



Inversion of a perturbed matrix

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Abstract

An efficient formula is developed for computing the inverse of a given matrix perturbed by any diverting matrix. The matrix entries may be scattering within this diverting matrix.

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1. Introduction

This work presents a simple and straightforward approach for computing the inverse of a given matrix altered by perturbation. The formulas developed by Sherman and Morrison and later expanded by Woodbury [1] are very useful. However, in their formulas, the perturbing entries must be confined within the product of UV^T , where U and V are column vectors (Sherman and Morrison) or any dimensionally compatible matrices (Woodbury), while in this work, the perturbing entries may be a single element, a row of elements, a column of elements, blocked elements, or even scattered elements without any restriction at all. Besides this, the formula is concise and exact. In addition, the technique may be used to convert any non-singular matrix into a singular matrix by replacing any one or several entries in the original matrix.

A typical example is provided to show the merit of the approach presented.

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2. Formulation

For a given $n \times n$ non-singular matrix A , its inverse matrix A^{-1} is first evaluated. If the original matrix A is perturbed by an $n \times n$ diverting matrix D , the inverse of this perturbed matrix $(A + D)$ may be found from [1–3]

$$(A + D)^{-1} = A^{-1} - A^{-1}(A^{-1} + D^{-1})^{-1}A^{-1}. \quad (1)$$

Obviously, (1) is not a feasible formula for computing $(A + D)^{-1}$, even though A^{-1} is already known beforehand. Besides, it requires D and $(A^{-1} + D^{-1})$ to be non-singular. For convenience, let

$$A^{-1} = B; \quad (2)$$

then

$$\begin{aligned} (A + D)^{-1} &= B - B(I + DB)^{-1}DB \\ &= B - BD(I + BD)^{-1}B. \end{aligned} \quad (3)$$

The non-singularity requirement for D in (1) is thus removed.

Let the matrices D and B be partitioned as

$$[D] = \begin{bmatrix} \overline{D} & 0 \\ 0 & 0 \end{bmatrix}, \quad [B] = \begin{bmatrix} \overline{B} & B_2 \\ B_1 & B_3 \end{bmatrix} \quad (4)$$

and

$$\overline{B} = \begin{bmatrix} \overline{B} \\ B_1 \end{bmatrix}, \quad \underline{B} = [\underline{B} \quad B_2] \quad (5)$$

where the element positions of \overline{B} are the transport element positions of \overline{D} . That is, the partitioned matrices \overline{D} , \overline{B} , \underline{B} , and \underline{B} are of order of $m_1 \times m_2$, $m_2 \times m_1$, $n \times m_2$, and $m_1 \times n$, respectively. Then (3) may be rewritten as

$$(A + D)^{-1} = B + H \quad (6)$$

where

$$\begin{aligned} H &= -\overline{B}(I + \overline{D}\overline{B})^{-1}\overline{D}\underline{B} \\ &= -\overline{B}\underline{D}(I + \underline{B}\underline{D})^{-1}\underline{B}. \end{aligned} \quad (7)$$

It is noted that $(I + \overline{D}\overline{B})^{-1}$ and $(I + \underline{B}\underline{D})^{-1}$ are respectively the inverses of square matrices of order m_1 and m_2 , where $m_1, m_2 \leq n$. In particular, if $m_1 = 1$ or $m_2 = 1$, one of these inversion factors becomes a scalar quantity (a single element matrix), that can be easily evaluated directly without going through the usual matrix inversion procedure.

It is noted also that \underline{D} does not necessary have to be a solid matrix located at the upper left corner of $[D]$. If non-zero elements are scattering within $[D]$, then \underline{D} is formed through the selected rows and columns that must cover these elements entirely. It follows that \overline{B} as well as \underline{B} and \underline{B} are determined from $[B]$ accordingly.

3. Example

Given a matrix A , and its inverse A^{-1} which has been computed:

$$[A] = \begin{bmatrix} 1.5 & -0.5 & -1.5 & 2.0 & -3.0 \\ -3.0 & 1.5 & 2.0 & -3.0 & 4.0 \\ -1.0 & 0.5 & 1.0 & -1.0 & 1.0 \\ 2.0 & -0.5 & -1.0 & 2.0 & -2.0 \\ -1.0 & 0.5 & 0.0 & -0.5 & 0.5 \end{bmatrix}$$

$$[A]^{-1} = [B] = \begin{bmatrix} 4.0 & 4.0 & -3.0 & -1.0 & -6.0 \\ 4.0 & 4.0 & -2.0 & 0.0 & -4.0 \\ -2.0 & -2.0 & 3.0 & 1.0 & 2.0 \\ -6.0 & -5.0 & 4.0 & 3.0 & 8.0 \\ -2.0 & -1.0 & 0.0 & 1.0 & 2.0 \end{bmatrix},$$

we want to find the inverse matrix $[A + D]^{-1}$, where A is perturbed by various matrices D .

