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A family of subdivision schemes with cubic precision

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Abstract

The four-point subdivision scheme is well known as an interpolating subdivision scheme, but it has recently come to our notice that it is but the first scheme in a family all of whose members have the property that if all the control points lie equally spaced along the same cubic polynomial, the limit curve is exactly that polynomial. Other members of the family have higher smoothness.

We study these schemes as functions, where the ordinate is given by the scheme, while the abscissae of the control points are equally spaced. Because all schemes include linear functions in their precision set, this may be regarded as a particular case of the parametric setting, rather than as a special case.

This paper introduces the family and determines how the support, the Hölder regularity, the precision set, the degree of polynomials spanned by the limit curves, and the artifact behavior vary with the integer parameter that identifies the members of the family.

For the family members with an even parameter value, most of these properties have also been shown by Dong and Shen (Dong, B., Shen, Z., 2007. Pseudo-splines, wavelets and framelets. Appl. Comput. Harmon. Anal. 22 (1), 78–104), as they turn out to be a particular kind of pseudo-splines. But regarding the regularity exponent of the limit functions, we derive the exact values and thus improve the lower bounds given by Dong and Shen in that paper. Moreover, our analysis also covers the family members with an odd parameter value which do not seem to fit into the general framework of pseudo-splines.

Just before this paper was submitted, (Choi, S.W., Lee, B.-G., Lee, Y.J., Yoon, J., 2006. Stationary subdivision schemes reproducing polynomials. Comput. Aided Geom. Design 23 (4), 351–360) appeared, which also discusses a family of subdivision schemes. The high order members of that family achieve higher degrees of polynomial reproduction, whereas ours aim only at cubic reproduction. This allows us to gain higher continuity for a given mask width.

Keywords: Subdivision; Smoothness; Approximation; Quasi-interpolation

1. Introduction

Subdivision is the process of generating curves and surfaces by iteratively refining a given set of control points according to certain refinement rules. In the simplest case of linear, uniform, and stationary subdivision, these rules

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are the same in every refinement step and the set of refined control points can be generated by computing affine combinations of the current control points.

This paper is about binary subdivision schemes for curves and it is common to formalize the refinement process as follows (Dyn and Levin, 2002). If we denote the initial data at level zero by f_i^0 for $i \in \mathbb{Z}$, then the refined data at level $\ell+1$ for any $\ell \in \mathbb{N}_0$ is given by the refinement equation

$$f_i^{\ell+1} = \sum_{j\in\mathbb{Z}} a_{i-2j} f_j^\ell, \quad i \in \mathbb{Z}, \tag{1}$$

where $\boldsymbol{a} = \{a_i\colon i \in \mathbb{Z}\}$ is called the *mask* of the scheme. A necessary condition for the convergence of the scheme is that the even as well as the odd mask coefficients a_i each sum up to 1 (Dyn, 1992), so that the sum in Eq. (1) is an affine combination. Furthermore, it is usually the case that only a finite number of the a_i are non-zero, so that this sum can be computed efficiently and the basic limit function has finite support.

For the analysis of subdivision schemes it is very practical to also consider the *z-transform* of the mask,

$$a(z) = \sum_{i\in\mathbb{Z}} a_i z^i,$$

which is usually called the *symbol* of the scheme. This enables us to write the refinement step in an algebraic way,

$$F^{\ell+1}(z) = a(z) F^\ell\bigl(z^2\bigr),$$

where F^ℓ is the z-transform of the data at level ℓ,

$$F^\ell(z) = \sum_{i\in\mathbb{Z}} f_i^\ell z^i.$$

Detailed information on this mechanism of Laurent polynomials can be found, for example, in the survey by Dyn and Levin (2002) and the tutorial by Dyn (2002a).

The simplest binary subdivision schemes are the ones that generate uniform B-splines. The mask $\boldsymbol{a}_k$ that gives the B-spline of order k (i.e. degree $k-1$) is just the kth row of Pascal's triangle, normalized such that all coefficients sum up to 2. In the case of B-splines, it is possible to decompose each refinement step from level ℓ to level $\ell+1$ into a *sampling step*,

$$g_{2i}^0 = 2 f_i^\ell \quad \text{and} \quad g_{2i+1}^0 = 0, \tag{2}$$

followed by k *smoothing steps*,

$$g_i^{j+1} = \bigl(g_{i-1}^j + g_i^j\bigr)/2, \quad j = 0, 1, \ldots, k-1, \tag{3}$$

and finally setting $f_i^{\ell+1} = g_i^k$ (Lane and Riesenfeld, 1980). This interpretation nicely reflects the fact that the symbol of the order-k B-spline scheme is just

$$a_k(z) = 2\sigma(z)^k$$

with the *averaging* or *smoothing operator*

$$\sigma(z) = (1+z)/2.$$

The number of σ-factors in the symbol plays a central role for the smoothness of a subdivision scheme as multiplying the symbol by σ increases the smoothness by one (Dyn, 1992). This is one way to explain why the continuity of the B-splines goes up with the order k.

