Supplementary Material to "On the Spherical Laplace Distribution"

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1 Proof of theoretical results

On the p-dimensional unit hypersphere $\mathbb{S}^p = \{\mathbf{x} \in \mathbb{R}^{p+1} : \|\mathbf{x}\| = 1\}$, the spherical Laplace (SL) distribution is an isotropic location-scale family distribution. Characterized by a location parameter $\boldsymbol{\mu} \in \mathbb{S}^p$ and a scale parameter $\sigma \in \mathbb{R}^+$, the density function of the SL distribution is defined by

$$f_{\mathrm{SL}}(\mathbf{x} \mid \boldsymbol{\mu}, \sigma) = \frac{1}{C_p(\boldsymbol{\mu}, \sigma)} \exp\left(-\frac{d(\mathbf{x}, \boldsymbol{\mu})}{\sigma}\right),$$

for a normalizing constant $C_p(\boldsymbol{\mu}, \sigma)$,

$$C_p(\boldsymbol{\mu}, \sigma) = \int_{\mathbb{S}^p} \exp\left(-\frac{d(\mathbf{x}, \boldsymbol{\mu})}{\sigma}\right) d\mathbf{x}.$$

For the sake of readers' convenience, we reiterate theoretical results presented in the paper and address their proofs accordingly.

Proposition 1. *The normalizing constant $C_p(\boldsymbol{\mu}, \sigma)$ can be written as a univariate integral as*

$$C_p(\boldsymbol{\mu}, \sigma) = A_{p-1} \int_{r=0}^{\pi} \exp\left(-\frac{r}{\sigma}\right) \sin^{p-1}(r) dr, \tag{1}$$

where $A_{p-1} = 2\pi^{p/2}/\Gamma(p/2)$ is the hypervolume or surface area of $\mathbb{S}^{p-1}$ and $\Gamma(\cdot)$ is the standard Gamma function.

Proof. On a complete Riemannian manifold, the geodesic distance between two points is characterized in terms of logarithmic maps and Riemannian metric as $d^2(\mathbf{x}, \boldsymbol{\mu}) = g_{\boldsymbol{\mu}}(\mathrm{Log}_{\boldsymbol{\mu}}(\mathbf{x}), \mathrm{Log}_{\boldsymbol{\mu}}(\mathbf{x}))$, which is equivalent to $\mathrm{Log}_{\boldsymbol{\mu}}(\mathbf{x})^\top \mathrm{Log}_{\boldsymbol{\mu}}(\mathbf{x})$ on $\mathbb{S}^p$ with a canonical metric. By a change of variable $\mathbf{u} = \mathrm{Log}_{\boldsymbol{\mu}}(\mathbf{x})$, a corresponding Jacobian determinant $|\mathbf{J}| = (\sin\|\mathbf{u}\|/\|\mathbf{u}\|)^{p-1}$ is obtained so that

$$\begin{aligned} C_p(\boldsymbol{\mu}, \sigma) &= \int_{\mathbb{S}^p} \exp\left(-\frac{d(\mathbf{x}, \boldsymbol{\mu})}{\sigma}\right) d\mathbf{x} = \int_{\mathbb{S}^p} \exp\left(-\frac{\|\mathbf{u}\|}{\sigma}\right) d\mathbf{x} \\ &= \int_{\|\mathbf{u}\|<\pi} \exp\left(-\frac{\|\mathbf{u}\|}{\sigma}\right) \left(\frac{\sin\|\mathbf{u}\|}{\|\mathbf{u}\|}\right)^{p-1} d\mathbf{u}. \end{aligned} \tag{2}$$

We can express the above using spherical coordinate system,

$$\begin{aligned} (2) &= \int_{r=0}^{\pi} \int_{\varphi_1=0}^{\pi} \cdots \int_{\varphi_{p-2}=0}^{\pi} \int_{\varphi_{p-1}=0}^{2\pi} \exp\left(-\frac{r}{\sigma}\right) \left(\frac{\sin r}{r}\right)^{p-1} r^{p-1} \sin^{p-2}(\varphi_1) \cdots \sin(\varphi_{p-2}) dr d\varphi_1 \cdots d\varphi_{p-1} \\ &= \int_{\varphi_1=0}^{\pi} \cdots \int_{\varphi_{p-2}=0}^{\pi} \int_{\varphi_{p-1}=0}^{2\pi} \sin^{p-2}(\varphi_1) \cdots \sin(\varphi_{p-2}) d\varphi_1 \cdots d\varphi_{p-1} \cdot \int_{r=0}^{\pi} \exp\left(-\frac{r}{\sigma}\right) \sin^{p-1}(r) dr, \end{aligned} \tag{3}$$

where the first term with nested integrals in (3) reduces to the hypervolume A_{p-1} of $\mathbb{S}^{p-1}$. $\square$

