uOttawa

MAT 2377 - Probability and Statistics for Engineers

WINTER 2022 SECTION A

PROFESSOR WANGJUN YUAN

Assignment 2

Kien Do (ID: 300163370)

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1. **Answer**

   This is a normal distribution question.

   Let $E[X] = 12.08$ inches be the mean of a normal variable.
   Let $SD[X] = 3.1$ inches be the standard deviation of a normal variable.
   Let $VAR[X] = SD[X]^2 = 3.1^2$.
   The precipitation of the years 2023 and 2024 are independent.

   Let X be the yearly precipitation,
   $X \sim N(E[X], \text{VAR}[X])$
   $X \sim N(12.08, 3.1^2) \quad \xleftarrow{\text{by substitution}}$
   Let X_1 and X_2 be 2023 and 2024's precipitation.

   Since the precipitation for 2023 and 2024 are independent (the probability of precipitation in year 1 doesn't affect year 2, and vice versa), we have that,

$$X_1 + X_2 \sim N(12.08 \times 2, 3.1^2 \times 2)$$
$$X_1 + X_2 \sim N(24.16, 19.22)$$

   Therefore, the E[X] for both years is 24.16, and the VAR[X] for both years is 19.22.

   We want to find the probability that the total precipitation of both years exceed 25 inches, which is $P((X_1 + X_2) > 25)$. Let $Z = X_1 + X_2$, we have,

$$\begin{aligned}
P((X_1 + X_2) > 25) &= P\left(Z > \frac{X - E[X]}{SD[X]}\right) \\
&= P\left(Z > \frac{X - E[X]}{\sqrt{\text{VAR[X]}}}\right) \\
&= P\left(Z > \frac{25 - 24.16}{\sqrt{19.22}}\right) \\
&= P(Z > 0.1916) \\
&= 1 - P(Z < 0.19) \\
&= 1 - 0.5753 \quad \xleftarrow[\text{for } Z < 0.19]{\text{See Normal Curve table}} \\
&= 0.4247
\end{aligned}$$

   $\therefore$ The probability that the total precipitation during the 2 years will exceed 25 inches is 0.4247.

2. **Answer**

   This is a standard error question.

   We know the standard deviation and the number of runs to be 1.20 and 40, respectively. So,

$$\begin{aligned}
\text{Standard error} &= \frac{SD[X]}{\sqrt{n}} \\
0.18973 &= \frac{1.20}{\sqrt{40}} \quad \xleftarrow{\text{by substitution}}
\end{aligned}$$

   Since the runs will be from 1.65 hours to 2.04 hours, we need to find Z for 1.65 and 2.04, $P(1.65 < E[X] < 2.04)$, then find $P(Z_1 < Z < Z_2)$.

$$\begin{aligned}
Z_1 &= \frac{1.65 - 1.82}{0.18973} \\
Z_1 &= -0.896
\end{aligned}
\qquad\qquad
\begin{aligned}
Z_2 &= \frac{2.04 - 1.82}{0.18973} \\
Z_2 &= 1.1595
\end{aligned}$$

   Calculate the probability of the sample mean, $P(Z_1 < Z < Z_2) = P(Z < Z_2) - P(Z < Z_1)$.

$$\begin{aligned}
P(-0.896 < Z < 1.1595) &= P(1.1595) - P(-0.896) \\
P(-0.896 < Z < 1.1595) &= P(1.16) - P(-0.90) \quad \xleftarrow{\text{rounding}} \\
P(-0.90 < Z < 1.16) &= 0.8770 - 0.1841 \\
P(-0.90 < Z < 1.16) &= 0.6929
\end{aligned}$$

   Therefore, the probability that the sample mean of the next 40 runs will be from 1.65 to 2.04 hours is 0.6929.

3. **Answer**

We know the sample mean of 20 observations is $\bar{x} = 46$ and the population standard deviation is $\sigma = 9.4$. We also know that the individual measurements follow a normal distribution.

