

## Utility Theorem

Let  $\Omega$  be the set of outcomes

$$\Omega = \mathbb{Z} \cup \Delta(\Omega)$$

$\mathbb{Z} \rightarrow$  some set of outcomes

$\Delta(\Omega) \rightarrow$  set of lotteries over finite subsets of  $X$

$[p_1: x_1, \dots, p_k: x_k] \rightarrow$  List / dist. of pairs of prob. and outcome with that prob.  
with  $x_1, \dots, x_k \in X$  and  $\sum_{j=1}^k p_j = 1$

A preference relation compares the relatively desirability of outcome

- $\omega_1 \geq \omega_2 \rightarrow$  agent weakly prefers  $\omega_1$  to  $\omega_2$
- $\omega_1 > \omega_2 \rightarrow$  agent strictly prefers  $\omega_1$  to  $\omega_2$
- $\omega_1 \sim \omega_2 \rightarrow$  agent is indifferent b/w  $\omega_1$  and  $\omega_2$

A utility function is a function  $u: \Omega \rightarrow \mathbb{R}$

• It represents a preference relation  $\geq$  iff

- $\forall \omega_1, \omega_2 \in \Omega: \omega_1 \geq \omega_2 \Leftrightarrow u(\omega_1) \geq u(\omega_2)$ , and
- $\forall [p_1: \omega_1, \dots, p_k: \omega_k] \in \Omega: u([p_1: \omega_1, \dots, p_k: \omega_k]) = \sum_{j=1}^k p_j u(\omega_j)$

$\rightarrow$  If preference relation  $\geq$  satisfies the axioms Completeness, Transitivity, Monotonicity, Substitutability, Decomposability, and continuity.

Then there exists a utility fn  $u$  that represents  $\geq$

## Axioms

- $\forall \omega_1, \omega_2 \in \Omega: (\omega_1 > \omega_2) \vee (\omega_1 \sim \omega_2) \vee (\omega_1 \leq \omega_2) \rightarrow$  Completeness
- $\forall \omega_1, \omega_2, \omega_3 \in \Omega: (\omega_1 \geq \omega_2) \wedge (\omega_2 \geq \omega_3) \Rightarrow \omega_1 \geq \omega_3 \rightarrow$  Transitivity
- $(\omega_1 > \omega_2) \Rightarrow [p_1: \omega_1, (1-p): \omega_2] > [q: \omega_1, (1-q): \omega_2] \rightarrow$  Monotonicity  
 $\hookrightarrow$  prefer 90% chance of getting \$1000 to 50% chance of getting \$1000
- $p + \sum_{j=3}^k p_j = 1, \text{ if } \omega_1 \sim \omega_2$   
 $[p: \omega_1, p_3: \omega_3, \dots, p_k: \omega_k] \sim [p: \omega_1, p_3: \omega_3, \dots, p_k: \omega_k] \rightarrow$  substitutability  
 $\hookrightarrow$  If I like apples & bananas equally, then shouldn't care about 30% chance of getting apples or bananas
- $(\forall \omega \in \Omega: p_{L_1}(\omega) = p_{L_2}(\omega)) \Rightarrow L_1 \sim L_2 \rightarrow$  Decomposability (No fun in Gambling)  
 $\hookrightarrow$  prob. that outcome  $\omega$  is selected  
 $L_1 = [0.5: [\omega_1, \omega_2], 0.5: \omega_3], L_2 = [0.25: \omega_1, 0.25: \omega_2, 0.5: \omega_3]$   
 $L_1 \sim L_2 \because$   
 $p_{L_1}(\omega_1) = 0.5 \times 0.5 = 0.25 = p_{L_2}(\omega_1)$
- $\omega_1 > \omega_2 > \omega_3 \Rightarrow \exists p \in [0, 1]: \omega_2 \sim [p: \omega_1, (1-p): \omega_3] \rightarrow$  continuity

## Proof Sketch (von Neumann & Morgenstern)

- Choose  $\bar{\omega}^+, \bar{\omega}^-$  such that  $\bar{\omega}^- \leq \omega \leq \bar{\omega}^+$  for all  $\omega$
- Construct  $u(\omega) = p$  such that  $u(\omega) = p$  for  $\omega \in [p: \bar{\omega}^+, (1-p): \bar{\omega}^-]$
- Substitutability lets us replace everything with these canonical lotteries.  
Monotonicity lets us assert the ordering b/w them

$\rightarrow$  For a given set of preferences, the utility function is not uniquely defined.

Comparisons of expected values are invariant to positive affine transformations

$$\begin{aligned} X \succeq Y &\Leftrightarrow E[u(X)] \geq E[u(Y)] \\ &\Leftrightarrow cE[u(X)] + b \geq cE[u(Y)] + b \\ &\Leftrightarrow E[cu(X) + b] \geq E[cu(Y) + b] \end{aligned}$$

for all  $b \in \mathbb{R}$  and  $c > 0$

Von Neumann & Morgenstern  $\rightarrow$  risk neutral

$\Rightarrow$  Utility theory  $\rightarrow$  requires you to know what the probabilities are.  
 ↳ assumes known, objective probabilities

Other theorems state that  $\Rightarrow$  rational agents must (a) have prob. beliefs  
 (b) update those beliefs as if by conditioning (c) max. the expected value w.r.t them.

### Summary

- Utility theory proves that agents whose preferences obey certain simple axioms about preferences over lotteries must act as if they were maximizing the expected value of scalar func.
- Rational agents  $\rightarrow$  who satisfy the axioms.

### Game Representations

Finite, n-person game:  $\langle N, A, u \rangle \rightarrow \text{Game}$

$N \rightarrow$  finite set of n players

$A = \langle A_1, \dots, A_n \rangle$  is a set of action sets for each player  $i$   
 .  $a \in A$  is an action profile

$u = \langle u_1, \dots, u_n \rangle$ , a utility function for each player, where  $u_i: A \rightarrow \mathbb{R}$

$\bullet$  Normal form  $\rightarrow$  2 player game as a matrix

↳ table rep. of simultaneous move game

	C	D
C	a, a	b, c
D	c, b	d, d

→ prisoner's dilemma where this is satisfied

$$c > a > d > b$$

Players have exactly opposed interests

→ Exactly two players

→ for all action profiles  $a \in A$ ,  $u_1(a) + u_2(a) \xrightarrow{\text{constant sum}} c$   
 - Special case: zero sum

→ Thus only need to store utility fn for one player

Games of pure competition

### Matching Pennies

↳ zero sum game

	H	T
H	1, -1	-1, 1
T	-1, 1	1, -1

### Rock Paper Scissors

	R	P	S
R	0, 0	-1, 1	1, -1
P	1, -1	0, 0	-1, 1
S	-1, 1	1, -1	0, 0

zero sum simultaneous move game

### Game of Cooperation

Players have exactly same interests

- no conflict: all players want the same things

$$\forall a \in A, \forall i, j, u_i(a) = u_j(a)$$

- single payoff per cell

What side of road to drive on?

