Determine the impulse response h(t) of the following system. Assume zero initial conditions.

$$\frac{d^2y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 3\,y(t) = \frac{dx(t)}{dt} + 2\,x(t)$$

On taking Laplace transform with zero initial condition we get,

$$s^2\,Y(s) + 4s\,Y(s) + 3\,Y(s) = s\,X(s) + 2\,X(s)$$

$$(s^2 + 4s + 3)\,Y(s) = [s + 2]\,X(s)$$

Input, $x(t) = \delta(t)$

$$\therefore X(s) = \mathcal{L}\{x(t)\} = \mathcal{L}\{\delta(t)\} = 1$$

When the input is impulse, the output is denoted by h(t). Let $\mathcal{L}\{h(t)\} = H(s)$.

$$\therefore (s^2 + 4s + 3)\,Y(s) = (s + 2)\,X(s) \;\Rightarrow\; (s^2 + 4s + 3)\,H(s) = s + 2$$

$$\boxed{Y(s) = H(s) \;\;;\;\; X(s) = 1}$$

$$\therefore H(s) = \frac{s + 2}{s^2 + 4s + 3} = \frac{s + 2}{(s + 1)(s + 3)}$$

The roots of quadratic $s^2 + 4s + 3 = 0$ are,

$$s = \frac{-4 \pm \sqrt{4^2 - 4 \times 3}}{2} = \frac{-4 \pm 2}{2} = -1, -3$$

By partial fraction expansion technique, H(s) can be expressed as,

$$H(s) = \frac{s + 2}{(s + 1)(s + 3)} = \frac{A}{s + 1} + \frac{B}{s + 3}$$

$$A = \frac{s + 2}{(s + 1)(s + 3)} \times (s + 1)\bigg|_{s = -1} = \frac{-1 + 2}{-1 + 3} = \frac{1}{2} = 0.5$$

$$B = \frac{s + 2}{(s + 1)(s + 3)} \times (s + 3)\bigg|_{s = -3} = \frac{-3 + 2}{-3 + 1} = \frac{1}{2} = 0.5$$

$$\therefore H(s) = 0.5\frac{1}{s + 1} + 0.5\frac{1}{s + 3} = 0.5\left(\frac{1}{s + 1} + \frac{1}{s + 3}\right)$$

$$\therefore \text{Impulse response,}\quad h(t) = \mathcal{L}^{-1}\{H(s)\} = \mathcal{L}^{-1}\left\{0.5\left(\frac{1}{s + 1} + \frac{1}{s + 3}\right)\right\}$$

$$= 0.5\left[\mathcal{L}^{-1}\left\{\frac{1}{s + 1}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s + 3}\right\}\right]$$

$$\boxed{\mathcal{L}\left\{e^{-at}\,u(t)\right\} = \frac{1}{s + a}}$$

$$= 0.5\left[e^{-t}\,u(t) + e^{-3t}\,u(t)\right] = 0.5\left(e^{-t} + e^{-3t}\right)u(t)$$

Poles and Zeros of Rational Function of s

Let X(s) be Laplace transform of x(t).

When X(s) is expressed as a ratio of two polynomials in s, then the s-domain signal X(s) is called a rational function of s.

The zeros and poles are two critical complex frequencies at which a rational function of s takes two extreme values, such as zero and infinity respectively.

Let X(s) is expressed as a ratio of two polynomials in s as shown in equation

X(s) = P(s)/ Q(s)

$$= \frac{b_0 s^M + b_1 s^{M-1} + b_2 s^{M-2} + \ldots\ldots + b_{M-1} s + b_M}{a_0 s^N + a_1 s^{N-1} + a_2 s^{N-2} + \ldots\ldots + a_{N-1} s + a_N}$$

where, P(s) = Numerator polynomial of X(s) Q(s) = Denominator polynomial of X(s)

let us scale the coefficients of numerator polynomial by b0 and the coefficients of denominator polynomial by a0 , and the equation can be expressed in factorized form as shown in equation

$$X(S) = \frac{b_0\left(s^M + \frac{b_1}{b_0} s^{M-1} + \frac{b_2}{b_0} s^{M-2} + \ldots\ldots + \frac{b_{M-1}}{b_0} s + \frac{b_M}{b_0}\right)}{a_0\left(s^N + \frac{a_1}{a_0} s^{N-1} + \frac{a_2}{a_0} s^{N-2} + \ldots\ldots + \frac{a_{N-1}}{a_0} s + \frac{a_N}{a_0}\right)}$$

$$= G \frac{(s - z_1)(s - z_2) \ldots\ldots (s - z_M)}{(s - p_1)(s - p_2) \ldots\ldots (s - p_N)}$$

$$\text{where, } G = \frac{b_0}{a_0} = \text{Scaling factor}$$

eqn 1

z1 , z2 , ..., zM = Roots of numerator polynomial,
P(s) p1 , p2 , ..., pN = Roots of denominator polynomial.