We want to know how likely it is that one can obtain $\bar{x} \geq 46$ with $n = 20$ if the population mean is $\mu = 40.38$. If this probability suggests that $\bar{x} = 40.38$ is reasonable, the claim is supported. If the probability is low, one can argue that the data do not support support the claim that $\mu = 40.38$.

The probability that we need to compute is $P(|\bar{X} - 40.38|) \geq 40.38$. In other words, if the mean is $\mu = 40.38$, what is the chance that $\bar{X}$ will deviate by as much as $46 - 40.38 = 5.62$?

According to CLT, we know that $Z = \dfrac{\bar{X} - \mu}{\sigma/\sqrt{n}}$ has a standard normal distribution, approximately. The desired probability is thus,

$$
\begin{aligned}
P(|\bar{X} - \mu| \geq 5.62) &= P(\bar{X} - \mu \geq 5.62) + P(\bar{X} - \mu \leq 5.62) \\
&= P\left(\frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \geq \frac{5.62}{9.4/\sqrt{20}}\right) + P\left(\frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \leq \frac{-5.62}{9.4/\sqrt{20}}\right) \\
&= P\left(Z \geq 2.67\right) + P\left(Z \leq -2.67\right) \\
&= 2P(Z \geq 2.67) \\
&\approx 2(0.0038) \xleftarrow[\text{-2.67}]{\text{z value for}} \\
&= \frac{19}{2500} \\
&= 0.0076
\end{aligned}
$$

Therefore, the data refutes the treatment plant's claim as one would expect by chance that an $\bar{x}$ would be 40.38 in 19 out of 2500 days.

4. **Answer**

(a) Sample mean and sample median.

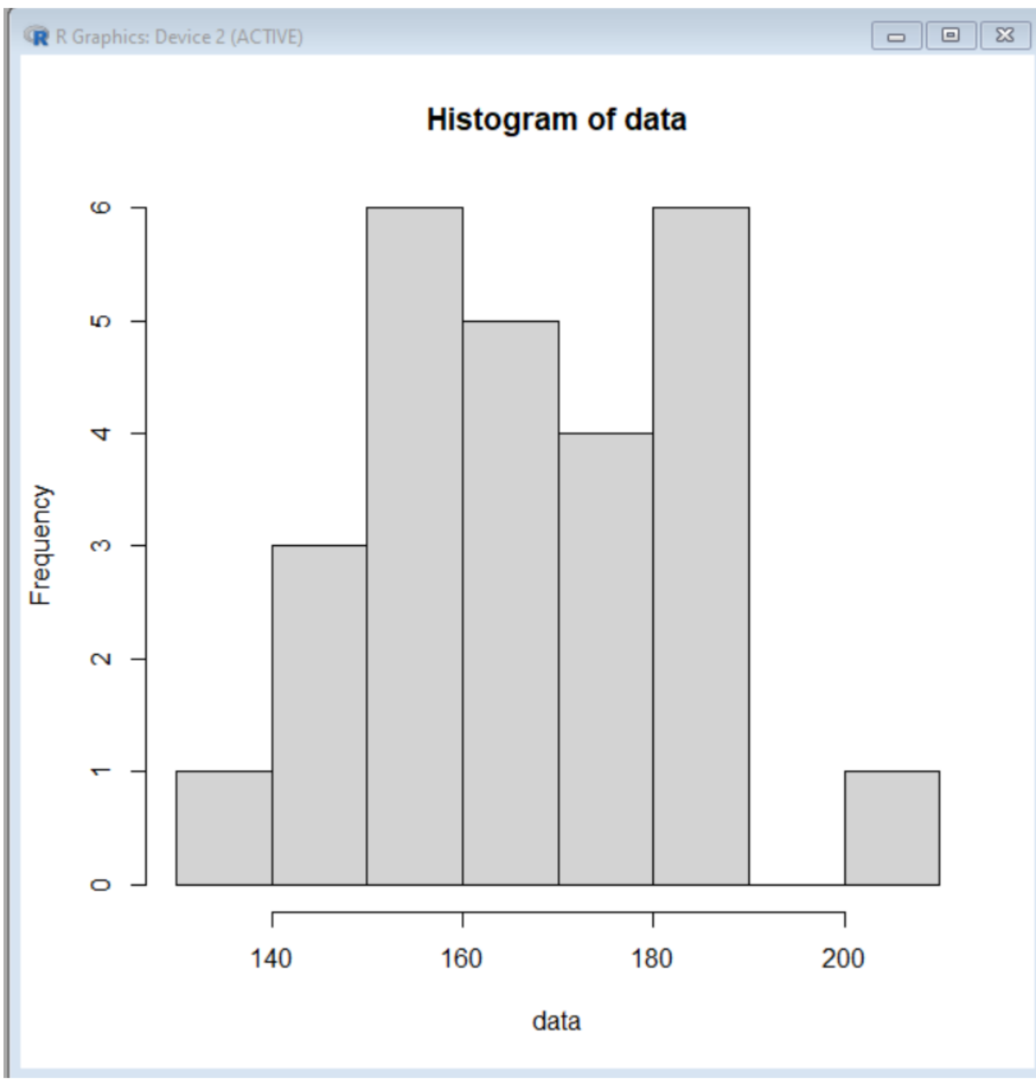
```
> data <- c(136, 182, 143, 147, 173, 151, 158, 160, 161, 171, 163, 155, 165,
↪  167, 173, 174, 152, 181, 144, 181, 185, 156, 169, 188, 190, 205)
> data_mean <- mean(data)
> data_median <- median(data)
> print(data_mean)
[1] 166.5385
> print(data_median)
[1] 166
```

