

Hybrid model of Vlasov-Poisson equations and comparison of Hamiltonian method and Lawson method

A. Crestetto ¹ N. Crouseilles ^{2,3} J. Massot ^{3,2}

¹LMJL, Université de Nantes

²Inria Rennes – Bretagne Atlantique

³IRMAR, Université de Rennes

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- 1 Modelization
- 2 Numerical scheme
 - Splitting method
 - Lawson method
- 3 Adaptive time step methods
 - Splitting method
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- 4 Relation of dispersion
- 5 Numerical results
- 6 Conclusion

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Vlasov-Poisson-Ampère equations $1D_x \times 1D_v$

Our model: transport of electron density distribution $f = f(t, x, v)$:

$$\begin{cases} \partial_t f + v \partial_x f + E \partial_v f = 0 \\ \partial_x E = \int_{\mathbb{R}} f \, dv - 1 \\ \partial_t E = - \int_{\mathbb{R}} v f \, dv + \frac{1}{|\Omega|} \int_{\Omega} \int_{\mathbb{R}} v f \, dv dx \end{cases} \quad (x, v) \in \Omega \times \mathbb{R}$$

Motivations:

- We consider an initial condition of the form: $f = f_c + f_h$ with:
 $f_c(t=0, x, v) = \mathcal{M}_{\rho_c, u_c, T_c}(v) \underset{T_c \rightarrow 0}{=} \delta_{v-u_c}(v) \rho_c(t=0, x)$
- We want high order methods in (x, v)
 - FFT in x + WENO in v
- We want high order methods in time t
 - splitting method vs exponential integrator

- Grid methods can't have an initial condition like :

$$f_0(x, v) = \rho_{c,0}(x)\delta_{v-u_c}(v) + f_{h,0}(x, v)$$

- Idea is to derive an hybrid model :

- Cold plasma approximation: $\frac{T_c}{T_h} \ll 1 \rightarrow f_c(t, x, v) = \rho_c(t, x)\delta_{v-u_c(t,x)}(v)$:
 - Fluid dynamic for cold particles (no velocity grid)
- Hypothesis on hot particles: $\int_{\mathbb{R}} f_h(t, x, v) dv \ll \rho_c(t, x)$:
 - Kinetic dynamic for hot particles

Derivation of hybrid model

$$\begin{cases} \int_{\mathbb{R}} \begin{pmatrix} 1 \\ v \end{pmatrix} \left(\partial_t f_c + v \partial_x f_c + E \partial_v f_c = 0 \right) dv \\ \partial_t f_h + v \partial_x f_h + E \partial_v f_h = 0 \\ \partial_x E = \int_{\mathbb{R}} f_c dv + \int_{\mathbb{R}} f_h dv - 1 \\ \partial_t E = - \int_{\mathbb{R}} v f_c dv - \int_{\mathbb{R}} v f_h dv + \frac{1}{|\Omega|} \int_{\Omega} \int_{\mathbb{R}} v (f_c + f_h) dv dx \end{cases}$$

we note:

$$\begin{pmatrix} \rho_c(t, x) \\ \rho_c(t, x) u_c(t, x) \end{pmatrix} = \int_{\mathbb{R}} \begin{pmatrix} 1 \\ v \end{pmatrix} f_c(t, x, v) dv$$

and *cold plasma approximation*: $f_c(t, x, v) = \rho_c(t, x) \delta_{v=u_c(t, x)}(v)$

Non-linear hybrid fluid-kinetic model

$$\begin{cases} \partial_t \rho_c + \partial_x(\rho_c u_c) = 0 \\ \partial_t(\rho_c u_c) + \partial_x(\rho_c u_c^2) - \rho_c E = 0 \\ \partial_t f_h + v \partial_x f_h + E \partial_v f_h = 0 \\ \partial_t E = -\rho_c u_c - \int_{\mathbb{R}} v f_h dv + \frac{1}{|\Omega|} \left(\int_{\Omega} \int_{\mathbb{R}} v f_h dv dx + \int_{\Omega} \rho_c u_c dx \right) \\ \left(\partial_x E = \rho_c + \int_{\mathbb{R}} f_h dv - 1 \right) \end{cases}$$

- If Poisson equation satisfied initially, the equation is propagated with time.

