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Polynomial reproduction by symmetric subdivision schemes

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Abstract

We first present necessary and sufficient conditions for a linear, binary, uniform, and stationary subdivision scheme to have polynomial reproduction of degree d and thus approximation order $d + 1$. Our conditions are partly algebraic and easy to check by considering the symbol of a subdivision scheme, but also relate to the parameterization of the scheme. After discussing some special properties that hold for symmetric schemes, we then use our conditions to derive the maximum degree of polynomial reproduction for two families of symmetric schemes, the family of pseudo-splines and a new family of dual pseudo-splines.

Keywords: Subdivision schemes; Polynomial reproduction; Polynomial generation; Approximation order; Quasi-interpolation

1. Introduction

This paper investigates certain aspects of subdivision schemes in the functional setting. We follow the notation of [1] and consider uniform and stationary subdivision schemes $S_{\boldsymbol{a}}$ that are determined by their *masks* $\boldsymbol{a} = (a_i)_{i\in\mathbb{Z}}$. Starting from some initial data $\boldsymbol{f}^0 = (f_i^0)_{i\in\mathbb{Z}}$ with

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$f_i^0 \in \mathbb{R}$ at level zero, such a scheme generates refined data $\boldsymbol{f}^{k+1} = (f_i^{k+1})_{i\in\mathbb{Z}}$ at subsequent levels $k+1$ for any $k \in \mathbb{N}_0$ according to the refinement equation

$$f_i^{k+1} = \sum_{j\in\mathbb{Z}} a_{i-2j} f_j^k, \quad i \in \mathbb{Z}. \tag{1}$$

The refinement rule (1) can be split into an *even* and an *odd* rule,

$$f_{2i}^{k+1} = \sum_{j\in\mathbb{Z}} a_{2(i-j)} f_j^k \quad \text{and} \quad f_{2i+1}^{k+1} = \sum_{j\in\mathbb{Z}} a_{2(i-j)+1} f_j^k, \tag{2}$$

to emphasize the fact that only the mask coefficients a_i with even indices are used to compute the new data with even indices, and that the new data with odd indices depends only on the mask coefficients a_i with odd indices. In this paper we consider only schemes with a finite number of non-zero coefficients in their masks.

It is also common to attach the data f_i^k to some parameter values t_i^k with $t_i^k < t_{i+1}^k$ such that $t_{i+1}^k - t_i^k = 2^{-k}$ for $i \in \mathbb{Z}$ and to define F^k to be the piecewise linear function that interpolates the data, namely

$$F^k(t_i^k) = f_i^k, \qquad F^k|_{[t_i^k, t_{i+1}^k]} \in \pi_1, \quad i \in \mathbb{Z}, k \in \mathbb{N}_0,$$

where π_d denotes the space of polynomials of degree d. If the sequence $(F^k)_{k\in\mathbb{N}_0}$ converges, then we denote its limit by

$$S_{\boldsymbol{a}}^\infty \boldsymbol{f}^0 = \lim_{k\to\infty} F^k$$

and say that $S_{\boldsymbol{a}}^\infty \boldsymbol{f}^0$ is the *limit function* of the subdivision scheme $S_{\boldsymbol{a}}$ for the data $\boldsymbol{f}^0$. If $S_{\boldsymbol{a}}^\infty \boldsymbol{f}^0$ exists for any $\boldsymbol{f}^0$, then $S_{\boldsymbol{a}}$ is termed *convergent*. We restrict most of our discussion to *non-singular* schemes for which $S_{\boldsymbol{a}}^\infty \boldsymbol{f}^0 \equiv 0$ if and only if $\boldsymbol{f}^0 \equiv 0$.

The main contribution of this paper is twofold. In Section 4 we first derive necessary and sufficient conditions for a subdivision scheme to have *polynomial reproduction* in the following sense.

Definition 1.1 (*Polynomial Reproduction*). A subdivision scheme $S_{\boldsymbol{a}}$ reproduces polynomials of degree d if it is convergent and if $S_{\boldsymbol{a}}^\infty \boldsymbol{f}^0 = p$ for any polynomial $p \in \pi_d$ and initial data $f_i^0 = p(t_i^0)$, $i \in \mathbb{Z}$.

For interpolatory subdivision schemes, these conditions simplify to those given by Dyn and Levin [2].

In Section 6 we then use these conditions to derive the maximum degree of polynomial reproduction for the members of two general families of subdivision schemes. One is the family of pseudo-splines (of type II) [3] that contains the schemes for uniform B-splines with odd degree and the $2n$-point interpolatory schemes of [4] as special cases. The other is a new family that we call *dual pseudo-splines*. It nicely complements the family of pseudo-splines and contains the even degree B-splines and the dual $2n$-point schemes [5] as special cases. While we require only simple algebraic considerations when dealing with polynomial reproduction in this paper, we plan to use Fourier analysis in a forthcoming paper to derive further properties of these subdivision schemes like smoothness and non-singularity.

Polynomial reproduction is a desirable property because any convergent subdivision scheme that reproduces polynomials of degree d has *approximation order* $d+1$. That is, if we take the values of any function $f \in \mathcal{C}^{d+1}$ with $\|f^{(d+1)}\|_\infty < \infty$ at uniform grids of width h, then the limit functions generated by the subdivision scheme from such initial data converge to f as $h \to 0$ and the rate of convergence is $O(h^{d+1})$ [6]. In fact, pseudo-splines (of type I) were first introduced by Daubechies et al. [7] to obtain tight framelet systems with a desirable approximation order.

A simple observation regarding polynomial reproduction is that any convergent scheme reproduces constant functions. In fact, it was shown by Cavaretta et al. [8] and Dyn [9] that if $S_{\boldsymbol{a}}$ is convergent then

$$\sum_{i\in\mathbb{Z}} a_{2i} = \sum_{i\in\mathbb{Z}} a_{2i+1} = 1. \tag{3}$$

Therefore, any initial constant data $\boldsymbol{f}^0 \equiv c$ is reproduced by the refinement rules (2) and hence $F^k \equiv c$ for all $k \in \mathbb{N}_0$. While the choice of parameter values t_i^k does not matter in this particular case, we shall see in Section 2 that it plays a crucial role for polynomial reproduction of higher degree.