(b) Sample variance and sample standard deviation.

```
> var(data)
[1] 275.5385
> sd(data)
[1] 16.59935
```

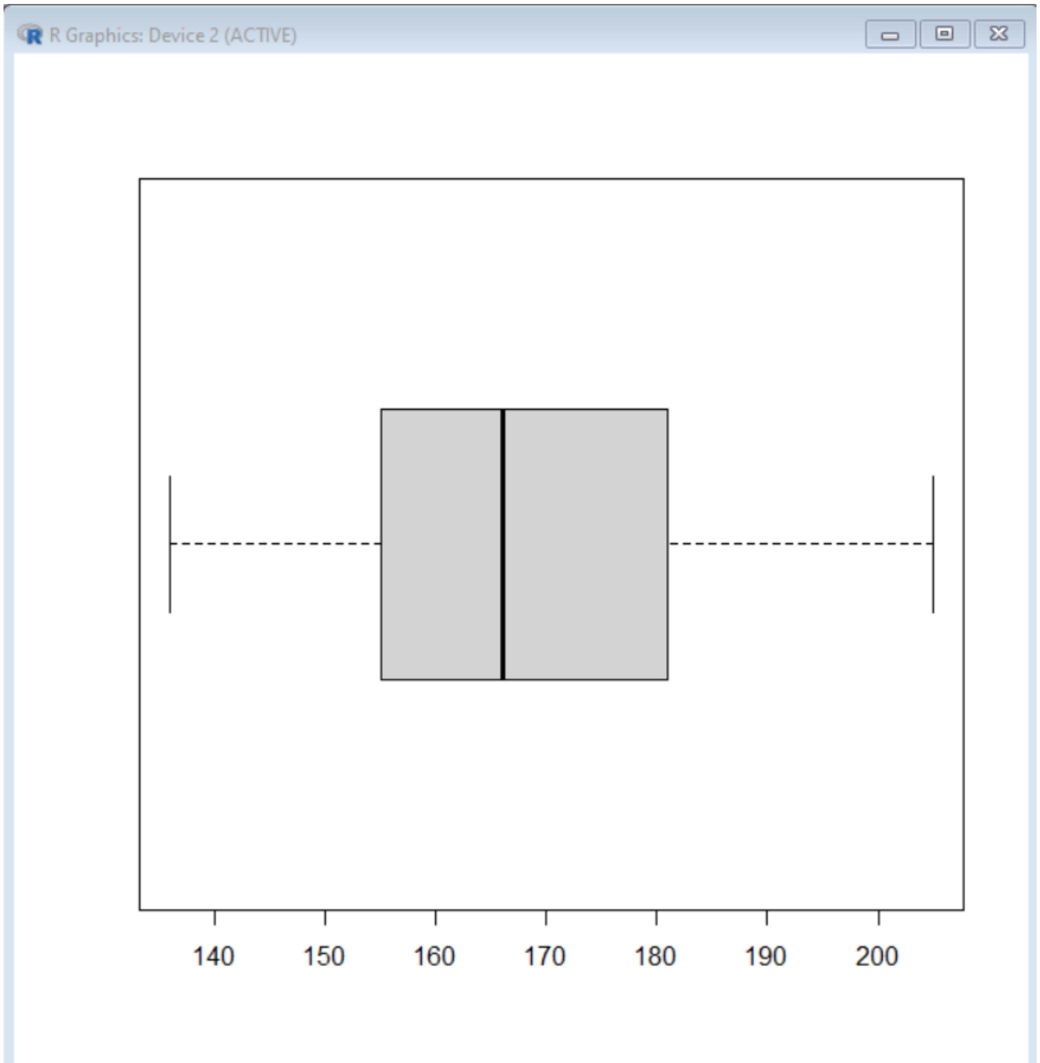
(c) Produce a histogram.

```
> hist(data)
```