In equation 1, if the value of s is equal to any one of the root of numerator polynomial then the signal X(s) will become zero.

Therefore the roots of numerator polynomial z1 , z2 , ..., zM are called zeros of X(s).

Since s is complex frequency, the zeros can be defined as values of complex frequencies at which the signal X(s) becomes zero.

In equation 1, if the value of s is equal to any one of the roots of the denominator polynomial then the signal X(s) will become infinite.

Therefore the roots of denominator polynomial p1 , p2 , ..., pN are called poles of X(s).

Since s is complex frequency, the poles can be defined as values of complex frequencies at which the signal X(s) become infinite.

Since the signal X(s) attains infinte value at poles, the ROC of X(s) does not include poles.

$$X(s) = \frac{(s+2)(s+5)}{s(s^2+6s+13)}$$

The roots of quadratic $s^2 + 6s + 13 = 0$ are,

$$s = \frac{-6 \pm \sqrt{36 - 4 \times 13}}{2} = \frac{-6 \pm j4}{2} = -3 + j\,2, \quad -3 - j\,2$$

$$\therefore s^2 + 6s + 13 = (s + 3 - j2)(s + 3 + j2)$$

$$\therefore X(s) = \frac{(s+2)(s+5)}{s(s^2+6s+13)} = \frac{(s+2)(s+5)}{s(s+3-j2)(s+3+j2)}$$

Stability in s-Domain

ROC of a Stable LTI System

Location of Poles for Stability of Causal Systems

Let h(t) be impulse response of an LTI causal system.

$$\boxed{\int_0^{\infty} h(t)\,dt < \infty}$$

The transfer function of a continuous time system is given by Laplace transform of its impulse response.

Let, $h(t) = e^{at}\,u(t)$

$$\therefore \text{Transfer function, } H(s) = \mathcal{L}\{h(t)\} = \mathcal{L}\left\{e^{at}\,u(t)\right\} = \frac{1}{s - a}$$

Here, the transfer function H(s) has pole at s = a.

If, a < 0, (i.e., if a is negative), then the pole will lie on the left half of s-plane, and from equation (3.46) we can say that the causal system is stable.

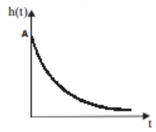
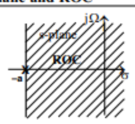
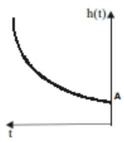
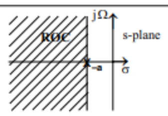
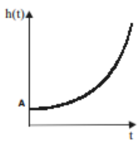
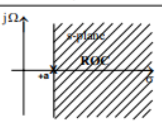
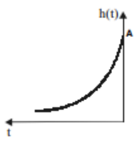
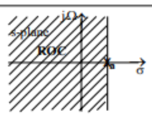
If, a > 0, (i.e., if a is positive), then the pole will lie on the right half of s-plane. and from equation (3.47) we can say that the causal system is unstable.

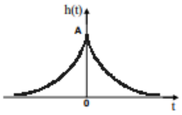
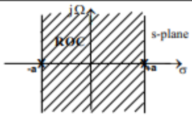
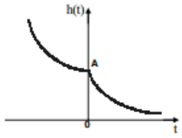
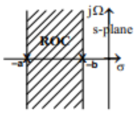
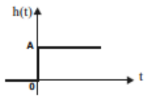
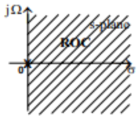
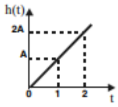
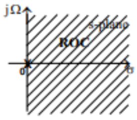
Therefore we can say that, for a stable LTI continuous time causal system the poles should lie on the left half of s-plane.

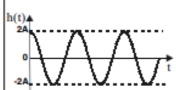
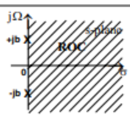
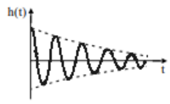
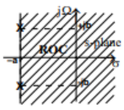
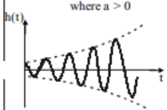
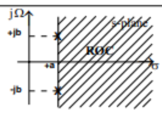
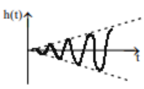
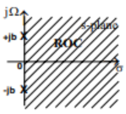
General Condition for Stability in s-Plane On combining the condition for location of poles and the ROC we can say that,

1. For a stable LTI continuous time causal system, the poles should lie on the left half of s-plane and the imaginary axis should be included in the ROC.

