MCV4UR : Advanced Placement Calculus and Vectors

Assignment #4

Reference Declaration

Complete the Reference Declaration section below in order for your assignment to be graded.

If you used any references beyond the course text and lectures (such as other texts, discussions with colleagues or online resources), indicate this information in the space below. If you did not use any aids, state this in the space provided.

Be sure to cite appropriate theorems throughout your work. You may use shorthand for well-known theorems like the MVT, IVT, etc.

Note: Your submitted work must be **your original work**.

Family Name: Do
First Name: Kien

Declared References:
Reading on Math Is Fun for question 14. https://www.mathsisfun.com/algebra/matrix-inverse.html

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1. Determine whether the following statements are True (T) or False (F).

   (a) If $\vec{u} \cdot \vec{v} = 0$ then either $\vec{u} = \vec{0}$ or $\vec{v} = \vec{0}$.
   False

   (b) $(\vec{u} \cdot \vec{v})\vec{u} \times (\vec{u} \times \vec{v})$ is a meaningful expression.
   True

   (c) Force and velocity are scalar quantities.
   False

   (d) If $|\vec{a}| = 1$ then $\vec{a}$ is a unit vector.
   True

   (e) If $\vec{a} = \vec{b} \times \vec{c}$ and $\vec{a}$ is perpendicular to both $\vec{b}$ and $\vec{c}$, then $\vec{b}$ and $\vec{c}$ are collinear.
   True

   (f) $\left\{\vec{i}, \left(\begin{smallmatrix} 0 \\ 1 \\ 0 \end{smallmatrix}\right), \vec{i} \times \vec{j}\right\}$ is a spanning set of $\mathbb{R}^3$.
   True

   (g) Given the standard basis vectors $\vec{i}, \vec{j}, \vec{k} \in \mathbb{R}^3$, we can express $\vec{k}$ as a linear combination of $\vec{i}$ and $\vec{j}$.
   True

   (h) If $\vec{a}$ and $\vec{b}$ are opposite vectors and $|\vec{a}| = n$ then $|\vec{b}| = -n$.
   False

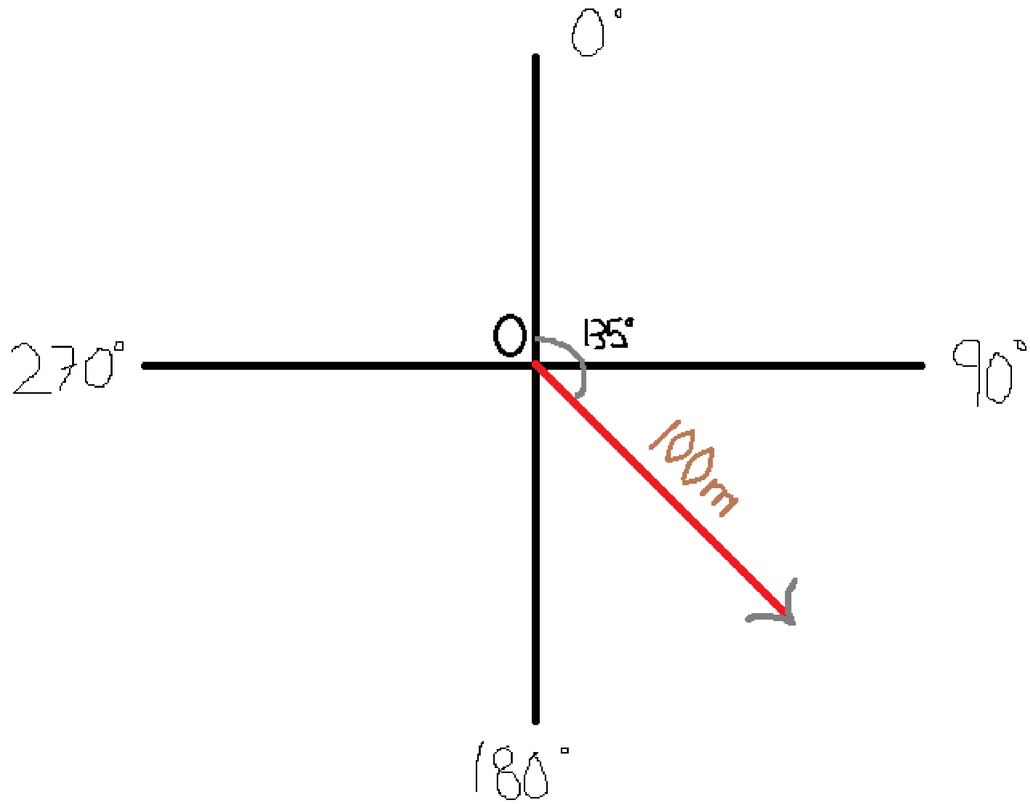
   (i) $\forall \vec{a}, \vec{b} \in \mathbb{R}^3$, $\vec{a} \times \vec{b} = \vec{b} \times \vec{a}$.
   False

   (j) $\exists \vec{a}, \vec{b} \in \mathbb{R}^3$, $\vec{a} \times \vec{b} = \vec{b} \times \vec{a}$.
   True

   (k) If $\forall m, n \in \mathbb{R}$, $|m\vec{a}| = |n\vec{a}|$ then $\vec{a} = \vec{0}$.
   True

2. Captain Picard walks 100 metres, bearing 135° to reach a campsite where there are four lights. Draw a lebeled geometric vector that represents his movement in this situation.

Solution:



3. An **orthogonal spanning set of** $\mathbb{R}^3$ is a set of 3 non-zero vectors, all of which are orthogonal to each other and span $\mathbb{R}^3$. Given $\vec{a} = \left(\begin{smallmatrix} 1 \\ -1 \\ 2 \end{smallmatrix}\right)$, determine vectors $\vec{b}$ and $\vec{c}$ such that $\{\vec{a}, \vec{b}, \vec{c}\}$ is an orthogonal spanning set of $\mathbb{R}^3$. Express the vector $\vec{a} = \left(\begin{smallmatrix} 1 \\ 0 \\ 1 \end{smallmatrix}\right)$ as a linear combination of the vectors in this spanning set.

Solution:
Recall the dot product as well as its definition and purpose. Recall that for some $\vec{a}, \vec{b}$, if $\vec{a} \cdot \vec{b} = 0$, we have that $\vec{a}$ and $\vec{b}$ are perpendicular to each other.

