Stability of Multiscale Transformations

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Abstract

After briefly reviewing the interrelation between Riesz-bases, biorthogonality and a certain stability notion for multiscale basis transformations we establish a basic stability criterion for a general Hilbert space setting. An important tool in this context is a strengthened Cauchy inequality. It is based on direct and inverse estimates for a certain scale of spaces induced by the underlying multiresolution sequence. Furthermore, we highlight some properties of these spaces pertaining to duality, interpolation, and applications to norm equivalences for Sobolev spaces.

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1 Background and Motivation

A standard framework for approximately recapturing a function v in some infinite dimensional separable Hilbert space V, say, either from explicitly given data or as a solution of an operator equation, is a *nested dense* sequence $\mathcal{S}$ of closed subspaces S_j of V, i.e.,

$$S_0 \subset S_1 \subset S_2 \subset \ldots \subset \mathcal{F}, \quad \overline{\bigcup_{j=0}^{\infty} S_j} = V, \tag{1.1}$$

where the closure is taken with respect to the norm $\|\cdot\|_V$ on V. Typical examples of interest are the space $L_2(\Omega)$ of square integrable functions on some measure space Ω or Sobolev spaces defined over Ω. A common feature of multiscale methods defined in such a framework, such as multigrid methods (see e.g. [BPX, DK, H, O1, O2, Y]), or wavelet expansions (see e.g. [M, Dau, C]) is to work in one way or another with successive *fine scale corrections* of current approximate solutions. One possible way to formulate this is to seek for appropriate decompositions of each space S_j into a direct sum of its coarser predecessor S_{j-1} and some complement W_j

$$S_j = S_{j-1} \oplus W_j. \tag{1.2}$$

In the following we will assume that each space S_j is spanned by some *Riesz basis* $\Phi_j = \{\phi_{j,k} : k \in I_j\}$ and moreover that the Φ_j are *uniformly stable*, i.e.,

$$\|\mathbf{c}\|_{\ell_2(I_j)} \sim \| \sum_{k\in I_j} c_k \phi_{j,k} \|_V. \tag{1.3}$$

Here we define for $\mathbf{c} = \{c_k\}_{k\in I_j}$ as usual $\|\mathbf{c}\|^2_{\ell_2(I_j)} := \sum_{k\in I_j} |c_k|^2$ and $a \sim b$ means that a can be bounded from below and above by a constant multiple of b where the constants are independent of any parameters a and b may depend on. In order to analyse the fine scale correction of a current approximation one is interested in a Riesz basis $\Psi_j = \{\psi_{j,k} : k \in J_j\}$ of W_j. Setting for simplicity $W_0 := S_0$, $\psi_{0,k} := \phi_{0,k}$, $k \in I_0 = J_0$, any $f_n \in S_n$ may then be written as

$$f_n = \sum_{k\in I_n} c_k \phi_{n,k}$$

in terms of the *single scale basis* Φ_n or in terms of a *multiscale expansion* as

$$f_n = \sum_{j=0}^{n} \sum_{k\in J_j} d_{j,k} \psi_{j,k}.$$

Since both representations are needed in many applications, the transformation $\mathbf{T}_n$ that takes the *multiscale coefficients* $\boldsymbol{d}^n = (d_{j,k} : k \in J_j, j = 0, \ldots, n)$ into the *single scale coefficients* $\mathbf{c}^n = (c_k : k \in I_n)$ plays an important role. Firstly, it should be efficiently executable, which is closely related to the locality of the basis functions, an issue discussed e.g. in [D1]. To be a bit more specific, note that when the Φ_j are Riesz bases, nestedness of the S_j implies the validity of *refinement relations*

$$\phi_{j,k} = \sum_{l\in I_j} a^j_{l,k} \phi_{j+1,l}, \tag{1.4}$$

while the basis functions of the complement spaces W_j must have the form

$$\psi_{j,k} = \sum_{l\in J_j} a^{j-1}_{l,k} \phi_{j,l}. \tag{1.5}$$

Locality means roughly that rows and columns of these refinement matrices contain a uniformly bounded number of nonvanishing entries. If this is the case and if $N_j := \dim S_j$ is finite and grows exponentially, the application of $\mathbf{T}_j$ requires only $\mathcal{O}(N_j)$ operations.

Secondly, the $\mathbf{T}_j$ should be *well conditioned*, which is the central theme of this paper. Again keeping the case $\dim S_j < \infty$ in mind, a natural interpretation of $\mathbf{T}_n$ being well-conditioned is to require that the spectral condition numbers of the transformations $\mathbf{T}_n$ remain uniformly bounded. Thus, in general, it makes sense to require that

$$\|\mathbf{T}_n\|, \ \left\|\mathbf{T}_n^{-1}\right\| = \mathcal{O}(1), \quad n \to \infty , \tag{1.6}$$

where $\|\cdot\|$ denotes the spectral norm of $\mathbf{T}_n$ as a mapping from $\ell_2(I_n)$ into itself.

One easily verifies the following observation (see e.g. [D1]).

Remark 1.1 *The $\mathbf{T}_n$ are well conditioned or stable in the sense of (1.6) if and only if*

$$\Psi = \bigcup_{j=0}^{\infty} \Psi_j = \{\psi_{j,k} : k \in J_j, j \in \mathbb{N}_0\}, \quad \mathbb{N}_0 = \{0, 1, 2, \ldots\},$$

is a Riesz-Basis of V, i.e., every $f \in V$ has a unique expansion

$$f = \sum_{j=0}^{\infty} \sum_{k \in J_j} d_{j,k}(f) \psi_{j,k}, \tag{1.7}$$

such that

$$\|f\|_V \sim \left(\sum_{j=0}^{\infty} \sum_{k \in J_j} |d_{j,k}(f)|^2 \right)^{\frac{1}{2}}. \tag{1.8}$$

The following familiar fact is a consequence of the Riesz Representation Theorem (see e.g. [D1]).

Remark 1.2 *(1.7) and (1.8) are equivalent to the existence of another Riesz-Basis $\tilde{\Psi} = \{\tilde{\psi}_{j,k} : k \in J_j, j \in \mathbb{N}_0\}$ which is biorthogonal to Ψ, i.e.*

$$\left\langle \psi_{j,k}, \tilde{\psi}_{j',k'} \right\rangle = \delta_{(j,k),(j',k')}, \quad (j,k), (j',k') \in J := \{(j,k) : k \in J_j, j \in \mathbb{N}_0\}, \tag{1.9}$$

where $\langle f, g \rangle$ denotes the inner product on V.

Of course, if Ψ is an orthonormal system, (1.8) is trivially satisfied. One then has $\Psi = \tilde{\Psi}$, and the W_j are orthogonal complements. Elegant explicit constructions of such orthonormal bases are available when $V = L_2(\mathbb{R}^s)$ and the elements $f_j \in S_j$ are dilates $f_j = f(2^j \cdot)$ of elements $f \in S_0$ which in turn is typically generated by the integer translates $\phi(\cdot - k)$, $k \in \mathbb{Z}^s$, of some function $\phi \in L_2(\mathbb{R}^s)$. Such spaces are often referred to as *principal shift-invariant spaces* [BDR]. In this case the refinement relation (1.4) takes the simpler form

$$\phi = \sum_{k \in \mathbb{Z}^s} a_k \phi(2 \cdot -k) \tag{1.10}$$

for some sequence $\boldsymbol{a} = (a_k)_{k \in \mathbb{Z}^s}$. Thus, one only has to determine a basis for W_1 whose dilates will then generate all the other orthogonal complements as well. This choice gives rise to the concept of wavelets (see e.g. [M, Dau, C]).

But even for $V = L_2(\mathbb{R}^s)$, the construction of orthonormal wavelet bases consisting of (non tensor product) compactly supported functions is known to be rather difficult. So when having to dispense with the structural advantages of shift-invariant spaces in the above sense, it is particularly important to note that for (1.8) to hold it would suffice to require that the $\psi_{j,k}$ be only orthogonal between levels, i.e.

$$\langle \psi_{j,k}, \psi_{j',k} \rangle = 0, \quad j \neq j', \tag{1.11}$$

as long as the Ψ_j are uniformly stable, i.e.,

$$\|\boldsymbol{c}\|_{\ell_2(J_j)} \sim \left\| \sum_{k \in J_j} c_k \psi_{j,k} \right\|_V . \tag{1.12}$$

Of course, one no longer has $\Psi = \tilde{\Psi}$. For shift-invariant subspaces S_j of $V = L_2(\mathbb{R}^s)$ many such examples can be found in the literature [CSW, SR].

Remark 1.1 also says that even orthogonality between levels is not necessary, but all possible leeway is exhausted by the biorthogonality described in Remark 1.2. Again examples of such biorthogonal Riesz-bases have been constructed for the principal shift-invariant multiresolution framework for $L_2(\mathbb{R}^s)$ ([CDF]) and a complete characterization of stability in the above sense is given in [CD1]. These results make crucial use of Fourier techniques (see also [CD2, V]). Unfortunately, the application of multiscale techniques in connection with integral or differential equations usually requires working with bounded domains or even with closed manifolds which excludes shift-invariance and corresponding Fourier techniques.

Therefore the central objective of this paper is to establish stability criteria which are not based on Fourier techniques and might thus open up a wider range of applications. Brief indications of such applications will be given later below. In particular, the question arises whether biorthogonality which, by the above remarks, comes in automatically, is also sufficient to ensure stability.

