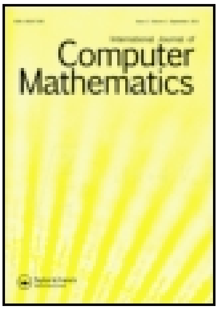

International Journal of Computer Mathematics
Publication details, including instructions for authors and subscription information:
http://www.tandfonline.com/loi/gcom20

A new fifth order weighted runge kutta formula

D.J. Evans [a] & A.R. Yaakub [a]
[a] Parallel Algorithms Research Centre, Loughborough University of Technology, Loughborough, Leicestershire, LE11 3TU, UK
Version of record first published: 19 Mar 2007.

Intern. J. Computer Math., Vol. 59, pp. 227–243
Reprints available directly from the publisher
Photocopying permitted by license only

A NEW FIFTH ORDER WEIGHTED RUNGE KUTTA FORMULA

D. J. EVANS and A. R. YAAKUB

Parallel Algorithms Research Centre,
Loughborough University of Technology,
Loughborough, Leicestershire, LE11 3TU, UK

(Received August 9 1995)

In this paper a new weighted Runge Kutta fifth order method with 5 stages is introduced. By comparison with the currently recommended RK4(5) Merson and RK5(6) Nystrom methods, the new method gives improved results.

KEY WORDS: Runge Kutta, Merson, Nystrom, weighted mean.

C. R. CATEGORIES: G.1.7.

1 INTRODUCTION

From recent publications (see Evans & Yaakub [1993] and Evans & Sanugi [1987], [1993]), fourth order linear and non-linear methods using a variety of means for solving initial value problems of the form $y' = f(x, y)$ have been shown to have the form

$$y_{n+1} = y_n + \frac{h}{3}\left[\sum_{i=1}^{3} \text{Means}\right], \tag{1.1}$$

where *Means* = principal means include arithmetic mean (AM), geometric mean (GM) [5], contraharmonic mean (C_0M) [2], centroidal mean (C_eM) [3], root mean square (RMS) [11], harmonic mean (H_aM) [10] and heronian mean (H_eM) [4] which involve k_i, $1 \leqslant i \leqslant 4$

where

$$k_1 = f(x_n, y_n)$$

$$k_2 = f(x_n + a_1h, y_n + a_1hk_1)$$

$$k_3 = f(x_n + (a_2 + a_3)h, y_n + a_2hk_1 + a_3hk_2)$$

$$k_4 = f(x_n + (a_4 + a_5 + a_6)h, y_n + a_4hk_1 + a_5hk_2 + a_6hk_3) \tag{1.2}$$

From the above discussion, a comparison of the parameters $a_i, 1 \leqslant i \leqslant 6$ in equation (1.2), show that a_1 is fixed, a_2, a_4, a_6 are decreasing and a_3, a_5 as increasing as shown in Table 1.

From Table 1, we can see that $a_2 + a_3 = 1/2$ and $a_4 + a_5 + a_6 = 1$. In this paper, our concern is to establish a new fifth order formula based on the last two k values i.e., k_3, and k_4, which involve the parameters $a_2 + a_3 = 1/2$ and $a_4 + a_5 + a_6 = 1$ and weighted so we establish a new weighted Runge-Kutta formula (WRK) similar to [12].

2 THE FOURTH ORDER ARITHMETIC MEAN WEIGHTED RUNGE-KUTTA FORMULA

The standard fourth order arithmetic mean (AM) Runge-Kutta formula for solving IVPs may be written in the form

$$y_{n+1} = y_n + \frac{h}{3}\left[\sum_{i=1}^{3}\left(\frac{k_i + k_{i+1}}{2}\right)\right] \tag{2.1}$$

or

$$y_{n+1} = y_n + \frac{h}{3}\left[\frac{k_1 + k_2}{2} + \frac{k_2 + k_3}{2} + \frac{k_3 + k_4}{2}\right]$$

where $k_i, 1, \leqslant i \leqslant 4$ as we mentioned above.

Now, based on the parameters involved in the last two k values i.e., k_3 and k_4, we attempt to develop a new fourth order method called the weighted Runge-Kutta (WRK) formula in the form

$$y_{n+1} = y_n + h\left[\sum_{i=1}^{3} w_i\left(\frac{k_i + k_{i+1}}{2}\right)\right] \tag{2.2}$$

Table 1 The values of parameters $a_i, 1 \leqslant i \leqslant 6$ based on the various integration formulas.

	C_0M	C_eM	RMS	AM	GM	H_eM	H_aM
a_1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
a_2	$\frac{1}{8}$	$\frac{1}{24}$	$\frac{1}{16}$	0	$-\frac{1}{16}$	$-\frac{1}{48}$	$-\frac{1}{8}$
a_3	$\frac{3}{8}$	$\frac{11}{24}$	$\frac{7}{16}$	$\frac{1}{2}$	$\frac{9}{16}$	$\frac{25}{48}$	$\frac{5}{8}$
a_4	$\frac{1}{4}$	$\frac{1}{12}$	$\frac{1}{8}$	0	$-\frac{1}{8}$	$-\frac{1}{24}$	$-\frac{1}{4}$
a_5	$-\frac{3}{4}$	$-\frac{25}{132}$	$-\frac{17}{56}$	0	$\frac{5}{24}$	$\frac{47}{600}$	$\frac{7}{20}$
a_6	$\frac{3}{2}$	$\frac{73}{66}$	$\frac{33}{28}$	1	$\frac{11}{12}$	$\frac{289}{300}$	$\frac{9}{10}$

where $\sum_{i=1}^{3} w_i = 1$ and $k_1 = f(y_n)$

$$k_2 = f(y_n + a_1 hk_1)$$

$$k_3 = f(y_n + a_2 hk_1 + (\tfrac{1}{2} - a_2)hk_2)$$

$$k_4 = f(y_n + a_3 hk_1 + a_4 hk_2 + (1 - a_3 - a_4)hk_3). \tag{2.3}$$

By use of the standard procedure of adjustment of the parameters, a fourth order accuracy formula is obtained from solving the 7 non-linear equations involving 7 variables obtained by comparison with terms of specified order in the Taylor series of expansion, i.e.,

$$hf:\quad 1.\ -x(1) - x(2) - x(3) = 0, \tag{2.4-i}$$

$$h^2ff_y:\quad 2.\ -2.*x(4)*x(1) - x(2) - 2.*x(4)*x(2) - 3.*x(3) = 0, \tag{2.4-ii}$$

