Hybrid model of Vlasov-Poisson equations and comparison of Hamiltonian method and Lawson method

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Outline

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Vlasov-Poisson-Ampère equations 1Dx $\times$ 1Dv

Our model: transport of electron density distribution $f = f(t, x, v)$:

$$\begin{cases} \partial_t f + v\partial_x f + E\partial_v f = 0 \\ \partial_x E = \int_{\mathbb{R}} f \,\mathrm{d}v - 1 \\ \partial_t E = -\int_{\mathbb{R}} vf \,\mathrm{d}v + \frac{1}{|\Omega|}\int_{\Omega}\int_{\mathbb{R}} vf \,\mathrm{d}v\mathrm{d}x \end{cases} \qquad (x, v) \in \Omega \times \mathbb{R}$$

Motivations:

- We consider an initial condition of the form: $f = f_c + f_h$ with: $f_c(t = 0, x, v) = \mathcal{M}_{\rho_c, u_c, T_c}(v) \underset{T_c \to 0}{=} \delta_{v - u_c}(v)\rho_c(t = 0, x)$
- We want high order methods in (x, v)
  - FFT in x + WENO in v
- We want high order methods in time t
  - splitting method *vs* exponential integrator

The idea

- Grid methods can't have an initial condition like :
  $f_0(x,v) = \rho_{c,0}(x)\delta_{v-u_c}(v) + f_{h,0}(x,v)$
- Idea is to derive an hybrid model :
  - Cold plasma approximation: $T_c \approx 0$ ➜
    $f_c(t,x,v) = \rho_c(t,x)\delta_{v-u_c(t,x)}(v)$: fluid dynamic for cold particles (no velocity grid)
  - Hypothesis on hot particles: $\int_{\mathbb{R}} f_h(t,x,v)\,\mathrm{d}v \ll \rho_c(t,x)$: kinetic dynamic for hot particles (still velocity grid without constraints on cold particles)

Derivation of hybrid model

$$\begin{cases} \int_{\mathbb{R}} \begin{pmatrix} 1 \\ v \end{pmatrix} \left(\partial_t f_c + v\partial_x f_c + E\partial_v f_c = 0 \right) \mathrm{d}v \\ \partial_t f_h + v\partial_x f_h + E\partial_v f_h = 0 \\ \partial_x E = \int_{\mathbb{R}} f_c \,\mathrm{d}v + \int_{\mathbb{R}} f_h \,\mathrm{d}v - 1 \\ \partial_t E = -\int_{\mathbb{R}} v f_c \,\mathrm{d}v - \int_{\mathbb{R}} v f_h \,\mathrm{d}v + \frac{1}{|\Omega|} \int_{\Omega} \int_{\mathbb{R}} v(f_c + f_h) \,\mathrm{d}v\mathrm{d}x \end{cases}$$

we note:

$$\begin{pmatrix} \rho_c(t,x) \\ \rho_c(t,x)u_c(t,x) \end{pmatrix} = \int_{\mathbb{R}} \begin{pmatrix} 1 \\ v \end{pmatrix} f_c(t,x,v) \,\mathrm{d}v$$

and *cold plasma approximation*: $f_c(t,x,v) = \rho_c(t,x)\delta_{v=u_c(t,x)}(v)$

Non-linear hybrid fluid-kinetic model

$$
\begin{cases}
\partial_t \rho_c + \partial_x(\rho_c u_c) = 0 \\
\partial_t(\rho_c u_c) + \partial_x(\rho_c u_c^2) - \rho_c E = 0 \\
\partial_t f_h + v\partial_x f_h + E\partial_v f_h = 0 \\
\partial_t E = -\rho_c u_c - \int_{\mathbb{R}} v f_h \,\mathrm{d}v + \frac{1}{|\Omega|}\left(\int_\Omega \int_{\mathbb{R}} v f_h \,\mathrm{d}v\mathrm{d}x + \int_\Omega \rho_c u_c \,\mathrm{d}x\right) \\
\color{red}{(}\ \partial_x E = \rho_c + \int_{\mathbb{R}} f_h \,\mathrm{d}v - 1\ \color{red}{)}
\end{cases}
$$