2. For a stable LTI continuous time noncausal system, the imaginary axis should be included in the ROC.

Impulse response h(t)	Transfer function $H(s) = \mathcal{L}\{h(t)\}$	Location of poles in s-plane and ROC
$h(t) = A\, e^{-at}\, u(t)\ ;\ a > 0$ 	$H(s) = \frac{A}{s+a}$ Pole at s = –a. ROC is $\sigma > -a$, where σ is real part of s.	 The pole at s = –a, lies on left half of s-plane. ROC includes imaginary axis. Causal system. Since pole lies on LHP and the imaginary axis is included in ROC, the system is stable.
$h(t) = A\, e^{-at}\, u(-t)\ ;\ a > 0$ 	$H(s) = -\frac{A}{s+a}$ Pole at s = –a. ROC is $\sigma < -a$, where σ is real part of s.	 The pole at s = –a, lies on left half of s-plane. The ROC does not include imaginary axis. Noncausal system. Since imaginary axis is not included in ROC, the system is unstable.
$h(t) = A\, e^{at}\, u(t)\ ;\ a > 0$ 	$H(s) = \frac{A}{s-a}$ Pole at s = +a. ROC is $\sigma > +a$, where σ is real part of s.	 The pole at s = +a, lies on right half of s-plane. ROC does not include imaginary axis. Causal system. Since pole lies on RHP and imaginary axis is not included in ROC, the system is unstable.
$h(t) = A\, e^{at}\, u(-t)\ ;\ a > 0$ 	$H(s) = -\frac{A}{s-a}$ Pole at s = +a. ROC is $\sigma < +a$, where σ is real part of s.	 The pole at s = +a, lies on right half of s-plane. The ROC includes imaginary axis. Noncausal system. Since imaginary axis is included in ROC, the system is stable.

Impulse response h(t)	Transfer function $H(s) = \mathcal{L}\{h(t)\}$	Location of poles in s-plane and ROC
$h(t) = A\,e^{-a\|t\|}$; $a > 0$ 	$H(s) = \frac{A}{s+a} - \frac{A}{s-a}$ Poles at $s = -a, +a$. ROC is $-a < \sigma < +a$, where σ is real part of s.	 The pole at $s = +a$, lies on RHP and pole at $s = -a$, lies on LHP. ROC includes imaginary axis. Noncausal system. Since the imaginary axis is included in ROC, the system is stable.
$h(t) = A\,e^{-at}u(t) + A\,e^{-bt}u(-t)$ where $a > 0$, $b > 0$, $a > b$ 	$H(s) = \frac{A}{s+a} - \frac{A}{s+b}$ Poles at $s = -a, -b$. ROC is $-a < \sigma < -b$, where σ is real part of s.	 The poles at $s = -a, -b$, lie on LHP. The ROC does not include imaginary axis. Noncausal system. Since the imaginary axis is not included in ROC, the system is unstable.
$h(t) = A\,u(t)$ 	$H(s) = \frac{A}{s}$ Pole at $s = 0$. ROC is $\sigma > 0$, where σ is real part of s.	 The pole at $s = 0$ lies on imaginary axis. The ROC does not include the imaginary axis. Causal system. Since the imaginary axis is not included in ROC, the system is unstable.
$h(t) = A\,t\,u(t)$ 	$H(s) = \frac{A}{s^2}$ Double pole at $s = 0$. ROC is $\sigma > 0$, where σ is real part of s.	 The poles at $s = 0$ lie on imaginary axis. The ROC does not include the imaginary axis. Causal system. Since the imaginary axis is not included in ROC, the system is unstable.

Impulse response h(t)	Transfer function $H(s) = \mathcal{L}\{h(t)\}$	Location of poles in s-plane and ROC
$h(t) = 2A\cos bt\,u(t)$ 	$H(s) = \frac{A}{s+jb} + \frac{A}{s-jb}$ Poles at $s = -jb, +jb$. ROC is $\sigma > 0$, where σ is real part of s.	 The poles at $s = -jb, +jb$, lie on imaginary axis. The ROC does not include the imaginary axis. Causal system. Since the imaginary axis is not included in ROC, the system is unstable.
$h(t) = 2A\,e^{-at}\cos bt\,u(t)$, where $a > 0$ 	$H(s) = \frac{A}{s+a+jb} + \frac{A}{s+a-jb}$ Poles at $s = -a - jb, -a + jb$. ROC is $\sigma > -a$, where σ is real part of s.	 The poles at $s = -a-jb, -a+jb$, lie on left half of s-plane. The ROC includes the imaginary axis. Causal system. Since poles lie on LHP and the imaginary axis is included in ROC, the system is stable.
$h(t) = 2A\,e^{at}\cos bt\,u(t)$, where $a > 0$ 	$H(s) = \frac{A}{s-a+jb} + \frac{A}{s-a-jb}$ Poles at $s = a - jb, a + jb$. ROC is $\sigma > a$, where σ is real part of s.	 The poles at $s = a-jb, a+jb$, lie on right half of s-plane. The ROC does not include imaginary axis. Causal system. Since poles lie on RHP and the imaginary axis is not included in ROC, the system is unstable.
$h(t) = 2A\,t\cos bt\,u(t)$ 	$H(s) = \frac{A}{(s+jb)^2} + \frac{A}{(s-jb)^2}$ Double poles at $s = -jb, +jb$. ROC is $\sigma > 0$, where σ is real part of s.	 The poles at $s = -jb, +jb$, lie on imaginary axis. The ROC does not include the imaginary axis. Causal system. Since the imaginary axis is not included in ROC, the system is unstable.