$$\begin{aligned} h^3ff_y^2:\quad 2.\ &-3.*x(4)*x(2) + 6.*x(4)*x(5)*x(2) - 3.*x(3) \\ &-3.*x(4)*x(3) + 6.*x(4)*x(5)*x(3) + 3.*x(6)*x(3) \\ &+3.*x(7)*x(3) - 6.*x(4)*x(7)*x(3) = 0, \end{aligned} \tag{2.4-iii}$$

$$\begin{aligned} h^3f^2f_{yy}:\quad 8.\ &-12.*x(4)**2*x(1) - 3.*x(2) - 12.*x(4)**2*x(2) \\ &-15.*x(3) = 0, \end{aligned} \tag{2.4-iv}$$

$$\begin{aligned} h^4ff_y^3:\quad 1.\ &-6.*x(4)*x(3) + 12.*x(4)*x(5)*x(3) \\ &-12.*x(4)*x(5)*x(6)*x(3) + 6.*x(4)*x(7)*x(3) \\ &-12.*x(4)*x(5)*x(7)*x(3) + 6.*x(4)*x(6)*x(3) = 0, \end{aligned} \tag{2.4-v}$$

$$\begin{aligned} h^4f^2f_yf_{yy}:\quad 8.\ &-6.*x(4)*x(2) - 6.*x(4)**2*x(2) \\ &+12.*x(4)*x(5)*x(2) + 12.*x(4)**2*x(5)*x(2) - 15.*x(3) \\ &-6.*x(4)*x(3) - 6.*x(4)**2*x(3) + 12.*x(4)*x(5)*x(3) \\ &+12.*x(4)**2*x(5)*x(3) + 15.*x(6)*x(3) + 15.*x(7)*x(3) \\ &-24.*x(4)*x(7)*x(3) - 12.*x(4)**2*x(7)*x(3) = 0, \end{aligned} \tag{2.4-vi}$$

$$h^4f^3f_{yyy}:\quad 4.\ -8.*x(4)**3*x(1) - x(2) - 9.*x(3) - 8.*x(4)**3*x(2) = 0 \tag{2.4-vii}$$

where $x(1) = w_1$, $x(2) = w_2$, $x(3) = w_3$, $x(4) = a_1$, $x(5) = a_2$, $x(6) = a_3$ and $x(7) = a_4$.

Equations (2.4-i)–(2.4-vii) are then solved simultaneously using the NAG routine (Subroutine C05NBF) for solving a system of non-linear equations to give the required parameters, i.e.,

$$x(1) = w_1 = 0.3333333333, \quad x(2) = w_2 = 0.3333333333, \quad x(3) = w_3 = 0.3333333333$$

$$x(4) = a_1 = 0.5000000000, \quad x(5) = a_2 = 0, \quad x(6) = a_3 = 0, \quad x(7) = a_4 = 0$$

Thus, the WRK method gives the same result as the standard fourth order arithmetic mean (AM) Runge-Kutta method i.e.,

$$y_{n+1} = y_n + \frac{h}{3}\left[\frac{k_1 + k_2}{2} + \frac{k_2 + k_3}{2} + \frac{k_3 + k_4}{2}\right]$$

where

$$\begin{aligned} k_1 &= f(y_n) \\ k_2 &= f\left(y_n + \frac{1}{2}hk_1\right) \\ k_3 &= f\left(y_n + \frac{1}{2}hk_2\right) \\ k_4 &= f(y_n + hk_3). \end{aligned}$$

3 A NEW FIFTH ORDER ARITHMETIC MEAN WEIGHTED RUNGE- KUTTA FORMULA

We now extend the same procedure to obtain the fifth order formula in the form

$$y_{n+1} = y_n + h\left[\sum_{i=1}^{4} w_i\left(\frac{k_i + k_{i+1}}{2}\right)\right] \tag{3.1}$$

where

$$\sum_{i=1}^{4} w_i = 1$$

and

$$\begin{aligned} k_1 &= f(y_n) \\ k_2 &= f(y_n + a_1 hk_1) \\ k_3 &= f(y_n + a_2 hk_1 + a_3 hk_2) \\ k_4 &= f(y_n + a_4 hk_1 + a_5 hk_2 + (\tfrac{1}{2} - a_4 - a_5) hk_3) \\ k_5 &= f(y_n + a_6 hk_1 + a_7 hk_2 + a_8 hk_3 + (1 - a_6 - a_7 - a_8) hk_4). \end{aligned} \tag{3.2}$$

The Taylor series expansion of $y(x_{n+h})$ up to sixth order is given by

$$y(x_n+h)=y_n+hf+\frac{1}{2}h^2ff_y+\frac{1}{6}h^3(ff_y^2+f^2f_{yy})$$
$$+\frac{1}{24}h^4(f^3f_{yyy}+4f^2f_yf_{yy}+ff_y^3)$$
$$+\frac{1}{120}h^5(ff_y^4+11f^2f_y^2f_{yy}+4f^3f_{yy}^2+7f^3f_yf_{yyy}+f^4f_{yyyy})$$
$$+\frac{1}{720}h^6(f^5f_{yyyyy}+11f^4f_yf_{yyyy}+15f^4f_{yy}f_{yyy}+32f^3f_y^2f_{yyy}$$
$$+34f^3f_yf_{yy}^2+26f^2f_y^3f_{yy}^2+ff_y^5)+0(h^7)\cdots \tag{3.3}$$

By substituting equation (3.2) into (3.1) and subtract from equation (3.3), we obtain 12 equations with 12 parameters, i.e.,

$$hf:\quad 1-x(1)-x(2)-x(3)-x(4)=0, \tag{3.4-i}$$

$$\begin{aligned}h^2ff_y:\quad & 2-2*x(5)*x(1)-2*x(5)*x(2)-2*x(6)*x(2)\\ & -2*x(7)*x(2)-x(3)-2*x(6)*x(3)-2*x(7)*x(3)-3*x(4)=0,\end{aligned} \tag{3.4-ii}$$