- The initial condition $(u_c^0, f_h^0, E^0, \rho_c^0)$ satisfies Poisson equation
  - $\partial_x E_0(x) = \rho_c^0(x) + \int_{\mathbb{R}} f_h^0(x, v)\,\mathrm{d}v$
- We can compute cold density ρ_c: $\forall t \geq 0$ with Poisson equation
- In other words, we don't need to compute ρ_c at each time

Following physicists framework, we linearize this non-linear model

Linearization of hybrid fluid-kinetic model

Linearization near equilibrium :

$$\begin{aligned}
\rho_c(t,x) &= \rho_c^{(0)}(x) + & \varepsilon \rho_c^{(1)}(t,x) \\
u_c(t,x) &= & \varepsilon u_c^{(1)}(t,x) \\
E(t,x) &= & \varepsilon E^{(1)}(t,x) \\
f_h(t,x,v) &= f_h^{(0)}(v) + & \varepsilon f_h^{(1)}(t,x,v)
\end{aligned}$$

We obtain **Linear hybrid model** (LHM):

$$\begin{cases}
\partial_t {u_c}^{(1)} = E^{(1)} + \mathcal{O}(\varepsilon) \\
\partial_t E^{(1)} = -\rho_c^{(0)} {u_c}^{(1)} - \int_{\mathbb{R}} v {f_h}^{(1)} \, \mathrm{d}v + \mathcal{O}(\varepsilon) \\
\partial_t {f_h}^{(1)} + v \partial_x {f_h}^{(1)} + E \partial_v {f_h}^{(1)} = 0 + \mathcal{O}(\varepsilon)
\end{cases}$$

Properties: conservation of mass and total energy ✔

All this derivation can be generalized to 3Dx-3Dv

Holderied et al. (2020)

Poisson bracket

LHM have an hamiltonian structure. The Poisson bracket is define by:

$$\begin{aligned}\{\mathcal{F},\mathcal{G}\}(u,E,f) = &\int_{\mathbb{R}}\int_{\mathbb{R}} f\left(\partial_x \frac{\delta\mathcal{F}}{\delta f}\partial_v \frac{\delta\mathcal{G}}{\delta f} - \partial_v \frac{\delta\mathcal{F}}{\delta f}\partial_x \frac{\delta\mathcal{G}}{\delta f}\right)\mathrm{d}v\mathrm{d}x \\ &+ \int_{\mathbb{R}}\left(\frac{\delta\mathcal{F}}{\delta u}\frac{\delta\mathcal{G}}{\delta E} - \frac{\delta\mathcal{F}}{\delta E}\frac{\delta\mathcal{G}}{\delta u}\right)\mathrm{d}x \\ &+ \int_{\mathbb{R}}\int_{\mathbb{R}}\left(\frac{\delta\mathcal{F}}{\delta E}\partial_v f\frac{\delta\mathcal{G}}{\delta f} - \frac{\delta\mathcal{G}}{\delta E}\partial_v f\frac{\delta\mathcal{F}}{\delta f}\right)\mathrm{d}v\mathrm{d}x\end{aligned}$$

and Hamiltonian by:

$$\mathcal{H}(t) = \underbrace{\frac{1}{2}\int_{\mathbb{R}} E^2\,\mathrm{d}x}_{\mathcal{H}_E} + \underbrace{\frac{1}{2}\int_{\mathbb{R}} \rho_c^{(0)} u_c^2\,\mathrm{d}x}_{\mathcal{H}_{u_c}} + \underbrace{\frac{1}{2}\int_{\Omega}\int_{\mathbb{R}} v^2 f_h\,\mathrm{d}v\mathrm{d}x}_{\mathcal{H}_{f_h}}$$

Outline

Hamiltonian splitting

To preserve the hamiltonian structure, we would like implement a splitting method inspired a hamiltonian splitting.