In this paper, we restrict our discussion to primal and dual parameterizations (see Section 2) and the results of Section 4 allow us to conclude that for symmetric subdivision schemes the maximum degree of polynomial reproduction is achieved by using the primal parameterization in case of odd symmetry, whereas the dual parameterization has to be used if the symmetry is even (see Section 5). For non-symmetric schemes, although the results of Sections 3 and 4 hold, it is possible to achieve a higher degree of polynomial reproduction by other parameterizations. This will be investigated elsewhere.

2. Parameterization

As the choice of specific parameter values t_i^k affects neither the convergence of a subdivision scheme $S_{\boldsymbol{a}}$ nor the smoothness of its limit functions, most standard tools for analysing both properties [8,1] simply use the parameterization that we refer to as the *primal parameterization*.

Definition 2.1 (*Primal Parameterization*). The primal parameterization of a subdivision scheme is based on the parameter values

$$t_i^k = i/2^k, \quad i \in \mathbb{Z}, k \in \mathbb{N}_0, \tag{4}$$

so that $t_{2i}^{k+1} = t_i^k$ and $t_{2i+1}^{k+1} = (t_i^k + t_{i+1}^k)/2$. Accordingly, we can say that each subdivision step replaces the old data f_i^k by the new data f_{2i}^{k+1} with even indices and the new data f_{2i+1}^{k+1} with odd indices is added halfway between the old data f_i^k and f_{i+1}^k (see Fig. 1).

But in so far as the polynomial reproduction property of $S_{\boldsymbol{a}}$ is concerned, this parameterization does not always yield the highest degree possible. Motivated by the following example, we also consider the *dual parameterization* in this paper.

Definition 2.2 (*Dual Parameterization*). The dual parameterization of a subdivision scheme attaches the data f_i^k to the parameter values

$$t_i^k = \left(i - \frac{1}{2}\right) \Big/ 2^k, \quad i \in \mathbb{Z}, k \in \mathbb{N}_0, \tag{5}$$

with $t_{2i-1}^{k+1} = (t_{i-1}^k + 3t_i^k)/4$ and $t_{2i}^{k+1} = (3t_i^k + t_{i+1}^k)/4$. In this setting, each subdivision step replaces the old data f_i^k by the new data f_{2i-1}^{k+1} and f_{2i}^{k+1}, one to the left, the other to the right, and both at one quarter the distance to the neighbours f_{i-1}^k and f_{i+1}^k (see Fig. 2).

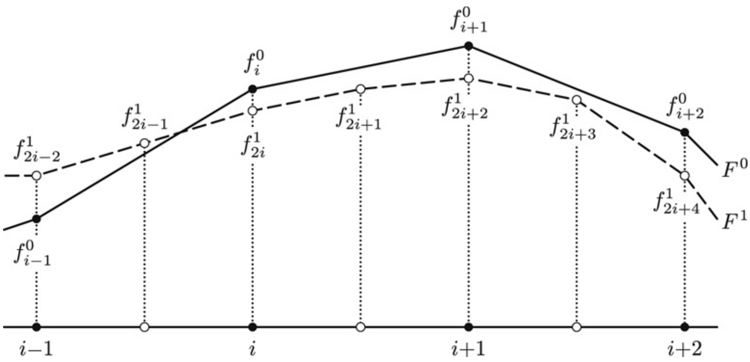

Fig. 1. Primal parameterization.

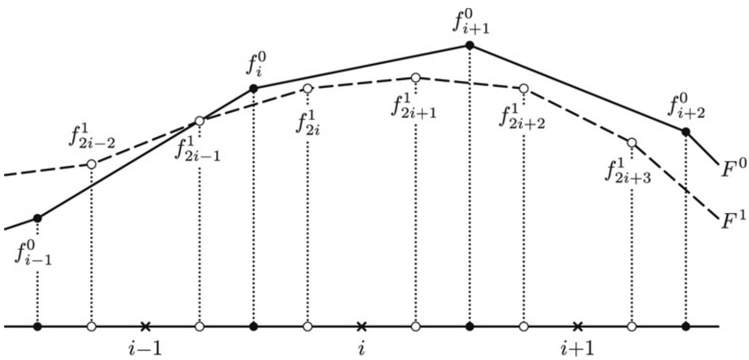

Fig. 2. Dual parameterization.

Note that the parameter values in (4) and (5) differ only by a shift of $1/2^{k+1}$ that vanishes as $k \to \infty$, so that the limit function $S_{\boldsymbol{a}}^{\infty}\boldsymbol{f}^0$ for any fixed initial data $\boldsymbol{f}^0$ is the same, no matter which of the two parameterizations is used. However, in the context of polynomial reproduction there still remains an important difference, because the initial data with respect to the primal parameterization is $f_i^0 = p(i)$, whereas $f_i^0 = p(i - 1/2)$ is used in the case of the dual parameterization.

For example, let us consider the uniform linear B-spline scheme with mask $[a_{-1}, a_0, a_1] = [\frac{1}{2}, 1, \frac{1}{2}]$ and assume the initial data to be sampled from the linear polynomial $p(x) = x$, that is, $f_i^0 = t_i^0$. If the primal parameter values in (4) are used, then it is easy to see that $(S_{\boldsymbol{a}}^{\infty}\boldsymbol{f}^0)(x) = x$, whereas the dual parameter values in (5) give the limit function $x - 1/2$. On the other hand, the limit function of the uniform quadratic B-spline scheme with mask $[a_{-2}, a_{-1}, a_0, a_1] = [\frac{1}{4}, \frac{3}{4}, \frac{3}{4}, \frac{1}{4}]$ is $x + 1/2$ for the primal and x for the dual parameterization. Both examples are special cases of schemes with a symmetric mask, and we shall come back to such schemes in Section 5.