$$\begin{aligned}h^3ff_y^2:\quad & 2-6*x(5)*x(7)*x(2)-3*x(6)*x(3)-3*x(7)*x(3)-6*x(5)*x(7)*x(3)\\ & +6*x(6)*x(8)*x(3)+6*x(7)*x(8)*x(3)-6*x(5)*x(9)*x(3)\\ & +6*x(6)*x(9)*x(3)\\ & +6*x(6)*x((7)*x(3)-3*x(4)-3*x(6)*x(4)-3*x(7)*x(4)\\ & +6*x(6)*x(8)*x(4)+6*x(7)*x(8)*x(4)-6*x(5)*x(9)*x(4)\\ & +6*x(6)*x(9)*x(4)+6*x(7)*x(9)*x(4)\\ & +3*x(10)*x(4)+3*x(11)*x(4)-6*x(5)*x(11)*x(4)\\ & +3*x(12)*x(4)-6*x(6)*x(12)*x(4)-6*x(7)*x(12)*x(4)=0,\end{aligned} \tag{3.4-iii}$$

$$\begin{aligned}h^3f^2f_{yy}:\; & 8-12*x(5)**2*x(1)-12*x(5)**2*x(2)-12*x(6)**2*x(2)\\ & -24*x(6)*x(7)*x(2)-12*x(7)**2*x(2)-3*x(3)\\ & -12*x(6)**2*x(3)-24*x(6)*x(7)*x(3)-12*x(7)**2*x(3)-15*x(4)=0,\\ & \ldots\end{aligned} \tag{3.4-iv}$$

$$\begin{aligned}h^4ff_y^3:\quad & 1-6*x(5)*x(7)*x(3)+12*x(5)*x(7)*x(8)*x(3)\\ & +12*x(5)*x(7)*x(9)*x(3)-6*x(6)*x(4)-6*x(7)*x(4)\\ & -6*x(5)*x(7)*x(4)+12*x(6)*x(8)*x(4)+12*x(7)*x(8)*x(4)\end{aligned}$$

$$
\begin{aligned}
&+ 12{*}x(5){*}x(7){*}x(8){*}x(4) - 12{*}x(5){*}x(9){*}x(4)\\
&+ 12{*}x(6){*}x(9){*}x(4) + 12{*}x(7){*}x(9){*}x(4) + 12{*}x(5){*}x(7){*}x(9){*}x(4)\\
&+ 6{*}x(6){*}x(10){*}x(4) + 6{*}x(7){*}x(10){*}x(4)\\
&- 12{*}x(6){*}x(8){*}x(10){*}x(4) - 12{*}x(7){*}x(8){*}x(10){*}x(4)\\
&+ 12{*}x(5){*}x(9){*}x(10){*}x(4) - 12{*}x(6){*}x(9){*}x(10){*}x(4)\\
&- 12{*}x(7){*}x(9){*}x(10){*}x(4) + 6{*}x(6){*}x(11){*}x(4)\\
&+ 6{*}x(7){*}x(11){*}x(4) - 12{*}x(6){*}x(8){*}x(11){*}x(4)\\
&- 12{*}x(7){*}x(8){*}x(11){*}x(4) + 12{*}x(5){*}x(9){*}x(11){*}x(4)\\
&- 12{*}x(6){*}x(9){*}x(11){*}x(4) - 12{*}x(7){*}x(9){*}x(11){*}x(4)\\
&+ 6{*}x(6){*}x(12){*}x(4) + 6{*}x(7){*}x(12){*}x(4)\\
&- 12{*}x(5){*}x(7){*}x(12){*}x(4) - 12{*}x(6){*}x(8){*}x(12){*}x(4)\\
&- 12{*}x(7){*}x(8){*}x(12){*}x(4) + 12{*}x(5){*}x(9){*}x(12){*}x(4)\\
&- 12{*}x(6){*}x(9){*}x(12){*}x(4) - 12{*}x(7){*}x(9){*}x(12){*}x(4) = 0, \qquad \text{(3.4-v)}
\end{aligned}
$$

$$
\begin{aligned}
h^4 f^2 f_y f_{yy}:\quad & 8 - 12{*}x(5){*}{*}2{*}x(7){*}x(2) - 24{*}x(5){*}x(6){*}x(7){*}x(2)\\
&- 24{*}x(5){*}x(7){*}{*}2{*}x(2) - 6{*}x(6){*}x(3) - 6{*}x(6){*}{*}2{*}x(3)\\
&- 6{*}x(7){*}x(3) - 12{*}x(5){*}{*}2{*}x(7){*}x(3) - 12{*}x(6){*}x(7){*}x(3)\\
&- 24{*}x(5){*}x(6){*}x(7){*}x(3) - 6{*}x(7){*}{*}2{*}x(3)\\
&- 24{*}x(5){*}x(7){*}{*}2{*}x(3) + 12{*}x(6){*}x(8){*}x(3)\\
&+ 12{*}x(6){*}{*}2{*}x(8){*}x(3) + 12{*}x(7){*}x(8){*}x(3)\\
&+ 24{*}x(6){*}x(7){*}x(8){*}x(3) + 12{*}x(7){*}{*}2{*}x(8){*}x(3)\\
&- 12{*}x(5){*}x(9){*}x(3) - 12{*}x(5){*}{*}2{*}x(9){*}x(3)\\
&+ 12{*}x(6){*}x(9){*}x(3) + 12{*}x(6){*}{*}2{*}x(9){*}x(3) + 12{*}x(7){*}x(9){*}x(3)\\
&+ 24{*}x(6){*}x(7){*}x(9){*}x(3)\\
&+ 12{*}x(7){*}{*}2{*}x(9){*}x(3) - 15{*}x(4) - 6{*}x(6){*}x(4)\\
&- 6{*}x(6){*}{*}2{*}x(4) - 6{*}x(7){*}x(4) - 12{*}x(6){*}x(7){*}x(4)\\
&- 6{*}x(7){*}{*}2{*}x(4) + 12{*}x(6){*}x(8){*}x(4) + 12{*}x(6){*}{*}2{*}x(8){*}x(4)\\
&+ 12{*}x(7){*}x(8){*}x(4) + 24{*}x(6){*}x(7){*}x(8){*}x(4)\\
&+ 12{*}x(7){*}{*}2{*}x(8){*}x(4) - 12{*}x(5){*}x(9){*}x(4)\\
&- 12{*}x(5){*}{*}2{*}x(9){*}x(4) + 12{*}x(6){*}x(9){*}x(4)\\
&+ 12{*}x(6){*}{*}2{*}x(9){*}x(4) + 12{*}x(7){*}x(9){*}x(4)
\end{aligned}
$$

$$+ 24*x(6)*x(7)*x(9)*x(4) + 12*x(7)**2*x(9)*x(4) + 15*x(10)*x(4)$$
$$+ 15*x(11)*x(4) - 24*x(5)*x(11)*x(4) - 12*x(5)**2*x(11)*x(4)$$
$$+ 15*x(12)*x(4) - 24*x(6)*x(12)*x(4)$$
$$- 12*x(6)**2*x(12)*x(4) - 24*x(7)*x(12)*x(4)$$
$$- 24*x(6)*x(7)*x(12)*x(4) - 12*x(7)**2*x(12)*x(4) = 0, \qquad (3.4\text{-vi})$$

