uOttawa

CSI 3105 - Design and Analysis of Algorithms

Fall 2022 Section A

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Assignment 2

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1. (a) **Answer**

Proof by contradiction:

Assume there exists a directed graph that has two black holes.

Let $G = (V, E)$ be a directed graph with two black hole vertices, a and b. That means, a and b both have no outgoing edges. This is a contradiction, because if a does not have an outgoing edge, then b cannot be a black hole. By definition, b can only be a black hole if all other vertices has an outgoing edge that leads to b. Since a does not have an outgoing edge, a cannot lead to b.

Similarly, assume that there exists a directed graph with n number of black holes, where $n \geq 2$. That is, let graph $G = V, E$, where $V = \{a_1, a_2, ..., a_n, b_0, b_1, ...b_k\}$, $\{a_1, a_2, ...a_n\}$ are black holes, and $b_0, b_1, ...b_k$ are not black holes (where $k \in Z, k \geq 1$).

Since $\{a_1, a_2, ..., a_n\}$ are black holes, they do not have any outgoing edges. However, that means graph G is not a graph that contains a black hole because not all vertices have an edge, let alone an outgoing edge that leads to $v \in \{a_1, a_2, ..., a_n\}$.

$\therefore$ Since a graph cannot contain two or more black holes, we have that the statement "in a directed graph, there is at most one black hole" is true.

(b) **Answer**

Let $G = (V, E)$ be a directed graph. Let algorithm $hasBlackhole(G)$, which takes graph G as argument, return true if a black hole is present inside the graph G, and false if not.

Algorithm $hasBlackhole(G)$:
- Represent the directed graph, G, in adjacency lists.
- Initialize an empty list A.
- Iterate through each vertex in G. If a vertex's adjacency list is empty, add that vertex to list A. $\leftarrow \boldsymbol{O(|V|)}$

—— This step essentially finds any vertex whose *outdegree* is 0.
- If the length of list A is not equal to 1, return false. Otherwise, continue. $\leftarrow \boldsymbol{O(1)}$

—— This is because if the length of A is 0, then there are no vertices with an *outdegree* of 0, which means there are no black holes. And if the length of A is greater than 1, a black hole cannot exist, as we have just proven in part a) that a directed graph can only have at most 1 black hole.
- Iterate through each vertex's adjacency list (except for the vertex with an empty adjacency list that we have identified in the previous step, let's call it vertex v); if every vertices' adjacency list contain the vertex v, then return true. Otherwise, return false. $\leftarrow \boldsymbol{O(|V| + |E|)}$

—— If every vertices' adjacency list contain the vertex v, that means all of those vertices have an edge that leads to vertex v and satisfies the definition of a black hole. Otherwise, it does not satisfy the definition of a black hole, and therefore means that a black hole does not exist in this graph, G.

$\therefore$ Since the baromometer instruction is $O(|V| + |E|)$, we have that the time complexity of $hasBlackhole(G)$ is $O(|V| + |E|)$.