} = (z-2) \quad \dots (2)$$

$$\Rightarrow 2\,(x-1) - 4\,(y-2) = 0$$

$$\Rightarrow (x-1) = 2\,(y-2)$$

$$\Rightarrow \frac{x-1}{2} = (y-2) \quad \dots (3)$$

From $(1), (2)\ and\ (3)$ we get $\quad \frac{x-1}{2} = \frac{y-2}{1} = \frac{z-2}{1}$

Which is the required equation of the normal line.

Exercise: 2.1

1. Find $\nabla\varphi$ if $\varphi = \frac{1}{2}\log(x^2 + y^2 + z^2)$ **Ans:** $\frac{\vec{r}}{r^2}$

2. Find the directional derivative of

   (i) $\varphi = 2xy + z^2$ at the point $(1, -1, 3)$ in the direction $\vec{i} + 2\vec{j} + 2\vec{k}$. **Ans:** $\frac{14}{3}$

   (ii) $\varphi = xy^2 + yz^3$ at the point $(2, -1, 1)$ in the direction of PQ where Q is the point $(3, 1, 3)$. **Ans:** $\frac{-11}{3}$

3. Prove that the directional derivative of $\varphi = x^3y^2z$ at $(1, 2, 3)$is maximum along the direction $9\vec{i} + 3\vec{j} + \vec{k}$. Also, find the maximum directional derivative. **Ans:** $4\sqrt{91}$

4. Find the unit tangent vector to the curve $\vec{r} = (t^2+1)\vec{i} + (4t-3)\vec{j} + (2t^2 - 65)\vec{k}$ at $t = 1$. **Ans:** $\frac{\vec{i}+2\vec{j}-\vec{k}}{\sqrt{6}}$

5. Find a unit normal to the following surfaces at the specified points.

   (i) $x^2y + 2xz = 4$ at $(2, -2, 3)$ **Ans:** $\pm\frac{1}{3}\left(\vec{i} - 2\vec{j} - 2\vec{k}\right)$

   (ii) $x^2 + y^2 = z$ at $(1, -2, 5)$ **Ans:** $\frac{1}{\sqrt{21}}\left(2\vec{i} - 4\vec{j} - \vec{k}\right)$

   (ii) $xy^3z^2 = 4$ at $(-1, -1, 2)$ **Ans:** $\frac{1}{\sqrt{11}}\left(-\vec{i} - 3\vec{j} + \vec{k}\right)$

   (iv) $x^2 + y^2 = z$ at $(1, 1, 2)$ **Ans:** $\frac{1}{3}\left(2\vec{i} + 2\vec{j} - \vec{k}\right)$

6. Find the angle between the surfaces $x^2 - y^2 - z^2 = z$ and $xy + yz - zx - 18 = 0$ at the

point $(6, 4, 3)$. **Ans:** $cos^{-1}\left[\frac{-24}{\sqrt{86}\sqrt{61}}\right]$

7. Find the angle between the surfaces $xy^2z = 3x + z^2$and $3x^2 - y^2 + 2z = 1$ at the point $(1, -2, 1)$. **Ans:** $cos^{-1}\left[\frac{-3}{7\sqrt{6}}\right]$

8. Find the equation of the tangent plane to the surfaces $2xz^2 - 3xy - 4x = 7$ at the point $(1, -1, 2)$. **Ans:** $7x - 3y + 8z - 26 = 0$

9. Find the equation of the tangent plane to the surfaces $xz^2 + x^2y = z - 1$ at the point $(1, -3, 2)$. **Ans:** $2x - y - 3z + 1 = 0$

10. Find the angle between the surfaces $x \log z = y^2 - 1$ and $x^2y = 2 - z$ at the point $(1, 1, 1)$. **Ans:** $cos^{-1}\left[\frac{1}{\sqrt{30}}\right]$

2.2 DIVERGENCE, CURL – IRROTATIONAL AND SOLENOIDAL VECTORS

Divergence of a vector function

If $\vec{F}(x, y, z)$ is a continuously differentiable vector point function in a given region of space, then the divergence of $\vec{F}$ is defined by

$$\nabla.\vec{F} = \text{div}\,\vec{F} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right).(F_1\vec{i} + F_2\vec{j} + F_3\vec{k})$$

$$\text{div}\vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \quad \text{where}\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$$

Note: $\nabla.\vec{F}$ Is a scalar point function.

Solenoidal vector

A vector $\vec{F}$ is said to be solenoidal if $div\,\vec{F} = 0$ (i.e)$\nabla.\vec{F} = 0$

Curl of a vector function

If $\vec{F}(x, y, z)$ is a differentiable vector point function defines at each point (x, y, z) in some region of space, then the curl of $\vec{F}$ is defined by

$$\text{Curl}\,\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= \vec{i}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) - \vec{j}\left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right) + \vec{k}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$$

Where $\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$

Note: $\nabla \times \vec{F}$ Is a vector point function.

Irrotational vector

A vector is said to be irrotational if Curl $\vec{F} = 0$ $(i.e)$ $\nabla \times \vec{F} = 0$

Scalar potential

If $\vec{F}$ is an irrotational vector, then there exists a scalar function φ such that $\vec{F} = \nabla\varphi$. Such a scalar function is called scalar potential of $\vec{F}$.

Problems based on Divergence and Curl of a vector

Example: 2.21 If $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ then find div $\vec{r}$ and curl$\vec{r}$

Solution:

Given $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

Now div $\vec{r} = \nabla \cdot \vec{r}$

$$= \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z)$$

$$= 1 + 1 + 1 = 3$$

And curl $\vec{r} = \nabla \times \vec{r}$

$$\nabla \times \vec{r} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix}$$

$$= \vec{i}\left(\frac{\partial}{\partial y}(z) - \frac{\partial}{\partial z}(y)\right) - \vec{j}\left(\frac{\partial}{\partial x}(z) - \frac{\partial}{\partial z}(x)\right) + \vec{k}\left(\frac{\partial}{\partial x}(y) - \frac{\partial}{\partial y}(x)\right)$$

$$= \vec{i}(0) + \vec{j}(0) + \vec{k}(0) = \vec{0}.$$

Example: 2.22 If $\vec{F} = xy^2\vec{i} + 2x^2yz\vec{j} - 3yz^2\vec{k}$ find $\nabla.\vec{F}$ and $\nabla \times \vec{F}$ at the point (1,-1, 1).

Solution:

Given $\vec{F} = xy^2\vec{i} + 2x^2yz\vec{j} - 3yz^2\vec{k}$

(i) $\nabla.\vec{F} = \frac{\partial}{\partial x}(xy^2) + \frac{\partial}{\partial y}(2x^2yz) + \frac{\partial}{\partial z}(-3yz^2)$

$$= y^2 + 2x^2z - 6yz$$

$$\nabla.\vec{F}_{(1,-1,1)} = 1 + 2 + 6 = 9$$

(ii) $$\nabla \times \vec{r} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy^2 & 2x^2yz & 3yz^2 \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial(-3yz^2)}{\partial y} - \frac{\partial(2x^2yz)}{\partial z}\right] - \vec{j}\left[\frac{\partial(-3yz^2)}{\partial x} - \frac{\partial(xy^2)}{\partial z}\right] + \vec{k}\left[\frac{\partial(2x^2yz)}{\partial x} - \frac{\partial(xy^2)}{\partial y}\right]$$

$$= \vec{i}(-3z^2 - 2x^2y) - \vec{j}(0) + \vec{k}(4xyz - 2xy)$$

$$\nabla \times \vec{F}_{(1,-1,1)} = \vec{i}(-3 + 2) + \vec{k}(-4 + 2)$$

$$= -\vec{i} - 2\vec{k}$$

Example: 2.23 If $\vec{F} = (x^2 - y^2 + 2xz)\vec{i} + (xz - xy + yz)\vec{j} + (z^2 + x^2)\vec{k}$, then find $\nabla \cdot \vec{F}$, $\nabla(\nabla \cdot \vec{F})$, $\nabla \times \vec{F}$, $\nabla \cdot (\nabla \times \vec{F})$, and $\nabla \times (\nabla \times \vec{F})$ at the point (1,1,1).

Solution:

Given $\vec{F} = (x^2 - y^2 + 2xz)\vec{i} + (xz - xy + yz)\vec{j} + (z^2 + x^2)\vec{k}$

(i) $\nabla \cdot \vec{F} = \frac{\partial}{\partial x}(x^2 - y^2 + 2xz) + \frac{\partial}{\partial y}(xz - xy + yz) + \frac{\partial}{\partial z}(z^2 + x^2)$

$$= (2x + 2z) + (-x + z) + 2z$$

$$= x + 5z$$

$\therefore \nabla \cdot \vec{F}_{(1,1,1)} = 6$

(ii) $\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 - y^2 + 2xz & xz - xy + yz & z^2 + x^2 \end{vmatrix}$

$$= \vec{i}\left[\frac{\partial(z^2+x^2)}{\partial y} - \frac{\partial(xz-xy+yz)}{\partial z}\right] - \vec{j}\left[\frac{\partial(z^2+x^2)}{\partial x} - \frac{\partial(x^2-y^2+2xz)}{\partial z}\right] + \vec{k}\left[\frac{\partial(xz-xy+yz)}{\partial x} - \frac{\partial(x^2-y^2+2xz)}{\partial y}\right]$$

$$= -(x + y)\vec{\imath} - (2x - 2x)\vec{\jmath} + (y + z)\vec{k}$$

$\therefore \nabla \times \vec{F}_{(1,1,1)} = -2\vec{\imath} + 2\vec{k}$

(iii) $\nabla(\nabla \cdot \vec{F}) = \vec{i}\frac{\partial}{\partial x}(x + 5z) + \vec{j}\frac{\partial}{\partial y}(x + 5z) + \vec{k}\frac{\partial}{\partial z}(x + 5z)$

$$= \vec{\imath} + 5\vec{k}$$

$\therefore \nabla(\nabla \cdot \vec{F})_{(1,1,1)} = \vec{\imath} + 5\vec{k}$

(iv) $\nabla \cdot (\nabla \times \vec{F}) = \frac{\partial}{\partial x}(-(x + y)) + \frac{\partial}{\partial y}(0) + \frac{\partial}{\partial z}(y + z)$

$$= -1 + 0 + 1$$

$\nabla \cdot (\nabla \times \vec{F})_{(1,1,1)} = 0$

(v) $\nabla \times (\nabla \times \vec{F}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -(x + y) & 0 & y + z \end{vmatrix}$

$\therefore \nabla \times (\nabla \times \vec{F})_{(1,1,1)} = \vec{\imath} + \vec{k}$

Example: 2.24 Find div $\vec{F}$ and curl $\vec{F}$, where $\vec{F}$ = grad(x^3+y^3+z^3−3xyz)

Solution:

Given $\vec{F}$ = grad(x^3+y^3+z^3−3xyz)

$$= \vec{i}\frac{\partial}{\partial x}(x^3 + y^3 + z^3 - 3xyz) + \vec{j}\frac{\partial}{\partial y}(x^3 + y^3 + z^3 - 3xyz) + \vec{k}\frac{\partial}{\partial z}(x^3 + y^3 + z^3 - 3xyz)$$

$$\vec{F} = \vec{i}(3x^2 - 3yz) + \vec{j}(3y^2 - 3xz) + \vec{k}(3z^2 - 3xy)$$

Now div $\vec{F} = \nabla \cdot \vec{F} = \frac{\partial}{\partial x}(3x^2 - 3yz) + \frac{\partial}{\partial y}(3y^2 - 3xz) + \frac{\partial}{\partial z}(3z^2 - 3xy)$

$$= 6x + 6y + 6z$$

$$= 6(x + y + z)$$

Curl $\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x^2 - 3yz & 3y^2 - 3xz & 3z^2 - 3xy \end{vmatrix}$

$$= \vec{i}[-3x + 3x] - \vec{j}[-3y + 3y] + \vec{k}\,[-3z + 3z]$$

$= \vec{0}$

Example: 2.25 Find div(grad φ) and curl(grad φ) at (1,1,1) for $\boldsymbol{\varphi = x^2y^3z^4}$

Solution:

Given $\varphi = x^2y^3z^4$

$$\text{grad } \varphi = \nabla \varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

$$= \vec{i}(2xy^3z^4) + \vec{j}(x^2 3y^2z^4) + \vec{k}(x^2y^3 4z^3)$$

$$\text{Div(grad } \varphi) = \nabla \cdot (\text{grad } \varphi)$$

$$= \frac{\partial}{\partial x}(2xy^3z^4) + \frac{\partial}{\partial y}(x^2 3y^2z^4) + \frac{\partial}{\partial z}(x^2y^3 4z^3)$$

$$= 2y^3z^4 + 6x^2yz^4 + 12x^2y^3z^4$$

$$\therefore \text{Div(grad } \varphi)_{(1,1,1)} = 2 + 6 + 12 = 20$$

$$\text{Curl(grad } \varphi) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy^3z^4 & x^2 3y^2z^4 & x^2y^3 4z^3 \end{vmatrix}$$

$$= \vec{i}(12x^2y^2z^3 - 12x^2y^2z^3) - \vec{j}(8xy^3z^3 - 8xy^3z^3) + \vec{k}(6xy^2z^4 - 6xy^2z^4)$$

$$= \vec{0}$$

$$\therefore \text{Curl grad}\varphi_{(1,1,1)} = \vec{0}$$

Vector Identities

1) $\nabla \cdot (\varphi \vec{F}) = \varphi(\nabla \cdot \vec{F}) + \vec{F} \cdot \nabla\varphi$
2) $\nabla \times (\varphi \vec{F}) = \varphi(\nabla \times \vec{F}) + (\nabla\varphi) \times \vec{F}$
3) $\nabla \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\nabla \times \vec{A}) - \vec{A} \cdot (\nabla \times \vec{B})$
4) $\nabla \times (\vec{A} \times \vec{B}) = \vec{A}(\nabla \cdot \vec{B}) - \vec{B}(\nabla \cdot \vec{A}) + (\vec{B} \cdot \nabla)\vec{A} - (\vec{A} \cdot \nabla)\vec{B}$
5) $\nabla(\vec{A} \cdot \vec{B}) = \vec{A} \times (\nabla \times \vec{B}) - (\vec{A} \cdot \nabla)\vec{B} + \vec{B} \times (\nabla \times \vec{A}) - (\vec{B} \cdot \nabla)\vec{A}$
6) $\nabla \cdot (\nabla\varphi) = \vec{0}$
7) $\nabla \cdot (\nabla \times \vec{F}) = 0$
8) $\nabla \times (\nabla \times \vec{F}) = \nabla(\nabla \cdot \vec{F}) - \nabla^2\vec{F}$
9) $\nabla \cdot \nabla\varphi = (\nabla \cdot \nabla)\varphi = \nabla^2\varphi$ where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is a laplacian operator

1) If φ is a scalar point function, $\vec{F}$ is a vector point function, then $\boldsymbol{\nabla . (\varphi \vec{F}) = \varphi(\nabla \cdot \vec{F}) + \vec{F} \cdot \nabla\varphi}$

Proof:

$$\nabla \cdot (\varphi \vec{F}) = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \cdot (\varphi \vec{F})$$

$$= \sum \vec{i} \cdot \frac{\partial}{\partial x}(\varphi \vec{F})$$

$$= \sum \vec{i} \cdot \left(\varphi\frac{\partial\vec{F}}{\partial x} + \vec{F}\frac{\partial\varphi}{\partial x}\right)$$

$$= \varphi\left(\sum \vec{\imath}\cdot\frac{\partial \vec{F}}{\partial x} + \vec{F}\frac{\partial \varphi}{\partial x}\right) + \vec{F}\cdot\left(\sum \vec{\imath}\frac{\partial \varphi}{\partial x}\right)$$

$$\therefore \nabla\cdot(\varphi\,\vec{F}) = \varphi\left(\nabla\cdot\vec{F}\right) + \vec{F}\cdot\nabla\varphi$$

2) If $\boldsymbol{\varphi}$ is a scalar point fuction, $\vec{\mathbf{F}}$ is a vector point function, then $\nabla\times(\boldsymbol{\varphi}\,\vec{\mathbf{F}}) = \boldsymbol{\varphi}\left(\nabla\times\vec{\mathbf{F}}\right) + (\nabla\boldsymbol{\varphi})\times\vec{\mathbf{F}}$

Proof:

$$\nabla\times(\varphi\,\vec{F}) = \sum \vec{\imath}\times\frac{\partial}{\partial x}(\varphi\,\vec{F})$$

$$= \sum \vec{\imath}\times\left[\varphi\frac{\partial \vec{F}}{\partial x} + \vec{F}\frac{\partial \varphi}{\partial x}\right]$$

$$= \sum \vec{\imath}\times\left(\frac{\partial \varphi}{\partial x}\vec{F} + \varphi\frac{\partial \vec{F}}{\partial x}\right)$$

$$= \left(\sum \vec{\imath}\frac{\partial \varphi}{\partial x}\right)\times\vec{F} + \varphi\left[\sum \vec{\imath}\times\frac{\partial \vec{F}}{\partial x}\right]$$

$$\therefore \nabla\times(\varphi\,\vec{F}) = \nabla\varphi\times\vec{F} + \varphi(\nabla\times\vec{F})$$

3) If $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ are vector point functions, then $\nabla\cdot(\vec{\mathbf{A}}\times\vec{\mathbf{B}}) = \vec{\mathbf{B}}\cdot(\nabla\times\vec{\mathbf{A}}) - \vec{\mathbf{A}}\cdot(\nabla\times\vec{\mathbf{B}})$

Proof:

$$\nabla.(\vec{A}\times\vec{B}) = \sum \vec{\imath}\cdot\frac{\partial}{\partial x}(\vec{A}\times\vec{B})$$

$$= \sum \vec{\imath}\cdot\left(\vec{A}\times\frac{\partial \vec{B}}{\partial x} + \frac{\partial \vec{A}}{\partial x}\times\vec{B}\right)$$

$$= \sum \vec{\imath}\cdot\left(\vec{A}\times\frac{\partial \vec{B}}{\partial x}\right) + \sum \vec{\imath}\cdot\left(\frac{\partial \vec{A}}{\partial x}\times\vec{B}\right)$$

$$= -\left(\sum \vec{\imath}\times\frac{\partial \vec{B}}{\partial x}\right)\cdot\vec{A} + \left(\sum \vec{\imath}\times\frac{\partial \vec{A}}{\partial x}\right).\vec{B}$$

$$= -\left(\nabla\times\vec{B}\right).\vec{A} + (\nabla\times\vec{A})\cdot\vec{B}$$

$$\therefore \nabla\cdot(\vec{A}\times\vec{B}) = \vec{B}\cdot(\nabla\times\vec{A}) - \vec{A}\cdot(\nabla\times\vec{B}) \qquad \left[\because \left(\nabla\times\vec{A}\right)\cdot\vec{B} = \vec{B}\cdot(\nabla\times\vec{A})\right]$$

(4) If $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ are vector point functions, then

$$\boldsymbol{\nabla\times(\vec{A}\times\vec{B}) = \vec{A}(\nabla\cdot\vec{B}) - \vec{B}(\nabla\cdot\vec{A}) + \left(\vec{B}\cdot\nabla\right)\vec{A} - (\vec{A}\cdot\nabla)\vec{B}}$$

Proof:

$$\nabla\times(\vec{A}\times\vec{B}) = \sum \vec{\imath}\times\frac{\partial}{\partial x}(\vec{A}\times\vec{B})$$

$$= \sum \vec{\imath}\times\left(\frac{\partial \vec{A}}{\partial x}\times\vec{B} + \vec{A}\times\frac{\partial \vec{B}}{\partial x}\right)$$

$$= \sum \vec{\imath}\times\left(\frac{\partial \vec{A}}{\partial x}\times\vec{B}\right) + \sum \vec{\imath}\times\left(\vec{A}\times\frac{\partial \vec{B}}{\partial x}\right)$$

We know that $\vec{a}\times\left(\vec{b}\times\vec{c}\right) = (\vec{a}\cdot\vec{c})\,\vec{b} - \left(\vec{a}\cdot\vec{b}\right)\vec{c}$

$$\nabla\times(\vec{A}\times\vec{B}) = \sum\left[\left(\vec{\imath}\cdot\vec{B}\right)\frac{\partial \vec{A}}{\partial x} - \left(\vec{\imath}\cdot\frac{\partial \vec{A}}{\partial x}\right)\vec{B}\right] + \sum\left[\left(\vec{\imath}\cdot\frac{\partial \vec{B}}{\partial x}\right)\vec{A} - \left(\vec{\imath}\cdot\vec{A}\right)\frac{\partial \vec{B}}{\partial x}\right]$$

$$= \left(\sum \vec{\imath}\cdot\frac{\partial \vec{B}}{\partial x}\right)\vec{A} - \left(\sum \vec{\imath}\cdot\frac{\partial \vec{A}}{\partial x}\right)\vec{B} + \sum\left(\vec{B}\cdot\vec{\imath}\right)\frac{\partial \vec{A}}{\partial x} - \sum\left(\vec{A}\cdot\vec{\imath}\right)\frac{\partial \vec{B}}{\partial x}$$

$$= \left(\sum \vec{\imath}\cdot\frac{\partial \vec{B}}{\partial x}\right)\vec{A} - \left(\sum \vec{\imath}\cdot\frac{\partial \vec{A}}{\partial x}\right)\vec{B} + \left(\vec{B}\cdot\sum \vec{\imath}\frac{\partial}{\partial x}\right)\vec{A} - \left(\vec{A}\cdot\sum \vec{\imath}\frac{\partial}{\partial x}\right)\vec{B}$$

$$\therefore \nabla\times(\vec{A}\times\vec{B}) = \vec{A}(\nabla\cdot\vec{B}) - \vec{B}(\nabla\cdot\vec{A}) + \left(\vec{B}\cdot\nabla\right)\vec{A} - (\vec{A}\cdot\nabla)\vec{B}$$

(5) If $\vec{A}$ and $\vec{B}$ are vector point functions, then

$$\nabla(\vec{A}\cdot\vec{B}) = \vec{A}\times(\nabla\times\vec{B}) + (\vec{A}\cdot\nabla)\vec{B} + \vec{B}\times(\nabla\times\vec{A}) + (\vec{B}\cdot\nabla)\vec{A}$$

Proof:

$$\nabla(\vec{A}\cdot\vec{B}) = \sum \vec{i}\,\frac{\partial}{\partial x}(\vec{A}\cdot\vec{B})$$

$$= \sum \vec{i}\left(\frac{\partial\vec{A}}{\partial x}\cdot\vec{B} + \vec{A}\cdot\frac{\partial\vec{B}}{\partial x}\right)$$

$$= \sum \vec{i}\left(\vec{B}\cdot\frac{\partial\vec{A}}{\partial x}\right) + \sum \vec{i}\left(\vec{A}\cdot\frac{\partial\vec{B}}{\partial x}\right)$$

$$= \sum\left(\vec{B}\cdot\frac{\partial\vec{A}}{\partial x}\right)\vec{i} + \sum\left(\vec{A}\cdot\frac{\partial\vec{B}}{\partial x}\right)\vec{i} \qquad \ldots (1)$$

We know that $\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\,\vec{b} - (\vec{a}\cdot\vec{b})\,\vec{c}$

$\therefore (\vec{a}\cdot\vec{b})\,\vec{c} = (\vec{a}\cdot\vec{c})\,\vec{b} - \vec{a}\times(\vec{b}\times\vec{c})$

Consider $\sum\left(\vec{B}\cdot\frac{\partial\vec{A}}{\partial x}\right)\vec{i} = \sum\left[(\vec{B}\cdot\vec{i})\frac{\partial\vec{A}}{\partial x} - \vec{B}\times\left(\frac{\partial\vec{A}}{\partial x}\times\vec{i}\right)\right]$

$$= \sum\left(\vec{B}\cdot\vec{i}\,\frac{\partial}{\partial x}\right)\vec{A} + \sum\left[\vec{B}\times\left(\vec{i}\times\frac{\partial\vec{A}}{\partial x}\right)\right]$$

$$= (\vec{B}\cdot\nabla)\,\vec{A} + \sum\left[\vec{B}\times\left(\vec{i}\,\frac{\partial}{\partial x}\times\vec{A}\right)\right]$$

$$= (\vec{B}\cdot\nabla)\,\vec{A} + \vec{B}\times(\nabla\times\vec{A}) \qquad \ldots (2)$$

In (2) interchanging $\vec{A}$ and $\vec{B}$ we get,

$$\sum\left(\vec{A}\cdot\frac{\partial\vec{B}}{\partial x}\right)\vec{i} = (\vec{A}\cdot\nabla)\,\vec{B} + \vec{A}\times(\nabla\times\vec{B}) \qquad \ldots (3)$$

Substitute in equation (1)

$$(1) \Rightarrow \nabla(\vec{A}\cdot\vec{B}) = (\vec{B}\cdot\nabla)\,\vec{A} + \vec{B}\times(\nabla\times\vec{A}) + (\vec{A}\cdot\nabla)\,\vec{B} + \vec{A}\times(\nabla\times\vec{B})$$

(6) If φ is a scalar point function, then $\nabla\times(\nabla\varphi) = \vec{0}$.

(or)

Prove that $curl(grad\ \varphi) = 0$.

Solution:

$$\nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

$$\nabla\times\nabla\varphi = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial\varphi}{\partial x} & \frac{\partial\varphi}{\partial y} & \frac{\partial\varphi}{\partial z} \end{vmatrix}$$

$$= \sum \vec{i}\left[\frac{\partial^2\varphi}{\partial y\partial z} - \frac{\partial^2\varphi}{\partial z\partial y}\right]$$

$$= \sum \vec{i}\,(\vec{0}) = \vec{0}$$

(7) If $\vec{F}$ is a vector point function, then $\nabla\cdot(\nabla\times\vec{F}) = 0$.

(or)

Prove that $div(curl\,\vec{F}) = 0$.