Following physicists framework, we linearize this non-linear model

Linearization of hybrid fluid-kinetic model

Linearization near equilibrium :

$$\begin{aligned}\rho_c(t, x) &= \rho_c^{(0)}(x) + \varepsilon \rho_c^{(1)}(t, x) \\ u_c(t, x) &= \varepsilon u_c^{(1)}(t, x) \\ E(t, x) &= \varepsilon E^{(1)}(t, x) \\ f_h(t, x, v) &= f_h^{(0)}(v) + \varepsilon f_h^{(1)}(t, x, v)\end{aligned}$$

We obtain **Linear hybrid model** (LHM):

$$\begin{cases} \partial_t u_c^{(1)} = E^{(1)} + \mathcal{O}(\varepsilon) \\ \partial_t E^{(1)} = -\rho_c^{(0)} u_c^{(1)} - \int_{\mathbb{R}} v f_h^{(1)} dv + \mathcal{O}(\varepsilon) \\ \partial_t f_h^{(1)} + v \partial_x f_h^{(1)} + E \partial_v f_h^{(1)} = 0 + \mathcal{O}(\varepsilon) \end{cases}$$

Properties: conservation of mass and total energy ✓

All this derivation can be generalized to 3Dx-3Dv



Holderied et al. (2020)

LHM has an hamiltonian structure. The Poisson bracket is define by:

$$\begin{aligned}\{\mathcal{F}, \mathcal{G}\}(u, E, f) &= \int_{\mathbb{R}} \int_{\mathbb{R}} f \left(\partial_x \frac{\delta \mathcal{F}}{\delta f} \partial_v \frac{\delta \mathcal{G}}{\delta f} - \partial_v \frac{\delta \mathcal{F}}{\delta f} \partial_x \frac{\delta \mathcal{G}}{\delta f} \right) dv dx \\ &+ \int_{\mathbb{R}} \left(\frac{\delta \mathcal{F}}{\delta u} \frac{\delta \mathcal{G}}{\delta E} - \frac{\delta \mathcal{F}}{\delta E} \frac{\delta \mathcal{G}}{\delta u} \right) dx \\ &+ \int_{\mathbb{R}} \int_{\mathbb{R}} \left(\frac{\delta \mathcal{F}}{\delta E} \partial_v f \frac{\delta \mathcal{G}}{\delta f} - \frac{\delta \mathcal{G}}{\delta E} \partial_v f \frac{\delta \mathcal{F}}{\delta f} \right) dv dx\end{aligned}$$

and Hamiltonian by:

$$\mathcal{H}(t) = \underbrace{\frac{1}{2} \int_{\mathbb{R}} E^2 dx}_{\mathcal{H}_E} + \underbrace{\frac{1}{2} \int_{\mathbb{R}} \rho_c^{(0)} u_c^2 dx}_{\mathcal{H}_{u_c}} + \underbrace{\frac{1}{2} \int_{\Omega} \int_{\mathbb{R}} v^2 f_h dv dx}_{\mathcal{H}_{f_h}}$$

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Hamiltonian splitting

We would like to implement a splitting method inspired by a hamiltonian splitting.

With $U = \begin{pmatrix} u_c \\ E \\ f_h \end{pmatrix}$ problem can be written as:

$$\begin{aligned}\dot{U} &= \{U, \mathcal{H}\} \\ &= \{U, \mathcal{H}_E\} + \{U, \mathcal{H}_{u_c}\} + \{U, \mathcal{H}_{f_h}\}\end{aligned}$$

We obtain the splitting:

- $\dot{U} = \{U, \mathcal{H}_E\}$ \rightarrow solution is $\varphi_t^{[E]}(U^0)$
- $\dot{U} = \{U, \mathcal{H}_{u_c}\}$ \rightarrow solution is $\varphi_t^{[u_c]}(U^0)$
- $\dot{U} = \{U, \mathcal{H}_{f_h}\}$ \rightarrow solution is $\varphi_t^{[f_h]}(U^0)$

Hamiltonian structure paves the way of a splitting method.