When analyzing a general subdivision scheme with symbol $a(z)$, it is often useful to factor out any σ-factors that it may contain and consider the reduced symbol $b(z) = a(z)/\sigma(z)^k$. While revising the four-point scheme (Dubuc, 1986), the dual four-point scheme (Dyn et al., 2005), and the scheme of Sabin and Dodgson (2005), which share the property of reproducing polynomials up to degree three, we noticed a striking similarity between the reduced symbols of these schemes.

In an attempt to generalize this pattern, we found a new family of subdivision schemes whose structure and properties are very similar to those of the B-spline schemes. The members of this family also depend on a parameter k (Section 2) and higher values of k give schemes with wider masks and support (Section 3), higher continuity (Section 4), and smaller artifacts (Section 6). The main difference is that all schemes from our family with $k \geqslant 4$ reproduce cubic polynomials (Section 5) whereas the B-spline schemes have only linear precision.

2. The family

Let us now consider the family of subdivision schemes $\boldsymbol{S} = \{S_k\colon\ k \in \mathbb{N}\}$ where the general member S_k has the symbol

$$a_k(z) = 2\sigma(z)^k K_k(z) \tag{4}$$

and can thus be understood as a convolution of the order-k B-spline with the *kernel*

$$K_k(z) = \bigl(-k + (8+2k)z - kz^2\bigr)/8. \tag{5}$$

A single refinement step with S_k can be decomposed analogously to Eqs. (2) and (3), except that the sampling step needs to be replaced by

$$g_{2i}^0 = -k\bigl(f_{i-1}^\ell + f_i^\ell\bigr)/4 \quad \text{and} \quad g_{2i+1}^0 = (8+2k)f_i^\ell/4.$$

Before we start discussing the properties of the scheme S_k for general k, let us take a closer look at the particular members for $k = 3, 4, 5, 6$.

2.1. The four-point scheme

The mask and the symbol of the scheme S_4 are

$$[-1, 0, 9, 16, 9, 0, -1]/16 \quad \text{and} \quad (1+z)^4\bigl(-1 + 4z - z^2\bigr)/16$$

and we recognize this as the well-known four-point scheme. This was one of the first interpolating subdivision schemes to be discovered, independently by Dubuc (1986) and Dyn et al. (1987). It also provided the basic test case for the development of sufficient conditions for derivative continuity. The rationale is that an interpolating scheme is required and so the points in the old polygon are retained in the new one. This property is maintained through all iterations and therefore the limit curve contains the original polygon points. The intermediate points are constructed by fitting a cubic through each set of four consecutive points, and sampling that cubic at the center of the configuration; see Fig. 1. Clearly, if all the points of the old polygon lie on the same cubic, the same will be true of the new polygon. This property holds at every iteration and thus will also be true of the limit curve.

2.2. The dual four-point scheme

For $k = 5$, the mask and the symbol of the scheme S_k are

$$[-5, -7, 35, 105, 105, 35, -7, -5]/128 \quad \text{and} \quad (1+z)^5\bigl(-5 + 18z - 5z^2\bigr)/128.$$

This method was published by Dyn et al. (2005) and follows the logic of the four-point scheme in that new points lie on a cubic through four points. But instead of being at the old points and midway between, they are at the *dual* points of $1/4$ and $3/4$ of the way along each span; see Fig. 2. The argument for cubic precision follows the same logic.

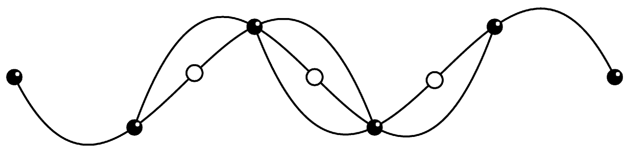

Fig. 1. Construction of S_4 by sampling cubics.

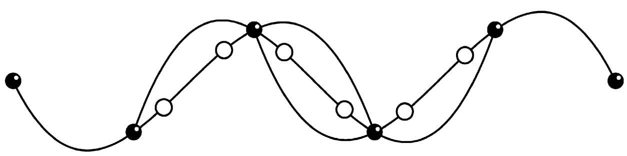

Fig. 2. Construction of S_5 by sampling cubics.

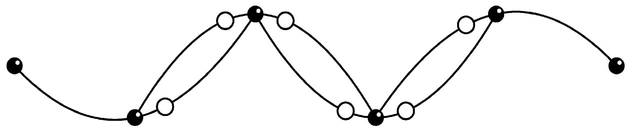

Fig. 3. Construction of S_3 by sampling quadratics.