Solution:

Let $\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= \vec{i}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) - \vec{j}\left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right) + \vec{k}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$$

$$\nabla \cdot (\nabla \times \vec{F}) = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \cdot$$

$$\left[\vec{i}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) - \vec{j}\left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right) + \vec{k}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\right]$$

$$= \frac{\partial^2 F_3}{\partial x\partial y} - \frac{\partial^2 F_2}{\partial x\partial z} - \frac{\partial^2 F_3}{\partial y\partial x} + \frac{\partial^2 F_1}{\partial y\partial z} + \frac{\partial^2 F_2}{\partial z\partial x} - \frac{\partial^2 F_1}{\partial z\partial y}$$

$$= 0$$

(8) If is a vector point function, then $\nabla \times (\nabla \times \vec{F}) = \nabla(\nabla \cdot \vec{F}) - \nabla^2\vec{F}$

(or)

Prove that $curl\,(curl\,\vec{F}) = grad\,(div\,\vec{F}) - \nabla^2\vec{F}$

Solution:

Let $\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$

$$\nabla \times (\nabla \times \vec{F}) = \vec{i}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) - \vec{j}\left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right) + \vec{k}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$$

And $\nabla \cdot \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$

L.H.S $\quad \nabla \times (\nabla \times \vec{F}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} & -\frac{\partial F_3}{\partial x} + \frac{\partial F_1}{\partial z} & \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \end{vmatrix}$

$$= \vec{i}\left[\frac{\partial^2 F_2}{\partial y\partial x} - \frac{\partial^2 F_1}{\partial y^2} - \frac{\partial^2 F_3}{\partial z\partial x} + \frac{\partial^2 F_1}{\partial z^2}\right] - \vec{j}\left[\frac{\partial^2 F_2}{\partial x^2} - \frac{\partial^2 F_1}{\partial x\partial y} - \frac{\partial^2 F_3}{\partial z\partial y} + \frac{\partial^2 F_2}{\partial z^2}\right]$$

$$+\vec{k}\left[-\frac{\partial^2 F_3}{\partial x^2} + \frac{\partial^2 F_1}{\partial x\partial z} - \frac{\partial^2 F_3}{\partial y^2} + \frac{\partial^2 F_2}{\partial y\partial z}\right]$$

R.H.S $\quad \nabla(\nabla \cdot \vec{F}) - \nabla^2\vec{F}$

$$= \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) - \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)(F_1\vec{i} + F_2\vec{j} + F_3\vec{k})$$

$$= \vec{i}\left[\frac{\partial^2 F_1}{\partial x^2} + \frac{\partial^2 F_2}{\partial x\partial y} + \frac{\partial^2 F_3}{\partial x\partial z}\right] + \vec{j}\left[\frac{\partial^2 F_1}{\partial y\partial x} + \frac{\partial^2 F_2}{\partial y^2} + \frac{\partial^2 F_3}{\partial y\partial z}\right] + \vec{k}\left[\frac{\partial^2 F_1}{\partial z\partial x} + \frac{\partial^2 F_2}{\partial z\partial y} + \frac{\partial^2 F_3}{\partial z^2}\right]$$

$$- \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)(F_1\vec{i} + F_2\vec{j} + F_3\vec{k})$$

$$= \vec{i}\left[\frac{\partial^2 F_2}{\partial x\partial y} + \frac{\partial^2 F_3}{\partial x\partial z} - \frac{\partial^2 F_1}{\partial y^2} - \frac{\partial^2 F_1}{\partial z^2}\right] - \vec{j}\left[\frac{\partial^2 F_2}{\partial x^2} - \frac{\partial^2 F_1}{\partial x\partial y} - \frac{\partial^2 F_3}{\partial z\partial y} + \frac{\partial^2 F_2}{\partial z^2}\right] +$$

$$\vec{k}\left[-\frac{\partial^2 F_3}{\partial x^2} + \frac{\partial^2 F_1}{\partial x\partial z} - \frac{\partial^2 F_3}{\partial y^2} + \frac{\partial^2 F_2}{\partial y\partial z}\right]$$

$$\text{L.H.S} = \text{R.H.S}$$

$$\therefore \nabla \times (\nabla \times \vec{F}) = \nabla(\nabla \cdot \vec{F}) - \nabla^2 \vec{F}$$

(9) $\nabla \cdot (\nabla\varphi) = (\nabla \cdot \nabla)\,\varphi = \nabla^2\varphi$

Proof:

$$\nabla\varphi = \vec{\imath}\frac{\partial\varphi}{\partial x} + \vec{\jmath}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

$$\therefore \nabla \cdot (\nabla\varphi) = \frac{\partial}{\partial x}\left(\frac{\partial\varphi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial\varphi}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial\varphi}{\partial z}\right)$$

$$= \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2}$$

$$\nabla \cdot \nabla = \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$$\nabla \cdot (\nabla\varphi) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\varphi = \nabla^2\varphi$$

Example: 2.26 Find (i) $\nabla \cdot \vec{r}$ (ii) $\nabla \times \vec{r}$

Solution:

Let $\vec{r} = x\,\vec{\imath} + y\,\vec{\jmath} + z\,\vec{k}$

(i) $\nabla \cdot \vec{r} = \left(\vec{\imath}\,\frac{\partial}{\partial x} + \vec{\jmath}\,\frac{\partial}{\partial y} + \vec{k}\,\frac{\partial}{\partial z}\right) \cdot \left(x\,\vec{\imath} + y\,\vec{\jmath} + z\,\vec{k}\right)$

$$= \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z)$$

$$= 1 + 1 + 1 = 3$$

(ii) $\nabla \times \vec{r} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix}$

$$= \vec{\imath}(0) + \vec{\jmath}\,(0) + \vec{k}\,(0) = \vec{0}$$

Example: 2.27 Find $\nabla \cdot \left(\frac{1}{r}\,\vec{r}\right)$ where $\vec{r} = x\,\vec{\imath} + y\,\vec{\jmath} + z\,\vec{k}$

Solution:

$$\nabla \cdot \left(\frac{1}{r}\,\vec{r}\right) = \nabla \cdot \left[\frac{1}{r}\left(x\,\vec{\imath} + y\,\vec{\jmath} + z\,\vec{k}\right)\right]$$

$$= \left(\vec{\imath}\,\frac{\partial}{\partial x} + \vec{\jmath}\frac{\partial}{\partial y} + \vec{k}\,\frac{\partial}{\partial z}\right) \cdot \left(\frac{x}{r}\,\vec{\imath} + \frac{y}{r}\,\vec{\jmath} + \frac{z}{r}\,\vec{k}\right)$$

$$= \sum \frac{\partial}{\partial x}\left(\frac{x}{r}\right)$$

$$= \sum\left[\frac{1}{r}(1) + x\left(-\frac{1}{r^2}\right)\frac{\partial r}{\partial x}\right]$$

$$= \sum\left[\frac{1}{r} - \frac{x}{r^2}\left(\frac{x}{r}\right)\right] \qquad \left(\because \frac{\partial r}{\partial x} = \frac{x}{r}\right)$$

` $= \sum\left[\frac{1}{r} - \frac{x^2}{r^3}\right]$

$$= \frac{3}{r} - \frac{1}{r^3}\left(x^2 + y^2 + z^2\right)$$

$$= \frac{3}{r} - \frac{r^2}{r^3} \qquad \because r^2 = (x^2 + y^2 + z^2)$$

$$= \frac{3}{r} - \frac{1}{r} = \frac{2}{r}$$

Example: 2.28 If is a constant vector and is the position vector of any point, prove that

(i) $\nabla \cdot (\vec{a} \times \vec{r}) = 0$ (ii) $\nabla \times (\vec{a} \times \vec{r}) = 2\vec{a}$

Solution:

Let $\vec{r} = x\,\vec{\imath} + y\,\vec{\jmath} + z\,\vec{k}$

$\vec{a} = a_1\,\vec{\imath} + a_2\,\vec{\jmath} + a_3\,\vec{k}$

$$\vec{a} \times \vec{r} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ a_1 & a_2 & a_3 \\ x & y & z \end{vmatrix}$$

$$= \vec{\imath}(a_2 z - a_3 y) - \vec{\jmath}(a_1 z - a_3 x) + \vec{k}(a_1 y - a_2 x)$$

(i) $\nabla \cdot (\vec{a} \times \vec{r}) = \frac{\partial}{\partial x}(a_2 z - a_3 y) + \frac{\partial}{\partial y}(-a_1 z + a_3 x) + \frac{\partial}{\partial z}(a_1 y - a_2 x)$

$$= 0 + 0 + 0 = 0$$

(ii) $\nabla \times (\vec{a} \times \vec{r}) = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_2 z - a_3 y & -a_1 z + a_3 x & a_1 y - a_3 x \end{vmatrix}$

`$= \vec{\imath}(a_1 + a_1) - \vec{\jmath}(-a_2 - a_2) + \vec{k}(a_3 + a_3)$

$$= 2a_1\vec{\imath} + 2a_2\vec{\jmath} + 2a_3\vec{k}$$

$$= 2\left(a_1\vec{\imath} + a_2\vec{\jmath} + a_3\vec{k}\right) = 2\vec{a}$$

Example: 2.29 Prove that $\boldsymbol{curl}(\boldsymbol{f}(\boldsymbol{r})\vec{\boldsymbol{r}}) = \vec{\boldsymbol{0}}$

Solution:

Let $f(r)\vec{r} = f(r)\left[x\,\vec{\imath} + y\,\vec{\jmath} + z\,\vec{k}\right]$

$= xf(r)\vec{\imath} + yf(r)\vec{\jmath} + zf(r)\vec{k}$

$$\nabla \times (f(r)\vec{r}) = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xf(r) & yf(r) & zf(r) \end{vmatrix}$$

$$= \sum \vec{\imath}\left[zf'(r)\frac{\partial r}{\partial y} - yf'(r)\frac{\partial r}{\partial z}\right]$$

$$= \sum \vec{\imath}\left[zf'(r)\left(\frac{y}{r}\right) - yf'(r)\left(\frac{z}{r}\right)\right]$$

$$= \sum \vec{\imath}\left[\frac{zy}{r}f'(r) - \frac{zy}{r}f'(r)\right]$$

$$= \sum \vec{\imath}\,(0)$$

$$= 0\,\vec{\imath} + 0\,\vec{\jmath} + 0\,\vec{k} = \vec{0}$$

Example: 2.30 Prove that $\boldsymbol{curl}[\boldsymbol{\varphi}\,\nabla\boldsymbol{\varphi}] = \vec{\boldsymbol{0}}$

(or)

Prove that $\nabla \times [\boldsymbol{\varphi}\,\nabla\boldsymbol{\varphi}] = \vec{\boldsymbol{0}}$

Solution:

$$\varphi\nabla\varphi = \varphi\left[\vec{\imath}\frac{\partial\varphi}{\partial x} + \vec{\jmath}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}\right]$$

$$= \vec{\imath}\left(\varphi\frac{\partial\varphi}{\partial x}\right) + \vec{\jmath}\left(\varphi\frac{\partial\varphi}{\partial y}\right) + \vec{k}\left(\varphi\frac{\partial\varphi}{\partial z}\right)$$

$$\nabla \times (\varphi\nabla\varphi) = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \varphi\frac{\partial\varphi}{\partial x} & \varphi\frac{\partial\varphi}{\partial y} & \varphi\frac{\partial\varphi}{\partial z} \end{vmatrix}$$

$$= \sum\vec{\imath}\left[\frac{\partial}{\partial y}\left(\varphi\frac{\partial\varphi}{\partial z}\right) - \frac{\partial}{\partial z}\left(\varphi\frac{\partial\varphi}{\partial y}\right)\right]$$

$$= \sum\vec{\imath}\left[\varphi\frac{\partial^2\varphi}{\partial y\partial z} + \frac{\partial\varphi}{\partial y}\cdot\frac{\partial\varphi}{\partial z} - \varphi\frac{\partial^2\varphi}{\partial z\partial y} - \frac{\partial\varphi}{\partial y}\cdot\frac{\partial\varphi}{\partial z}\right]$$

$$= \sum\vec{\imath}\,(0)$$

$$= 0\,\vec{\imath} + 0\,\vec{\jmath} + 0\,\vec{k} = \vec{0}$$

Example: 2.31 If $\vec{\boldsymbol{\omega}}$ is a constant vector and $\vec{\boldsymbol{v}} = \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}$, then prove that $\vec{\boldsymbol{\omega}} = \frac{1}{2}(\nabla \times \vec{\boldsymbol{v}})$.

Solution:

Let $\vec{r} = x\,\vec{\imath} + y\,\vec{\jmath} + z\,\vec{k}$

$$\vec{\omega} = \omega_1\,\vec{\imath} + \omega_2\,\vec{\jmath} + \omega_3\,\vec{k}$$

$$\vec{\omega} \times \vec{r} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \omega_1 & \omega_2 & \omega_3 \\ x & y & z \end{vmatrix}$$

$$= \vec{\imath}(\omega_2 z - \omega_3 y) - \vec{\jmath}(\omega_1 z - \omega_3 x) + \vec{k}(\omega_1 y - \omega_2 x)$$

$$\nabla \times \vec{v} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \omega_2 z - \omega_3 y & -\omega_1 z + \omega_3 x & \omega_1 y - \omega_2 x \end{vmatrix}$$

$$= \vec{\imath}(\omega_1 + \omega_1) - \vec{\jmath}(-\omega_2 - \omega_2) + \vec{k}(\omega_3 + \omega_3)$$

$$= 2\omega_1\vec{\imath} + 2\omega_2\vec{\jmath} + 2\omega_3\vec{k}$$

$$= 2\left(\omega_1\vec{\imath} + \omega_2\vec{\jmath} + \omega_3\vec{k}\right) = 2\vec{\omega}$$

$$\vec{\omega} = \frac{1}{2}(\nabla \times \vec{v})$$

Problems based on solenoidal vector and irrotational vector and scalar potential

Example: 2.32 Prove that the vector $\vec{\boldsymbol{F}} = \boldsymbol{z}\,\vec{\boldsymbol{\imath}} + \boldsymbol{x}\,\vec{\boldsymbol{\jmath}} + \boldsymbol{y}\,\vec{\boldsymbol{k}}$ is solenoidal.

Solution:

Given $\vec{F} = z\,\vec{\imath} + x\,\vec{\jmath} + y\,\vec{k}$

To prove $\nabla\cdot\vec{F} = 0$

$$\nabla\cdot\vec{F} = \frac{\partial}{\partial x}(z) + \frac{\partial}{\partial y}(x) + \frac{\partial}{\partial z}(y)$$

$$= 0$$

$\therefore\ \vec{F}$ is solenoidal.

Example: 2.33 Show that the vector $\vec{F} = 3y^4z^2\vec{i} + 4x^3z^2\vec{j} - 3x^2y^2\vec{k}$ is solenoidal.

Solution:

Given $\vec{F} = 3y^4z^2\vec{i} + 4x^3z^2\vec{j} - 3x^2y^2\vec{k}$

To prove $\nabla \cdot \vec{F} = 0$

$$\nabla \cdot \vec{F} = \frac{\partial}{\partial x}(3y^4z^2) + \frac{\partial}{\partial y}(4x^3z^2) + \frac{\partial}{\partial z}(3x^2y^2)$$

$$= 0 + 0 + 0 = 0$$

$\therefore\ \vec{F}$ is solenoidal.

Example: 2.34 If $\vec{F} = (x+3y)\vec{i} + (y-2z)\vec{j} + (x+\lambda z)\vec{k}$ is solenoidal, then find the value of λ.

Solution:

Given $\vec{F}$ is solenoidal.

$$(ie)\nabla \cdot \vec{F} = 0$$

$$\Rightarrow \frac{\partial}{\partial x}(x+3y) + \frac{\partial}{\partial y}(y-2z) + \frac{\partial}{\partial z}(x+\lambda z) = 0$$

$$\Rightarrow 1 + 1 + \lambda = 0$$

$$\therefore\ \lambda = -2$$

Example: 2.35 Find a such that $(3x-2y+z)\vec{i} + (4x+ay-z)\vec{j} + (x-y+2z)\vec{k}$ is solenoidal.

Solution:

Given $(3x-2y+z)\vec{i} + (4x+ay-z)\vec{j} + (x-y+2z)\vec{k}$ is solenoidal.

$$(ie)\nabla \cdot \vec{F} = 0$$

$$\Rightarrow \frac{\partial}{\partial x}(3x-2y+z) + \frac{\partial}{\partial y}(4x+ay-z) + \frac{\partial}{\partial z}(x-y+2z) = 0$$

$$\Rightarrow 3 + a + 2 = 0$$

$$\therefore\ a = -5$$

Example: 2.36 Show that the vector $\vec{F} = (6xy+z^3)\vec{i} + (3x^2-z)\vec{j} + (3xz^2-y)\vec{k}$ is irrotational.

Solution:

Given $\vec{F} = (6xy+z^3)\vec{i} + (3x^2-z)\vec{j} + (3xz^2-y)\vec{k}$

To prove $curl\ \vec{F} = 0$

$$(i.e) To\ prove\ \nabla \times \vec{F} = 0$$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 6xy+z^3 & 3x^2-3z^2 & 3xz^2-y \end{vmatrix}$$

$$= \vec{i}(-1+1) - \vec{j}(3z^2 - 3z^2) + \vec{k}(6x-6x) = \vec{0}$$

$\therefore\ \vec{F}$ is irrotational.

Example: 2.37 Find the constants a, b, c so that the vectors is irrotational

$\vec{F} = (x + 2y + az)\vec{i} + (bx + 3y - z)\vec{j} + (4x + cy + 2z)\vec{k}$.

Solution:

Given $\vec{F} = (x + 2y + az)\vec{i} + (bx + 3y - z)\vec{j} + (4x + cy + 2z)\vec{k}$ **is irrotational.**

$$(ie)\nabla \times \vec{F} = 0$$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x + 2y + az & bx + 3y - z & 4x + cy + 2z \end{vmatrix} = \vec{0}$$

$$\Rightarrow \vec{i}(c + 1) - \vec{j}(4 - a) + \vec{k}(b - 2) = \vec{0}$$

$$\Rightarrow c + 1 = 0 \ ; \qquad 4 - a = 0 \ ; \qquad b - 2 = 0$$

$$\Rightarrow c = -1 ; \qquad 4 = a ; \qquad b = 2$$

Example: 2.38 Prove that $\vec{F} = (6xy + z^3)\vec{i} + (3x^2 - z)\vec{j} + (3xz^2 - y)\vec{k}$ **is irrotational and find** φ **such that** $\vec{F} = \nabla\varphi$**.**

Solution:

Given $\vec{F} = (6xy + z^3)\vec{i} + (3x^2 - z)\vec{j} + (3xz^2 - y)\vec{k}$

To prove $\nabla \times \vec{F} = 0$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 6xy + z^3 & 3x^2 - z & 3xz^2 - y \end{vmatrix}$$

$$= \vec{i}(-1 + 1) - \vec{j}(3z^2 - 3z^2) + \vec{k}(6x - 6x)$$

$$= \vec{0}$$

$\therefore$ $\vec{F}$ is irrotational.

To find φ such that $\vec{F} = \nabla\varphi$.

$$\nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

Equating the coefficients of $\vec{i}, \vec{j}$ and $\vec{k}$ we get,

$$\frac{\partial\varphi}{\partial x} = 6xy + z^3; \qquad \frac{\partial\varphi}{\partial y} = 3x^2 - z; \qquad \frac{\partial\varphi}{\partial z} = 3xz^2 - y$$

Integrating the above equations partially with respect to x, y, z respectively

$$\varphi = 3x^2y + xz^3 + f_1(y, z)$$

$$\varphi = 3x^2y - yz + f_2(x, z)$$

$$\varphi = xz^3 - yz + f_3(x, y)$$

$\therefore \varphi = 3x^2y + xz^3 - yz + c$ where c is constant.

Example: 2.39 Prove that $\vec{F} = (y^2 \cos x + z^3)\vec{i} + (2y \sin z - 4)\vec{j} + (3xz^2)\vec{k}$ **is irrotational and find its scalar potential.**

Solution:

Given $\vec{F} = (y^2 cos\, x + z^3)\vec{\imath} + (2y\, sinz - 4)\vec{\jmath} + (3xz^2)\vec{k}$

To prove $\nabla \times \vec{F} = 0$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 cos\, x + z^3 & 2y\, sinz - 4 & 3xz^2 \end{vmatrix}$$

$$= \vec{\imath}(0-0) - \vec{\jmath}(3z^2 - 3z^2) + \vec{k}(2y\cos x - 2ycos\, x)$$

$$= \vec{0}$$

$\therefore$ $\vec{F}$ is irrotational.

To find φ such that $\vec{F} = \nabla\varphi$.

$$\nabla\varphi = \vec{\imath}\frac{\partial\varphi}{\partial x} + \vec{\jmath}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

Equating the coefficients of $\vec{\imath}, \vec{\jmath}$ and $\vec{k}$ we get,

$\frac{\partial\varphi}{\partial x} = y^2\cos x + z^3$; $\quad \frac{\partial\varphi}{\partial y} = 2ysin\, x - 4$; $\quad \frac{\partial\varphi}{\partial z} = 3xz^2$

Integrating the above equations partially with respect to x, y, z respectively

$\varphi = y^2\sin x + z^3x + f_1(y,z)$

$\varphi = y^2\sin x - 4y + f_2(x,z)$

$\varphi = xz^3 + f_3(x,y)$

$\therefore \varphi = y^2\sin x + z^3x - 4y + c$ is scalar potential.

Example: 2.40 Prove that $\vec{F} = (2x+yz)\vec{\imath} + (4y+zx)\vec{\jmath} + (6z-xy)\vec{k}$ is solenoidal as well as irrotational also find the scalar potential of $\vec{F}$.

Solution:

Given $\vec{F} = (2x+yz)\vec{\imath} + (4y+zx)\vec{\jmath} + (6z-xy)\vec{k}$

(i) To prove $\vec{F}$ is solenoidal.

$(ie)\, To\, prove\, \nabla\cdot\vec{F} = 0$

$$\nabla\cdot\vec{F} = \frac{\partial}{\partial x}(2x+yz) + \frac{\partial}{\partial y}(4y+zx) + \frac{\partial}{\partial z}(-6z+xy)$$

$$= 2 + 4 - 6 = 0$$

$\therefore$ $\vec{F}$ is solenoidal.

(ii) To prove $\vec{F}$ is irrotational.

$(ie)\, To\, prove\, \nabla\times\vec{F} = 0$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x+yz & 4y+zx & -6z+xy \end{vmatrix}$$

$$= \vec{\imath}(x-x) - \vec{\jmath}(y-y) + \vec{k}(z-z)$$

$$= \vec{0}$$

$\therefore\ \vec{F}$ is irrotational.

(iii) To find φ such that $\vec{F} = \nabla\varphi$.

$$(2x+yz)\vec{i} + (4y+zx)\vec{j} + (6z-xy)\vec{k} = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

Equating the coefficients of $\vec{i}, \vec{j}$ and $\vec{k}$ we get,

$\frac{\partial\varphi}{\partial x} = 2x + yz;\qquad \frac{\partial\varphi}{\partial y} = 4y + zx;\qquad \frac{\partial\varphi}{\partial z} = -6z + xy$

Integrating the above equations partially with respect to x, y, z respectively

$\varphi = x^2 + xyz + f_1(y,z)$

$\varphi = 2y^2 + xyz + f_2(x,z)$

$\varphi = -3z^2 + xyz + f_3(x,y)$

$\therefore \varphi = x^2 + 2y^2 - 3z^2 + xyz + c$ where c is a constant.

$\therefore\ \varphi$ is a scalar potential of $\vec{F}$.

Example: 2.41 If $\nabla\varphi = 2xyz^3\vec{i} + x^2z^3\vec{j} + 3x^2yz^2\vec{k}$ find φ if $\varphi(-1,2,2) = 4$

Solution:

Given $\nabla\varphi = 2xyz^3\vec{i} + x^2z^3\vec{j} + 3x^2yz^2\vec{k}$ ... (1)

We know that $\nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$ ... (2)

Comparing (1) and (2)

$\frac{\partial\varphi}{\partial x} = 2xyz^3;\qquad \frac{\partial\varphi}{\partial y} = x^2z^3;\qquad \frac{\partial\varphi}{\partial z} = 3x^2yz^2$

Integrating the above equations partially with respect to x, y, z respectively

$\varphi = x^2yz^3 + f_1(y,z)$

$\varphi = x^2yz^3 + f_2(x,z)$

$\varphi = x^2yz^3 + f_3(x,y)$

$\therefore \varphi = x^2yz^3 + c$ where c is a constant.

Given $\varphi(-1,2,2) = 4$

$\Rightarrow 16 + c = 4$

$\Rightarrow c = -12$

$\therefore\ \varphi = x^2yz^3 - 12$

Example: 2.42 If $\vec{A}$ and $\vec{B}$ are irrotational, then prove that $\vec{A} \times \vec{B}$ is solenoidal.

Solution:

Given $\vec{A}$ and $\vec{B}$ are irrotational.

$(ie) \nabla \times \vec{A} = 0$ and $\nabla \times \vec{B} = 0$

We know that $\nabla \cdot (\vec{A} \times \vec{B}) = (\nabla \times \vec{A}) \cdot \vec{B} - (\nabla \times \vec{B}) \cdot \vec{A}$

$$= 0 \cdot \vec{A} - 0 \cdot \vec{B}$$

$$= 0$$

Hence $\vec{A} \times \vec{B}$ is solenoidal.

Example: 2.43 if $\vec{A}$ is a constant vector, then prove that (i) $div\,\vec{A} = 0$ and (ii) $curl\,\vec{A} = 0$

Solution:

Let $\vec{A} = A_1\vec{i} + A_2\vec{j} + A_3\vec{k}$

$\frac{\partial A_1}{\partial x} = 0; \quad \frac{\partial A_2}{\partial y} = 0; \quad \frac{\partial A_3}{\partial z} = 0$

(i) $\nabla \cdot \vec{A} = \frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}$

$= 0 + 0 + 0 = 0$

Hence $div\,\vec{A} = 0$.

(ii) $\nabla \times \vec{A} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix}$

$= \vec{i}(0-0) - \vec{j}(0-0) + \vec{k}(0-0)$

$= \vec{0}$

$\therefore\ curl\,\vec{F} = \vec{0}$

Example: 2.44 If φ and χ are differentiable scalar fields, prove $\nabla\varphi \times \nabla\chi$ is solenoidal.

Solution:

Consider $\nabla \cdot (\nabla\varphi \times \nabla\chi)$

$= \nabla\chi \cdot \nabla \times (\nabla\varphi) - \nabla\varphi \cdot [\nabla \times (\nabla\chi)] \qquad [\because \nabla \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\nabla \times \vec{A}) - \vec{A} \cdot (\nabla \times \vec{B})]$

$= \nabla\chi \cdot 0 - \nabla\varphi \cdot 0$

$= 0$

$\therefore \nabla\varphi \times \nabla\chi$ is solenoidal.

Example: 2.45 Find $f(r)$ if the vector $f(r)\vec{r}$ is both solenoidal and irrotational.

Solution:

(i) Given $f(r)\vec{r}$ is solenoidal.