$$h^4 f^3 f_{yyy}\text{:}\ 4 - 8*x(5)**3*x(1) - 8*x(5)**3*x(2) - 8*x(6)**3*x(2)$$
$$- 24*x(6)**2*x(7)*x(2) - 24*x(6)*x(7)**2*x(2)$$
$$- 8*x(7)**3*x(2) - x(3) - 8*x(6)**3*x(3) - 24*x(6)**2*x(7)*x(3)$$
$$- 24*x(6)*x(7)**2*x(3) - 8*x(7)**3*x(3) - 9*x(4) = 0, \qquad (3.4\text{-vii})$$

$$h^5 f f_y^4\text{:}\ 1 - 30*x(5)*x(7)*x(4) + 60*x(5)*x(7)*x(8)*x(4)$$
$$+ 60*x(5)*x(7)*x(9)*x(4) + 30*x(5)*x(7)*x(10)*x(4)$$
$$- 60*x(5)*x(7)*x(8)*x(10)*x(4) - 60*x(5)*x(7)*x(9)*x(10)*x(4)$$
$$+ 30*x(5)*x(7)*x(11)*x(4) + 30*x(5)*x(7)*x(12)*x(4)$$
$$- 60*x(5)*x(7)*x(8)*x(11)*x(4) - 60*x(5)*x(7)*x(9)*x(11)*x(4)$$
$$- 60*x(5)*x(7)*x(8)*x(12)*x(4) - 60*x(5)*x(7)*x(9)*x(12)*x(4) = 0, \qquad (3.4\text{-viii})$$

$$\cdots$$

$$h^5 f^2 f_y^2 f_{yy}\text{:}\ 22 - 60*x(5)**2*x(7)**2*x(2) - 15*x(6)**2*x(3)$$
$$- 30*x(5)*x(7)*x(3) - 30*x(5)**2*x(7)*x(3)$$
$$- 30*x(6)*x(7)*x(3) - 60*x(5)*x(6)*x(7)*x(3)$$
$$- 15*x(7)**2*x(3) - 60*x(5)*x(7)**2*x(3)$$
$$- 60*x(5)**2*x(7)**2*x(3) + 60*x(6)**2*x(8)*x(3)$$
$$+ 60*x(5)*x(7)*x(8)*x(3) + 60*x(5)**2*x(7)*x(8)*x(3)$$
$$+ 120*x(6)*x(7)*x(8)*x(3) + 120*x(5)*x(6)*x(7)*x(8)*x(3)$$
$$+ 60*x(7)**2*x(8)*x(3)$$
$$+ 120*x(5)*x(7)**2*x(8)*x(3) - 60*x(6)**2*x(8)**2*x(3)$$
$$- 120*x(6)*x(7)*x(8)**2*x(3)$$
$$- 60*x(7)**2*x(8)**2*x(3) - 60*x(5)*x(6)*x(9)*x(3)$$
$$+ 60*x(6)**2*x(9)*x(3) + 60*x(5)**2*x(7)*x(9)*x(3)$$
$$+ 120*x(6)*x(7)*x(9)*x(3) + 120*x(5)*x(6)*x(7)*x(9)*x(3)$$
$$+ 60*x(7)**2*x(9)*x(3)$$

$$+ 120*x(5)*x(7)**2*x(9)*x(3) + 120*x(5)*x(6)*x(8)*x(9)*x(3)$$
$$- 120*x(6)**2*x(8)*x(9)*x(3)$$
$$+ 120*x(5)*x(7)*x(8)*x(9)*x(3) - 240*x(6)*x(7)*x(8)*x(9)*x(3)$$
$$- 120*x(7)**2*x(8)*x(9)*x(3)$$
$$- 60*x(5)**2*x(9)**2*x(3) + 120*x(5)*x(6)*x(9)**2*x(3)$$
$$- 60*x(6)**2*x(9)**2*x(3)$$
$$+ 120*x(5)*x(7)*x(9)**2*x(3) - 120*x(6)*x(7)*x(9)**2*x(3)$$
$$- 60*x(7)**2*x(9)**2*x(3) - 15*x(4)$$
$$- 90*x(6)*x(4) - 45*x(6)**2*x(4) - 90*x(7)*x(4)$$
$$- 30*x(5)*x(7)*x(4) - 30*x(5)**2*x(7)*x(4)$$
$$- 90*x(6)*x(7)*x(4) - 60*x(5)*x(6)*x(7)*x(4) - 45*x(7)**2*x(4)$$
$$- 60*x(5)*x(7)**2*x(4)$$
$$+ 180*x(6)*x(8)*x(4) + 120*x(6)**2*x(8)*x(4) + 180*x(7)*x(8)*x(4)$$
$$+ 60*x(5)*x(7)*x(8)*x(4)$$
$$+ 60*x(5)**2*x(7)*x(8)*x(4) + 240*x(6)*x(7)*x(8)*x(4)$$
$$+ 120*x(5)*x(6)*x(7)*x(8)*x(4)$$
$$+ 120*x(7)**2*x(8)*x(4) + 120*x(5)*x(7)**2*x(8)*x(4)$$
$$- 60*x(6)**2*x(8)**2*x(4)$$
$$- 120*x(6)*x(7)*x(8)**2*x(4) - 60*x(7)**2*x(8)**2*x(4)$$
$$- 180*x(5)*x(9)*x(4) - 60*x(5)**2*x(9)*x(4)$$
$$+ 180*x(6)*x(9)*x(4) - 60*x(5)*x(6)*x(9)*x(4)$$
$$+ 120*x(6)**2*x(9)*x(4) + 180*x(7)*x(9)*x(4)$$
$$+ 60*x(5)**2*x(7)*x(9)*x(4) + 240*x(6)*x(7)*x(9)*x(4)$$
$$+ 120*x(5)*x(6)*x(7)*x(9)*x(4)$$
$$+ 120*x(7)**2*x(9)*x(4) + 120*x(5)*x(7)**2*x(9)*x(4)$$
$$+ 120*x(5)*x(6)*x(8)*x(9)*x(4)$$
$$- 120*x(6)**2*x(8)*x(9)*x(4) + 120*x(5)*x(7)*x(8)*x(9)*x(4)$$
$$- 240*x(6)*x(7)*x(8)*x(9)*x(4)$$
$$- 120*x(7)**2*x(8)*x(9)*x(4) - 60*x(5)**2*x(9)**2*x(4)$$
$$+ 120*x(5)*x(6)*x(9)**2*x(4)$$
$$- 60*x(6)**2*x(9)**2*x(4) + 120*x(5)*x(7)*x(9)**2*x(4)$$

