MCR3UR : Pre Advanced Placement Functions, Enriched

Assignment #3

Reference Declaration

Complete the Reference Declaration section below in order for your assignment to be graded.

If you used any references beyond the course text and lectures (such as other texts, discussions with your peers or online resources), indicate this information in the space below. If you did not use any aids, state this in the space provided.

Be sure to cite appropriate theorems or identities throughout your work. Justify conclusions with clear and logical explanations.

Note: Your submitted work must be **your original work**.

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Declared References: Some Latex help from online discussion threads. Discussions with classmates about question 4.

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1. For both sequences below determine the fourth term t_4, and the general term t_n both *explicitly* and *recursively*, and identify the sequence as arithmetic, geometric or neither.

   (a) $12, 8, 4, \ldots$

   Identifying the sequence type
   As n increases by 1, t_n decreases by 4. Therefore, this is an arithmetic sequence with a common difference of -4.

   Determining the general formula
   To determine the general formula, I will use the equation $t_n = cn + b$ where t_n is the term, c is the common difference, n is the term number and b is a constant.
   Sub in $c = -4$, $n = 1$, $t_n = 12$ to determine b.

   $$12 = -4(1) + b$$
   $$b = 12 + 4$$
   $$b = 16$$

   Therefore, the general formula is

   $$t_n = -4n + 16$$

   Determining the recursive formula
   The first term is 12, the second term equals $12 - 4 = 8$, the third term equals $8 - 4 = 4$ and so on ... This means that term $t_n =$ *the previous term* $- 4$. The term previous to t_n can be represented as $t_{(n-1)}$. Therefore, the recursive formula is

   $$t_1 = 12, \; t_n = t_{(n-1)} - 4$$

   (b) $5, 10, 20, \ldots$

   Identifying the sequence type
   As n increases by 1, t_n doubles. Therefore this is a geometric sequence with a common ratio of 2.

   Determining the general formula
   To determine the general formula for this geometric sequence, I will use the equation $t_n = c^n \times b$ where c is the common ratio, t_n, n *and* b remain the same as question 1.a).
   Sub in $c = 2$, $t_n = 5$, $n = 1$ to determine b.

   $$5 = 2^1 \times b$$
   $$b = \frac{5}{2}$$

   Therefore, the general formula for this arithmetic sequence is

   $$t_n = 2^n \times \frac{5}{2}$$

   Determining the recursive formula
   The first term equals 5. The second term equals the first term times two and so on ... This means that term $t_n = 2 \times$ *the previous term.*
   Therefore, the recursive formula for this geometric sequence is

   $$t_1 = 5, \; t_n = 2 \times t_{(n-1)}$$

2. Given the 4th and 6th terms in a sequence are 4 and 12 respectively determine the general term, *recursively*, assuming the sequence is:

   (a) arithmetic

   Determining the general term
   To determine the recursive formula for this arithmetic sequence, t_1 and the common difference is required. First, find the common difference, c. Consider the following:

$$\begin{aligned} t_5 &= t_4 + c \\ t_6 &= t_5 + c \\ t_6 &= (t_4 + c) + c \\ t_6 &= t_4 + 2c \end{aligned}$$

   Move c to one side of the equation and sub in $t_6 = 12,\ t_4 = 4.$

$$\begin{aligned} c &= \frac{t_6 - t_4}{2} \\ c &= \frac{12 - 4}{2} \\ c &= 4 \end{aligned}$$

   Determine the value of the constant, b, using $t_n = cn + b$.
   Sub in $t_n = 4,\ n = 4,\ c = 4.$

$$\begin{aligned} 4 &= 4 \times 4 + b \\ b &= 4 - 16 \\ b &= -12 \end{aligned}$$

   Therefore the general formula is

$$t_n = 4n - 12$$

   Determining the recursive formula
   Using the general formula, sub in $n = 1$ to determine t_1.

$$\begin{aligned} t_1 &= 4 \times 1 - 12 \\ t_1 &= -8 \end{aligned}$$

   Knowing that the first term is -8 and the common difference is 4, the recursive formula is

$$t_1 = -8,\ t_n = t_{(n-1)} + 4$$

   (b) geometric

   Determine the general formula
   Similar to 2.a), I must find t_1 and the common ratio to find the recursive formula. To do so, the general formula is required. Consider the following:

$$\begin{aligned} t_5 &= t_4 \times c \\ t_6 &= t_5 \times c \\ t_6 &= (t_4 \times c)c \\ t_6 &= t_4 \times c^2 \end{aligned}$$