We can start by determining $\vec{b}$ such that $\vec{a} \perp \vec{b}$ by determining the coordinates of $\vec{b}$ such that $\vec{a} \cdot \vec{b} = 0$.

$$0 = \vec{a} \cdot \vec{b} \tag{1}$$
$$0 = (1, -1, 2) \cdot (b_x, b_y, b_z) \quad \xleftarrow{\text{sub in given coordinates of } \vec{a}} \tag{2}$$
$$0 = b_x - b_y + 2b_z \tag{3}$$

Let $b_x = 2,\ b_y = 2,\ 2b_z = 0$. Sub these values into line (3) to verify that the equation is true.

$$0 \stackrel{?}{=} (2) - (2) + 2(0) \tag{4}$$
$$0 = 0 \tag{5}$$

Therefore, we have that $\vec{b} = (2, 2, 0)$ is a vector that is perpendicular to $\vec{a}$. Since $\vec{a} \perp \vec{b}$, it is also true that $\vec{a}$ and $\vec{b}$ are orthogonal to each other.

Now, recall the cross product and its definition. Recall that the cross product of any two vectors will yield another vector that is orthogonal to those two vectors. That is to say, for some $\vec{a} \times \vec{b} = \vec{c}$, we have that $\vec{c}$ is orthogonal to $\vec{a}$ and $\vec{b}$. So, we can simply determine $\vec{c}$ that is orthogonal to $\vec{a}$ and $\vec{b}$ by taking their cross product.

$$\vec{c} = \vec{a} \times \vec{b} \tag{6}$$
$$\vec{c} = \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -1 & 2 \\ 2 & 2 & 0 \end{pmatrix} \tag{7}$$
$$\vec{c} = (0 - 4)\vec{i} - (0 - 4)\vec{j} + (2 - (-2))\vec{k} \tag{8}$$
$$\vec{c} = -4\vec{j} + 4\vec{j} + 4\vec{k} \tag{9}$$

Therefore, we have that $\vec{c} = (-4, 4, 4)$ is orthogonal to $\vec{a}$ and $\vec{b}$.

Therefore, $\{\vec{a}, \vec{b}, \vec{c}\}$ **is an orthogonal spanning set of** $\mathbb{R}^3$ **for** $\vec{a} = \left(\begin{smallmatrix} 1 \\ -1 \\ 2 \end{smallmatrix}\right)$, $\vec{b} = \left(\begin{smallmatrix} 2 \\ 2 \\ 0 \end{smallmatrix}\right)$ **and** $\vec{c} = \left(\begin{smallmatrix} -4 \\ 4 \\ 4 \end{smallmatrix}\right)$.

Express the vector $\vec{v} = \left(\begin{smallmatrix}1\\0\\1\end{smallmatrix}\right)$ as a linear combination of the vectors in this spanning set.
We have $\vec{v} = \left(\begin{smallmatrix}1\\0\\1\end{smallmatrix}\right)$, $\vec{a} = \left(\begin{smallmatrix}1\\-1\\2\end{smallmatrix}\right)$, $\vec{b} = \left(\begin{smallmatrix}2\\2\\0\end{smallmatrix}\right)$ and $\vec{c} = \left(\begin{smallmatrix}-4\\4\\4\end{smallmatrix}\right)$.

Consider $p, q, r \in \mathbb{R}$ such that $\vec{v} = p\vec{a} + q\vec{b} + r\vec{z}$. Solve for p, q, r by creating an augmented matrix and reducing it into Row Echelon Form(REF).

$$\left(\begin{array}{ccc|c} 1 & 2 & -4 & 1 \\ -1 & 2 & 4 & 0 \\ 2 & 0 & 4 & 1 \end{array}\right) \tag{10}$$

$$\equiv_{\substack{R_2{}'=R_1+R_2 \\ R_3{}'=2R_1-R_3}} \left(\begin{array}{ccc|c} 1 & 2 & -4 & 1 \\ 0 & 4 & 0 & 1 \\ 0 & 4 & -12 & 1 \end{array}\right) \tag{11}$$

$$\equiv_{R_3'=R_2-R_3} \left(\begin{array}{ccc|c} 1 & 2 & -4 & 1 \\ 0 & 4 & 0 & 1 \\ 0 & 0 & 12 & 0 \end{array}\right) \tag{12}$$

Now that the augmented matrix is in REF, we can now solve for p, q, r.
Solve for r,

$$0 = 12r \tag{13}$$
$$r = 0 \tag{14}$$

Solve for q,

$$1 = 4q \tag{15}$$
$$q = \frac{1}{4} \tag{16}$$

Solve for p,

$$1 = 1p + 2q - 4r \tag{17}$$
$$p = 1 + 4r - 2q \tag{18}$$
$$p = 1 + 4(0) - 2\left(\frac{1}{4}\right) \tag{19}$$
$$p = \frac{1}{2} \tag{20}$$

So, we have that

$$\vec{v} = \frac{1}{2}\vec{a} + \frac{1}{4}\vec{b}$$

Therefore, we have that $\vec{v} = \left(\begin{smallmatrix}1\\0\\1\end{smallmatrix}\right)$ as a linear combination of the vectors in this spanning set is $\left(\begin{smallmatrix}\frac{1}{2}\\\frac{1}{4}\\0\end{smallmatrix}\right)$

4. Verify *Lagrange's Identity*, $|\vec{u} \times \vec{v}|^2 = |\vec{u}|^2|\vec{v}|^2 - (\vec{u} \cdot \vec{v})^2$, for $\vec{u} = \left(\begin{smallmatrix} 0 \\ 3 \\ 4 \end{smallmatrix}\right)$ and $\vec{v} = \left(\begin{smallmatrix} 4 \\ 0 \\ 3 \end{smallmatrix}\right)$.

Solution:
Solve the left side and the right side separately and if both sides yield the same outcome, Lagrange's Identity would be verified.