Splitting method

- **Lie:** order 1 method, composition of substeps:

$$U(t^{n+1}) \approx U^{n+1} = \varphi_{\Delta t}^{[E]} \circ \varphi_{\Delta t}^{[u_c]} \circ \varphi_{\Delta t}^{[f_h]}(U^n)$$

- **Strang:** order 2 method, for a 3 steps formulation:

$$U^{n+1} = S_{\Delta t}(U^n) = \varphi_{\Delta t/2}^{[f_h]} \circ \varphi_{\Delta t/2}^{[u_c]} \circ \varphi_{\Delta t}^{[E]} \circ \varphi_{\Delta t/2}^{[u_c]} \circ \varphi_{\Delta t/2}^{[f_h]}(U^n)$$



Strang (1968)

- **Suzuki:** order 4 method, composition of 5 Strang methods:

$$U^{n+1} = \mathcal{S}_{\Delta t}(U^n) = S_{\alpha_1 \Delta t} \circ S_{\alpha_2 \Delta t} \circ S_{\alpha_3 \Delta t} \circ S_{\alpha_2 \Delta t} \circ S_{\alpha_1 \Delta t}(U^n)$$

with:

$$\alpha_1 = \alpha_2 = \frac{1}{4 - \sqrt[3]{4}} \quad \alpha_3 = \frac{1}{1 - 4^{\frac{2}{3}}}$$



Suzuki (1990)



Casas and Escorihuela-Tomàs (2020)

$$\varphi^{[E]}(U) = \begin{cases} \partial_t u_c = E \\ \partial_t E = 0 \\ \partial_t f_h = -E \partial_v f_h \end{cases} \quad \rightarrow \quad \varphi_{\Delta t}^{[E]}(U^n) = \begin{pmatrix} u_c^n + \Delta t E^n \\ E^n \\ f_h^n(x, v - \Delta t E^n) \end{pmatrix}$$

Numerical tools:

- Lagrange 5 interpolation to approximate $f_h(x, v - \Delta t E^n)$
- More costly step, so we keep it in the middle of Strang method

Numerical resolution of each step

$$\varphi^{[u_c]}(U) = \begin{cases} \partial_t u_c = 0 \\ \partial_t E = -\rho_c^{(0)} u_c \\ \partial_t f_h = 0 \end{cases} \rightarrow \varphi_{\Delta t}^{[u_c]}(U^n) = \begin{pmatrix} u_c^n \\ E^n - \Delta t \rho_c^{(0)} u_c^n \\ f_h^n \end{pmatrix}$$

Numerical tools:

- Fastest step

Numerical resolution of each step

$$\varphi^{[f_h]}(U) = \begin{cases} \partial_t u_c = 0 \\ \partial_t E = - \int_{\mathbb{R}} v f_h \, dv \rightarrow \varphi_{\Delta t}^{[f_h]}(U^n) = \left(\hat{E}^n - \frac{i}{k} \int_{\mathbb{R}} (e^{-ikv\Delta t} - 1) \hat{f}_h^n \, dv \right) \\ \partial_t f_h = -v \partial_x f_h \end{cases}$$

Numerical tools:

- FFT and iFFT during this step
- Fast with `fftw` if you reuse allocated memory

Lawson method

Fluid part is linear, we want to solve it exactly → Lawson method

$$\underbrace{\partial_t \begin{pmatrix} u_c \\ E \\ \hat{f}_h \end{pmatrix}}_U + \underbrace{\begin{pmatrix} 0 & -1 & 0 \\ \rho_c^{(0)} & 0 & 0 \\ 0 & 0 & ikv \end{pmatrix}}_A \underbrace{\begin{pmatrix} u_c \\ E \\ \hat{f}_h \end{pmatrix}}_U + \underbrace{\begin{pmatrix} 0 \\ \int_{\mathbb{R}} v f_h dv \\ \widehat{E \partial_v f_h} \end{pmatrix}}_{F(U)} = 0$$

We rewrite as:

$$\underbrace{\partial_t \left(e^{tA} U \right)}_V + e^{tA} F \left(\underbrace{U}_{e^{-tA} V} \right) = 0$$

and now we solve with a RK method: $\partial_t V = -e^{tA} F(e^{-tA} V)$ and next rewrite with the U variable. For example with Lawson Euler method:

$$V(t^n + \Delta t) \approx V^{n+1} = V^n - \Delta t e^{t^n A} F(e^{-t^n A} V^n)$$

or as an expression of U :

$$U^{n+1} = e^{-\Delta t A} U^n - \Delta t e^{-\Delta t A} F(U^n)$$

Numerical tools:

- FFT in x direction
- WENO5 in v direction to approximate $\widehat{E\partial_v f_h}$

CFL: Lawson(RK(4,4))–WENO5 : $\Delta t \leq \frac{\sigma}{\|E^n\|_\infty}$, with $\sigma = 1.433$

 Crouseilles, Einkemmer, and Massot (2019)

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Main idea of adaptive time step methods (error estimate)

For a generic ODE $\dot{u} = f(t, u(t))$, adaptive time step method needs 2 numerical estimations of solution $u(t^{n+1})$ of different order, p and $p + 1$:

$$u_{[p]}^{n+1} = u(t^{n+1}) + \mathcal{O}(\Delta t^{p+1}) \quad u_{[p+1]}^{n+1} = u(t^{n+1}) + \mathcal{O}(\Delta t^{p+2})$$

Estimate of the local error:

$$L_{[p]}^{n+1} = \left| u_{[p+1]}^{n+1} - u_{[p]}^{n+1} \right|$$

If $L_{[p]}^{n+1} > \text{tol}$: we reject the step and start again from time t^n . Else we accept the step. In both cases, the optimal new time step is:

$$\Delta t_{\text{opt}} = \sqrt[p]{\frac{\text{tol}}{L_{[p]}^{n+1}}} \Delta t^n$$

In practice we don't want volatile time step:

$$\Delta t^{n+1} = \max(0.5\Delta t^n, \min(2\Delta t^n, \Delta t_{\text{opt}}))$$

Adaptive time step method for splitting method

For the Suzuki splitting method:



Blanes, Casas, and Thalhammer (2019)

$$U_{[4]}^{n+1} = \mathcal{S}_{\Delta t}(U^n) = S_{\alpha_1 \Delta t} \circ S_{\alpha_2 \Delta t} \circ S_{\alpha_3 \Delta t} \circ S_{\alpha_2 \Delta t} \circ \underbrace{S_{\alpha_1 \Delta t}(U^n)}_{U^{(1)}}.$$
$$\underbrace{\hspace{10em}}_{U^{(2)}}$$
$$\underbrace{\hspace{15em}}_{U^{(3)}}$$
$$\underbrace{\hspace{20em}}_{U^{(4)}}$$

We compute an order 3 approximation from U^n and $U^{(s)}$, $s = 1, 2, 3, 4$:

$$U_{[3]}^{n+1} = -U^n + w_1(U^{(1)} + U^{(4)}) + w_2(U^{(2)} + U^{(3)})$$

with:

$$w_1 = \frac{g_2(1 - g_2)}{g_1(g_1 - 1) - g_2(g_2 - 1)}, \quad w_2 = 1 - w_1, \quad \begin{aligned} g_1 &= \alpha_1 \\ g_2 &= \alpha_1 + \alpha_2 \end{aligned}$$

$$\text{and } L_{[3]}^n = \left\| U_{[4]}^{n+1} - U_{[3]}^{n+1} \right\|_2$$

Adaptive time step method for Lawson method

Lawson methods are built on Runge-Kutta method, embedded Lawson method are written with an underlying embedded Runge-Kutta method.



Dormand and Prince (1978)

With DP4(3) (Dormand-Prince method of order 4, with embedded 3 method):

0						
$\frac{1}{2}$	$\frac{1}{2}$					
$\frac{1}{2}$	0	$\frac{1}{2}$				
$\frac{1}{2}$	0	0	1			
1	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{6}$		
	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{2}{30}$	$\frac{1}{10}$	

} Classical RK(4,4)

We compute a 3rd order approximation from U^n , $U^{(s)}$, $s = 1, 2, 3, 4$ done by the last line of Butcher tableau.

$$\text{And } L_{[3]}^n = \left\| U_{[4]}^{n+1} - U_{[3]}^{n+1} \right\|_2$$

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Sketch of computing

We linearize around unstable equilibrium state (TSI type):