3. Polynomial generation

An obvious necessary condition for a subdivision scheme $S_{\boldsymbol{a}}$ to reproduce polynomials of degree d is that it must be able to *generate* polynomials of the same degree as limit functions for some initial data. For the kind of subdivision schemes that we consider, this property is equivalent to a simple condition on the mask $\boldsymbol{a}$ that can best be stated by using the algebraic formalism of *z-transforms*.

Definition 3.1 (*z-Transform*). For any sequence $\boldsymbol{c} = (c_i)_{i\in\mathbb{Z}}$ we denote by

$$c(z) = \sum_{i\in\mathbb{Z}} c_i z^i$$

its z-transform and the even and odd components of the z-transform by

$$c_{\mathrm{e}}(z) = \sum_{i\in\mathbb{Z}} c_{2i} z^{2i} \quad \text{and} \quad c_{\mathrm{o}}(z) = \sum_{i\in\mathbb{Z}} c_{2i+1} z^{2i+1}.$$

Obviously,

$$\begin{aligned} c(z) &= c_{\mathrm{e}}(z) + c_{\mathrm{o}}(z), \qquad & c_{\mathrm{e}}(z) &= (c(z) + c(-z))/2, \\ c(-z) &= c_{\mathrm{e}}(z) - c_{\mathrm{o}}(z), \qquad & c_{\mathrm{o}}(z) &= (c(z) - c(-z))/2. \end{aligned} \tag{6}$$

Moreover, we can now write the refinement rule (1) as

$$f^{k+1}(z) = a(z) f^k(z^2) \tag{7}$$

and the even and odd rules (2) as

$$f_{\mathrm{e}}^{k+1}(z) = a_{\mathrm{e}}(z) f^k(z^2) \quad \text{and} \quad f_{\mathrm{o}}^{k+1}(z) = a_{\mathrm{o}}(z) f^k(z^2).$$

Note that the z-transform $a(z)$ of the mask $\boldsymbol{a}$ is usually called the *symbol* of the scheme $S_{\boldsymbol{a}}$ and that $a(z)$ is a Laurent polynomial, as we consider only schemes with masks consisting of a finite number of non-zero coefficients.

Theorem 3.2 (*Polynomial Generation*). *For a non-singular subdivision scheme $S_{\boldsymbol{a}}$ the condition*

(PG) *$a(z)$ is divisible by $(1+z)^{d_{\mathrm{G}}+1}$*

is equivalent to the property that for any polynomial p of degree $d \le d_{\mathrm{G}}$ there exists some initial data $\boldsymbol{f}^0$ such that $S_{\boldsymbol{a}}^\infty \boldsymbol{f}^0 = p$. Moreover, $\boldsymbol{f}^0$ is sampled from a polynomial of the same degree and with the same leading coefficient. In other words, there exists some $q \in \pi_d$ such that $f_i^0 = q(t_i^0)$ for $i \in \mathbb{Z}$ and $p - q \in \pi_{d-1}$.

This theorem is proved in a more general setting by Cavaretta et al. [8, Chapter 6].

Remark 3.3. The non-singularity of the scheme $S_{\boldsymbol{a}}$ is actually not required for the sufficiency of condition **(PG)** for polynomial generation, but needed only to show its necessity (see also [1] and [10, Theorem 3.7]).

Levin [6] showed that any subdivision scheme $S_{\boldsymbol{a}}$ that generates polynomials of degree d can also reproduce polynomials of same degree if the initial data is pre-processed by a suitable linear operator Q, so that the combination of $S_{\boldsymbol{a}}^\infty$ and Q gives a quasi-interpolation operator with optimal approximation order $d+1$. In the next section, however, we derive conditions on the symbol $a(z)$ that guarantee $S_{\boldsymbol{a}}$ reproduces polynomials up to degree d without the need for any pre-processing.

4. Polynomial reproduction

Let us start by introducing the following definition of data that is generated by uniformly sampling a polynomial.

Definition 4.1 (*Polynomial Data*). A sequence $\boldsymbol{g} = (g_i)_{i\in\mathbb{Z}}$ is called polynomial data of degree d if there exists a polynomial $p \in \pi_d$ such that $g_i = p(i)$ for all $i \in \mathbb{Z}$.

If we denote by Δ^ℓ the ℓth order finite difference operator on sequences,

$$\Delta^\ell \boldsymbol{g} = (\Delta^\ell g_i)_{i\in\mathbb{Z}} \quad \text{with } \Delta^\ell g_i = \sum_{j=0}^{\ell} (-1)^j \binom{\ell}{j} g_{i-j},$$

then such polynomial data is characterized by having vanishing finite differences of order $d+1$, namely

$$\Delta^{d+1}\boldsymbol{g} \equiv 0,$$

which in terms of z-transforms translates to the condition

$$(1-z)^{d+1} g(z) = 0. \tag{8}$$

Interestingly, $(1-z)^{d+1}$ is essentially the only Laurent polynomial that annihilates the z-transforms of all polynomial data of degree d.

Lemma 4.2. *The Laurent polynomial $b(z)$ is divisible by $(1-z)^{d+1}$ if and only if*

$$b(z)g(z) = 0 \tag{9}$$

for any polynomial data $\mathbf{g}$ of degree d.

Proof. The necessity of condition (9) follows immediately from (8). In order to show the sufficiency we will prove by induction that there exist Laurent polynomials $r_0, \ldots, r_d$ such that

$$b(z) = (1-z)^{k+1} r_k(z) \tag{10}$$

for $k = 0, \ldots, d$. We start with $k = 0$ and let $\boldsymbol{g}$ be any polynomial data of degree 0 so that its z-transform is

$$g(z) = c \sum_{i\in\mathbb{Z}} z^i$$

for some $c \in \mathbb{R}$. Then

$$b(z)g(z) = \left(\sum_{i\in\mathbb{Z}} b_i z^i\right)\left(c\sum_{j\in\mathbb{Z}} z^j\right) = c\sum_{j\in\mathbb{Z}} z^j \left(\sum_{i\in\mathbb{Z}} b_i\right) = 0$$

for any $c \in \mathbb{R}$ and therefore

$$\sum_{i\in\mathbb{Z}} b_i = b(1) = 0.$$

In other words, $b(z)$ has a root at $z = 1$ and there exists some $r_0(z)$ with $b(z) = (1-z)r_0(z)$.