Let's begin with the left side, we have,

$$= |\vec{u} \times \vec{v}|^2 \tag{1}$$

$$= [(0,3,6) \times (4,0,3)]^2 \xleftarrow{\text{sub in given values of } \vec{u} \text{ and } \vec{v}} \tag{2}$$

Evaluate $\vec{u} \times \vec{v}$ or $\left(\begin{smallmatrix} 0 \\ 3 \\ 4 \end{smallmatrix}\right) \times \left(\begin{smallmatrix} 4 \\ 0 \\ 3 \end{smallmatrix}\right)$ separately,

$$= \vec{u} \times \vec{v} \tag{3}$$

$$= \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 3 & 4 \\ 4 & 0 & 3 \end{pmatrix} \tag{4}$$

$$= 9\vec{i} + 16\vec{j} - 12\vec{k} \tag{5}$$

$$= (9, 16, -12) \tag{6}$$

Sub $(9, 16, -12)$ from line (6) into line (2) to continue solving the left side,

$$= |(9, 16, -12)|^2 \tag{7}$$

$$= \left(\sqrt{9^2 + 16^2 + (-12)^2}\right)^2 \tag{8}$$

$$= \sqrt{481}^2 \tag{9}$$

$$= 481 \tag{10}$$

Now, let's evaluate the right side of the equation,

$$= |\vec{u}|^2|\vec{v}|^2 - (\vec{u} \cdot \vec{v})^2 \tag{11}$$

$$= \sqrt{(0^2 + 3^2 + 4^2)}^2 \sqrt{(4^2 + 0^2 + 3^2)}^2 - [(0,3,4) \cdot (4,0,3)] \tag{12}$$

$$= (0^2 + 3^2 + 4^2)(4^2 + 0^2 + 3^2) - [(0,3,4) \cdot (4,0,3)] \tag{13}$$

$$= 625 - 12^2 \tag{14}$$

$$= 625 - 144 \tag{15}$$

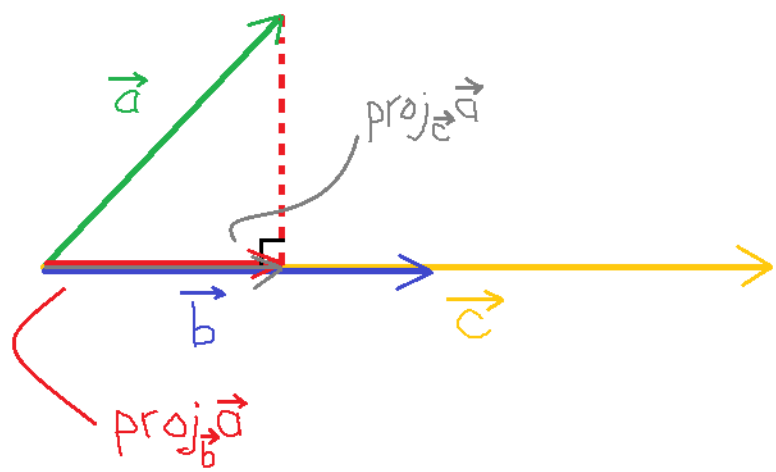
$$= 481 \tag{16}$$

Since the left side of the equation equals 481 and the right side of the equation also equals 481, both sides yield the same value therefore Lagrange's Identity is verified.

5. Consider non-zero vectors $\vec{a}, \vec{b}, \vec{c} \in \mathbb{R}^2$ such that $\vec{b}$ and $\vec{c}$ are collinear.

   (a) Sketch a possible graphical representation of $proj_{\vec{b}}\vec{a}$ and $proj_{\vec{c}}\vec{a}$.

   Solution:



   (b) Prove $proj_{\vec{b}}\vec{a} = proj_{\vec{c}}\vec{a}$.

   Solution:
   Recall that

$$proj_{\vec{b}}\vec{a} = \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|^2}\,\vec{b}$$

Also, recall Theorem 3.2 (pg. 714) states that for some angle θ between nonzero $\vec{a}$ and $\vec{b}$. Then,

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta$$

Combining these two theorems, we have,

$$proj_{\vec{b}}\vec{a} = \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|^2}\,\vec{b} \tag{1}$$

$$proj_{\vec{b}}\vec{a} = \frac{|\vec{a}||\vec{b}| \cos\theta}{|\vec{b}|^2}\,\vec{b} \qquad \xleftarrow{\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta} \tag{2}$$

$$proj_{\vec{b}}\vec{a} = \frac{|\vec{a}| \cos\theta}{|\vec{b}|}\,\vec{b} \tag{3}$$

Recall that $\dfrac{\vec{b}}{|\vec{b}|}$ represents a unit vector in the direction of $\vec{b}$. Therefore, we can rewrite line (3) as follows,

$$proj_{\vec{b}}\vec{a} = |\vec{a}| \cos\theta\,\frac{\vec{b}}{|\vec{b}|} \tag{4}$$

$$proj_{\vec{b}}\vec{a} = |\vec{a}| \cos\theta\,\hat{b} \tag{5}$$

By the same token, we can perform the same process for $proj_{\vec{c}}\vec{a}$.

$$proj_{\vec{c}}\vec{a} = \frac{\vec{a}\cdot\vec{c}}{|\vec{c}|^2}\,\vec{c} \tag{6}$$

$$proj_{\vec{c}}\vec{a} = \frac{|\vec{a}||\vec{c}|\cos\theta}{|\vec{c}|^2}\,\vec{c} \qquad \xleftarrow{\vec{a}\cdot\vec{c}=|\vec{a}||\vec{c}|\cos\theta} \tag{7}$$

$$proj_{\vec{c}}\vec{a} = \frac{|\vec{a}|\cos\theta}{|\vec{c}|}\,\vec{c} \tag{8}$$

$$proj_{\vec{c}}\vec{a} = |\vec{a}|\cos\theta\,\frac{\vec{c}}{|\vec{c}|} \tag{9}$$

$$proj_{\vec{c}}\vec{a} = |\vec{a}|\cos\theta\,\hat{c} \tag{10}$$

Since $\vec{a}$ and $\vec{b}$ are collinear, we know that they are parallel to each other. In terms of direction, we have two cases: both projection vectors point in the same direction, the projections vectors point in opposite directions. Consider and prove $proj_{\vec{b}}\vec{a} = proj_{\vec{c}}\vec{a}$ in each case, separately.