To this end, it is helpful to reformulate the problem somewhat which will lead us to a convenient reinterpretation of biorthogonality without actually specifying bases beforehand. Suppose for a moment that Ψ is a Riesz-basis. By Remark 1.2, this gives rise to mappings

$$Q_n f := \sum_{j=0}^{n} \sum_{k \in J_j} \left\langle f, \tilde{\psi}_{j,k} \right\rangle \psi_{j,k}, \quad Q'_n f := \sum_{j=0}^{n} \sum_{k \in J_j} \left\langle f, \psi_{j,k} \right\rangle \tilde{\psi}_{j,k} \tag{1.13}$$

which are obviously projectors and dual to each other. Evidently, (1.8) is equivalent to

$$\|f\|_V \sim \left(\sum_{j=0}^{\infty} \|(Q_j - Q_{j-1}), f\|_V^2 \right)^2 \tag{1.14}$$

where here and in the sequel we set for convenience $Q_{-1} := 0$.

Thus the problem of constructing well conditioned multiscale transformations may be conceptually reformulated in two steps as follows. Given a dense nested sequence $\mathcal{S}$, the first step consists in finding an associated uniformly bounded sequence $\mathcal{Q}$ of linear projectors $Q_j : V \to S_j$, $j \in \mathbb{N}_0$, such that (1.14) holds. The second step is then to identify stable bases $\Psi_j = \{\psi_{j,k} : k \in J_j\}$ of the *detail spaces*

$$W_j := (Q_j - Q_{j-1})V = (Q_j - Q_{j-1})S_j.$$

The degree of difficulty of the second step depends very much on the particular situation at hand. In some applications like preconditioning symmetric positive definite matrices stemming from Galerkin discretizations of elliptic boundary value problems, it suffices to know (1.14) without ever making explicit use of the bases Ψ_j of W_j ([Y, X]). In other situations, Ψ_j can be constructed once the Q_j are known and *some* stable basis $\hat{\Psi}_j$ of *some (initial)* complement $\hat{W}_j$ of S_{j-1} in S_j is given (see e.g. [CDP]).

The first step, i.e., the question when do projectors Q_j yield the norm equivalence (1.14), will be seen to be tractable in much greater generality and in a completely basis free context. It will be this issue and some consequences which we will focus on for the rest of the paper.

This problem has been already addressed in [D1]. However, A. Cohen recently pointed out to me that the proof of a corresponding claim formulated there contained a flaw which turned out to be serious. In fact, as grateful as I am for A. Cohen's comments as embarrassed I am to admit that Theorem 3.2 as stated in [D1] is not correct. In the following a completely different approach will be employed to establish stability results for a general Hilbert space setting whose present formulation has benefitted to a great deal from private communications with A. Cohen and Y. Meyer.

2 Some Further Prerequisites

To see which properties of the Q_j are relevant for an equivalence of the form (1.14) note that (1.7) and (1.8) entail the following obvious properties:

- The Q_j and hence the adjoints Q'_j are *uniformly bounded.*
- The ranges $\tilde{S}_j$ of the Q'_j are also nested.

At this point it is useful to keep the following facts in mind which can actually be established in a somewhat more general setting where the role of the Hilbert space V is played by a reflexive Banach space (see e.g. [D1]). Nevertheless, for the remainder of this section $\mathcal{S}$ will be as before a dense nested sequence of closed subspaces of some Hilbert space V and $\mathcal{Q}$ will be an associated sequence of uniformly bounded linear projectors Q_j from V onto S_j.

Remark 2.1 *The following facts are equivalent:*

(i) The ranges $\tilde{S}_j$ of the adjoints Q'_j are also nested.

(ii) The Q_j commute, i.e.,

$$Q_\ell Q_j = Q_\ell \quad \textit{for all} \quad \ell \leq j. \tag{2.1}$$

(iii) The operators $Q_j - Q_{j-1}$ are also projectors.

As an immediate consequence of (i) and (ii) one has

Remark 2.2 *The sequence $\mathcal{Q}'$ of adjoints Q'_j of the Q_j also satisfies (2.1).*

Setting

$$\|Q_j\|_V = \sup_{v\in V, \|v\|_V=1} \|Q_j v\|_V,$$

the classical Lebesgue estimate gives

$$\|Q_n v - v\|_V \leq (1 + \|Q_n\|_V) \inf_{v_n \in S_n} \|v - v_n\|_V \lesssim \inf_{v_n\in S_n} \|v - v_n\|_V \tag{2.2}$$

for any $v \in V$ where $a \lesssim b$ means that a can be bounded by a constant multiple of b where the constant is independent of any parameter a and b may depend on. Denseness of $\mathcal{S}$ combined with (2.2) ensures that $\lim_{n\to\infty} \|Q_n v - v\|_V = 0$ for $v \in V$. This leads to the following observation.

Remark 2.3 *Suppose now that $\mathcal{Q}$ satisfies (2.1).Since $\|Q_n\|_V = \|Q_n'\|_V$, a relation analogous to (2.2) is also valid for the sequence $\tilde{\mathcal{S}}$ of ranges $\tilde{S}_j$ of the adjoints Q_j' which, by Remark 4.4 and (2.1), are also nested. Combining the reflexivity of V with a separation argument, reveals that $\tilde{\mathcal{S}}$ is dense as well (see [D1]) so that the expansions*

$$v = \sum_{j=0}^{\infty} (Q_j - Q_{j-1})v, \quad v = \sum_{j=0}^{\infty} (Q_j' - Q_{j-1}')v, \quad Q_{-1} = Q_{-1}' := 0, \tag{2.3}$$

converge strongly in V.

When trying to construct suitable multiscale bases Ψ through a sequence $\mathcal{Q}$ one has to define the projectors Q_j first in terms of the single scale bases Φ_j. It is therefore useful to reinterpret condition (2.1) in this context [D1].

Remark 2.4 *Suppose that the collections $\tilde{\Phi}_j \subset V$ are biorthogonal to the Φ_j, i.e.,*

$$\langle \phi_{j,k}, \tilde{\phi}_{j,k'} \rangle = \delta_{k,k'}, \quad k, k' \in I_j.$$

If the projectors

$$Q_j f = \sum_{k\in I_j} \langle f, \tilde{\phi}_{j,k}\rangle \phi_{j,k},$$

are uniformly bounded then the $\tilde{\Phi}_j$ are also uniformly stable. Moreover, condition (2.1) is equivalent to the fact that the $\tilde{\Phi}_j$ are also refinable, i.e.,

$$\tilde{\phi}_{j,k} = \sum_{l\in I_{j+1}} \tilde{a}^j_{l,k} \tilde{\phi}_{j+1,l}, \quad k \in I_j,$$

and the matrices $\mathbf{A}_j = (a^j_{l,k})_{l\in I_{j+1}, k\in I_j}$, $\tilde{\mathbf{A}} = (\tilde{a}^j_{l,k})_{l\in I_{j+1}, k\in I_j}$ satisfy $\tilde{\mathbf{A}}_j'\mathbf{A}_j = I$ where $\tilde{\mathbf{A}}'$ denotes the adjoint, i.e., the complex conjugate transpose of $\mathbf{A}_j$.

The fact that Φ_j, $\tilde{\Phi}_j$ are both uniformly stable bases ensures that the $\mathbf{A}_j, \tilde{\mathbf{A}}_j$ are matrix representations of uniformly bounded operators from $\ell_2(I_j)$ into $\ell_2(I_{j+1})$. One can deduce from these facts that the products $\tilde{\mathbf{A}}_j'\mathbf{A}_j$ are still well-defined when $\#I_j = \infty$ (see [CDP] for details).

3 Main Results

As before let V be a Hilbert space endowed with a norm $\|\cdot\|_V$ which is induced in a canonical way by the inner product $\langle\cdot,\cdot\rangle$ on V. We will assume that V is continuously imbedded into a larger topological linear Hausdorff space $\mathcal{H}$. We will consider *Gelfand triples* $U \subset V \subset U'$ where for any dense Banach subspace U of V the space U' is its dual relative to the dual form $\langle\cdot,\cdot\rangle_{U\times U'}$, induced by $\langle\cdot,\cdot\rangle_{V\times V} = \langle\cdot,\cdot\rangle$. Therefore, if there is no risk of confusion, we will simply write $\langle\cdot,\cdot\rangle$ instead of $\langle\cdot,\cdot\rangle_{U\times U'}$.

Adhering to the above notation, $\mathcal{S}$ will be an ascending dense sequence of nested closed subspaces S_j of V and $\mathcal{Q}$ will stand for an associated sequence of linear projectors Q_j from V onto S_j. In view of the previous observations, we will always assume that the Q_j are uniformly bounded relative to $\|\cdot\|_V$. Setting

$$N_{\mathcal{Q}}(v) := \left(\sum_{j=0}^{\infty} \|(Q_j - Q_{j-1})v\|_V^2\right)^{\frac{1}{2}},$$

our objective is to investigate the validity of the norm equivalence

$$\|\cdot\|_V \sim N_{\mathcal{Q}}(\cdot) \tag{3.1}$$

in the present general context. The first useful observation is that (3.1) can be reduced to the validity of pairs of *one sided* estimates.

Theorem 3.1 *Let $\mathcal{S}$ be a dense nested sequence of closed subspaces of the Hilbert space V and let $\mathcal{Q}$ be an associated sequence of uniformly bounded linear projectors satisfying (2.1), i.e.*

$$Q_l Q_j = Q_l \quad \textit{for all} \quad l \le j.$$

Then the following statements are equivalent:

(i)

$$N_{\mathcal{Q}}(v) \lesssim \|v\|_V, \quad N_{\mathcal{Q}'}(v) \lesssim \|v\|_V, \quad v \in V;$$

(ii)

$$\|v\|_V \lesssim N_{\mathcal{Q}}(v), \quad \|v\|_V \lesssim N_{\mathcal{Q}'}(v), \quad v \in V;$$

(iii)

$$\|\cdot\|_V \sim N_{\mathcal{Q}}(\cdot) \quad \textit{or} \quad \|\cdot\|_V \sim N_{\mathcal{Q}'}(\cdot);$$

(iv)

$$N_{\mathcal{Q}'}(\cdot) \sim \|\cdot\|_V \sim N_{\mathcal{Q}}(\cdot).$$

Proof: In an earlier version of the paper a proof of Theorem 3.1 was arranged to emerge as a by product of duality (see Remark 5.1 below in Section 5.1.) Although duality will be used later on anyway it is worth presenting at this point the following more direct and much shorter argument due to A. Cohen. It was independently brought to my attention by H. Esser. The key is the following observation.