$\therefore\ \nabla \cdot (f(r)\vec{r}) = 0$

We know that $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$\therefore\ f(r)\vec{r} = f(r)\,x\vec{i} + f(r)\,y\vec{j} + f(r)\,z\vec{k}$

Now $\nabla \cdot (f(r)\vec{r}) = 0$

$\Rightarrow \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \cdot \left(f(r)\,x\,\vec{i} + f(r)\,y\,\vec{j} + f(r)\,z\,\vec{k}\right) = 0$

$\Rightarrow \frac{\partial}{\partial x}(f(r)\,x) + \frac{\partial}{\partial y}(f(r)\,y) + \frac{\partial}{\partial z}(f(r)\,z) = 0$

$\Rightarrow \sum \frac{\partial}{\partial x}(f(r)\,x) = 0$

$\Rightarrow \sum \left[f(r) \ \cdot 1 + x\, f'(r) \frac{\partial r}{\partial x} \right] = 0$

$\Rightarrow \sum \left[f(r) + x\, f'(r) \frac{x}{r} \right] = 0$

$\Rightarrow \sum \left[f(r) + \frac{x^2}{r}\, f'(r) \right] = 0$

$\Rightarrow 3f(r) + f'(r) \left[\frac{x^2}{r} + \frac{y^2}{r} + \frac{z^2}{r} \right] = 0$

$\Rightarrow 3f(r) + \frac{f'(r)}{r} [r^2] = 0 \quad [\because\ x^2 + y^2 + z^2 = r^2]$

$\Rightarrow 3f(r) + f'(r)\, r = 0$

$\Rightarrow f'(r)\, r = -3f(r)$

$\Rightarrow \frac{f'(r)}{f(r)} = \frac{-3}{r}$

Integrating with respect to r, we get

$$\Rightarrow \int \frac{f'(r)}{f(r)}\, dr = \int \frac{-3}{r}\, dr$$

$$\Rightarrow \log f(r) = -3 \log r + \log c$$

$$= \log r^{-3} + \log c$$

$$= \log \left(\frac{1}{r^3} \right) + \log c$$

$$= \log \left(\frac{c}{r^3} \right)$$

$$\therefore f(r) = \frac{c}{r^3}$$

(ii) Given $f(r)\vec{r}$ is irrotational.

$$\nabla \times f(r)\vec{r} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xf(r) & yf(r) & zf(r) \end{vmatrix}$$

$$= \sum \vec{i} \left[z \frac{\partial}{\partial y} f(r) - y \frac{\partial}{\partial z} f(r) \right]$$

$$= \sum \vec{i} \left[z f'(r) \frac{\partial r}{\partial y} - y\, f'(r) \frac{\partial r}{\partial z} \right]$$

$$= \sum \vec{i} \left[z f'(r) \frac{y}{r} - y\, f'(r) \frac{z}{r} \right]$$

$$= \sum \vec{i} f'(r) \left[\frac{zy}{r} - \frac{zy}{r} \right]$$

$$= \vec{0} \text{ for all } f(r)$$

Example: 2.46 Prove that $r^n\vec{r}$ is irrotational for every n and solenoidal only for $n = -3$.

Solution:

We know that $\vec{r} = x\vec{i} + y\,\vec{j} + z\,\vec{k}$

$\therefore\ r^n\vec{r} = r^n\, x\vec{i} + r^n\, y\,\vec{j} + r^n\, z\,\vec{k}$

(i) To prove $r^n\vec{r}$ is irrotational.

$$\nabla \times (r^n \vec{r}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ r^n\, x & r^n\, y & r^n\, z \end{vmatrix}$$

$$= \sum \vec{i}\left[z\, nr^{n-1}\frac{\partial r}{\partial y} - y\, nr^{n-1}\,\frac{\partial r}{\partial z}\right]$$

$$= \sum \vec{i}\left[z\, nr^{n-1}\frac{y}{r} - y\, nr^{n-1}\,\frac{z}{r}\right]$$

$$= \sum \vec{i}\left[\, nr^{n-1}\frac{zy}{r} - \, nr^{n-1}\,\frac{zy}{r}\right]$$

$$= \sum \vec{i}\,(0)$$

$$= 0\,\vec{i} + \,0\,\vec{j} + \,0\,\vec{k} = \,\vec{0}$$

$\therefore\ r^n\vec{r}$ is irrotational for every n.

(ii) To prove $r^n\vec{r}$ is solenoidal.

$$\nabla \cdot (r^n\vec{r}) = \,\nabla\,\cdot\left(r^n\, x\vec{i} + r^n\, y\,\vec{j} + r^n\, z\,\vec{k}\right)$$

$$= \sum \frac{\partial}{\partial x}\,(r^n\, x)$$

$$= \sum\left[r^n\,(1) + \,xnr^{n-1}\,\frac{\partial r}{\partial x}\right]$$

$$= \sum\left[r^n\, + \,xnr^{n-1}\,\frac{x}{r}\right]$$

$$= \sum[r^n\, + \,x^2nr^{n-2}\,]$$

$$= 3r^n\, + nr^{n-2}(x^2 + \,y^2 + \,z^2)$$

$$= 3r^n\, + nr^{n-2}(r^2)$$

$$= 3r^n\, + nr^n$$

$$= r^n(3\, + n)$$

When $n\, = -3$, we get $\nabla \cdot (r^n\vec{r}) = 0$

$\therefore\ r^n\vec{r}$ is solenoidal only if $\,n\, = -3$.

Problems based on Laplace operator

Example: 2.47 Find $\nabla^2(\log r)$

Solution:

$$\nabla^2(\log r) = \,\sum \frac{\partial^2}{\partial x^2}(\log r)$$

$$= \,\sum \frac{\partial}{\partial x}\left(\frac{1}{r}\,\frac{\partial r}{\partial x}\right)$$

$$= \,\sum \frac{\partial}{\partial x}\left(\frac{1}{r^2}\,x\right)$$

$$= \sum\left[\frac{1}{r^2}(1) + \,x\left(-\frac{2}{r^3}\right)\,\frac{\partial r}{\partial x}\right]$$

$$= \sum\left[\frac{1}{r^2} - \,x\left(\frac{2}{r^3}\right)\,\frac{x}{r}\right]$$

$$= \sum\left[\frac{1}{r^2} - \,\frac{2x^2}{r^4}\right]$$

$$= \,\frac{3}{r^2} - \,\frac{2}{r^4}(x^2 + \,y^2 + \,z^2)$$

$$= \frac{3}{r^2} - \frac{2}{r^4}(r^2)$$

$$= \frac{3}{r^2} - \frac{2}{r^2} = \frac{1}{r^2}$$

Example: 2.48 Prove that $\nabla^2(r^n) = n(n+1)r^{n-2}$, where $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ and $r = |\vec{r}|$ and hence deduce $\nabla^2\left(\frac{1}{r}\right)$.

(or)

Prove that $div\,(grad\, r^n) = n(n+1)\, r^{n-2}$

Solution:

Let $r = |\vec{r}| = \sqrt{x^2 + y^2 + z^2}$

Hence $\frac{\partial r}{\partial x} = \frac{x}{r}$; $\quad \frac{\partial r}{\partial y} = \frac{y}{r}$; $\quad \frac{\partial r}{\partial z} = \frac{z}{r}$

$$\nabla^2(r^n) = \sum \frac{\partial^2}{\partial x^2}(r^n)$$

$$= \sum \frac{\partial}{\partial x}\left[n r^{n-1}\frac{\partial r}{\partial x}\right]$$

$$= \sum \frac{\partial}{\partial x}\left[n r^{n-1}\frac{x}{r}\right]$$

$$= \sum \frac{\partial}{\partial x}[n\, x r^{n-2}]$$

$$= \sum n\left[x(n-2)r^{n-3}\frac{\partial r}{\partial x} + r^{n-2}\,(1)\right]$$

$$= \sum n\left[x(n-2)r^{n-3}\frac{x}{r} + r^{n-2}\right]$$

$$= \sum[n[(n-2)r^{n-4}\,x^2 + r^{n-2}]]$$

$$= \sum[n(n-2)r^{n-4}\,x^2 + n\,r^{n-2}]$$

$$= n(n-2)r^{n-4}\,(x^2 + y^2 + z^2) + 3\,n\,r^{n-2}$$

$$= n(n-2)r^{n-4}\,r^2 + 3\,n\,r^{n-2}$$

$$= n(n-2)r^{n-2} + 3\,n\,r^{n-2}$$

$$= n\,r^{n-2}\,(n-2+3)$$

$$= n\,r^{n-2}\,(n+1) \qquad \ldots (1)$$

(ii) $\nabla^2\left(\frac{1}{r}\right) = \nabla^2(r^{-1})$

$$= (-1)\,(-1+1)\,r^{-1-2} \quad \text{by (1)}$$

$$= (-1)\,(0)\,r^{-3} = 0$$

Example: 2.49 Prove that $\nabla^2(r^n\vec{r}) = n\,(n+3)r^{n-2}\vec{r}$

Solution:

We have $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

Hence $\frac{\partial \vec{r}}{\partial x} = \vec{i}$; $\quad \frac{\partial \vec{r}}{\partial y} = \vec{j}$; $\quad \frac{\partial \vec{r}}{\partial z} = \vec{k}$

Also $r = |\vec{r}| = \sqrt{x^2 + y^2 + z^2}$

Hence $\frac{\partial r}{\partial x} = \frac{x}{r}$; $\quad \frac{\partial r}{\partial y} = \frac{y}{r}$; $\quad \frac{\partial r}{\partial z} = \frac{z}{r}$

$$\nabla^2(r^n\vec{r}) = \sum \frac{\partial^2}{\partial x^2}(r^n\vec{r})$$

$$= \sum \frac{\partial}{\partial x}\left[r^n \frac{\partial \vec{r}}{\partial x} + n\,r^{n-1}\frac{\partial r}{\partial x}\vec{r}\right]$$

$$= \sum \frac{\partial}{\partial x}\left[r^n\,\vec{i} + n\,r^{n-1}\frac{x}{r}\vec{r}\right]$$

$$= \sum \frac{\partial}{\partial x}\left[r^n\,\vec{i} + n\,r^{n-2}\,x\,\vec{r}\right]$$

$$= \sum\left[n\,r^{n-1}\frac{\partial r}{\partial x}\vec{i} + n\left[r^{n-2}x\left(\frac{\partial \vec{r}}{\partial x}\right) + r^{n-2}(1)\vec{r} + \left[(n-2)\,r^{n-3}\frac{\partial r}{\partial x}\right]x\vec{r}\right]\right]$$

$$= \sum\left[n\,r^{n-1}\frac{x}{r}\vec{i} + n\,r^{n-2}\,x\vec{i} + n\,r^{n-2}\vec{r} + n(n-2)\,r^{n-3}\frac{x}{r}\,x\vec{r}\right]$$

$$= \sum[n\,r^{n-2}x\vec{i} + n\,r^{n-2}\,x\vec{i} + n\,r^{n-2}\vec{r} + n(n-2)\,r^{n-4}\,x^2\vec{r}]$$

$$= n\,r^{n-2}\left(x\vec{i} + y\,\vec{j} + z\,\vec{k}\right) + n\,r^{n-2}\left(x\vec{i} + y\,\vec{j} + z\,\vec{k}\right) + 3n\,r^{n-2}\vec{r}$$

$$n(n-2)\,r^{n-4}\vec{r}(x^2 + y^2 + z^2)$$

$$= n\,r^{n-2}\vec{r} + n\,r^{n-2}\vec{r} + 3n\,r^{n-2}\vec{r} + n(n-2)r^{n-4}\vec{r}r^2$$

$$= 5n\,r^{n-2}\vec{r} + n(n-2)r^{n-2}\vec{r}$$

$$= n\,r^{n-2}\vec{r}(5 + n - 2)$$

$$= n\,r^{n-2}\vec{r}(n+3)$$

$$= n(n+3)r^{n-2}\vec{r}$$

Example: 2.50 Prove that $\nabla^2 \boldsymbol{f}(\boldsymbol{r}) = \boldsymbol{f}''(\boldsymbol{r}) + \left(\frac{2}{r}\right)\boldsymbol{f}'(\boldsymbol{r})$

Solution:

$$\nabla^2 f(r) = \sum \frac{\partial^2}{\partial x^2} f(r)$$

$$= \sum \frac{\partial}{\partial x}\left[f'(r)\,\frac{\partial r}{\partial x}\right]$$

$$= \sum \frac{\partial}{\partial x}\left[f'(r)\,\frac{x}{r}\right]$$

$$= \sum \frac{\partial}{\partial x}\left[f'(r)x\,\frac{1}{r}\right]$$

$$= \sum\left[f'(r)x\left[\frac{-1}{r^2}\frac{\partial r}{\partial x}\right] + f'(r)\,(1)\,\frac{1}{r} + f''(r)\,\frac{\partial r}{\partial x}\,x\frac{1}{r}\right]$$

$$= \sum\left[f'(r)x\,\frac{-1}{r^2}\frac{x}{r} + f'(r)\,\frac{1}{r} + f''(r)\,\frac{x}{r}\,x\frac{1}{r}\right]$$

$$= \sum\left[f'(r)\,\frac{-1}{r^3}x^2 + f'(r)\,\frac{1}{r} + f''(r)\,\frac{1}{r^2}\,x^2\right]$$

$$= f'(r)\,\frac{-1}{r^3}(x^2 + y^2 + z^2) + \frac{3}{r}f'(r) + f''(r)\,\frac{1}{r^2}\,(x^2 + y^2 + z^2)$$

$$= -f'(r)\,\frac{1}{r^3}(r^2) + \frac{3}{r}f'(r) + f''(r)\,\frac{1}{r^2}\,(r^2)$$

$$= -f'(r)\,\frac{1}{r} + \frac{3}{r}f'(r) + f''(r)$$

$$= f''(r) + \frac{2}{r}f'(r)$$

Exercise: 2.2

1. When $\varphi = x^3 + y^3 + z^3 - 3xyz$, find $\nabla\varphi, \nabla \cdot \nabla\varphi, \nabla \times \nabla\varphi$ at the point $(1, 2, 3)$.

Ans: $(\nabla\varphi)_{(1,2,3)} = -15\vec{\imath} + 3\vec{\jmath} + 21\vec{k}$

$(\nabla \cdot \varphi)_{(1,2,3)} = 36$

$(\nabla \times \nabla\varphi)_{(1,2,3)} = \vec{0}$

2. Show that, $div\left(\frac{\vec{r}}{r}\right) = \frac{2}{r}$

3. Find $\nabla \cdot \vec{F}$ and $\nabla \times \vec{F}$ of the vector point function $\vec{F} = xz^3\,\vec{\imath} - 2x^2yz\,\vec{\jmath} + 2yz^4\,\vec{k}$ at $(1, -1, 1)$. **Ans:** $\left(\nabla \cdot \vec{F}\right)_{(1,-1,1)} = -9, \left(\nabla \times \vec{F}\right)_{(1,-1,1)} = 3\vec{\jmath} + 4\vec{k}$

4. Show that the vector $\vec{F} = (\sin y + z)\vec{\imath} + (x\cos y - z)\,\vec{\jmath} + (x - y)\,\vec{k}$ is irrotational.

5. Show that the vector $\vec{F} = (2xy - z^2)\vec{\imath} + (x^2 + 2yz)\,\vec{\jmath} + (y^2 - 2zx)\,\vec{k}$ is irrotational and find its scalar potential. **Ans:** $x^2y - xz^2 + y^2z + c$

6. Show that the vector $\vec{F} = (3x^2 + 2y^2 + 1)\vec{\imath} + (4xy - 3y^2z - 3)\,\vec{\jmath} + (2 - y^3)\,\vec{k}$ is irrotational and find its scalar potential. **Ans:** $x^3 + 2y^2x + x - y^3z - 3y + 2z + c$

7. Show that the vector $\vec{F} = (y^2 + 2xz^2)\vec{\imath} + (2xy - z)\,\vec{\jmath} + (2x^2z - y + 2z)\,\vec{k}$ is irrotational and find its scalar potential. **Ans:** $xy^2 + x^2z^2 - yz + z^2 + c$

8. Prove that $\vec{F} = (x^2 - y^2 + x)\vec{\imath} - (2xy + z)\,\vec{\jmath}$ is irrrotational and hence, find its scalar potential. **Ans:** $\frac{x^3}{3} - xy^2 + \frac{x^2}{2} - \frac{y^2}{2} + c$

9. Find the constants a, b, c so that the following vector is irrotational.

(i) $\vec{F} = (axy + bz^3)\vec{\imath} + (3x^2 - cz)\,\vec{\jmath} + (3xz^2 - y)\,\vec{k}$ **Ans:** $a = 6, b = 1, c = 1$

(ii) $\vec{A} = (axy - z^3)\vec{\imath} + (a - 2)x^2\,\vec{\jmath} + (1 - a)xz^2\,\vec{k}$ **Ans:** $a = 4$

10. Show that the following vectors are solenoidal.

(i) $\vec{a} = (x + 3y)\vec{\imath} + (y - 3z)\,\vec{\jmath} + (x - 2z)\,\vec{k}$

(ii) $\vec{a} = 5y^4z^3\vec{\imath} + 8xz^2\,\vec{\jmath} - y^2x\,\vec{k}$

2.3 VECTOR INTEGRATION

Line Integral

An integral which is evaluated along a curve then it is called line integral.

Let C be the curve in same region of space described by a vector valued function $\vec{r} = x\vec{\imath} + y\,\vec{\jmath} + z\,\vec{k}$ of a point (x, y, z) and let $\vec{F} = F_1\vec{\imath} + F_2\,\vec{\jmath} + F_3\,\vec{k}$ be a continuous vector valued function defined along a curve C. Then the line integral $\vec{F}$ over C is denoted by

$$\int_c \vec{F} \cdot d\vec{r}.$$

Work done by a Force

If $\vec{F}(x, y, z)$ is a force acting on a particle which moves along a given curve C, then

$\int_c \vec{F} \cdot d\vec{r}$ gives the total work done by the force $\vec{F}$ in the displacement along C.

Thus work done by force $\vec{F} = \int_c \vec{F} \cdot d\vec{r}$

Conservative force field

The line integral $\int_A^B \vec{F} \cdot d\vec{r}$ depends not only on the path C but also on the end points A and B.

If the integral depends only on the end points but not on the path C, then $\vec{F}$ is said to be conservative vector field.

If $\vec{F}$ is conservative force field, then it can be expressed as the gradient of some scalar function φ.

$(ie)\ \vec{F} = \nabla\varphi$

$\vec{F} = \nabla\varphi = \left(\vec{\imath}\,\frac{\partial\varphi}{\partial x} + \vec{\jmath}\,\frac{\partial\varphi}{\partial y} + \vec{k}\,\frac{\partial\varphi}{\partial z}\right)$

$\vec{F} \cdot d\vec{r} = \left(\vec{\imath}\,\frac{\partial\varphi}{\partial x} + \vec{\jmath}\,\frac{\partial\varphi}{\partial y} + \vec{k}\,\frac{\partial\varphi}{\partial z}\right) \cdot \left(dx\,\vec{\imath} + dy\,\vec{\jmath} + dz\,\vec{k}\right)$

$= \frac{\partial\varphi}{\partial x}\,dx + \frac{\partial\varphi}{\partial y}\,dy + \frac{\partial\varphi}{\partial z}\,dz = \partial\varphi$

$\int_c \vec{F} \cdot d\vec{r} = \int_A^B \partial\varphi$

$= [\varphi]_A^B$

$= \varphi[B] - \varphi[A]$

$\therefore$ work done by $\vec{F} = \varphi[B] - \varphi[A]$

Note:

If $\vec{F}$ is conservative, then $\nabla \times \vec{F} = \nabla \times (\nabla\varphi) = \vec{0}$ and hence $\vec{F}$ is irrotational.

Problems based on line integral

Example: 2.51 If $\vec{F} = 3xy\vec{\imath} - y^2\vec{\jmath}$, evaluate $\int_c \vec{F} \cdot d\vec{r}$ where c is the curve $y = 2x^2$ from $(0,0)$ to $(1,2)$.

Solution:

Given $\vec{F} = 3xy\vec{\imath} - y^2\vec{\jmath}$

$d\vec{r} = dx\vec{\imath} + dy\,\vec{\jmath}$

$\vec{F} \cdot d\vec{r} = 3xy\,dx - y^2dy$

Given C is $y = 2x^2$

$\therefore dy = 4xdx$

Along C, x varies from 0 to 1.

$\int_c \vec{F} \cdot d\vec{r} = \int_0^1 3x\,(2x^2)dx - 4x^4(4xdx)$

$= \int_0^1 6x^3 - 16x^5\,dx$

$$= \left[6\frac{x^4}{4} - 16\ \frac{x^6}{6}\right]$$

$$= \frac{6}{4} - \frac{16}{6} = -\frac{7}{6} \text{ units.}$$

Example: 2.52 Find the work done, when a force $\vec{F} = (x^2 - y^2 + x)\vec{\imath} - (2xy + y)\vec{\jmath}$ moves a particle from the origin to the point $(1, 1)$ along $y^2 = x$.

Solution:

Given $\vec{F} = (x^2 - y^2 + x)\vec{\imath} - (2xy + y)\vec{\jmath}$

$$d\vec{r} = dx\vec{\imath} + dy\vec{\jmath}$$

$\vec{F} \cdot d\vec{r} = (x^2 - y^2 + x)dx - (2xy + y)dy$

Given $y^2 = x \Rightarrow 2ydy = dx$

Along the curve C, y varies from 0 to 1.

$$\int_c \vec{F} \cdot d\vec{r} = \int_0^1 ((y^2)^2 - y^2 + y^2)\ 2ydy - (2(y^2)y + y)dy$$

$$= \int_0^1 (2y^5 - 2y^3 + 2y^3 - 2y^3 - y)\ dy$$

$$= \int_0^1 (2y^5 - 2y^3 - y)\ dy$$

$$= \left[2\frac{y^6}{6} - 2\frac{y^4}{4} - \frac{y^2}{2}\right]_0^1$$

$$= \frac{2}{6} - \frac{2}{4} - \frac{1}{2} = -\frac{2}{3}$$

Example: 2.53 Find the work done in moving a particle in the force field $\vec{F} = 3x^2\vec{\imath} + (2xz - y)\vec{\jmath} - z\vec{k}$ from $t = 0$ to $t = 1$ along the curve $x = 2t^2$, $y = t, z = 4t^3$.

Solution:

Given $\vec{F} = 3x^2\vec{\imath} + (2xz - y)\vec{\jmath} - z\vec{k}$

$$d\vec{r} = dx\vec{\imath} + dy\vec{\jmath} + dz\vec{k}$$

$\vec{F} \cdot d\vec{r} = 3x^2dx + (2xz - y)dy - zdz$

Given $x = 2t^2$, $\quad y = t, \quad z = 4t^3$

$$dx = 4tdt, \quad dy = dt, \quad dz = 12t^2dt$$

$$\int_c \vec{F} \cdot d\vec{r} = \int_0^1 48t^5dt + (16t^5 - t)dt - 48t^5\ dt$$

$$= \int_0^1 (16t^5 - t)dt$$

$$= \left[\frac{16\ t^6}{6} - \frac{t^2}{2}\right]_0^1 = \frac{16}{6} - \frac{1}{2} = \frac{13}{6}$$

Example: 2.54 If $\vec{F} = (3x^2 + 6y)\vec{\imath} + 14yz\vec{\jmath} + 20xz^2\vec{k}$, evaluate $\int_c \vec{F} \cdot d\vec{r}$ from $(0, 0, 0)$ to $(1, 1, 1)$ along the curve $x = t$, $y = t^2$, $z = t^3$.

Solution:

Given $\vec{F} = (3x^2 + 6y)\vec{i} + 14yz\vec{j} + 20xz^2\vec{k}$

$d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$

$\vec{F} \cdot d\vec{r} = (3x^2 + 6y)dx + 14yzdy + 20xz^2dz$

Given $x = t$, $y = t^2$, $z = t^3$

$dx = dt$, $d\,y = 2tdt$, $dz = 3t^2dt$

The point $(0, 0, 0)$ to $(1, 1,1)$ on the curve correspond to $t = 0$ and $t = 1$.

$$\int_c \vec{F} \cdot d\vec{r} = \int_0^1 (3t^2 + 6t^2)dt + 14t^5(2t\,dt) + 20t^7(3t^2)dt$$

$$= \int_0^1 (9t^2 + 28\,t^6 + 60t^9)\,dt$$

$$= \left[9\frac{t^3}{3} + 28\frac{t^7}{7} + 60\,\frac{t^9}{9}\right]_0^1$$

$$= \frac{9}{3} + \frac{28}{7} + \frac{60}{10} = 3 + 4 + 6 = 13\text{units.}$$

Example: 2.55 Find $\int_c \vec{F} \cdot d\vec{r}$ where $\vec{F} = (2y + 3)\vec{i} + xz\vec{j} + (yz - x)\vec{k}$ along the line joining the points $(0, 0, 0)$ to $(2, 1, 1)$.

Solution:

Given $\vec{F} = (2y + 3)\vec{i} + xz\vec{j} + (yz - x)\vec{k}$

$d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$

$\vec{F} \cdot d\vec{r} = (2y + 3)dx + xzdy + (yz - x)dz$

Equation of Straight line $\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$

The equation of the straight line joining $(0, 0, 0)$ to $(2, 1, 1)$.

$$\Rightarrow \frac{x - 0}{2 - 0} = \frac{y - 0}{1 - 0} = \frac{z - 0}{1 - 0}$$

$$\Rightarrow \frac{x}{2} = \frac{y}{1} = \frac{z}{1} = t \text{ (say)}$$

$x = 2t$, $y = t$, $z = t$

$dx = 2dt$, $d\,y = dt$, $dz = dt$

When t = 0 we get $(0, 0, 0)$

When t = 1 we get $(2, 1, 1)$

$\therefore t$ varies from 0 to 1.

$$\int_c \vec{F} \cdot d\vec{r} = \int_0^1 (2t + 3)2dt + (2t)tdt + (t^2 - 2t)dt$$

$$= \int_0^1 (4t + 6 + 2t^2 + t^2 - 2t)\,dt$$

$$= \int_0^1 (3t^2 + 2t + 6)\,dt$$

$$= \left[3\frac{t^3}{3} + 2\frac{t^2}{2} + 6t\right]_0^1$$

$$= \frac{3}{3} + \frac{2}{2} + 6 = 8 \text{ units}$$

Example: 2.56 Find the work done in moving a particle in the force field

$\vec{F} = 3x^2\vec{i} + (2xz - y)\vec{j} + z\vec{k}$ **along the straight line** $(0,0,0)$ **to** $(2,1,3)$**.**

Solution:

Given $\vec{F} = 3x^2\vec{i} + (2xz - y)\vec{j} + z\vec{k}$

$$d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$$

$$\vec{F} \cdot d\vec{r} = 3x^2 dx + (2xz - y)dy + z dz$$

Equation of Straight line $\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$

The equation of the line joining two points $(0,0,0)$ to $(2,1,3)$ is

$$\Rightarrow \frac{x - 0}{2 - 0} = \frac{y - 0}{1 - 0} = \frac{z - 0}{3 - 0}$$

$$\Rightarrow \frac{x}{2} = \frac{y}{1} = \frac{z}{3} = t \text{ (say)}$$

$$x = 2t, \quad y = t, \quad z = 3t$$

$$dx = 2dt, \quad dy = dt, \quad dz = 3dt$$

When $t = 0$ we get $(0,0,0)$

When $t = 1$ we get $(2,1,3)$

$$\int_c \vec{F} \cdot d\vec{r} = \int_0^1 3(4t^2)2dt + [2(2t)(3t) - t]dt + (3t)3dt$$

$$= \int_0^1 (24t^2 + 12t^2 - t + 9t)\, dt$$

$$= \int_0^1 (36t^2 + 8t)\, dt$$

$$= \left[36\frac{t^3}{3} + 8\frac{t^2}{2}\right]_0^1$$

$$= 12 + 4 = 16 \text{ units}$$

Example: 2.57 Find $\int_c \vec{F} \cdot d\vec{r}$ **where c is the circle** $x^2 + y^2 = 4$ **in the** xy **plane where**

$\vec{F} = (2xy + z^3)\vec{i} + x^2\vec{j} + 3xz^2\vec{k}$**.**

Solution:

Given $\vec{F} = (2xy + z^3)\vec{i} + x^2\vec{j} + 3xz^2\vec{k}$

In xy plane $z = 0 \Rightarrow dz = 0$

$$d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$$

$$\vec{F} \cdot d\vec{r} = 2xy dx + x^2 dy$$

Given C is $x^2 + y^2 = 4$

The parametric form of circle is

$$x = 2\cos\theta, \qquad y = 2\sin\theta$$

$$dx = -2\sin\theta d\theta, \quad dy = 2\cos\theta d\theta$$

And θ varies from 0 to 2π

$$\int_c \vec{F} \cdot d\vec{r} = \int_0^{2\pi} [2(2\cos\theta)(2\sin\theta)]\,(-2\sin\theta d\theta) + (2\cos\theta)^2 2\cos\theta\, d\theta$$

$$= \int_0^{2\pi} -16\cos\theta sin^2\theta + 8\cos^3\theta\, d\theta$$

$$= \int_0^{2\pi} -16\cos\theta(1 - cos^2\theta) + 8\cos^3\theta\, d\theta$$

$$= \int_0^{2\pi} -16\cos\theta + 16\, cos^3\theta + 8\cos^3\theta\, d\theta$$

$$= -16\int_0^{2\pi} \cos\theta\, d\theta + 24\int_0^{2\pi} \cos^3\theta\, d\theta$$

$$= -16\int_0^{2\pi} \cos\theta\, d\theta + 24\int_0^{2\pi} \frac{3\cos\theta + \cos 3\theta}{4}\, d\theta$$

$$= 16\,[\sin\theta]_0^{2\pi} + \frac{24}{4}\left[3\sin\theta + \frac{sin3\theta}{3}\right]_0^{2\pi}$$

$$= 0 \qquad [\because \sin n\pi = 0, sin0 = 0]$$

Example: 2.58 State the physical interpretation of the line integral $\int_A^B \vec{F} \cdot d\vec{r}$.