Theorem 2. *Let $\mathbf{x}_1, \ldots, \mathbf{x}_N$ be an i.i.d sample on a p-dimensional unit hypersphere $\mathbb{S}^p$. If the sample is contained in an open geodesic ball $B(\mathbf{x}, \pi/4)$ for some $\mathbf{x} \in \mathbb{S}^p$ and not totally contained in any geodesic, maximum likelihood estimates $(\hat{\boldsymbol{\mu}}_{MLE}, \hat{\sigma}_{MLE})$ uniquely exist.*

Proof: We first use the characterization of maximum likelihood estimate for location parameter of the SL distribution as the Fréchet median. Conditions for existence and uniqueness have been extensively studied on a general Riemannian manifold $\mathcal{M}$ [1, 2], whose statements are phrased as follows for completeness. Let $\bar{B}(\mathbf{x},\rho)$ and Δ denote a closed geodesic ball of radius ρ centered at $\mathbf{x} \in \mathcal{M}$ and an upper bound of sectional curvatures in $\bar{B}(\mathbf{x},\rho)$, respectively. Theorem 3.1 of [1] states that if a sample is not totally contained in any geodesic and the radius satisfies

$$\rho < \min\left\{\frac{\pi}{4\sqrt{\Delta}}, \frac{\mathrm{inj}(\bar{B}(\mathbf{x},\rho))}{2}\right\},$$

the Fréchet median exists and is unique because the objective function becomes strictly convex under the stated conditions. The unit hypersphere has a constant sectional curvature so that $\Delta = 1$ and an injectivity radius is π at all points. Therefore, the maximal convexity radius ρ is $\min\{\pi/4, \pi/2\} = \pi/4$ on $\mathbb{S}^p$. On top of a support condition of a random sample being not totally contained in any geodesic, this establishes the existence and uniqueness of $\hat{\boldsymbol{\mu}}_{\mathrm{MLE}}$.

We now turn to examine the scale parameter. Let $g(\sigma) = S/\sigma + \log C_p(\sigma)$ with a known constant $S \in [0,\pi]$ as the geodesic distance between any two points on the sphere is bounded above by the injectivity radius π. It needs to be shown that $g(\sigma)$ admits a unique critical point. The first-order condition $g'(\sigma) = 0$ is equivalent to whether $-SC_p(\sigma) + \sigma^2 C_p'(\sigma) = 0$. Hence, it is sufficient to show if $G(S) = -SC_p(\sigma) + \sigma^2 C_p'(\sigma)$ has a unique zero in $(0,\pi)$ for any choice of σ. This implies a bijection between S and σ, establishing the existence and uniqueness for a critical point of $g(\sigma)$ equivalently.

First, it is trivial that $G(S)$ is a continuous function since every term consists of smooth, bounded functions on a finite interval. Second, we have $G(0) > 0$ as

$$\begin{aligned}
G(0) &= \sigma^2 C_p'(\sigma) \\
&= \sigma^2 \cdot A_{p-1} \int_{r=0}^{\pi} \frac{r}{\sigma^2} \exp\left(-\frac{r}{\sigma}\right) \sin^{p-1}(r) dr \\
&= A_{p-1} \int_{r=0}^{\pi} r \exp\left(-\frac{r}{\sigma}\right) \sin^{p-1}(r) dr,
\end{aligned}$$

where all terms in the integrand are positive on $(0,\pi)$. On the other end, $G(\pi) < 0$ since

$$\begin{aligned}
G(\pi) &= -\pi C_p(\sigma) + \sigma^2 C_p'(\sigma) \\
&= A_{p-1}\left\{-\pi \int_{r=0}^{\pi} \exp\left(-\frac{r}{\sigma}\right) \sin^{p-1}(r) dr + \sigma^2 \int_{r=0}^{\pi} \frac{r}{\sigma^2} \exp\left(-\frac{r}{\sigma}\right) \sin^{p-1}(r) dr\right\} \\
&= A_{p-1} \int_{r=0}^{\pi} (-\pi + r) \exp\left(-\frac{r}{\sigma}\right) \sin^{p-1}(r) dr,
\end{aligned}$$

and $-\pi + r < 0$ while the other terms in the integrand are strictly positive except for a measure zero set. Lastly, $G(S)$ is a monotonically decreasing function. Take $S' \in (0,\pi)$ such that $S < S'$, then

$$\begin{aligned}
G(S') - G(S) &= -S' C_p(\sigma) + \sigma^2 C_p'(\sigma) - (-SC_p(\sigma) + \sigma^2 C_p'(\sigma)) \\
&= (S - S') C_p(\sigma) < 0.
\end{aligned}$$