(1) For

$$[D] = \begin{bmatrix} 0. & 0. & 0. & 0. & 0. \\ 0. & -1.5 & 0. & -1.0 & +2.0 \\ 0. & 0. & 0. & 0. & 0. \\ 0. & +1.5 & 0. & 0. & +1.0 \\ 0. & 0. & 0. & 0. & 0. \end{bmatrix},$$

we have

$$\underline{\overline{D}} = \begin{bmatrix} -1.5 & -1.0 & +2.0 \\ +1.5 & 0.0 & +1.0 \end{bmatrix}, \quad \underline{\overline{B}} = \begin{bmatrix} 4.0 & 0.0 \\ -5.0 & 3.0 \\ -1.0 & 1.0 \end{bmatrix},$$

$$\overline{B} = \begin{bmatrix} 4.0 & -1.0 \\ 4.0 & 0.0 \\ -2.0 & 1.0 \\ -5.0 & 3.0 \\ -1.0 & 1.0 \end{bmatrix}, \quad \underline{B} = \begin{bmatrix} 4.0 & 4.0 & -2.0 & 0.0 & -4.0 \\ -6.0 & -5.0 & 4.0 & 3.0 & 8.0 \\ -2.0 & -1.0 & 0.0 & 1.0 & 2.0 \end{bmatrix}$$

and

$$[I + \underline{\overline{D}} \underline{\overline{B}}]^{-1} = \begin{bmatrix} -2.0 & -1.0 \\ 5.0 & 2.0 \end{bmatrix}^{-1} = \begin{bmatrix} 2.0 & 1.0 \\ -5.0 & -2.0 \end{bmatrix}$$

$$[I + \underline{\overline{B}} \underline{D}]^{-1} = \begin{bmatrix} -5.0 & -4.0 & 8.0 \\ 12.0 & 6.0 & -7.0 \\ 3.0 & 1.0 & 0.0 \end{bmatrix}^{-1} = \begin{bmatrix} 7.0 & 8.0 & -20.0 \\ -21.0 & -24.0 & 61.0 \\ -6.0 & -7.0 & 18.0 \end{bmatrix}.$$

Then

$$\begin{aligned}
[H] &= -\overline{B}(I + \overline{D}\overline{B})^{-1}\overline{D}B = -\overline{B}\overline{D}(I + \overline{B}\overline{D})^{-1}B \\
&= \begin{bmatrix} 28.0 & 9.0 & 31.0 & 7.0 & -2.0 \\ 16.0 & 4.0 & 20.0 & 4.0 & 0.0 \\ -20.0 & -7.0 & -21.0 & -5.0 & 2.0 \\ -56.0 & -20.0 & -58.0 & -14.0 & 6.0 \\ -16.0 & -6.0 & -16.0 & -4.0 & 2.0 \end{bmatrix}.
\end{aligned}$$

Thus

$$\begin{aligned}
[A + D]^{-1} &= [B + H] \\
&= \begin{bmatrix} 32.0 & 13.0 & 28.0 & 6.0 & -8.0 \\ 20.0 & 8.0 & 18.0 & 4.0 & -4.0 \\ -22.0 & -9.0 & -18.0 & -4.0 & 4.0 \\ -62.0 & -25.0 & -54.0 & -11.0 & 14.0 \\ -18.0 & -7.0 & -16.0 & -3.0 & 4.0 \end{bmatrix}.
\end{aligned}$$

(2) For

$$[D] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & +0.6 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

we then have

$$\begin{aligned}
[H] &= -\frac{(+0.6)}{[1 + (+0.6)(-5.0)]} \begin{bmatrix} 4.0 \\ 4.0 \\ -2.0 \\ -5.0 \\ -1.0 \end{bmatrix} [-6.0 \quad -5.0 \quad 4.0 \quad 3.0 \quad 8.0] \\
&= \begin{bmatrix} -7.2 & -6.0 & 4.8 & 3.6 & 9.6 \\ -7.2 & -6.0 & 4.8 & 3.6 & 9.6 \\ 3.6 & 3.0 & -2.4 & -1.8 & -4.8 \\ 9.0 & 7.5 & -6.0 & -4.5 & -12.0 \\ 1.8 & 1.5 & -1.2 & -0.9 & -2.4 \end{bmatrix}.
\end{aligned}$$

Thus

$$\begin{aligned}
[A + D]^{-1} &= [B + H] \\
&= \begin{bmatrix} -3.2 & -2.0 & 1.8 & 2.6 & 3.6 \\ -3.2 & -2.0 & 2.8 & 3.6 & 5.6 \\ 1.6 & 1.0 & 0.6 & -0.8 & -2.8 \\ 3.0 & 2.5 & -2.0 & -1.5 & -4.0 \\ -0.2 & 0.5 & -1.2 & 0.1 & -0.4 \end{bmatrix}.
\end{aligned}$$

It is noted that if the perturbing element +0.6 is replaced by +0.2, then the perturbed matrix $(A + D)$ becomes a singular matrix.

4. Conclusion

An efficient formula has been derived for finding the inverse of a given matrix altered by a perturbing matrix whose entries may be scattering around. If these scattering entries are specially formed into a single row or column, the result can be easily obtained by simple matrix multiplication without going through the usual matrix inversion procedure.

The application of the technique may be used to convert any non-singular matrix into a singular matrix. It may also lead to the formulation of a recurrence scheme for finding the inversion of any arbitrary square matrix of high order [4].

References

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