$- 120*x(6)*x(7)*x(9)**2*x(4)$

$- 60*x(7)**2*x(9)**2*x(4) + 30*x(10)*x(4) + 90*x(6)*x(10)*x(4)$

$+ 30*x(6)**2*x(10)*x(4) + 90*x(7)*x(10)*x(4)$

$+ 60*x(6)*x(7)*x(10)*x(4) + 30*x(7)**2*x(10)*x(4)$

$- 180*x(6)*x(8)*x(10)*x(4) - 60*x(6)**2*x(8)*x(10)*x(4)$

$- 180*x(7)*x(8)*x(10)*x(4) - 120*x(6)*x(7)*x(8)*x(10)*x(4)$

$- 60*x(7)**2*x(8)*x(10)*x(4)$

$+ 180*x(5)*x(9)*x(10)*x(4) + 60*x(5)**2*x(9)*x(10)*x(4)$

$- 180*x(6)*x(9)*x(10)*x(4) - 60*x(6)**2*x(9)*x(10)*x(4)$

$- 180*x(7)*x(9)*x(10)*x(4) - 120*x(6)*x(7)*x(9)*x(10)*x(4)$

$- 60*x(7)**2*x(9)*x(10)*x(4) - 15*x(10)**2*x(4)$

$+ 30*x(11)*x(4) - 60*x(5)*x(11)*x(4) + 90*x(6)*x(11)*x(4)$

$+ 30*x(6)**2*x(11)*x(4) + 90*x(7)*x(11)*x(4)$

$+ 60*x(6)*x(7)*x(11)*x(4) + 30*x(7)**2*x(11)*x(4)$

$- 180*x(6)*x(8)*x(11)*x(4) - 60*x(6)**2*x(8)*x(11)*x(4)$

$- 180*x(7)*x(8)*x(11)*x(4) - 120*x(6)*x(7)*x(8)*x(11)*x(4)$

$- 60*x(7)**2*x(8)*x(11)*x(4)$

$+ 180*x(5)*x(9)*x(11)*x(4) + 60*x(5)**2*x(9)*x(11)*x(4)$

$- 180*x(6)*x(9)*x(11)*x(4) - 60*x(6)**2*x(9)*x(11)*x(4)$

$- 180*x(7)*x(9)*x(11)*x(4) - 120*x(6)*x(7)*x(9)*x(11)*x(4)$

$- 60*x(7)**2*x(9)*x(11)*x(4) - 30*x(10)*x(11)*x(4)$

$+ 60*x(5)*x(10)*x(11)*x(4) - 15*x(11)**2*x(4)$

$+ 60*x(5)*x(11)**2*x(4) - 60*x(5)**2*x(11)**2*x(4)$

$+ 30*x(12)*x(4) + 30*x(6)*x(12)*x(4) + 30*x(6)**2*x(12)*x(4)$

$+ 30*x(7)*x(12)*x(4) - 120*x(5)*x(7)*x(12)*x(4)$

$- 60*x(5)**2*x(7)*x(12)*x(4) + 60*x(6)*x(7)*x(12)*x(4)$

$- 120*x(5)*x(6)*x(7)*x(12)*x(4) + 30*x(7)**2*x(12)*x(4)$

$- 120*x(5)*x(7)**2*x(12)*x(4) - 180*x(6)*x(8)*x(12)*x(4)$

$- 60*x(6)**2*x(8)*x(12)*x(4)$

$- 180*x(7)*x(8)*x(12)*x(4) - 120*x(6)*x(7)*x(8)*x(12)*x(4)$

$- 60*x(7)**2*x(8)*x(12)*x(4)$

$$+ 180*x(5)*x(9)*x(12)*x(4) + 60*x(5)**2*x(9)*x(12)*x(4)$$
$$- 180*x(6)*x(9)*x(12)*x(4) - 60*x(6)**2*x(9)*x(12)*x(4)$$
$$- 180*x(7)*x(9)*x(12)*x(4) - 120*x(6)*x(7)*x(9)*x(12)*x(4)$$
$$- 60*x(7)**2*x(9)*x(12)*x(4) - 30*x(10)*x(12)*x(4)$$
$$+ 60*x(6)*x(10)*x(12)*x(4) + 60*x(7)*x(10)*x(12)*x(4)$$
$$- 30*x(11)*x(12)*x(4) + 60*x(5)*x(11)*x(12)*x(4)$$
$$+ 60*x(6)*x(11)*x(12)*x(4) - 120*x(5)*x(6)*x(11)*x(12)*x(4)$$
$$+ 60*x(7)*x(11)*x(12)*x(4)$$
$$- 120*x(5)*x(7)*x(11)*x(12)*x(4) - 15*x(12)**2*x(4)$$
$$+ 60*x(6)*x(12)**2*x(4) - 60*x(6)**2*x(12)**2*x(4)$$
$$+ 60*x(7)*x(12)**2*x(4) - 120*x(6)*x(7)*x(12)**2*x(4)$$
$$- 60*x(7)**2*x(12)**2*x(4) = 0, \qquad (3.4\text{-ix})$$

$$h^5 f^3 f_{yy}^2\colon 8 - 60*x(5)**2*x(6)*x(7)*x(2) - 60*x(5)**2*x(7)**2*x(2)$$
$$- 15*x(6)**2*x(3) - 30*x(6)*x(7)*x(3)$$
$$- 60*x(5)**2*x(6)*x(7)*x(3) - 15*x(7)**2*x(3)$$
$$- 60*x(5)**2*x(7)**2*x(3) + 30*x(6)**2*x(8)*x(3)$$
$$+ 60*x(6)*x(7)*x(8)*x(3) + 30*x(7)**2*x(8)*x(3)$$
$$- 30*x(5)**2*x(9)*x(3) + 30*x(6)**2*x(9)*x(3)$$
$$+ 60*x(6)*x(7)*x(9)*x(3) + 30*x(7)**2*x(9)*x(3) - 15*x(4)$$
$$- 15*x(6)**2*x(4) - 30*x(6)*x(7)*x(4)$$
$$- 15*x(7)**2*x(4) + 30*x(6)**2*x(8)*x(4)$$
$$+ 60*x(6)*x(7)*x(8)*x(4) + 30*x(7)**2*x(8)*x(4)$$
$$- 30*x(5)**2*x(9)*x(4) + 30*x(6)**2*x(9)*x(4)$$
$$+ 60*x(6)*x(7)*x(9)*x(4) + 30*x(7)**2*x(9)*x(4)$$
$$+ 15*x(10)*x(4) + 15*x(11)*x(4) - 60*x(5)**2*x(11)*x(4)$$
$$+ 15*x(12)*x(4) - 60*x(6)**2*x(12)*x(4)$$
$$- 120*x(6)*x(7)*x(12)*x(4) - 60*x(7)**2*x(12)*x(4) = 0, \qquad (3.4\text{-x})$$