By the intermediate value theorem and monotonicity, $G(S)$ has a unique zero in $(0,\pi)$ and so does $g'(\sigma)$, which completes the proof. □

2 Details on the rejection sampler for the SL distribution

A standard rejection sampler aims at drawing random samples from a target density $f(\mathbf{x})$ by samples from a proposal distribution $g(\mathbf{x})$, assuming the likelihood ratio $f(\mathbf{x})/g(\mathbf{x})$ is upper bounded by some constant $M \in (1, \infty)$ over the entire domain. A draw from $g(\mathbf{x})$ is accepted as a random sample from $f(\mathbf{x})$ with probability $f(\mathbf{x})/Mg(\mathbf{x})$ and the process is repeated until a successful draw. For the SL distribution with parameters $(\boldsymbol{\mu}, \sigma) \in \mathbb{S}^p \times \mathbb{R}^+$, we consider the SN distribution of parameters $(\boldsymbol{\mu}, 1/\sigma) \in \mathbb{S}^p \times \mathbb{R}^+$ as a proposal density. We note that density functions for SN and SL distributions are explicitly written as

$$f_{\mathrm{SN}}(\mathbf{x} \mid \boldsymbol{\mu}, \lambda) = \frac{1}{Z_p(\lambda)} \exp\left(-\frac{\lambda}{2} d^2(\mathbf{x}, \boldsymbol{\mu})\right) \quad \text{and} \quad f_{\mathrm{SL}}(\mathbf{x} \mid \boldsymbol{\mu}, \sigma) = \frac{1}{C_p(\sigma)} \exp\left(-\frac{1}{\sigma} d(\mathbf{x}, \boldsymbol{\mu})\right),$$

for some normalizing constant Z_p for the SN distribution [3]. Under the scenario, the likelihood ratio is given by

$$\begin{aligned}
\frac{f_{\mathrm{SL}}(\mathbf{x} \mid \boldsymbol{\mu}, \sigma)}{f_{\mathrm{SN}}(\mathbf{x} \mid \boldsymbol{\mu}, \lambda)} &= \frac{Z_p(\lambda)}{C_p(\sigma)} \exp\left(-\frac{1}{\sigma} d(\mathbf{x}, \boldsymbol{\mu}) + \frac{\lambda}{2} d^2(\mathbf{x}, \boldsymbol{\mu})\right) \\
&= \frac{Z_p(1/\sigma)}{C_p(\sigma)} \exp\left(-\frac{1}{\sigma} d(\mathbf{x}, \boldsymbol{\mu}) + \frac{1}{2\sigma} d^2(\mathbf{x}, \boldsymbol{\mu})\right) \\
&= \frac{Z_p(1/\sigma)}{C_p(\sigma)} \exp\left(\frac{(d(\mathbf{x}, \boldsymbol{\mu}) - 1)^2}{2\sigma}\right) \cdot \exp\left(-\frac{1}{2\sigma}\right) \\
&\leq \frac{Z_p(1/\sigma)}{C_p(\sigma)} \exp\left(\frac{((\pi - 1)^2}{2\sigma}\right) \cdot \exp\left(-\frac{1}{2\sigma}\right) =: M,
\end{aligned}$$

where the constant M is an upper bound of the ratio. An inequality comes from the fact that injectivity radius of a unit hypersphere is π, i.e., any two points on the unit hypersphere have distance of π at most. Then, the corresponding probabilistic acceptance threshold τ is as follows;

