Solving population growth equations in R

Discrete vs. continuous time

Discrete example

First we will consider **discrete-time** models. That is, models where time advances in chunks called **timesteps**. These models are often appropriate for systems with a strong biological rhythm, such as populations with a single annual birth pulse. Consider the logistic growth model:

$$N_{t+1} = N_t + rN_t(1 - \frac{N_t}{K}) \tag{0.1}$$

Or equivalently:

$$N_{t+1} = N_t(1 + r(1 - \frac{N_t}{K})) \tag{0.2}$$

In this second case, we've simply factored out the N_t from both terms on the right hand side.

Equation 0.1 shows how discrete time models update **recursively** at each successive timestep (N_{t+1}). They take the population size at the previous time step (N_t), and then add or subtract a time-specific growth rate, in this case, $rN_t(1 - \frac{N_t}{K})$. Note that the population growth rate changes through time as a function of population size.

Continuous comparison

Recall that in continuous time, the logistic growth model is given by the ODE:

$$\frac{dN}{dt} = rN(1 - \frac{N}{K}) \tag{0.3}$$

On your own

1. Identify the population growth rate in discrete time equation 0.1 and continuous time equation 0.3 (underline it).

Notice that the formula for **the growth rate is the same in both equations**. We can re-write equation 0.1 to define the growth rate by moving N_t to the left-hand side:

$$N_{t+1} - N_t = rN_t(1 - \frac{N_t}{K}) \tag{0.4}$$

Quite literally, this equation tells us that change in population size over a single time step (i.e., $N_{t+1} - N_t$) is equal to $rN_t(1 - \frac{N_t}{K})$. Thus, $rN_t(1 - \frac{N_t}{K})$ gives the rate of change over a single time step in this model.

Using the same logic, **to convert from continuous time to discrete time**:

1. For each state variable in the continuous time equation(s), replace the derivative, $\frac{dN}{dt}$ with a discrete time statement of the single-time-step rate of change: $N_{t+1} - N_t$.
2. Then solve for N_{t+1}, or the appropriate state variable. (i.e. for state variable N, move everything but N_{t+1} to the right hand side of the equation.

With a partner

Convert the simple SIR model to discrete time. Recall from last week that the SIR model is give by:

$$\frac{dS}{dt} = -\beta SI \tag{0.5}$$
$$\frac{dI}{dt} = \beta SI - \gamma I \tag{0.6}$$
$$\frac{dR}{dt} = \gamma I \tag{0.7}$$

Write your equations here:

Takeaways

- Discrete time models have pre-determined, time steps. These models consider the rate of population change only at the boundaries between time steps.
- Continuous time models have infinitely small time steps. Thus, these models tell us the population growth rate (slope) at any point in time. (Think back to the definition of the derivative from calculus).
- Discrete time models are easy to solve with for loops. Their equations typically return population size, rather than a derivative.
- Continuous time ODE models represent only the derivative (continuous rate of change) of each state variable. Solving these systems for actual population sizes usually requires an ODE solver (a function in R or other language that approximates the analytical solution using vanishingly small time steps).

Cautions

- If you choose too large a time step in discrete time models, you can encounter complicated oscillatory behavior or chaos.
- In continuous time models, remember that individual animals/humans/plants are discrete, so if your population size drops below 1 (or some small threshold), your population has gone extinct. Remember the "atto fox”!