2.3. The dual three-point scheme

The mask and the symbol of the scheme S_3 are

$$[-3, 5, 30, 30, 5, -3]/32 \quad \text{and} \quad (1+z)^3\left(-3+14z-3z^2\right)/32.$$

This is the "black sheep" of the family because it does not have cubic precision. However, in all other respects it is a full member of the family. It is similar to the dual four-point scheme, but instead of the two new points in each span coming from a cubic through four points, the two new points adjacent to a given old point are taken by sampling a quadratic through three adjacent old points; see Fig. 3. It therefore has quadratic precision by construction.

2.4. A relaxation of the four-point scheme

It was observed by Sabin and Dodgson (2005) that the four point scheme could be expressed in terms of placing the new points such that the second divided difference at each new point would be the mean of the second divided differences of the adjacent old points. However, the second divided difference at the old points then changes. If these points are relaxed to make the new second divided difference a little closer to that of the old points, the number of σ-factors in the symbol can be increased from four to six. This approach turns out to give the scheme S_6 whose mask and symbol are

$$[-3, -8, 12, 72, 110, 72, 12, -8, -3]/128 \quad \text{and} \quad (1+z)^6\left(-3+10z-3z^2\right)/128.$$

If all the old points lie on the same cubic, the old point's divided differences do not change, and so the limit curve does not differ from that of the four-point scheme. Therefore, cubic precision is again given by construction.

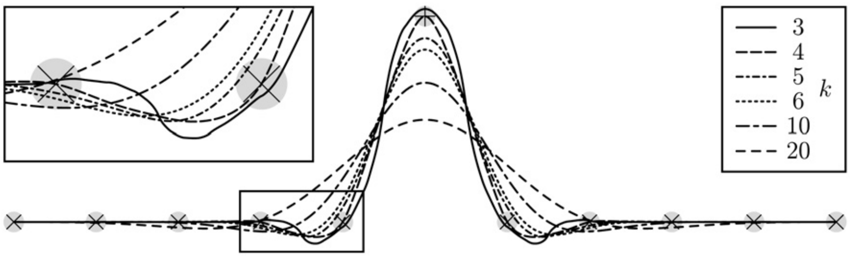

Fig. 4. The basic limit functions of some family members S_k.

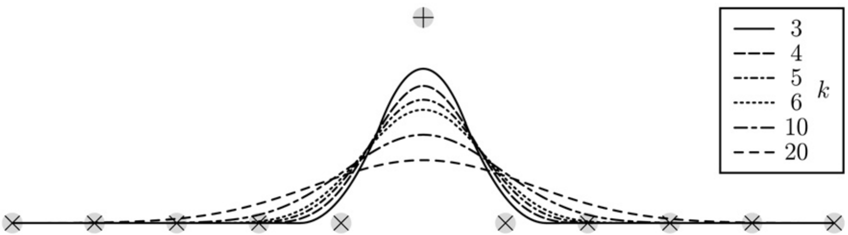

Fig. 5. The basic limit functions of B-splines with the same numbers of σ-factors.

3. Support

It is obvious from Eqs. (4) and (5) that the number of non-zero entries in the mask of the general scheme S_k is $k+3$. Following Ivrissimtzis et al. (2004), we conclude that the *support* of the basic limit function is $k+2$. This is equal to the number of spans of the curve affected when one control point is moved, or to the number of control points influencing a given point or a given span of the limit curve.

Fig. 4 shows the basic limit functions for some of the schemes S_k. Comparing the plots to those of the B-splines with the same number of σ-factors (see Fig. 5), it can be seen that although the family members have wider support, the width at one half of the maximum height tends to be narrower and the maximum height itself is closer to 1. This implies that the control is slightly sharper when curves are edited by moving control points.

4. Hölder regularity

Hölder regularity is an extension of the notion of continuity which gives more information for fractal schemes like these than just quoting the number of derivatives which converge. A function $\phi:\mathbb{R}\to\mathbb{R}$ is defined to be regular of order $n+\alpha$ (for $n\in\mathbb{N}_0$ and $0<\alpha\leqslant 1$) if it is n times continuously differentiable and $\phi^{(n)}$ is Lipschitz of order α, i.e.

$$\left|\phi^{(n)}(x+h)-\phi^{(n)}(x)\right|\leqslant c|h|^{\alpha}$$

for all x and h in $\mathbb{R}$ and some constant c.