Case 1: Projection vectors point in the same direction

Since $\vec{b}$ and $\vec{c}$ are collinear and point in the same direction, their normal vectors must also point in the same direction. Recall that the normal vector of any vector is just that vector with a magnitude of one. Since $\vec{a}$ and $\vec{b}$ are collinear, we have that their normal vectors are equivalent in magnitude and direction. Therefore, we can say that,

$$\hat{b} = \hat{c}$$

from line (5) and line (10). Since $\hat{b} = \hat{c}$, we can also say that,

$$|\vec{a}|\cos\theta\,\hat{b} = |\vec{a}|\cos\theta\,\hat{c}$$

And since $proj_{\vec{b}}\vec{a} = |\vec{a}|\cos\theta\,\hat{b}$ and $proj_{\vec{c}}\vec{a} = |\vec{a}|\cos\theta\,\hat{c}$, it is true that

$$proj_{\vec{b}}\vec{a} = proj_{\vec{c}}\vec{a}$$

Case 2: Projection vectors point in opposite directions

Earlier, we mentioned that $\hat{b}$ and $\hat{c}$ are of equal magnitude. Since $\vec{b}$ and $\vec{c}$ point in opposite directions, we can say that,

$$\hat{b} = -\hat{c}$$

Since $\hat{b}$ and $\hat{c}$ are parallel and point in opposite directions, if we place the initial points of $\hat{b}$ and $\hat{c}$ together, we effectively have created a 180°. We have that the cosine of angle θ created by $\hat{b}$ will equal the negative cosine of the angle γ created by $\hat{c}$. Therefore, we have that,

$$|\vec{a}|\cos\theta\,\hat{b} = |\vec{a}|(-\cos\theta)\,(-\hat{c}) \tag{11}$$

$$|\vec{a}|\cos\theta\,\hat{b} = |\vec{a}|\cos\theta\,\hat{c} \tag{12}$$

$$proj_{\vec{b}}\vec{a} = proj_{\vec{c}}\vec{a} \tag{13}$$

Therefore, regardless of the direction and magnitude of $\vec{a}, \vec{b}$ and $\vec{c}$, as long as $\vec{b}$ and $\vec{c}$ are collinear, we have that

$$proj_{\vec{b}}\vec{a} = proj_{\vec{c}}\vec{a}$$

(c) Explain *briefly* in words why $proj_{\vec{b}}\vec{a} = proj_{\vec{c}}\vec{a}$.

Solution:
Since $\vec{b}$ and $\vec{c}$ are collinear, $\vec{b}$ and $\vec{c}$ are parallel and can either: go in the same direction or go in opposite directions. I will explain each case separately starting with the first case.

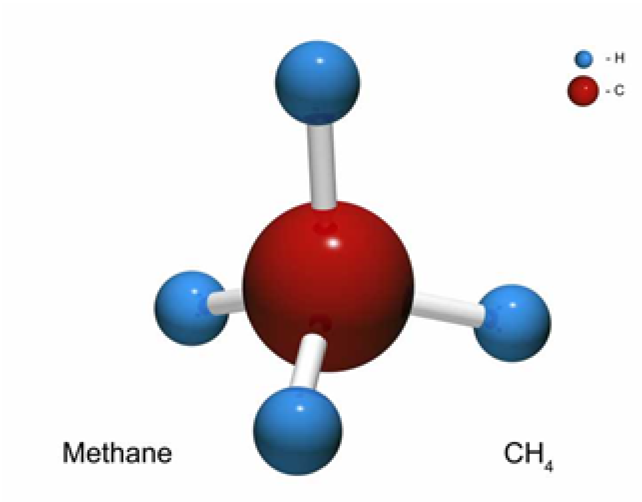
Case 1: $\vec{b}$ and $\vec{c}$ point in the same direction
Refer back to the figure in part (a), we can see that $proj_{\vec{b}}\vec{a}$ and $proj_{\vec{c}}\vec{a}$ depend on the magnitude of $\vec{a}$ and the angle created between $\vec{a}$ and $\vec{b}$, $\vec{a}$ and $\vec{c}$, respectively, but does not depend on the magnitude of $\vec{b}$ nor $\vec{c}$. We can also see that the magnitude of $proj_{\vec{b}}\vec{a} = proj_{\vec{c}}\vec{a}$.

Since the $\vec{b}$ and $\vec{c}$ are collinear, we have that the angle between $\vec{a}$, $\vec{b}$ and $\vec{a}$, $\vec{c}$ are the same. Therefore, $proj_{\vec{b}}\vec{a} = proj_{\vec{c}}\vec{a}$.

Case 2: $\vec{b}$ and $\vec{c}$ point in opposite directions
This case is the same as case 1 except $\vec{b}$ and $\vec{c}$ point in opposite directions. This means we can say that $\vec{b}$ and $\vec{c}$ are opposite in signs.
Let the angle created between $\vec{a}$ and $\vec{b}$ be θ. Since $\vec{b}$ and $\vec{c}$ are collinear and point in opposite directions, we have that that the angle created between $\vec{a}$ and $\vec{c}$ is just $\gamma = 180° - \theta$.
Therefore, the product of $\vec{b}$ and $\cos(\theta)$ has the same sign as the product of $\vec{c}$ and $\cos(180° - \theta)$.

Since these two factors are the same, we have that $proj_{\vec{b}}\vec{a} = proj_{\vec{c}}\vec{a}$.

6. In a methane molecule (CH_4), a carbon atom is surrounded by four hydrogen atoms. Assume that the hydrogen atoms are located in space at $A(0,0,0), B(1,1,0), C(1,0,1)$ and $D(0,1,1)$ and that the carbon atom is located at $E\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)$. Determine the *bond angle*, the angle from some hydrogen atom to the carbon atom, to another hydrogen atom.



Solution:
We will use point $A(0,0,0)$ as our first hydrogen atom, point $E\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)$ as our carbon atom, and $B(1,1,0)$ as our third hydrogen atom. From here, we can create two vectors, $\vec{AE}$ and $\vec{EB}$.

$$\vec{AE} = \vec{E} - \vec{A} \tag{1}$$
$$\vec{AE} = \left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right) - (0,0,0) \tag{2}$$
$$\vec{AE} = \left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right) \tag{3}$$

$$\vec{EB} = \vec{B} - \vec{E} \tag{4}$$
$$\vec{EB} = (1,1,0) - \left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right) \tag{5}$$
$$\vec{EB} = \left(\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right) \tag{6}$$

Recall Theorem 3.2 on pg 714,

$$a \cdot b = |a||b|\cos\theta$$

Isolate θ on one side by rearranging this equation.