Remark 3.1 *One has*

$$N_{\mathcal{Q}'}(v) \lesssim \|v\|_V, \quad v \in V, \tag{3.2}$$

if and only if

$$\|v\|_V \lesssim N_{\mathcal{Q}}(v), \quad v \in V. \tag{3.3}$$

The fact that (3.2) implies (3.3) has been used in numerous proofs of the Riesz basis property of wavelet bases (see e.g. [CDF, V]). In fact, one infers from (2.1), Remark 2.3, and Cauchy-Schwarz' inequality that

$$\begin{aligned}
\|v\|_V^2 &= \langle \sum_{j=0}^{\infty}(Q_j - Q_{j-1})v, \sum_{j=0}^{\infty}(Q_j' - Q_{j-1}')v\rangle = \sum_{j=0}^{\infty}\langle (Q_j - Q_{j-1})v, (Q_j' - Q_{j-1}')v\rangle \\
&\leq \sum_{j=0}^{\infty} \|(Q_j - Q_{j-1})v\|_V \left\|(Q_j' - Q_{j-1}')v\right\|_V \\
&\leq \left(\sum_{j=0}^{\infty} \|(Q_j - Q_{j-1})v\|_V^2\right)^{1/2} \left(\sum_{j=0}^{\infty} \left\|(Q_j' - Q_{j-1}')v\right\|_V^2\right)^{1/2}.
\end{aligned}$$

Using (3.2) and dividing by $\|v\|_V$, yields (3.3).

Now suppose that (3.3) holds. Keeping Remark 2.3 in mind and using Cauchy Schwarz' inequality, yields

$$\begin{aligned}
N_{\mathcal{Q}'}^2(v) &= \sum_{j=0}^{\infty}\langle (Q_j - Q_{j-1})(Q_j' - Q_{j-1}')v, v\rangle = \langle \sum_{j=0}^{\infty}(Q_j - Q_{j-1})(Q_j' - Q_{j-1}')v, v\rangle \\
&\leq \|\sum_{j=0}^{\infty}(Q_j - Q_{j-1})(Q_j' - Q_{j-1}')v\|_V \|v\|_V. \qquad (3.4)
\end{aligned}$$

Since by (2.1), $(Q_j - Q_{j-1})(Q_l - Q_{l-1}) = (Q_j - Q_{j-1})\delta_{j,l}$ our assumption means that whenever $\sum_{j=0}^{\infty}(Q_j - Q_{j-1})(Q_j' - Q_{j-1}')v$ converges one has

$$\|\sum_{j=0}^{\infty}(Q_j - Q_{j-1})(Q_j' - Q_{j-1}')v\|_V \lesssim \left(\sum_{j=0}^{\infty} \|(Q_j - Q_{j-1})(Q_j' - Q_{j-1}')v\|_V^2\right)^{1/2}.$$

Using the uniform boundedness of $\mathcal{Q}$, one readily infers now from (3.4) that

$$N_{\mathcal{Q}'}^2(v) \lesssim N_{\mathcal{Q}'}(v)\|v\|_V,$$

whence the claimed equivalence of (3.2) and (3.3) follows.

The assertion of Theorem 3.1 follows now directly from Remark 3.1. □

The following comment by Y. Meyer shows that condition (2.1) alone is not sufficient to imply (3.1). In every separable Hilbert space one can find a Schauder basis which is not a Riesz basis. Let $\{v_j\}_{j\in\mathbb{N}}$ be such basis and let v_j' the corresponding biorthogonal coefficient functionals. Set $S_j := \text{span}\,\{v_1, \ldots, v_j\}$ and let Q_j denote the corresponding expansion operator, defined by $Q_n v := \sum_{j=1}^{n}\langle v, v_j'\rangle v_j$, it is clear that the S_j form a dense nested sequence of closed subspaces of V, that $\mathcal{Q}$ is uniformly bounded and that

(2.1) holds. However, since by assumption, the v_j do not form a Riesz basis it is clear that (3.1) cannot hold.

Thus the question arises under which additional assumptions imbeddings of the form (i) or (ii) in Theorem 3.1 are valid. To this end, recall that in the principal shift-invariant case the Riesz basis property implies that the spaces in $\mathcal{S}$ and $\tilde{\mathcal{S}}$ do not only belong to $L_2(I\!R)$ but must lie in some Sobolev space with positive exponent [CD2, V] which in turn can be seen to imply the validity of certain *inverse* inequalities for the elements in the spaces S_j (see e.g. [D2]). Moreover, refinability is known to imply a certain minimum approximation power [CDM]. The relevance of these facts becomes clear in the light of the following set of conditions which imply the validity of the imbedding (ii) in Theorem 3.1.

Theorem 3.2 *Suppose that for $V, \mathcal{S}$ and $\mathcal{Q}$ as above there exists a dense Banach space $U \subset V$ and some $\varrho > 1$ such that for some $\delta > 0$*

$$\inf_{v_n \in S_n} \|v - v_n\|_{U'} \lesssim \varrho^{-n\delta} \|v\|_V, \quad \inf_{v_n \in S_n} \|v - v_n\|_V \lesssim \varrho^{-n\delta} \|v\|_U, \tag{3.5}$$

as well as

$$\|v_n\|_U \lesssim \varrho^{n\delta} \|v_n\|_V, \quad \|v_n\|_V \lesssim \varrho^{n\delta} \|v_n\|_{U'}, \tag{3.6}$$

and that $\mathcal{Q}$ is uniformly bounded on U'. Then

$$\|\cdot\|_V \lesssim N_{\mathcal{Q}}(\cdot), \quad \|\cdot\|_V \lesssim N_{\mathcal{Q}'}(\cdot),$$

and hence, by Theorem 3.1,

$$\|\cdot\|_V \sim N_{\mathcal{Q}}(\cdot) \sim N_{\mathcal{Q}'}(\cdot).$$

Proof: We employ a familiar technique from multilevel finite element analysis often referred to as *strenghtened Cauchy inequality* (see e.g. [X]). Estimating

$$\langle (Q_j - Q_{j-1})v, (Q_i - Q_{i-1})v \rangle \le \begin{cases} \|(Q_j - Q_{j-1})v\|_{U'} \|(Q_i - Q_{i-1})v\|_U & \text{if} \quad i \le j, \\ \|(Q_j - Q_{j-1})v\|_U \|(Q_i - Q_{i-1})v\|_{U'} & \text{if} \quad i > j, \end{cases}$$

and employing (3.5), (3.6), yields

$$\begin{aligned} \|v\|_V^2 &= \sum_{i,j=0}^{\infty} \langle (Q_j - Q_{j-1})v, (Q_i - Q_{i-1})v \rangle \\ &\lesssim \sum_{i,j=0}^{\infty} \varrho^{-|i-j|\delta} \|(Q_i - Q_{i-1})v\|_V \|(Q_j - Q_{j-1})v\|_V \\ &\lesssim \sum_{j=0}^{\infty} \|(Q_j - Q_{j-1})v\|_V^2. \end{aligned}$$

This proves the first part of the claim.

Moreover, taking $\|v\|_{U'} = \sup_{\|w\|_U \le 1} |\langle w, v \rangle|$ into account, combining (2.2) with straightforward duality arguments, and using the uniform boundedness of $\mathcal{Q}$ on U', one easily shows that the validity of (3.5) and (3.6) for $\mathcal{S}$ implies the validity of (3.5) and (3.6) for $\tilde{\mathcal{S}}$ too. Thus the above reasoning can be applied to $\mathcal{Q}'$ as well which completes the proof. $\square$

The assumptions made in Theorem 3.2 are still not very practical. Our main objective is therefore to find next a criterion which poses conditions only on $\mathcal{S}$ and $\tilde{\mathcal{S}}$. To this end, the classical *Bernstein* and *Jackson* inequalities suggest the following setup for measuring regularity and approximation power without specifying a particular subspace beforehand. Let

$$\omega : V \times \mathbb{R}_+ \to \mathbb{R}$$

have the following properties:

$$\begin{aligned} \omega(v,t) &\lesssim \|v\|_V, \quad v \in V, \\ \lim_{t\to 0^+} \omega(v,t) &= 0, \quad v \in V, \\ \omega(u+v,t) &\leq \omega(u,t) + \omega(v,t), \quad u,v \in V. \end{aligned} \tag{3.7}$$

We will refer to ω satisfying (3.7), as a *modulus*. $\mathcal{S}$ is said to satisfy a Jackson inequality (relative to ω) if there exists some $\varrho > 1$ such that

$$\inf_{v_n \in S_n} \|v - v_n\|_V \lesssim \omega(v, \varrho^{-n}), \quad v \in V. \tag{3.8}$$

$\mathcal{S}$ satisfies a *Bernstein-inequality* if for some $\gamma > 0$

$$\omega(v_n, t) \lesssim (\min\{1, t\varrho^n\})^\gamma \|v_n\|_V, \quad v_n \in S_n. \tag{3.9}$$

It will be seen that the validity of such inverse and direct estimates imply the imbeddings (ii) in Theorem 3.1.