With $U = \begin{pmatrix} u_c \\ E \\ f_h \end{pmatrix}$ problem can be write as:

$$\begin{aligned} \dot{U} &= \{U, \mathcal{H}\} \\ &= \{U, \mathcal{H}_E\} + \{U, \mathcal{H}_{u_c}\} + \{U, \mathcal{H}_{f_h}\} \end{aligned}$$

We obtain the splitting:

- $\dot{U} = \{U, \mathcal{H}_E\}$ ➜ solution is $\varphi_t^{[E]}(U^0)$
- $\dot{U} = \{U, \mathcal{H}_{u_c}\}$ ➜ solution is $\varphi_t^{[u_c]}(U^0)$
- $\dot{U} = \{U, \mathcal{H}_{f_h}\}$ ➜ solution is $\varphi_t^{[f_h]}(U^0)$

Splitting method

- **Lie:** order 1 method, to compute an approximation at time t^n:

$$U(t^{n+1}) \approx U^{n+1} = \varphi_{\Delta t}^{[E]} \circ \varphi_{\Delta t}^{[u_c]} \circ \varphi_{\Delta t}^{[f_h]}(U^n)$$

- **Strang:** order 2 method, for a 3 steps formulation:

$$U^{n+1} = S_{\Delta t}(U^n) = \varphi_{\Delta t/2}^{[E]} \circ \varphi_{\Delta t/2}^{[u_c]} \circ \varphi_{\Delta t}^{[f_h]} \circ \varphi_{\Delta t/2}^{[E]} \circ \varphi_{\Delta t/2}^{[u_c]}(U^n)$$

Strang (1968)

- **Suzuki:** order 4 method, composition of 5 Strang methods:

$$U^{n+1} = \mathcal{S}_{\Delta t}(U^n) = S_{\alpha_1 \Delta t} \circ S_{\alpha_2 \Delta t} \circ S_{\alpha_3 \Delta t} \circ S_{\alpha_2 \Delta t} \circ S_{\alpha_1 \Delta t}(U^n)$$

with:

$$\alpha_1 = \alpha_2 = \frac{1}{4 - \sqrt[3]{4}} \qquad \alpha_3 = \frac{1}{1 - 4^{\frac{2}{3}}}$$

Suzuki (1990)

Casas and Escorihuela-Tomàs (2020)

Numerical resolution of each step

$$\varphi^{[E]}(U) = \begin{cases} \partial_t u_c = E \\ \partial_t E = 0 \\ \partial_t f_h = -E \partial_v f_h \end{cases} \quad \rightarrow \varphi^{[E]}_{\Delta t}(U^n) = \begin{pmatrix} u_c^n + \Delta t E^n \\ E^n \\ f_h(x, v - \Delta t E^n) \end{pmatrix}$$

Numerical tools:

- Lagrange 5 interpolation to approximate $f_h(x, v - \Delta t E^n)$
- Longest step, so we keep it in the middle of Strang method

Numerical resolution of each step

$$\varphi^{[u_c]}(U) = \begin{cases} \partial_t u_c = 0 \\ \partial_t E = -\rho_c^{(0)} u_c \\ \partial_t f_h = 0 \end{cases} \quad \rightarrow \varphi_{\Delta t}^{[u_c]}(U^n) = \begin{pmatrix} u_c^n \\ E^n - \Delta t \rho_c^{(0)} u_c^n \\ f_h^n \end{pmatrix}$$

Numerical tools:

- Fastest step

Numerical resolution of each step

$$\varphi^{[f_h]}(U) = \begin{cases} \partial_t u_c = 0 \\ \partial_t E = -\int_{\mathbb{R}} v f_h \,\mathrm{d}v \\ \partial_t f_h = -v\partial_x f_h \end{cases} \to \varphi^{[f_h]}_{\Delta t}(U^n) = \begin{pmatrix} u_c^n \\ \hat{E}^n - \frac{i}{k}\int_{\mathbb{R}}(e^{-ikv\Delta t} - 1)\hat{f}_h^n \,\mathrm{d}v \\ e^{-ikv\Delta t}\hat{f}_h^n \end{pmatrix}$$