According to Rioul (1992) and Dyn and Levin (2002), the Hölder regularity of a subdivision scheme with symbol $a(z)$ can be computed in the following way. Let k be the maximum number of σ-factors contained in the symbol and $b(z)=a(z)/\sigma(z)^k$ be the reduced symbol after dividing out all σ-factors. Without loss of generality we can assume $b_0,\dots,b_n$ to be the non-zero coefficients of $b(z)$ and let A_0 and A_1 be the $n\times n$ matrices with elements

$$\begin{aligned}(A_0)_{ij}&=b_{n+i-2j},\\(A_1)_{ij}&=b_{n+i-2j+1}\end{aligned}\tag{6}$$

for $i,j=1,\dots,n$. Then the Hölder regularity is given by $r=k-\log_2(\mu)$ where μ is the *joint spectral radius* of the matrices A_0 and A_1,

$$\mu=\rho(A_0,A_1)=\limsup_{m\to\infty}\bigl(\max\bigl\{\|A_{\epsilon_m}\cdots A_{\epsilon_2}A_{\epsilon_1}\|_\infty^{1/m}\colon\ \epsilon_i\in\{0,1\},\ i=1,\dots,m\bigr\}\bigr).$$

Note that it is easy to conclude (Lagarias and Wang, 1995) that μ can be bounded from below by the spectral radii and from above by the norms of the matrices A_0 and A_1,

$$\max\bigl\{\rho(A_0),\rho(A_1)\bigr\}\leqslant\mu\leqslant\max\bigl\{\|A_0\|_\infty,\|A_1\|_\infty\bigr\}.\tag{7}$$

Theorem 1. *The Hölder regularity of the scheme S_k is $k-\log_2(2+k/2)$.*

Proof. According to Eqs. (5) and (6) the matrices A_0 and A_1 for the reduced symbol $2K_k(z)$ of the scheme S_k are

$$A_0=\begin{pmatrix}2+k/2 & 0\\ -k/4 & -k/4\end{pmatrix}\quad\text{and}\quad A_1=\begin{pmatrix}-k/4 & -k/4\\ 0 & 2+k/2\end{pmatrix}.$$

As the largest eigenvalue and the max-norm of both matrices are $2+k/2$, the statement follows directly from Eq. (7). □

Likewise, it follows that the Hölder regularity of the order-k B-spline is $k-1$. Fig. 6 compares the regularity of the schemes S_k to those of the B-splines with the same mask-size, i.e. the B-splines of order $k+2$. Note that the smoothness of S_k increases almost linearly with k.

To avoid confusion, we should stress again that if the Hölder regularity is an integer r (as it is for all B-splines and for S_k with $k=4,12,28,60,\dots$), then the scheme is not C^r but only C^{r-1}. Moreover, the $(r-1)$th derivative is in general only *almost* Lipschitz of order 1,

$$\left|\phi^{(r-1)}(x+h)-\phi^{(r-1)}(x)\right|\leqslant c|h|\bigl|\log|h|\bigr|,$$

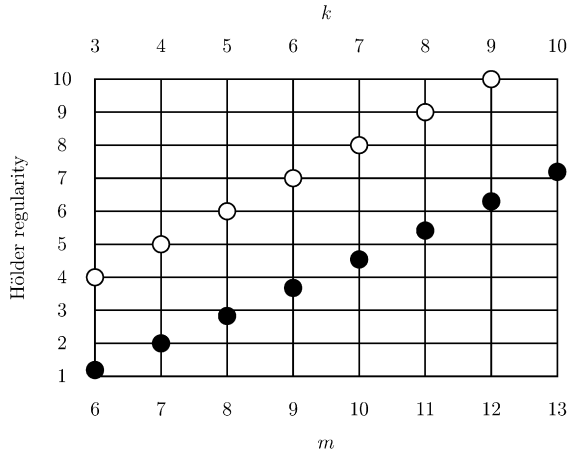

Fig. 6. Hölder regularity plotted against k and m, where m is the number of entries in the mask. Values for family members are shown in black, B-splines of the same mask width in white.

and we shall more precisely say that the Hölder regularity is $r - \varepsilon$ for an arbitrarily small $\varepsilon > 0$ in that case; see (Rioul, 1992, Theorem 10.3). For example, the regularity of the four-point scheme S_4 is only $2 - \varepsilon$ and that of S_{12} is only $9 - \varepsilon$.

Notice that the B-splines are a special case, because the $(k - 1)$th derivative of the B-spline of order k is piecewise constant and therefore bounded. It follows that the $(k - 2)$th derivative is truly Lipschitz of order 1 and we can omit the ε and say that its Hölder regularity is $k - 1$.

5. Response to polynomial data

There are three numerical quantities which characterize the behavior of a subdivision scheme when the original control points lie on a polynomial. These are that for all polynomials up to a certain degree, the limit curve

(1) is the same polynomial,
(2) is a polynomial of the same degree and with the same leading term,
(3) passes through the original control points.

These three behaviors define three maximal degrees, which we shall call *reproduction degree*, *generation degree*, and *interpolation degree* respectively. In the following, we give more precise definitions of these quantities, derive their values for the subdivision schemes S_k, and compare them to those of the B-splines.