Now assume that (10) holds for some $k < d$ and let $\boldsymbol{g}$ be any polynomial data of degree $k+1$. By taking the finite differences of degree $k+1$ of $\boldsymbol{g}$ we get the constant sequence $\boldsymbol{f} = \Delta^{k+1}\boldsymbol{g}$ with z-transform

$$f(z) = (1-z)^{k+1} g(z).$$

From (9) and (10) we then have

$$b(z)g(z) = r_k(z)(1-z)^{k+1} g(z) = r_k(z) f(z) = 0,$$

and with the same arguments as in the case $k = 0$ we conclude that there exists some $r_{k+1}(z)$ with $r_k(z) = (1-z)r_{k+1}(z)$. Therefore,

$$b(z) = (1-z)^{k+1} r_k(z) = (1-z)^{k+2} r_{k+1}(z),$$

which completes the induction step. □

The following equivalence then is an immediate consequence of Theorem 3.2 and Lemma 4.2.

Corollary 4.3. *A subdivision scheme $S_{\boldsymbol{a}}$ generates polynomials of degree d if and only if*

$$a(z)g(-z) = 0 \tag{11}$$

for any polynomial data $\boldsymbol{g}$ of degree d.

Proof. Theorem 3.2 states that the symbol $a(z)$ of a subdivision scheme that generates polynomials of degree d is divisible by $(1+z)^{d+1}$ so that the Laurent polynomial $b(z) = a(-z)$ is divisible by $(1-z)^{d+1}$. By Lemma 4.2 this is equivalent to the property that $b(z)g(z) = a(-z)g(z) = 0$ for any polynomial data $\boldsymbol{g}$ of degree d, hence the statement follows by replacing z with $-z$. □

We further need the notion of *stepwise polynomial reproduction*, which is in fact equivalent to polynomial reproduction in the limit for non-singular subdivision schemes.

Definition 4.4 (*Stepwise Polynomial Reproduction*). We say that $S_{\boldsymbol{a}}$ reproduces polynomial data of degree d in each subdivision step if the data $\boldsymbol{f}^k$ with $f_i^k = p(t_i^k)$ is refined to $\boldsymbol{f}^{k+1}$ with $f_i^{k+1} = p(t_i^{k+1})$, $i \in \mathbb{Z}$ for any $p \in \pi_d$ and $k \in \mathbb{N}_0$.

Corollary 4.5. *A subdivision scheme $S_{\boldsymbol{a}}$ that reproduces polynomial data of degree d in each subdivision step also reproduces polynomials of degree d and vice versa.*

Proof. For any $p \in \pi_d$ let $\boldsymbol{f}^0$ be the initial data with $f_i^0 = p(t_i^0)$, $i \in \mathbb{Z}$. If $S_{\boldsymbol{a}}$ reproduces this data in each subdivision step, then

$$F^k(t_i^k) = f_i^k = p(t_i^k), \quad i \in \mathbb{Z}, k \in \mathbb{N}_0,$$

so that $(F^k)_{k\in\mathbb{N}_0}$ is a sequence of piecewise linear approximations to p over uniform grids of width $h(k) = 1/2^k$ and thus clearly converges to p as $k \to \infty$.

We now assume that $S_{\boldsymbol{a}}$ reproduces polynomials of degree d and let $k \in \mathbb{N}_0$. On the one hand, applying the subdivision scheme to the data $\boldsymbol{f}^k$ with $f_i^k = p(t_i^k)$ gives $p = S_{\boldsymbol{a}}^\infty \boldsymbol{f}^k = S_{\boldsymbol{a}}^\infty \boldsymbol{f}^{k+1}$, on the other hand we also get $p = S_{\boldsymbol{a}}^\infty \boldsymbol{g}^{k+1}$ as the limit function for the data $\boldsymbol{g}^{k+1}$ with $g_i^{k+1} = p(t_i^{k+1})$. By the linearity of the operator $S_{\boldsymbol{a}}^\infty$ we then have $S_{\boldsymbol{a}}^\infty(\boldsymbol{f}^{k+1} - \boldsymbol{g}^{k+1}) \equiv 0$ and as we consider only non-singular schemes, this implies $\boldsymbol{f}^{k+1} = \boldsymbol{g}^{k+1}$. □

Note that a similar equivalence holds between stepwise polynomial generation and the generation of polynomials in the limit. We can now establish our conditions for the reproduction of polynomials that are similar to the one for polynomial generation in Theorem 3.2.

Theorem 4.6 (*Primal Polynomial Reproduction*). *If $S_{\boldsymbol{a}}$ is a subdivision scheme that generates polynomials of degree d_G, then it reproduces polynomials of degree $d_R \le d_G$ with respect to the primal parameterization if and only if*

(PR1) $a(z) - 2$ *is divisible by* $(1-z)^{d_R+1}$.