Theorem 3.3 *Assume that in addition to the hypotheses of Theorem 3.1 there exists a modulus ω such that $\mathcal{S}$ and $\tilde{\mathcal{S}}$ satisfy a Jackson inequality (3.8) and a Bernstein inequality (3.9) with some constants $\gamma, \gamma' > 0$, respectively. Then*

$$\|\cdot\|_V \sim N_{\mathcal{Q}}(\cdot) \sim N_{\mathcal{Q}'}(\cdot).$$

Before proving Theorem 3.3 we pause to make a few comments. A possible strategy for constructing projectors Q_j satisfying (2.1), is based on Remark 2.4. Solving first the discrete problems $\tilde{\mathbf{A}}_j'\mathbf{A}_j$ to obtain candidates for the dual refinement matrices, one attempts to build the dual bases $\tilde{\Phi}_j$ via a limit process which is often called *subdivision scheme* [D1]. However, in many cases a biorthogonal system with the above properties can be constructed by modifying an initially given system defined on all of $\mathbb{R}^n$, say. In this case one gets around a possibly difficult convergence analysis and Theorem 3.1 or Theorem 3.3 applies directly. Examples of such circumstances are encountered when establishing stability for wavelet bases adapted to bounded domains [CDD], when constructing compactly supported divergence free wavelets (see e.g. [U]), or when constructing pressure and velocity trial spaces with built in Ladyšenskaja-Babuška-Brezzi condition for the Stokes problem [DKU].

4 Proof of Theorem 3.3

Our goal is to show that under the assumptions of Theorem 3.3 one can always construct a subspace U of V for which the hypotheses of Theorem 3.2 are satisfied so that the assertion of Theorem 3.3 follows from Theorem 3.2.

4.1 Spaces Induced by $\mathcal{Q}$

As candidates for U in Theorem 3.2 we introduce the following scale of auxiliary spaces. Fix some $\varrho > 1$ and let $A^\tau_{\mathcal{Q}}$ for $\tau > 0$ denote the space of those elements in V for which

$$\|v\|_{A^\tau_{\mathcal{Q}}} := \left(\sum_{j=0}^{\infty} \varrho^{2\tau j} \, \|(Q_j - Q_{j-1})v\|_V^2\right)^{\frac{1}{2}} \tag{4.1}$$

is finite. Standard arguments confirm that $A^\tau_{\mathcal{Q}}$ is a Banach space (see e.g. [BS] or [BL], Lemma 2.2.1).

Remark 4.1 *The spaces $A^\tau_{\mathcal{Q}}$ are reflexive (relative to the duality pairing induced by the inner product $\langle\cdot,\cdot\rangle$ on V).*

Proof: Defining

$$\langle v, w\rangle_\tau := \sum_{j=0}^{\infty} \varrho^{2\tau j} \, \langle (Q_j - Q_{j-1})v, (Q_j - Q_{j-1})w\rangle \, ,$$

this can be deduced from the fact that $A^\tau_{\mathcal{Q}}$ is a Hilbert space relative to $\langle\cdot,\cdot\rangle_\tau$. Alternativeley, one can argue as follows. Since by Lemma 1.11.1 in [T], one has for

$$\ell_2(V) := \{\mathbf{v} = \{v_j\}_{j\in\mathbb{N}_0} : v_j \in V, \ \|\mathbf{v}\|_{\ell_2(V)} := \left(\sum_{j=0}^{\infty} \|v_j\|_V^2\right)^{1/2} < \infty\},$$

that $(\ell_2(V))' = \ell_2(V)$, the spaces $\ell_2(V)$ are reflexive. Next note that the mapping

$$\sigma_\tau : v \ \mapsto \ \{\varrho^{j\tau}(Q_j - Q_{j-1})v\}_{j=0}^{\infty}$$

is an injective bounded linear operator from $A^\tau_{\mathcal{Q}}$ into $\ell_2(V)$. Since $A^\tau_{\mathcal{Q}}$ is a Banach space

$$U_\tau := \sigma_\tau(A^\tau_{\mathcal{Q}})$$

is a closed subspace of $\ell_2(V)$ and hence also reflexive. Since σ_τ is an isometry onto U_τ the assertion follows. $\square$

Remark 4.2 *For any $\tau > 0$ one has*

$$A^\tau_{\mathcal{Q}} \hookrightarrow V, \tag{4.2}$$

as well as

$$A^\alpha_{\mathcal{Q}} \hookrightarrow A^\beta_{\mathcal{Q}}, \quad 0 < \beta \le \alpha. \tag{4.3}$$

Proof: (4.2) follows from

$$\begin{aligned}\|v\|_V &\le \sum_{j=0}^{\infty} \|(Q_j - Q_{j-1})v\|_V \\ &\le \left(\sum_{j=0}^{\infty} \varrho^{-2\tau j}\right)^{1/2} \left(\sum_{j=0}^{\infty} \varrho^{2\tau j}\|(Q_j - Q_{j-1})v\|_V^2\right)^{1/2} \lesssim \|v\|_{A_{\mathcal{Q}}^{\tau}},\end{aligned}$$

while (4.3) is trivial. □

Remark 4.3 *For each fixed $\tau > 0$, the spaces $A_{\mathcal{Q}}^{\tau}$ are dense in V. Moreover, one has*

$$V \hookrightarrow \left(A_{\mathcal{Q}}^{\tau}\right)', \quad V \hookrightarrow \left(A_{\mathcal{Q}'}^{\tau}\right)' \tag{4.4}$$

and V is dense in $\left(A_{\mathcal{Q}}^{\tau}\right)'$ and $\left(A_{\mathcal{Q}'}^{\tau}\right)'$.

Proof: Since obviously each S_j is contained in $A_{\mathcal{Q}}^{\tau}$ the first part of the assertion is a consequence of the denseness of $\mathcal{S}$. The rest of the assertion follows by duality and Remark 4.1. □

In the following it is useful to keep in mind that, by Remarks 2.1-2.3,

Remark 4.4 *whatever will be said about the spaces $A_{\mathcal{Q}}^{\tau}$ has an analogous counterpart for the spaces $A_{\mathcal{Q}'}^{\tau}$.*

Proposition 4.1 *For $0 < \tau$ one has $\|Q_n\|_{A_{\mathcal{Q}}^{\tau}} \le 1$, where as usual* $\|Q_n\|_U := \sup_{\|v\|_U \le 1} \|Q_n v\|_U$.

Proof: By (2.1), one has

$$\|Q_n v\|_{A_{\mathcal{Q}}^{\tau}}^2 = \sum_{j=0}^{\infty} \varrho^{2j\tau} \|(Q_j - Q_{j-1})Q_n v\|_V^2 = \sum_{j=0}^{n} \varrho^{2j\tau} \|(Q_j - Q_{j-1})v\|_V^2 \le \|v\|_{A_{\mathcal{Q}}^{\tau}}^2 .$$

□

We will make use of the following counterpart for the dual spaces.

Proposition 4.2 *The projectors Q_n and Q_n' possess bounded extensions to $\left(A_{\mathcal{Q}'}^{\delta}\right)'$, $\left(A_{\mathcal{Q}}^{\delta}\right)'$, respectively, i.e., denoting these extensions again by Q_n, Q_n', respectively, one has*

$$\|Q_n\|_{(A_{\mathcal{Q}'}^{\delta})'},\ \|Q_n'\|_{(A_{\mathcal{Q}}^{\delta})'} \le 1.$$

Proof: Let $v \in V$. By definition of the norms of a dual space, one has

$$\begin{aligned}\|Q_n v\|_{\left(A_{\mathcal{Q}'}^{\delta}\right)'} &= \sup_{\|u\|_{A_{\mathcal{Q}'}^{\delta}}=1} |\langle Q_n v, u\rangle| = \sup_{\|u\|_{A_{\mathcal{Q}'}^{\delta}}=1} |\langle v, Q_n' u\rangle| \\ &\le \sup_{\|u\|_{A_{\mathcal{Q}'}^{\delta}}=1} \|v\|_{\left(A_{\mathcal{Q}'}^{\delta}\right)'} \|Q_n' u\|_{A_{\mathcal{Q}'}^{\delta}} .\end{aligned}$$

In view of Remark 4.3, the assertion follows from Proposition 4.1 applied to $\mathcal{Q}'$ and continuous extension. □

4.2 Direct and Inverse Estimates

It will be seen next that natural pairs of direct and inverse estimates hold with respect to the spaces $A^\tau_{\mathcal{Q}}$.

Proposition 4.3 *For $0 < \tau$ one has*

$$\|Q_n v - v\|_V \lesssim \varrho^{-n\tau} \|v\|_{A^\tau_{\mathcal{Q}}}, \quad v \in A^\tau_{\mathcal{Q}}.$$

Proof: Since by (2.1),

$$v - Q_n v = \sum_{j=0}^{\infty}(Q_j - Q_{j-1})(v - Q_n v) = \sum_{j=n+1}^{\infty} \varrho^{-j\tau} \varrho^{j\tau}(Q_j - Q_{j-1})v,$$

triangle and Schwarz inequality provide

$$\|v - Q_n v\|_V \leq \left(\sum_{j=n+1}^{\infty} \varrho^{-j\tau}\right) \|v\|_{A^\tau_{\mathcal{Q}}},$$

whence the assertion follows. □

Proposition 4.4 *For $0 < \tau$ one has*

$$\|v_n\|_{A^\tau_{\mathcal{Q}}} \lesssim \varrho^{n\tau} \|v_n\|_V, \qquad v_n \in S_n, \tag{4.5}$$

where as in Proposition 4.3 the constants may depend on τ.