Solution:

Physically $\int_A^B \vec{F} \cdot d\vec{r}$ denotes the total work done by the force $\vec{F}$, displacing a particle from A to B along the curve C.

Example: 2.59 If $\vec{F} = (4xy - 3x^2z^2)\vec{i} + 2x^2\vec{j} - 2x^2z\vec{k}$, check whether the integral $\int_c \vec{F} \cdot d\vec{r}$ is independent of the path C.

Solution:

Given $\vec{F} = (4xy - 3x^2z^2)\vec{i} + 2x^2\vec{j} - 2x^2z\vec{k}$

$$d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$$

$$\vec{F} \cdot d\vec{r} = (4xy - 3x^2z^2)dx + 2x^2dy - 2x^2zdz$$

Then $\int_c \vec{F} \cdot d\vec{r}$ is independent of path C if $\nabla \times \vec{F} = 0$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 4xy - 3x^2z^2 & 2x^2 & -2x^3z \end{vmatrix}$$

$$= \vec{i}(0 - 0) - \vec{j}(-6x^2z + 6x^2z) + \vec{k}(4x - 4x)$$

$$= \vec{0}$$

Hence the line integral is independent of path.

Example: 2.60 Show that $\vec{F} = x^2\vec{i} + y^2\vec{j} + z^2\vec{k}$ is a conservative vector field.

Solution:

If $\vec{F}$ is conservative, then $\nabla \times \vec{F} = \vec{0}$.

$$\text{Now, } \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & y^2 & z^2 \end{vmatrix}$$

$$= \vec{i}(0-0) - \vec{j}(0-0) + \vec{k}(0-0)$$

$$= \vec{0}$$

$\therefore$ $\vec{F}$ is a conservative vector field.

Surface Integral

An integral which is evaluated over a surface is called a surface integral.

Consider a surface S. Let $\vec{F}$ be a vector valued function which is defined at each point on the surface and let P be any point on the surface and $\vec{n}$ be the unit outward normal to the surface at P. The normal component of $\vec{F}$ at P is $\vec{F} \cdot \vec{n}$.

The integral of the normal component of $\vec{F}$ is denoted by $\iint_S \vec{F} \cdot \vec{n}\, ds$ and is called the surface integral.

Evaluation of surface integral

Let R_1 be the projection of S on the $xy-$ plane, $\vec{k}$ is the unit vector normal to the $xy-$ plane then $ds = \frac{dx\,dy}{|\vec{n}\cdot\vec{k}|}$

$$\therefore \iint_S \vec{F} \cdot \vec{n}\, ds = \iint_{R1} \vec{F} \cdot \vec{n} \frac{dx\,dy}{|\vec{n}\cdot\vec{k}|}$$

If R_2 be the projection of s on $yz-$ plane

$$\therefore \iint_S \vec{F} \cdot \vec{n}\, ds = \iint_{R2} \vec{F} \cdot \vec{n} \frac{dx\,dy}{|\vec{n}\cdot\vec{i}|}$$

If R_3 be the projection of s on $xz-$ plane

$$\therefore \iint_S \vec{F} \cdot \vec{n}\, ds = \iint_{R3} \vec{F} \cdot \vec{n} \frac{dx\,dy}{|\vec{n}\cdot\vec{j}|}$$

Problems based on surface integral

Example: 2.61 Evaluate $\iint_S \vec{F} \cdot \vec{n}\, ds$ if $\vec{F} = (x+y^2)\vec{i} - 2x\vec{j} + 2yz\vec{k}$ and s is the surface of the plane $2x + y + 2z = 6$ in the first octant.

Solution:

Given $\vec{F} = (x+y^2)\vec{i} - 2x\vec{j} + 2yz\vec{k}$

Let $\varphi = 2x + y + 2z - 6$

Then $\nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$

$$= 2\vec{i} + 1\vec{j} + 2\vec{k}$$

$$|\nabla\varphi| = \sqrt{4+1+4} = \sqrt{9} = 3$$

$\hat{n} = \frac{\nabla\varphi}{|\nabla\varphi|} = \frac{2\vec{\imath}+ 1\vec{\jmath}+ 2\vec{k}}{3}$

$\vec{F} \cdot \hat{n} = \left[(x + y^2)\vec{\imath} - 2x\vec{\jmath} + 2yz\,\vec{k}\right] \cdot \left(\frac{2\vec{\imath}+ 1\vec{\jmath}+ 2\vec{k}}{3}\right)$

$= \frac{1}{3}\,[2(x + y^2) - 2x + 4yz]$

$= \frac{2}{3}\,[y^2 + 2yz]$

$= \frac{2}{3}\,y[y + 2z]$

$= \frac{2}{3}\,y[y + 6 - 2x - y]$ $\qquad\qquad [\because 2z = 6 - 2x - y]$

$= \frac{2}{3}\,y[6 - 2x]$

$= \frac{4}{3}\,y[3 - x]$

Let R be the projection of S on the $xy -$ plane

$$\therefore ds = \frac{dx\,dy}{\left|\hat{n} \cdot \vec{\mathrm{k}}\right|}$$

$\hat{n} \cdot \vec{\mathrm{k}} = \left(\frac{2\vec{\imath}+ 1\vec{\jmath}+ 2\vec{k}}{3}\right) \cdot \vec{\mathrm{k}} = \frac{2}{3}$

$\therefore \iint\limits_S \vec{F} \cdot \hat{n}\, ds = \iint\limits_R \vec{F} \cdot \hat{n} \frac{dx\,dy}{\left|\hat{n} \cdot \vec{\mathrm{k}}\right|}$

$= \iint\limits_R \frac{4}{3}\,y(3 - x)\,\frac{dx\,dy}{\left(\frac{2}{3}\right)}$

$= 2\int\int (3 - x)y\,dxdy$

In $R_1(2x + y = 6)$, x varies from 0 to $\frac{6-y}{2}$

y varies from 0 to 6

$= 2\int_0^6\int_0^{\frac{6-y}{2}} y\,(3 - x)\,dxdy$

$= 2\int_0^6 y\left[3x - \frac{x^2}{2}\right]_0^{\frac{6-y}{2}} dy$

$= 2\int_0^6 y\left[3\left(\frac{6-y}{2}\right) - \frac{1}{2}\left(\frac{6-y}{2}\right)^2\right] dy$

$= 2\int_0^6 \frac{1}{2}(18y - 3y^2) - \frac{1}{8}(6 - y)^2\,dy$

$= \frac{2}{2}\left[18\,\frac{y^2}{2} - \frac{3y^3}{3} - \frac{1}{8}\frac{(6-y)^3}{3(-1)}\right]$

$= \left[9(6)^2 - (6)^3 + \frac{1}{12}(0)\right] - \left[0 - 0 + \frac{1}{12}\,(6)^3\right]$

$= 81$ units

Example: 2.62 Show that $\iint\limits_S \left(yz\,\vec{\imath} + zx\,\vec{\jmath} + xy\,\vec{k}\right) \cdot \hat{n}\,ds = \frac{3}{8}$ **where s is the surface of the sphere** $x^2 + y^2 + z^2 = 1$ **in the first octant.**

Solution:

Given $\vec{F} = yz\,\vec{i} + zx\,\vec{j} + xy\,\vec{k}$

Let $\varphi = x^2 + y^2 + z^2 - 1$

$$\nabla\varphi = \vec{i}\,\frac{\partial\varphi}{\partial x} + \vec{j}\,\frac{\partial\varphi}{\partial y} + \vec{k}\,\frac{\partial\varphi}{\partial z}$$

$$= 2x\vec{i} + 2y\vec{j} + 2z\vec{k}$$

$$|\nabla\varphi| = \sqrt{4x^2 + 4y^2 + 4z^2} = 2(1)$$

$\therefore$ The unit outward normal is $\hat{n} = \frac{\nabla\varphi}{|\nabla\varphi|} = \frac{2(x\vec{i} + y\vec{j} + z\vec{k})}{2}$

$\vec{F}\cdot\hat{n} = \left[yz\vec{i} + zx\vec{j} + xy\,\vec{k}\right]\cdot\left(x\vec{i} + y\vec{j} + z\vec{k}\right)$

$= 3xyz$

Let R be the projection of S on xy −plane

$$\therefore ds = \frac{dx\,dy}{\left|\hat{n}\cdot\vec{\mathrm{k}}\right|}$$

$\left|\hat{n}\cdot\vec{\mathrm{k}}\right| = \left(x\vec{i} + y\vec{j} + z\vec{k}\right)\cdot\vec{k} \quad = z$

$$\therefore \iint_S \vec{F}\cdot\hat{n}\,ds = \iint_R \vec{F}\cdot\hat{n}\,\frac{dx\,dy}{\left|\hat{n}\cdot\vec{\mathrm{k}}\right|}$$

$$= \int\int 3xyz\,\frac{dxdy}{z}$$

$$= \int\int 3xy\,\,dxdy$$

In $R_1(x^2 + y^2 = 1)$, x varies from 0 to $\sqrt{1 - y^2}$

y varies from 0 to 1

$$= \int_0^1\int_0^{\sqrt{1-y^2}} 3xy\,dxdy$$

$$= 3\int_0^6 \left[y\,\frac{x^2}{2}\right]_0^{\sqrt{1-y^2}} dy$$

$$= \frac{3}{2}\int_0^1 y(1 - y^2)dy$$

$$= \frac{3}{2}\int_0^1 y - y^3 dy$$

$$= \frac{3}{2}\left[\frac{y^2}{2} - \frac{y^4}{4}\right]_0^1$$

$$= \frac{3}{2}\left(\frac{1}{2} - \frac{1}{4}\right) = \frac{3}{8}$$

Volume integral

An integral which is evaluated over a volume bounded by a surface is called a volume integral.

If $\vec{F} = F_1\vec{i} + F_2\,\vec{j} + F_3\,\vec{k}$ is a vector field in V, then the volume integral is defined by

$$\iiint_V \vec{F}\,\,dv$$

Problems based on volume integral

Example: 2.63 If $\vec{F} = (2x^2 - 3z)\vec{i} - 2xy\vec{j} - 4x\vec{k}$, evaluate $\iiint_V \nabla \times \vec{F}\, dv$ where v is the volume of the region bounded by $x = 0, y = 0, z = 0$ and $2x + 2y + z = 4$.

Solution:

Given $\vec{F} = (2x^2 - 3z)\vec{i} - 2xy\vec{j} - 4x\vec{k}$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x^2 - 3z & -2xy & -4x \end{vmatrix}$$

$$= \vec{i}(0 - 0) - \vec{j}(-4 + 3) + \vec{k}(-2y - 0)$$

$$= \vec{j} - 2y\vec{k}$$

For limits

Given $x = 0, y = 0, z = 0$ and $2x + 2y + z = 4$

$\therefore z : 0 \to 4 - 2x - 2y$

Put $z = 0 \Rightarrow 2x + 2y = 4$ (or) $x + y = 4$

$\therefore y : 0 \to 2 - x$

Put $z = 0, y = 0 \Rightarrow 2x = 4$ (or) $x = 2$

$\therefore x : 0 \to 2$

$$\therefore \int\int\int \nabla \times \vec{F}\, dv = \int_0^2 \int_0^{2-x} \int_0^{4-2x-2y} (\vec{j} - 2y\vec{k}) dz dy dx$$

$$= \int_0^2 \int_0^{2-x} (\vec{j} - 2y\vec{k})[z]_0^{4-2x-2y} dy dx$$

$$= \int_0^2 \int_0^{2-x} \left[(4 - 2x - 2y)\vec{j} - 2y(4 - 2x - 2y)\vec{k}\right] dy dx$$

$$= \int_0^2 \left\{\left[4y - 2xy - \frac{2y^2}{2}\right]\vec{j} - \left[4y^2 - 2xy^2 - \frac{4y^3}{3}\right]\vec{k}\right\}_0^{2-x} dx$$

$$= \int_0^2 \{[4(2 - x) - 2x(2 - x) - (2 - x)^2]\,\vec{j} - \left[4(2 - x)^2 - 2x(2 - x)^2 - \frac{4}{3}(2 - x)^3\vec{k}\right]\} dx$$

$$= \int_0^2 [8 - 4x - 4x + 2x^2 - 4 + 4x - x^2]\,\vec{j} - \left[16 - 16x + 4x^2 - 8x + 8x^2 - 2x^3 - \frac{4}{3}(8 - 12x + 6x^2 - x^3)\,\vec{k}\right] dx$$

$$= \int_0^2 \left[(4 - 4x + x^2)\,\vec{j} - \frac{\vec{k}}{3}(16 - 24x + 12x^2 - 2x^3)\right] dx$$

$$= \left[4x - 2x^2 + \frac{x^3}{3}\right]_0^2 \vec{j} + \frac{\vec{k}}{3}\left[16x - 12x^2 + 4x^3 - \frac{x^4}{2}\right]_0^2$$

$$= \left(8 - 8 + \frac{8}{3}\right)\vec{j} - \frac{\vec{k}}{3}(32 - 48 + 32 - 8)$$

$$= \frac{8}{3}(\vec{j} - \vec{k})$$

Exercise: 2.3

1. If $\vec{F} = x^2\vec{\imath} + xy^2\,\vec{\jmath}$, evaluate the line integral $\int_c \vec{F} \cdot d\vec{r}$ from $(0,0)$ to $(1,1)$along the

   path $y = x$. **Ans:** $\frac{1}{2}$

2. Evaluate $\int_c \vec{F} \cdot d\vec{r}$,where $\vec{F} = x^2y^2\vec{\imath} + y\,\vec{\jmath}$and C is $y^2 = 4x$ in the XY plane from

   $(0,0)$ to $(4,4)$. **Ans:** 264

3. If $\vec{F} = xy\vec{\imath} + (x^2 + y^2)\,\vec{\jmath}$, then find $\int_c \vec{F} \cdot d\vec{r}$, where C is the arc of the parabola

   $y = x^2 - 4$ from $(2,0)$ to $(4,12)$ **Ans:** 732

4. If $\vec{F} = xy\vec{\imath} + z\,\vec{\jmath} - xyz\,\vec{k}$, then evaluate $\int_c \vec{F} \cdot d\vec{r}$, from the point $(0,0,0)$ to $(1,1,1)$ where C is the

   curve $x = t, y = t^2, z = t$ **Ans:** $\frac{67}{60}$

5. Find the work done in moving a particle in the field

   $\vec{F} = 3x^2\vec{\imath} + (2xz - y)\,\vec{\jmath} + (x^2 + y^2)\,\vec{k}$ along the straight line from $(0,0,0)$ to $(2,1,3)$.

   Ans: 16

6. Evaluate the line integral $\int_C (x^2 + xy)dx + (x^2 + y^2)dy$, where C is the square formed

   by the lines $x = \pm 1$ and $y = \pm 1$. **Ans:** 0

7. Find the total work done in moving a particle by a force field $\vec{F} = yz\vec{\imath} + xz\,\vec{\jmath} + xy\,\vec{k}$

   along the curve $x = t, y = t^2, z = t^3$ from $(0,0,0)$ to $(2,4,8)$ **Ans:** 64

8. Evaluate $\iint_S \vec{F} \cdot \hat{n}\, ds$ where $\vec{F} = 18z\vec{\imath} - 12\,\vec{\jmath} + 3y\,\vec{k}$ and S is the part of the plane

   $2x + 3y + 6z = 12$ which is in the first order. **Ans:**24

9. Evaluate $\iint_S \vec{F} \cdot \hat{n}\, ds$ where $\vec{F} = (x + y^2)\vec{\imath} - 2x\,\vec{\jmath} + 2yz\,\vec{k}$ and S is the surface of the

   plane $2x + y + 2z = 6$ which is in the first order. **Ans:**24

10. Evaluate $\iiint_V \nabla \times \vec{F}\, dv$ where $\vec{F} = (2x^2 - 3z)\vec{\imath} - 2y\,\vec{\jmath} - 4xz\,\vec{k}$ and V is bounded

    by the planes $x = 0, y = 0, z = 0$ and $2x + 2y + z = 4$ **Ans:** $\frac{8}{3}$

2.4 Green's Theorem

Green's theorem relates a line integral to the double integral taken over the region bounded by the closed curve.

Statement

If $M(x,y)$ and $N(x,y)$ are continuous functions with continuous, partial derivatives in a region R of the xy – plane bounded by a simple closed curve C, then

$\oint_c Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy$,where C is the curve described in the positive direction.

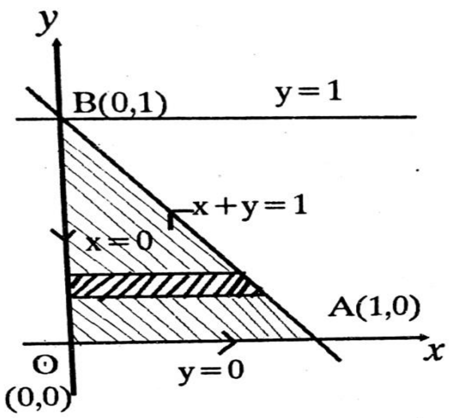
Vector form of Green's theorem

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_R (\nabla \times \vec{F}) \cdot \vec{k}\, dR$$

Problems based on Green's theorem

Example: 2.64 Verify Green's theorem in the plane for $\int_C (3x^2 - 8y^2)dx + (4y - 6xy)dy$ where C is the boundary of the region defined by $x = 0, y = 0, x + y = 1$.

Solution:



We have to prove that $\int_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$

Here, $M = 3x^2 - 8y^2$ and $N = 4y - 6xy$

$$\Rightarrow \frac{\partial M}{\partial y} = -16y \qquad \Rightarrow \frac{\partial N}{\partial x} = -6y$$

$$\therefore \int_C (3x^2 - 8y^2)dx + (4y - 6xy)dy = \int_C M\,dx + N\,dy$$

By Green's theorem in the plane,

$$\int_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

$$= \int_0^1 \int_0^{1-x} (10y)\,dy\,dx$$

$$= 10 \int_0^1 \left[\frac{y^2}{2}\right]_0^{1-x} dx$$

$$= 5 \int_0^1 (1-x)^2\,dx$$

$$= 5 \left[\frac{(1-x)^3}{-3}\right]_0^1 = \frac{5}{3} \dots (1)$$

Consider $\int M\,dx + N\,dy = \int_{OA} + \int_{AB} + \int_{BO}$

Along $OA, y = 0 \Rightarrow dy = 0, x$ varies from 0 to 1

$$\therefore \int_{OA} M\,dx + N\,dy = \int_0^1 3x^2 dx = [x^3]_0^1 = 1$$

Along $AB, y = 1 - x \Rightarrow dy = -dx$ and x varies from 1 to 0

$$\therefore \int_{AB} M\,dx + N\,dy = \int_1^0 [3x^2 - 8(1-x)^2 - 4(1-x) + 6x(1-x)]dx$$

$$= \left[\frac{3x^3}{3} - \frac{8(1-x)^3}{-3} - \frac{4(1-x)^2}{-2} + 3x^2 - 2x^3\right]_1^0$$

$$= \frac{8}{3} + 2 - 1 - 3 + 2 = \frac{8}{3}$$

Along $BO, x = 0 \Rightarrow dx = 0$ and y varies from 1 to 0

$$\therefore \int_{BO} M\,dx + N\,dy = \int_1^0 4y\,dy = [2y^2]_1^0 = -2$$

$$\therefore \int_c M\,dx + N\,dy = 1 + \frac{8}{3} - 2 = \frac{5}{3} \quad \dots(2)$$

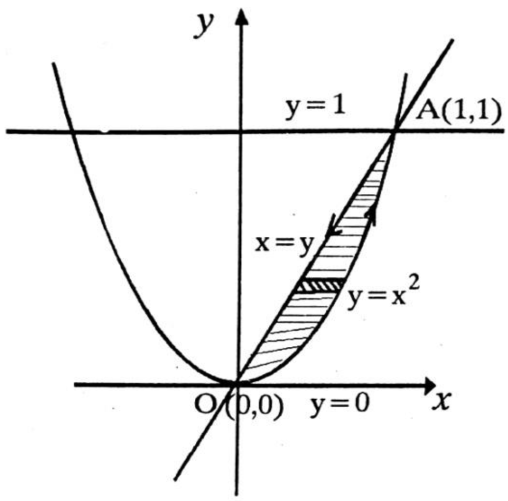
$\therefore$ From (1) and (2)

$$\therefore \int_c M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

Hence Green's theorem is verified.

Example: 2.65 Verify Green's theorem in the XY −plane for $\int_c (xy + y^2)dx + x^2 dy$ where C is the closed curve of the region bounded by $y = x, y = x^2$.

Solution:



We have to prove that $\int_c M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$

Here, $M = xy + y^2$ and $N = x^2$

$$\Rightarrow \frac{\partial M}{\partial y} = x + 2y \qquad \Rightarrow \frac{\partial N}{\partial x} = 2x$$

R.H.S = $\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$

Limits:

x varies from y to $\sqrt{y}$

y varies from 0 to 1

$$\therefore \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy = \int_0^1 \int_y^{\sqrt{y}} 2x - (x + 2y)\, dx\, dy$$

$$= \int_0^1 \left[\frac{x^2}{2} - 2xy\right]_y^{\sqrt{y}} dy$$

$$= \int_0^1 \left(\frac{y}{2} - 2y\sqrt{y}\right) - \left(\frac{y^2}{2} - 2y^2\right) dy$$

$$= \int_0^1 \left(\frac{y}{2} - 2y^{\frac{3}{2}} + 3\frac{y^2}{2}\right) dy$$

$$= \left[\frac{y^2}{2} - \frac{4y^{\frac{5}{2}}}{5} + \frac{y^3}{2}\right]_0^1$$

$$= \frac{1}{4} - \frac{4}{5} + \frac{1}{2} = -\frac{1}{20}$$

L.H.S = $\int_c M\, dx + N\, dy$

Consider $\int M\, dx + N\, dy = \int_{OA} + \int_{AO}$

Along $OA, y = x^2 \Rightarrow dy = 2x\, dx, x$ varies from 0 to 1

$$\therefore \int_{OA} M\, dx + N\, dy = \int_0^1 [(x(x^2) + (x^2)^2)dx + x^2 \cdot 2x\, dx]$$

$$= \int_0^1 (3x^3 + x^4)\, dx$$

$$= \left[\frac{3x^4}{4} + \frac{x^5}{5}\right]_0^1$$

$$= \frac{3}{4} + \frac{1}{5} = \frac{19}{20}$$

Along $AO, y = x \Rightarrow dy = dx\ and\ x$ varies from 1 to 0

$$\therefore \int_{A0} M\, dx + N\, dy = \int_1^0 (x^2 + x^2)dx + x^2\, dx$$

$$= \int_1^0 3x^2 dx = [x^3]_1^0 = -1$$

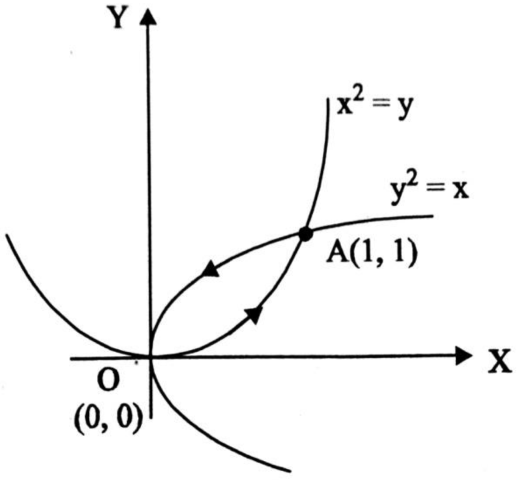
L.H.S = $\int_c M\, dx + N\, dy = \frac{19}{20} - 1 = -\frac{1}{20}$

$\therefore$ L.H.S = R.H.S

Hence Green's theorem is verified.

Example: 2.66 Verify Green's theorem in the plane for $\int_c (3x^2 - 8y^2)dx + (4y - 6xy)dy$ where C is the boundary of the region defined by $y = x^2, x = y^2$.