Proof. Because of Corollary 4.5, it suffices to show that condition **(PR1)** is equivalent to the property that $S_{\boldsymbol{a}}$ reproduces polynomial data of degree d_R in each subdivision step. To this end, let t_i^k be the parameter values from (4) and $p \in \pi_{d_R}$, so that the sequences $\boldsymbol{f}^k$ and $\boldsymbol{g}$ with $f_i^k = p(t_i^k)$ and $g_i = p(t_i^{k+1})$, $i \in \mathbb{Z}$ are both polynomial data of degree d_R. Since $f_i^k = g_{2i}$, we have

$$f^k(z^2) = \sum_{i\in\mathbb{Z}} f_i^k z^{2i} = \sum_{i\in\mathbb{Z}} g_{2i} z^{2i} = g_e(z) = (g(z) + g(-z))/2$$

and refining the data $\boldsymbol{f}^k$ with the subdivision scheme gives, in view of (7),

$$f^{k+1}(z) = a(z) f^k(z^2) = a(z)g(z)/2 + a(z)g(-z)/2 = a(z)g(z)/2,$$

where the last identity follows from (11). On the other hand, $g(z) = f^{k+1}(z)$, and hence the data $\boldsymbol{f}^k$ is reproduced by subdivision with $S_{\boldsymbol{a}}$ if and only if

$$a(z)g(z) = 2g(z),$$

which, according to Lemma 4.2, is equivalent to **(PR1)**. □

Theorem 4.7 (*Dual Polynomial Reproduction*). *If $S_{\boldsymbol{a}}$ is a subdivision scheme that generates polynomials of degree d_G, then it reproduces polynomials of degree $d_R \le d_G$ with respect to the dual parameterization if and only if*

(PR2) $a(z^2)z - 2$ *is divisible by* $(1-z)^{d_R+1}$.

Proof. Due to Corollary 4.5, it is again sufficient to show that condition **(PR2)** is equivalent to the property of stepwise polynomial reproduction. Let t_i^k be the parameter values from (5) and $p \in \pi_{d_R}$, so that the sequences $\boldsymbol{f}^k$, $\boldsymbol{g}$, and $\boldsymbol{h}$ with $f_i^k = p(t_i^k)$, $g_i = p((i-1)/2^{k+1})$, and $h_i = p((i-1)/2^{k+2})$ for $i \in \mathbb{Z}$ are all polynomial data of degree d_R. Since $f_i^k = g_{2i}$, we conclude as in the proof of Theorem 4.6 that

$$f^{k+1}(z) = a(z)g(z)/2. \tag{12}$$

Noting that $g_i = h_{2i-1}$, we further have

$$g(z^2) = \sum_{i\in\mathbb{Z}} g_i z^{2i} = \sum_{i\in\mathbb{Z}} h_{2i-1} z^{2i-1} z = h_o(z) z \tag{13}$$

and therefore

$$f^{k+1}(z^2) = a(z^2) h_o(z) z/2.$$

If **(PR2)** holds, then we know from Lemma 4.2 that

$$a(z^2)h(z)z = 2h(z)$$

and therefore

$$a(z^2)h(-z)z = -2h(-z)$$

for any polynomial data $\boldsymbol{h}$ of degree d_{R}. Thus

$$\begin{aligned} f^{k+1}(z^2) &= a(z^2)h_{\mathrm{o}}(z)z/2 = \left(a(z^2)h(z)z - a(z^2)h(-z)z\right)/4 \\ &= \left(h(z) + h(-z)\right)/2 = h_{\mathrm{e}}(z). \end{aligned}$$

Comparing the coefficients of $f^{k+1}(z^2)$ and $h_{\mathrm{e}}(z)$, we see that $f_i^{k+1} = h_{2i} = p(t_i^{k+1})$ for all $i \in \mathbb{Z}$, hence $S_{\boldsymbol{a}}$ reproduces polynomials of degree d_{R}. On the other hand, if the scheme has the property that $f_i^{k+1} = h_{2i}$, then by (12) and (13) we have

$$a(z^2)h_{\mathrm{o}}(z)z/2 = h_{\mathrm{e}}(z)$$

for any polynomial data $\boldsymbol{h}$ of degree d_{R} and in particular for the data $\tilde{\boldsymbol{h}}$ with $\tilde{h}_i = h_{i+1}$, so that

$$a(z^2)h_{\mathrm{e}}(z)z/2 = a(z^2)\tilde{h}_{\mathrm{o}}(z)z^2/2 = \tilde{h}_{\mathrm{e}}(z)z = h_{\mathrm{o}}(z).$$

Combining both identities then gives

$$a(z^2)h(z)z = \left(a(z^2)h_{\mathrm{e}}(z)z + a(z^2)h_{\mathrm{o}}(z)z\right) = 2\left(h_{\mathrm{o}}(z) + h_{\mathrm{e}}(z)\right) = 2h(z)$$

and condition **(PR2)** follows from Lemma 4.2. □

Remark 4.8. Note that the non-singularity of the scheme $S_{\boldsymbol{a}}$ is only needed in the second half of the proof of Corollary 4.5 and is thus not required for the sufficiency of the conditions **(PR1)** and **(PR2)** for polynomial reproduction.

As mentioned above, the degree d_{R} of polynomial reproduction can never exceed the degree d_{G} of polynomial generation. We shall now derive an interesting observation in the case that $d_{\mathrm{G}} > d_{\mathrm{R}}$. From Theorem 3.2 we know that for any polynomial p of degree d with $d_{\mathrm{G}} \geq d > d_{\mathrm{R}}$ there exists some polynomial $q \in \pi_d$ such that p is the limit function for the initial data $\boldsymbol{f}^0$ sampled from q, and that $p - q \in \pi_{d-1}$. The examples from [6] further suggest that even the two leading coefficients of p and q agree, that is, $p - q \in \pi_{d-2}$. This is in fact confirmed by the following more general statement.

Corollary 4.9. *Let $S_{\boldsymbol{a}}$ be a convergent subdivision scheme with generation degree d_{G} and reproduction degree d_{R}. If p and q are polynomials of degree $d \leq d_{\mathrm{G}}$ such that $p = S_{\boldsymbol{a}}^{\infty}\boldsymbol{f}^0$ for $f_i^0 = q(t_i^0)$, $i \in \mathbb{Z}$, then p and q have the same $d_{\mathrm{R}} + 1$ leading coefficients.*

Proof. We start by extending the definition of the finite difference operator Δ^ℓ to functions, that is,

$$(\Delta^\ell f)(x) = \sum_{j=0}^{\ell} (-1)^j \binom{\ell}{j} f(x-j).$$

A useful identity that follows immediately from the linearity of the operator $S_{\boldsymbol{a}}^{\infty}$ and the relation

$$(S_{\boldsymbol{a}}^{\infty}\boldsymbol{f}_{\cdot+i}^0)(x) = (S_{\boldsymbol{a}}^{\infty}\boldsymbol{f}^0)(x+i)$$

is that the operators $S_{\boldsymbol{a}}^{\infty}$ and Δ^ℓ commute [1],

$$\Delta^\ell(S_{\boldsymbol{a}}^{\infty}\boldsymbol{f}^0) = S_{\boldsymbol{a}}^{\infty}(\Delta^\ell\boldsymbol{f}^0). \tag{14}$$