Proof: Since the Q_j are uniformly bounded projectors one obtains

$$\|v_n\|^2_{A^\tau_{\mathcal{Q}}} = \sum_{j=0}^{n} \varrho^{2j\tau} \|(Q_j - Q_{j-1})v_n\|^2_V \lesssim \left(\sum_{j=0}^{n} \varrho^{2j\tau}\right) \|v_n\|^2_V.$$

□

The above direct and inverse estimates have counterparts relative to the dual spaces $\left(A^\tau_{\mathcal{Q}}\right)'$ and $\left(A^\tau_{\mathcal{Q}'}\right)'$.

Proposition 4.5 *The following inverse estimate holds*

$$\|v_n\|_V \lesssim \varrho^{n\tau} \|v_n\|_{\left(A^\tau_{\mathcal{Q}'}\right)'}, \quad v_n \in S_n.$$

Proof: By Remark 4.4, Proposition 4.4, and the uniform boundedness of $\mathcal{Q}'$, we obtain

$$\begin{aligned}
\|v_n\|_V &= \sup_{\|u\|_V=1} |\langle v_n, u\rangle| = \sup_{\|u\|_V=1} |\langle v_n, Q'_n u\rangle| \\
&\leq \sup_{\|u\|_V=1} \|v_n\|_{\left(A^\tau_{\mathcal{Q}'}\right)'} \|Q'_n u\|_{A^\tau_{\mathcal{Q}'}} \lesssim \sup_{\|u\|_V=1} \|v_n\|_{\left(A^\tau_{\mathcal{Q}'}\right)'} \varrho^{n\tau} \|u\|_V.
\end{aligned}$$

□

In the same fashion one deduces from Proposition 4.3 the following fact.

Proposition 4.6 *One has the direct estimate*

$$\|Q_n v - v\|_{(A^\tau_{\mathcal{Q}'})'} \lesssim \varrho^{-n\tau} \|v\|_V , \quad v \in V.$$

In view of the above direct and inverse estimates, it is clear that for each fixed $\tau > 0$ the space $U = A^\tau_{\mathcal{Q}}$ is a natural candidate in the context of Theorem 3.2. However, the estimates in Propositions 4.5, 4.6 do not quite fit yet since they involve the spaces $\left(A^\tau_{\mathcal{Q}'}\right)'$ instead of $\left(A^\tau_{\mathcal{Q}}\right)'$. We will show next that under the assumptions of Theorem 3.3 these spaces turn out to be equivalent for a certain range of τ. The key is to compare them with yet another scale of spaces.

4.3 Generalized Besov Spaces

Let ω be a modulus on V as defined in (3.7). For any $\tau > 0$ consider the spaces B^τ_ω of those $v \in V$ for which

$$\|v\|_{B^\tau_\omega} := \|v\|_V + \left(\sum_{j=0}^{\infty} \varrho^{2j\tau} \omega(v, \varrho^{-j})^2\right)^{\frac{1}{2}} \tag{4.6}$$

is finite. It will be pointed out below that the spaces B^τ_ω agree for appropriate choices of V with Besov- and Sobolev spaces. In order to interrelate the spaces B^τ_ω and $A^\tau_{\mathcal{Q}}$ we merely need a discrete *Hardy-type-inequality* which, in principle, is known. Since it will be used later on again and since it is needed in a slightly modified form here, we briefly recall next the relevant version and include a proof for the convenience of the reader.

Lemma 4.1 *For any (complex) sequence* $\mathbf{a} = \{a_j\}_{j \in \mathbb{N}_0}$ *let*

$$\|\mathbf{a}\|_{l_2^s} := \left(\sum_{j \in \mathbb{N}_0} \varrho^{2sj} |a_j|^2\right)^{1/2}.$$

(i) If $\mathbf{b} = \{b_j\}_{j \in \mathbb{N}_0}$ *satisfies for some* $t > 0$

$$|b_j| \lesssim \varrho^{-jt} \left(\sum_{n=0}^{j} \varrho^{nt} |a_n|\right),$$

then $\|\mathbf{b}\|_{l_2^s} \lesssim \|\mathbf{a}\|_{l_2^s}$ *for every (fixed)* $0 \le s < t$.

(ii) If

$$|b_j| \lesssim \sum_{n=j}^{\infty} |a_n|,$$

one has for each $s > 0$ *that* $\|\mathbf{b}\|_{l_2^s} \lesssim \|\mathbf{a}\|_{l_2^s}$.

Proof: (i): Cauchy-Schwarz' inequality yields

$$
\begin{aligned}
\|\mathbf{b}\|_{l_2^s}^2 &\lesssim \sum_{j=0}^{\infty} \varrho^{2j(s-t)} \left(\sum_{n=0}^{j} \varrho^{n(t-s)/2} \varrho^{n(t-s)/2} \varrho^{ns} |a_n| \right)^2 \\
&\lesssim \sum_{j=0}^{\infty} \varrho^{-j(t-s)} \left(\sum_{n=0}^{j} \varrho^{n(t-s)} \varrho^{2ns} |a_n|^2 \right) \\
&= \sum_{j=0}^{\infty} \left(\sum_{n=0}^{\infty} \varrho^{-n(t-s)} \right) \varrho^{2js} |a_j|^2 \lesssim \|\mathbf{a}\|_{l_2^s}^2.
\end{aligned}
$$

(ii): For $s > 0$ one obtains

$$
\begin{aligned}
\|\mathbf{b}\|_{l_2^s}^2 &\lesssim \sum_{j=0}^{\infty} \varrho^{2sj} \left(\sum_{n=j}^{\infty} |a_n| \right)^2 \le \sum_{j=0}^{\infty} \varrho^{2sj} \left(\sum_{n=j}^{\infty} \varrho^{-ns} \right) \left(\sum_{n=j}^{\infty} \varrho^{ns} |a_n|^2 \right) \\
&\lesssim \sum_{j=0}^{\infty} \varrho^{sj} \left(\sum_{n=j}^{\infty} \varrho^{ns} |a_n|^2 \right) = \sum_{j=0}^{\infty} \left(\sum_{m=0}^{j} \varrho^{ms} \right) \varrho^{js} |a_j|^2 \lesssim \|\mathbf{a}\|_{l_2^s}^2.
\end{aligned}
$$

□

Proposition 4.7 *Suppose that $\mathcal{S}$ satisfies (3.8) and (3.9). Let $\mathcal{Q}$ be a uniformly bounded sequence of linear projectors Q_j on V with ranges S_j. Then for $0 < \tau < \gamma$ one has*

$$A_{\mathcal{Q}}^{\tau} \cong B_{\omega}^{\tau}.$$

Proof: By (2.2) and (3.8), one has

$$\|(Q_j - Q_{j-1})v\|_V \lesssim \omega(v, \varrho^{-j}) + \omega(v, \varrho^{-j+1}).$$

Hence $\|v\|_{A_{\mathcal{Q}}^{\tau}} \lesssim \|v\|_{B_{\omega}^{\tau}}$ which confirms the continuous imbedding

$$B_{\omega}^{\tau} \hookrightarrow A_{\mathcal{Q}}^{\tau}.$$

Conversely, let $v = \sum_{n=0}^{\infty} v_n$ with $v_n \in S_n$. By (3.9), one has for $0 < \tau < \gamma$

$$
\begin{aligned}
\sum_{j=0}^{\infty} \varrho^{j\tau 2} \omega(v, \varrho^{-j})^2 &\le \sum_{j=0}^{\infty} \varrho^{j\tau 2} \left(\sum_{n=0}^{j} \omega(v_n, \varrho^{-j}) + \sum_{n=j+1}^{\infty} \omega(v_n, \varrho^{-j}) \right)^2 \\
&\lesssim \sum_{j=0}^{\infty} \varrho^{j\tau 2} \left(\sum_{n=0}^{j} \varrho^{(n-j)\gamma} \|v_n\|_V + \sum_{n=j+1}^{\infty} \|v_n\|_V \right)^2 \\
&\lesssim \sum_{j=0}^{\infty} \varrho^{j\tau 2} \left(\sum_{n=0}^{j} \varrho^{(n-j)\gamma} \|v_n\|_V \right)^2 + \varrho^{j\tau 2} \left(\sum_{n=j+1}^{\infty} \|v_n\|_V \right)^2 \\
&= \sum_{j=0}^{\infty} \varrho^{2j(\tau-\gamma)} \left(\sum_{n=0}^{j} \varrho^{n(\gamma-\tau)} \varrho^{n\tau} \|v_n\|_V \right)^2 + \sum_{j=0}^{\infty} \varrho^{j\tau 2} \left(\sum_{n=j+1}^{\infty} \varrho^{-n\tau} \varrho^{n\tau} \|v_n\|_V \right)^2 \\
&\lesssim \left(\sum_{j=0}^{\infty} \varrho^{2j\tau} \|v_j\|_V^2 \right)^{\frac{1}{2}},
\end{aligned}
$$

where we have used the discrete Hardy-type inequality from Lemma 4.1 (ii) in the last step ([DK, DJP]). Choosing $v_j = (Q_j - Q_{j-1})v$ proves the imbedding $A^\tau_{\mathcal{Q}} \hookrightarrow B^\tau_\omega$ whence the assertion follows. $\square$

We are now in a position to complete the proof of Theorem 3.3. To this end, let in the following δ be any fixed positive number satisfying

$$0 < \delta < \min\{\gamma, \gamma'\}. \tag{4.7}$$

As an immediate consequence of Proposition 4.7 one has, under the assumptions of Theorem 3.3,

$$\left(A^\delta_{\mathcal{Q}}\right)' \cong \left(A^\delta_{\mathcal{Q}'}\right)' \cong \left(B^\delta_\omega\right)'. \tag{4.8}$$