- for kinetic model

$$f_{\text{eq}}(v) = \mathcal{M}_{\rho_c, 0, \textcolor{red}{T}_c}(v) + \mathcal{M}_{\alpha/2, v_0, 1}(v) + \mathcal{M}_{\alpha/2, -v_0, 1}(v)$$

- for LHM

$$\begin{aligned} U_{\text{eq}} &= (u_{c,\text{eq}}, E_{\text{eq}}, f_{h,\text{eq}})^T \\ &= (0, 0, \mathcal{M}_{\alpha/2, v_0, 1} + \mathcal{M}_{\alpha/2, -v_0, 1})^T \end{aligned}$$

We obtain 2 relations of dispersion:

- One for kinetic model, depends on T_c : $D_{[T_c]}^K(k, \omega)$
- One for linear hybrid model (LHM): $D^{LHM}(k, \omega)$

Relations of dispersion

- **Kinetic model:**

$$D_{[T_c]}^K(k, \omega) = 1 - \frac{1}{k^2} \left[-\frac{1-\alpha}{T_c} \left(1 + \frac{1}{\sqrt{2T_c}} \frac{\omega}{k} Z \left(\frac{1}{\sqrt{2T_c}} \frac{\omega}{k} \right) \right) \right. \\ \left. - \frac{\alpha}{2} \left(1 + \frac{1}{\sqrt{2}} \left(\frac{\omega}{k} - v_0 \right) Z \left(\frac{1}{\sqrt{2}} \left(\frac{\omega}{k} - v_0 \right) \right) \right) \right. \\ \left. - \frac{\alpha}{2} \left(1 + \frac{1}{\sqrt{2}} \left(\frac{\omega}{k} + v_0 \right) Z \left(\frac{1}{\sqrt{2}} \left(\frac{\omega}{k} + v_0 \right) \right) \right) \right]$$

- **LHM:**

$$D^{LHM}(k, \omega) = 1 - \frac{1}{k^2} \left[(1-\alpha) \frac{k^2}{\omega^2} - \frac{\alpha}{2} \left(1 + \frac{1}{\sqrt{2}} \left(\frac{\omega}{k} - v_0 \right) Z \left(\frac{1}{\sqrt{2}} \left(\frac{\omega}{k} - v_0 \right) \right) \right) \right. \\ \left. - \frac{\alpha}{2} \left(1 + \frac{1}{\sqrt{2}} \left(\frac{\omega}{k} + v_0 \right) Z \left(\frac{1}{\sqrt{2}} \left(\frac{\omega}{k} + v_0 \right) \right) \right) \right]$$

where $Z(z) = \sqrt{\pi} e^{-z^2} (i - \operatorname{erfi}(z))$ (Fried and Conte function)

Convergence of relations of dispersion

Since

$$Z(z) \underset{z \rightarrow +\infty}{\sim} -\frac{1}{z} - \frac{1}{2z^3} - \frac{3}{4z^5} + \mathcal{O}(z^{-7})$$

with $z = \frac{1}{\sqrt{2T_c}} \frac{\omega}{k}$ when $T_c \rightarrow 0$ we get:

$$-\frac{1-\alpha}{T_c} \left(1 + \frac{1}{\sqrt{2T_c}} \frac{\omega}{k} Z \left(\frac{1}{\sqrt{2T_c}} \frac{\omega}{k} \right) \right) = -\frac{1-\alpha}{T_c} \left(-\frac{1}{2 \left(\frac{1}{\sqrt{2T_c}} \frac{\omega}{k} \right)} + \mathcal{O}(z^{-4}) \right)$$

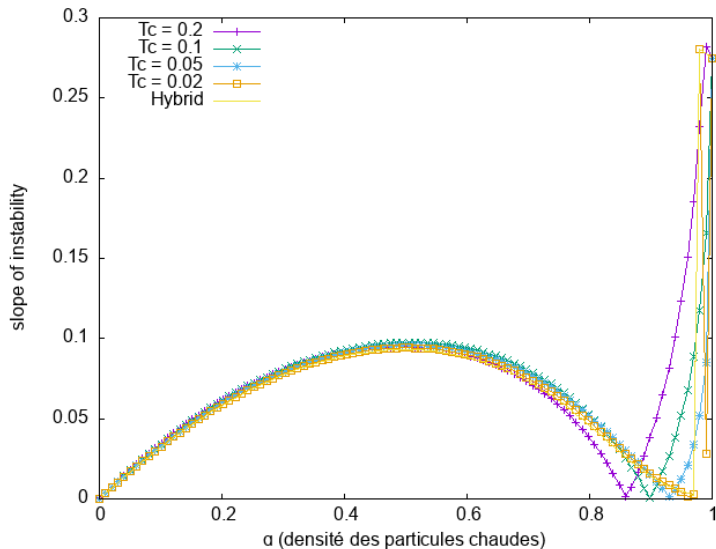
which is equivalent to:

$$\frac{1-\alpha}{T_c} \frac{k^2 T_c}{\omega^2} = (1-\alpha) \frac{k^2}{\omega^2}$$

We have:

$$\lim_{T_c \rightarrow 0} D_{[T_c]}^K(k, \omega) = D^{LHM}(k, \omega) \quad \checkmark$$

Slope of both relations of dispersion

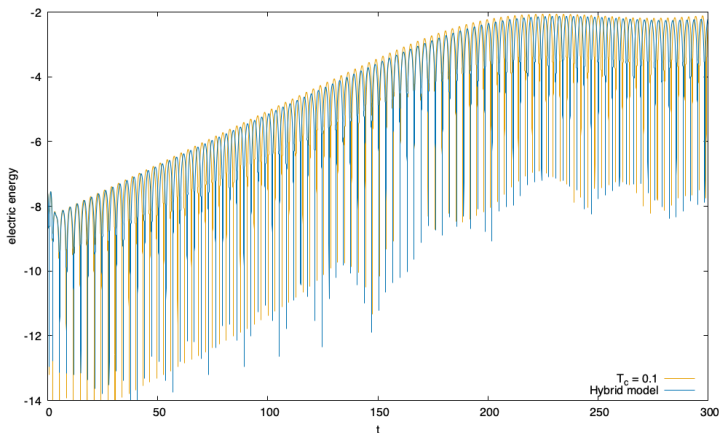


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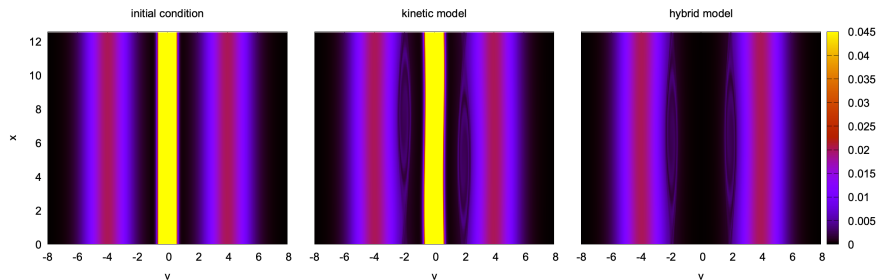
Validation of hybrid model

$\alpha = 0.2, \epsilon = 10^{-2}, k = 0.5, x \in [0, \frac{2\pi}{k}], v \in [-12, 12], v_0 = 4$
 $N_x = 135, N_v = 1200, \Delta t = 0.5\Delta v$ for kinetic and LHM



Validation of hybrid model

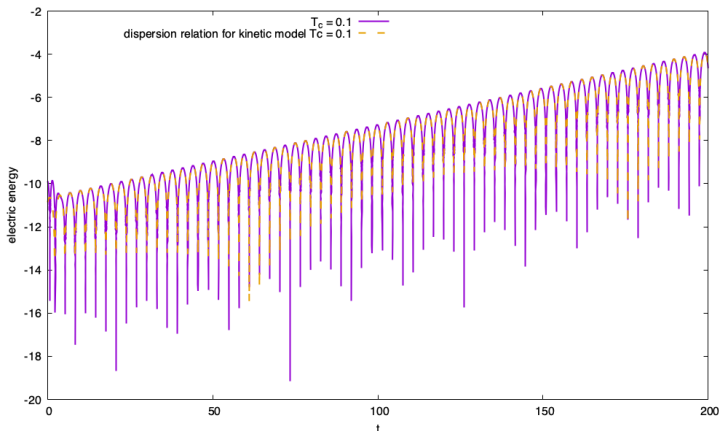
$\alpha = 0.2, \epsilon = 10^{-2}, k = 0.5, x \in [0, \frac{2\pi}{k}], v \in [-12, 12], v_0 = 4$
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Validation with relations of dispersion