$$\begin{aligned}
\tau &= \frac{f_{\mathrm{SL}}(\mathbf{x} \mid \boldsymbol{\mu}, \sigma)}{M f_{\mathrm{SN}}(\mathbf{x} \mid \boldsymbol{\mu}, 1/\sigma)} \\
&= \frac{C_p(\sigma)}{Z_p(1/\sigma)} \exp\left(-\frac{(\pi - 1)^2}{2\sigma}\right) \cdot \exp\left(\frac{1}{2\sigma}\right) \\
&\quad \times \frac{1}{C_p(\sigma)} \exp\left(-\frac{d(\mathbf{x}, \boldsymbol{\mu})}{\sigma}\right) \cdot Z_p(1/\sigma) \cdot \exp\left(\frac{1}{2\sigma} d^2(\mathbf{x}, \boldsymbol{\mu})\right) \\
&= \exp\left(\frac{1}{2\sigma} d^2(\mathbf{x}, \boldsymbol{\mu}) - \frac{1}{\sigma} d(\mathbf{x}, \boldsymbol{\mu}) + \frac{1}{2\sigma}\right) \cdot \exp\left(-\frac{(\pi - 1)^2}{2\sigma}\right) \\
&= \exp\left(\frac{(d(\mathbf{x}, \boldsymbol{\mu}) - 1)^2 - (\pi - 1)^2}{2\sigma}\right).
\end{aligned}$$

3 Computation for the mixture model of SL distributions

In this section, the pseudocode for fitting a SL mixture model with a finite number of components is described. In order to acquire computational speedups when the number of observations is large, we adopt the following heuristic assignment schematics, hard and $\mathsf{stochastic}$ assignments. Note that the references within the following algorithm are referred to the main paper.

Algorithm EM algorithm for the finite mixture of SL distributions.

Input: a random sample $\mathbf{x}_1, \dots, \mathbf{x}_N \in \mathbb{S}^p$, number of clusters K.

Output: a soft clustering/membership matrix $\mathbf{\Gamma}$.

  Initialize $\mathbf{\Theta}^{(0)} = \{\pi_k^{(0)}, \boldsymbol{\mu}_k^{(0)}, \sigma_k^{(0)}\}_{k=1}^K$.
  repeat
    {E-step}
    for $n = 1 : N$ **do**
      for $k = 1 : K$ **do**
        $\mathbf{\Gamma}(n,k) = \pi_k^{(t)} f_{\mathrm{SL}}(\mathbf{x}_n \mid \boldsymbol{\mu}_k^{(t)}, \sigma_k^{(t)})$
      end for
      $\mathbf{\Gamma}(n,:) = \mathbf{\Gamma}(n,:)/\sum_{k=1}^K \mathbf{\Gamma}(n,k)$
    end for
    {Heuristics}
    if hard assignment **then**
      $\mathbf{\Gamma} \leftarrow \mathsf{hard}(\mathbf{\Gamma})$ by (13).
    else if stochastic assignment **then**
      $\mathbf{\Gamma} \leftarrow \mathsf{stochastic}(\mathbf{\Gamma})$ by (14).
    end if
    {M-step}
    for $k = 1 : K$ **do**
      $\pi_k^{(t+1)} = \sum_{n=1}^N \gamma_{nk}/N$.
      $\boldsymbol{\mu}_k^{(t+1)} = \underset{\boldsymbol{\mu} \in \mathbb{S}^p}{\operatorname{argmin}} \ \sum_{n=1}^N \gamma_{nk} \cdot d(\mathbf{x}_n, \boldsymbol{\mu})$ by Algorithm 2.
    end for
    if homogeneous model **then**
      Update $\sigma^{(t+1)}$ using (12) by Algorithm 3.
    else
      for $k = 1 : K$ **do**
        Update $\sigma_k^{(t+1)}$ using (11) by Algorithm 3.
      end for
    end if
  until convergence.

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4 Clustering results of the **small-mix** and **household** examples

Table 1: Average of clustering quality indices from 100 runs for the small-mix experiment. The cells with bold-face numerics indicate that the corresponding row is the best performing model given the cluster number K and quality index.

	Jaccard			Rand			NMI		
	$K=2$	$K=3$	$K=4$	$K=2$	$K=3$	$K=4$	$K=2$	$K=3$	$K=4$
moSL-soft	**0.9862**	0.7521	0.6107	**0.9930**	0.8756	0.8052	**0.9712**	0.7972	0.7194
moSL-hard	0.9689	0.7430	0.5861	0.9841	0.8708	0.7926	0.9422	0.7848	0.7015
kmeans	0.9575	0.7463	0.5588	0.9782	0.8726	0.7789	0.9270	0.7822	0.6930
spkmeans	0.9465	0.7427	0.5997	0.9724	0.8709	0.7996	0.9118	0.7935	0.7104
movMF	0.9852	**0.7927**	**0.6841**	0.9825	**0.8963**	**0.8424**	0.9607	0.8193	**0.7582**
moSN	0.9853	0.7907	0.6325	0.9830	0.8953	0.8167	0.9627	**0.8225**	0.7335

Table 2: Clustering quality indices for the household data projected onto $\mathbb{S}^2$. The cells with bold-face numerics indicate that the corresponding row is the best performing algorithm given the cluster number K and quality index.