Now let $\boldsymbol{g}^0 = \Delta^{d-d_{\mathrm{R}}}\boldsymbol{f}^0$ be the polynomial data of degree d_{R} that is sampled from the polynomial $\Delta^{d-d_{\mathrm{R}}}q$. It then follows from the reproduction property of $S_{\boldsymbol{a}}$ that the corresponding limit function is

$$S_{\boldsymbol{a}}^{\infty}\boldsymbol{g}^0 = \Delta^{d-d_{\mathrm{R}}}q.$$

Due to (14) we also have

$$\Delta^{d-d_{\mathrm{R}}}p = \Delta^{d-d_{\mathrm{R}}}(S_{\boldsymbol{a}}^{\infty}\boldsymbol{f}^0) = S_{\boldsymbol{a}}^{\infty}(\Delta^{d-d_{\mathrm{R}}}\boldsymbol{f}^0) = S_{\boldsymbol{a}}^{\infty}\boldsymbol{g}^0,$$

and conclude that $\Delta^{d-d_{\mathrm{R}}}(p-q) \equiv 0$. This implies that the degree of the polynomial $p-q$ is at most $d-d_{\mathrm{R}}-1$ and so the first $d_{\mathrm{R}}+1$ leading coefficients of p and q must be identical. □

5. Symmetric schemes

Let us now investigate the conditions for reproduction of polynomials of low degree in more detail. According to condition **(PG)**, the generation of constant functions requires the symbol $a(z)$ to have a zero at $z=-1$, and it follows from conditions **(PR1)** and **(PR2)** that the scheme further reproduces these functions with respect to the primal as well as the dual parameterization if and only if $a(1)=2$. Combining both conditions, a subdivision scheme $S_{\boldsymbol{a}}$ reproduces constant polynomials if and only if

$$a(-1)=0 \quad \text{and} \quad a(1)=2 \iff a_{\mathrm{e}}(1)=a_{\mathrm{o}}(1)=1,$$

where the equivalence to the conditions on the right follows from (6). Note that these latter conditions are further equivalent to the ones in Eq. (3) and thus confirm our previous observation that any convergent subdivision scheme reproduces constant functions, regardless of the chosen parameterization.

To check reproduction of linear polynomials we have to find out if the roots $z=-1$ and $z=1$ are double roots of $a(z)$ and $a(z)-2$ (or $a(z^2)z-2$), respectively. It then follows that any scheme with constant reproduction also reproduces linear functions if and only if

$$a'(-1)=0 \quad \text{and} \quad a'(1)=0 \quad (\text{or } a'(1)=-1), \tag{15}$$

where the two options in the second condition refer to the primal and the dual parameterization, respectively. Obviously, a scheme cannot reproduce linear functions with respect to both parameterizations, and the value $a'(1)$ actually tells which of the two should be chosen.

For example, the uniform degree m B-spline schemes all reproduce constant functions, because their general symbol

$$b_m(z) = 2^{-m}(1+z)^{m+1}z^n, \quad n \in \mathbb{Z},$$

clearly fulfills the conditions $b_m(-1)=0$ and $b_m(1)=2$. For $m>0$ we further have $b_m'(-1)=0$ and $b_m'(1)=m+1+2n$. Now, by appropriately shifting the symbol with the choice $n=-\lceil\frac{m+1}{2}\rceil$, $b_m'(1)$ evaluates to 0 for odd m and to -1 for even m, thus confirming the known fact that all but the piecewise constant B-splines reproduce linear functions with respect to the appropriate parameterization.

The B-spline schemes are particular examples of *odd* and *even symmetric* subdivision schemes, and we can show more generally which parameterization to choose in order to have at least linear reproduction.

Definition 5.1 (*Symmetric Schemes*). A subdivision scheme $S_{\boldsymbol{a}}$ is called odd symmetric if

$$a_{-i} = a_i, \quad i \in \mathbb{Z},$$

and even symmetric if

$$a_{-i} = a_{i-1}, \quad i \in \mathbb{Z}.$$

In terms of Laurent polynomials, these conditions translate to $a(z) = a(1/z)$ and $a(z)z = a(1/z)$, respectively.

Corollary 5.2. *In order to achieve as high degrees of polynomial reproduction as possible, the primal parameterization should be used for odd symmetric schemes and the dual parameterization for schemes with even symmetry.*

Proof. If $S_{\boldsymbol{a}}$ is odd symmetric, then by taking the derivative on both sides of the condition $a(z) = a(1/z)$ we get

$$a'(z) = -a'(1/z)/z^2,$$

which implies $a'(-1) = a'(1) = 0$. Thus, according to (15), it is impossible for an odd symmetric scheme to reproduce linear functions with respect to the dual parameterization. However, linear reproduction with respect to the primal parameterization comes for free for any such scheme that reproduces constants.

If $S_{\boldsymbol{a}}$ is even symmetric, then the condition $a(z)z = a(1/z)$ gives

$$a'(z)z + a(z) = -a'(1/z)/z^2.$$

In particular $a(-1) = 0$ and $a'(1) = -a(1)/2$. Hence, if the scheme reproduces constants then $a'(1) = -1$, so that linear functions with respect to the primal parameterization cannot be reproduced. On the other hand, linear reproduction with respect to the dual parameterization is guaranteed for all even symmetric schemes that reproduce constants and generate linear polynomials. $\square$

These observations encourage us to always use the appropriate parameterization for odd and even symmetric schemes by default and call them *primal* and *dual schemes*, respectively.