5.1. Reproduction degree

Let Π_d denote the space of polynomials up to degree d. A general approximation operator A is said to reproduce polynomials up to degree d, if $Af = f$ for every $f \in \Pi_d$. Following Levin (2003), the stationary uniform subdivision schemes that we consider are approximation operators of the form $A = S^\infty Q$, with an operator Q that generates the initial data f_i^0 by uniformly sampling the function f and a subdivision operator S that successively refines the data.

For convergent subdivision schemes, a sufficient condition for polynomial reproduction is the following. If the subdivision operator is applied to a set of control points that were sampled at equal intervals from some polynomial, then the new control points after refinement lie on the same polynomial, but now with half the sampling distance.

Note that it is easy to characterize such "polynomial data". In the same way that a polynomial of degree d has vanishing $(d + 1)$th *derivative*, uniform samplings of that polynomial have vanishing $(d + 1)$th *differences*. In terms of Laurent polynomials, this translates to

$$\delta(z)^{d+1} F(z) = 0,$$

with the *difference operator*

$$\delta(z) = 1 - z.$$

It is well known that the B-spline subdivision schemes reproduce linear polynomials only, whereas the first few members of the family $\boldsymbol{S}$ have quadratic ($k = 3$) or even cubic precision ($k = 4, 5, 6$) by construction (see Section 2). We assert that the latter also holds for all members of the family with larger values of k.

Theorem 2. *The reproduction degree of S_k is 2 for $k = 3$ and 3 for $k \geqslant 4$.*

Proof. The first statement follows because S_3 is based on sampling local quadratic interpolants and therefore has at least quadratic precision by construction. In the next section we will see that this is the maximal reproduction degree.

To prove the second statement, let us consider the difference between the schemes S_{k+2} and S_k. With the difference operator δ, the symbol a_k from Eq. (4) can be written as

$$a_k(z) = \sigma^k\big(8z - k\delta^2\big)/4.$$

Using the identity $\sigma(z)^2 - z = \delta(z)^2/4$ it follows that

$$\begin{aligned}
a_{k+2}(z) - a_k(z)z &= \sigma^k\big(\sigma^2\big(8z - (k+2)\delta^2\big) - \big(8z - k\delta^2\big)z\big)/4\\
&= \sigma^k\big(\sigma^2\big(8z - k\delta^2 - 2\delta^2\big) - \big(8z - k\delta^2\big)z\big)/4\\
&= \sigma^k\big(\big(\sigma^2 - z\big)\big(8z - k\delta^2\big) - 2\sigma^2\delta^2\big)/4\\
&= \sigma^k\big(\big(\delta^2/4\big)\big(8z - k\delta^2\big) - 2\sigma^2\delta^2\big)/4\\
&= \sigma^k\delta^2\big(8z - k\delta^2 - 8\sigma^2\big)/16\\
&= \sigma^k\delta^2\big(8\big(z - \sigma^2\big) - k\delta^2\big)/16\\
&= \sigma^k\delta^2\big(-2\delta^2 - k\delta^2\big)/16\\
&= -\sigma^k\delta^4(2+k)/16,
\end{aligned}$$

and by further noticing that $\sigma(z)\delta(z) = (1 - z^2)/2 = \delta(z^2)/2$, we have

$$a_{k+2}(z)F\big(z^2\big) = a_k(z)F\big(z^2\big)z - \sigma(z)^{k-4}(k+2)/256 \cdot \delta\big(z^2\big)^4 F\big(z^2\big).$$

Provided that $k \geqslant 4$ and that the data F is sampled uniformly from a cubic polynomial, the second term on the right hand side vanishes and so the new data after refinement with the scheme S_{k+2} is equal to that after refinement with S_k, except for an index shift by $+1$. Therefore, if S_k has cubic precision then so also does S_{k+2}.

As S_4 is guaranteed to reproduce cubic polynomials by construction, we conclude by induction that this property holds for all even $k \geqslant 4$. Similarly, the statement follows for all odd k above 4 because S_5 has cubic precision by construction. □

5.2. Generation degree

Besides the polynomial precision, which in itself may be a desirable property, the reproduction degree of a convergent subdivision scheme also tells how well the limit function approximates the function f from which the initial data may have been sampled. In fact, due to the compact support and the boundedness of the basic limit function, it follows that a scheme that reproduces polynomials up to degree d has an *approximation order* of $d + 1$ (Dyn, 2002b).

Levin (2003), however, showed that the approximation order derived from the reproduction degree is usually not optimal and that a simple modification of the initial data leads to an approximation order of one larger than the generation degree. In the notation of the previous section, this is realized by adapting the operator Q.