		L	R
		1, 1	0, 0
L	L	1, 1	0, 0
	R	0, 0	1, 1

↳ canonical game of cooperation

- Most interesting games combine elements of cooperation and competition

### Battle of Sexes

		B	F
		2, 1	0, 0
B	B	2, 1	0, 0
	F	0, 0	1, 2

→ General sum game

### Repeated Games

- Play the same game over and over again

### Perfect-Information Extensive Form Game

- Game unfolds over time

- Player can see each other's moves

- Can be written as trees, leaves labelled as payoffs

### Stochastic Game

- Combines perfect-information extensive form game with repeated game

- Multiplayer generalization of a Markov Decision Process (MDP)

  - state: which game is being played

  - actions: set of alternatives in that game

  - reward: payoffs in that game

  - transition: mapping from players' actions to next state  
fn

### Imperfect-Information Extensive Form Game

- generalizes perfect-information extensive-form game games by allowing for imperfect observation of previous player's moves

- Some actions might be observed perfectly

### Stackelberg Games → player commits to drawing action from a prob. distribution

- ↳ special case of imperfect-info games

- One player commits to a strategy which is observed by the second player
  - If strategy is randomized, the second player can't see random draws

- Same game played over & over again, the second player can see the first player's actions and figure out her strategy

### Bayesian Games

- Uncertainty about payoffs (either one's own or others')

  - prior → over which game is played

## Nash Equilibrium

Agents simultaneously make a decision.

Then they receive an outcome depending on profile of actions

$$G = (N, A, u)$$

utility profile  
↳ set of action profile  
of n players per agent

} Normal form game recap

## Optimal Decisions in Games

In single-agent decision theory, the key notion is optimal decision:

→ the action that maximizes the agent's expected utility.

$$a^* = \underset{a \in A}{\operatorname{argmax}} E[u(a)]$$

In multi-agent, the optimality notion is ill-defined

$$a_i^* = \underset{a_i \in A_i}{\operatorname{argmax}} E[u_i(a_i, a_{-i})]$$

issue because same for other agents

- The best strategy depends on actions of others.

## Solution Concepts

→ From an outside perspective, some outcomes of a game be considered better

→ Solution concepts pick sets of things that are interesting/good/desirable.

→ We cannot compare utilities of the agents to each other, because of affine invariance.

## Pareto Optimality → weak

↓ suppose outcome  $\sigma$  is at least as good as  $\sigma'$  for every agent  $i$   
 all companions are within single within agent →  $\sigma$  Pareto dominates  $\sigma'$  → whenever  $\sigma \geq \sigma'$  for all  $i \in N$  and  $\sigma > \sigma'$  for some  $i \in N$

→ An outcome  $\sigma^*$  is Pareto optimal if no other outcome Pareto dominates it.

→ Question is asked for a single agent and is done for all.

→ A game can have multiple pareto optimal outcomes.

Example:

$\sigma' =$  Everyone gets pie

$\sigma =$  Everyone gets pie & Alice gets also cake

}  $\sigma \rightarrow$  seems better than  $\sigma'$  because everyone is okay with  $\sigma$  and  $\sigma'$ .

	Coop.	Defect
Coop.	-1, -1	-5, 0
Defect	0, -5	-3, -3

## Nash Equilibrium

↳ more individual approach

$a_{-i} = (a_1, a_2, \dots, a_{i-1}, a_{i+1}, \dots, a_n) \rightarrow$  every agent except for agent  $i$

$$a = (a_i, a_{-i})$$

Best Response → which actions are better from an individual agent's viewpoint.

$$BR_i(a_{-i}) = \{a_i^* \in A_i | u(a_i^*, a_{-i}) \geq u(a_i, a_{-i}) \forall a_i \in A_i\}$$

↳  $i$ 's best response to  $a_{-i}$

In Nash Equilibrium, stable outcome one where no agent regrets their action.

An action profile  $a \in A$  is a (pure strategy) Nash Equilibrium iff.

$$\forall i \in N: a_i \in BR_i(a_{-i}) \rightarrow \text{for every } i$$

- A game can have more than one pure strategy Nash equilibrium.

↳ Nihilistic Game e.g. → all payoffs  $(0,0)$

- Can be the case where there is no P. Nash eq.

↳ rock paper scissors (Yes)

		Coop.	Defect
Coop.	Coop.	-1, -1	-5, 0
	Defect	0, -5	-3, -3

→ Nash Equilibrium

		A	B
A	0, 0	0, 0	
	0, 0	0, 0	

• Matching Pennies has no pure strategy

→ Pareto dominates the Nash equilibrium

### Mixed Strategies

→ consider randomize → strategy

A strategy  $s_i$  for agent  $i$  is any probability dist. over the set  $A_i$ , where each action  $a_i$  is played with prob.  $s_i(a_i)$ .

Pure strategy:  $s_i(a_i) = 1$  for some  $a_i$  only one action played

Mixed strategy:  $s_i(a_i) < 1$  for all  $a_i$  for all  $a_i$  randomized over multiple actions

Set of  $i$ 's strategies:  $S_i = \Delta(A_i)$

Strategy profiles:  $S = S_1 \times \dots \times S_n$

→ The utility of mixed strategy profile is its expected utility.

Assume that agents are decision theoretically rational

→ Agents randomize independently

$$\Delta(A_1) \times \dots \times \Delta(A_n)$$

$$u_i(s) = \sum_{a \in A} \Pr(a|s) u_i(a) \rightarrow \text{Prob. of profile a given strategy profile}$$

$$\begin{aligned} \text{utility} \\ \text{at strategy} \\ \text{profile} \end{aligned} = \sum_{a \in A} \left( \prod_{j \neq i} s_j(a_j) \right) u_i(a)$$

$$\rightarrow \Pr(a|s) = \prod_{j \neq i} s_j(a_j) \rightarrow \text{independence assumption}$$

The set of  $i$ 's best responses to a strategy profile  $s_{-i} \in S_{-i}$  is

$$BR_i(s_{-i}) = \{a_i^* \in A_i | u_i(a_i^*, s_{-i}) \geq u_i(a_i, s_{-i}) \quad \forall a_i \in A_i\}$$

A strategy profile is Nash Eq. iff,

$$\forall i \in N, s_i \in S_i: u_i(s) \geq u_i(s'_i, s_{-i})$$

Equivalently,  $\Pr(a_i|s) > 0 \Leftrightarrow a_i \in BR_i(s_{-i})$  ↗ prob. that action  $a_i$  is played under strategy  $s_i$

$$\forall i \in N, a_i \in S_i: s_i(a_i) > 0 \Leftrightarrow a_i \in BR_i(s_{-i})$$

When at least one  $s_i$  is mixed,  $s$  is a mixed strategy Nash eq.

Theorem  $\Rightarrow$  Every game with finite number of players and action profiles has at least one Nash eq.

Is it possible/rational for an agent to play anything other than Nash eq.? Yes

↳ Nash Eq. only optimal if believe that other agents will play their parts of the same Nash equilibrium.

## Computing Mixed Nash Equilibrium

	B	F
B	2,1	0,0
F	0,0	1,2

- Let player 2 play B with p, F with 1-p
  - If player 1 best-responds with mixed strategy, player 2 must make her indifferent b/w F and B.
- ↳ because if there is preference, then it is not a mixed strategy as best-response.

- Likewise, player 1 must randomize to make player 2 indifferent.

- Player 1 is willing to randomize because she is being indifferent.

Let player 1 play B with  $q_1$ , F with  $1-q_1$

$$u_1(B) = u_1(F)$$

$$2 + 0(1-q_1) = 0 + 1(1-q_1)$$

$$q_1 = \frac{2}{3}$$

$$u_1(B) = u_1(F)$$

$$2p + 0(1-p) = 0p + 1(1-p)$$

$$p = \frac{1}{3}$$

↳ If p is -ve or bigger than 1 then there is no way of picking p so that player is indifferent.

⇒ Each randomize b/w all the things such that it makes the other person indifferent.

Strategies:  $(\frac{2}{3}, \frac{1}{3}), (\frac{1}{3}, \frac{2}{3})$  are Nash equilibrium.

### Maxmin Strategy → defensive

Player i's maxmin strategy is a strategy that maximizes i's worst-case payoff, in the situation where all other players (-i) happen to play the strategies which cause greatest harm to i.

