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Symmetric Iterative Interpolation Processes

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Abstract. Using a base b and an even number of knots, we define a symmetric iterative interpolation process. The main properties of this process come from an associated function F. The basic functional equation for F is that $F(t/b)=\sum_n F(n/b)F(t-n)$. We prove that F is a continuous positive definite function. We find almost precisely in which Lipschitz classes derivatives of F belong. If a function y is defined only on integers, this process extends y continuously to the real axis as $y(t)=\sum_n y(n)F(t-n)$. Error bounds for this iterative interpolation are given.

1. Introduction

We introduce a family of interpolation processes. This family uses two positive integral parameters: b, a base, and $2N$, an even number of moving nodes. Given a function y which is defined on integers, we extend this function to all integral multiples of $1/b$: if r is an integer between 0 and b, and if n is an integer, $y(n+r/b)$ is defined as the value $p(n+r/b)$ where p is the Lagrange polynomial of degree smaller than $2N$ such that $p(k)=y(k)$ for every integer k of $[n-N+1, n+N]$. This construction can be iterated. In this way, an extension $y(t)$ is found for any rational number t whose denominator is an integral power of b.

Our main concern is to prove that the extension $y(t)$ is uniformly continuous on any finite interval whatever the base b, the even number $2N$ of nodes, and the initial values $y(n)$. For each base b and each number N, there is a fundamental function $F(t)$, the iterative interpolation of the sequence $F(n)$, which is equal to 1 at $n=0$ and equal to 0 elsewhere. The main properties of the interpolation process come from its fundamental function. The basic functional equation for F is

$$F(t/b)=\sum_n F(n/b)F(t-n).$$

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Analysis of F is carried through to the study of two mathematical companions of the process: the trigonometric polynomial

$P(\theta)=\sum_k F(k/b)\, e^{ik\theta}$ and the infinite matrix $A=(F(k/b-j))_{-\infty<k<\infty,-\infty<j<\infty}$.

We see that for a symmetric iterative interpolation process which uses Lagrange polynomials, $P(\theta)$ is always nonnegative and is divisible by another trigonometric polynomial: $\sin^{2N}(b\theta/2)/\sin^{2N}(\theta/2)$. The Lipschitz classes to which derivatives of F belong are related to the eigenvalues of matrix A. We compute derivatives of F using linear equations and interpolation theory. We derive error bounds for iterative interpolation when applied to a specific function.

2. Iterative Interpolation with a Base b

In [3] we described a new method of interpolation. The starting point is a function $y(n)$ which is defined on integers. We would like to extend this function to the whole real axis if possible. First, an integer larger than one, b, is used; this will be the base of the interpolation process. $\mathbf{B}_n$ is the set of b-adic rational numbers m/b^n where m is integral, $n=0, 1, 2, \ldots,$ and $y(t)$ is already defined over $\mathbf{B}_0$. The basic idea is to extend the function y to $\mathbf{B}_1, \mathbf{B}_2, \mathbf{B}_3, \ldots$ by induction. In order to extend y from $\mathbf{B}_0$ to $\mathbf{B}_1$, a sequence of parameters $\{p_k\}_{k=-\infty,\infty}$ are used. Some conditions are assumed about this sequence. First, $p_0=1$ and $p_k=0$ for any other integral multiple of b: $k=\pm b, \pm 2b, \pm 3b, \ldots$. In order to simplify the discussion, we also assume that only a finite number of the parameters p_k are different from zero. With these hypotheses, the extension of y is $y(j/b)=\sum_k p_{j-kb}y(k)$. The extension to $\mathbf{B}_2, \mathbf{B}_3, \ldots$ can be carried through in the same way: if j is an integer, we set

$$y(j/b^{n+1})=\sum_k p_{j-kb}y(k/b^n). \tag{2.1}$$

The properties of the parameters allow a consistent extension with no restriction on the original sequence $y(n)$. This extension of the sequence $y(n)$ to the set of b-adic rational numbers is simply written as $y(t)$ and is called the iterative interpolation with base b.