$$h^5 f^3 f_y f_{yyy}\colon 28 - 40*x(5)**3*x(7)*x(2) - 120*x(5)*x(6)**2*x(7)*x(2)$$
$$- 240*x(5)*x(6)*x(7)**2*x(2)$$
$$- 120*x(5)*x(7)**3*x(2) - 15*x(6)*x(3) - 20*x(6)**3*x(3)$$

$$
\begin{aligned}
&- 15*x(7)*x(3) - 40*x(5)**3*x(7)*x(3)\\
&- 60*x(6)**2*x(7)*x(3) - 120*x(5)*x(6)**2*x(7)*x(3)\\
&- 60*x(6)*x(7)**2*x(3) - 240*x(5)*x(6)*x(7)**2*x(3)\\
&- 20*x(7)**3*x(3) - 120*x(5)*x(7)**3*x(3) + 30*x(6)*x(8)*x(3)\\
&+ 40*x(6)**3*x(8)*x(3) + 30*x(7)*x(8)*x(3)\\
&+ 120*x(6)**2*x(7)*x(8)*x(3) + 120*x(6)*x(7)**2*x(8)*x(3)\\
&+ 40*x(7)**3*x(8)*x(3) - 30*x(5)*x(9)*x(3)\\
&- 40*x(5)**3*x(9)*x(3) + 30*x(6)*x(9)*x(3)\\
&+ 40*x(6)**3*x(9)*x(3) + 30*x(7)*x(9)*x(3)\\
&+ 120*x(6)**2*x(7)*x(9)*x(3) + 120*x(6)*x(7)**2*x(9)*x(3)\\
&+ 40*x(7)**3*x(9)*x(3) - 65*x(4)\\
&- 15*x(6)*x(4) - 20*x(6)**3*x(4) - 15*x(7)*x(4)\\
&- 60*x(6)**2*x(7)*x(4) - 60*x(6)*x(7)**2*x(4)\\
&- 20*x(7)**3*x(4) + 30*x(6)*x(8)*x(4) + 40*x(6)**3*x(8)*x(4)\\
&+ 30*x(7)*x(8)*x(4)\\
&+ 120*x(6)**2*x(7)*x(8)*x(4) + 120*x(6)*x(7)**2*x(8)*x(4)\\
&+ 40*x(7)**3*x(8)*x(4) - 30*x(5)*x(9)*x(4)\\
&- 40*x(5)**3*x(9)*x(4) + 30*x(6)*x(9)*x(4)\\
&+ 40*x(6)**3*x(9)*x(4) + 30*x(7)*x(9)*x(4)\\
&+ 120*x(6)**2*x(7)*x(9)*x(4) + 120*x(6)*x(7)**2*x(9)*x(4)\\
&+ 40*x(7)**3*x(9)*x(4) + 65*x(10)*x(4)\\
&+ 65*x(11)*x(4) - 120*x(5)*x(11)*x(4) - 40*x(5)**3*x(11)*x(4)\\
&+ 65*x(12)*x(4) - 120*x(6)*x(12)*x(4)\\
&- 40*x(6)**3*x(12)*x(4) - 120*x(7)*x(12)*x(4)\\
&- 120*x(6)**2*x(7)*x(12)*x(4) - 120*x(6)*x(7)**2*x(12)*x(4)\\
&- 40*x(7)**3*x(12)*x(4) = 0, \qquad (3.4\text{-xi})
\end{aligned}
$$

$$
\begin{aligned}
h^5 f^4 f_{yyyy}:\ & 32 - 80*x(5)**4*x(1) - 80*x(5)**4*x(2)\\
&- 80*x(6)**4*x(2) - 320*x(6)**3*x(7)*x(2)\\
&- 480*x(6)**2*x(7)**2*x(2) - 320*x(6)*x(7)**3*x(2)\\
&- 80*x(7)**4*x(2) - 5*x(3) - 80*x(6)**4*x(3)
\end{aligned}
$$

$$- 320{*}x(6){*}{*}3{*}x(7){*}x(3) - 480{*}x(6){*}{*}2{*}x(7){*}{*}2{*}x(3)$$

$$- 320{*}x(6){*}x(7){*}{*}3{*}x(3) - 80{*}x(7){*}{*}4{*}x(3) - 85{*}x(4) = 0 \qquad \text{(3.4-xii)}$$

where

$$x(1) = w_1,\ x(2) = w_2,\ x(3) = w_3,\ x(4) = w_4,\ x(5) = a_1,\ x(6) = a_2.$$

$$x(7) = a_3,\ x(8) = a_4,\ x(9) = a_5,\ x(10) = a_6,\ x(11) = a_7 \text{ and } x(12) = a_8.$$

Similarly, equations (3.4-i)–(3.4-xii) are solved simultaneously by using the NAG routine (Subroutine C05NBF) for solving a system of non-linear equations to give the required parameters, i.e.,

$$\begin{aligned}
&w_1 = x(1) = 0.2615038147, w_2 = x(2) = -0.2765809214, w_3 = x(3) = 0.5947141647\\
&w_4 = x(4) = 0.4203629420, a_1 = x(5) = 1.5471214403, a_2 = x(6) = 0.1756458393\\
&a_3 = x(7) = 0.1243059001, a_4 = x(8) = 0.1009316694, a_5 = x(9) = 0.1100539630\\
&a_6 = x(10) = 0.99974318,\ x_7 = x(11) = -0.0928890403, x_8 = x(12) = -0.6201812828
\end{aligned}$$

$$\cdots \qquad (3.5)$$

Thus, the new fifth order WRK method can be written as follows

$$y_{n+1} = y_n + h\left[0.2615038147\left(\frac{k_1+k_2}{2}\right) - 0.2765809214\left(\frac{k_2+k_3}{2}\right) + 0.5947141647\left(\frac{k_3+k_4}{2}\right) + 0.4203629420\left(\frac{k_4+k_5}{2}\right)\right] \qquad (3.6)$$

where

$$\begin{aligned}
k_1 &= f(y_n)\\
k_2 &= f(y_n + 1.5471214403hk_1)\\
k_3 &= f(y_n + 0.1756458393hk_1 + 0.1243059001hk_2)\\
k_4 &= f(y_n + 0.1009316694hk_1 + 0.1100539630hk_2 + 0.2890143692hk_3)\\
k_5 &= f(y_n + 0.9997431862hk_1 - 0.0928890403hk_2 - 0.6201812828hk_3\\
&\quad + 0.7133271396hk_4). \qquad \cdots \qquad (3.7)
\end{aligned}$$