	Jaccard			Rand			NMI		
	$K=2$	$K=3$	$K=4$	$K=2$	$K=3$	$K=4$	$K=2$	$K=3$	$K=4$
moSL-soft	**1.0000**	0.7275	0.6342	**1.0000**	0.8603	**0.8218**	**1.0000**	0.7244	**0.7524**
moSL-hard	0.5920	0.7275	0.5363	0.7385	0.8603	0.7705	0.5105	0.7244	0.6546
kmeans	0.5920	**0.7789**	0.5363	0.7385	**0.8923**	0.7705	0.5105	**0.8331**	0.6546
spkmeans	0.5920	**0.7789**	0.5363	0.7385	**0.8923**	0.7705	0.5105	**0.8331**	0.6546
movMF	0.9025	0.7275	**0.6450**	0.9500	0.8603	0.8179	0.8558	0.7244	0.6645
moSN	0.5920	0.7275	0.5363	0.7385	0.8603	0.7705	0.5105	0.7244	0.6546

5 Performance measures from the additional simulations for parameter estimation

In this section, we report performance measures of the proposed algorithms for parameter estimation under varying simulation settings such as dimensionality, scale for the data-generating distribution, and the number of observations of a random sample.

Table 3: Average accuracy and elapsed time for the location parameter estimation example from 100 repeats per setting. Accuracy is measured by the geodesic distance $d(\hat{\boldsymbol{\mu}}_{\text{MLE}}, \boldsymbol{\mu}_0)$ and elapsed time is measured in seconds.

dimension	σ_0	n	Weiszfeld		RGD	
			accuracy	time	accuracy	time
$p = 5$	0.01	50	0.00672	0.00031	0.00693	0.00046
		100	0.00445	0.00053	0.00457	0.00075
		250	0.00277	0.00125	0.00284	0.00177
		500	0.00204	0.00227	0.00209	0.00340
	0.05	50	0.03223	0.00029	0.03301	0.00042
		100	0.02189	0.00059	0.02258	0.00090
		250	0.01318	0.00141	0.01349	0.00206
		500	0.00925	0.00243	0.00949	0.00356
	0.1	50	0.06844	0.00032	0.07044	0.00044
		100	0.04683	0.00062	0.04803	0.00086
		250	0.03116	0.00140	0.03195	0.00196
		500	0.02404	0.00264	0.02471	0.00400
	0.5	50	0.33819	0.00086	0.34528	0.00116
		100	0.22558	0.00114	0.23096	0.00167
		250	0.13930	0.00276	0.14277	0.00401
		500	0.08994	0.00519	0.09231	0.00777
	1	50	0.56018	0.00102	0.57682	0.00139
		100	0.44617	0.00199	0.45898	0.00285
		250	0.27722	0.00441	0.28263	0.00662
		500	0.19769	0.00862	0.20139	0.01300
	5	50	1.22967	0.00152	1.26201	0.00220
		100	1.09516	0.00338	1.12680	0.00483
		250	0.82620	0.00891	0.84687	0.01346
		500	0.73339	0.01674	0.75288	0.02417
	10	50	1.13413	0.00145	1.16390	0.00207
		100	1.15891	0.00366	1.19725	0.00559
		250	0.97991	0.00864	1.00427	0.01178
		500	0.87189	0.01705	0.88945	0.02503