Thus, choosing $U = B^\delta_\omega$ and taking (4.8) into account, Propositions 4.3, 4.6 and Propositions 4.4, 4.5, respectively, imply that (3.5) and (3.6) hold, while Proposition 4.2 ensures that $\mathcal{Q}$ is bounded on U'. The assertion of Theorem 3.3 follows now from Theorem 3.2. $\square$

5 More about the Spaces $A^\tau_{\mathcal{Q}}$

It is worth collecting further information about the spaces $A^\tau_{\mathcal{Q}}$. We will always assume that (2.1) holds. To this end, let us first extend the definition (4.1) of $\|\cdot\|_{A^\tau_{\mathcal{Q}}}$ to $\tau \leq 0$. Note that for $\tau > 0$ and $v \in \left(A^\tau_{\mathcal{Q}}\right)'$

$$\begin{aligned}\|v\|_{\left(A^\tau_{\mathcal{Q}}\right)'} &= \sup_{\|w\|_{A^\tau_{\mathcal{Q}}} \leq 1} |\langle w, v\rangle| = \sup_{\|w\|_{A^\tau_{\mathcal{Q}}} \leq 1} |\langle \sum_{j=0}^{\infty} (Q_j - Q_{j-1})w, v\rangle| \\ &\leq \sup_{\|w\|_{A^\tau_{\mathcal{Q}}} \leq 1} \sum_{j=0}^{\infty} \varrho^{2j\tau} \|(Q_j - Q_{j-1})w\|_V \varrho^{-2j\tau} \|(Q'_j - Q'_{j-1})v\|_V\end{aligned}$$

so that Hölder's inequality yields

$$\|v\|_{\left(A^\tau_{\mathcal{Q}}\right)'} \leq \|v\|_{A^{-\tau}_{\mathcal{Q}'}}. \tag{5.1}$$

On account of Remark 4.3, we may therefore define for any $\tau \geq 0$ the space $A^{-\tau}_{\mathcal{Q}}$ as the set of those elements v in $\left(A^\alpha_{\mathcal{Q}'}\right)'$, $\alpha \geq \tau$, for which $\|v\|_{A^{-\tau}_{\mathcal{Q}}}$ is finite. Again $A^{-\tau}_{\mathcal{Q}}$ is easily seen to be a Banach space.

5.1 Duality

The usefulness of admitting indices $\tau \in \mathbb{R}$ lies in the following fact which is already suggested by (5.1).

Theorem 5.1 *For $\mathcal{S}$ and $\mathcal{Q}$ as above and any $\tau \in I\!R$ one has*

$$\left(A^\tau_{\mathcal{Q}}\right)' = A^{-\tau}_{\mathcal{Q}'}, \tag{5.2}$$

i.e., $\|\cdot\|_{\left(A^\tau_{\mathcal{Q}}\right)'} = \|\cdot\|_{A^{-\tau}_{\mathcal{Q}'}}$.

Remark 5.1 *Since $A^0_{\mathcal{Q}} \hookrightarrow V$, $\left(A^0_{\mathcal{Q}}\right)' \hookrightarrow V$ implies $V \cong A^0_{\mathcal{Q}}$, Theorem 3.1 follows also from (5.2) for $\tau = 0$.*

To prove Theorem 5.1 it remains to establish the converse estimate to (5.1). This can be confirmed by adapting an argument in [T], Section 1.11.1. (see also [D2]). We dispense here with giving the details since the assertion of Theorem 5.1 will also follow from later considerations concerning the important fact that the spaces $A^\tau_{\mathcal{Q}}$ are *interpolation spaces.*

Before addressing this issue, we note that canonical direct and inverse estimates hold uniformly in the scale of spaces $A^\tau_{\mathcal{Q}}$, $\tau \in I\!R$, assuming only (2.1).

Remark 5.2 *For any $\tau, \delta \in I\!R$, $\tau \le \delta$ one has*

$$\begin{aligned} \|Q_n v - v\|_{A^\tau_{\mathcal{Q}}} &\le \varrho^{n(\tau-\delta)}\|v\|_{A^\delta_{\mathcal{Q}}}, \quad v \in A^\delta_{\mathcal{Q}}, \\ \|v_n\|_{A^\delta_{\mathcal{Q}}} &\le \varrho^{n(\delta-\tau)}\|v_n\|_{A^\tau_{\mathcal{Q}}}, \quad v_n \in S_n. \end{aligned}$$

Proof: By (2.1), one has

$$\begin{aligned} \|Q_n v - v\|^2_{A^\tau_{\mathcal{Q}}} &= \sum_{j=n+1}^{\infty} \varrho^{2j\tau}\|(Q_j - Q_{j-1})v\|^2_V \\ &\le \varrho^{2n(\tau-\delta)}\|v\|^2_{A^\delta_{\mathcal{Q}}}. \end{aligned}$$

Similarly

$$\|v_n\|^2_{A^\delta_{\mathcal{Q}}} = \sum_{j=0}^{n} \varrho^{2j(\delta-\tau)}\varrho^{2j\tau}\|(Q_j - Q_{j-1})v_n\|^2_V \le \varrho^{2n(\delta-\tau)}\|v_n\|^2_{A^\tau_{\mathcal{Q}}},$$

which completes the proof. □

Remark 5.3 *If in addition $\|\cdot\|_V \sim \|\cdot\|_{A^0_{\mathcal{Q}}}$, i.e., when for instance the additional assumptions in Theorem 3.3 hold, the estimates in Propositions 4.3, 4.4, 4.5, and 4.6 are covered as special cases where, in view of Remark 5.2, the constants are now seen to be independent of τ.*

It will be pointed out later that the auxiliary spaces $A^\tau_{\mathcal{Q}}$ agree for certain choices of V and a certain range of τ with Sobolev spaces. Thus corresponding direct and inverse estimates are readily available for that range.

5.2 Interpolation

We will make use next of a few standard facts about interpolation of Banach spaces.

Let X_1 and X_2 be Banach spaces which are continuously imbedded in some larger topological Hausdorff space. In view of the situation at hand, it suffices to assume that $X_2 \subseteq X_1$. As usual the corresponding K-functional is then defined by

$$K(v,t,X_1,X_2) = \inf_{y\in X_2}\{\|v-y\|_{X_1} + t\,\|y\|_{X_2}\}.$$

It is well-known that the spaces

$$[X_1,X_2]_\theta := \{v\in X_1+X_2 : \|v\|_\theta^2 := \sum_{j=0}^{\infty}\varrho^{j\theta 2}K(v,\varrho^{-j},X_1,X_2)^2 < \infty\}, \tag{5.3}$$

form interpolation spaces between X_1 and X_2 (see e.g. [BL, BS, P, T]). On account of Remark 4.3, we can define interpolation spaces with $X_1 = \left(A^\delta_{\mathcal{Q}'}\right)'$ and $X_2 = A^\tau_{\mathcal{Q}}$, noting that for $\tau+\delta \geq 0$ one has $X_2 \hookrightarrow X_1$. For convenience, we will briefly write

$$K(v,t) := K(v,t^{\delta+\tau},\left(A^\delta_{\mathcal{Q}'}\right)',A^\tau_{\mathcal{Q}}). \tag{5.4}$$

Proposition 5.1 *For $\tau+\delta>0$, $0\leq\theta\leq 1$, and $\eta := \theta\tau-(1-\theta)\delta$ one has*

$$A^\eta_{\mathcal{Q}} \cong \left[\left(A^\delta_{\mathcal{Q}'}\right)', A^\tau_{\mathcal{Q}}\right]_\theta .$$

Proof: We show first that $A^\eta_{\mathcal{Q}} \hookrightarrow \left[\left(A^\delta_{\mathcal{Q}'}\right)', A^\tau_{\mathcal{Q}}\right]_\theta$. By Proposition 4.5, one has for every $y\in A^\tau_{\mathcal{Q}}$, $v\in V$

$$\begin{aligned}
\|(Q_j-Q_{j-1})v\|_V &\lesssim \varrho^{j\delta}\,\|(Q_j-Q_{j-1})v\|_{\left(A^\delta_{\mathcal{Q}'}\right)'}\\
&\leq \varrho^{j\delta}\left(\|(Q_j-Q_{j-1})(v-y)\|_{\left(A^\delta_{\mathcal{Q}'}\right)'} + \left\|(Q_j-Q_{j-1})^2 y\right\|_{\left(A^\delta_{\mathcal{Q}'}\right)'}\right),
\end{aligned}$$

where we have also used (2.1) in the last step. Propositions 4.2, 4.3 and 4.6 yield now

$$\begin{aligned}
\|(Q_j-Q_{j-1})v\|_V &\lesssim \varrho^{j\delta}\left(\|v-y\|_{\left(A^\delta_{\mathcal{Q}'}\right)'} + \varrho^{-j\delta}\,\|(Q_j-Q_{j-1})y\|_V\right)\\
&\lesssim \varrho^{j\delta}\left(\|v-y\|_{\left(A^\delta_{\mathcal{Q}'}\right)'} + \varrho^{-j(\delta+\tau)}\,\|y\|_{A^\tau_{\mathcal{Q}}}\right).
\end{aligned}$$

Since $y\in A^\tau_{\mathcal{Q}}$ was arbitrary we obtain

$$\varrho^{j\eta}\,\|(Q_j-Q_{j-1})v\|_V \lesssim \varrho^{j\theta(\tau+\delta)}K(v,\varrho^{-j})$$

so that squaring and summing over j yields

$$\left[\left(A^\delta_{\mathcal{Q}'}\right)', A^\tau_{\mathcal{Q}}\right]_\theta \hookrightarrow A^\eta_{\mathcal{Q}}.$$