By rationalizing the coefficients in equation (3.5) the fifth order WRK method can also be written in rational form as given in Appendix 1.

3.1 *Error Analysis*

By substituting the values of $a_i, 1 \leqslant i \leqslant 8$ and $w_i, 1 \leqslant i \leqslant 4$ in (3.5) into (3.1) and (3.2) using Mathematica and evaluating all the terms up to (h^6) to represent the local truncation error (LTE) for this method we have

$$LTE = h^6\left[\frac{1}{720}ff_y^5 + 0.0018022816f^2f_y^3f_{yy} - 0.0166861138f^3f_yf_{yy}^2 + 0.0082646021f^3f_y^2f_{yyy} + 0.0041171137f^4f_{yy}f_{yyy} - 0.0023096163f^4f_yf_{yyy} + 0.0000588245f^5f_{yyyyy}\right] \quad \cdots \quad (3.8)$$

4 STABILITY ANALYSIS OF THE FIFTH ORDER WRK METHOD

We examine the stability region for the fifth order arithmetic mean WRK method with the test equation $y' = \lambda y$ and we obtain

$$\begin{aligned}
k_1 &= \lambda(y_n)\\
k_2 &= \lambda(y_n + 1.5471214403hk_1)\\
k_3 &= \lambda(y_n + 0.1756458393hk_1 + 0.1243059001hk_2)\\
k_4 &= \lambda(y_n + 0.1009316694hk_1 + 0.1100539630hk_2 + 0.2890143692hk_3)\\
k_5 &= \lambda(y_n + 0.9997431862hk_1 - 0.0928890403hk_2 - 0.6201812828hk_3\\
&\quad + 0.7133271396hk_4). \quad \cdots \quad (4.1)
\end{aligned}$$

By substituting $k_i, 1 \leqslant i \leqslant 5$ in equation (4.1) and $w_i, 1 \leqslant i \leqslant 4$ in equation (3.5) into the fifth order WRK formula, i.e.,

$$y_{n+1} = y_n + h\left[\sum_{i=1}^{4} w_i\left(\frac{k_i + k_{i+1}}{2}\right)\right] \quad (4.2)$$

we obtain

$$y_{n+1} = y_n + (h\lambda)y_n + 0.5(h\lambda)^2y_n + 0.166667(h\lambda)^3y_n + 0.0416667(h\lambda)^4y_n + 0.00833333(h\lambda)^5y_n + 0(h^6). \quad \cdots \quad (4.3)$$

By rationalizing the coefficients in equation (4.3) we have

$$y_{n+1} = y_n + (h\lambda)y_n + \frac{1}{2}(h\lambda)^2y_n + \frac{1}{6}(h\lambda)^3y_n + \frac{1}{24}(h\lambda)^4y_n + \frac{1}{120}(h\lambda)^5 + 0(h^6) \quad \cdots \quad (4.4)$$

By substituting $h\lambda = z$ in (4.4), we can show that

$$y_{n+1} = y_n + y_n\left[z + \frac{z^2}{2} + \frac{z^3}{6} + \frac{z^4}{24} + \frac{z^5}{120}\right] + 0(z^6). \tag{4.5}$$

Following Evans and Sanugi [1991], we write

$$\frac{y_{n+1}}{y_n} = Q$$

in the equation (4.5), to obtain

$$Q = 1 + z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \frac{1}{24}z^4 + \frac{1}{120}z^5 + 0(z^6). \tag{4.6}$$

To determine the stability region of the fifth order WRK formula in the complex plane that satisfy the condition

$$\left|\frac{y_{n+1}}{y_n}\right| = |Q| < 1$$

i.e.,

$$\left|1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \frac{z^4}{24} + \frac{z^5}{120}\right| < 1 \tag{4.7}$$

by the use of Mathematica, we can plot the graphic surface defined by equation (4.7) as shown in Figure 1 can also plot the stability region defined by the formula in equation (4.7) as shown in Figure 2.

5 NUMERICAL EXAMPLE

We consider the IVP

$$y' = x - y + 1, \quad y(0) = 1, \quad 0 \leqslant x \leqslant 1 \tag{5.1}$$

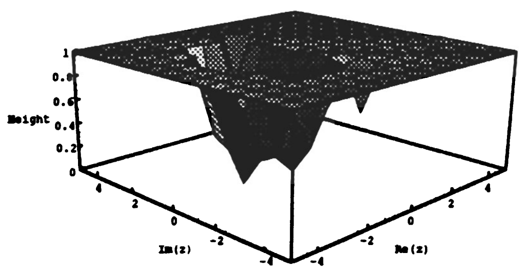

Figure 1 Graphic surface defined by fifth order WRK formula.

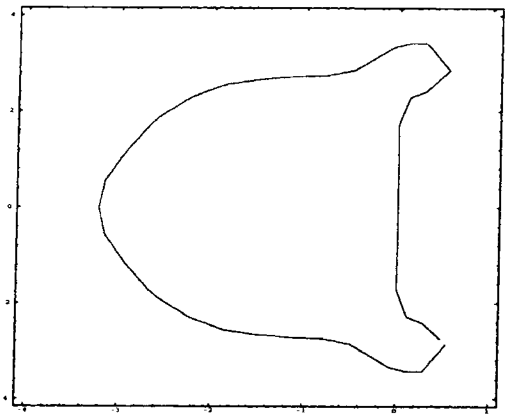

Figure 2 Stability region for fifth order WRK formula.