Relations of dispersion give the electric energy approximation in the linear phase

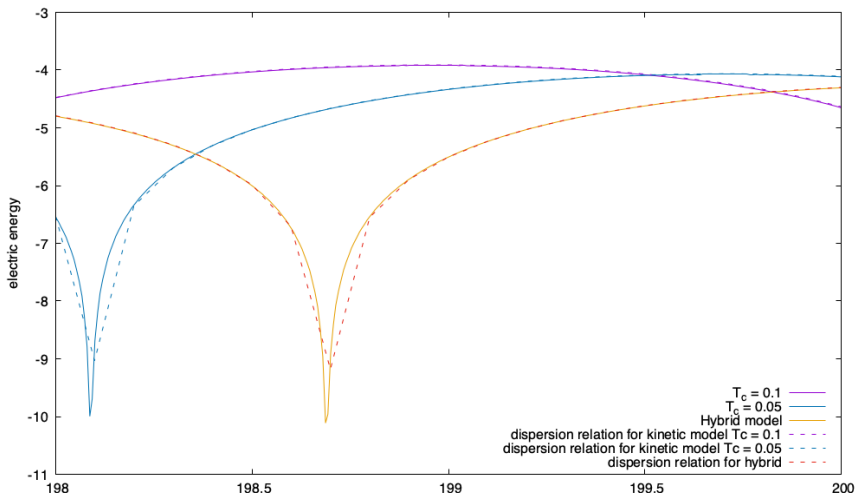
Perturbation $\epsilon = 10^{-4}$, $\alpha = 0.1$, $N_x = 135$, $N_v = 512$, $\Delta t = 0.5\Delta v$



Validation with relations of dispersion

Relations of dispersion give the electric energy approximation in the linear phase

Perturbation $\epsilon = 10^{-4}$, $\alpha = 0.1$, $N_x = 135$, $N_v = 512$, $\Delta t = 0.5\Delta v$



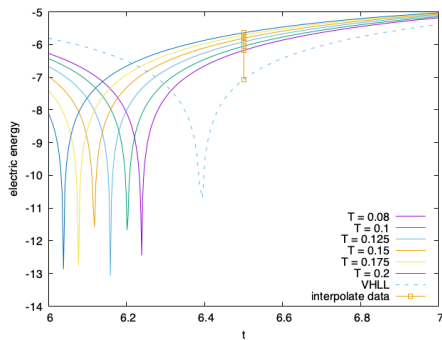
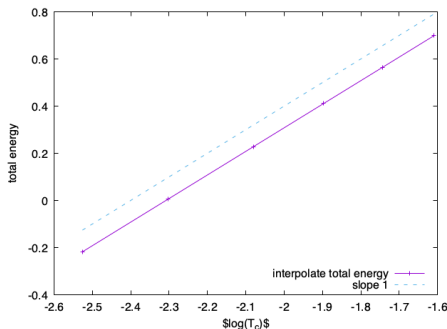
Convergence of kinetic model to linear hybrid model