The maxmin value of the game for player i is that minimum amount of payoff guaranteed by a maxmin strategy.

$$\operatorname{argmax}_{s_i} \min_{s_{-i}} u_i(s_i, s_{-i}) \rightarrow \text{maxmin strategy}$$

$$\text{maxmin value} = \max_{s_i} \min_{s_{-i}} u_i(s_i, s_{-i}) \rightarrow \text{maxmin value}$$

could play ⇒ conservative agent maximizing worst-case payoff  
paranoid agent believing everyone is out to get him.

### Minmax Strategy → road-rage of GT

Player i's minmax strategy against player -i in a 2-player game is a strategy that minimizes -i's best-case payoff and the minmax value for i against -i is payoff.

$$\begin{aligned} \operatorname{argmin}_{s_i} \max_{s_{-i}} u_i(s_i, s_{-i}) &\rightarrow \text{strategy minmax} \\ \min_{s_i} \max_{s_{-i}} u_i(s_i, s_{-i}) &\rightarrow \text{value of minmax} \end{aligned}$$

could play ⇒ burn the world, revenge, threat

Theorem (Von Neumann) ⇒ In any finite, two-player zero-sum game, in any Nash equilibrium each player receives a payoff that equals both his minmax value and maxmin value.

- Each player's maxmin value is equal to his minmax value. By convention, the maxmin player for player 1 is called the value of the game.
- Any maxmin strategy profile, minmax strategy profile is a Nash equilibrium. All Nash equilibrium have same payoff vector.
- For both players, the set of maxmin strategies coincides with set of minmax strategies.

### Iterated Removal of Dominated Strategies

- No equilibrium can involve a strictly dominated strategy
- Remove it and end up with strategically equal game
- Allows us to remove another strategy that wasn't dominated before

→ strategy where no matter what the other player does, my choice is always better  
→ can do this over & over to termination

- Used as a pre-processing step before computing an equilibrium  
 $\rightarrow$  In traveler's dilemma, doing this can end up with just  $(180, 180) \rightarrow$  equilibrium

Rationalizability  $\rightarrow$  In 2-player games, those that survive iterated removal of dominated strategies

- Rather than ask what's irrational, ask what is a best response to some beliefs about opponent.

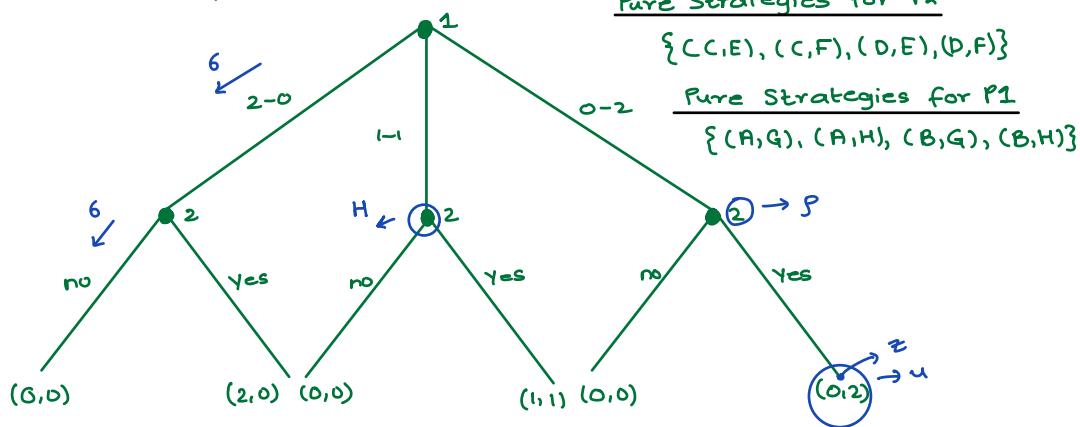
$\rightarrow$  Equilibrium strategies are always rationalizable

Furthermore, in two-player games, rationalizable  $\Leftrightarrow$  survives iterated removal of dominated strategies

### Extensive Form Games

Normal-form games do not have any notion of sequence: all actions happen simultaneously.

The extensive form is a game representation that explicitly includes temporal structure (tree representation).



Perfect-Information  $\rightarrow$  Every agent sees all actions of the other players (including any special "Chance" player)

A finite perfect-information game in extensive form is a tuple  $G = (N, A, H, Z, X, p, \sigma, u)$  where

$N \rightarrow$  set of  $n$  players

$A \rightarrow$  set of actions

$H \rightarrow$  set of non-terminal choice nodes

$Z \rightarrow$  set of terminal nodes (disjoint from  $H$ )

$X: H \rightarrow 2^A$  is the action func. (actions available for choice nodes)

$p: H \rightarrow N$  is the player function (turn of the player)

$\sigma: H \times A \rightarrow H \cup Z$  is the successor function (what to do next, deterministic)

$u: (U_1, \dots, U_n) \rightarrow$  profile of utility functions

$u_i: Z \rightarrow \mathbb{R}$  for each player  $i$

Pure Strategies (in perfect-information in EFG)

$\prod_{h \in H} X(h) \rightarrow$  cross-product of actions available to  $i$  at each of their choice nodes

associate an action with every choice-node, even those that will never be reached.

### Induced Normal Form

- Any pair of pure strategies uniquely identifies a terminal node, which identifies a utility for each agent.

	C,E	C,F	D,E	D,F
A,G	3,8	3,8	8,3	8,3
A,H	3,8	3,8	8,3	8,3
B,G	5,5	2,10	5,5	2,10
B,H	5,5	1,0	5,5	1,0

→ induced normal form

- Any perfect-information extensive-form game defines a corresponding induced-normal form game.

Mixed Strategy → prob. distribution over pure strategies

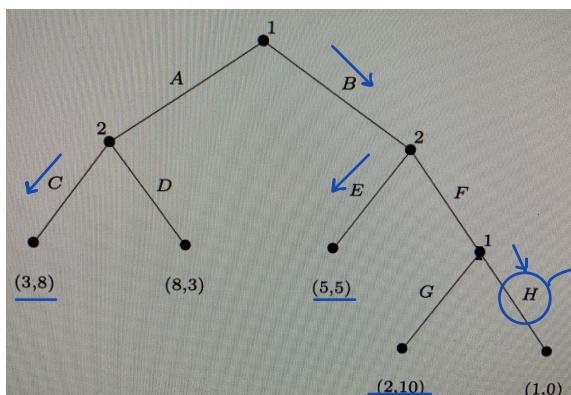
Best Response → set of mixed strategies that is utility maximizer against some other set of mixed strategies.

Theorem (Zermelo) → Every finite perfect-information

game in extensive form has atleast one Nash equilibrium (pure)

every normal form game has NE, and every extensive form game has induced normal game.

- Solve by backward induction
- Start from the bottom of the tree, no agent needs to randomize because deterministic best response.
  - Replace these nodes with resulting utility vector.
  - Repeat until an action is assigned for all choice nodes.



→ Any one implausible?

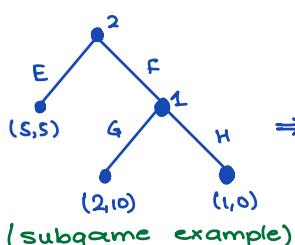
- (A,H), (C,F) } both are
- (B,H), (C,E) } implausible

### Subgame Perfection

- Some equilibria are better than others

Subgame → The subgame of G rooted at h is the restriction of G to the descendants of h.

The subgames of G are the subgames of G rooted at h for every choice node  $h \in H$ .