Numerical tools:

- FFT and iFFT during this step
- Fast with `fftw` if you reuse allocated memory

Lawson method

Fluid part is linear, we want to solve it exactly ➜Lawson method

$$\partial_t \underbrace{\begin{pmatrix} u_c \\ E \\ \hat{f}_h \end{pmatrix}}_{U} + \underbrace{\begin{pmatrix} 0 & -1 & 0 \\ \rho_c^{(0)} & 0 & 0 \\ 0 & 0 & ikv \end{pmatrix}}_{A} \begin{pmatrix} u_c \\ E \\ \hat{f}_h \end{pmatrix} + \underbrace{\begin{pmatrix} 0 \\ \int_{\mathbb{R}} v f_h \,\mathrm{d}v \\ \widehat{E \partial_v f_h} \end{pmatrix}}_{F(U)} = 0$$

We rewrite as:

$$\partial_t \underbrace{\left(e^{tA} U\right)}_{V} + e^{tA} N(\underbrace{U}_{e^{-tA}V}) = 0$$

and now we solve with a RK method: $\partial_t V = -e^{tA} N(e^{-tA} V)$ and next rewrite with the U variable. For example with Lawson Euler method:

$$V(t^n + \Delta t) \approx V^{n+1} = V^n - \Delta t e^{-t^n A} F(e^{t^n A} V^n)$$

or as an expression of U:

$$U^{n+1} = e^{\Delta t A} U^n - \Delta t e^{-\Delta t A} F(U^n)$$

Space integrators with Lawson method

Numerical tools:

- FFT in x direction
- WENO5 in v direction to approximate $\widehat{E\partial_v f_h}$

CFL: Lawson(DP4(3))–WENO5 : $\Delta t \leq \frac{\sigma}{\|E^n\|_\infty}$, with $\sigma = 1.433$

Crouseilles, Einkemmer, and Massot (2019)

Outline

Main idea of adaptive time step methods (error estimate)

For a generic ODE $\dot{u} = f(t, u(t))$, adaptive time step method needs 2 numerical estimations of solution $u(t^{n+1})$ of different order, p and $p+1$:

$$u_{[p]}^{n+1} = u(t^{n+1}) + \mathcal{O}\left(\Delta t^{p+1}\right) \qquad u_{[p+1]}^{n+1} = u(t^{n+1}) + \mathcal{O}\left(\Delta t^{p+2}\right)$$

Estimate of the local error:

$$L_{[p]}^{n+1} = \left| u_{[p+1]}^{n+1} - u_{[p]}^{n+1} \right|$$

If $L_{[p]}^{n+1} > \text{tol}$: we reject the step and start again from time t^n. Else we accept the step. In two cases, the optimal new time step size is:

$$\Delta t_{\text{opt}} = \sqrt[p]{\frac{\text{tol}}{L_{[p]}^{n+1}}}\Delta t^n$$

In practice we don't want volatile time step:
$\Delta t^{n+1} = \max\left(0.5\Delta t^n, \min\left(2\Delta t^n, \Delta t_{\text{opt}}\right)\right)$

Adaptive time step method for splitting method

For the Suzuki splitting method:

Blanes, Casas, and Thalhammer (2019)

$$U_{[4]}^{n+1} = \mathcal{S}_{\Delta t}(U^n) = S_{\alpha_1\Delta t}\circ \underbrace{S_{\alpha_2\Delta t}\circ\underbrace{S_{\alpha_3\Delta t}\circ\underbrace{S_{\alpha_2\Delta t}\circ\underbrace{S_{\alpha_1\Delta t}(U^n)}_{U^{(1)}}}_{U^{(2)}}}_{U^{(3)}}}_{U^{(4)}}.$$

We compute an order 3 approximation from U^n and $U^{(s)}$, $s = 1, 2, 3, 4$:

$$U_{[3]}^{n+1} = -U^n + w_1(U^{(1)} + U^{(4)}) + w_2(U^{(2)} + U^{(3)})$$

with:

$$w_1 = \frac{g_2(1-g_2)}{g_1(g_1-1) - g_2(g_2-1)}, \quad w_2 = 1 - w_1, \quad \begin{aligned} g_1 &= \alpha_1 \\ g_2 &= \alpha_1 + \alpha_2 \end{aligned}$$

and $L_{[3]}^n = \left\| U_{[4]}^{n+1} - U_{[3]}^{n+1} \right\|_2$

Adaptive time step method for Lawson method

Lawson methods are built on Runge-Kutta method, embedded Lawson method are write with an underlying embedded Runge-Kutta method.