Solution:



We have to prove that $\int_c M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$

Here, $M = 3x^2 - 8y^2$ and $N = 4y - 6xy$

$\Rightarrow \frac{\partial M}{\partial y} = -16y \qquad \Rightarrow \frac{\partial N}{\partial x} = -6y$

R.H.S = $\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$

Limits:

x varies from y^2 to $\sqrt{y}$

y varies from 0 to 1

$$\therefore \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \int_0^1 \int_{y^2}^{\sqrt{y}} (-6y + 16y)\,dx\,dy$$

$$= \int_0^1 [10xy]_{y^2}^{\sqrt{y}} dy$$

$$= 10\int_0^1 \left(y\sqrt{y} - y^3\right) dy$$

$$= 10\left[\frac{y^{\frac{5}{2}}}{\frac{5}{2}} - \frac{y^4}{4}\right]_0^1$$

$$= 10\left(\frac{2}{5} - \frac{1}{4}\right) = \frac{3}{2}$$

L.H.S = $\int_c M\,dx + N\,dy$

Consider $\int M\,dx + N\,dy = \int_{OA} + \int_{AO}$

Along $OA, y = x^2 \Rightarrow dy = 2x\,dx, x$ varies from 0 to 1

$$\therefore \int_{OA} M\,dx + N\,dy = \int_0^1 (3x^2 - 8x^4)\,dx + (4x^2 - 6x^3)(2x)dx$$

$$= \int_0^1 (3x^2 - 8x^4 + 8x^3 - 12x^4)dx$$

$$= \int_0^1 (-20x^4 + 8x^3 + 3x^2)\,dx$$

$$= \left[-20\frac{x^5}{5} + 8\,\frac{x^4}{4} + 3\,\frac{x^3}{3}\right]_0^1$$

$$= -4 + 2 + 1 = -1$$

Along $AO, x = y^2 \Rightarrow dx = 2ydy \; and \; y$ varies from 1 to 0

$$\therefore \int_{Ao} M\,dx + N\,dy = \int_1^0 (3y^4 - 8y^2)2ydy + (4y - 6yy^2)\,dy$$

$$= \int_1^0 (6y^5 - 16y^3 + 4y - 6y^3)dx$$

$$= \int_1^0 (6y^5 - 22y^3 + 4y)dx$$

$$= \left[6\frac{y^6}{6} - 22\frac{y^4}{4} + 4\frac{y^2}{2}\right]_1^0$$

$$= 0 - \left[1 - \frac{11}{2} + 2\right]$$

$$= -\left(3 - \frac{11}{2}\right) = \frac{5}{2}$$

L.H.S = $\int_c M\,dx + N\,dy = -1 + \frac{5}{2} = \frac{3}{2}$

$\therefore$ L.H.S = R.H.S

Hence Green's theorem is verified.

Example: 2.67 Verify Green's theorem in the plane for the integral $\int_c (x - 2y)dx + xdy$ taken around the circle $x^2 + y^2 = 1$.

Solution:

We have to prove that $\int_c M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$

Here, $M = x - 2y \; and \; N = x$

$$\Rightarrow \frac{\partial M}{\partial y} = -2 \qquad \Rightarrow \frac{\partial N}{\partial x} = 1$$

R.H.S = $\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$

$$\therefore \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \iint_R (1 + 2)dxdy$$

$$= 3\iint_R dxdy$$

$= 3$ (Area of the circle)

$$= 3\pi r^2$$

$$= 3\pi \qquad (\because radius = 1)$$

L.H.S = $\int_c M\,dx + N\,dy$

Given C is $x^2 + y^2 = 1$

The parametric equation of circle is

$x = \cos\theta,\ y = \sin\theta$

$dx = -\sin\theta d\theta,\ dy = \cos\theta\, d\theta$

Where θ varies from 0 to 2π

$$\therefore \int_c M\,dx + N\,dy = \int_0^{2\pi} (\cos\theta - 2\sin\theta)\,(-\sin\theta\, d\theta) + \cos\theta\,(\cos\theta\, d\theta)$$

$$= \int_0^{2\pi} (-\sin\theta\cos\theta + 2\sin^2\theta + \cos^2\theta)\, d\theta$$

$$= \int_0^{2\pi} (-\sin\theta\cos\theta + \sin^2\theta + 1)\, d\theta \quad (\because \sin^2\theta + \cos^2\theta = 1)$$

$$= \int_0^{2\pi} \left(-\frac{\sin 2\theta}{2} + \frac{1-\cos 2\theta}{2} + 1\right) d\theta$$

$$= \left[-\frac{1}{2}\left(-\frac{\cos 2\theta}{2}\right) + \frac{\theta}{2} - \frac{1}{2}\left(\frac{\sin 2\theta}{2}\right) + \theta\right]_0^{2\pi}$$

$$= \left[\frac{\cos(4\pi)}{4} + \frac{2\pi}{2} - \frac{\sin 4\pi}{4} + 2\pi\right] - \left[\frac{\cos 0}{4} + \frac{0}{2} - \frac{\sin 0}{4} + 0\right]$$

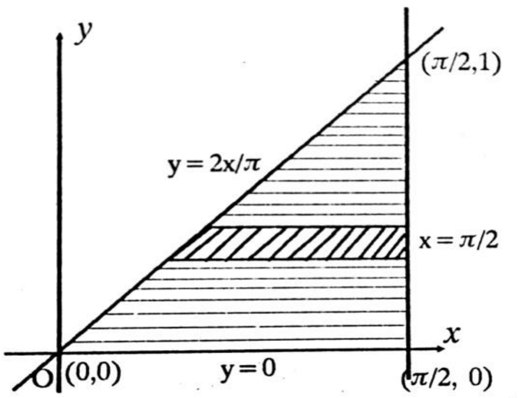
$$= \frac{1}{4} + \pi + 2\pi - \frac{1}{4} = 3\pi \quad [\because \sin n\pi = 0, \sin 0 = 0, \cos 0 = 1], [\cos n\pi = (-1)^n]$$

$\therefore$ L.H.S = R.H.S

Hence Green's theorem is verified.

Example: 2.68 Using Green's theorem evaluate $\int_c (y - \sin x)dx + \cos x\, dy$ where C is the triangle bounded by $y = 0, x = \frac{\pi}{2}, y = \frac{2x}{\pi}$.

Solution:



We have to prove that $\int_c M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$

Here, $M = y - \sin x$ *and* $N = \cos x$

$\Rightarrow \frac{\partial M}{\partial y} = 1 - 0 \qquad \Rightarrow \frac{\partial N}{\partial x} = -\sin x$

Limits:

x varies from $\frac{y\pi}{2}$ to $\frac{\pi}{2}$

y varies from 0 to 1

Hence $\int_c (y - \sin x)dx + \cos x\, dy = \int_0^1 \int_{\frac{y\pi}{2}}^{\frac{\pi}{2}} (-\sin x - 1)\, dx\, dy$

$$= \int_0^1 (\cos x - x)_{\frac{y\pi}{2}}^{\frac{\pi}{2}}\, dy$$

$$= \int_0^1 \left[\left(\cos\frac{\pi}{2} - \frac{\pi}{2}\right) - \left(\cos\left(\frac{y\pi}{2}\right) - \frac{y\pi}{2}\right)\right] dy$$

$$= \int_0^1 \left[0 - \frac{\pi}{2} - \cos\frac{y\pi}{2} + \frac{y\pi}{2}\right] dy$$

$$= \left[-\frac{\pi}{2}y - \frac{\sin\frac{y\pi}{2}}{\frac{\pi}{2}} + \frac{\pi}{2}\frac{y^2}{2}\right]_0^1$$

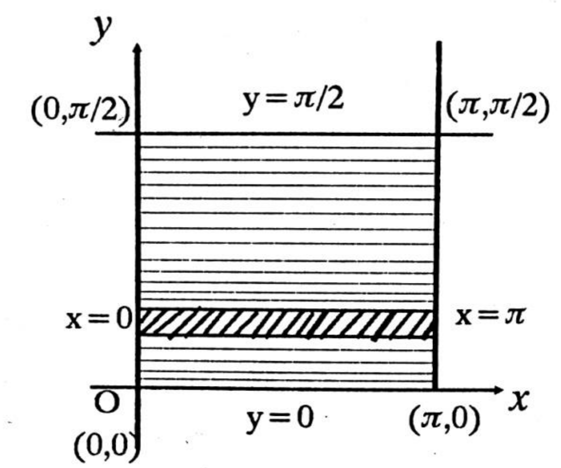
$$= -\frac{\pi}{2} - \frac{2}{\pi}\sin\left(\frac{\pi}{2}\right) + \frac{\pi}{4}$$

$$= -\frac{\pi}{2} - \frac{2}{\pi} + \frac{\pi}{4}$$

$$= -\frac{\pi}{4} - \frac{2}{\pi} = -\left[\frac{\pi}{4} + \frac{2}{\pi}\right]$$

Example: 2.69 Evaluate by Green's theorem $\int_c [e^{-x}(\sin y\, dx + \cos y\, dy)]$ where C being the rectangle with vertices $(0,0), (\pi,0), \left(\pi,\frac{\pi}{2}\right)$ and $\left(0,\frac{\pi}{2}\right)$.

Solution:



We have to prove that $\int_c M\, dx + N\, dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy$

Here, $M = e^{-x}\sin y$ and $N = e^{-x}\cos y$

$\Rightarrow \frac{\partial M}{\partial y} = e^{-x}\cos y$ $\qquad \Rightarrow \frac{\partial N}{\partial x} = -e^{-x}\cos y$

Limits:

x varies from 0 to π

y varies from 0 to $\frac{\pi}{2}$

$\therefore \int_c [e^{-x}(\sin y\, dx + \cos y\, dy)] = \int_0^{\frac{\pi}{2}} \int_0^{\pi} (-e^{-x}\cos y - e^{-x}\cos y)\, dxdy$

$$= \int_0^{\frac{\pi}{2}} \int_0^{\pi} -2\, e^{-x} \cos y \; dx\, dy$$

$$= -2 \int_0^{\frac{\pi}{2}} \left[\frac{e^{-x} \cos y}{-1}\right]_0^{\pi} dy$$

$$= 2 \int_0^{\frac{\pi}{2}} [e^{-\pi} \cos y - e^{0} \cos y] dy$$

$$= 2 \int_0^{\frac{\pi}{2}} [e^{-\pi} \cos y - \cos y] dy$$

$$= 2\, [e^{-\pi} \sin y - \sin y]_0^{\frac{\pi}{2}}$$

$$= 2 \left[\left(e^{-\pi} \sin\frac{\pi}{2} - \sin\frac{\pi}{2}\right) - (e^{-\pi} \sin 0 - \sin 0)\right]$$

$$= 2\, [e^{-\pi} - 1]$$

Example: 2.70 Prove that the area bounded by a simple closed curve C is given by $\frac{1}{2} \int_c (xdy - ydx)$. Hence find the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ by using Green's theorem.

Solution:

By Green theorem, $\int_c M\, dx + N\, dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy$

Let $M = -y \; and \; N = x$

$$\Rightarrow \frac{\partial M}{\partial y} = -1 \qquad \Rightarrow \frac{\partial N}{\partial x} = 1$$

$$\therefore \int_c (xdy - ydx) = \iint_R (1+1)\, dx\, dy$$

$$= 2 \iint_R dx\, dy = 2 \quad \text{(Area enclosed by C)}$$

$$\therefore \text{Area enclosed by } C = \frac{1}{2} \int_c (xdy - ydx)$$

Equation of ellipse in parametric form is $x = a \cos\theta \; and \; y = b \sin\theta$ where $0 \le \theta \le 2\pi$.

$$\therefore \text{Area of the ellipse } = \frac{1}{2} \int_0^{2\pi} (a\cos\theta)(b\cos\theta) - (b\sin\theta)(-a\sin\theta)\; d\theta$$

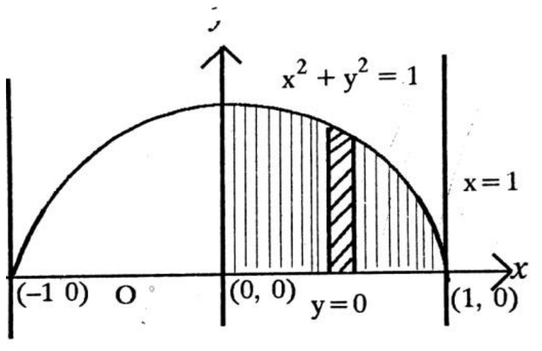
$$= \frac{1}{2}\, ab \int_0^{2\pi} (cos^2\theta + sin^2\theta)\, d\theta$$

$$= \frac{1}{2}\, ab \int_0^{2\pi} d\theta = \frac{1}{2}\, ab\, [\theta]_0^{2\pi} = \pi ab$$

Example: 2.71 Evaluate the integral using Green's theorem

$\int_c (2x^2 - y^2)dx + (x^2 + y^2)dy$ where C is the boundary in the xy – plane of the area enclosed by the x – axis and the semicircle $x^2 + y^2 = a^2$ in the upper half xy – plane.

Solution:



In this figure $'a'$ is represented as 1

By Green theorem, $\int_c M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$

Let $M = 2x^2 - y^2$ and $N = x^2 + y^2$

$$\Rightarrow \frac{\partial M}{\partial y} = -2y \qquad \Rightarrow \frac{\partial N}{\partial x} = 2x$$

Limits:

y varies from 0 to $\sqrt{a^2 - x^2}$

x varies from $-a$ to a

$$\therefore \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dy\,dx = \int_{-a}^{a}\int_0^{\sqrt{a^2-x^2}} (2x + 2y)\,dy\,dx$$

$$= 2\int_{-a}^{a}\left[xy + \frac{y^2}{2}\right]_0^{\sqrt{a^2-x^2}} dx$$

$$= 2\int_{-a}^{a}\left[x\sqrt{a^2 - x^2} + \frac{a^2 - x^2}{2}\right] dx$$

In the first integral, the function is odd function.

$\therefore$ The value is zero.

$\therefore$ we get $2\int_{-a}^{a}\frac{a^2 - x^2}{2}\,dx$

$$= \left[a^2x - \frac{x^3}{3}\right]_{-a}^{a}$$

$$= \left(a^3 - \frac{a^3}{3}\right) - \left(-a^3 + \frac{a^3}{3}\right)$$

$$= \frac{4a^3}{3}$$

Exercise: 2.4

1. Using Green's theorem in the plane, evaluate $\int_c (x^2 - y^2)dx + 2xydy$ where C is the closed curve of the region bounded by $y = x^2 and\ y^2 = x$ **Ans:** $\frac{3}{5}$

2. Find by Green's theorem the value of $\int_c (x^2ydx + ydy)$ along the closed curve formed by $y = x^2 and\ y^2 = x$ between $(0,0)\ to\ (1,1)$ **Ans:** $\frac{1}{28}$

3. Verify Green's theorem for the integral $\int_c [(x-y)dx+(x+y)dy]$ taken around the boundary area in the first quadrant between the curves $y = x^2 and\ y^2 = x$.

Ans: Common value $= \frac{2}{3}$

4. Find the area of a circle of radius 'a' using Green's theorem. **Ans:** πa^2

5. Evaluate $\int_c [(sinx - y)dx - \cos x\, dy]$, where C is the triangle with vertices $(0,0), \left(\frac{\pi}{2}, 0\right)\ and\ \left(\frac{\pi}{2}, 1\right)$ **Ans:** $\frac{2}{\pi} + \frac{\pi}{4}$

6. Using Green's theorem, find the value of $\int_c [(xy - x^2)dx + x^2 y dy]$ along the closed curve C formed by $y = 0, x = 1\ and\ y = x$ **Ans:** $-\frac{1}{12}$

7. Verify Green's theorem for $\int_c [(x^2 - y^2)dx + 2xydy]$, where C is the boundary of the rectangle in the xoy – plane bounded by the lines $x = 0, x = a, y = 0\ and\ y = b$.

Ans: Common value $= 2ab^2$

8. Verify Green's theorem for $\int_c [(2x - y)dx + (x + y)dy]$, where C is the boundary of the Circle $x^2 + y^2 = a^2$ in the xoy – plane. **Ans:** $2\pi a^2$

2.5 STOKE'S THEOREM

Statement of Stoke's theorem

If S is an open surface bounded by a simple closed curve C if $\vec{F}$ is continuous having continuous partial derivatives in S and C, then

$$\int_c \vec{F} \cdot d\vec{r} = \iint_S curl\,\vec{F} \cdot \hat{n}\, ds$$

(or)

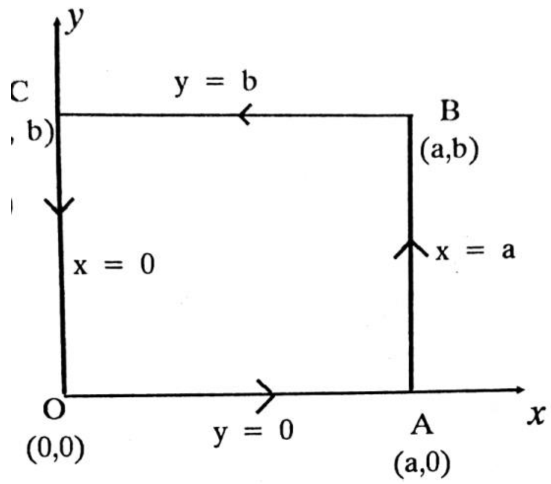
$$\int_c \vec{F} \cdot d\vec{r} = \iint_S \nabla \times \vec{F} \cdot \hat{n}\, ds$$

$\hat{n}$ is the outward unit normal vector and C is traversed in the anti – clockwise direction.

Problems based on Stoke's theorem

Example: 2.72 Verify stokes theorem for a vector field defined by $\vec{F} = (x^2 - y^2)\vec{i} + 2xy\vec{j}$ in a rectangular region in the xoy plane bounded by the lines $x = 0, x = a, y = 0, y = b$.

Solution:



By Stokes theorem, $\int_c \vec{F} \cdot d\vec{r} = \iint_S Curl\,\vec{F} \cdot \hat{n}\, dS$

To evaluate: $\iint_S Curl\,\vec{F} \cdot \hat{n}\, dS$

Given $\vec{F} = (x^2 - y^2)\vec{i} + 2\,x\,y\vec{j}$

Curl $\vec{F} = \nabla \times \vec{F}$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 - y^2 & 2xy & 0 \end{vmatrix}$$

$= \vec{i}\,(0) - \vec{j}\,(0 - 0) + \vec{k}[\,2y - (0 - 2y)]$

$= 4y\,\vec{k}$

Since the surface is a rectangle in the xy plane, $\hat{n} = \vec{k}$, $dS = dxdy$

Curl $\vec{F} \cdot \hat{n} = 4y\ \vec{k} \cdot \vec{k} = 4y$

Order of integration is $dxdy$

x varies from $x = 0$ to $x = a$

y varies from $y = 0$ to $y = b$

$$\Rightarrow \iint_S Curl\,\vec{F} \cdot \hat{n}\, dS = \int_0^b \int_0^a 4y\, dx\, dy$$

$$= \int_0^b 4y\,[x]_0^a dy$$

$$= \int_0^b 4ay dy$$

$$= \left[\frac{4ay^2}{2}\right]_0^b$$

$$= 2ab^2$$

$$\Rightarrow \iint_S Curl\,\vec{F} \cdot \hat{n}\, dS = 2ab^2 \qquad \dots (1)$$

Here the line integral over the simple closed curve C bounding the surface $OABCO$ consisting of the edges OA, AB ,BC and CO.

Curve	Equation	Limit
OA	$y = 0$	$x = 0$ to $x = a$
AB	$x = a$	$y = 0$ to $y = b$
BC	$y = b$	$x = a$ to $x = 0$
CO	$x = 0$	$y = b$ to $y = 0$

Therefore, $\int_{c} \vec{F} \cdot d\vec{r} = \int_{OABCO} \vec{F} \cdot d\vec{r}$

$$\int_{c} \vec{F} \cdot d\vec{r} = \int_{OA} + \int_{AB} + \int_{BC} + \int_{CO}$$

$$\vec{F} \cdot d\vec{r} = (x^2 - y^2) + 2xy\mathrm{d}y \qquad \dots (2)$$

On OA: $y = 0, dy = 0, x$ varies from $0\ to\ a$

(2) $\Rightarrow \vec{F} . \mathrm{d}\vec{r} = x^2 dx$

$$\int_{OA} \vec{F} \cdot d\vec{r} = \int_0^a x^2\, dx$$

$$= \left[\frac{x^3}{3}\right]_0^a = \frac{a^3}{3}$$

On AB: $x = a, dx = 0, y$ varies from $0\ to\ b$

(2) $\Rightarrow \vec{F} . \mathrm{d}\vec{r} = 2ay\, dy$

$$\int_{AB} \vec{F} \cdot d\vec{r} = \int_0^b 2ay\, dy$$

$$= \left[\frac{2ay^2}{2}\right]_0^b = ab^2$$

On BC: $y = b, dy = 0, x$ varies from $a\ to\ 0$

(2) $\Rightarrow \vec{F} . \mathrm{d}\vec{r} = (x^2 - b^2)dx$

$$\int_{BC} \vec{F} \cdot d\vec{r} = \int_a^0 x^2 - b^2\, dx$$

$$= \left[\frac{x^3}{3} - b^2 x\right]_a^0$$

$$= -\frac{a^3}{3} + a\, b^2$$

On CO: $x = 0, dx = 0, y$ varies from $b\ to\ 0$

(2) $\Rightarrow \vec{F} . \mathrm{d}\vec{r} = 0$

$$\int_{CO} \vec{F} \cdot d\vec{r} = 0$$

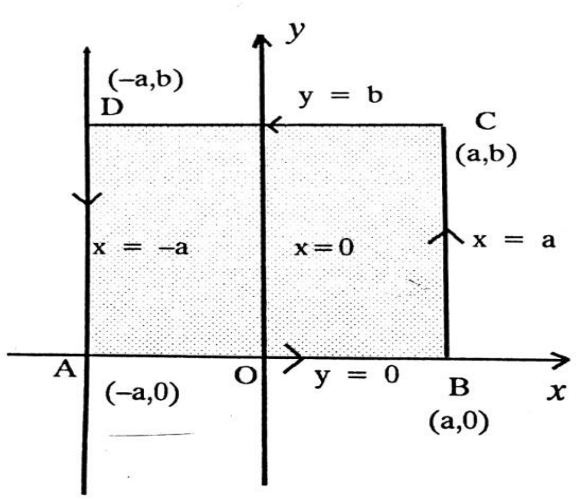
(2) $\Rightarrow \vec{F} . \mathrm{d}\vec{r} = \frac{a^3}{3} + ab^2 - \frac{a^3}{3} + ab^2 = 2ab^2 \qquad \dots (3)$

From (3) and (1) $\int_c \vec{F} \cdot d\vec{r} = \iint_S Curl\,\vec{F} \cdot \hat{n}\, dS$

Hence Stokes theorem is verified.

Example: 2.73 Verify Stoke's theorem for $\vec{F} = (x^2 + y^2)\vec{i} - 2xy\,\vec{j}$ taken around the rectangle bounded by the lines $x = \pm a, y = 0, y = b$.

Solution:



By Stokes theorem, $\int_c \vec{F} \cdot d\vec{r} = \iint_S Curl\,\vec{F} \cdot \hat{n}\, dS$

Given $\vec{F} = (x^2 + y^2)\vec{i} - 2xy\,\vec{j}$

$$Curl\,\vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 + y^2 & -2xy & 0 \end{vmatrix}$$

$$= \vec{i}[0-0] - \vec{j}\,[0-0] + \vec{k}\,[-2y - 2y]$$

$$= -4y\,\vec{k}$$

Since the region is in xoy plane we can take $\hat{n} = \vec{k}$ and $dS = dx\,dy$

Limits:

x varies from $-a$ to a.

y varies from 0 to b.

$$\therefore \iint_S Curl\,\vec{F} \cdot \hat{n}\, dS = -4\int_0^b \int_{-a}^{a} y\, dx\, dy$$

$$= -4\int_0^b [xy]_{-a}^{a}\, dy$$

$$= -8a\left[\frac{y^2}{2}\right]_0^b = -4ab^2 \qquad \ldots(1)$$

$$\int_c \vec{F} \cdot d\vec{r} = \int_{AB} + \int_{BC} + \int_{CD} + \int_{DA}$$

Along AB: $y = 0, dy = 0, x$ varies from $-a$ to a

$$d\vec{r} = dx\,\vec{i} + dy\,\vec{j}$$

$$\int_{AB} \vec{F} \cdot d\vec{r} = \int_{-a}^{a} x^2\, dx$$

$$= \left[\frac{x^3}{3}\right]_{-a}^{a} = \frac{2a^3}{3}$$

Along $BC, x = a, dx = 0,$ y varies from $0\ to\ b$

$$\int_{BC} \vec{F} \cdot d\vec{r} = \int_0^b (-2ay)\, dy$$

$$= -a[y^2]_0^b = -ab^2$$

Along CD: $y = b, dy = 0, x$ varies from $a\ to - a$

$$\int_{CD} \vec{F} \cdot d\vec{r} = \int_a^{-a} (x^2 + b^2)\, dx = \left[\frac{x^3}{3} + b^2 x\right]_a^{-a}$$

$$= -\frac{a^3}{3} - ab^2 - \frac{a^3}{3} - ab^2 = -\frac{2a^3}{3} - 2ab^2$$

Along DC: $x = -a, dx = 0, y$ varies from $b\ to\ 0$

$$\int_{DC} \vec{F} \cdot d\vec{r} = \int_b^0 2ay\, dy$$

$$= a[y^2]_b^0 = -b^2 a$$

$$\therefore \int_c \vec{F} \cdot d\vec{r} = \frac{2a^3}{3} - ab^2 - \frac{2a^3}{3} - 2ab^2 - b^2 a$$

$$= -4ab^2 \quad \ldots.(2)$$

From (1) and (2) $\int_c \vec{F} \cdot d\vec{r} = \iint_S Curl\,\vec{F} \cdot \vec{n}\, dS$

Hence Stoke's theorem is verified.

Example: 2.74 Verify Stoke's theorem for $\vec{F} = (2x - y)\vec{i} - yz^2\,\vec{j} - y^2 z\vec{k}$ where S is the upper half of the sphere $x^2 + y^2 + z^2 = 1$ and c is the Circular boundary on $z = 0$ plane.

Solution:

By Stokes theorem, $\int_c \vec{F} \cdot d\vec{r} = \iint_S Curl\,\vec{F} \cdot \hat{n}\, dS$

Given $\vec{F} = (2x - y)\vec{i} - yz^2\,\vec{j} - y^2 z\vec{k}$

$$Curl\,\vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x - y & -yz^2 & -y^2 z \end{vmatrix}$$

$$= \vec{i}[-2yz + 2yz] - \vec{j}\,[0 - 0] + \vec{k}\,[0 + 1]$$

$$= \vec{k}$$

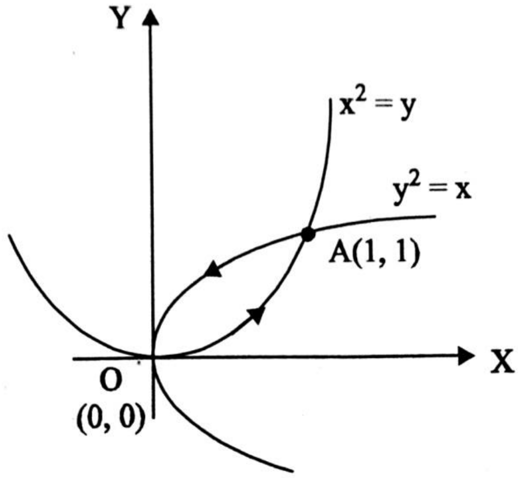
Here $\vec{n} = \vec{k}$ since C is the Circular boundary on $z = 0$ plane.