The generation degree of a subdivision scheme is the maximum degree of polynomials that can potentially be generated by the scheme, provided that the initial data is chosen "correctly". Obviously, it is not less than the reproduction degree. It turns out (Cavaretta et al., 1991, Chapter 6) that in our setting the generation degree is just one less than the number of σ-factors contained in the symbol of the scheme and that the limit function is a polynomial if and only if the initial data lies on a (potentially different) polynomial of the same degree. Moreover, both the initial and the resulting polynomial have the same leading coefficient (Levin, 2003). We conclude the following statement.

Theorem 3. *The generation degree of S_k is $k-1$.*

In other words, the subdivision scheme S_k has the same potential approximation power as the B-spline of order k, simply because its symbol contains the same number of smoothing factors σ. Note that the generation degree is equal to the reproduction degree for $k=3$ and $k=4$ and truly greater for $k \geqslant 5$.

Following Levin (2003), the optimal approximation order of the scheme S_k can be achieved by preprocessing the initial data. For example, if the subdivision scheme for the cubic B-spline is applied to the data after convolving it once with the mask

$$[-1, 8, -1]/6,$$

then the overall operator has an approximation order of 4 whereas the scheme has only quadratic approximation order if used without that preprocessing. Similarly, we can treat the given data before using the scheme S_k for $k>4$ and raise the approximation order from 4 to k. A simple computation that involves computing the unit row eigenvector of the subdivision matrix shows that the correct preprocessing mask for S_5 is

$$[5, 866, -3509, 54428, -3509, 866, 5]/49152$$

and for S_6

$$[-3, 136, -517, 5488, -517, 136, -3]/4720.$$

For greater values of k, the mask can be computed analogously with the algorithm given by Levin (2003, Section 2.5), but we did not succeed in finding a general representation in a closed form.

5.3. Interpolation degree

The third polynomial degree that we consider is the largest integer d such that the limit curve interpolates the given data whenever it has been sampled from a polynomial in Π_d. All interpolating subdivision schemes obviously have an interpolation degree of ∞ and in general, it is not less than the reproduction degree. In most cases, however, the interpolation and the reproduction degree are the same.

For primal schemes, it is straightforward to determine the values of the limit curve that correspond to the initial data, because the values of the basic limit function at the integers are given by the entries of the unit row eigenvector of the subdivision matrix. Denoting the z-transform of this eigenvector by $\ell(z)$, it follows that the subdivision scheme maps the initial data f_i^0 to the coefficients of the Laurent polynomial

$$F^\infty(z) = \ell(z)F^0(z).$$

For example, we have $\ell(z) = (1+4z+z^2)/6 = z + \delta(z)^2/6$ for the cubic B-spline scheme, and as the second term annihilates data that has been sampled from a linear polynomial, the limit values f_i^∞ are equal to the initial data f_i^0 in that case, except for an index shift by $+1$.

For dual schemes, the analysis is slightly more complicated as the unit row eigenvector of the subdivision matrix does not give the values of the basic limit function at the integers but at the odd half-integers. In order to get the values at the integers, we need to first apply the eigenvector to the data after one subdivision step. This gives the values at all half-integers and those at the integers can then be found by subtracting the values at the odd half-integers. Algebraically, this translates to

$$\tilde{\ell}(z^2) = \ell(z)a(z) - z\ell(z^2),$$

and we conclude that the initial data f_i^0 is mapped to the coefficients of

$$F^\infty(z) = \tilde{\ell}(z)F^0(z)$$

for dual schemes. For example, we have $\ell(z) = \sigma(z)$ for the quadratic B-spline with symbol $a(z) = 2\sigma(z)^3$, so that $\tilde{\ell}(z) = (1+6z+z^2)/8 = z + \delta(z)^2/8$. Again, it follows that the interpolation degree is 1 and this is actually true for all B-spline schemes.

Likewise, the interpolation degree for the members of the family $\boldsymbol{S}$ matches their reproduction degree, with only two exceptions.

Theorem 4. *The interpolation degree of S_k is ∞ for $k = 4$ and 3 for all other $k \geqslant 3$.*

Proof. The scheme S_4 is interpolating for any data, and for all $k > 4$, the statement follows because S_k reproduces cubic polynomials. For $k = 3$, the z-transform of the unit row eigenvector is

$$\ell_3(z) = \left(-1 + 9z + 9z^2 - z^3\right)/16$$

and a straightforward computation yields

$$\tilde{\ell}_3(z) = z^2 + 3/512 \cdot \delta(z)^4.$$

Therefore, the limit values f_i^∞ are equal to the initial data f_i^0, except for an index shift by $+2$, if the latter has been sampled from a polynomial of degree not larger than 3. □

This shows that the interpolation degree of the scheme S_3 is actually higher than its reproduction degree and that this scheme interpolates initial data sampled from a cubic polynomial without reproducing that cubic. We found this quite remarkable and do not know of any other non-interpolating scheme with a similar behavior, so far.