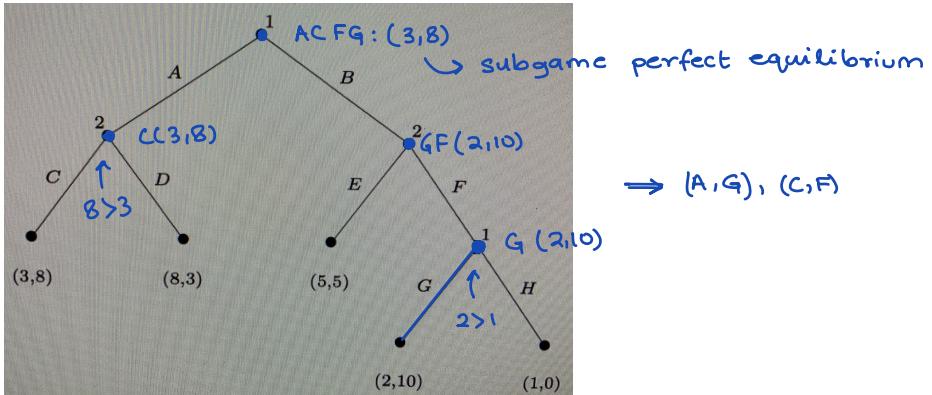
Strategy Profile  $s \in S$  is a subgame perfect equilibrium of G iff, for every subgame  $G'$  of G, the restriction of s to  $G'$  is a Nash eq. of  $G'$ .

→ So, any equilibrium computed by backward induction will be subgame perfect.

because it goes back up as the backward induction is done.

→ So every finite perfect-info game in EF has atleast one subgame perfect equilibrium.

Backward Induction → finds the subgame perfect equilibrium  
 ↳ replace subgames with their equilibrium values



### Imperfect Information Extensive Form Games

↳ model games with sequential actions some of which are hidden

$$g = (N, A, H, Z, \chi, p, S, u, I)$$

N → set of n players

A → set of actions

H → set of non-terminal choice nodes

Z → set of terminal nodes (disjoint from H)

$\chi: H \rightarrow 2^A$  is the action func (actions available for choice nodes)

$p: H \rightarrow N$  is the player function (turn of the player)

$s: H \times A \rightarrow H \cup Z$  is the successor function (what to do next, deterministic)

U:  $(u_1, \dots, u_n) \rightarrow$  profile of utility functions

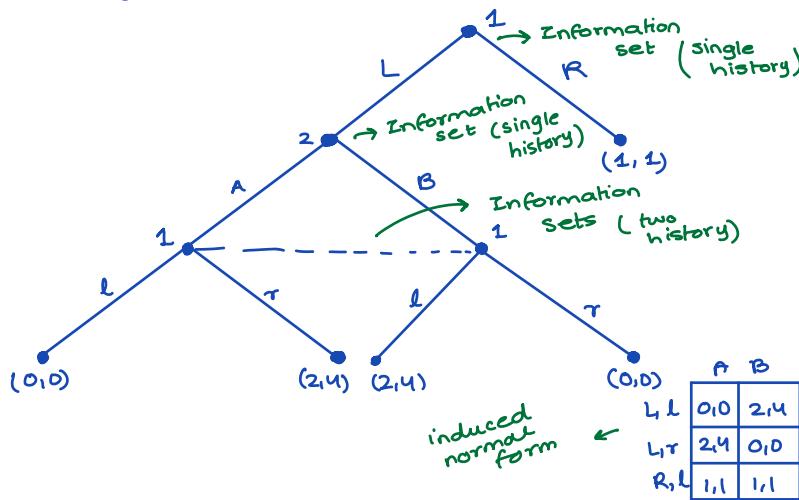
$u_i: Z \rightarrow \mathbb{R}$  for each player i

I:  $(I_1, \dots, I_n)$ , where for each  $i \in N$ ,  $I_i = (I_{i,1}, \dots, I_{i,k_i})$  is an equivalence relation on  $\{h \in H \mid p(h)=i\}$  and

For every  $h, h'$  such that  $h \in I$  and  $h' \in I$  for some  $i \in N$  and  $I \in I_i$ ,  
 $p(h)=p(h')=i$  and  $\chi(h)=\chi(h')$

The elements of partition are often called information sets.

Players cannot distinguish which history they are in within an information set.



### Pure Strategies

Need something to take information set / choice node and map it to the action.

$\prod_{I \in I} \chi(I)$  ↳ actions available at each information set

where  $\chi(I) = \chi(h)$  for any arbitrary  $h \in I$

### Mixed Strategies

Distribution over pure strategies.

### Best Response

A set of strategies maximizing utility given another set of strategy.

	A	B
L	0,0	2,4
R	2,4	0,0
L, R	1,1	1,1
R, L	1,1	1,1
R, R	1,1	1,1

Q- Can you represent an arbitrary perfect information EFG as an imperfect information EFG? Yes. Every single information set will contain one choice node.

→ Any normal form game can be represented as imperfect information EFG.

→ Players can play in any order

because none of the player that goes second can see what the first player did. ↴ one player plays first, second player doesn't get to see what first player did and it plays and game ends.

Q- What happens if we run this NFG → EFG translation on induced normal form game of any arbitrary EFG? → lose all the structure

both will have same Nash Eq. ↴ cannot recover EFG structure and nothing is gained.

Nash Eq.

	C	D
C	(-1, -1)	(-4, 0)
D	(0, -4)	(-3, -3)

G (NFG)

↳ cannot recover EFG structure

and nothing is gained.

↳ perfect-recall

Imperfect Information EFG

1

c

D

2

c

D

2

c

D

### Behavioural vs. Mixed Strategies

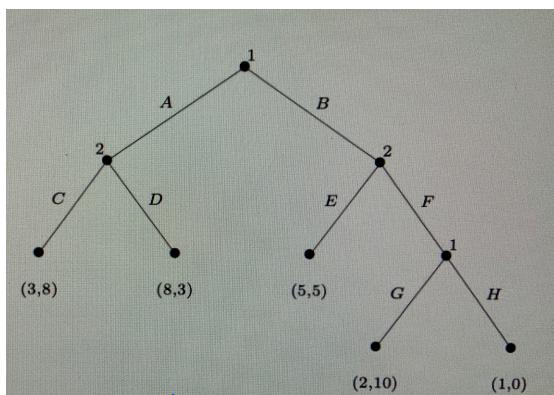
Mixed strategy  $s_i$  (in an imperfect-information EFG) is any distribution over an agent's pure strategies:

$$s_i \in \Delta(A^{I_i})$$

Behavioural strategy  $b_i$  is a mapping from an agent's information sets to a distribution over the actions at that information set, which is sample independently each time the agent arrives at the information set

$$b_i \in [\Delta(X(I))]_{I \in I_i}$$

↳ randomize at the information set



perfect-information

game has to be perfect-recall for this to be true

Behavioural Strategy:  $([.6:A, .4:B], [.6:G, .4:H])$

Mixed Strategy:  $[.6:(A,G), .4:(B,H)]$

Are they equivalent? No.

Two strategies are equivalent if they induce same dist. over outcomes for all strategies of other player.

These are not equivalent.

Person 2: chooses F then these two give different results.

Can you construct a mixed strategy that is equivalent to the behavioral strategy above? Yes.

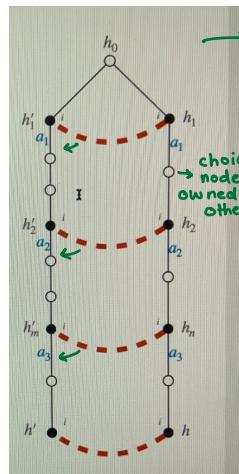
Can you construct a behavior strategy that is equivalent to the mixed strategy above? Yes.