In the proof of the previous corollary some of the conditions for the generation and reproduction of linear functions follow directly from the symmetry of the schemes. These are in fact special cases of two more general propositions regarding the degrees of polynomial generation and reproduction of symmetric schemes.

Corollary 5.3. *A symmetric subdivision scheme $S_{\boldsymbol{a}}$ generates polynomials up to a degree of the same parity as the parity of its symmetry.*

Proof. Let d_{G} be the maximal degree of polynomial generation of the scheme $S_{\boldsymbol{a}}$. Then, according to condition **(PG)**, there exists a Laurent polynomial $r(z)$ such that

$$a(z) = (1+z)^{d_{\mathrm{G}}+1} r(z).$$

For a scheme with odd symmetry we have

$$a(z) = a(1/z) = (1 + 1/z)^{d_{\mathrm{G}}+1} r(1/z) = (1+z)^{d_{\mathrm{G}}+1} z^{-d_{\mathrm{G}}-1} r(1/z),$$

so that

$$z^{d_{\mathrm{G}}+1} r(z) = r(1/z).$$

If we assume d_{G} to be even, then substituting $z = -1$ gives

$$-r(-1) = r(-1),$$

showing that $r(z)$ contains $1 + z$ as a factor, which in turn contradicts the assumption that d_{G} is maximal. Therefore, d_{G} is always odd for schemes with odd symmetry and a similar argument shows that d_{G} is always even for schemes with even symmetry. $\square$

Corollary 5.4. *Let $S_{\boldsymbol{a}}$ be a symmetric subdivision scheme with the appropriate parameterization. Then $S_{\boldsymbol{a}}$ reproduces polynomials up to an odd degree, provided that it generates polynomials up to that degree.*

Proof. Let d_{R} be the maximal degree of polynomial reproduction of $S_{\boldsymbol{a}}$. Conditions **(PR1)** and **(PR2)** then imply the existence of a Laurent polynomial $r(z)$ with

$$a(z) - 2 = (1-z)^{d_{\mathrm{R}}+1} r(z)$$

if $S_{\boldsymbol{a}}$ is odd symmetric and with

$$a(z^2)z - 2 = (1-z)^{d_{\mathrm{R}}+1} r(z)$$

in case of even symmetry. Using the properties that $a(z) = a(1/z)$ for odd and $a(z^2)z = a(1/z^2)/z$ for even symmetric schemes, we conclude in both cases that

$$(1-z)^{d_{\mathrm{R}}+1} r(z) = (1-1/z)^{d_{\mathrm{R}}+1} r(1/z),$$

leading to

$$(-z)^{d_{\mathrm{R}}+1} r(z) = r(1/z).$$

Assuming d_{R} to be even and substituting $z = 1$ then yields

$$-r(1) = r(1),$$

so that $r(z)$ is divisible by $1 - z$, contradicting the assumption that d_{R} is maximal. $\square$

6. Two families of symmetric subdivision schemes

As an application of our results, we shall now derive the degree of polynomial reproduction for the members of a known family of primal subdivision schemes $S_{\boldsymbol{a}_m^l}$ and a new family of dual subdivision schemes $S_{\tilde{\boldsymbol{a}}_m^l}$. We define the Laurent polynomials

$$\sigma(z) = \frac{(1+z)^2}{4z}, \qquad \delta(z) = -\frac{(1-z)^2}{4z}, \tag{16}$$

and note that $\sigma(z)$ and $\delta(z)$ satisfy the two identities

$$\sigma(z) + \delta(z) = 1 \quad \text{and} \quad \delta(z^2) = 4\sigma(z)\delta(z). \tag{17}$$

Then the symbols of the primal schemes are

$$a_m^l(z) = 2\,\sigma(z)^m \sum_{i=0}^{l} \binom{m+l}{i} \delta(z)^i \sigma(z)^{l-i}, \tag{18}$$

whereas those of the dual schemes are

$$\tilde{a}_m^l(z) = \frac{1+z}{z}\,\sigma(z)^m \sum_{i=0}^{l} \binom{m+1/2+l}{i} \delta(z)^i \sigma(z)^{l-i}, \tag{19}$$

with $m, l \geq 0$. It follows directly from $\sigma(1/z) = \sigma(z)$ and $\delta(1/z) = \delta(z)$ that the schemes $S_{\boldsymbol{a}_m^l(z)}$ and $S_{\tilde{\boldsymbol{a}}_m^l(z)}$ are odd and even symmetric, respectively.

We note that as shown in [11, Equation (2.5)], the primal schemes are equivalent to the pseudo-splines of type II that were introduced by Dong and Shen [3] for the construction of symmetric framelets whose truncated framelet series has a desirable approximation order.

Theorem 6.1. *The primal subdivision schemes with symbols $a_m^l(z)$ reproduce polynomials up to degree* $\min(2m-1, 2l+1)$.

Proof. It follows directly from (18) that $a_m^l(z)$ is divisible by $(1+z)^{2m}$, hence the scheme generates polynomials of degree $2m-1$. It is further clear that this is the maximal degree of polynomial generation because the remainder $r(z) = a_m^l(z)/(1+z)^{2m}$ evaluates to $r(-1) = 2(-1/4)^m \binom{m+l}{l} \neq 0$ at $z = -1$. According to (17) we have $(\sigma+\delta)^{m+l} = 1$ and by applying the binomial theorem to the left-hand side, we can write $a_m^l(z)$ as

$$\begin{aligned} a_m^l(z) &= 2 - 2\sum_{i=l+1}^{m+l} \binom{m+l}{i} \delta(z)^i \sigma(z)^{m+l-i} \\ &= 2 - 2\,\delta(z)^{l+1} \sum_{i=1}^{m} \binom{m+l}{i+l} \delta(z)^{i-1} \sigma(z)^{m-i}, \end{aligned}$$

showing that $a_m^l(z) - 2$ is clearly divisible by $(1-z)^{2l+2}$. This is again maximal, because the remainder $\tilde{r}(z) = (a_m^l(z)-2)/(1-z)^{2l+2}$ evaluates to $\tilde{r}(1) = -2(-1/4)^{l+1}\binom{m+l}{1+l} \neq 0$. The statement then follows from Theorems 3.2 and 4.6. $\square$