Conversely, note that

$$
\begin{aligned}
K(v, \varrho^{-j}) &\leq \|v - Q_j v\|_{\left(A^\delta_{\mathcal{Q}'}\right)'} + \varrho^{-j(\tau+\delta)} \|Q_j v\|_{A^\tau_{\mathcal{Q}}} \\
&= \left\| \sum_{n=j+1}^{\infty} (Q_n - Q_{n-1}) v \right\|_{\left(A^\delta_{\mathcal{Q}'}\right)'} + \varrho^{-j(\tau+\delta)} \left\| \sum_{n=0}^{j} (Q_n - Q_{n-1}) v \right\|_{A^\tau_{\mathcal{Q}}} \\
&\lesssim \sum_{n=j+1}^{\infty} \varrho^{-n\delta} \|(Q_n - Q_{n-1}) v\|_V + \varrho^{-j(\tau+\delta)} \sum_{n=0}^{j} \varrho^{n\tau} \|(Q_n - Q_{n-1}) v\|_V ,
\end{aligned}
$$

where we have used Propositions 4.6 and 4.4 in the last step. Thus one has

$$
\begin{aligned}
\|v\|^2_\theta &\leq \sum_{j=0}^{\infty} \varrho^{2j\theta(\tau+\delta)} \left\{ \sum_{n=j+1}^{\infty} \varrho^{-n\delta} \|(Q_n - Q_{n-1}) v\|_V \right. \\
&+ \left. \varrho^{-j(\tau+\delta)} \sum_{n=0}^{j} \varrho^{n\tau} \|(Q_n - Q_{n-1}) v\|_V \right\}^2 \\
&\lesssim \sum_{j=0}^{\infty} \varrho^{2j\theta(\tau+\delta)} \left\{ \left(\sum_{n=j+1}^{\infty} \varrho^{-n\delta} \|(Q_n - Q_{n-1}) v\|_V \right)^2 \right. \\
&+ \left. \left(\varrho^{-j(\tau+\delta)} \sum_{n=0}^{j} \varrho^{n\tau} \|(Q_n - Q_{n-1}) v\|_V \right)^2 \right\}.
\end{aligned}
$$

Choosing $a_n = \varrho^{-n\delta} \|(Q_n - Q_{n-1}) v\|_V$ and $s = \theta(\delta + \tau)$ in Lemma 4.1 (ii), yields

$$
\sum_{j=0}^{\infty} \varrho^{2j\theta(\tau+\delta)} \left\{ \sum_{n=j+1}^{\infty} \varrho^{-n\delta} \|(Q_n - Q_{n-1}) v\|_V \right\}^2 \lesssim \|\mathbf{a}\|^2_{l_2^s} = \|v\|^2_{A^\eta_{\mathcal{Q}}}.
$$

Taking $t = \tau + \delta$, $s = \theta(\tau + \delta)$, and $a_j = \varrho^{-j\delta} \|(Q_j - Q_{j-1}) v\|_V$ in Lemma 4.1 (i), one obtains

$$
\sum_{j=0}^{\infty} \varrho^{2j\theta(\tau+\delta)} \left\{ \varrho^{-j(\tau+\delta)} \sum_{n=0}^{j} \varrho^{n\tau} \|(Q_n - Q_{n-1}) v\|_V \right\}^2 \lesssim \|\mathbf{a}\|^2_{l_2^{\theta(\tau+\delta)}} = \|v\|^2_{A^\eta_{\mathcal{Q}}},
$$

which finishes the proof of Proposition 5.1. $\square$

We are now in a position to confirm the assertion of Theorem 5.1 (up to equivalence of norms). Since, by Remark 4.1, the spaces $A^\delta_{\mathcal{Q}}, A^\delta_{\mathcal{Q}'}$ are reflexive we can apply a well-known duality theorem from interpolation theory (see e.g. [BL, Theorem 3.7.1] or [T], pp 69) which asserts that

$$
\left[\left(A^\delta_{\mathcal{Q}'} \right)', A^\tau_{\mathcal{Q}} \right]'_\theta = \left[\left(A^\tau_{\mathcal{Q}} \right)' A^\delta_{\mathcal{Q}'} \right]_{1-\theta}. \tag{5.5}
$$

Setting again $\eta := \theta\tau - (1-\theta)\delta$, and applying Proposition 5.1 to both sides of (5.5), provides

$$\left(A_{\mathcal{Q}}^{\eta}\right)' \cong A_{\mathcal{Q}'}^{-\eta}, \tag{5.6}$$

as claimed. □

Of course, in view of Theorem 5.1 and Proposition 5.1, Remark 5.2 follows also by interpolation from Propositions 4.3, 4.4, 4.5, and 4.6.

5.3 Shifting Scales

Interpolation can conveniently be carried out explicitly by means of the mapping (cf. [K])

$$\Lambda_\tau : v \mapsto \sum_{j=0}^{\infty} \varrho^{j\tau}(Q_j - Q_{j-1})v \tag{5.7}$$

whose adjoint is, of course, given by

$$\Lambda_\tau' = \sum_{j=0}^{\infty} \varrho^{j\tau}(Q_j' - Q_{j-1}').$$

Remark 5.4 *For $\tau, \delta \in I\!R$ one has*

$$\|\Lambda_\tau v\|_{A_{\mathcal{Q}}^{\delta}} = \|v\|_{A_{\mathcal{Q}}^{\tau+\delta}},$$

and

$$\|v\|_{A_{\mathcal{Q}}^{\tau}} = \left(\sum_{j=0}^{\infty} \varrho^{2j(\tau-\delta)} \|(Q_j - Q_{j-1})v\|_{A_{\mathcal{Q}}^{\delta}}^2\right)^{1/2}.$$

Proof: Since by (2.1),

$$(Q_j - Q_{j-1})\Lambda_\tau v = \varrho^{j\tau}(Q_j - Q_{j-1})v, \tag{5.8}$$

one obtains

$$\|\Lambda_\tau v\|_{A_{\mathcal{Q}}^{\delta}}^2 = \sum_{j=0}^{\infty} \varrho^{2j\delta} \|(Q_j - Q_{j-1})\Lambda_\tau v\|_V^2 = \sum_{j=0}^{\infty} \varrho^{2j(\tau+\delta)} \|(Q_j - Q_{j-1})v\|_V^2 .$$

Moreover, noting that

$$\|(Q_j - Q_{j-1})v\|_{A_{\mathcal{Q}}^{\tau}} = \varrho^{\tau j} \|(Q_j - Q_{j-1})v\|_V,$$

the assertion follows. □

The above considerations can be carried further, replacing the Hilbert space V by a Banach space B and considering norms of the type

$$\|v\|_{A_q^{\tau}(B,\mathcal{Q})} := \left(\sum_{j=0}^{\infty} \varrho^{jq\tau} \|(Q_j - Q_{j-1})v\|_B^q\right)^{1/q}$$

for some $1 \leq q \leq \infty$. Following essentially the above lines, this issue is taken up in [D2]. The point is that this offers a unified treatment of approximation properties, interpolation and duality for a general class of spaces which turn out to agree in many cases with classical Besov and Sobolev spaces. The multiresolution $\mathcal{S}$ and the projectors $\mathcal{Q}$ defining the multiscale decompositions replace in some sense Fourier transform techniques, offering, in principle, a more flexible tool that can be adapted to function spaces defined over different types of domains. Here we contend ourselves with exemplifying such a program for the case of Sobolev spaces.

5.4 Sobolev Norms

We have already seen that, under the hypotheses of Theorem 3.3, for a certain range of parameters τ the spaces $A^\tau_{\mathcal{Q}}$ can be identified with with spaces B^τ_ω which have a meaning independently of $\mathcal{S}$ and $\mathcal{Q}$.

Of course, $\omega(v,t)$ plays the role of a modulus of smoothness and the B^τ_w are essentially Besov-spaces. Typical examples can be described as follows. Defining for $h \in \mathbb{R}^d$,

$$\Delta_h^m f = \sum_{j=0}^{m} \binom{m}{j} (-1)^{m-j} f(\cdot + jh)$$

the m-th order forward difference in the direction h and setting for a given domain $\Omega \subset \mathbb{R}^d$

$$\Omega_{h,m} = \{x \in \Omega : x + jh \in \Omega, j = 0, \ldots, m\},$$

the m-th order L_2-modulus is defined by

$$\omega_m(v,t,)_{L_2(\Omega)} := \sup_{|h| \leq t} \|\Delta_h^m f\|_{L_2(\Omega_{k,m})}. \tag{5.9}$$

It is clear that $\omega_m(\cdot,\cdot)_{L_2(\Omega)}$ satisfies (3.7). Let $B^t_\omega(\Omega)$ be defined by (4.6) relative to the L_2-modulus $\omega_m(\cdot,\cdot)_{L^2(\Omega)}$. Under certain rather weak regularity assumptions on the boundary $\partial\Omega$ of Ω Sobolev and Besov spaces on Ω can be defined by extension techniques. We will assume in the following that Ω has this property. It is well-known that then

$$B^t_\omega(\Omega) \cong H^t(\Omega), \quad 0 < t < m, \tag{5.10}$$

where $H^t(\Omega)$ denotes the usual Sobolev space on Ω (see e.g. [DK, DP]).