(d) Produce a boxplot.

```
> boxplot(data, horizontal = TRUE)
```



(e) Find the 95% and 99% CIs, assuming that the measurements follow a normal distribution.

```
# Normal population, population standard deviation unknown
> n = 26 # size of sample data set, n
> error <- qt((1+0.95)/(2), n-1) * sd(data)/sqrt(n)
> mean(data) - error
[1] 159.8338
> mean(data) + error
[1] 173.2431 # therefore the 95% CI is (159.8338, 173.2431)
> error <- qt((1+0.99)/(2), n-1) * sd(data)/sqrt(n)
> mean(data) - error
[1] 157.4642
> mean(data) + error
[1] 175.6127 # therefore the 99% CI is (157.4642, 175.6127)
```

5. **Answer**

   (a) Calculate minimum sample size needed, given $\sigma = 1.6$, margin of error $d = 0.50$, and confidence level $CL = 99\%$.

   First, we need to find the z-value for 95%.

$$A = \frac{1 + CL}{2}$$
$$A = \frac{1 + 0.95}{2}$$
$$A = 0.975$$

   On the Normal Probability Table, that corresponds to a z-value of $z_{a/2} = 1.96$.
   The minimum sample size, n, for a $100(1-\alpha)\%$ confidence interval

$$\left(\bar{x} - z_{a/2} \times \frac{\sigma}{\sqrt{n}}, \bar{x} + z_{a/2} \times \frac{\sigma}{\sqrt{n}}\right)$$

   with a margin of error d is

$$n = \left(\frac{z_{a/2}\,\sigma}{d}\right)^2$$

   Therefore, by substitution, we have,

$$n = \left(\frac{z_{a/2}\,\sigma}{d}\right)^2$$
$$n = \left(\frac{1.96 \times 1.6}{0.50}\right)^2$$
$$n \approx 40 \quad \xleftarrow{\text{rounded up}}$$

   Therefore, a minimum sample size she needs to take is 40.

   (b) How large is the sample if the confidence level is 99%?

   First, we need to find the z-value for 99%.

$$A = \frac{1 + CL}{2}$$
$$A = \frac{1 + 0.99}{2}$$
$$A = 0.995$$

   On the Normal Probability Table, that corresponds to a z-value of $z_{a/2} = 2.57$.

Therefore, by substitution, we have,

$$n = \left(\frac{z_{a/2}\,\sigma}{d}\right)^2$$
$$n = \left(\frac{2.57 \times 1.6}{0.50}\right)^2$$
$$n \approx 68 \quad \xleftarrow{\text{rounded up}}$$

Therefore, a minimum sample size she needs to take is 68.