Dormand and Prince (1978)

With DP4(3) (Dormand-Prince method of order 4, with embedded 3 method):

$$
\begin{array}{c|ccccc}
0 & & & & & \\
\frac{1}{2} & \frac{1}{2} & & & & \\
\frac{1}{2} & 0 & \frac{1}{2} & & & \\
1 & 0 & 0 & 1 & & \\
\hline
1 & \frac{1}{6} & \frac{1}{3} & \frac{1}{3} & \frac{1}{6} & \\
\hline
 & \frac{1}{6} & \frac{1}{3} & \frac{1}{3} & \frac{2}{30} & \frac{1}{10}
\end{array}
$$

We compute a 3rd order approximation from U^n, $U^{(s)}$, $s = 1, 2, 3, 4$ done by the last line of Butcher tableau.

and $L^n_{[3]} = \left\| U^{n+1}_{[4]} - U^{n+1}_{[3]} \right\|_2$

Outline

Sketch of computing

We linearize around unstable equilibrium state (TSI type):

- for kinetic model

$$f_{\mathrm{eq}}(v) = \mathcal{M}_{\rho_c,0,T_c}(v) + \mathcal{M}_{\alpha/2,v_0,1}(v) + \mathcal{M}_{\alpha/2,-v_0,1}(v)$$

- for LHM

$$\begin{aligned} U_{\mathrm{eq}} &= \left(u_{c,\mathrm{eq}}, E_{\mathrm{eq}}, f_{h,\mathrm{eq}}\right)^{\mathsf{T}} \\ &= \left(0, 0, \mathcal{M}_{\alpha/2,v_0,1} + \mathcal{M}_{\alpha/2,-v_0,1}\right)^{\mathsf{T}} \end{aligned}$$

We obtain 2 relations of dispersion:

- One for kinetic model, depends on T_c: $D^{K}_{[T_c]}(k,\omega)$
- One for linear hybrid model (LHM): $D^{LHM}(k,\omega)$

Relations of dispersion

- **Kinetic model:**

$$D^K_{[T_c]}(k,\omega) = 1 - \frac{1}{k^2}\Bigg[-\frac{1-\alpha}{T_c}\left(1+\frac{1}{\sqrt{2T_c}}\frac{\omega}{k}Z\left(\frac{1}{\sqrt{2T_c}}\frac{\omega}{k}\right)\right)$$
$$-\frac{\alpha}{2}\left(1+\frac{1}{\sqrt{2}}\left(\frac{\omega}{k}-v_0\right)Z\left(\frac{1}{\sqrt{2}}\left(\frac{\omega}{k}-v_0\right)\right)\right)$$
$$-\frac{\alpha}{2}\left(1+\frac{1}{\sqrt{2}}\left(\frac{\omega}{k}+v_0\right)Z\left(\frac{1}{\sqrt{2}}\left(\frac{\omega}{k}+v_0\right)\right)\right)\Bigg]$$

- **LHM:**

$$D^{LHM}(k,\omega) = 1 - \frac{1}{k^2}\Bigg[(1-\alpha)\frac{k^2}{\omega^2} - \frac{\alpha}{2}\left(1+\frac{1}{\sqrt{2}}\left(\frac{\omega}{k}-v_0\right)Z\left(\frac{1}{\sqrt{2}}\left(\frac{\omega}{k}-v_0\right)\right)\right)$$
$$-\frac{\alpha}{2}\left(1+\frac{1}{\sqrt{2}}\left(\frac{\omega}{k}+v_0\right)Z\left(\frac{1}{\sqrt{2}}\left(\frac{\omega}{k}+v_0\right)\right)\right)\Bigg]$$

where $Z(z) = \sqrt{\pi}e^{-z^2}(i - \mathrm{erfi}(z))$ (Fried and Conte function)