### Perfect Recall

If for any pair of choice nodes in same information set and for any path  $h_0, a_0, h_1, a_1, \dots, h_m$  from the root of the game to  $h$  and for any path  $h'_0, a'_0, h'_1, a'_1, \dots, h'_m, h'$  from the root of the game to  $h'$ , it must be the case that:

- 1)  $n = m$  (same nodes)
- 2) for all  $0 \leq j \leq n$ , if  $p(h_j) = i$ , then exists an information set  $I' \in I$  such that  $h_j, h'_j \in I'$  (both are in same information set)
- 3) for all  $0 \leq j \leq n$ , if  $p(h_j) = i$ , then  $a_j = a'_j$  (same action)

→ A game of perfect-information is a game of perfect-recall.

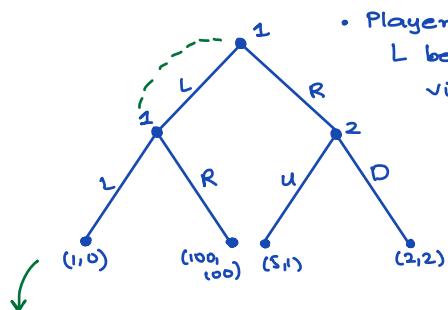


never forget where we are in the tree

IF I can't tell where I am in the tree then that is not because I forgot something.

⇒ G is a game of perfect recall if every player has perfect recall in G.

### Imperfect Recall



• Player 1 does not remember whether they played L before or not. In this case, because the visit same information set multiple times.

↳ don't have to always visit same information set for imperfect recall.

Mixed Strategy → has ability to do the same thing at a given info. set

Behavior Strategy → allows randomization at the given info. set.

### Mixed Strategy Equilibrium

→ convert to induced normal form  
and there is always a MSE.

Equilibrium in Behavioural Strategy → not guaranteed always

### Model as Imperfect Recall?

→ when actual agents may forget previous history  
• cases where the agent strategies are executed by proxies  
→ approximation technique

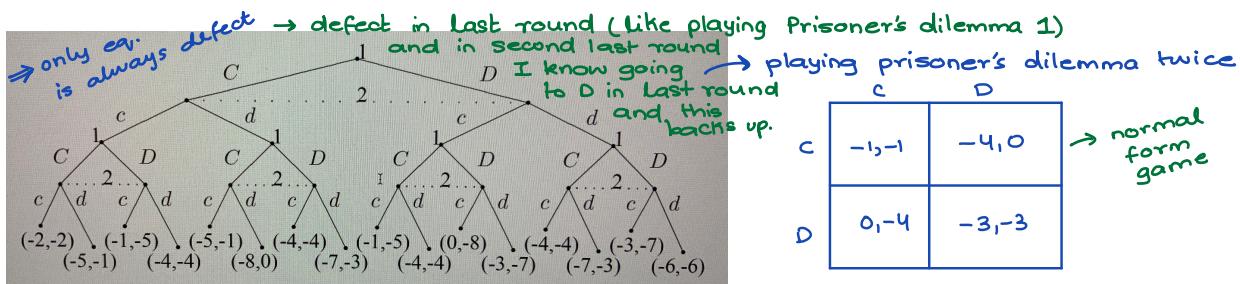
Theorem (Kuhn's) ⇒ If game has perfect recall, then any mixed strategy has equivalent behavior strategy and vice versa.

### Repeated Games

Play the same NFG over and over game  
- each round is called stage game

### Finitely Repeated Game

↳ written as extensive form game with imperfect information  
- at each round players don't know what other have done  
- overall payoff ftn is additive: sum of payoffs in stage games



- Repeating a Nash strategy in each stage game will be an equilibrium in behavioural strategies (called stationary strategy).
- When NFG has dominant strategy, then we can apply backward induction.

### Ininitely Repeated Game

↳ infinite deep game tree  
 can't have payoffs at the terminal node  
 can't add stage game payoffs (divergent series)

Given an infinite sequence of payoffs  $r_1, r_2, \dots$ , for player  $i$ , the average reward of  $i$  is

$$\lim_{k \rightarrow \infty} \frac{r_1 + r_2 + \dots + r_k}{k}$$

Discounted Reward  $\rightarrow 0 < \beta < 1$

$$\downarrow \quad \sum_{j=1}^{\infty} \beta^j r_j$$

- near term well-being is more important
- agent cares about future just as much as present, but with probability  $1-\beta$  the game will end.

### Folk Theorem

Consider any  $n$  player game  $G = (N, A, u)$  and payoff vector  $r = (r_1, r_2, \dots, r_n)$

↳ real number payoff to each player of game

$$v_i = \min_{S_i \in S_{-i}} \max_{S_i \in S_i} u_i(s_{-i}, s_i)$$

guaranteed for  $i$  ↳  $v_i$ 's minmax value: the amount of utility  $i$  can get when  $-i$  play minmax against  $i$ .

- payoff profile is feasible if it is a convex, rational combination of outcomes in  $G$ .

### Theorem (Folk's)

Consider any  $n$ -player game  $G$  and any payoff vector  $(r_1, r_2, \dots, r_n)$

- If  $r$  is the payoff in any Nash eq. of the infinitely repeated  $G$  with average rewards, then for each player  $i$ ,  $r_i$  is enforceable.
- If  $r$  is both feasible and enforceable, then  $r$  is the payoff in some Nash eq. of the infinitely repeated  $G$  with average rewards. ↳ there is a Nash eq. with an average payoff vector  $r$  iff.  $r$  is enforceable & feasible.
- ↳ feasible & enforceable  $\rightarrow$  Nash equilibrium  
 never accept anything lower than what others are forcing you to get when BR.

### Pure Strategy in Infinitely Repeated Game

→ choice of action at every decision point  
 action at every stage game  $\rightarrow$  infinite number of actions  
 ↳ set of pure strategies

### Nash Equilibria

- with infinite number of equilibria, can't construct an induced normal form
- Nash's theorem only applies to finite game
- Could be infinite number of pure-strategy equilibria
- Can characterize a set of payoffs that are achievable under equilibrium, without having to enumerate equilibria

- A payoff profile  $r$  is enforceable if  $r_i \geq v_i$  minmax value
- A payoff profile  $r$  is feasible if there exist rational, non-negative values  $\alpha_a$  such that for all  $i$ , we can express  $r_i$  as  $\sum_{a \in A} \alpha_a u_i(a)$  with  $\sum_{a \in A} \alpha_a = 1$

↳ expressed as weighted sum

## Bayesian Games (Def #1)

→ randomness about choice of game, other person's and/or own utility fn.

→ a set of games that differ only in their payoffs, a common prior defined over them, and a partition over the games for each agent.

$(N, G, P_i, I)$  where

$N \rightarrow$  set of agents

$G \rightarrow$  set of games with  $N$  agents each such that if  $g, g' \in G$  then for each

agent  $i \in N$  the strategy space in  $g$  is identical to strategy space in  $g'$

$P \in \Pi(G) \rightarrow$  common prior over games, where  $\Pi(G)$  is a set of all prob. dist.

over  $G \rightarrow$  update prior based on partition

$I = (I_1, \dots, I_N) \rightarrow$  set of partitions of  $G$ , one for each agent

		Player 2 cqu	
		only distinguishable b/w $I_{2,1}$ or $I_{2,2}$	
		$I_{2,1}$	
		L	MP R
	U/T	2, 0	0, 2
	D/B	0, 2	2, 0
		$p = 0.3 \rightarrow$ played 30%	
		L	PD R
	U/T	2, 2	0, 3
	D/B	3, 0	1, 1
		$p = 0.1 \rightarrow$ played 10%	
		$I_{1,2}$	
		Coord R	
	U/T	2, 2	0, 0
	D/B	0, 0	1, 1
		$p = 0.2 \rightarrow$ played 20%	
		BoS R	
	U/T	2, 1	0, 0
	D/B	0, 0	1, 2
		$p = 0.4 \rightarrow$ played 40%	