Any sequence $y(n)$ can be extended to the set $\mathbf{B}$ of b-adic numbers according to scheme (2.1). This interpolation process satisfies some simple properties. First, it is a *linear process*. Another basic property is the *translation property of the interpolation*: if $y(n)$ is a given sequence with $y(t)$ as interpolation and if m is a given integer, the interpolation of the sequence $y(n+m)$ is the function $y(t+m)$. The following property is called the *homogeneity of the process of interpolation*: if $y(t)$ is the interpolation of the sequence $y(n)$ defined over $\mathbf{B}_0$ and if h is a negative integral power of b, then the interpolation of the sequence $y(nh)$ is the function $y(th)$. These three properties can be proven rather easily.

3. The Fundamental Function of an Iterative Process

Let us start with the following sequence: $y(0)=1$ and $y(n)=0$ when n is any other integer. Using scheme (2.1), this sequence can be extended to $\mathbf{B}$, the b-adic numbers. Let us denote by $F(t)$ this extension, F is called the *fundamental function* of the interpolation process. There is a very close connection between the weights p_k and the values of F on $\mathbf{B}_1$: $p_k = F(k/b)$. This is a direct consequence from (2.1). Let us look for places where F vanishes.

Lemma 3.1. *If N is the smallest index k for which $p_k \neq 0$ and if N^* is the largest index k for which $p_k \neq 0$, then the fundamental function of the interpolation process vanishes outside* $(N/(b-1), N^*/(b-1))$.

Proof. We set $t_0 = 0$ and t_n is defined through the recurrence $t_n = t_{n-1} + Nb^{-n}$. The sequence t_n is decreasing and converges to $N/(b-1)$. As is easily verified from relation (2.1), the restriction of F to $\mathbf{B}_n$ vanishes on $(-\infty, t_n)$, where $n = 1, 2, 3, \ldots$. For any b-adic number $t \leq N/(b-1)$, $F(t) = 0$. A similar argument shows that $F(t^*)$ is also zero if $t^* \geq N^*/(b-1)$. ■

As in [3], the next two theorems can be proved.

Theorem 3.2. *If $y(n)$ is a function defined on integers, the extension $y(t)$ to b-adic numbers according to scheme* (2.1) *is $\sum_n y(n)F(t-n)$ which is basically a finite series.*

Theorem 3.3. *The fundamental function F fulfils the functional equation*

$$F(t/b^m) = \sum_n F(n/b^m)F(t-n). \tag{3.1}$$

4. Some Examples of Iterative Interpolation

We describe a class of iterative interpolation connected with polynomial interpolation. If $\mathbf{S}$ is a given finite set of integers and if $y(n)$ is a function already defined over the set of integers, then it is possible to extend y to the set $\mathbf{B}_1$ by using polynomial interpolation over translates of $\mathbf{S}$ in the following way. If $t = n + r/b$ (where integer r is between 1 and $b-1$) is a point of $\mathbf{B}_1$ where the extension has to be done and if $P(x)$ is the interpolating polynomial such that $P(n+k) = y(n+k)$ for every integer k of $\mathbf{S}$, then we set $y(t) = P(t)$. If $\{L_k\}_{k\in\mathbf{S}}$ areLagrange polynomials when the integers of $\mathbf{S}$ are used as nodal points, then the extension is

$$y(n + r/b) = \sum_{k\in\mathbf{S}} L_k(r/b)y(n+k). \tag{4.1}$$

The fundamental function F has the following basic values: if $-n$ belongs to $\mathbf{S}$ and if r is an integer between 1 and $b-1$, then

$$F(n+r/b) = L_{-n}(r/b); \qquad F(0) = 1 \quad \text{and, for all other values of } \mathbf{B}_1, \quad F = 0. \tag{4.2}$$

Definition. The interpolation process (4.1) is called a *Lagrange iterative interpolation.* The main property of this interpolation process is provided by the following statement whose proof is easily obtained.