	0.01	50	0.01113	0.00026	0.01139	0.00040
		100	0.00796	0.00043	0.00818	0.00060
		250	0.00502	0.00113	0.00515	0.00171
		500	0.00348	0.00216	0.00357	0.00311
	0.05	50	0.05798	0.00029	0.05930	0.00041
		100	0.04115	0.00051	0.04210	0.00069
		250	0.02366	0.00140	0.02434	0.00198
		500	0.01828	0.00244	0.01878	0.00363
	0.1	50	0.12417	0.00036	0.12668	0.00051
		100	0.08499	0.00077	0.08700	0.00116
		250	0.05368	0.00159	0.05505	0.00246
		500	0.03857	0.00336	0.03967	0.00499
$p = 10$	0.5	50	0.64652	0.00099	0.66424	0.00139
		100	0.43624	0.00214	0.44816	0.00305
		250	0.29295	0.00635	0.29953	0.00917
		500	0.20830	0.00949	0.21271	0.01310
	1	50	0.92133	0.00152	0.94644	0.00224
		100	0.72675	0.00308	0.74565	0.00444
		250	0.50497	0.00815	0.51890	0.01147
		500	0.36499	0.01601	0.37349	0.02377
	5	50	1.34914	0.00164	1.39048	0.00243
		100	1.32430	0.00396	1.35234	0.00588
		250	1.27650	0.00999	1.31860	0.01441
		500	1.23680	0.02009	1.27001	0.02874
	10	50	1.36467	0.00159	1.39984	0.00229
		100	1.37816	0.00389	1.40600	0.00547
		250	1.32753	0.01000	1.35787	0.01466
		500	1.23180	0.01968	1.25773	0.02876
	0.01	50	0.01796	0.00056	0.01828	0.00088
		100	0.01295	0.00090	0.01328	0.00132
		250	0.00780	0.00201	0.00797	0.00288
		500	0.00555	0.00410	0.00567	0.00608
	0.05	50	0.09916	0.00091	0.10187	0.00132
		100	0.06678	0.00143	0.06853	0.00200
		250	0.04320	0.00291	0.04438	0.00410
		500	0.02917	0.00558	0.02998	0.00817
	0.1	50	0.22941	0.00119	0.23764	0.00176
		100	0.16278	0.00210	0.16719	0.00300
		250	0.10096	0.00503	0.10376	0.00702
		500	0.06847	0.00908	0.07014	0.01326
$p = 20$	0.5	50	0.91954	0.00339	0.94349	0.00483

		100	0.75020	0.00758	0.76419	0.01086
		250	0.56608	0.01865	0.58265	0.02786
		500	0.41351	0.03322	0.42388	0.04756
	1	50	1.15131	0.00434	1.17831	0.00617
		100	1.06114	0.00926	1.08457	0.01355
		250	0.91941	0.02432	0.94822	0.03475
		500	0.76804	0.04435	0.79063	0.06720
	5	50	1.36798	0.00419	1.39901	0.00582
		100	1.30432	0.01059	1.33375	0.01629
		250	1.19988	0.02431	1.22636	0.03481
		500	1.13442	0.04344	1.15883	0.06197
	10	50	1.39784	0.00435	1.43111	0.00648
		100	1.44322	0.01143	1.47934	0.01659
		250	1.37432	0.02568	1.40912	0.03777
		500	1.37062	0.04267	1.40315	0.06225

Table 4: Average accuracy and elapsed time for the scale parameter estimation example from 100 repeats per setting. Accuracy is measured by the relative error $|\hat{\sigma}_{\text{MLE}} - \sigma_0|/\sigma_0$ and elapsed time is measured in seconds.

σ_0	p	n	accuracy				time			
			NewtonE	NewtonA	Roptim	DE	NewtonE	NewtonA	Roptim	DE
0.01	5	50	0.08476	0.07592	0.08391	0.08397	0.00302	0.00286	0.00578	0.05060
		100	0.07152	0.06170	0.07239	0.07233	0.00345	0.00356	0.00774	0.04363
		250	0.07278	0.06250	0.07434	0.07423	0.00603	0.00630	0.01593	0.04498
		500	0.07289	0.06235	0.07437	0.07427	0.01044	0.01047	0.02882	0.04939
	10	50	0.18025	0.18320	0.18324	0.21126	0.00219	0.00270	0.00692	0.04245
		100	0.17433	0.17645	0.17413	0.19400	0.00326	0.00358	0.00921	0.04334
		250	0.17728	0.18015	0.17720	0.20104	0.00566	0.00656	0.01609	0.04625
		500	0.17432	0.17604	0.17460	0.19928	0.00993	0.01045	0.03076	0.05010
	20	50	0.33431	0.33743	0.34380	0.34421	0.00608	0.01412	0.00962	0.05629
		100	0.33646	0.33305	0.33467	0.33383	0.00355	0.00588	0.00977	0.04512
		250	0.33514	0.33203	0.33707	0.33742	0.00176	0.00392	0.01693	0.04721
		500	0.33799	0.33263	0.33543	0.33527	0.00074	0.00279	0.03125	0.05078
0.05	5	50	0.06596	0.06595	0.06582	0.06589	0.00244	0.00271	0.00574	0.05216
		100	0.04714	0.04706	0.04710	0.04699	0.00370	0.00424	0.00870	0.05252
		250	0.04827	0.04822	0.04823	0.04828	0.00634	0.00599	0.01692	0.05054
		500	0.04797	0.04800	0.04797	0.04794	0.00980	0.01013	0.02931	0.05446
	10	50	0.17251	0.17250	0.17253	0.17255	0.00211	0.00283	0.00544	0.04097
		100	0.15935	0.15957	0.15945	0.15929	0.00316	0.00333	0.00822	0.04137