Sub in $t_4 = 4,\ t_6 = 12$ *to determine the common ratio.*

$$\begin{aligned} 12 &= 4c^2 \\ c^2 &= 3 \\ c &= \sqrt{3} \end{aligned}$$

Determine the value of constant, b, using $t_n = c^n \times b$.
Sub in $t_n = 4,\ n = 4,\ c = \sqrt{3}$.

$$\begin{aligned} 4 &= \sqrt{3}^4 \times b. \\ b &= \frac{4}{\sqrt{3}^4} \\ b &= \frac{4}{9} \end{aligned}$$

Therefore the general formula is

$$t_n = \sqrt{3}^n \times \frac{4}{9}$$

Determine the recursive formula
Using the general formula, sub in $n = 1$ to determine t_1.

$$\begin{aligned} t_1 &= \sqrt{3}^1 \times \frac{4}{9} \\ t_1 &= \frac{\sqrt{3} \times 4}{9} \\ t_1 &= \frac{\sqrt{3} \times 4}{\sqrt{3}^4} \\ t_1 &= 4\sqrt{3}^{-3} \\ t_1 &\approx 0.769800358 \end{aligned}$$

Knowing that the first term is $t_1 = 4\sqrt{3}^{-3}$ and the common ratio is $\sqrt{3}$, the recursive formula is

$$t_1 = 4\sqrt{3}^{-3},\ t_n = t_{(n-1)} \times \sqrt{3}$$

3. Express $(2x - 3y)^5$ as a summation based on the binomial theorem; then express it as the sum of six terms demonstrating your understanding of the summation; then simplify the sum.

e.g.

$$\begin{aligned}
(x+y)^3 &= \sum_{r=0}^{3} \binom{3}{r} x^{3-r} y^r \\
&= \binom{3}{0} x^3 y^0 + \binom{3}{1} x^2 y^1 + \binom{3}{2} x^1 y^2 + \binom{3}{3} x^0 y^3 \\
&= x^3 + 3x^2 y + 3xy^2 + y^3
\end{aligned}$$

$$\begin{aligned}
(2x - 3y)^5 &= \sum_{r=0}^{5} \binom{5}{r} 2x^{5-r} (-3y)^r \\
&= \binom{5}{r} 2^{5-r} (-3)^r x^{5-r} y^r \\
&= \binom{5}{0} 2^{5-0} (-3)^0 x^{5-0} y^0 + \binom{5}{1} 2^{5-1} (-3)^1 x^{5-1} y^1 + \binom{5}{2} 2^{5-2} (-3)^2 x^{5-2} y^2 \\
&+ \binom{5}{3} 2^{5-3} (-3)^3 x^{5-3} y^3 + \binom{5}{4} 2^{5-4} (-3)^4 x^{5-4} y^4 + \binom{5}{5} 2^{5-5} (-3)^5 x^{5-5} y^5 \\
&= 2^5 x^5 + 5(2)^4(-3)x^4 y + 10(2)^3(-3)^2 x^3 y^2 + 10(2)^2(-3)^3 x^2 y^3 + 5(2)(-3)^4 x y^4 \\
&+ (-3)^5 y^5 \\
&= 32x^5 - 240x^4 y + 720x^3 y^2 - 1080x^2 y^3 + 810xy^4 - 243y^5
\end{aligned}$$

Therefore, the simplified sum of $(2x - 3y)^5$ is $32x^5 - 240x^4y + 720x^3y^2 - 1080x^2y^3 + 810xy^4 - 243y^5$

4. A Sierpinski Carpet is a fractal design created by starting with a square of area 1 and dividing it into 9 congruent squares and removing the centre one. The same procedure is applied to the remaining 8 squares, and so on, forever. Determine the first 5 terms in the sequence b_n which describes the remaining black area at each step as we go along as well as the general term b_n. Describe what will happen to the area if this process continues forever.

To determine the first 5 terms of the sequence and describe what will happen to the area if it were to continue forever, I will represent the area of the black square(s) in each figure as a fraction. The area of the black squares of the first three figures are

$$\frac{1}{1}, \frac{8}{9}, \frac{64}{81}$$

From the first three terms, I can see that the numerator increases 8 times and the denominator increases 9 times after each figure.
With this, I can determine the area of the black squares in the fourth and fifth figure using the area of black squares in the third figure.