6. Artifact behavior

It is useful to consider the behavior of a curve scheme when the data has specific patterns. For example, we used cardinal data (where the value of one control point is unity and the rest zero) for analysis of the support and of the general shape of the basis function. Polynomial data (where the values of the control points are taken from a polynomial in their indexes) is used for analysis of precision and thence approximation order.

When looking at the way in which the curvature of the limit curve varies along its length, it is convenient to consider data in which the control points have values taken from a sinusoidal function. For example, the initial data could be set to

$$f_i^0 = \cos(2\pi\omega i + \eta)$$

for some positive ω and arbitrary η. In the 2D setting, if the control points lie evenly spaced around a circle, then both coordinates are sinusoidal. Note that by the Shannon sampling theorem we are considering only $\omega < 1/2$ and typically $\omega \ll 1/2$. The limit curve, however, exhibits a variation of curvature which has components at twice the Shannon limit. These cannot be corrected by movement of the control points and they are therefore an artifact of the scheme. Theory whereby the amplitude A of this artifact can be determined from the mask of the scheme is elaborated in (Sabin et al., 2005).

We can determine A as follows. Let p be the biggest power of z in the product $\ell(z)a(z)$ of the unit row eigenvector and the mask; for the family members S_k, this is just $2k + 2$ and $2k - 2$ for the order-k B-spline schemes. Then we let $\hat{P}(z) = \ell(z)a(z)z^{-p/2}$ be the symmetrized version of this Laurent polynomial and express it as a polynomial Q in the symmetrized smoothing factor $\hat{\sigma}(z) = \sigma(z)z^{-1/2}$, so that $Q(\hat{\sigma}) = \hat{P}$. The amount of artifact present in the limit curve when the data is sampled from a sinusoid with $n = 1/\omega$ samples per cycle is given by $A(\omega) = Q(\sin(\pi\omega/2))/2$; see (Sabin et al., 2005).

As an example, let us consider the scheme S_3. With $\ell_3(z)$ from the proof of Theorem 4 and $a_3(z)$ from Eq. (4), we find

$$\hat{P}_3(z) = \left(3z^4 - 32z^3 - 12z^2 + 288z + 530 + 288z^{-1} - 12z^{-2} - 32z^{-3} + 3z^{-4}\right)/512$$

and $Q_3(x) = (3x^8 - 14x^6 + 15x^4)/2$. Likewise, it is easy to see that we have $Q_4(x) = -4x^6 + 6x^4$ for the 4-point scheme S_4, and that the quadratic B-spline has $Q(x) = 2x^4$ and therefore $A(\omega) = \sin^4(\pi\omega/2)$.

The artifact magnitude in the limit curve is plotted against n for several values of k in Fig. 7 and an example for curves generated with four control points is shown in Fig. 8.

7. Summary and discussion

Besides choosing between interpolating and approximating methods, deciding on a particular subdivision scheme requires to trade off the size of the support against the level of continuity and the degree of polynomial reproduction, which in turn determines the approximation order.

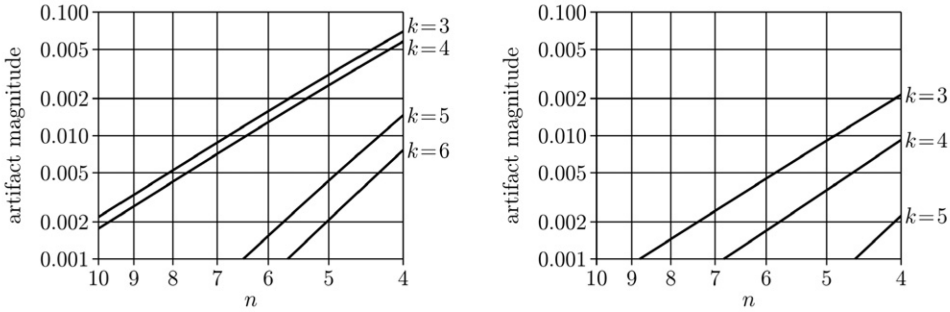

Fig. 7. Artifact magnitudes for some family members S_k (left) and B-splines of order k (right).

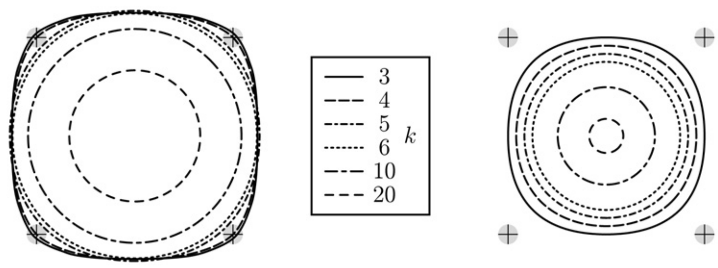

Fig. 8. Fits at $\omega = 1/4$ for some family members S_k (left) and B-splines of order k (right).