Table 2 Absolute error by various formula for solving equation (5.1).

x	*Classical AM-RK4*	*RK4(5) Merson*	*RK5(6) Nystrom*	*WRK-RK5(5)*
0.1	0.8196E-07	0.1252E-07	0.1707E-08	0.1380E-08
0.2	0.1483E-06	0.2266E-07	0.3089E-08	0.2498E-08
0.3	0.2013E-06	0.3075E-07	0.4192E-08	0.3391E-08
0.4	0.2429E-06	0.3710E-07	0.5058E-08	0.4091E-08
0.5	0.2747E-06	0.4196E-07	0.5721E-08	0.4627E-08
0.6	0.2983E-06	0.4556E-07	0.6211E-08	0.5024E-08
0.7	0.3149E-06	0.4810E-07	0.6557E-08	0.5303E-08
0.8	0.3256E-06	0.4974E-07	0.6781E-08	0.5484E-08
0.9	0.3315E-06	0.5063E-07	0.6902E-08	0.5583E-08
1.0	0.3332E-06	0.5090E-07	0.6939E-08	0.5613E-08

and the exact solution is $y(x) = x + \exp(-x)$. The absolute errors in the numerical solution using formula (3.1)–(3.2) compared with the fourth order RK formula (classical formula), RK5(6)-Nystrom [7] and RK4(5)-Merson [9] are shown in Table 2.

From Table 2, we can see that the accuracy obtained from using WRK-RK5(5) is better than the RK4, RK4(5)-Merson and RK5(6)-Nystrom methods. When we make a work comparison with the fifth order RK5(6)-Nystrom method, then this new fifth order method saves one function evaluation. From the above discussion we can conclude that contrary to Butcher [1] a fifth order arithmetic mean weighted Runge-Kutta method with five stages does exist. The study of fifth order weighted Runge-Kutta methods for a variety of means are under further investigation.

References

[1] J. C. Butcher, *The numerical analysis of ordinary differential equations: Runge-Kutta and general linear methods*, John Wiley & Sons, Chichester, (1987).
[2] D. J. Evans and A. R. Yaakub, A New Fourth Order Runge-Kutta Method Based On The Contraharmonic Mean Formula, Loughborough University of Technology, Department of Computer Studies, *Report No. 849* (1993).
[*Contributed Talks* at the VI-th SERC Numerical Analysis Summer School, Leicester University, 25–29 July 1994]
[3] D. J. Evans and A. R. Yaakub, A New 4th Order Runge-Kutta Method Based On The Centroidal Mean Formula, Loughborough University of Technology, Department of Computer Studies, *Report No. 851.* (1993)
[4] D. J. Evans and N. Yaacob, A Fourth Order Runge-Kutta Method Based On The Heronian Mean Formula, Loughborough University of Technology, Department of Computer Studies, *Report No. 879* (1994).
[5] D. J. Evans and B. B. Sanugi, A New 4th Order Runge-Kutta Method for Initial Value Problems, in S.O. Fatunla (Ed.), *Computational Mathematics II*, Boole Press, Dublin (1987).
[6] D. J. Evans and B. B. Sanugi, A Comparison of Nonlinear Trapezoidal Formulae for Solving Initial Value Problems, *Intern. J. Computer Math.*, **41**, (1991) 65–79.
[7] J. D. Lambert, *Computational Methods In Ordinary Differential Equations*, John Wiley & Sons, London (1973).
[8] J. D. Lambert, *Numerical Methods For Ordinary Differential Systems*, John Wiley & Sons, London (1991).
[9] R. H. Merson, An Operational Method For The Study of Integrating processes, *Proc. Symp. Data Processing*, Weapon Research Establishment, Salisbury, S. Australia (1957).
[10] B. B. Sanugi and D. J. Evans, A New Fourth Order Runge-Kutta Formula Based on Harmonic Mean, Loughborough University of Technology, Department of Computer Studies, *Report No. 799* (1993).
[11] A. R. Yaakub and D. J. Evans, A New Fourth Order Runge-Kutta Methods Based On The Root Mean Square Formula, Loughborough University of Technology, Department of Computer Studies, *Report No. 862* (1993).
[12] Bahrom Sanugi and Nazeeruddin Yaacob, A New 5th Order 5 stage Runge-Kutta Method for Initial Value Type Problems in ODE's. *Int. Jour. Comp. Maths* (1995) (in press).

APPENDIX 1

By use of Mathematica to rationalize the coefficients in equation (3.5), we obtain

$$w_1 = 0.2615038147 = \frac{28449}{108790}, \quad w_2 = -0.2765809214 = \frac{40890}{147841}$$

$$w_3 = 0.5947141647 = \frac{211318}{355327}, \quad w_4 = 0.4203629420 = \frac{65601}{156058}$$

$$a_1 = 1.5471214403 = \frac{479767}{310103}, \quad a_2 = 0.1756458393 = \frac{14081}{80167}$$

$$a_3 = 0.1243059001 = \frac{7768}{62491}, \quad a_4 = 0.1009316694 = \frac{20551}{203613}$$

$$a_5 = 0.1100539630 = \frac{15153}{137687}, \quad a_6 = 0.9997431862 = \frac{151822}{151861}$$

$$a_7 = -0.0928890403 = \frac{4909}{52848}, \quad a_8 = -0.6201812828 = \frac{91411}{147394}$$

$$a_{11} = \frac{1}{2} - a_4 - a_5 = 0.2890143692 = \frac{32604}{112811}$$

$$a_{22} = 1 - a_6 - a_7 - a_8 = 0.7133271396 = \frac{548973}{769595}$$

and this new fifth order WRK method can be written in rational form as

$$y_{n+1} = y_n + h\left[\frac{28449}{108790}\left(\frac{k_1 + k_2}{2}\right) - \frac{40890}{147841}\left(\frac{k_2 + k_3}{2}\right) + \frac{211318}{355327}\left(\frac{k_3 + k_4}{2}\right) + \frac{65601}{156058}\left(\frac{k_4 + k_5}{2}\right)\right]$$

where

$$k_1 = f(y_n)$$

$$k_2 = f\left(y_n + \frac{479767}{310103}hk_1\right)$$

$$k_3 = f\left(y_n + \frac{14081}{80167}hk_1 + \frac{7768}{62491}hk_2\right)$$

$$k_4 = f\left(y_n + \frac{20551}{203613}hk_1 + \frac{15153}{137687}hk_2 + \frac{32604}{112811}hk_3\right)$$

$$k_5 = f\left(y_n + \frac{151822}{151861}hk_1 - \frac{4909}{52848}hk_2 - \frac{91411}{147394}hk_3 + \frac{548973}{769595}hk_4\right).$$