6. **Answer**

(a) Compute a 95% confidence interval for μ.

We need to use the confidence interval formula for

$$\mu : \bar{X} \pm t_{a/2}(n-1)\frac{S}{\sqrt{n}} \quad \longleftarrow \frac{\text{the whole } t_{a/2}(n-1)}{\text{is the t-value}}$$

However, before using that formula, we need to find sample mean $\bar{X}$, sample SD S, and $t_{a/2}$. We can calculate the mean $\bar{E}$ to be

$$\bar{X} = \frac{5+8.5+12+15+7+9+7.5+6.5+10.5}{9} = \frac{81}{9} = 9$$

Find S^2 then find S.

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}\left(X_i - \bar{X}\right)^2$$
$$S^2 = \frac{1}{9-1}\sum_{i=1}^{n}\left(X_i - 9\right)^2$$
$$S^2 = 9.5$$
$$S = 3.08$$

Find $t_{a/2}$ by finding $a/2$ and r on the t-distribution Probability Table. Recall $a = 1 - CL$ and $r = n - 1$.

$$a/2 = (1-CL)/2 \qquad\qquad r = n-1$$
$$a/2 = (1-0.95)/2 \qquad\qquad r = 9-1$$
$$a/2 = 0.025 \qquad\qquad r = 8$$

Referring to the t-distribution probability table, we have that $t_{a/2} = 2.306$.
Now, calculate a 95% confidence interval for μ by substituting in relevant values.

$$\mu : \bar{X} \pm t_{a/2}(n-1)\frac{S}{\sqrt{n}}$$
$$\mu : (9) \pm (2.306)\frac{3.08}{\sqrt{(9)}}$$
$$\mu : 9 \pm 2.367$$

(b) Compute a 95% confidence interval for μ, if we know $\sigma^2 = 9$.

Now that we are given the value of the variance $\sigma^2 = 9$, we can use the following symmetric $100(1-a)\%$ confidence interval formula,

$$\bar{X} \pm z_{a/2}\frac{\sigma}{\sqrt{n}}$$

We have that $\sigma = \sqrt{\sigma^2} = \sqrt{9} = 3$, and that $z_{a/2} = 1.96$ for a 95% confidence interval. Therefore, by substitution,

$$\mu : \bar{X} \pm z_{a/2}\frac{\sigma}{\sqrt{n}}$$
$$\mu : 9 \pm (1.96)\frac{3}{\sqrt{3}}$$
$$\mu : 9 \pm 1.96$$

7. **Answer**

Find the sample mean of A.

$$\mu_A = \frac{36+54+44+52+41+37+53+51}{8} = 46$$

Find the sample mean of B.

$$\mu_B = \frac{52+60+64+44+38+48+68}{7} = 53.43$$

Find the variance for type A, s_A^2.

$$\begin{aligned} s_A^2 &= \frac{1}{n-1}\sum_{i=1}^{N}(x_i-\bar{x})^2 \\ s_A^2 &= \frac{(36-46)^2+...+(51-46)^2}{8-1} \\ s_A^2 &= 54.85 \end{aligned}$$

Find the variance for type B, s_B^2.

$$\begin{aligned} s_B^2 &= \frac{1}{n-1}\sum_{i=1}^{N}(x_i-\bar{x})^2 \\ s_B^2 &= \frac{(52-53.43)^2+...+(68-53.43)^2}{7-1} \\ s_B^2 &= 120.95 \end{aligned}$$

(a) Since types A and B are normally distributed, and the population variances are equal, $\sigma_A^2 = \sigma_B^2$, the degrees of freedom is then $df = (n_A-1)+(n_B-1) = (8-1)+(7-1) = 13$.

The t-value for a 95% confidence level is $t_{0.025} = 2.16$ and the t-value for a 98% confidence level is $t_{0.01} = 2.65$.