$$a \cdot b = |a||b|\cos\theta \tag{7}$$
$$cos\theta = \frac{a \cdot b}{|a||b|} \tag{8}$$
$$\theta = \cos^{-1}\left(\frac{a \cdot b}{|a||b|}\right) \tag{9}$$

Sub in equivalent values for our question. We have,

$$\theta = \cos^{-1}\left(\frac{\vec{AE} \cdot \vec{EB}}{|\vec{AE}||\vec{EB}|}\right)$$

Determine $\vec{AE} \cdot \vec{EB}$, $|\vec{AE}|$ and $|\vec{EB}|$ separately.

$$\vec{AE} \cdot \vec{EB} = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) \cdot \left(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right) \tag{10}$$

$$\vec{AE} \cdot \vec{EB} = \frac{1}{4} + \frac{1}{4} - \frac{1}{4} \tag{11}$$

$$\vec{AE} \cdot \vec{EB} = \frac{1}{4} \tag{12}$$

$$|\vec{AE}| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} \tag{13}$$

$$|\vec{AE}| = \sqrt{\frac{3}{4}} \tag{14}$$

$$|\vec{EB}| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right)^2} \tag{15}$$

$$|\vec{EB}| = \sqrt{\frac{3}{4}} \tag{16}$$

Sub in all relevant values into the equation $\theta = \cos^{-1}\left(\dfrac{\vec{AE} \cdot \vec{EB}}{|\vec{AE}||\vec{EB}|}\right)$ and solve for θ.

$$\theta = \cos^{-1}\left(\frac{\vec{AE} \cdot \vec{EB}}{|\vec{AE}||\vec{EB}|}\right) \tag{17}$$

$$\theta = \cos^{-1}\left(\frac{\frac{1}{4}}{\frac{3}{4}}\right) \tag{18}$$

$$\theta = \cos^{-1}\left(\frac{1}{3}\right) \tag{19}$$

$$\theta \approx 70.5^\circ \tag{20}$$

However, since we have an obtuse angle, we have that,

$$\theta = 180^\circ - 70.5^\circ \tag{21}$$

$$\theta = 109.5^\circ \tag{22}$$

7. If θ is the acute angle between $\vec{a}$ and $\vec{b}$ then prove $proj_{\vec{a}}\vec{b} \cdot proj_{\vec{b}}\vec{a} = (\vec{a} \cdot \vec{b})\cos^2(\theta)$.

Solution:
Prove that the left side equals the right side,

$$= proj_{\vec{a}}\vec{b} \cdot proj_{\vec{b}}\vec{a} \tag{1}$$

$$= \left(\frac{\vec{b} \cdot \vec{a}}{|\vec{a}|^2}\vec{a}\right) \cdot \left(\frac{\vec{a} \cdot \vec{b}}{|\vec{b}|^2}\vec{b}\right) \quad \xleftarrow{\text{definition of vector projection}} \tag{2}$$

$$= (\vec{a} \cdot \vec{b}) \cdot \frac{(\vec{a} \cdot \vec{b})^2}{|\vec{a}|^2|\vec{b}|^2} \tag{3}$$

$$= (\vec{a} \cdot \vec{b}) \cdot \frac{\left(|\vec{a}||\vec{b}|\cos(\theta)\right)^2}{|\vec{a}|^2|\vec{b}|^2} \quad \xleftarrow{\text{Theorem 3.2 pg. 714}} \tag{4}$$

$$= (\vec{a} \cdot \vec{b}) \cdot \frac{|\vec{a}|^2|\vec{b}|^2\cos^2(\theta)}{|\vec{a}|^2|\vec{b}|^2} \tag{5}$$

$$= (\vec{a} \cdot \vec{b})\cos^2(\theta) \tag{6}$$

Therefore, if θ is the acute angle between $\vec{a}$ and $\vec{b}$,

$$proj_{\vec{a}}\vec{b} \cdot proj_{\vec{b}}\vec{a} = (\vec{a} \cdot \vec{b})\cos^2(\theta)$$

8. If $\vec{a}+\vec{b}+\vec{c}=\vec{0}$ then prove $\vec{a}\times\vec{b}=\vec{b}\times\vec{c}=\vec{c}\times\vec{a}$.

Solution:
Start by manipulating the given equation $\vec{a}+\vec{b}+\vec{c}=\vec{0}$.

$$\vec{a}+\vec{b}+\vec{c}=\vec{0} \tag{1}$$
$$(\vec{a}+\vec{b}+\vec{c})\times\vec{b}=\vec{0}\times\vec{b} \tag{2}$$
$$\vec{a}\times\vec{b}+\vec{b}\times\vec{b}+\vec{c}\times\vec{b}=0 \quad \xleftarrow{\text{Recall } \vec{b}\times\vec{b}=0} \tag{3}$$
$$\vec{a}\times\vec{b}+\vec{c}\times\vec{b}=0 \tag{4}$$
$$\vec{a}\times\vec{b}=-(\vec{c}\times\vec{b}) \tag{5}$$

Recall Theorem 4.3 on pg. 726, for some $\vec{a},\vec{b}$, we have that $\vec{a}\times\vec{b}=-(\vec{a}\times\vec{b})$, or, that the cross product operation is anti-commutative. By the anti-commutative law of the cross product, we have that

$$-(\vec{c}\times\vec{b})=\vec{b}\times\vec{c}$$

Therefore, we can express line (5) as follows,

$$\vec{a}\times\vec{b}=-(\vec{c}\times\vec{b}) \tag{6}$$
$$\vec{a}\times\vec{b}=\vec{b}\times\vec{c} \tag{7}$$

Now, repeat the same process, this time manipulating the given equation $\vec{a}+\vec{b}+\vec{c}=\vec{0}$ by multiplying both sides by $\vec{c}$.

$$\vec{a}+\vec{b}+\vec{c}=\vec{0} \tag{8}$$
$$(\vec{a}+\vec{b}+\vec{c})\times\vec{c}=\vec{0}\times\vec{c} \tag{9}$$
$$\vec{a}\times\vec{c}+\vec{b}\times\vec{c}+\vec{c}\times\vec{c}=0 \tag{10}$$
$$\vec{a}\times\vec{c}+\vec{b}\times\vec{c}=0 \tag{11}$$
$$\vec{b}\times\vec{c}=-(\vec{a}\times\vec{c}) \tag{12}$$
$$\vec{b}\times\vec{c}=\vec{c}\times\vec{a} \tag{13}$$