$a_1$	$a_2$	$\theta_1$	$\theta_2$	$u_1$	$u_2$
U	L	$\theta_{1,1}$	$\theta_{2,1}$	2	0
U	L	$\theta_{1,1}$	$\theta_{2,2}$	2	2
U	L	$\theta_{1,2}$	$\theta_{2,1}$	2	2
U	L	$\theta_{1,2}$	$\theta_{2,2}$	2	1
U	R	$\theta_{1,1}$	$\theta_{2,1}$	0	2
U	R	$\theta_{1,1}$	$\theta_{2,2}$	0	3
U	R	$\theta_{1,2}$	$\theta_{2,1}$	0	0
U	R	$\theta_{1,2}$	$\theta_{2,2}$	0	0
D	L	$\theta_{1,1}$	$\theta_{2,1}$	0	2

→ repeat for others similarly

### Pure Strategies

→ a mapping from every type agent observes it has to the action it would play if he had that type

$s_i: \Theta_i \rightarrow A_i \rightarrow$  action for each type

### Mixed Strategy

→ a mapping from all the types agent might have to distribution over action choices

$s_i: \Theta_i \rightarrow \Pi(A_i) \rightarrow$  prob. dist. over the actions

→ Players can see their info set before making a move

→ Game is sampled based on a common prior

→ Information is updated when game is selected

→ Two players get to see different things

### Epistemic Types (Def #2)

↳ directly represent uncertainty over utility function using the notion of epistemic type

Bayesian Game =  $(N, A, \Theta, p, u)$

$N \rightarrow$  set of agents

$A \rightarrow (A_1, \dots, A_N)$ ,  $A_i$  set of actions for  $i$

$\Theta \rightarrow (\Theta_1, \dots, \Theta_N)$ , where  $\Theta_i$  type space of  $i$

$p \rightarrow \Theta: [0, 1]$  common prior over types

$u \rightarrow (u_1, \dots, u_N)$ ,  $u_i: A \times \Theta \rightarrow \mathbb{R}$  utility fn for  $i$

so agents have posterior abt other types

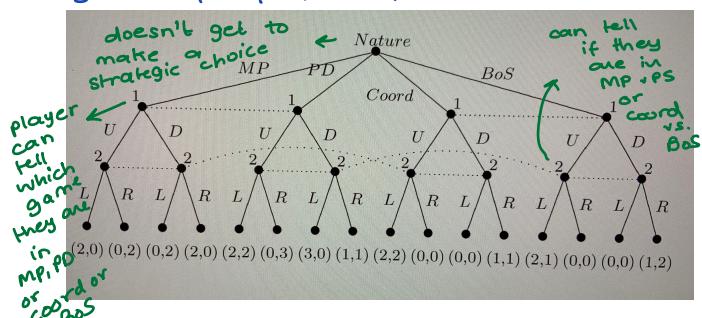
↳ depends on action taken and type of everyone

• Agent can't see type of others, can only see own type

### Extensive Form with Chance Moves

• Add agent 'Nature' following a known mixed strategy based on prior

• Reduce Bayesian games to extensive form games of imperfect information



$s_j(A_j | \theta_j)$

→ the prob. under mixed strategy  $s_j$  that agent  $j$  plays  $a_j$  given they observed type  $\theta_j$

### Ex-interim

→ an agent knows type of own but not of others  
belief about other agent's types

$$EU_i(s|\theta_i) = \sum_{\theta_{-i} \in \Theta_{-i}} p(\theta_{-i}|\theta_i) \sum_{a \in A} \left( \prod_{j \in N} s_j(a_j|\theta_j) \right) u_i(a, \theta_{-i}, \theta_i)$$

→ agent's strategies given by mixed strategies  
 $u_i$  weighted by

prob. that  $a$  would be realized given all player's mixed strategies and types

prob. that other players type would be  $\theta_{-i}$  given own type  $\theta_i$

$i$  must consider every  $\theta_{-i}$  and every  $a$  in order to evaluate  $u_i(a, \theta_i, \theta_{-i})$

### Ex-Ante

→ the agent knows nothing about anyone's type

$$EU_i(s) = \sum_{\theta \in \Theta} p(\theta) EU_i(s|\theta)$$

$$EU_i(s) = \sum_{\theta \in \Theta} p(\theta) \sum_{a \in A} \left( \prod_{j \in N} s_j(a_j|\theta_j) \right) u_i(a, \theta)$$

common prior      ↴ sum and weight by the prior over types      utility weighted by all prob. under mixed strategies

### Ex-post

→ an agent knows all agent's type

$$EU_i(s, \theta) = \sum_{a \in A} \left( \prod_{j \in N} s_j(a_j|\theta_j) \right) u_i(a, \theta)$$

• The only uncertainty here concerns the other agent's mixed strategies

### Best Response

→ best response to mixed profile  $s_{-i}$  of  $i$

$$BR_i(s_{-i}) = \underset{s'_i \in S_i}{\operatorname{argmax}} EU_i(s'_i, s_{-i})$$

↳ based on  $i$ 's ex-ante expected utility

• Equivalent to performing

indep. max of  $i$ 's ex-interim expected utility conditioned on actions of other players each type that  $i$  could have.

→ expectation over types of other players

Nash Equilibrium → maximize each type's expected utility

A Bayes-Nash eq. is a mixed strategy profile  $s$  that satisfies

↳ map from types to actions

based on Ex-interim ↳  $\forall i \quad s_i \in BR_i(s_{-i}) \rightarrow$  every body is best responding to each other

- Can also construct an induced normal form for Bayesian games
- the numbers in the cell correspond to ex-ante expected utilities

↳ as long as the strategy space is unchanged, best responses don't change b/w ex-ante & ex-interim cases

### Ex-post Equilibrium

$$\left( \forall \theta, \forall i \quad s_i \in \underset{s'_i \in S_i}{\operatorname{argmax}} EU_i(s'_i, s_{-i}, \theta) \right)$$

↳ agents do not have accurate beliefs about type distribution

\* can construct induced normal form for Bayesian game

→ the numbers in cell correspond to ex-ante expected utilities

→ As long as the strategy space is unchanged, best responses don't change b/w ex-ante & ex-interim cases.

## Analyzing Bayesian Games

### A Sheriff's Dilemma

- ↳ a sheriff faces an armed suspect and they each must (simultaneously) decide whether to shoot the other or not.
- suspect is either a criminal with prob.  $p$  or not with  $(1-p)$
- sheriff would rather shoot if suspect shoots, but does not if the suspect does not
- the criminal would shoot regardless of sheriff, as the suspect would be caught if does not shoot
- an innocent suspect would not shoot regardless of sheriff

		Sheriff ↓	
		Good	Shoot
Suspect	→ $\theta_I$ $1-p$	Shoot	-3, -1 ↓ -2 > -3 0 > -1
		Not	-2, -1 0, 0 ✓

strictly dominant strategy for the good player  
 If the player sees that he is innocent, he would not shoot since that is his dominant strategy.  
 So can remove shoot row for the suspect

		Sheriff ↓	
		Good	Shoot
Suspect	→ $\theta_I$ $1-p$	Good	-2, -1
		Not	0, 0

-1(1-p) → if sheriff shoots

{ Bayesian Equilibria in Innocent Type  
 ↳ always not shoot

{ Bayesian Equilibria in Guilty Type  
 ↳ always shoot

for the suspect

- Strategic uncertainty about what other player is going to do
- pay-off uncertainty about what the best response will be (value to their actions)