$\therefore \iint_S Curl\,\vec{F} \cdot \hat{n}\, dS = \iint_S dx\, dy$ = area of the circle

$$= \pi(1)^2 = \pi$$

Example: 2.75 Verify stokes theorem in a plane for $\vec{F} = (2xy - x^2)\vec{i} - (x^2 - y^2)\vec{j}$ Where C is the boundary of the region bounded by the parabolas $y^2 = x$ and $x^2 = y$.

Solution:



By Stokes theorem, $\int_c \vec{F} \cdot d\vec{r} = \iint_S Curl\,\vec{F} \cdot \hat{n}\, dS$

To evaluate: $\iint_S Curl\,\vec{F} \cdot \hat{n}\, dS$

Given $\vec{F} = (2xy - x^2)\vec{i} - (x^2 - y^2)\vec{j}$

Curl $\vec{F} = \nabla \times \vec{F}$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (2xy - x^2) & -(x^2 - y^2) & 0 \end{vmatrix}$$

$$= \vec{i}\,[\tfrac{\partial}{\partial y}(0) - \tfrac{\partial}{\partial z}(-(x^2 - y^2))\,] - \vec{j}\,[\tfrac{\partial}{\partial x}(0) - \tfrac{\partial}{\partial z}(2xy - x^2)]$$

$$+ \vec{k}[\tfrac{\partial}{\partial x}(-(x^2 - y^2)) - \tfrac{\partial}{\partial y}(2xy - x^2)]$$

$$= \vec{i}\,(0) - \vec{j}\,(0 - 0) + \vec{k}\,(-2x - 2x)]$$

$$= -4x\vec{k}$$

Since the surface is a rectangle in the xy −plane, $\hat{n} = \vec{k}$, $dS = dxdy$

Curl $\vec{F} \cdot \hat{n} = -4x\vec{k} \cdot \vec{k} = -4x$

Order of integration is $dxdy$

Limits:

x varies from y^2 to $\sqrt{y}$.

y varies from 0 to 1

$$\Rightarrow \iint_S Curl\,\vec{F} \cdot \hat{n}\, dS = \int_0^1 \int_{y^2}^{\sqrt{y}} -4x\, dx\, dy$$

$$= -4\int_0^1 \left[\frac{x^2}{2}\right]_{y^2}^{\sqrt{y}} dy$$

$$= -2\int_0^1 (y - y^4)dy$$

$$= -2\left[\frac{y^2}{2} - \frac{y^5}{2}\right]_0^1$$

$$= -2\left(\frac{1}{2} - \frac{1}{5}\right)$$

$$= -\frac{3}{5}$$

$$\therefore \iint_S Curl\,\vec{F} \cdot \hat{n}\,dS = -\frac{3}{5} \qquad \ldots (1)$$

To evaluate: $\iint_S Curl\,\vec{F} \cdot \hat{n}\,dS$

Here the line integral over the simple closed curve C bounding the surface OAO consisting of the curves OA and AO.

$$\int_c \vec{F} \cdot d\vec{r} = \int_{OA} + \int_{AO} \quad \ldots \ (2)$$

$$\vec{F} \cdot d\vec{r} = (2xy - x^2)\vec{\imath} - (x^2 - y^2)\vec{\jmath} \qquad \ldots (3)$$

On OA: $y = x^2, dy = 2xdx,$ x varies from 0 to 1

(3) $\Rightarrow \vec{F}\,.\,d\vec{r} = (2xx^2 - x^2)dx - (x^2 - x^4)2xdx$

$$= (2x^3 - x^2 - 2x^3 + 2x^5)dx$$

$$= (2x^5 - x^2)dx$$

$$\int_{OA} \vec{F} \cdot d\vec{r} = \int_0^1 (2x^5 - x^2)dx$$

$$= \left[\frac{2x^6}{3} - \frac{x^3}{3}\right]_0^1 = \frac{1}{3} - \frac{1}{3} = 0$$

On AO: $x = y^2, dx = 2ydy,$ y varies from 1 to 0

(3) $\Rightarrow \vec{F}\,.\,d\vec{r} = (2y^2y - y^4)2ydy - (y^4 - y^2)dy$

$$= (4y^4 - 2y^5)dy - (y^4 - y^2)dy$$

$$= (4y^4 - 2y^5 - y^4 + y^2)dy$$

$$= (3y^4 - 2y^5 + y^2)dy$$

$$\int_{AO} \vec{F} \cdot d\vec{r} = \int_1^0 (3y^4 - 2y^5 + y^2)dy$$

$$= \left[\frac{3y^5}{5} - \frac{2y^6}{6} + \frac{y^3}{3}\right]_1^0 = -\frac{3}{5} + \frac{1}{3} - \frac{1}{3} = -\frac{3}{5}$$

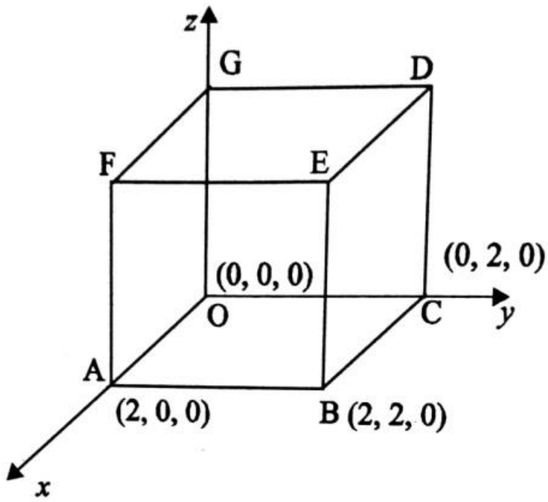
(2) $\Rightarrow \int_c \vec{F} \cdot d\vec{r} = 0 - \frac{3}{5} = -\frac{3}{5} \qquad \ldots \ (3)$

From (3) and (1) $\int_c \vec{F} \cdot d\vec{r} = \iint_S Curl\,\vec{F} \cdot \hat{n}\,dS$

Hence Stokes theorem is verified.

Example: 2.76 Verify Stoke's theorem in a plane for $\vec{F} = (y - z + 2)\vec{i} - (yz + 4)\vec{j} - xz\,\vec{k}$, where S is the open surface of the cube formed by the planes $x = 0, x = 2, y = 0,\ \ y = 2,$ $z = 0, z = 2$ above the xy – plane.

Solution:



Stoke's theorem is $\int_c \vec{F} \cdot d\vec{r} = \iint_S curl\,\vec{F} \cdot \hat{n}\ ds$

Given $\vec{F} = (y - z + 2)\vec{i} - (yz + 4)\vec{j} - xz\,\vec{k}$

$\vec{F} \cdot d\vec{r} = (y - z + 2)dx - (yz + 4)dy - xz\,dz$

L.H.S $= \int_c \vec{F} \cdot d\vec{r} = \int_{OA} + \int_{AB} + \int_{BC} + \int_{CO}$

In xy plane $z = 0 \Rightarrow dz = 0$

$\vec{F} \cdot d\vec{r} = (y + 2x)dx + 4dy$

On OA: $y = 0 \Rightarrow dy = 0$, x varies from 0 to 2.

$\Rightarrow \int_{OA} \vec{F} \cdot d\vec{r} = \int_0^2 2\,dx$

$= 2[x]_0^2 = 4$

On AB: $x = 2 \Rightarrow dx = 0$, y varies from 0 to 2.

$\Rightarrow \int_{AB} \vec{F} \cdot d\vec{r} = \int_0^2 4\,dy$

$= 4[y]_0^2 = 8$

On BC: $y = 2 \Rightarrow dy = 0$, x varies from 2 to 0.

$\Rightarrow \int_{BC} \vec{F} \cdot d\vec{r} = \int_2^0 4\,dx$

$= 4[x]_2^0 = -8$

On CO: $x = 0 \Rightarrow dx = 0$, y varies from 2 to 0.

$$\Rightarrow \int_{CO} \vec{F} \cdot d\vec{r} = \int_0^2 4\, dy$$

$$= 4[y]_0^2 = -8$$

$$\therefore \int_c \vec{F} \cdot d\vec{r} = 4 + 8 - 8 - 8 = -4 \quad \dots (1)$$

R.H.S = $\iint_S curl\, \vec{F} \cdot \hat{n}\, ds$

$$curl\, \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y - z + 2 & yz + 4 & -xz \end{vmatrix}$$

$$= \vec{i}\,(0 - y) - \vec{j}\,(-z + 1) + \vec{k}\,(0 - 1)$$

$$= y\vec{i} - (z - 1)\,\vec{j} - \vec{k}$$

Given S is an open surface consisting of the 5 faces of the cube except, $xy -$ plane.

$$\int\int curl\, \vec{F} \cdot \hat{n}\, ds = \iint_{S1} + \iint_{S2} + \cdots + \iint_{S5}$$

$$curl\, \vec{F} = y\vec{i} - (z - 1)\,\vec{j} - \vec{k}$$

Faces	Plane	ds	$\hat{n}$	$curl\, \vec{F} \cdot \hat{n}$	
Top (S_1)	xy	$dxdy$	$\vec{k}$	-1	$\int_0^2 \int_0^2 -1\, dxdy$
Left (S_2)	xz	$dxdz$	$-\vec{j}$	$-(z-1)$	$\int_0^2 \int_0^2 (-z+1)\, dxdz$
Right (S_3)	xz	$dxdz$	$\vec{j}$	$(z-1)$	$\int_0^2 \int_0^2 (z-1)\, dxdz$
Back (S_4)	yz	$dydz$	$-\vec{i}$	y	$\int_0^2 \int_0^2 y\, dydz$
Front (S_5)	yz	$dydz$	$\vec{i}$	$-y$	$\int_0^2 \int_0^2 -y\, dydz$

On S_1: $\int_0^2 \int_0^2 (-1)\, dxdy$

$$= -\int_0^2 [x]_0^2\, dy$$

$$= 2\int_0^2 dy$$

$$= -2[y]_0^2 = -4$$

On S_2: $\int_0^2 \int_0^2 (-z + 1)\, dxdz$

$$= \int_0^2 (-z+1)[x]_0^2 \, dz$$

$$= 2\int_0^2 (-z+1)\, dz$$

$$= 2\left[-\frac{z^2}{2} + z\right]_0^2 = 2(0) = 0$$

On S_3: $\int_0^2 \int_0^2 (z-1)\, dxdz$

$$= \int_0^2 (z-1)[x]_0^2 \, dz$$

$$= 2\int_0^2 (z-1)\, dz$$

$$= 2\left[\frac{z^2}{2} - z\right]_0^2 = 2(0) = 0$$

On S_4: $\int_0^2 \int_0^2 y \, dydz$

$$= \int_0^2 \left[\frac{y^2}{2}\right]_0^2 dy$$

$$= 2\int_0^2 dz$$

$$= 2[z]_0^2 = 4$$

On S_5: $\int_0^2 \int_0^2 -y \, dydz$

$$= -\int_0^2 \left[\frac{y^2}{2}\right]_0^2 dy$$

$$= -2\int_0^2 dz$$

$$= -2[z]_0^2 = -4$$

$$\therefore \iint_S curl\,\vec{F} \cdot \hat{n}\, ds = -4+0+0+4-4 = -4 \quad \dots (2)$$

From (1) and (2) $\int_c \vec{F} \cdot d\vec{r} = \iint_S curl\,\vec{F} \cdot \hat{n}\, ds$

Hence Stoke's theorem is verified.

Example: 2.77 Verify Stoke's theorem in a plane for $\vec{F} = xy\vec{i} - 2yz\vec{j} - zx\,\vec{k}$, where S is the open surface of the rectangular parallelopiped formed by the planes $x=0, x=1, y=0, y=2, z=0, z=3$ above the xoy – plane.

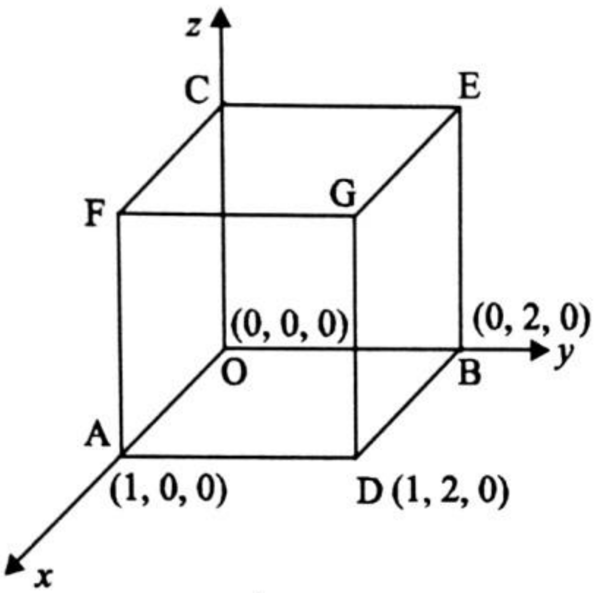
Solution:

Stoke's theorem is $\int_c \vec{F} \cdot d\vec{r} = \iint_S curl\,\vec{F} \cdot \hat{n}\, ds$

Given $\vec{F} = xy\vec{i} - 2yz\vec{j} - zx\,\vec{k}$

$\vec{F} \cdot d\vec{r} = xydx - 2yzdy - zx\,dz$

L.H.S $= \int_c \vec{F} \cdot d\vec{r} = \int_{OA} + \int_{AB} + \int_{BC} + \int_{CO}$



In xy plane $z = 0 \Rightarrow dz = 0$

$\vec{F} \cdot d\vec{r} = xydx$

On OA: $y = 0 \Rightarrow dy = 0$, x varies from 0 to 1.

$$\Rightarrow \int_{OA} \vec{F} \cdot d\vec{r} = \int_0^1 0\, dx$$

$$= 0$$

On AB: $x = 1 \Rightarrow dx = 0$, y varies from 0 to 2.

$$\Rightarrow \int_{AB} \vec{F} \cdot d\vec{r} = \int_0^2 0\, dy$$

$$= 0$$

On BC: $y = 2 \Rightarrow dy = 0$, x varies from 1 to 0.

$$\Rightarrow \int_{BC} \vec{F} \cdot d\vec{r} = \int_1^0 2x\, dx$$

$$= 2\left[\frac{x^2}{2}\right]_1^0 = -1$$

On CO: $x = 0 \Rightarrow dx = 0$, y varies from 2 to 0.

$$\Rightarrow \int_{CO} \vec{F} \cdot d\vec{r} = \int_2^0 0\, dy$$

$$= 0$$

$$\therefore \int_c \vec{F} \cdot d\vec{r} = 0 + 0 - 1 + 0 = -1 \quad \dots (1)$$

R.H.S $= \iint_S curl\, \vec{F} \cdot \hat{n}\, ds$

$$curl\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & -2yz & -xz \end{vmatrix}$$

$$= \vec{i}\,(0 + 2y) - \vec{j}\,(-z - 0) + \vec{k}\,(0 - x)$$

$$= 2y\vec{i} + z\,\vec{j} - x\,\vec{k}$$

Given S is an open surface consisting of the 5 faces of the cube except, $xy -$ plane.

$$\int\int curl\,\vec{F}\,\cdot\,\hat{n}\;ds = \iint_{S1} + \iint_{S2} + \cdots + \iint_{S5}$$

$$curl\,\vec{F} = 2y\vec{i} + z\,\vec{j} - x\,\vec{k}$$

Faces	Plane	ds	$\hat{n}$	$curl\,\vec{F}\,\cdot\,\hat{n}$	
Top (S_1)	xy	$dxdy$	$\vec{k}$	$-x$	$\int_0^2\int_0^1 -x\;dxdy$
Left (S_2)	xz	$dxdz$	$-\vec{j}$	$-z$	$\int_0^3\int_0^1 -z\;dxdz$
Right (S_3)	xz	$dxdz$	$\vec{j}$	z	$\int_0^3\int_0^1 z\;dxdz$
Back (S_4)	yz	$dydz$	$-\vec{i}$	$-2y$	$\int_0^3\int_0^2 -2y\;dydz$
Front (S_5)	yz	$dydz$	$\vec{i}$	$2y$	$\int_0^3\int_0^2 2y\;dydz$

On S_1: $\int_0^2\int_0^1(-1)\;dxdy$

$$= -\int_0^2\left[\frac{x^2}{2}\right]_0^1 dy$$

$$= -\frac{1}{2}\int_0^2 dy$$

$$= -\frac{1}{2}\,[y]_0^2 \;=\; -1$$

On S_2: $\int_0^3\int_0^1 -z\;dxdz$

$$= -\int_0^3[zx]_0^1\;dz$$

$$= -\int_0^2 z\;dz$$

$$= -\left[\frac{z^2}{2}\right]_0^3 = -\frac{9}{2}$$

On S_3: $\int_0^3\int_0^1 z\;dxdz$

$$= \int_0^3 [zx]_0^1 \, dz$$

$$= 2\int_0^3 z \, dz$$

$$= \left[\frac{z^2}{2}\right]_0^3 = \frac{9}{2}$$

On S_4: $\int_0^3 \int_0^2 -2y \, dydz$

$$= -2\int_0^3 \left[\frac{y^2}{2}\right]_0^2 dz$$

$$= -4\int_0^3 dz$$

$$= -4[z]_0^3 = -12$$

On S_5: $\int_0^3 \int_0^2 2y \, dydz$

$$= 2\int_0^3 \left[\frac{y^2}{2}\right]_0^2 dz$$

$$= 4\int_0^3 dz$$

$$= 4[z]_0^3 = 12$$

$$\therefore \iint_S curl\,\vec{F} \cdot \hat{n} \, ds = -1 - \frac{9}{2} + \frac{9}{2} - 12 + 12 = -1 \quad \ldots (2)$$

From (1) and (2) $\int_c \vec{F} \cdot d\vec{r} = \iint_S curl\,\vec{F} \cdot \hat{n} \, ds$

Hence Stoke's theorem is verified.

Example: 2.78 Verify Stoke's theorem for $\vec{F} = y^2z\vec{i} + z^2x\vec{j} + x^2y\,\vec{k}$, where S is the open surface of the cube formed by the planes $x = \pm a$, $y = \pm a$, and $z = \pm a$ in which the plane $z = -a$ is a cut.

Solution:

Stoke's theorem is $\int_c \vec{F} \cdot d\vec{r} = \iint_S curl\,\vec{F} \cdot \hat{n} \, ds$

Given $\vec{F} = y^2z\vec{i} + z^2x\vec{j} + x^2y\,\vec{k}$

$\vec{F} \cdot d\vec{r} = y^2zdx + z^2xdy + x^2y\,dz$

This square ABCD lies in the plane $z = -a \Rightarrow dz = 0$

$\therefore \vec{F} \cdot d\vec{r} = -ay^2dx + a^2x\,dy$

L.H.S $= \int_c \vec{F} \cdot d\vec{r} = \int_{AB} + \int_{BC} + \int_{CD} + \int_{DA}$

On AB: $y = -a \Rightarrow dy = 0$, x varies from $-a$ to a.

$$\Rightarrow \int_{AB} \vec{F} \cdot d\vec{r} = \int_{-a}^{a} -a^3 \, dx$$

$$= -a^3 [x]_{-a}^{a}$$

$$= -a^3(2a) = -2a^4$$

On BC: $x = a \Rightarrow dx = 0$, y varies from $-a$ to a.

$$\Rightarrow \int_{BC} \vec{F} \cdot d\vec{r} = \int_{-a}^{a} a^3 \, dy$$

$$= a^3 \, [y]_{-a}^{a}$$

$$= a^3(2a) = 2a^4$$

On CD: $y = a \Rightarrow dy = 0$, x varies from a to $-a$.

$$\Rightarrow \int_{CD} \vec{F} \cdot d\vec{r} = \int_{a}^{-a} -a^3 \, dx$$

$$= -a^3 \, [x]_{a}^{-a}$$

$$= -a^3(-2a) = 2a^4$$

On DA: $x = -a \Rightarrow dx = 0$, y varies from a to $-a$.

$$\Rightarrow \int_{DA} \vec{F} \cdot d\vec{r} = \int_{a}^{-a} -a^3 \, dy$$

$$= -a^3 \, [y]_{a}^{-a}$$

$$= -a^3(-2a) = 2a^4$$

$$\therefore \int_{c} \vec{F} \cdot d\vec{r} = -2a^4 + 2a^4 + 2a^4 + 2a^4 = 4a^4 \quad \dots (1)$$

R.H.S = $\iint_S \, curl\,\vec{F} \cdot \hat{n} \, ds$

$$curl\,\vec{F} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 z & z^2 x & x^2 y \end{vmatrix}$$

$$= \vec{\imath}\,(x^2 - 2xz) - \vec{\jmath}\,(y^2 - 2xy) + \vec{k}\,(z^2 - 2yz)$$

Given S is an open surface consisting of the 5 faces of the cube except, $z = -a$.

$$\int\int curl\,\vec{F} \cdot \hat{n} \, ds = \iint_{S1} + \iint_{S2} + \cdots + \iint_{S5}$$

$$curl\,\vec{F} = 2y\vec{\imath} + z\,\vec{\jmath} - x\,\vec{k}$$

Faces	Plane	ds	$\hat{n}$	Eqn	$curl\,\vec{F} \cdot \hat{n}$	$\nabla \times \vec{F} \cdot \hat{n}$
Top (S_1)	xy	$dxdy$	$\vec{k}$	$z = a$	$z^2 - 2yz$	$a^2 - 2ay$
Left (S_2)	xz	$dxdz$	$-\vec{\jmath}$	$y = -a$	$y^2 - 2xy$	$a^2 + 2ax$
Right (S_3)	xz	$dxdz$	$\vec{\jmath}$	$y = a$	$-(y^2 - 2xy)$	$-(a^2 - 2ax)$
Back (S_4)	yz	$dydz$	$-\vec{\imath}$	$x = -a$	$-(x^2 - 2xz)$	$-(a^2 + 2az)$
Front (S_5)	yz	$dydz$	$\vec{\imath}$	$x = a$	$x^2 - 2xz$	$a^2 - 2az$

On S_1: $\int_{-a}^{a}\int_{-a}^{a}(a^2 - 2ay)\,dxdy$

$$= \int_{-a}^{a}[(a^2x - 2ayx)]_{-a}^{a}\,dy$$

$$= \int_{-a}^{a} (a^3 - 2a^2 y) - (-a^3 + 2a^2 y)\, dy$$

$$= \int_{-a}^{a} 2a^3 - 4a^2 y\, dy$$

$$= \left[2a^3 y - 4a^2 \frac{y^2}{2}\right]_{-a}^{a}$$

$$= (2a^4 - 2a^4) - (-2a^4 - 2a^4)$$

$$= 2a^4 - 2a^4 + 2a^4 + 2a^4$$

$$= 4a^4$$

On $S_2 + S_3$: $\int_{-a}^{a}\int_{-a}^{a} (a^2 + 2ax)\, dxdz + \int_{-a}^{a}\int_{-a}^{a} -(a^2 - 2ax)\, dxdz$

$$= \int_{-a}^{a}\int_{-a}^{a} (a^2 + 2ax - a^2 + 2ax)\, dx\, dz$$

$$= \int_{-a}^{a}\int_{-a}^{a} 4ax\, dx\, dz$$

$$= 4a \int_{-a}^{a} \left[\frac{x^2}{2}\right]_{-a}^{a} dz$$

$$= 2a^3 \int_{-a}^{a} dz$$

$$= 2a^3\, [z]_{-a}^{a}$$

$$= 2a^3(0) = 0$$

On $S_4 + S_5$: $\int_{-a}^{a}\int_{-a}^{a} -(a^2 + 2az)\, dydz + \int_{-a}^{a}\int_{-a}^{a} (a^2 - 2az)\, dydz$

$$= \int_{-a}^{a}\int_{-a}^{a} (-a^2 - 2az + a^2 - 2az)\, dy\, dz$$

$$= \int_{-a}^{a}\int_{-a}^{a} -4az\, dy\, dz$$

$$= -4a \int_{-a}^{a} [zy]_{-a}^{a} dz$$

$$= -4a \int_{-a}^{a} z(2a) dz$$

$$= -6a^2 \left[\frac{z^2}{2}\right]_{-a}^{a}$$

$$= -3a^2(a^2 - a^2) = 0$$

$$\therefore \iint_S curl\, \vec{F} \cdot \hat{n}\, ds = 4a^4 + 0 + 0 = 4a^4 \quad \ldots (2)$$

From (1) and (2) $\int_c \vec{F} \cdot d\vec{r} = \iint_S curl\, \vec{F} \cdot \hat{n}\, ds$

Hence Stoke's theorem is verified.

Example: 2.79 Evaluate $\int_c \vec{F} \cdot d\vec{r}$ by stoke's theorem, where $\vec{F} = y^2 \vec{i} + x^2 \vec{j} + (x + z)\vec{k}$, and C is the boundary of the triangle with vertices at $(0, 0, 0), (1, 0, 0)$ *and* $(1, 1, 0)$.

Solution:

Stoke's theorem is $\int_c \vec{F} \cdot d\vec{r} = \iint_S curl\, \vec{F} \cdot \hat{n}\, ds \quad \ldots (1)$

Given $\vec{F} = y^2 \vec{i} + x^2 \vec{j} + (x + z)\vec{k}$

And C is triangle $(0,0,0), (1,0,0)$ and $(1,1,0)$.

Since z −coordinate of each vertex is zero the triangle lies in xy − plane with corners $(0,0), (1,0)$ and $(1,1)$.

To evaluate : $\iint\limits_S \; curl\,\vec{F}\cdot\,\hat{n}\;ds$

$In\ xy\ - plane\,\hat{n} = \vec{k}, ds = dxdy$

$$curl\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & x^2 & -(x+z) \end{vmatrix}$$

$$= \vec{i}\,(0) - \vec{j}\,(-1) + \vec{k}\,(2x - 2y)$$

$$= \vec{j} + \ 2(x-y)\vec{k}$$

$$curl\,\vec{F}\cdot\,\hat{n} \ = \left(\vec{j} + \ 2(x-y)\vec{k}\right)\cdot\vec{k}$$

$$= 2(x-y)$$

Limits:

x varies from y to 1.

y varies from 0 to 1.

$$\therefore \iint\limits_S \; curl\,\vec{F}\cdot\,\hat{n}\;ds = \int_0^1\int_y^1 2(x-y)dxdy$$

$$= 2\int_0^1\left[\frac{x^2}{2} - \ xy\right]_y^1 dy$$

$$= 2\int_0^1\left(\frac{1}{2} - \ y - \frac{y^2}{2} + \ y^2\right) dy$$

$$= 2\left[\frac{y}{2} - \frac{y^2}{2} - \frac{y^3}{6} + \frac{y^3}{3}\right]_0^1$$

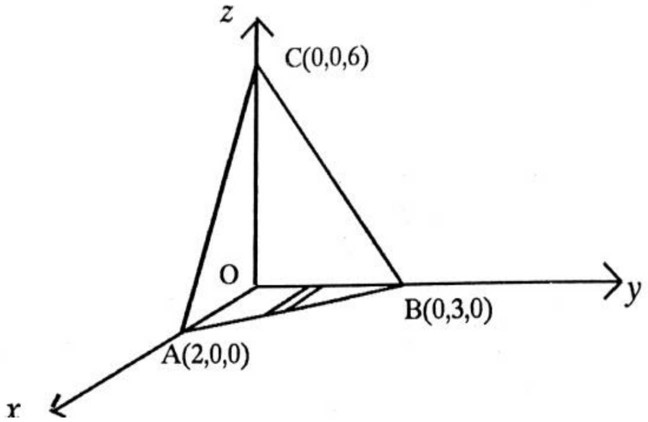
$$= 2\left[\frac{1}{2} - \frac{1}{2} - \frac{1}{6} + \frac{1}{3}\right]$$

$$= 2\left[\frac{1}{6}\right] = \frac{1}{3}$$

From (1), $\int\limits_c \; \vec{F}\cdot d\vec{r} = \frac{1}{3}$

Example: 2.80 Evaluate the integral $\int\limits_c \; (x+y)dx + \ (2x-z)dy + \ (y+z)dz$, where C is the boundary of the triangle with vertices $(2,0,0), (0,3,0)$ and $(0,0,6)$ using stoke's theorem.