B-splines are optimal in the sense that they offer the best smoothness for a fixed mask size. Moreover, maximal polynomial reproduction and thus optimal approximation order can be obtained by modifying the initial data in a suitable way, but without such a preprocessing step, only linear functions are reproduced.

The highest degree of polynomial reproduction for a given mask size without preprocessing is given by the family of interpolating $2n$-point schemes (Deslauriers and Dubuc, 1989), but this property is traded in for a considerable loss of smoothness. Choi et al. (2006) showed that increasing the mask size by two can improve the smoothness of these schemes by roughly two levels without changing the degree of polynomial reproduction.

Likewise, our family shows that increasing the mask size by two can improve the B-spline schemes in the sense that the degree of polynomial reproduction is raised from linear to cubic, so that the approximation order is 4. This modification does not change the potential to have optimal approximation order after preprocessing the initial data and comes at the cost of one order of continuity for $k < 12$ and two orders for $12 \leqslant k < 28$.

The properties of the subdivision schemes S_k are summarized in Table 1 and for a better comparison we have listed the corresponding properties of the B-spline schemes in Table 2.

We should note that there is an interesting link between our family and pseudo-splines, which have first been introduced by Selesnick (2001) and Daubechies et al. (2003) for the construction of tight wavelet frames. Dong and Shen (2007) later extended the idea and introduced pseudo-splines of type II. It turns out that this family of refinable functions not only includes the B-splines with even order and the basis functions of the interpolatory $2n$-point schemes, but also half of the members of our family. In fact, the schemes S_{2m} are the pseudo-splines of type II with order $(m, 1)$. Thus, our result from Section 5 on the approximation order of the schemes is just a special case of (Dong and Shen, 2007, Theorem 3.10). Dong and Shen also analyzed the regularity of pseudo-splines and gave $2m - \log_2(2 + 3m/2)$ (Dong and Shen, 2007, paragraph after Theorem 3.4) as a lower bound for the Hölder exponent of S_{2m}. This is slightly smaller than the exact value of $2m - \log_2(2 + m)$ that we proved in Theorem 1. Notice that our family is larger in the sense that it also contains schemes that lie between the pseudo-splines, namely the schemes S_k with an odd index k.

Finally, we would like to mention that the schemes S_k for $k = 3, 4, 5, 6$ are also members of the family $S_{L,\omega}$ described by Choi et al. (2006) if the parameters L and ω are chosen as listed in Table 3. For $k \geqslant 7$, this correspondence breaks down and the continuity per mask width of the schemes S_k grows roughly four times as fast as that of the schemes $S_{L,\omega}$.

Table 1
Properties of the family

Properties	k					
	3	4	5	6	10	20
No. of entries in mask	6	7	8	9	13	23
Support	5	6	7	8	12	22
Height at center	1.04	1.00	0.89	0.83	0.68	0.50
Width at half-height	1.03	1.10	1.20	1.28	1.60	2.16
Hölder regularity	1.193	$2-\varepsilon$	2.840	3.678	7.193	16.415
Continuity	1	1	2	3	7	16
Reproduction degree	2	3	3	3	3	3
Generation degree	2	3	4	5	9	19
Interpolation degree	3	∞	3	3	3	3
Artifact at $\omega = 1/4$	0.0698	0.0581	0.0146	0.00764	$1.4 \cdot 10^{-4}$	$3.3 \cdot 10^{-9}$
Artifact at $\omega = 1/6$	0.0158	0.0129	0.0015	0.00071	$2.0 \cdot 10^{-6}$	$5.7 \cdot 10^{-13}$

Table 2
Properties of the B-splines

Properties	Degree					
	2	3	4	5	9	19
Order / no. of σ-factors	3	4	5	6	10	20
No. of entries in mask	4	5	6	7	11	21
Support	3	4	5	6	10	20
Height at center	0.75	0.67	0.60	0.55	0.43	0.31
Width at half-height	1.24	1.40	1.56	1.68	2.20	3.16
Hölder regularity	2	3	4	5	9	19
Continuity	1	2	3	4	8	18
Reproduction degree	1	1	1	1	1	1
Generation degree	2	3	4	5	9	19
Interpolation degree	1	1	1	1	1	1
Artifact at $\omega = 1/4$	0.0214	0.0092	0.00225	0.00077	$6.0 \cdot 10^{-6}$	$3.5 \cdot 10^{-11}$
Artifact at $\omega = 1/6$	0.0045	0.0017	0.00021	$5.5 \cdot 10^{-5}$	$6.7 \cdot 10^{-8}$	$4.2 \cdot 10^{-15}$

Table 3
Correspondence between the schemes S_k and the family of Choi et al. (2006)

	S_3	S_4	S_5	S_6
L	3	4	3	4
ω	0	0	5/128	3/128

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