Convergence of relations of dispersion

Since

$$Z(z) \underset{z\to+\infty}{\sim} -\frac{1}{z} - \frac{1}{2z^3} - \frac{3}{4z^5} + \mathcal{O}(z^{-7})$$

with $z = \frac{1}{\sqrt{2T_c}}\frac{\omega}{k}$ when $T_c \to 0$ we get:

$$-\frac{1-\alpha}{T_c}\left(1 + \frac{1}{\sqrt{2T_c}}\frac{\omega}{k} Z\left(\frac{1}{\sqrt{2T_c}}\frac{\omega}{k}\right)\right) = -\frac{1-\alpha}{T_c}\left(-\frac{1}{2\left(\frac{1}{\sqrt{2T_c}}\frac{\omega}{k}\right)} + \mathcal{O}(z^{-4})\right)$$

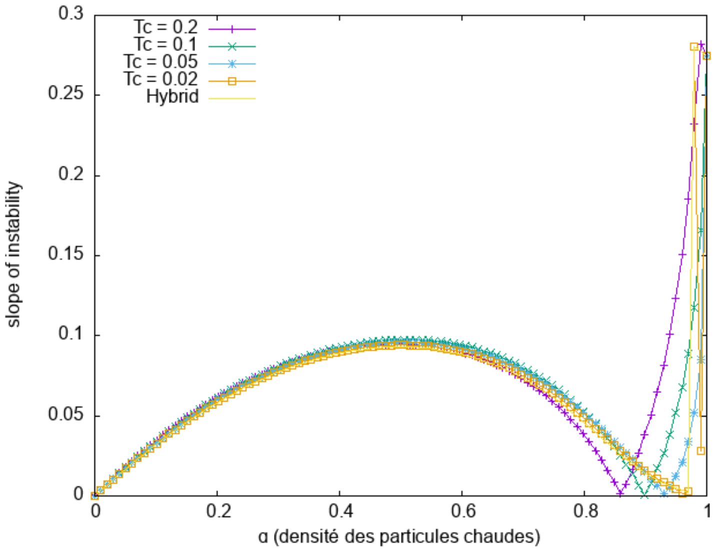
which is equivalent to:

$$\frac{1-\alpha}{T_c}\frac{k^2 T_c}{\omega^2} = (1-\alpha)\frac{k^2}{\omega^2}$$

We have:

$$\lim_{T_c\to 0} D^K_{[T_c]}(k,\omega) = D^{LHM}(k,\omega) \quad ✔$$

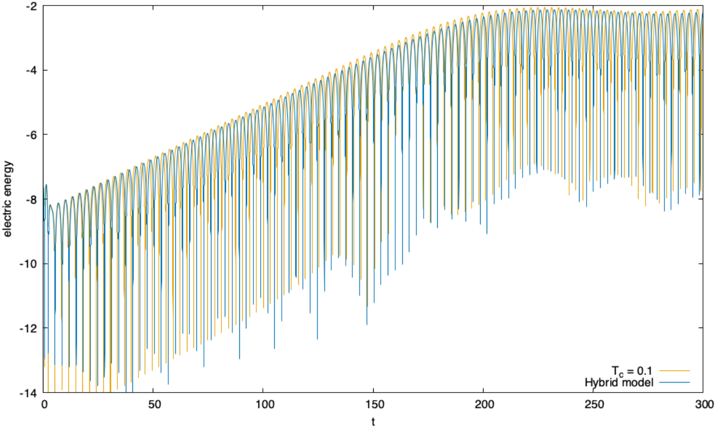
Slope of two relations of dispersion



Outline

Validation of hybrid model

$k = 0.5$, $x \in [0, \frac{2\pi}{k}]$, $v \in [-8, 8]$, $v_0 = 4$
$N_x = 135$, $N_v = 1200$, $\Delta t = 0.5\Delta v$ for LHM and kinetic