Let the area of black squares be 'a' in figure 4 and 'b' in figure 5.

$$a = \frac{64 \times 8}{81 \times 9}$$
$$a = \frac{512}{729}$$

Therefore, the area of the fourth figure is $\frac{512}{729}$

$$b = a \times \frac{8}{9}$$
$$b = \frac{512 \times 8}{729 \times 9}$$
$$b = \frac{4168}{6561}$$

The area of the fifth figure is $\frac{4168}{6561}$

Therefore, the first five terms of sequence b_n is

$$\frac{1}{1}, \frac{8}{9}, \frac{64}{81}, \frac{512}{729}, \frac{4168}{6561}$$

If this process continues forever, the area will be infinitely close to zero. Since the denominator increases by a larger value compared to the numerator, the fraction gets smaller as the term number increases. And since b_n is a geometric sequence, the fraction will never be zero, but infinitely close to zero.

5. Determine the 50th partial sum of the sequence $t_n = 3 + 5(n-1)$.

$$\begin{aligned}
S_{50} &= \sum_{n=1}^{50} 3 + 5(n-1) \\
&= \sum_{n=1}^{50} 3 + 5n - 5 \\
&= \sum_{n=1}^{50} 5n - 2 \\
&= 5\sum_{n=1}^{50} n + \sum_{n=1}^{50}(-2) \\
&= 5\left[\frac{50(51)}{2}\right] + 50(-2) \\
&= 6375 - 100 \\
&= 6275
\end{aligned}$$

Therefore, the 50^{th} partial sum of the sequence $t_n = 3 + 5(n-1)$ is 6275.

6. Define a sequence a_n whose 5th partial sum is 20 assuming a_n is:

   (a) arithmetic

Since this is an arithmetic sequence, the formula to determine the sum of an arithmetic sequence is

$$S_n = \frac{n[2a + d(n-1)]}{2}$$

where S_n = n^{th} partial sum, n = term number, a = value of the first term, d = common difference.

I know that the sequence has a 5th partial sum of 20, therefore $n = 5$ and $S_n = 20$.
Sub in $n = 5$ and $S_n = 20$ and simplify.

$$\begin{aligned} 20 &= \frac{5[2a + d(5-1)]}{2} \\ \frac{20 \times 2}{5} &= 2a + d(5-1) \\ 8 &= 2a + 4d \end{aligned}$$

Since there are two variables, a, the starting value, and d, the common difference, there is an unlimited amount of sequences where the 5th partial sum of a_n is 20.

Here is an example of arithmetic sequence a_n whose partial sum is 20.
Sub in d with a random number, 5. Determine a, and show the first 5 terms of the sequence

$$\begin{aligned} 8 &= 2a + 4d \\ a &= \frac{8 - 4d}{2} \\ a &= \frac{8 - 4 \times 5}{2} \\ a &= \frac{8 - 20}{2} \\ a &= \frac{-12}{2} \\ a &= -6 \end{aligned}$$

a = -6, d = 5. Therefore, the first five terms of sequence a_n is

$$-6, -1, 4, 9, 14$$

Verifying my work

$$\begin{aligned} S_5 &= -6 + (-1) + 4 + 9 + 14 \\ S_5 &= 20 \end{aligned}$$

Therefore, the general formula for this arithmetic sequence with a 5th partial sum of 20 is

$$S_k = 5n - 11$$

(b) geometric

Since this is a geometric sequence, the formula to determine the sum of a geometric sequence is

$$S_k = \frac{a(r^k - 1)}{r - 1}$$

where $S_k = k^{th}$ *partial term,* $a =$ *value of the first term,* $r =$ *ratio,* $k =$ *term number*
We know that $S_k = 20$ and $k = 5$.
Sub in $S_k = 20$*,* $k = 5$ *and simplify.*

$$\begin{aligned} S_k &= \frac{a(r^k - 1)}{r - 1} \\ 20 &= \frac{a(r^5 - 1)}{r - 1} \end{aligned}$$

Here is an example of geometric sequence a_n *whose partial sum is 20.*
Sub in r *with a random number, 2. Determine* a*, and show the first 5 terms of the sequence*

$$\begin{aligned} 20 &= \frac{a(r^5 - 1)}{r - 1} \\ 20 &= \frac{a(2^5 - 1)}{2 - 1} \\ 20 &= \frac{31a}{1} \\ a &= \frac{20}{31} \end{aligned}$$

$a = \frac{20}{31}, r = 2$. *Therefore, the first five terms of sequence* a_n *is*

$$\frac{20}{31}, \frac{40}{31}, \frac{80}{31}, \frac{160}{31}, \frac{320}{31}$$

Verifying my work

$$\begin{aligned} S_5 &= \frac{20}{31}, \frac{40}{31}, \frac{80}{31}, \frac{160}{31}, \frac{320}{31} \\ S_5 &= 20 \end{aligned}$$

Therefore, the general formula for this geometric sequence with the 5th partial sum of 20 is

$$S_k = 2^k \times \frac{10}{31}$$