		250	0.15437	0.15423	0.15414	0.15409	0.00555	0.00549	0.03549	0.04221
		500	0.15557	0.15546	0.15563	0.15567	0.00931	0.00957	0.02636	0.04556
	20	50	0.30124	0.30467	0.30743	0.30756	0.00493	0.00406	0.00617	0.03738
		100	0.29983	0.29687	0.30345	0.30363	0.00521	0.00608	0.00830	0.04097
		250	0.29234	0.29548	0.29978	0.29932	0.00828	0.00852	0.01636	0.04442
		500	0.29043	0.29635	0.29891	0.29896	0.01104	0.01163	0.03167	0.04748
0.1	5	50	0.08211	0.08203	0.08227	0.08214	0.00211	0.00256	0.00599	0.04470
		100	0.05459	0.05471	0.05462	0.05460	0.00297	0.00344	0.00975	0.04411
		250	0.04692	0.04697	0.04688	0.04689	0.00599	0.00588	0.01668	0.04658
		500	0.04376	0.04371	0.04369	0.04371	0.00954	0.01075	0.02782	0.04950
	10	50	0.12741	0.12717	0.12744	0.12733	0.00239	0.00257	0.00550	0.04220
		100	0.12658	0.12642	0.12655	0.12654	0.00308	0.00377	0.00842	0.04436
		250	0.11350	0.11333	0.11333	0.11311	0.00599	0.00691	0.01566	0.04757
		500	0.11697	0.11693	0.11705	0.11689	0.01026	0.01068	0.02884	0.05080
	20	50	0.15965	0.15970	0.15980	0.15966	0.00253	0.00264	0.00615	0.04245
		100	0.14674	0.14684	0.14691	0.14704	0.00308	0.00342	0.00947	0.04818
		250	0.14730	0.14706	0.14715	0.14731	0.00595	0.00640	0.01826	0.04716
		500	0.14029	0.14028	0.14010	0.14016	0.01050	0.01128	0.03038	0.05181
0.5	5	50	0.13775	0.13774	0.13769	0.13761	0.00207	0.00267	0.00547	0.03804
		100	0.09126	0.09133	0.09134	0.09146	0.00311	0.00383	0.00888	0.03870
		250	0.06190	0.06173	0.06183	0.06192	0.00567	0.00632	0.01571	0.04117
		500	0.03962	0.03956	0.03953	0.03954	0.01002	0.01068	0.03084	0.05024
	10	50	0.21176	0.21193	0.21203	0.21195	0.00234	0.00258	0.00557	0.04184
		100	0.13793	0.13808	0.13804	0.13784	0.00319	0.00346	0.00848	0.04343
		250	0.10370	0.10404	0.10392	0.10372	0.00541	0.00653	0.01501	0.04461
		500	0.06312	0.06310	0.06319	0.06321	0.01013	0.01039	0.02857	0.04812
	20	50	0.42899	0.42865	0.42862	0.42841	0.00254	0.00262	0.00613	0.04667
		100	0.28234	0.28228	0.28239	0.28221	0.00337	0.00378	0.00820	0.04645
		250	0.19248	0.19208	0.19251	0.19237	0.00555	0.00595	0.01591	0.04761
		500	0.11190	0.11188	0.11161	0.11188	0.00960	0.01537	0.02721	0.04991
1	5	50	0.22924	0.22915	0.22935	0.22958	0.00201	0.00266	0.00591	0.03786
		100	0.16745	0.16695	0.16710	0.16734	0.00277	0.00341	0.00870	0.03744
		250	0.12082	0.12102	0.12087	0.12087	0.00540	0.00613	0.01574	0.03923
		500	0.09411	0.09406	0.09396	0.09399	0.01006	0.01030	0.02743	0.04383
	10	50	0.44890	0.44828	0.44850	0.44842	0.00207	0.00308	0.00567	0.04217
		100	0.29815	0.29752	0.29777	0.29775	0.00302	0.00350	0.00930	0.04312
		250	0.19481	0.19477	0.19517	0.19503	0.00578	0.00640	0.01586	0.04965
		500	0.15321	0.15316	0.15329	0.15339	0.00941	0.00988	0.02685	0.04639
	20	50	0.65159	0.65101	0.65132	0.65158	0.00253	0.00288	0.00625	0.04664
		100	0.53600	0.53655	0.53708	0.53731	0.00310	0.00371	0.00841	0.04766
		250	0.33806	0.33847	0.33757	0.33878	0.00591	0.00575	0.01612	0.04525