Since $\vec{a}\times\vec{b}=\vec{b}\times\vec{c}$ and $\vec{b}\times\vec{c}=\vec{c}\times\vec{a}$. We have that,

$$\vec{a}\times\vec{b}=\vec{b}\times\vec{c}=\vec{c}\times\vec{a}$$

Therefore, if $\vec{a}+\vec{b}+\vec{c}=\vec{0}$, we have that $\vec{a}\times\vec{b}=\vec{b}\times\vec{c}=\vec{c}\times\vec{a}$

9. Determine the equation of a line through the point $A(1,2,-3)$ and lying within the plane $x-2y+2z+9=0$.

Solution:
Consider some point $B(-9,0,0)$, verify that this point lies within the plane $x-2y+2z+9=0$.

$$0 = x-2y+2z+9 \tag{1}$$
$$0 = (-9)-2(0)+2(0)+9 \tag{2}$$
$$0 = 0 \tag{3}$$

Since the equation above is true, I have verified that point B lies within the plane $x-2y+2z+9=0$.

Determine an equation of a line that passes through $(1,2,-3)$ and $(-9,0,0)$. Let's start with the vector form, we have,

$$\vec{r} = (1,2,-3)+t(a,b,c)$$

Determine the direction vector by subtracting point A from point B,

$$(a,b,c) = (-9-1, 0-2, 0-(-3)) \tag{4}$$
$$(a,b,c) = (-10,-2,3) \tag{5}$$

We have that the equation of the line from A to B is

$$\vec{r} = (1,2,-3)+t(-10,-2,3)$$

Since this line goes through A and B, and A and B are both on the plane $x-2y+2z+9=0$, we have that the line $\vec{r} = (1,2,-3)+t(-10,-2,3)$ also lies on the plane $x-2y+2z+9=0$.

Therefore, one possible equation of a line, in vector form, that goes through point $A(1,2,-3)$ and lies within the plane $x-2y+2z+9=0$ is

$$\vec{r} = (1,2,-3)+t(-10,-2,3)$$

10. Determine the equation of a plane through the origin and the points $P(3, 1, 2)$ and $Q(-1, 2, -3)$.

Solution:

Vector Form
First, we need to determine the direction vectors of this plane by selecting two pairs of non-zero, non-collinear vectors from the given points, O, P, Q.
Let's use $\vec{OP}$ and $\vec{OQ}$, we have,

$$\vec{OP} = (3-0, 1-0, 2-0) \tag{1}$$
$$\vec{OP} = (3, 1, 2) \tag{2}$$
$$\vec{OQ} = (-1-0, 2-0, -3-0) \tag{3}$$
$$\vec{OQ} = (-1, 2, -3) \tag{4}$$

Now that we have the direction vectors for this plane, we can determine the position vector.
Let's choose point $P(3, 1, 2)$ to be our position vector. We have that the equation of this plane in Vector Form is,

$$\vec{r} = s(3, 1, 2) + t(-1, 2, -3) + (3, 1, 2)$$

Cartesian Form
Determine the normal vectors,

$$\vec{n} = \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 1 & 2 \\ -1 & 2 & -3 \end{pmatrix} \tag{5}$$
$$\vec{n} = -7\vec{i} + 7\vec{j} + 7\vec{k} \tag{6}$$

From the normal vectors, we can sub in the corresponding coefficients of $\vec{i}, \vec{j}, \vec{k}$ into the cartesian form, $Ax + By + Cz + D = 0$. We have,

$$-7x + 7y + 7z + D = 0$$

I can determine D by subbing in the coordinates of the origin, $(0, 0, 0)$ into (x, y, z). By inspection, I can determine that $D = 0$. Therefore, the equation of the plane going through points O, P, Q in Cartesian Form is,

$$-7x + 7y + 7z = 0$$

Therefore, the equation of the plane that goes through the origin, the points $P(3, 1, 2)$ and $Q(-1, 2, -3)$, is

$$\vec{r} = s(3, 1, 2) + t(-1, 2, -3) + (3, 1, 2)$$

or

$$-7x + 7y + 7z = 0$$

11. Determine the (*least*) distance between the point $A(1, -2, 4)$ and the plane $3x + 2y + 6z - 5 = 0$.

12. Suppose that $\vec{v_1}$ and $\vec{v_2}$ are vectors such that $|\vec{v_1}| = 2$ and $|\vec{v_2}| = 3$, and $\vec{v_1} \cdot \vec{v_2} = 5$. Let $\vec{v_3} = proj_{\vec{v_1}}\vec{v_2}$, $\vec{v_4} = proj_{\vec{v_2}}\vec{v_3}$ and so on so that $\vec{v_{k+1}} = proj_{\vec{v_{k-1}}}\vec{v_k}$. Evaluate $\sum_{i=1}^{\infty} |\vec{v_i}|$.

Solution:
By definition, we have that,

$$\sum_{i=1}^{\infty} |\vec{v_i}| = |\vec{v_1}| + |\vec{v_2}| + |\vec{v_3}| + ... + |\vec{v_{\infty-1}}| + |\vec{v_\infty}| + ... \tag{1}$$

Recall the magnitude of some $proj_{\vec{b}}\vec{a}$ is

$$comp_{\vec{b}}\vec{a} = \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|}$$

Since $\vec{v_{k+1}} = proj_{\vec{v_{k-1}}}\vec{v_k}$, we have that

$$|\vec{v_{k+1}}| = \frac{\vec{v_k} \cdot \vec{v_{k-1}}}{|\vec{v_{k-1}}|}$$

Therefore, we have that,

$$\sum_{i=1}^{\infty} |\vec{v_i}| = |\vec{v_1}| + |\vec{v_2}| + |\vec{v_3}| + ... + |\vec{v_{\infty-1}}| + |\vec{v_\infty}| + ... \tag{2}$$

$$\sum_{i=1}^{\infty} |\vec{v_i}| = 2 + 3 + \frac{\vec{v_2} \cdot \vec{v_1}}{|\vec{v_1}|} + \frac{\vec{v_3} \cdot \vec{v_2}}{|\vec{v_2}|} + \frac{\vec{v_4} \cdot \vec{v_3}}{|\vec{v_3}|} + ... \tag{3}$$

$$\sum_{i=1}^{\infty} |\vec{v_i}| = 2 + 3 + \frac{|\vec{v_2}||\vec{v_1}|\cos\theta}{|\vec{v_1}|} + \frac{|\vec{v_3}||\vec{v_2}|\cos\theta}{|\vec{v_2}|} + \frac{|\vec{v_4}||\vec{v_3}|\cos\theta}{|\vec{v_3}|} + ... \quad \xleftarrow{\text{Theorem 3.2 pg. 714}} \tag{4}$$

$$\sum_{i=1}^{\infty} |\vec{v_i}| = 2 + 3 + |\vec{v_2}|\cos\theta + |\vec{v_3}|\cos\theta + |\vec{v_4}|\cos\theta + ... \tag{5}$$