### Compute Mixed Strategy Nash Equilibrium

		P2	
		RL	RR
P1	Up	* 2.4, 3.2 * 0, 2.4	0, 2.4
	Down	1.8, 0.4	* 1, 1.2

P2 play RL with  $p(RL)$  and RR with  $1-p(RL)$

$$u_1(\text{Up}) = u_1(\text{Down})$$

$$2.4 \times p(RL) + 0 \times (1-p(RL)) = 1.8 p(RL) + 1 \times (1-p(RL))$$

$$\Rightarrow p(RL) = 0.625$$

\* Best responses

P1 play Up with  $p(\text{Up})$  and down with  $(1-p(\text{Up}))$

$$u_2(\text{RL}) = u_2(\text{RR})$$

$$3.2 \times p + 0.4 \times (1-p) = 2.4p + 1.2(1-p)$$

$$3.2p + 0.4 - 0.4p = 2.4p + 1.2 - 1.2p$$

$$2.8p + 0.4 = 1.2p + 1.2$$

$$2.8p - 1.2p = 1.2 - 0.4$$

$$1.6p = 0.8$$

$$\Rightarrow p(\text{Up}) = \frac{0.8}{1.6} = \frac{1}{2}$$

	P2↓		
	L	R	
P1 →	T	80, 40      *	40, 80      *
	B	40, 80      *	80, 40      *

P1 play T with  $p(T)$  and B with  $p(1-p(T))$

$$u_1(L) = u_1(R)$$

$$40 \times p(T) + 80 \times (1-p(T)) = 80 \times p(T) + 40 \times (1-p(T))$$

$$40p + 80 - 80p = 80p + 40 - 40p$$

$$-40p + 80 = 40p + 40$$

P2 play L with  $p(L)$  and R with  $1-p(L)$

$$u_2(L) = u_2(R)$$

$$80 \times p(L) + 40 \times (1-p(L)) = 40 \times p(L) + 80 \times (1-p(L))$$

$$80p + 40(1-p) = 40p + 80(1-p)$$

$$80p + 40 - 40p = 40p + 80 - 80p$$

$$40p + 40 = -40p + 80$$

$$40p + 40p = 80 - 40$$

$$280p = 40$$

$$p = \frac{1}{7}$$

So, the strategies are  $(\frac{1}{7}, \frac{1}{7}), (\frac{1}{7}, \frac{1}{7})$

	P2↓		
	L	R	
P1 →	U/T	320, 40      *	40, 80      *
	D/B	40, 80      *	80, 40      *

P2 plays L with  $p(L)$  and R with  $1-p(L)$

$$p(T) = p(B)$$

$$320 \times p + 40 \times (1-p) = 40p + 80(1-p)$$

$$320p + 40 - 40p = 40p + 80 - 80p$$

$$(320 - 40)p + 40 = -40p + 80$$

$$280p + 40 = -40p + 80$$

$$280p + 40p = 80 - 40$$

$$320p = 40$$

$$p = \frac{40}{320} = \frac{1}{8} = 0.125$$

$$\text{Strategy: } (\frac{1}{2}, \frac{1}{2}), (\frac{1}{8}, \frac{7}{8})$$

	P2↓		
	L	R	
P1 →	U/T	44, 40      *	40, 80      *
	D/B	40, 80      *	80, 40      *

P2 plays L with  $p(L)$  and R with  $1-p(L)$

$$p(T) = p(B)$$

$$44p + 40 \times (1-p) = 40p + 80(1-p)$$

$$(44 - 40)p + 40 = -40p + 80$$

$$4p + 40p = 80 - 40$$

$$44p = 40$$

$$p = \frac{40}{44}$$

$$p = 0.9$$

$$\text{Strategy: } (\frac{1}{2}, \frac{1}{2}), (0.9, 0.1)$$

- A strategy profile consisting of dominant strategy for every player must be a Nash equilibrium.
- An equilibrium in strictly dominant strategies must be unique.

### Pareto Optimality

↳ from the outside observer

outcome  $\sigma$  is at least as good as  $\sigma'$  for everyone and some agent prefers  $\sigma > \sigma'$  (strictly)

→ so reasonable to say that  $\sigma$  is better than  $\sigma'$

→ we say that  $\sigma$  Pareto-dominates  $\sigma'$ .

⇒ An outcome  $\sigma$  is Pareto-optimal if there is no other outcome that Pareto-dominates it.

• A game can have more than one Pareto-optimal outcomes

• Every game must have at least one Pareto-optimal outcome.

• Every outcome in 0 sum game is Pareto-optimal.

## Mechanism Design

### Bayesian Game Setting

→ a Bayesian game with no actions, but instead with a set of outcomes over which agents have utilities that depend on their types.

( $N, O, \Theta, p, u$ ) where utility set  $u_i: O \times \Theta \rightarrow \mathbb{R}$   
 $\downarrow$  common prior (prob. dist over  $\Theta$ )  
 $\downarrow$   $\Theta = \Theta_1 \times \dots \times \Theta_n$   
finite set of outcomes  
set of possible joint type vectors  
n agents

Mechanism → creates a set of actions and this can be sequential, simultaneous, anything.

A mechanism (for Bayesian game  $(N, O, \Theta, p, u)$ ) is a pair  $(A, M)$  where

•  $A = A_1 \times \dots \times A_n$  where  $A_i$  set of actions available to agent  $i \in N$

•  $M: A \rightarrow \Pi(O)$  maps each action profile to a distribution over outcomes

The designer gets to specify,

- the action sets for agent (though these maybe constrained by the env)
- the mapping to outcomes, over which agents have utility
- can't change outcomes; agent's preferences or type spaces

The problem is to pick a mechanism that will cause rational agents to behave in a desired way, specifically maximizing the mechanism designer's own utility or objective func.

- each agent holds private information, in Bayesian game sense
- often we are interested in settings where agent's action space is identical to their type space, and an action can be interpreted as a declaration of the agent's type.
- perform an optimization problem, given that the values of (some of) the inputs are unknown
- choose the Bayesian game out of a set of possible Bayesian games that maximizes some performance measure.

### Auctions

↳ mechanisms for allocating resources among self-interested agents  
→ outcome consists of allocating scarce resources

English Auction → auctioneer starts the bidding at some "reservation price"  
bidders shout ascending price  
highest bidder gets the good

First price Auction → bidders write down bids on pieces of paper  
(silent auction) auctioneer awards the good to bidder with highest bid  
the bidder pays amount of bid

Second price Auction → same like first pay auction  
winner pays second-highest bid amount

All-pay Auction → everybody pays, winner gets awarded

### Representation as Bayesian Game

Agents: bidders

Actions: bid amounts  $b_i$

Types: valuations  $v_i$

Common prior: agent's types drawn independently (IPV)

Risk attitude: risk-neutral, money & utility relationship is linear

Allocations & payments: determined based on vector of bid amounts  $b$

Utility fn: If  $i$  is awarded  $v_i - p_i$  else 0

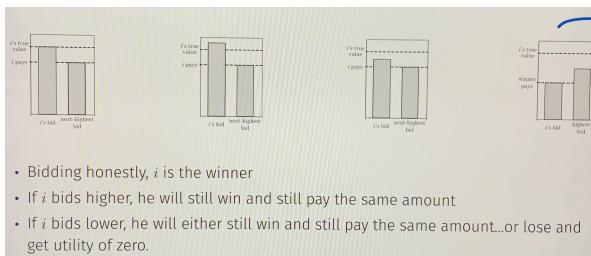
↳ (IPV, risk-neutral case)

outcomes: allocations & payments

↳ cartesian prod. of who gets what & what everyone pays

### Second Price Analysis

Theorem  $\Rightarrow$  Bidding one's value (truthful value) is a dominant strategy in a second price auction



case #1: bidding honestly,  $i$  is the winner

case #2: bidding honestly,  $i$  is the loser

( $\hookrightarrow$  can be thought of conversely as of the first.)