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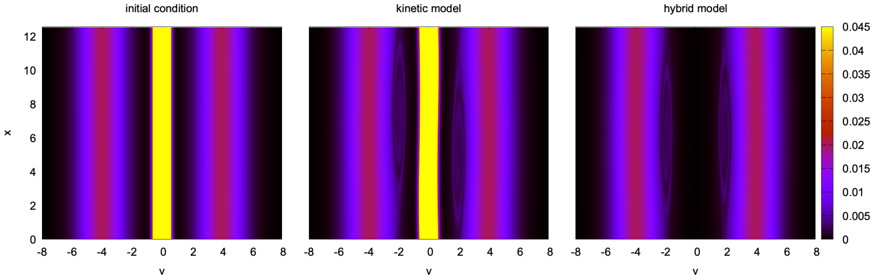
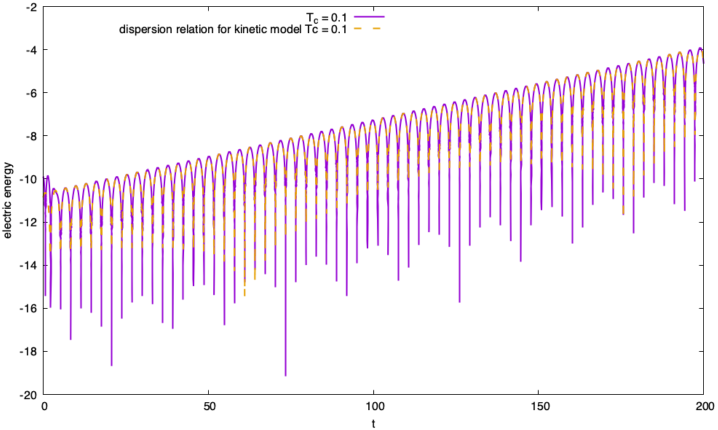


Validation with relations of dispersion

Relations of dispersion get the electric energy approximation in the linear phase

Perturbation $\epsilon = 10^{-4}$, $\alpha = 0.1$, $N_x = 135$, $N_v = 1200$, $\Delta t = 0.5\Delta v$



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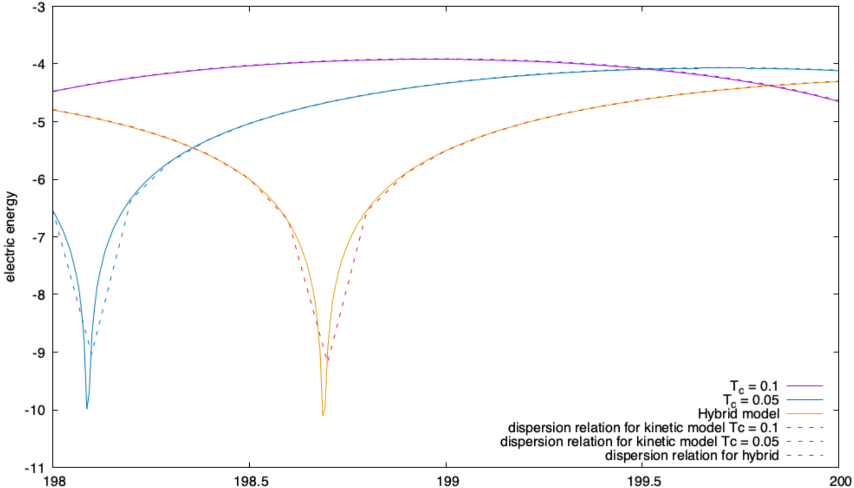
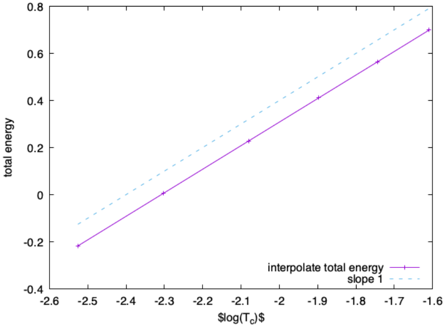
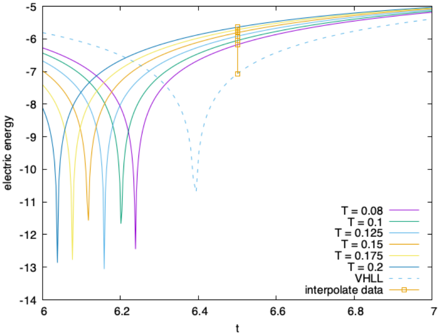


