Matrix Algegba

kgc

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1 Basic Notation

Matrx X_{mn} means X has m rows and n columns.
Square matrix(m == n):

$$K_{22} = \begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{pmatrix}$$

Rectangular matrix:

$$R_{23} = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \end{pmatrix}$$

A general matrix:

$$\mathrm{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

The main diagonal of a matrix A_{nm} is the entries $a_{ij}$where i == j.
Examples:

$$\mathrm{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mathrm{B} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \mathrm{C} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

2 Basic Algebra

Add by another matrix:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{pmatrix} \Rightarrow \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} & a_{13}+b_{13} \\ a_{21}+b_{21} & a_{22}+b_{22} & a_{23}+b_{23} \end{pmatrix} \tag{1}$$

Multiply by scaler:

$$\mathrm{k} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} \Rightarrow \begin{pmatrix} ka_{11} & ka_{12} & ka_{13} \\ ka_{21} & ka_{22} & ka_{23} \end{pmatrix} \tag{2}$$

Mulitply by another Matrix:

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{pmatrix} \Rightarrow \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} & a_{11}b_{13} + a_{12}b_{23} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} & a_{21}b_{23} + a_{22}b_{23} \end{pmatrix} \tag{3}$$

If $A_{mn} = B_{mk}C_{kn}$, for entry a_{ij} in A:

$$a_{ij} = \sum_{x=1}^{k} b_{ix}c_{xj}$$

Question: Prove the associative low for matrix multiplication. That is:

$$A_{mn}B_{nq}C_{qp} = A_{mn}(B_{nq}C_{qp})$$

Prove:

$$[ABC]_{ij} = \sum_{x=1}^{q} [AB]_{ix}c_{xj} = \sum_{x=1}^{q}\sum_{y=1}^{n} a_{iy}b_{yx}c_{xp} =^{*} \sum_{y=1}^{n} a_{iy}[BC]_{yj} = [A(BC)]_{ij}$$

* why this translation must be right

3 Some Special Matrixes

Zero Matrix: write as *0* Identify Matrix: write as I

$$AI = IA =$$

Diagonal Matrix: has only elements on the main diagonal, and zero everywhere else.

$$\begin{pmatrix} d1 & 0 & 0 \\ 0 & d2 & 0 \\ 0 & 0 & d3 \end{pmatrix}$$

Banded Matrix or called Tridiagonal Matrix: one main diagonal and two bands around it, up and down.

$$\begin{pmatrix} d1 & d2 & 0 & 0 \\ d3 & d4 & d5 & 0 \\ 0 & d6 & d7 & d8 \\ 0 & 0 & d9 & d10 \end{pmatrix}$$

Up Triangular Matrix:

$$\begin{pmatrix} d1 & d2 & d3 & d9 \\ 0 & d4 & d5 & d11 \\ 0 & 0 & d7 & d8 \\ 0 & 0 & 0 & d10 \end{pmatrix}$$

Low Triangular Matrix

$$\begin{pmatrix} d1 & 0 & 0 \\ d5 & d2 & 0 \\ d7 & d4 & d3 \end{pmatrix}$$

4 Transpose and Inverses

Transpose: entry a_{ij}^T in A^T is entry a_{ji} in A
Some equation:

- $(A^T)^T = A$
- $(A+B)^T = A^T + A^T$
- $(AB)^T = B^T A^T$

Some special matrices:

- symmetric matrix $A^T = A$
- skew-symmetric matrix$A^T = -A$
  So the diagonal elements must be **0**.

Inner Products & Outer Products: both involve to vectors

$$U = \begin{pmatrix} u1 \\ u2 \\ u3 \end{pmatrix} \quad V = \begin{pmatrix} v1 \\ v2 \\ v3 \end{pmatrix}$$

Inner Products:

$$V^T U = \begin{pmatrix} v1 & v2 & v3 \end{pmatrix} \begin{pmatrix} u1 \\ u2 \\ u3 \end{pmatrix} = v1 * u1 + v2 * u2 + v3 * u3 \tag{4}$$

Some relative concept:

- U and V is orthogonal if $U^T V = \mathbf{0}$
- norm.
  $$\|U\| = (U^T U)^{1/2} \tag{5}$$
  A vector is normalized if $\|U\| = 1$.
- If U and V are orthogonal and both are normalized, U and V are orthonormal.

5 Key Words

diagonal, commute under multiplication, associative low

6 Some proof

- $(AB)^T = B^T A^T$

$$
\begin{aligned}
|B^T A^T|_{ij} &= \sum_{k=1}^{n} b_{ik}^T a_{kj}^T \\
&= \sum_{k=1}^{n} b_{ki} ajk \\
&= \sum_{k=1}^{n} a_{jk} b_{kj} \\
&= |AB|_{ji} = |AB|_{ij}^T
\end{aligned}
\tag{6}
$$