### First Price Analysis $\rightarrow$ both bidders half their value

Theorem  $\Rightarrow$  In FPA, with two bidders (risk neutral) whose valuations are drawn independently and uniformly drawn at random from  $[0, 1]$ ,  $(\frac{1}{2}v_1, \frac{1}{2}v_2)$  is a Bayes-Nash equilibrium strategy profile.

If  $n$  risk-neutral agents then unique symmetric equilibrium is given by strategy profile  $(\frac{n-1}{n}v_1, \dots, \frac{n-1}{n}v_n)$

### Revenue Equivalence

The  $k^{\text{th}}$  order statistic of a distribution: the expected value of the  $k^{\text{th}}$ -largest of  $n$  draws.

For  $n$  IID draws from  $[0, v_{\max}]$ , the  $k^{\text{th}}$ -order statistic  
 $\downarrow$   
from uniform dist.  $\frac{n+1-k}{n+1} v_{\max}$

In second price auction, the seller's expected revenue is

$$\frac{n-1}{n+1} v_{\max}$$

- every symmetric game has symmetric equilibrium
- in a symmetric equilibrium of this auction game, higher bid  $\Leftrightarrow$  higher valuation

conditions of revenue equivalence theorem

### Computing Bayes Nash Equilibria

		L	MP	R
		U	2, 0	0, 2
		D	0, 2	2, 0

$(p=0.3)$

		L	PD	$(p=0.1)$
		U	2, 2	0, 3
		D	3, 0	1, 1

$(p=0.2)$

		L	Coord	R
		U	2, 2	0, 0
		D	0, 0	1, 1

$(p=0.2)$

		LL	LR	RL	RR	
		UU	2, 1	1, 0.7	1, 1.2	0, 0.9
		UO	0.8, 0.2	1, 1.1	0.4, 1	0.6, 1.9
		OU	1.8, 1.4	0.5, 1.1	1.7, 0.4	0.7, 0.1
		OO	0.3, 0.6	0.5, 1.5	1.1, 0.2	1.3, 1.1

$$\left. \begin{aligned} EU_1(UU, LL) &= 2(0.3) + 2(0.1) + 2(0.2) + 2(0.4) \\ &= 2 \\ EU_2(UU, LL) &= 0(0.3) + 2(0.1) + 2(0.2) + 1(0.4) \\ &= 1 \end{aligned} \right\} \text{Ex ante expected utility}$$

## Behavioral Game Theory

Best response: Max utility action is always played

Quantal response: High-utility action is played often than Low-utility actions

$$QBR(s_{-i}; \lambda)(a_i) = \frac{\exp(\lambda u_i(a_i; s_{-i}))}{\sum_{a'_i \in A} \exp(\lambda u_i(a'_i; s_{-i}))} \quad (\text{softmax})$$

Quantal Response Eq. → can have various eq. in utility

→ A strategy profile  $s$  is a quantal response eq. (QRE) with precisions  $\lambda$  if every agent is simultaneously quantally responding to the profile of the other agents' strategies i.e.

- not clear how agents would arrive at a QRE
- NFG have always a QRE per  $\lambda$

$$V_i \in N : S_i = QBR_i(s_{-i}; \lambda)$$

- can relax rational expectations

Note that agents still have rational expectations: they are responding to the real strategies of the other agents

- can relax rational expectations

## Level-k Model

Every agent performs some finite number of steps of strategic reasoning:

level-0: some default, won-strategic distribution of play (often uniform)

level-1: Best response to level-0 players

level-2: Best response to level-1 or to level-1 and level-0 players

⋮

level-k: Best response to level k-1 or to levels  $\{0, 1, \dots, k-1\}$

A strategy profile  $s$  is the prediction of a level-k model with parameters  $\alpha_0, \alpha_1, \dots, \alpha_k$  and level-0 strategies  $\Pi_{i,0}^{lk}$  if

$$V_i \in N : S_i = \sum_{k=0}^K \alpha_k \Pi_{i,0}^{lk}$$

$$\text{where } \Pi_{i,0}^{lk} = BR_i(\Pi_{-i,k-1}^{lk})$$

$\Pi_{i,0}^{lk} \rightarrow$  level-0 strategy for each agent

- Every agent has fixed level representing the number of steps of strategic reasoning they can perform
  - they assume that every agent performs exactly one step fewer

## Cognitive Hierarchy

- Agents know that other agents might perform any number of steps of reasoning (less than their own)

- Agents know the relative proportions of all lower levels

A strategy profile  $s$  is the prediction of a cog. hier. model with parameters  $\alpha_0, \alpha_1, \dots, \alpha_k$  and level-0 strategies  $\Pi_{i,0}^{ch}$  if

$$S_i = \sum_{k=0}^K \alpha_k \Pi_{i,k}^{ch}$$

$$\Pi_{i,k}^{ch} = BR_i(\Pi_{-i,k-1}^{ch}) \text{ for all } k > 0$$

$$\Pi_{i,0:k}^{ch} = \frac{\sum_{j=0}^K \alpha_j \Pi_{i,j}^{ch}}{\sum_{j=0}^K \alpha_j}$$

- All of these (level-k/CH/QCH) assume that  $\Pi_{i,0}$  is the uniform assumption.
- Or hand-pick default strategy
- Implausible? don't scale
- Ideally, want level-0 strategy to learn from data.

## Quantal Cognitive Hierarchy

A strategy profile  $s$  is the prediction of a cog. hier. model with parameters  $\alpha_0, \alpha_1, \dots, \alpha_k$  and level-0 strategies  $\Pi_{i,0}^{qch}$  if

$$S_i = \sum_{k=0}^K \alpha_k \Pi_{i,k}^{qch}, \quad \Pi_{i,k}^{qch} = QBR_i(\Pi_{-i,k-1}^{ch}; \lambda), \quad \Pi_{i,0:k}^{qch} = \frac{\sum_{j=0}^K \alpha_j \Pi_{i,j}^{qch}}{\sum_{j=0}^K \alpha_j}$$

Level-0 Model: Linear<sup>4</sup> → stick it on top of other iterative model (QCH e.g.)  
 $\pi_{i,0}$  is a linear<sup>4</sup> level-0 model with parameter  $\vec{w}$  if  
 $\pi_{i,0}(a_i) \propto \sum_{f \in F} w_f f(a_i)$  different weights for every feature

where  $F = \{f^{\text{maxmax}}, f^{\text{maxmin}}, f^{\text{eff}}, f^{\text{fair}}, f^{\text{unif}}\}$  and

$$f^{\text{maxmax}}(a_i) = 1 \text{ iff } a_i \in \operatorname{argmax}_{a'_i \in A} \max_{a'_{-i} \in A_{-i}} u_i(a') \rightarrow \text{max. of my and others actions}$$

$$f^{\text{maxmin}}(a_i) = 1 \text{ iff } a_i \in \operatorname{argmax}_{a'_i \in A} \min_{a'_{-i} \in A_{-i}} u_i(a') \rightarrow \text{max of my own? min of others}$$

$$f^{\text{eff}}(a_i) = 1 \text{ iff } a_i \in \operatorname{argmax}_{a'_i \in A} \max_{a'_{-i} \in A_{-i}} \sum_{j \in N} u_j(a')$$

$$f^{\text{fair}}(a_i) = 1 \text{ iff } a_i \in \operatorname{argmin}_{a'_i \in A} \min_{a'_{-i} \in A_{-i}} \max_{j, j' \in N} |u_j(a') - u_{j'}(a')|$$

$$f^{\text{unif}}(a_i) = 1 \text{ for all } a_i$$