Solution:



Stoke's theorem is $\int_c \vec{F} \cdot d\vec{r} = \iint_S curl\,\vec{F} \cdot \hat{n}\ ds$

Given $\vec{F} \cdot d\vec{r} = (x+y)dx + (2x-z)dy + (y+z)dz$

$\therefore \vec{F} = (x+y)\vec{i} + (2x-z)\vec{j} + (y+z)\vec{k}$

$$curl\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+y & 2x-z & y+z \end{vmatrix}$$

$$= \vec{i}\,(1-1) - \vec{j}\,(0) + \vec{k}\,(2-1)$$

$$= 2\vec{i} + \vec{k}$$

Given C is the triangle with vertices $(2,0,0), (0,3,0)\ and\ (0,0,6)$.

Equation of the plane is $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

$$\Rightarrow 3x + 2y + z = 6$$

Let $\varphi = 3x + 2y + z - 6$

$$\nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

$$= 3\vec{i} + 2\vec{j} + \vec{k}$$

$$|\nabla\varphi| = \sqrt{9+4+1} = \sqrt{14}$$

$$\hat{n} = \frac{\nabla\varphi}{|\nabla\varphi|} = \frac{3\vec{i}+2\vec{j}+\vec{k}}{\sqrt{14}}$$

Let R be the projection on XY −plane.

$$\therefore ds = \frac{dx\,dy}{\left|\hat{n} \cdot \vec{k}\right|} = \frac{dx\,dy}{\left(\frac{1}{\sqrt{14}}\right)}$$

Where $\hat{n} \cdot \vec{k} = \left(\frac{3\vec{i}+2\vec{j}+\vec{k}}{\sqrt{14}}\right) \cdot \vec{k}$

$$= \frac{1}{\sqrt{14}}$$

Now $curl\,\vec{F} \cdot \hat{n} = \left(2\vec{i} + \vec{k}\right) \cdot \left(\frac{3\vec{i}+2\vec{j}+\vec{k}}{\sqrt{14}}\right)$

$$= \frac{6+1}{\sqrt{14}} = \frac{7}{\sqrt{14}}$$

$$\Rightarrow \iint_S curl\, \vec{F} \cdot \hat{n}\, ds = \iint_R \frac{7}{\sqrt{14}} \frac{dxdy}{\left(\frac{1}{\sqrt{14}}\right)}$$

$$= 7 \iint_R dxdy$$

$$= 7\, [Area\ of\ the\ triangle]$$

$$= 7 \left[\frac{1}{2}\,(2)\,(3)\right] = 21 \quad \left[\because Area\ of\ the\ triangle = \frac{1}{2} bh\right]$$

Example: 2.81 Evaluate by Stoke's theorem $\int_c (\boldsymbol{e^x dx + 2ydy - dz})$, where C is the Curve $\boldsymbol{x^2 + y^2 = 4, z = 2}$.

Solution:

Stoke's theorem is $\int_c \vec{F} \cdot d\vec{r} = \iint_S curl\, \vec{F} \cdot \hat{n}\, ds$ ... (1)

Given $\vec{F} \cdot d\vec{r} = e^x dx + 2ydy - dz$

$\therefore \vec{F} = e^x \vec{\imath} + 2y\vec{\jmath} - \vec{k}$

$$curl\, \vec{F} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^x & 2y & -1 \end{vmatrix}$$

$$= \vec{\imath}\,(0-0) - \vec{\jmath}\,(0-0) + \vec{k}\,(0-1)$$

$$= \vec{0}$$

$\therefore$ (1) $\Rightarrow \int_c \vec{F} \cdot d\vec{r} = 0$

$(i.e) \int_c (e^x dx + 2ydy - dz) = 0$

Example: 2.82 Evaluate $\int_c \left(\boldsymbol{yz\vec{\imath} + xz\vec{\jmath} + xy\vec{k}}\right) \cdot \boldsymbol{d\vec{r}}$, where C is the boundary of the surface S.

Solution:

Given $\vec{F} = yz\vec{\imath} + xz\vec{\jmath} + xy\,\vec{k}$

Stoke's theorem is $\int_c \vec{F} \cdot d\vec{r} = \iint_S \left(\nabla \times \vec{F}\right) \cdot \hat{n}\, ds$ ... (1)

$$curl\, \vec{F} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & xz & xy \end{vmatrix}$$

$$= \vec{\imath}\,(x-x) - \vec{\jmath}\,(y-y) + \vec{k}\,(z-z)$$

$$= \vec{0}$$

$\therefore$ (1) $\Rightarrow \int_c \vec{F} \cdot d\vec{r} = 0$

Exercise: 2.5

1. Verify Stoke's theorem for the function $\vec{F} = x^2\vec{\imath} + xy\vec{\jmath}$, integrated round the square in the $z = 0$ plane whose sides are along the lines $x = 0, y = 0, x = a, y = a$. **Ans:** $\frac{a^3}{2}$

2. Verify Stoke's theorem for $\vec{F} = y\vec{\imath} + z\vec{\jmath} + x\vec{k}$, where S is the upper half surface of the sphere $x^2 + y^2 + z^2 = 1$ and C is its boundary. **Ans:** $-\pi$

3. Evaluate $\int_c [xydx + xy^2dy]$ by Stoke's theorem where C is the square in xy plane with vertices $(1,0), (-1,0), (0,1)$ *and* $(0,-1)$ **Ans:** $\frac{1}{2}$

4. Verify Stoke's theorem for $\vec{F} = (y - z + 2)\vec{\imath} + (yz + 4)\vec{\jmath} - xz\vec{k}$, where S is the open surface of the cube $x = 0, y = 0, z = 0, x = 2, y = 2, z = 2$ above the xy plane.

   Ans: Common value $= -4$

5. Verify Stoke's theorem for $\vec{F} = (x^2 - y^2)\vec{\imath} + 2xy\vec{\jmath} + xyz\vec{k}$, over the surface of the box bounded by the planes $x = 0, y = 0, x = a, y = b, z = c$ above the xy plane.

   Ans: Common value $2ab^2$

6. Verify Stoke's theorem for $\vec{F} = xy\vec{\imath} - 2yz\vec{\jmath} - zx\vec{k}$, where S is the open surface of the rectangular parallelepiped formed by the planes $x = 0, x = 1, y = 0, y = 2, z = 3$ above the xoy plane.

   Ans: Common value -1

7. Verify Stoke's theorem for $\vec{F} = -y\vec{\imath} + 2yz\vec{\jmath} + y^2\vec{k}$, where S is the half of the sphere $x^2 + y^2 + z^2 = a^2$ and C is the circular boundary on the xoy plane.

   Ans: Common value $= \pi a^2$

8. Using Stoke's theorem $\int_c \vec{F} \cdot d\vec{r}$ where $\vec{F} = (\sin x - y)\vec{\imath} - \cos x\,\vec{\jmath}$ and C is the boundary of the triangle whose vertices $(0,0), \left(\frac{\pi}{2}, 0\right)$ *and* $\left(\frac{\pi}{2}, 1\right)$ **Ans:** $\frac{\pi}{4} + \frac{2}{\pi}$

2.6 GAUSS DIVERGENCE THEOREM

This theorem enables us to convert a surface integral of a vector function on a closed surface into volume integral.

Statement of Gauss Divergence theorem

If V is the volume bounded by a closed surface S and if a vector function $\vec{F}$ is continuous and has continuous partial derivatives in V and on S, then

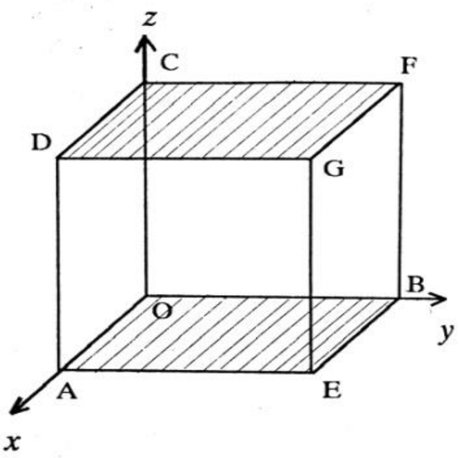
$$\iint_S \vec{F} \cdot \hat{n}\, ds = \iiint_V \nabla \cdot \vec{F}\, dv$$

Where $\hat{n}$ is the unit outward normal to the surface S and $dV = dxdydz$

Problems based on gauss Divergence theorem

Example: 2.83 Verify the G.D.T for $\vec{F} = 4xz\vec{i} - y^2\vec{j} + yz\vec{k}$ over the cube bounded by $x = 0,\ x = 1,$ $y = 0,\ y = 1,\ z = 0,\ z = 1.$

Solution:



Gauss divergence theorem is $\iint_S \vec{F} \cdot \hat{n}\, ds = \iiint_V \nabla \cdot \vec{F}\, dv$

Given $\vec{F} = 4xz\vec{i} - y^2\vec{j} + yz\vec{k}$

$$\nabla \cdot \vec{F} = 4z - 2y + y$$
$$= 4z - y$$

Now, R.H.S $= \iiint_V \nabla \cdot \vec{F}\, dv$

$$= \int_0^1 \int_0^1 \int_0^1 (4z - y)\, dxdydz$$
$$= \int_0^1 \int_0^1 [(4xz - yz)]_0^1\ dydz$$
$$= \int_0^1 \int_0^1 (4z - y)\, dydz$$
$$= \int_0^1 \left(4zy - \frac{y^2}{2}\right)_0^1\ dz$$
$$= \int_0^1 \left(4z - \frac{1}{2}\right)\ dz$$
$$= \left[4\frac{z^2}{2} - \frac{1}{2}z\right]_0^1 = \left(2 - \frac{1}{2}\right) - 0 = \frac{3}{2}$$

Now, L.H.S $= \iint_S \vec{F} \cdot \hat{n}\, ds = \iint_{S1} + \iint_{S2} + \iint_{S3} + \iint_{S4} + \iint_{S5} + \iint_{S6}$

Faces	Plane	dS	$\hat{n}$	$\vec{F} \cdot \hat{n}$	Equation	$\vec{F} \cdot \hat{n}$ on S	$= \iint_S \vec{F} \cdot \hat{n}\, ds$
S_1(Bottom)	xy	$dxdy$	$-\vec{k}$	$-yz$	$z = 0$	0	$\int_0^1 \int_0^1 0\, dxdy$
S_2(Top)	xy	$dxdy$	$\vec{k}$	yz	$z = 1$	y	$\int_0^1 \int_0^1 y\, dxdy$

$S_3(Left)$	xz	$dxdz$	$-\vec{j}$	y^2	$y=0$	0	$\int_0^1\int_0^1 0\,dxdz$
$S_4(Right)$	xz	$dxdz$	$\vec{j}$	$-y^2$	$y=1$	-1	$\int_0^1\int_0^1 -1dxdz$
$S_5(Back)$	yz	$dydz$	$-\vec{i}$	$-4xz$	$x=0$	0	$\int_0^1\int_0^1 0\,dydz$
$S_6(Front)$	yz	$dydz$	$\vec{i}$	$4xz$	$x=1$	$4z$	$\int_0^1\int_0^1 4z\,dydz$

$(i)\iint\limits_{S1} \vec{F}\cdot\hat{n}\,ds + \iint\limits_{S2} \vec{F}\cdot\hat{n}\,ds = \int_0^1\int_0^1 0\,dxdy + \int_0^1\int_0^1 y\,dxdy$

$= 0 + \int_0^1\int_0^1 y\,dxdy$

$= \int_0^1 [yx]_0^1\,dy$

$= \int_0^1 y\,dy$

$= \left[\frac{y^2}{2}\right]_0^1 \quad = \frac{1}{2} - 0 \quad = \frac{1}{2}$

$(ii)\iint\limits_{S3} \vec{F}\cdot\hat{n}\,ds + \iint\limits_{S4} \vec{F}\cdot\hat{n}\,ds = \int_0^1\int_0^1 0\,dxdz + \int_0^1\int_0^1 -1\,dxdz$

$= 0 + \int_0^1\int_0^1 -1\,dxdz$

$= -\int_0^1 [x]_0^1\,dz$

$= -\int_0^1 dz$

$= -[z]_0^1 \quad = -[1]$

$(iii)\iint\limits_{S5} \vec{F}\cdot\hat{n}\,ds + \iint\limits_{S6} \vec{F}\cdot\hat{n}\,ds = \int_0^1\int_0^1 0\,dydz + \int_0^1\int_0^1 4zdydz$

$= 0 + \int_0^1\int_0^1 4z\,dydz$

$= \int_0^1 [4zy]_0^1\,dz$

$= \int_0^1 4z\,dz$

$= 4\left[\frac{z^2}{2}\right]_0^1 \quad = 4\left(\frac{1}{2} - 0\right) \quad = 2$

$\therefore \iint\limits_{S} \vec{F}\cdot\hat{n}\,ds = \iint\limits_{S1} + \iint\limits_{S2} + \iint\limits_{S3} + \iint\limits_{S4} + \iint\limits_{S5} + \iint\limits_{S6}$

$= \quad (i) + (ii) + (iii)$

$= \frac{1}{2} - 1 + 2 \quad = \frac{3}{2}$

$\therefore \iint\limits_{S} \vec{F}\cdot\hat{n}\,ds = \iiint\limits_{V} \nabla\cdot\vec{F}\,dv$

Hence Gauss divergence theorem is verified.

Example: 2.84 Verify the G.D.T for $\vec{F} = (x^2 - yz)\vec{i} + (y^2 - xz)\vec{j} + (z^2 - xy)\vec{k}$ over the rectangular parallelopiped $0 \le x \le a,\ 0 \le y \le b, 0 \le z \le c$. (OR)

Verify the G.D.T for $\vec{F} = (x^2 - yz)\vec{i} + (y^2 - xz)\vec{j} + (z^2 - xy)\vec{k}$ over the rectangular parallelopiped bounded by $x = 0, x = a, y = 0, y = b, z = 0, z = c$.

Solution:

Gauss divergence theorem is $\iint_S \vec{F} \cdot \hat{n}\, ds = \iiint_V \nabla \cdot \vec{F}\, dv$

Given $\vec{F} = (x^2 - yz)\vec{i} + (y^2 - xz)\vec{j} + (z^2 - xy)\vec{k}$

$$\nabla \cdot \vec{F} = 2x + 2y + 2z \quad = 2(x + y + z)$$

Now, R.H.S $= \iiint_V \nabla \cdot \vec{F}\, dv$

$$= 2\int_0^c \int_0^b \int_0^a (x + y + z)dxdydz$$

`

$$= 2\int_0^c \int_0^b \left[\left(\frac{x^2}{2} + xy + xz\right)\right]_0^a dydz$$

$$= 2\int_0^c \int_0^b \left(\frac{a^2}{2} + ay + az\right) dydz$$

$$= 2\int_0^c \left(\frac{a^2 y}{2} + \frac{ay^2}{2} + azy\right)_0^b dz$$

$$= 2\int_0^c \left(\frac{a^2 b}{2} + \frac{ab^2}{2} + azb\right) dz$$

$$= 2\left[\frac{a^2 bz}{2} + \frac{ab^2 z}{2} + \frac{abz^2}{2}\right]_0^c$$

$$= 2\left(\frac{a^2 bc}{2} + \frac{ab^2 c}{2} + \frac{abc^2}{2}\right)$$

$$= abc(a + b + c)$$

Now, L.H.S $= \iint_S \vec{F} \cdot \hat{n}\, ds = \iint_{S1} + \iint_{S2} + \iint_{S3} + \iint_{S4} + \iint_{S5} + \iint_{S6}$

Faces	**Plane**	dS	$\hat{n}$	$\vec{F} \cdot \hat{n}$	**Eqn**	$\vec{F} \cdot \hat{n}$ **on S**	$= \iint_S \vec{F} \cdot \hat{n}\, ds$
S_1(Bottom)	xy	$dxdy$	$-\vec{k}$	$-(z^2 - xy)$	$z = 0$	xy	$\int_0^b \int_0^a xy\, dxdy$
S_2(Top)	xy	$dxdy$	$\vec{k}$	$(z^2 - xy)$	$z = c$	$c^2 - xy$	$\int_0^b \int_0^a c^2 - xy\, dxdy$
$S_3(Left)$	xz	$dxdz$	$-\vec{j}$	$-(y^2 - xz)$	$y = 0$	xz	$\int_0^c \int_0^a xz\, dxdz$
$S_4(Right)$	xz	$dxdz$	$\vec{j}$	$(y^2 - xz)$	$y = b$	$b^2 - xz$	$\int_0^c \int_0^a b^2 - xz\, dxdz$

$S_5(Back)$	yz	$dydz$	$-\vec{i}$	$-(x^2-yz)$	$x=0$	yz	$\int_0^c\int_0^b yz\,dydz$
$S_6(Front)$	yz	$dydz$	$\vec{i}$	(x^2-yz)	$x=a$	a^2-yz	$\int_0^c\int_0^b a^2-yz\,dydz$

$(i)\iint\limits_{S1} \vec{F}\cdot\hat{n}\,ds + \iint\limits_{S2} \vec{F}\cdot\hat{n}\,ds = \int_0^b\int_0^a xy\,dxdy + \int_0^b\int_0^a c^2 - xy\,dxdy$

$= \int_0^b\int_0^a c^2\,dxdy$

$= c^2\int_0^a dx\int_0^b dy$

$= c^2[x]_0^a[y]_0^b \quad = c^2ab$

$(ii)\iint\limits_{S3} \vec{F}\cdot\hat{n}\,ds + \iint\limits_{S4} \vec{F}\cdot\hat{n}\,ds = \int_0^c\int_0^a xz\,dxdz + \int_0^c\int_0^a b^2 - xz\,dxdz$

$= \int_0^c\int_0^a b^2\,dxdz$

$= b^2\int_0^a dx\int_0^c dz$

$= b^2[x]_0^a[z]_0^c \quad = b^2ac$

$(iii)\iint\limits_{S5} \vec{F}\cdot\hat{n}\,ds + \iint\limits_{S6} \vec{F}\cdot\hat{n}\,ds = \int_0^c\int_0^b yz\,dydz + \int_0^c\int_0^b a^2 - yz\,dydz$

$= \int_0^c\int_0^b a^2\,dydz$

$= a^2\int_0^b dy\int_0^c dz$

$= a^2[y]_0^b[z]_0^c \quad = a^2bc$

$\therefore \iint\limits_{S} \vec{F}\cdot\hat{n}\,ds = \iint\limits_{S1} + \iint\limits_{S2} + \iint\limits_{S3} + \iint\limits_{S4} + \iint\limits_{S5} + \iint\limits_{S6}$

$= \quad (i) + (ii) + (iii)$

$= abc(a+b+c)$

$\therefore \iint\limits_{S} \vec{F}\cdot\hat{n}\,ds = \iiint\limits_{V} \nabla\cdot\vec{F}\,dv$

Hence Gauss divergence theorem is verified.

Example: 2.85 Verify divergence theorem for $\vec{F} = (2x-z)\vec{i} + x^2y\vec{j} - xz^2\vec{k}$ over the cube bounded by $x=0,\ x=1,\ y=0,\ y=1,\ z=0,\ z=1$.

Solution:

Gauss divergence theorem is $\iint_S \vec{F} \cdot \hat{n}\, ds = \iiint_V \nabla \cdot \vec{F}\, dv$

Given $\vec{F} = (2x - z)\vec{i} + x^2 y\vec{j} - xz^2\vec{k}$

$\nabla \cdot \vec{F} = 2 + x^2 - 2xz$

Now, R.H.S $= \iiint_V \nabla \cdot \vec{F}\, dv$

$= \int_0^1 \int_0^1 \int_0^1 (2 + x^2 - 2xz)dxdydz$

$= \int_0^1 \int_0^1 \left[\left(2x + \frac{x^3}{3} - \frac{2zx^2}{2}\right)\right]_0^1 dydz$

$= \int_0^1 \int_0^1 \left(2 + \frac{1}{3} - z\right) dydz$

$= \int_0^1 \left(2y + \frac{1}{3}y - zy\right)_0^1 dz$

$= \int_0^1 \left(2 + \frac{1}{3} - z\right) dz$

$= \left[2z + \frac{1}{3}z - \frac{z^2}{2}\right]_0^1$

$= \left(2 + \frac{1}{3} - \frac{1}{2}\right) - 0 = \frac{11}{6}$

Now, L.H.S $= \iint_S \vec{F} \cdot \hat{n}\, ds = \iint_{S1} + \iint_{S2} + \iint_{S3} + \iint_{S4} + \iint_{S5} + \iint_{S6}$

Faces	**Plane**	***dS***	$\hat{n}$	$\vec{F} \cdot \hat{n}$	**Equation**	$\vec{F} \cdot \hat{n}$ **on S**	$= \iint_S \vec{F} \cdot \hat{n}\, ds$
S_1(Bottom)	xy	dxdy	$-\vec{k}$	xz^2	$z = 0$	0	$\int_0^1 \int_0^1 0\, dxdy$
S_2(Top)	xy	dxdy	$\vec{k}$	$-xz^2$	$z = 1$	$-x$	$\int_0^1 \int_0^1 (-x)\, dxdy$
$S_3(Left)$	xz	dxdz	$-\vec{j}$	$-x^2 y$	$y = 0$	0	$\int_0^1 \int_0^1 0\, dxdz$
$S_4(Right)$	xz	dxdz	$\vec{j}$	$x^2 y$	$y = 1$	x^2	$\int_0^1 \int_0^1 x^2 dxdz$
$S_5(Back)$	yz	dydz	$-\vec{i}$	$-(2x - z)$	$x = 0$	z	$\int_0^1 \int_0^1 z\, dydz$
$S_6(Front)$	yz	dydz	$\vec{i}$	$(2x - z)$	$x = 1$	$2 - z$	$\int_0^1 \int_0^1 2 - z\, dydz$

$(i) \iint_{S1} \vec{F} \cdot \hat{n}\, ds + \iint_{S2} \vec{F} \cdot \hat{n}\, ds = \int_0^1 \int_0^1 0\, dxdy + \int_0^1 \int_0^1 (-x)\, dxdy$

$= \int_0^1 \int_0^1 (-x)\, dxdy$

$$= -\int_0^1 \left[\frac{x^2}{2}\right]_0^1 \, dy$$

$$= -\int_0^1 \frac{1}{2} \, dy$$

$$= -\left[\frac{1}{2}y\right]_0^1 \quad = -\left(\frac{1}{2} - 0\right) \quad = \frac{-1}{2}$$

$$(ii) \iint_{S3} \vec{F} \cdot \hat{n}\, ds + \iint_{S4} \vec{F} \cdot \hat{n}\, ds = \int_0^1 \int_0^1 0 \, dxdz + \int_0^1 \int_0^1 x^2 \, dxdz$$

$$= \int_0^1 \int_0^1 x^2 \, dxdz$$

$$= \int_0^1 \left[\frac{x^3}{3}\right]_0^1 \, dz$$

$$= \int_0^1 \frac{1}{3} \, dz$$

$$= \left[\frac{1}{3}z\right]_0^1 \quad = \left(\frac{1}{3} - 0\right) \quad = \frac{1}{3}$$

$$(iii) \iint_{S5} \vec{F} \cdot \hat{n}\, ds + \iint_{S6} \vec{F} \cdot \hat{n}\, ds = \int_0^1 \int_0^1 z \, dydz + \int_0^1 \int_0^1 (2 - z) dydz$$

$$= \int_0^1 \int_0^1 2 \, dydz$$

$$= 2 \int_0^1 [y]_0^1 \, dz$$

$$= 2 \int_0^1 dz$$

$$= 2\,[z]_0^1 \quad = 2$$

$$\therefore \iint_S \vec{F} \cdot \hat{n}\, ds = \iint_{S1} + \iint_{S2} + \iint_{S3} + \iint_{S4} + \iint_{S5} + \iint_{S6}$$

$$= (i) + (ii) + (iii)$$

$$= -\frac{1}{2} + \frac{1}{3} + 2 \quad = \frac{11}{6}$$

$$\therefore \iint_S \vec{F} \cdot \hat{n}\, ds = \iiint_V \nabla \cdot \vec{F}\, dv$$

Hence Gauss divergence theorem is verified.

Example: 2.86 Verify divergence theorem for $\vec{F} = x^2\vec{i} + z\vec{j} + yz\vec{k}$ over the cube bounded by $x = \pm 1$, $y = \pm 1$, $z = \pm 1$.