This result was first shown by Dong and Shen [3, Theorem 3.10] using a Fourier analysis approach. Dong and Shen also noted that $a_m^0(z) = b_{2m-1}(z)$, $m \geq 1$ are the symbols of the odd degree B-splines and that $a_n^{n-1}(z)$, $n \geq 1$ are those of the $2n$-point interpolatory schemes of [4]. Moreover, it is straightforward to verify that the symbols of the schemes $S_{2L}(\omega)$, $L \geq 1$ in [12] are affine combinations of $a_{L+1}^{L-1}(z)$ and $a_L^{L-1}(z)$ with weights $\alpha_L(\omega) = \omega 16^L / \binom{2L}{L}$ and $1 - \alpha_L(\omega)$ and that $a_k^1(z)$ are the symbols of the schemes S_{2k}, $k \geq 2$ in [13].

Theorem 6.2. *The dual subdivision schemes with symbols $\tilde{a}_m^l(z)$ reproduce polynomials up to degree* $\min(2m, 2l+1)$.

Proof. For any real $\alpha > 0$ and $|x| \leq 1$,

$$(1+x)^\alpha = \sum_{i=0}^{\infty} \binom{\alpha}{i} x^i.$$

Now by (17),

$$1 = \left(\sigma(z^2) + \delta(z^2)\right)^{m+1/2+l} = \sigma(z^2)^{m+1/2+l} \left(1 + \frac{\delta(z^2)}{\sigma(z^2)}\right)^{m+1/2+l},$$

and since $\frac{\delta(z^2)}{\sigma(z^2)} \in [-1, 0]$ for real z, we have

$$1 = \frac{1+z^2}{2z} \sigma(z^2)^m \sum_{i=0}^{\infty} \binom{m+1/2+l}{i} \delta(z^2)^i \sigma(z^2)^{l-i}.$$

Using (19) we can rewrite the above equality as

$$2 - \tilde{a}_m^l(z^2)z = \frac{1+z^2}{z} \delta(z^2)^{l+1} \sum_{i=l+1}^{\infty} \binom{m+1/2+l}{i} \delta(z^2)^{i-l-1} \sigma(z^2)^{m+l-i}.$$

According to (17), $\delta(z^2)^{l+1} = 4^{l+1}\delta(z)^{l+1}\sigma(z)^{l+1}$, and we get

$$2 - \tilde{a}_m^l(z^2)z = \delta(z)^{l+1} R(z),$$

with

$$R(z) = \frac{1+z^2}{z} 4^{l+1} \sigma(z)^{l+1} \sum_{i=l+1}^{\infty} \binom{m+1/2+l}{i} \delta(z^2)^{i-l-1} \sigma(z^2)^{m+l-i}.$$

By (16), $\sigma(1) = 1$ and $\delta(1) = 0$, and thus

$$R(1) = 2^{2l+3} \binom{m+1/2+l}{l+1} \neq 0.$$

On the other hand, $R(z)$ is the rational function

$$R(z) = \frac{2 - \tilde{a}_m^l(z^2)z}{\delta(z)^{l+1}}.$$

These two properties of R imply that the numerator of R is divisible by exactly $2l+2$ factors $1-z$. The claim of the theorem now follows from Theorems 3.2 and 4.7.

Like the family of primal schemes, this new family of dual schemes also has some well-known special cases. The symbols of the even degree B-splines are $\tilde{a}_m^0(z) = b_{2m}(z)$, $m \geq 0$, those of the schemes $S_{2L-1}(\omega)$, $L \geq 1$ in [12] are affine combinations of $\tilde{a}_L^{L-1}(z)$ and $\tilde{a}_{L-1}^{L-1}(z)$ with weights $\tilde{\alpha}_L(\omega) = \omega 4^{2L-1} / \binom{2L-3/2}{L-1}$ and $1 - \tilde{\alpha}_L(\omega)$, and $\tilde{a}_k^1(z)$ are the symbols of the schemes S_{2k+1}, $k \geq 1$ in [13]. Moreover, $\tilde{a}_n^{n-1}$, $n \geq 1$ are the symbols of the dual $2n$-point schemes of [5], which are based on interpolating $2n$ successive data points $f_{i-n+1}^k, \ldots, f_{i+n}^k$ at the dual parameter values $t_{i-n+1}^k, \ldots, t_{i+n}^k$ from (5) by a polynomial of degree $2n-1$ and then evaluating this polynomial at t_{2i}^{k+1} and t_{2i+1}^{k+1} to determine the new data f_{2i}^{k+1} and f_{2i+1}^{k+1}. We found that a similar construction yields the symbols $\tilde{a}_{n-1}^{n-1}$ of the dual $(2n-1)$-point schemes for $n \geq 1$. Here a polynomial of degree $2n-2$, interpolating the $2n-1$ points (t_j^k, f_j^k), $|j-i| \leq n-1$ is constructed, and $f_{2i-1}^{k+1}, f_{2i}^{k+1}$ are the values of this polynomial at $t_{2i-1}^{k+1}, t_{2i}^{k+1}$, respectively. In this construction the parameterization is again the dual one.

Finally, we would like to note that by using the identity

$$\sum_{i=0}^{l} \binom{r+i}{i} x^i = \sum_{i=0}^{l} \binom{r+1+l}{i} x^i (1-x)^{l-i},$$

which can be proved straightforwardly by induction over l for any $r, x \in \mathbb{R}$, the symbols from both families can be expressed in a slightly more compact form, namely

$$a_m^l(z) = 2\,\sigma(z)^m \sum_{i=0}^{l} \binom{m-1+i}{i} \delta(z)^i$$

and

$$\tilde{a}_m^l(z) = \frac{1+z}{z}\,\sigma(z)^m \sum_{i=0}^{l} \binom{m-1/2+i}{i} \delta(z)^i.$$

This form of $a_m^l(z)$ also appears in the papers by Dong and Shen [14,3].

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