$$\alpha = 0.1, v \in [-12, 12]$$

$N_x = 135, N_v = 715, 764, 826, 1131, \Delta t = 0.5\Delta v$ for kinetic with

$T_c = 0.2, 0.175, 0.15, 0.08$

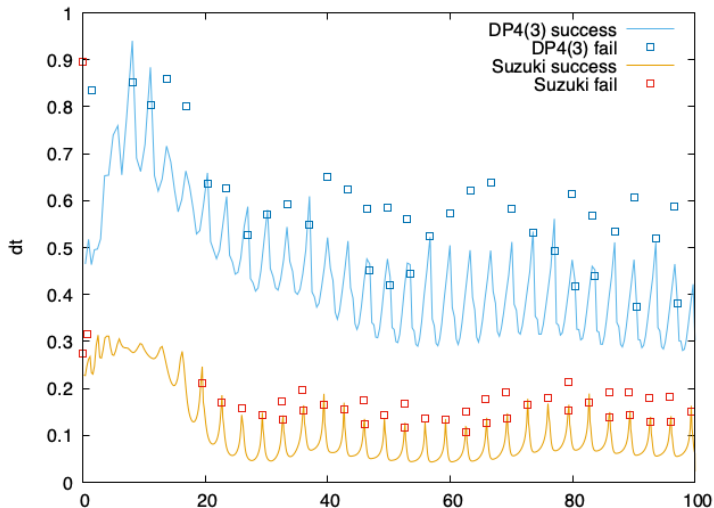
$N_v = 256$ for LHM



Test adaptive time step method

$$N_x = 81, N_v = 128$$

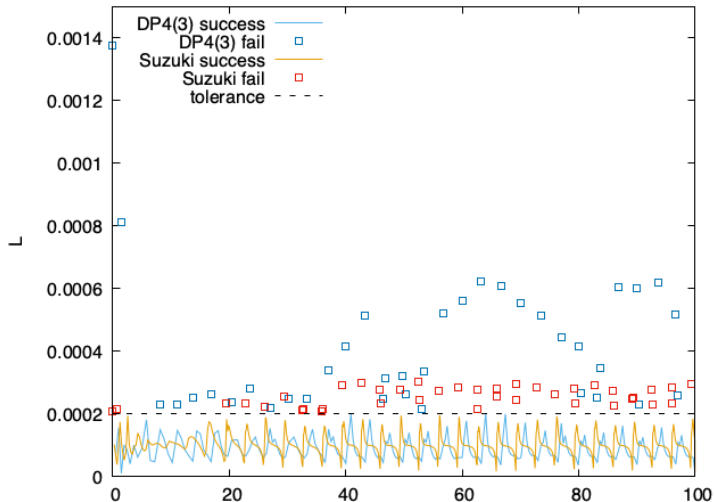
$\text{tol} = 2 \cdot 10^{-4}$, $L = \|U_{[4]}^{n+1} - U_{[3]}^{n+1}\|_2$ is the local error



Test adaptive time step method

$$N_x = 81, N_v = 128$$

$\text{tol} = 2 \cdot 10^{-4}$, $L = \|U_{[4]}^{n+1} - U_{[3]}^{n+1}\|_2$ is the local error



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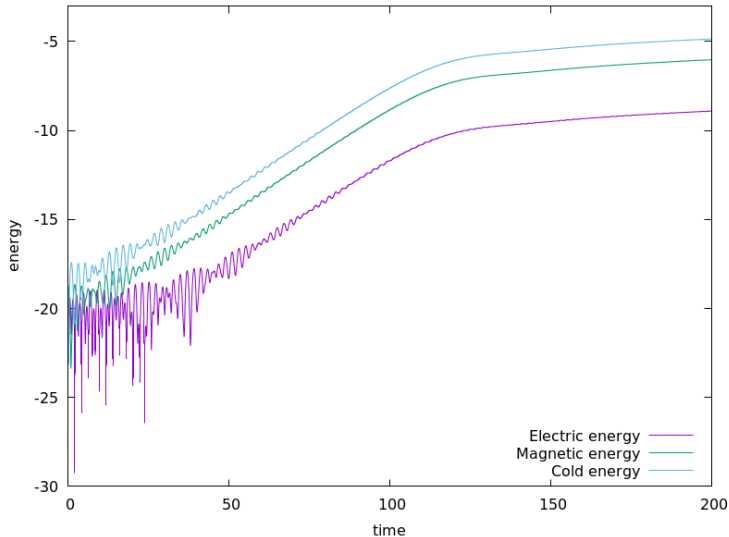
Summary

- Validation and robustness of Linear Hybrid Model
- Derivation of geometric structure for LHM
- Numerical cost for one Strang is equivalent of one stage of Lawson-RK method
 - 5 Strang for Suzuki
 - 4 stages for RK(4,4)

Future works

- Extension to $1D_x - 3D_v$, same framework of [Holderied et al. \(2020\)](#)
 - Splitting into 6 sub-steps → Strang in 11 steps
 - Lawson methods should be more efficient
 - Compare with PIC method

First results with splitting method for 1Dx – 3Dv



For Lawson method, I work on code generator from sympy expressions

Thank you for your attention

CPU time comparaison