Corollary 5.1 *Let $\mathcal{S}$ satisfy the Jackson estimate*

$$\inf_{v_n \in S_n} \|v - v_n\|_V \lesssim \omega_m(v, \varrho^{-n})_{L^2(\Omega)}, \qquad v \in L^2(\Omega), \tag{5.11}$$

as well as the Bernstein estimate

$$\omega_m(v_n, t)_{L^2(\Omega)} \lesssim (\min\{1, t\varrho^n\})^\gamma \|v_n\|_V, \qquad v_n \in S_n, \tag{5.12}$$

for some $\gamma > 0$. Moreover assume that for $\mathcal{Q}$ as above $\tilde{\mathcal{S}}$ satisfies analogous inequalities with respect to parameters m', γ'. Let $r := \min\{m,\gamma\}$, $r' := \min\{m',\gamma'\}$, then

$$A^t_{\mathcal{Q}} \cong H^t(\Omega), \qquad t \in (-r', r),$$

where it is to be understood here that $H^t(\Omega) = (H^{-t}(\Omega))'$ for $t \leq 0$.

Proof: Fix $t \in (0, r)$. By Proposition 4.7 and (5.10), we have $H^t(\Omega) \cong A^t_{\mathcal{Q}}$. By Theorem 3.3, $L_2(\Omega) = H^0(\Omega) \cong A^0_{\mathcal{Q}} \cong A^0_{\mathcal{Q}'}$. Since $H^s(\Omega)$ is for $0 \leq s \leq t$ obtained by interpolation between $L_2(\Omega)$ and $H^t(\Omega)$ (see e.g. [T]), we infer from Proposition 5.1 that $H^s(\Omega) \cong A^s_{\mathcal{Q}}$, $0 \leq s \leq t$. By Theorem 5.1, one has $A^{-t}_{\mathcal{Q}} \cong \left(A^t_{\mathcal{Q}'}\right)' \cong (H^t(\Omega))'$ for $0 < t < r'$, which completes the proof. □

The estimates (5.11), (5.12) are known to hold for all commonly used spline, finite element or multiresolution spaces. The above norm equivalences play a crucial role for preconditioning linear systems arising from Galerkin discretizations of elliptic boundary value problems, or more generally, of strongly elliptic pseudodifferential equations (see e.g. [DK, DPS]).

The Jackson and Bernstein inequalities are often given in a form anlogous to Remark 5.2. To this end, let

$$|u|^2_{H^m(\Omega)} = \sum_{|\alpha|=m} \|D^\alpha u\|^2_{L_2(\Omega)}$$

denote the usual Sobolev-Seminorm. For standard finite element or spline spaces S_j Bramble-Hilbert-type arguments typically provide error estimate of the form

$$\inf_{v_n \in S_n} \|u - v_n\|_{L_2(\Omega)} \lesssim \varrho^{-nm} \, |u|_{H^m(\Omega)} \tag{5.13}$$

provided that S_n contains all polynomials of degree $m-1$ on Ω. Likewise it is often known that

$$\|v_n\|_{H^t(\Omega)} \lesssim \varrho^{n(t-s)} \|v_n\|_{H^s(\Omega)}, \quad v_n \in S_n, \tag{5.14}$$

holds for $s < t < \gamma$.

Corollary 5.2 *Suppose $\mathcal{S}, \tilde{\mathcal{S}}$ satisfy (5.13) and (5.14) with m, γ and m', γ', respectively. Then the assertion of Corollary 5.1 remains valid, i.e., for $t \in (-r', r)$ one has*

$$\|v\|_{H^t(\Omega)} \sim \left(\sum_{j=0}^{\infty} \varrho^{2tj} \|(Q_j - Q_{j-1})v\|^2_{L^2(\Omega)} \right)^{1/2} \sim \left(\sum_{j=0}^{\infty} \|(Q_j - Q_{j-1})v\|^2_{H^t(\Omega)} \right)^{1/2}.$$

Proof: Note that by (5.13),

$$\begin{aligned} \|u - v_n\|_{L_2(\Omega)} &\leq \|u - v\|_{L_2(\Omega)} + \|v - v_n\|_{L_2(\Omega)} \\ &\lesssim \|u - v\|_{L_2(\Omega)} + \varrho^{-nm} \, |v|_{H^m(\Omega)} \, . \end{aligned}$$

Thus one obtains

$$\inf_{v_n \in S_n} \|u - v_n\|_{L_2(\Omega)} \lesssim K(u, \varrho^{-nm}, L_2(\Omega), H^m(\Omega)).$$

It is known that, under the above mentioned regularity assumptions on the boundary $\partial\Omega$, this K-functional is equivalent to the standard modulus of smoothness $\omega_m(\cdot, t)_{L^2(\Omega)}$

defined in (5.9) [JS]. Thus (5.13) implies (5.11). Likewise, for $\varrho^{-j-1} \leq t < \varrho^{-j}$, one has

$$\begin{aligned}
\omega_m(v_n, t)^2_{L^2(\Omega)} &\leq \omega_m(v_n, \varrho^{-j})^2_{L^2(\Omega)} \leq \varrho^{-2sj} \left(\sum_{i=0}^{\infty} \varrho^{2is} \omega_m(v_n, \varrho^{-i})^2_{L^2(\Omega)} \right) \\
&= \varrho^{-2js} \|v_n\|^2_{B^s_\omega(\Omega)} \sim \varrho^{-2js} \|v_n\|^2_{H^s(\Omega)} \lesssim (t\varrho^n)^{2s} \|v_n\|^2_{L^2(\Omega)},
\end{aligned}$$

where we have used (5.14) in the last step. The assertion follows now from Corollary 5.1 and Remark 5.4. □

We conclude with an application which arises in connection with collocation methods for boundary integral equations. Suppose that $S_j \subset C(\Omega) \subset H^{s'}(\Omega)$, for some $s' > d/2$, and let L_j denote Lagrange interpolation projectors relative to suitably chosen nodes in Ω such that

$$\|L_j\|_{H^s(\Omega)} = \mathcal{O}(1), \quad j \to \infty, \tag{5.15}$$

for $d/2 < s < s'$. For $\mathcal{S}$ satisfying (5.13) one can then show that

$$\|L_j v - v\|_{H^\tau(\Omega)} \lesssim \varrho^{-j(t-\tau)} \|v\|_{H^t(\Omega)}, \quad v \in H^t(\Omega), \tag{5.16}$$

holds for $0 \leq \tau < s'$, $\tau \leq t$, $d/2 < t \leq m$, say, (see e.g. [DPS1] for examples concerning cardinal interpolation on the torus).

Corollary 5.3 *Suppose that in addition to (5.14) the estimates (5.15) and (5.16) hold. Then for $d/2 < s < s'$ one has*

$$\|v\|^2_{H^s(\Omega)} \sim \sum_{j=0}^{\infty} \|(L_j - L_{j-1})v\|^2_{H^s(\Omega)} \sim \sum_{j=0}^{\infty} \varrho^{2js} \|(L_j - L_{j-1})v\|^2_{L^2(\Omega)}. \tag{5.17}$$

Proof: Clearly the L_j satisfy (2.1). Since by the inverse inequality (5.14),

$$\|(L_j - L_{j-1})v\|_{H^s(\Omega)} \lesssim \varrho^{sj} \|(L_j - L_{j-1})v\|_{L^2(\Omega)},$$

while (5.16) yields

$$\varrho^{sj} \|(L_j - L_{j-1})v\|_{L^2(\Omega)} \lesssim \|(L_j - L_{j-1})v\|_{H^s(\Omega)}$$

the middle and right hand terms in (5.17) are equivalent. However, note that since the L_j are not uniformly bounded on $L^2(\Omega)$ one cannot apply Proposition 4.7 with (5.10) directly to deduce the right most equivalence. Also one lacks sufficient information about the adjoints of L_j so that Theorem 3.3 does not seem to apply directly either. However, for $d/2 < s < s'$ and $\epsilon > 0$ such that $d/2 < s - \epsilon$, $s + \epsilon < s'$ the dual of $H^{s+\epsilon}(\Omega)$ relative to the inner product $\langle \cdot, \cdot \rangle_s$ on $H^s(\Omega)$ is contained in $H^{s-\epsilon}(\Omega)$. Thus, in view of (5.14), (5.16), and (5.15), one can apply 3.2 with $V = H^s(\Omega)$ and $U = H^{s+\epsilon}(\Omega)$ to deduce the first norm equivalence in (5.17). Alternatively one can argue as follows. On account of (5.15), $\mathcal{L} = \{L_j\}_{j=0}^{\infty}$ satisfies (2.2) for $V = H^s(\Omega)$. For ϵ as above let

$$K(v, t) := \inf_{w \in H^{s+\epsilon}(\Omega)} \{\|v - w\|_{H^s(\Omega)} + t^\epsilon \|w\|_{H^{s+\epsilon}(\Omega)}\}$$

which is clearly a modulus on $H^s(\Omega)$. While (3.8) follows as in the proof of Corollary 5.2, the Bernstein estimate (3.9) with $\gamma = \epsilon$ is an immediate consequence of (5.14) and the definition of K. By Proposition 4.7 (with $V = H^s(\Omega), \mathcal{Q} = \mathcal{L}, \omega = K$), one obtains for $0 < \delta < \epsilon$

$$\sum_{j=0}^{\infty} \varrho^{2j\delta} \|(L_j - L_{j-1})v\|_{H^s(\Omega)}^2 \sim \sum_{j=0} \varrho^{2j\delta} K(v, \varrho^{-j})^2. \tag{5.18}$$

While, by the above reasoning, the left hand side of (5.18) is equivalent to

$$\sum_{j=0}^{\infty} \|(L_j - L_{j-1})v\|_{H^{s+\delta}(\Omega)}^2$$

the right hand side is equivalent to $\|\cdot\|_{H^{s+\delta}(\Omega)}^2$ since the $H^s(\Omega)$ are interpolation spaces. Since ϵ was arbitrary the proof is complete. □

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