① $\begin{cases} \dfrac{\partial u}{\partial t} - \varepsilon \dfrac{\partial^2 u}{\partial x^2} + a \dfrac{\partial u}{\partial x} = 0, & x \in \mathbb{R} \\ u\big|_{t=0} = 1 - H(x) = \varphi(x) \end{cases}$

$$\frac{\partial u}{\partial t} = \varepsilon \frac{\partial^2 u}{\partial x^2} - a \frac{\partial u}{\partial x}$$

$$\exists \begin{cases} y = x - at \\ \tau = t \end{cases}$$

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial t}(u(y,\tau)) = \frac{\partial u}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial u}{\partial \tau}\cancelto{1}{\frac{\partial \tau}{\partial t}} = -a\frac{\partial u}{\partial y} + \frac{\partial u}{\partial \tau}$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial u}{\partial \tau}\cancelto{0}{\frac{\partial \tau}{\partial x}} = \frac{\partial u}{\partial y}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\frac{\partial u}{\partial y} = \frac{\partial^2 u}{\partial y^2}\cancelto{1}{\frac{\partial y}{\partial x}} + \frac{\partial^2 u}{\partial y \partial \tau}\cancelto{0}{\frac{\partial \tau}{\partial x}} = \frac{\partial^2 u}{\partial y^2}$$

было: $\begin{cases} \dfrac{\partial u}{\partial t} = \varepsilon \dfrac{\partial^2 u}{\partial x^2} - a \dfrac{\partial u}{\partial x}, & x \in \mathbb{R} \\ u\big|_{t=0} = \varphi(x) \end{cases}$

стало: $\begin{cases} -a\cancel{\dfrac{\partial u}{\partial y}} + \dfrac{\partial u}{\partial \tau} = \varepsilon \dfrac{\partial^2 u}{\partial y^2} - a\cancel{\dfrac{\partial u}{\partial y}}, & y \in \mathbb{R} \\ u\big|_{t=0} = \varphi(y) \end{cases}$ (т.к. $x = y$ при $t = 0$)

стало: $\begin{cases} \dfrac{\partial u}{\partial t} = \varepsilon \dfrac{\partial^2 u}{\partial y^2}, & y \in \mathbb{R} \\ u\big|_{t=0} = \varphi(y) \end{cases}$ – задача теплопроводности

Из курса урматов:

Если $\varepsilon > 0$

Задача теплопроводности

$\begin{cases} u_t = a^2 \Delta u + \cancel{f(x,t)} \to 0, \text{ т.к. } f(x,t) = 0 \\ u\big|_{t=0} = u^0(x) \end{cases}$

Решение: формула Пуассона:

$$u(x,t) = (4\pi a^2 t)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-\frac{(x-y)^2}{4a^2 t}} u^0(y)\,dy + \cancelto{0}{\int_0^t d\tau\, (4\pi a^2 \tau)^{-\frac{n}{2}} \int_{\mathbb{R}^n} dy\, e^{-\frac{(x-y)^2}{4a^2\tau}} f(y, t-\tau)} \text{ т.к. } f(x,t) = 0$$

$$u(y,t) = \frac{1}{2\sqrt{\pi\varepsilon\tau}} \int_{-\infty}^{+\infty} \varphi(\tilde{y}) e^{\frac{-(y-\tilde{y})^2}{4\varepsilon\tau}}\, d\tilde{y}$$

$$u(x,t) = \frac{1}{2\sqrt{\pi\varepsilon t}} \int_{-\infty}^{+\infty} \varphi(\tilde{y}) e^{\frac{-(x-at-\tilde{y})^2}{4\varepsilon t}}\, d\tilde{y}$$

если $\varepsilon = 0$

$\begin{cases} u'_t = 0 \\ u\big|_{t=0} = \varphi(y) \end{cases} \Rightarrow u(y,t) = u(y)$

$u = u(y) = u(y,t),\ \forall t$

$\Rightarrow u = u(y,0) = \varphi(y)$

$u = \varphi(x - at)$