Convergence of kinetic model to linear hybrid model

$\alpha = 0.1$

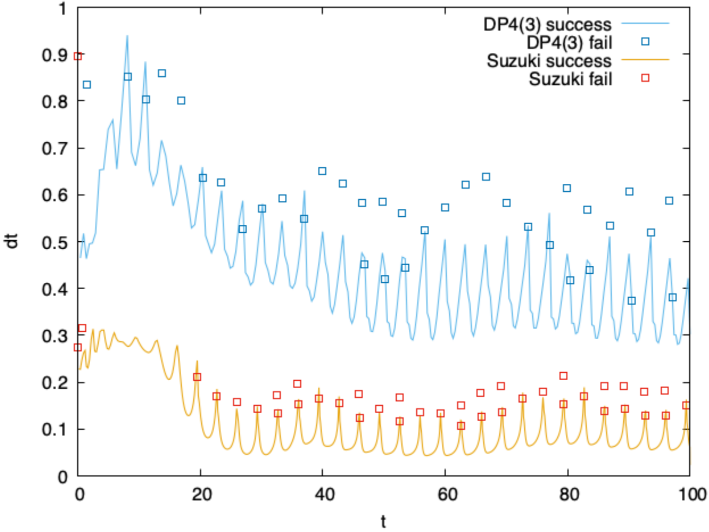
$N_x = 135,\ N_v = 1200,\ \Delta t = 0.5\Delta v$ for LHM and kinetic





Test adaptive time step method

tol $= 2 \cdot 10^{-4}$, $L = \|U^{n+1}_{[4]} - U^{n+1}_{[3]}\|_2$ is the local error



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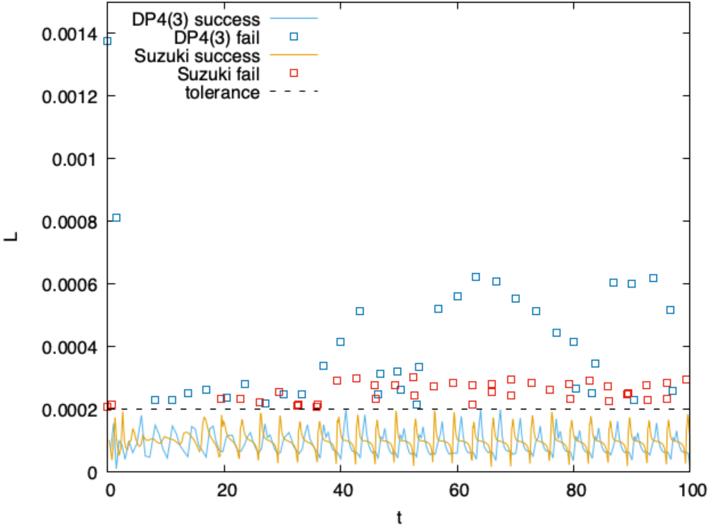

Outline

Summary

- Validation and robustness of Linear Hybrid Model
- Derivation of geometric structure for LHM
- Numerical cost for one Strang is equivalent of one stage of Lawson-RK method
  - 5 Strang for Suzuki
  - 4 stages for RK(4,4)

Future works

- Extension to $1\mathrm{D}x - 3\mathrm{D}v$
  - Splitting into 6 sub-steps ➜Strang in 11 steps
  - Lawson methods should be more efficient

Thank you for your attention