Solution:

Gauss divergence theorem is $\iint_S \vec{F} \cdot \hat{n}\, ds = \iiint_V \nabla \cdot \vec{F}\, dv$

Given $\vec{F} = x^2\vec{i} + z\vec{j} + yz\vec{k}$

$$\nabla \cdot \vec{F} = 2x + y$$

Now, R.H.S $= \iiint_V \nabla \cdot \vec{F}\, dv$

$$= \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 (2x + y) dxdydz$$

$$= \int_{-1}^{1}\int_{-1}^{1}\left[\left(2\frac{x^2}{2}+yx\right)\right]_{-1}^{1} dydz$$

$$= \int_{-1}^{1}\int_{-1}^{1}[(1+y)-(1-y)]dydz$$

$$= \int_{-1}^{1}\int_{-1}^{1}[2y]dydz$$

$$= \int_{-1}^{1}\left(2\frac{y^2}{2}\right)_{-1}^{1} dz$$

$$= \int_{-1}^{1}[(1)-((-1)^2)]\, dz$$

$$= \int_{-1}^{1}[0]\, dz$$

$$= 0$$

Now, L.H.S $= \iint_S \vec{F}\cdot\hat{n}\, ds = \iint_{S1} + \iint_{S2} + \iint_{S3} + \iint_{S4} + \iint_{S5} + \iint_{S6}$

Faces	**Plane**	$\boldsymbol{dS}$	$\hat{\boldsymbol{n}}$	$\vec{\boldsymbol{F}}\cdot\hat{\boldsymbol{n}}$	**Equation**	$\vec{\boldsymbol{F}}\cdot\hat{\boldsymbol{n}}$ **on S**	$= \iint_S \vec{\boldsymbol{F}}\cdot\hat{n}\,\boldsymbol{ds}$
S_1(Bottom)	xy	dxdy	$-\vec{k}$	$-yz$	$z=-1$	y	$\int_{-1}^{1}\int_{-1}^{1} y\, dxdy$
S_2(Top)	xy	dxdy	$\vec{k}$	yz	$z=1$	y	$\int_{-1}^{1}\int_{-1}^{1} y\, dxdy$
$S_3(Left)$	xz	dxdz	$-\vec{j}$	$-z$	$y=-1$	$-z$	$\int_{-1}^{1}\int_{-1}^{1} -z\, dxdz$
$S_4(Right)$	xz	dxdz	$\vec{j}$	z	$y=1$	z	$\int_{-1}^{1}\int_{-1}^{1} zdxdz$
$S_5(Back)$	yz	dydz	$-\vec{i}$	$-x^2$	$x=-1$	-1	$\int_{-1}^{1}\int_{-1}^{1} -1\, dydz$
$S_6(Front)$	yz	dydz	$\vec{i}$	x^2	$x=1$	1	$\int_{-1}^{1}\int_{-1}^{1} dydz$

$(i) \iint_{S1} \vec{F}\cdot\hat{n}\, ds + \iint_{S2} \vec{F}\cdot\hat{n}\, ds = \int_{-1}^{1}\int_{-1}^{1} y\, dxdy + \int_{-1}^{1}\int_{-1}^{1} y\, dxdy$

$$= \int_{-1}^{1}\int_{-1}^{1} 2y\, dxdy$$

$$= 2\int_{-1}^{1}[xy]_{-1}^{1}\, dy$$

$$= 2\int_{-1}^{1}[(y)-(-y)]\, dy$$

$$= 2\int_{-1}^{1} 2ydy$$

$$= 4\left[\frac{y^2}{2}\right]_{-1}^{1} = 4\left[\left(\frac{1}{2}\right)-\left(\frac{1}{2}\right)\right] = 0$$

$(ii) \iint_{S3} \vec{F}\cdot\hat{n}\, ds + \iint_{S4} \vec{F}\cdot\hat{n}\, ds = \int_{-1}^{1}\int_{-1}^{1} -z\, dxdz + \int_{-1}^{1}\int_{-1}^{1} z\, dxdz$

$$= \int_{-1}^{1}\int_{-1}^{1} 0\, dxdz$$

$$= 0$$

$$(iii) \iint_{S5} \vec{F} \cdot \hat{n}\, ds + \iint_{S6} \vec{F} \cdot \hat{n}\, ds = -\int_{-1}^{1}\int_{-1}^{1} dxdz + \int_{-1}^{1}\int_{-1}^{1} dxdz$$

$$= 0$$

$$\therefore \iint_{S} \vec{F} \cdot \hat{n}\, ds = \iint_{S1} + \iint_{S2} + \iint_{S3} + \iint_{S4} + \iint_{S5} + \iint_{S6}$$

$$= (i) + (ii) + (iii)$$

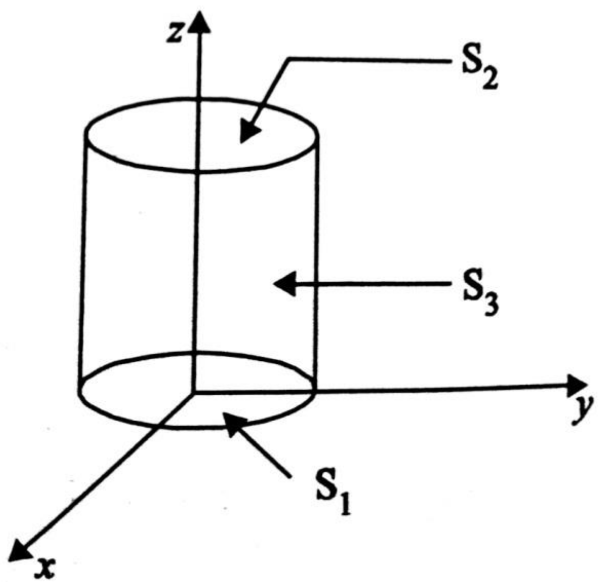
$$= 0$$

$$\therefore \iint_{S} \vec{F} \cdot \hat{n}\, ds = \iiint_{V} \nabla \cdot \vec{F}\, dv$$

Hence , Gauss divergence theorem is verified.

Example: 2.87 Verify divergence theorem for the function $\vec{F} = 4x\vec{i} - 2y^2\vec{j} + z^2\vec{k}$ taken over the surface bounded by the cylinder $x^2 + y^2 = 4$ and $z = 0, z = 3$.

Solution:



Gauss divergence theorem is $\iint_{S} \vec{F} \cdot \hat{n}\, ds = \iiint_{V} \nabla \cdot \vec{F}\, dv$

Given $\vec{F} = 4x\vec{i} - 2y^2\vec{j} + z^2\vec{k}$

$$\nabla \cdot \vec{F} = 4 - 4y + 2z$$

Limits:

$z = 0$ to 3

$x^2 + y^2 = 4 \Rightarrow y^2 = 4 - x^2$

$$\Rightarrow y = \pm\sqrt{4 - x^2}$$

$\therefore y = -\sqrt{4 - x^2}$ to $\sqrt{4 - x^2}$

Put $y = 0 \Rightarrow x^2 = 4$

$$\Rightarrow \; x = \pm 2$$

$\therefore y = \; -2 \; to \; 2$

$$\therefore \text{R.H.S} = \iiint_V \nabla \cdot \vec{F} \, dv$$

$$= \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_0^3 (4 - 4y + 2z) \, dzdydx$$

$$= \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \left[4z - 4yz + 2\frac{z^2}{2}\right]_0^3 dydx$$

$$= \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (12 - 12y + 9) dydx$$

$$= \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (21 - 12y) dydx$$

$$= 2 \int_{-2}^{2} \int_0^{\sqrt{4-x^2}} 21 \, dydx \qquad \left[\begin{array}{c} \because \int_{-a}^{a} f(x) \, dx = 2 \int_0^a f(x) \; if \; f(x) \; is \; even \\ = 0 \; if \; f(x) \; is \; odd \end{array}\right]$$

$$= 42 \int_{-2}^{2} [y]_0^{\sqrt{4-x^2}} dx$$

$$= 42 \int_{-2}^{2} \sqrt{4 - x^2} \; dx$$

$$= 42 \times 2 \int_0^2 \sqrt{4 - x^2} \, dx \qquad [\because even \; function]$$

$$= 84 \left[\frac{x}{2}\sqrt{4 - x^2} + \frac{4}{2} sin^{-1}\frac{x}{2}\right]_0^2$$

$$= 84 \, [0 + 2 \, sin^{-1}(1)]$$

$$= 84 \left[2 \times \frac{\pi}{2}\right]$$

$$= 84 \, \pi$$

$$\text{L.H.S} = \iint_S \vec{F} \cdot \hat{n} \, ds$$

$$= \iint_{S1} + \iint_{S2} + \iint_{S3}$$

Along S_1 (bottom):

xy −plane $\Rightarrow z = 0, dz = 0$

And $ds = dxdy, \;\; \hat{n} \; = \; -\vec{k}$

$$\therefore \vec{F} \cdot \hat{n} \; = \left(4x \, \vec{\imath} - \; 2y^2 \, \vec{\jmath} + \; z^2 \, \vec{k}\right) \cdot \left(-\vec{k}\right)$$

$$= \; -z^2 = 0$$

$$\therefore \iint_{S1} \vec{F} \cdot \hat{n} \, ds = \iint_{S1} 0 = 0$$

Along S_2 (top):

xy −plane $\Rightarrow z = 3, dz = 0$

And $ds = dxdy, \;\; \hat{n} \; = \; \vec{k}$

$$\therefore \vec{F} \cdot \hat{n} \; = \left(4x \, \vec{\imath} - \; 2y^2 \, \vec{\jmath} + \; z^2 \, \vec{k}\right) \cdot \left(-\vec{k}\right)$$

$$= \; z^2 = 9$$

$$\therefore \iint_{S2} \vec{F} \cdot \hat{n}\, ds = \iint_{S2} 9\, dxdy$$

$$= \iint_{R} 9\, dxdy$$

$$= 9\,(Area\ of\ the\ circle)$$

$$= 9\,(\pi r^2) \qquad\qquad [\because r = 2]$$

$$= 36\,\pi$$

Along S_3 (curved surface):

Given $x^2 + y^2 = 4$

Let $\varphi = x^2 + y^2 - 4$

$$\nabla\varphi = \vec{\imath}\frac{\partial\varphi}{\partial x} + \vec{\jmath}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

$$= 2x\vec{\imath} + 2y\vec{\jmath}$$

$$|\nabla\varphi| = \sqrt{4x^2 + 4y^2} = 2\sqrt{4} = 4$$

$$\hat{n} = \frac{\nabla\varphi}{|\nabla\varphi|} = \frac{2(x\vec{\imath}+y\vec{\jmath})}{4}$$

$$= \frac{x\vec{\imath}+y\vec{\jmath}}{2}$$

The cylindrical coordinates are

$x = 2\cos\theta,\ y = 2\sin\theta\ ds = 2dzd\theta$

Where z varies from 0 to 3

θ varies from 0 to 2π

Now $\vec{F} \cdot \hat{n} = \left(4x\,\vec{\imath} - 2y^2\,\vec{\jmath} + z^2\,\vec{k}\right) \cdot \left(\frac{x\vec{\imath}+y\vec{\jmath}}{2}\right)$

$$= 2x^2 - y^3$$

$$= 2(2\cos\theta)^2 - (2\sin\theta)^3$$

$$= 8\cos^2\theta - 8\sin^3\theta$$

$$= 8\left[\frac{1+\cos 2\theta}{2} - \left(\frac{3\sin\theta - \sin 3\theta}{4}\right)\right]$$

$$\therefore \iint_{S3} \vec{F} \cdot \hat{n}\, ds = 8\int_0^{2\pi}\int_0^3 \left(\frac{1}{2} + \frac{\cos 2\theta}{2} - \frac{3\sin\theta}{4} + \frac{\sin 3\theta}{4}\right) 2dzd\theta$$

$$= 16\int_0^{2\pi} \left(\frac{1}{2} + \frac{\cos 2\theta}{2} - \frac{3\sin\theta}{4} + \frac{\sin 3\theta}{4}\right) [z]_0^3\, d\theta$$

$$= 48\left[\frac{\theta}{2} + \frac{\sin 2\theta}{4} - \frac{3\cos\theta}{4} - \frac{\cos 3\theta}{12}\right]_0^{2\pi}$$

$$= 48\left[\left(\frac{2\pi}{2} + \frac{3}{4} - \frac{1}{12}\right) - \left(\frac{3}{4} - \frac{1}{12}\right)\right]$$

$$= 48\,\pi$$

L.H.S = $\iint_S \vec{F} \cdot \hat{n}\, ds = 0 + 36\,\pi + 48\pi$

$= 84\,\pi$

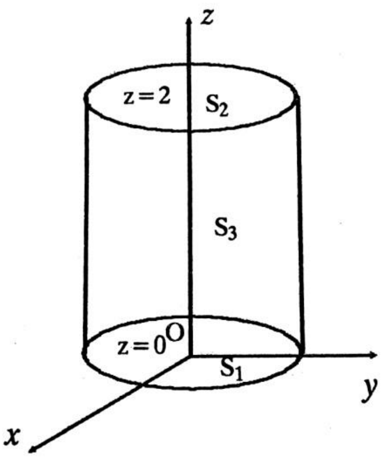
$\therefore$ L.H.S = R.H.S

$(i.e) \iint_S \vec{F} \cdot \hat{n}\, ds = \iiint_V \nabla \cdot \vec{F}\, dv$

Hence Gauss divergence theorem is verified.

Example: 2.88 Verify divergence theorem for the function $\vec{F} = y\vec{i} + x\vec{j} + z^2\vec{k}$ over the cylindrical region bounded by $x^2 + y^2 = 9$ and $z = 0, z = 2$.

Solution:



Gauss divergence theorem is $\iint_S \vec{F} \cdot \hat{n}\, ds = \iiint_V \nabla \cdot \vec{F}\, dv$

Given $\vec{F} = y\vec{i} + x\vec{j} + z^2\vec{k}$

$\nabla \cdot \vec{F} = 2z$

Limits:

$z = 0\ to\ 2$

$x^2 + y^2 = 9 \Rightarrow y^2 = 9 - x^2$

$\Rightarrow y = \pm\sqrt{9 - x^2}$

$\therefore y = -\sqrt{9 - x^2}\ to\ \sqrt{9 - x^2}$

Put $y = 0 \Rightarrow x^2 = 9$

$\Rightarrow x = \pm 3$

$\therefore y = -3\ to\ 3$

$\therefore$ R.H.S $= \iiint_V \nabla \cdot \vec{F}\, dv$

$= \int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_0^2 (2z)\, dzdydx$

$= \int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \left[2\frac{z^2}{2}\right]_0^2 dydx$

$$= 4\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} dydx$$

$$= 4\int_{-3}^{3}[y]_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\, dx$$

$$= 4\int_{-3}^{3} 2\sqrt{9-x^2}\; dx$$

$$= 8\times 2\int_{0}^{3}\sqrt{9-x^2}\, dx \qquad [\because even\ function]$$

$$= 16\left[\frac{x}{2}\sqrt{9-x^2}+\frac{9}{2}sin^{-1}\frac{x}{3}\right]_0^3$$

$$= 16\left[0+\frac{9}{2}\, sin^{-1}(1)\right]$$

$$= 16\left[\frac{9}{2}\times\frac{\pi}{2}\right]$$

$$= 36\,\pi$$

L.H.S $= \iint_S \vec{F}\cdot\hat{n}\, ds$

$$= \iint_{S1} + \iint_{S2} + \iint_{S3}$$

Along S_1 (bottom):

xy −plane $\Rightarrow z = 0, dz = 0$

And $ds = dxdy,\ \hat{n} = -\vec{k}$

$$\therefore \vec{F}\cdot\hat{n} = \left(y\,\vec{i}+x\,\vec{j}+z^2\,\vec{k}\right)\cdot\left(-\vec{k}\right)$$

$$= -z^2 = 0$$

$$\therefore \iint_{S1}\vec{F}\cdot\hat{n}\, ds = \iint_{S1} 0 = 0$$

Along S_2 (top):

xy −plane $\Rightarrow z = 2, dz = 0$

And $ds = dxdy,\ \hat{n} = \vec{k}$

$$\therefore \vec{F}\cdot\hat{n} = \left(y\,\vec{i}+x\,\vec{j}+z^2\,\vec{k}\right)\cdot\left(\vec{k}\right)$$

$$= z^2 = 4$$

$$\therefore \iint_{S2}\vec{F}\cdot\hat{n}\, ds = \iint_{S2} 4\, dxdy$$

$$= \iint_{R} 4\, dxdy$$

$$= 4\,(Area\ of\ the\ circle)$$

$$= 4\,(\pi r^2) \qquad [\because r = 2]$$

$$= 36\,\pi$$

Along S_3 (curved surface):

Given $x^2 + y^2 = 9$

Let $\varphi = x^2 + y^2 - 9$

$$\nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

$$= 2x\vec{i} + 2y\vec{j}$$

$$|\nabla\varphi| = \sqrt{4x^2 + 4y^2} = 2\sqrt{9} = 6$$

$$\hat{n} = \frac{\nabla\varphi}{|\nabla\varphi|} = \frac{2(x\vec{i} + y\vec{j})}{6}$$

$$= \frac{x\vec{i} + y\vec{j}}{3}$$

The cylindrical coordinates are

$x = 3\cos\theta,\ y = 3\sin\theta$

$ds = 3dzd\theta$

Where z varies from 0 to 2

θ varies from 0 to 2π

Now $\vec{F} \cdot \hat{n} = \left(y\,\vec{i} + x\,\vec{j} + z^2\,\vec{k}\right) \cdot \left(\frac{x\vec{i} + y\vec{j}}{3}\right)$

$$= \frac{xy}{3} + \frac{xy}{3} = \frac{2xy}{3}$$

$$= \frac{2}{3}(3\cos\theta)\,(3\sin\theta)$$

$$= 2 \times 3\sin\theta\cos\theta$$

$$= 3\sin 2\theta$$

$$\therefore \iint\limits_{S3} \vec{F} \cdot \hat{n}\,ds = 3\int_0^{2\pi}\int_0^{2}(\sin 2\theta)\,3dzd\theta$$

$$= 9\int_0^{2\pi}(\sin 2\theta)\,[z]_0^2\,d\theta$$

$$= 9\left[-\frac{\cos 2\theta}{2}\right]_0^{2\pi}$$

$$= -9\,[1 - 1]$$

$$= 0$$

$$\text{L.H.S} = \iint\limits_{S} \vec{F} \cdot \hat{n}\,ds = 0 + 36\,\pi + 0$$

$$= 36\,\pi$$

$\therefore$ L.H.S = R.H.S

$$(i.e) \iint\limits_{S} \vec{F} \cdot \hat{n}\,ds = \iiint\limits_{V} \nabla \cdot \vec{F}\,dv$$

Hence Gauss divergence theorem is verified.

Example: 2.89 If S is any closed surface enclosing a volume V and if $\vec{F} = ax\vec{i} + by\vec{j} + cz\vec{k}$, prove that $\iint_S \vec{F} \cdot \hat{n}\, ds = (a+b+c)V$. Deduce that $\iint_S \vec{F} \cdot \hat{n}\, ds = \frac{4\pi}{3}(a+b+c)$ if S is the surface of the sphere $x^2 + y^2 + z^2 = 1$.

Solution:

Gauss divergence theorem is $\iint_S \vec{F} \cdot \hat{n}\, ds = \iiint_V \nabla \cdot \vec{F}\, dv$

Given $\vec{F} = ax\vec{i} + by\vec{j} + cz\vec{k}$

$$\nabla \cdot \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = a + b + c$$

$$\therefore \iint_S \vec{F} \cdot \hat{n}\, ds = \iiint_V (\mathrm{a+b+c})\, dv$$

$$= (a+b+c)V$$

If S is the surface of the sphere $x^2 + y^2 + z^2 = 1$ then $V = \frac{4}{3}\pi(1)^3 = \frac{4\pi}{3}$

$$\therefore \iint_S \vec{F} \cdot \hat{n}\, ds = (\mathrm{a+b+c})\, \frac{4\pi}{3}$$

$$= \frac{4\pi}{3}\,(a+b+c)$$

Example: 2.90 Using the divergence theorem of Gauss evaluate $\iint_S \vec{F} \cdot \hat{n}\, ds$ where $\vec{F} = x^3\vec{i} + y^3\vec{j} + z^3\vec{k}$, and S is the sphere $x^2 + y^2 + z^2 = a^2$.

Solution:

Gauss divergence theorem is $\iint_S \vec{F} \cdot \hat{n}\, ds = \iiint_V \nabla \cdot \vec{F}\, dv$

Given $\vec{F} = x^3\vec{i} + y^3\vec{j} + z^3\vec{k}$

$$\nabla \cdot \vec{F} = 3x^2 + 3y^2 + 3z^2$$

$$= 3(x^2 + y^2 + z^2)$$

$$\therefore \iint_S \vec{F} \cdot \hat{n}\, ds = 3\iiint_V (x^2 + y^2 + z^2)\, dxdydz$$

Here we have to use spherical polar co – ordinates.

$x = r\sin\theta\cos\varphi\,,\ y = r\sin\theta\sin\varphi,\ \ z = r cos\,\theta$

$x^2 + y^2 + z^2 = a^2$ and $dxdydz = r^2 sin\,\theta\, dr\, d\theta\, d\varphi$

$$\therefore 3\iiint_V (x^2 + y^2 + z^2)\, dxdydz = 3\int_0^{2\pi}\int_0^{\pi}\int_0^{a} r^2 r^2 \sin\theta\, dr\, d\theta\, d\varphi$$

$$= 3\int_0^{2\pi}\int_0^{\pi}\left[\frac{r^5}{5}\, sin\,\theta\right]_0^a d\theta\, d\varphi$$

$$= \frac{3a^5}{5}\int_0^{2\pi}[-\cos\theta]_0^{\pi}\, d\varphi$$

$$= \frac{3a^5}{5}\int_0^{2\pi}(-\cos\pi + \cos 0)\, d\varphi$$

$$= \frac{6a^5}{5}[\varphi]_0^{2\pi}$$

$$= \frac{6a^5}{5}(2\pi) = \frac{12\pi a^5}{5}$$

Example: 2.91 Show that $\iint_S \boldsymbol{curl\,\vec{F}} \cdot \hat{n}\,\boldsymbol{ds} = \boldsymbol{0}$ where S is any closed surface.

Solution:

Gauss divergence theorem is $\iint_S \vec{F} \cdot \hat{n}\,ds = \iiint_V \nabla \cdot \vec{F}\,dv$

$$\therefore \iint_S \vec{F} \cdot \hat{n}\,ds = \iiint_V \nabla \cdot \left(\nabla \times \vec{\mathrm{F}}\right) dv$$

where V is the volume of the closed surface S.

Since $\nabla \cdot \left(\nabla \times \vec{F}\right) = 0, we\ get \iiint_V \nabla \cdot \left(\nabla \times \vec{\mathrm{F}}\right) dv = 0$

$$\therefore \iint_S \left(\nabla \times \vec{\mathrm{F}}\right) \cdot \hat{n}\ dv = 0\ (or) \iint_S curl\vec{F} \cdot \hat{n}\,ds = 0$$

Example: 2.92 Prove that $\iint_S \frac{\vec{r}\cdot\hat{n}}{r^2}\boldsymbol{ds} = \iiint_V \frac{\mathrm{dv}}{\mathrm{r}^2}$

Solution:

Gauss divergence theorem is $\iint_S \vec{F} \cdot \hat{n}\,ds = \iiint_V \nabla \cdot \vec{F}\,dv$

$$\therefore \iint_S \frac{\vec{r}}{r^2} \cdot \hat{n}\,ds = \iiint_V \nabla \cdot \left(\frac{\vec{r}}{r^2}\right) dv$$

Now $\nabla \cdot \frac{\vec{r}}{r^2} = \left(\vec{\imath}\frac{\partial}{\partial x} + \vec{\jmath}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \cdot \left(\frac{x\vec{\imath}+y\vec{\jmath}+z\vec{k}}{r^2}\right)$

$$= \frac{\partial}{\partial x}\left(\frac{x}{r^2}\right) + \frac{\partial}{\partial y}\left(\frac{y}{r^2}\right) + \frac{\partial}{\partial z}\left(\frac{z}{r^2}\right)$$

$$= \sum \frac{r^2(1) - x\,2r\,\frac{\partial r}{\partial x}}{r^4}$$

$$= \sum \frac{r^2 - 2xr\left(\frac{x}{r}\right)}{r^4}$$

$$= \sum \frac{r^2 - 2x^2}{r^4}$$

$$= \frac{3r^2 - 2(x^2+y^2+z^2)}{r^4}$$

$$= \frac{3r^2 - 2r^2}{r^4} = \frac{r^2}{r^4} = \frac{1}{r^2}$$

$$\therefore \iint_S \frac{\vec{r}}{r^2} \cdot \hat{n}\,ds = \iiint_V \frac{1}{r^2}\,dv$$

Example: 2.93 Evaluate $\iint_S \vec{\boldsymbol{r}} \cdot \hat{n}\,\boldsymbol{ds}$ where S is a closed surface using Gauss divergence theorem.

Solution:

Gauss divergence theorem is $\iint_S \vec{F} \cdot \hat{n}\, ds = \iiint_V \nabla \cdot \vec{F}\, dv$

$$\therefore \iint_S \vec{r} \cdot \hat{n}\, ds = \iiint_V (\nabla \cdot \vec{r})\, dv$$

$$= \iiint_V \left[\nabla \cdot \left(x\vec{i} + y\vec{j} + z\vec{k}\right)\right] dv$$

$$= \iiint_V (1 + 1 + 1)\, dv$$

$$= 3\iiint_V dv$$

$$= 3V$$

Exercise: 2.5

1. Verify divergence theorem for the function $\vec{F} = (x^2 - yz)\vec{i} - (y^2 - zx)\vec{j} + (z^2 - xy)\vec{k}$ over the surface bounded by $x = 0, x = 1, y = 0, y = 2, z = 0, z = 3$ **Ans:** 36
2. Verify divergence theorem for the function $\vec{F} = 4xz\vec{i} - y^2\vec{j} + yz\vec{k}$ over the cube $x = 0, x = 1, y = 0, y = 1, z = 0, z = 1$ **Ans:** Common value $= \frac{3}{2}$
3. Verify divergence theorem for the function $\vec{F} = (2x - z)\vec{i} - x^2y\vec{j} - xz^2\vec{k}$ over the cube bounded by $x = 0, x = 1, y = 0, y = 1, z = 0, z = 1$ **Ans:** Common value $= \frac{11}{6}$
4. Verify divergence theorem for $\vec{F} = xy^2\vec{i} + yz^2\vec{j} + zx^2\vec{k}$ over the region $x^2 + y^2 = 4$ and $z = 0, z = 3$ **Ans:** Common value $= 84\pi$
5. Using divergence theorem, prove that (i) $\iint_S \vec{R} \cdot d\vec{S} = 3V$ (ii) $\iint_S \nabla r^2 \cdot d\vec{S} = 6V$
6. $\vec{F} = x^2\vec{i} + z\vec{j} + yz\vec{k}$ over the cube bounded by $x = 0, x = a, y = 0, y = a, z = 0, z = a$ **Ans:** Common value $= \frac{3a}{2}$
7. $\vec{F} = (x^3 - yz)\vec{i} - 2x^2y\vec{j} + 2\vec{k}$ over the parallelopiped bounded by the planes $x = 0, x = 1, y = 0, y = 2, z = 0, z = 3$ **Ans:** Common value $= 2$
8. $\vec{F} = 2xy\vec{i} + y\,z^2\vec{j} + xz\vec{k}$ over the parallelopiped bounded by the planes $x = 0, y = 0, z = 0, x = 2, y = 1, z = 3$ **Ans:** Common value $= 20$
9. $\vec{F} = 2x^2y\vec{i} - y^2\vec{j} + 4\,xz^2\vec{k}$ taken over the region in the first octant bounded $x^2 + y^2 = 9$ and $x = 2$ **Ans:** Common value $= 180$
10. $\vec{F} = x^2\vec{i} + y^2\vec{j} + z^2\vec{k}$ taken over the cuboid formed by the planes $x = 0, x = a, y = 0, y = b, z = 0, z = c$ **Ans:** Common value $= abc(a + b + c)$