Let's take a closer look at line (5), specifically the individual terms on the right hand side of the equation,

$$\sum_{i=1}^{\infty} |\vec{v_i}| = \underbrace{2}_{|v_1|} + \underbrace{3}_{|v_2|} + \underbrace{|\vec{v_2}|\cos\theta}_{|v_3|} + \underbrace{|\vec{v_3}|\cos\theta}_{|v_4|} + |\underbrace{|\vec{v_4}|\cos\theta}_{|v_5|} + ...$$

Notice how $|\vec{v_3}|$ is defined by $|\vec{v_2}|$, $|\vec{v_4}|$ is defined by $|\vec{v_3}|$, $|\vec{v_5}|$ is defined by $|\vec{v_4}|$ and so forth. We can re-define $|\vec{v_3}|, |\vec{v_4}|, |\vec{v_5}|, ...$ such that they are all defined by $|\vec{v_2}|$,

$$|\vec{v_2}| = |\vec{v_2}| \tag{6}$$

$$|\vec{v_3}| = |\vec{v_2}|\cos\theta \tag{7}$$

$$|\vec{v_4}| = \underbrace{(|\vec{v_2}|\cos\theta)}_{|\vec{v_3}|}\cos\theta \tag{8}$$

$$|\vec{v_5}| = \underbrace{\left(\underbrace{(|\vec{v_2}|\cos\theta)}_{|\vec{v_3}|}\cos\theta\right)}_{|\vec{v_4}|}\cos\theta \tag{9}$$

With this, let's rewrite line (5),

$$\sum_{i=1}^{\infty}|\vec{v_i}| = 2+|\vec{v_2}|+|\vec{v_2}|\cos\theta+|\vec{v_2}|\cos\theta\cos\theta+|\vec{v_2}|\cos\theta\cos\theta\cos\theta+... \tag{10}$$

$$\sum_{i=1}^{\infty}|\vec{v_i}| = 2+|\vec{v_2}|+|\vec{v_2}|\cos\theta+|\vec{v_2}|\cos^2\theta+|\vec{v_2}|\cos^3\theta+... \tag{11}$$

$$\sum_{i=1}^{\infty}|\vec{v_i}| = 2+|\vec{v_2}|\left(\cos\theta+\cos^2\theta+\cos^3\theta+...\right) \tag{12}$$

$$\sum_{i=1}^{\infty}|\vec{v_i}| = 2+|\vec{v_2}|\sum_{i=0}^{\infty}\left(\cos^i(\theta)\right) \tag{13}$$

$$\sum_{i=1}^{\infty}|\vec{v_i}| = 2+3\sum_{i=0}^{\infty}\left(\cos^i(\theta)\right) \qquad \xleftarrow{|\vec{v_2}|=3} \tag{14}$$

Recall that for some $\vec{v_1}, \vec{v_2}$, we have that

$$\vec{v_1}\cdot\vec{v_2} = |\vec{v_1}||\vec{v_2}|\cos\theta \tag{15}$$

$$\cos\theta = \frac{\vec{v_1}\cdot\vec{v_2}}{|\vec{v_1}||\vec{v_2}|} \tag{16}$$

$$\cos\theta = \frac{(5)}{(2)(3)} \qquad \xleftarrow{\text{values given from the question}} \tag{17}$$

$$\cos\theta = \frac{5}{6} \tag{18}$$

Now that we know that $\cos\theta = \frac{5}{6}$, we can sub this into line (14),

$$\sum_{i=1}^{\infty}|\vec{v_i}| = 2+3\sum_{i=0}^{\infty}\left(\frac{5}{6}\right)^i \tag{19}$$

Recall that the sum of a geometric series is, $\sum_{i=0}^{n} r^i = \frac{1-r^n}{1-r}$

Let $r=\frac{5}{6}$, $n=\infty$, we have that,

$$\sum_{i=1}^{\infty}|\vec{v_i}| = 2+3\sum_{i=0}^{\infty}\left(\frac{5}{6}\right)^i \tag{20}$$

$$\sum_{i=1}^{\infty}|\vec{v_i}| = 2+3\left(\frac{1-\left(\frac{5}{6}\right)^n \xleftarrow{\text{approaches } 0}}{1-\left(\frac{5}{6}\right)}\right) \tag{21}$$

$$\sum_{i=1}^{\infty}|\vec{v_i}| = 2+3\left(\frac{1}{1-\left(\frac{5}{6}\right)}\right) \tag{22}$$

$$\sum_{i=1}^{\infty}|\vec{v_i}| = 20 \tag{23}$$

$$\therefore \sum_{i=1}^{\infty}|\vec{v_i}| = 20$$

13. Determine and classify the nature of the intersection between the following planes:

$$\Pi_1 : x - 5y + 2z = 10$$

$$\Pi_2 : x + 7y - 2z + 6 = 0$$

$$\Pi_3 : 8x + 5y + z = 20$$

LaTeX note: You may find the *amatrix* custom environment included in the code useful for creating the augmented matrix below.

$$\left(\begin{array}{ccc|c} 1 & -5 & 2 & 10 \\ 1 & 7 & -2 & -6 \\ 8 & 5 & 1 & 20 \end{array}\right)$$

Solution:
By inspection, I can see that the three planes have non-collinear normal vectors. Since they have non-collinear normals, we can determine the nature of the intersection by creating an augmented matrix system and reducing it in Row Echelon Form(REF).

$$\left(\begin{array}{ccc|c} 1 & -5 & 2 & 10 \\ 1 & 7 & -2 & -6 \\ 8 & 5 & 1 & 20 \end{array}\right) \tag{1}$$

$$\equiv_{\substack{R_2{}' = R_1 - R_2 \\ R_3{}' = 8R_1 - R_3}} \left(\begin{array}{ccc|c} 1 & -5 & 2 & 10 \\ 0 & -12 & 4 & 16 \\ 0 & -45 & 15 & 60 \end{array}\right) \tag{2}$$

$$\equiv R_2{}' = \frac{1}{12} R_2 \left(\begin{array}{ccc|c} 1 & -5 & 2 & 10 \\ 0 & -1 & \frac{1}{3} & \frac{4}{3} \\ 0 & -45 & 15 & 60 \end{array}\right) \tag{3}$$

$$\equiv R_3{}' = 45R_2 - R_3 \left(\begin{array}{ccc|c} 1 & -5 & 2 & 10 \\ 0 & -1 & \frac{1}{3} & \frac{4}{3} \\ 0 & 0 & 0 & 0 \end{array}\right) \tag{4}$$

We have a tautology case in the third row of line (4), therefore, z is a *free variable* and any value of z will satisfy R_3. So, we will need to parameterize this variable.