We know that the population variances are equal, therefore we can calculate the pooled SD,

$$\begin{aligned} sd_p &= \sqrt{\frac{(n_A-1)s_A^2+(n_B-1)s_B^2}{df}} \\ sd_p &= \sqrt{\frac{(8-1)(54.85)+(7-1)(120.95)}{13}} \\ sd_p &= 9.24 \end{aligned}$$

We can now calculate the standard error,

$$\begin{aligned} se &= sd_p\sqrt{\frac{1}{n_A}+\frac{1}{n_B}} \\ se &= (9.24)\sqrt{\frac{1}{8}+\frac{1}{7}} \\ se &= \frac{99}{40} \end{aligned}$$

Calculate the confidence interval for both confidence levels.

$$
\begin{aligned}
CI &= \left(\bar{X}_A - \bar{X}_B - t_c \times se, \quad \bar{X}_A - \bar{X}_B + t_c \times se\right) \\
CI_{95\%} &= \left(46 - 53.43 - 2.16 \times \frac{99}{40}, \quad 46 - 53.43 + 2.16 \times \frac{99}{40}\right) \\
CI_{95\%} &= (-12.776, \quad -2.084) \\
CI_{98\%} &= \left(46 - 53.43 - 2.65 \times \frac{99}{40}, \quad 46 - 53.43 + 2.65 \times \frac{99}{40}\right) \\
CI_{98\%} &= (-13.99, \quad -0.87)
\end{aligned}
$$

Therefore, $CI_{95\%} = (-12.776, -2.084)$ and $CI_{98\%} = (-13.99, -0.87)$.

(b) The z-value for a 95% confidence level is $\frac{1+0.95}{2} = 0.975 \rightarrow z_{a/2} = 1.96$ and a z-value for a 97% confidence level is $\frac{1+0.97}{2} = 0.985 \rightarrow z_{a/2} = 2.17$. Assume $\sigma_A^2 = 40$ and $\sigma_B^2 = 100$. The confidence interval for each confidence level,

$$
\begin{aligned}
CI &= \left(\bar{X}_A - \bar{X}_B - z_{a/2}\sqrt{\frac{\sigma_A^2}{n_A} + \frac{\sigma_B^2}{n_B}}, \quad \bar{X}_A - \bar{X}_B + z_{a/2}\sqrt{\frac{\sigma_A^2}{n_A} + \frac{\sigma_B^2}{n_B}}\right) \\
CI_{95\%} &= \left(46 - 53.43 - 1.96\sqrt{\frac{40}{8} + \frac{100}{7}}, \quad 46 - 53.43 + 1.96\sqrt{\frac{40}{8} + \frac{100}{7}}\right) \\
CI_{95\%} &= (-16.03, 1.18) \\
CI_{97\%} &= \left(46 - 53.43 - 2.17\sqrt{\frac{40}{8} + \frac{100}{7}}, \quad 46 - 53.43 + 2.17\sqrt{\frac{40}{8} + \frac{100}{7}}\right) \\
CI_{97\%} &= (-16.96, 2.09)
\end{aligned}
$$

Therefore, the CI for a 95% confidence is $(-16.03, 1.18)$ and the CI for a 97% confidence is $(-16.96, 2.09)$.

8. **Answer**

We know that $\bar{E} = 52$ percent with an error of ± 4 percent, and that the $CL = 0.95$. Recall the formula,

$$\hat{P} \pm z_{a/2}\sqrt{\frac{\hat{P}(1-\hat{P})}{n}}$$

Notice that the $\pm z_{a/2}\sqrt{\frac{\hat{P}(1-\hat{P})}{n}}$ part is the margin of error and we already know the margin of error is ± 4 percent. Therefore, by substitution, we have,

$$0.04 = z_{a/2}\sqrt{\frac{\hat{P}(1-\hat{P})}{n}}$$

Now rearrange the equation and solve for n.

$$\begin{aligned}
n &= P(1-P) \times \frac{z^2}{0.04^2} \\
n &= (0.52)(1-(0.52)) \times \frac{1.96^2}{0.04^2} \\
n &= 599.28 \\
n &\approx 600 \quad \xleftarrow[\text{people cannot be a decimal}]{\text{round to 600 because}}
\end{aligned}$$