		500	0.25429	0.25480	0.25491	0.25456	0.01012	0.00956	0.02610	0.04889
5	5	50	0.71740	0.71967	0.71801	0.71752	0.00212	0.00294	0.00662	0.04423
		100	0.63575	0.63764	0.63610	0.63647	0.00314	0.00399	0.00765	0.03694
		250	0.45104	0.45045	0.45059	0.45139	0.00573	0.00666	0.01527	0.03954
		500	0.35416	0.35469	0.35444	0.35491	0.01027	0.01108	0.02760	0.04411
	10	50	0.85163	0.85407	0.85222	0.85366	0.00254	0.00310	0.00600	0.04527
		100	0.80135	0.80051	0.80126	0.80144	0.00365	0.00376	0.00997	0.04395
		250	0.67648	0.67539	0.67657	0.67521	0.00605	0.00709	0.01707	0.04646
		500	0.51264	0.51321	0.51206	0.51318	0.01171	0.01271	0.03043	0.05196
	20	50	0.93176	0.93154	0.93079	0.93204	0.00238	0.00348	0.00687	0.04793
		100	0.90124	0.90096	0.89997	0.90214	0.00398	0.00370	0.00940	0.05832
		250	0.84475	0.84479	0.84456	0.84534	0.00609	0.00663	0.01450	0.04643
		500	0.78437	0.78493	0.78330	0.78365	0.01013	0.01049	0.02714	0.05347
10	5	50	0.85464	0.85690	0.85581	0.85660	0.00218	0.00363	0.00543	0.03934
		100	0.81485	0.81529	0.81429	0.81520	0.00334	0.00429	0.00816	0.03856
		250	0.70522	0.70388	0.70391	0.70506	0.00613	0.00733	0.01706	0.04160
		500	0.65070	0.64979	0.65008	0.65012	0.01006	0.01129	0.03010	0.04922
	10	50	0.92874	0.92971	0.93049	0.93003	0.00245	0.00264	0.00582	0.04183
		100	0.90104	0.90136	0.90001	0.90096	0.00348	0.00405	0.00821	0.04234
		250	0.84038	0.84082	0.84018	0.84047	0.00618	0.00677	0.01516	0.04610
		500	0.78663	0.78595	0.78569	0.78597	0.01078	0.01127	0.03031	0.05010
	20	50	0.96752	0.96667	0.96787	0.96733	0.00260	0.00281	0.00842	0.05209
		100	0.94684	0.94531	0.94769	0.94665	0.00334	0.00345	0.00868	0.04889
		250	0.91812	0.92019	0.91929	0.91853	0.00610	0.00553	0.01623	0.04735
		500	0.88335	0.88276	0.88389	0.88446	0.01002	0.01043	0.02840	0.05380

References

[1] L. Yang, "Riemannian median and its estimation," *LMS Journal of Computation and Mathematics*, vol. 13, pp. 461–479, Dec. 2010.

[2] M. Arnaudon, F. Barbaresco, and L. Yang, "Medians and Means in Riemannian Geometry: Existence, Uniqueness and Computation," in *Matrix Information Geometry* (F. Nielsen and R. Bhatia, eds.), pp. 169–197, Berlin, Heidelberg: Springer Berlin Heidelberg, 2013.

[3] S. Hauberg, "Directional Statistics with the Spherical Normal Distribution," in *2018 21st International Conference on Information Fusion (FUSION)*, (Cambridge, United Kingdom), pp. 704–711, IEEE, July 2018.