Let $z = t \quad (t \in \mathbb{R})$

From R_2, we have,

$$-1y + \frac{1}{3}z = \frac{4}{3} \tag{5}$$

$$-y + \frac{1}{3}t = \frac{4}{3} \quad \xleftarrow{z=t} \tag{6}$$

$$y = \frac{1}{3}t - \frac{4}{3} \tag{7}$$

From R_1, we have,

$$1x - 5y + 2z = 10 \tag{8}$$

$$x = 10 - 2z + 5y \tag{9}$$

$$x = 10 - 2t + 5\left(\frac{1}{3}t - \frac{4}{3}\right) \quad \xleftarrow{z=t \text{ and } y=\frac{1}{3}t-\frac{4}{3}} \tag{10}$$

$$x = 10 - 2t + \frac{5}{3}t - \frac{20}{3} \tag{11}$$

$$x = -\frac{1}{3}t + \frac{10}{3} \tag{12}$$

Therefore, the solution to this system is

$$(x, y, z) = \left(-\frac{1}{3}t + \frac{10}{3},\ \frac{1}{3}t - \frac{4}{3},\ t\right) \qquad (t \in \mathbb{R})$$

We can also express the solution as a line in vector form.

$$\vec{r} = t\vec{d} + \vec{r}_0 \tag{13}$$

$$\vec{r} = t\left(-\frac{1}{3}, \frac{1}{3}, 1\right) + \left(\frac{10}{3}, -\frac{4}{3}, 0\right) \tag{14}$$

Since the three planes are consistent and:

- Have infinite solutions in a linear systems
- Intersect along a line
- Do not have collinear normal vectors
- Required variable parameterization

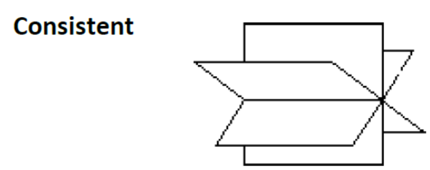


Figure 1: Nature of intersection of three planes

$\therefore$ Π_1, Π_2 **and** Π_3 **intersect along a line and are non-coincident planes (Figure 1).**

14. Given $A = \left(\begin{smallmatrix} -1 & \frac{3}{2} \\ 1 & -1 \end{smallmatrix}\right)$, demonstrate that $det(A) \neq 0$ which implies that A^{-1} exists. Then, determine A^{-1} and verify $AA^{-1} = I$.

Solution:
Begin by demonstrating that $A = \left(\begin{smallmatrix} -1 & \frac{3}{2} \\ 1 & -1 \end{smallmatrix}\right)$ does indeed have an inverse matrix, A^{-1}.
Determine $det(A)$.

$$A = \left(\begin{smallmatrix} -1 & \frac{3}{2} \\ 1 & -1 \end{smallmatrix}\right) \tag{1}$$

$$det(A) = (-1)(-1) - (1)\left(\frac{3}{2}\right) \tag{2}$$

$$det(A) = -\frac{1}{2} \tag{3}$$

Therefore, the inverse matrix, A^{-1} exists because $det(A) \neq 0$.

Now we can determine A^{-1}. Consider some 2x2 matrix $N = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, recall that its inverse matrix, N^{-1} is,

$$N^{-1} = \frac{1}{det(N)}\left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right)$$

By the same token, we have that A^{-1} is,

$$A^{-1} = \frac{1}{det(A)}\left(\begin{smallmatrix} -1 & -\frac{3}{2} \\ -1 & -1 \end{smallmatrix}\right) \tag{4}$$

$$A^{-1} = \frac{1}{-\frac{1}{2}}\left(\begin{smallmatrix} -1 & -\frac{3}{2} \\ -1 & -1 \end{smallmatrix}\right) \tag{5}$$

$$A^{-1} = -2\left(\begin{smallmatrix} -1 & -\frac{3}{2} \\ -1 & -1 \end{smallmatrix}\right) \tag{6}$$

$$A^{-1} = \left(\begin{smallmatrix} 2 & 3 \\ 2 & 2 \end{smallmatrix}\right) \tag{7}$$

Therefore, we have that $A^{-1} = \left(\begin{smallmatrix} 2 & 3 \\ 2 & 2 \end{smallmatrix}\right)$

Verify that $AA^{-1} = I$. Consider some matrix $P = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and some matrix $Q = \left(\begin{smallmatrix} x & y \\ z & k \end{smallmatrix}\right)$, the product of P and Q is $PQ = \left(\begin{smallmatrix} ax+bz & ay+bk \\ cx+dz & cy+dk \end{smallmatrix}\right)$. By the same token, we have that,

$$AA^{-1} = \left(\begin{smallmatrix} -1 & \frac{3}{2} \\ 1 & -1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 2 & 3 \\ 2 & 2 \end{smallmatrix}\right) \tag{8}$$

$$AA^{-1} = \left(\begin{smallmatrix} (-1)(2)+\left(\frac{3}{2}\right)(2) & (-1)(3)+\left(\frac{3}{2}\right)(2) \\ (1)(2)+(-1)(2) & (1)(3)+(-1)(2) \end{smallmatrix}\right) \tag{9}$$

$$AA^{-1} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \tag{10}$$

Therefore, the product of A and its inverse equals the identity matrix, $AA^{-1} = I$.

Therefore, I have finished MCV4UR.
Thank you for being a great Grade 11 and Grade 12 Math teacher, Mr. Blakely!
Have a great summer!
QED