Lemma 4.1. *For any polynomial P of degree smaller than the number of points of* S, *the Lagrange iterative interpolation of the sequence $y(n) = P(n)$ is precisely the function $y(t) = P(t)$.*

When the nodal points are $\mathbf{S} = \{-N+1, -N+2, \dots, N-1, N\}$ for some positive integer N, we speak of *symmetric* interpolation; indeed, symmetric data $y(n)$ produces symmetric interpolation $y(t)$, the fundamental function $F(t)$ is itself symmetric. For example if $\mathbf{S} = \{-1, 0, 1, 2\}$, we get four Lagrange polynomials:

$$L_{-1}(x) = -x(1-x)(2-x)/6, \qquad L_0(x) = (x+1)(1-x)(2-x)/2,$$

$$L_1(x) = (x+1)x(2-x)/2, \qquad L_2(x) = -(x+1)x(1-x)/6.$$

When base b is 2 or 3, the main values of function F are respectively:

t	1/2	3/2
$F(t)$	9/16	−1/16

t	1/3	2/3	4/3	5/3
$F(t)$	20/27	10/27	−5/81	−4/81

If $\mathbf{S} = \{-2, -1, 0, 1, 2, 3\}$, the six Lagrange polynomials are:

$$L_{-2}(x) = (x+1)x(1-x)(2-x)(3-x)/120,$$

$$L_{-1}(x) = -(x+2)x(1-x)(2-x)(3-x)/24,$$

$$L_0(x) = (x+2)(x+1)(1-x)(2-x)(3-x)/12,$$

$$L_1(x) = (x+2)(x+1)x(2-x)(3-x)/12,$$

$$L_2(x) = -(x+2)(x+1)x(1-x)(3-x)/24,$$

$$L_3(x) = (x+2)(x+1)x(1-x)(2-x)/120$$

When the base is 2 or 3, the basic values of F are respectively:

t	1/2	3/2	5/2
$F(t)$	75/128	−25/256	3/256

t	1/3	2/3	4/3	5/3	7/3	8/3
$F(t)$	560/729	280/729	−70/729	−56/729	8/729	7/729

Definition. In order to avoid a long expression, we speak of the *interpolation process of type* (b, N) instead of the Lagrange iterative symmetric interpolation process when the base is b and when the set of nodal points S is $\{-N+1, -N+2, \dots, N-1, N\}$.

5. Fourier Transform of an Iterative Interpolation

We recall some results we have already shown [3] about general iterative interpolation. These results will help us later in the study of properties of the fundamental interpolating function. A sequence of trigonometric polynomials is associated to the fundamental function $F(t)$: $P_n(\theta)=\sum_k F(k/b^n)\, e^{ik\theta}$. If we set $P(\theta)=P_1(\theta)$, we get a recurrence from formula (3.1): $P_{n+1}(\theta)=P_n(b\theta)P(\theta)$. Hence

$$P_n(\theta)=\prod_{0\le k\le n-1} P(b^k\theta). \tag{5.1}$$

Definition. The trigonometric polynomial $P(\theta)$ is said to be the *characteristic polynomial* of the interpolation scheme (2.1).

Under very weak assumption, there is always a Schwartz distribution accompanying the function F. This assumption is that $\sum_k F(k/b)=b$. In that case, if, for any C^∞-function φ, $T_n(\varphi)=\sum_m F(m/b^n)\varphi(m/b^n)/b^n$ (T_n is a linear combination of Dirac masses), then, in [3], we have proven that this sequence of distributions converges weakly to a distribution T.

Theorem 5.1. *If $\sum_k F(k/b)=b$, then the sequence of distributions T_n converges weakly toward a distribution T. If $G(\xi)$ is the Fourier transform of T, then $G(\xi)=\prod_{k\ge 1}[P(-\xi/b^k)/b]$. $G(0)=1$ and the following functional equation is satisfied*:

$$G(b\xi)=G(\xi)\left(\sum_k F(k/b)\, e^{-ik\xi}\right)\Big/ b. \tag{5.2}$$

The Fourier transform of T_n is in fact $G_n(\xi)=P_n(-\xi/b^n)/b^n=\prod_{1\le k\le n}[P(-\xi/b^k)/b]$. The sequence $G_n(\xi)$ converges pointwise to $G(\xi)$.

Definition. The function G is called the *Fourier transform* of the interpolation process.

6. Continuity of the Fundamental Function

We consider an interpolation process of type (b, N). We study the Lagrange iterative interpolation when nodal points are given as $\{-N+1, -N+2, \dots, 0, 1, \dots, N-1, N\}$. We prove that its fundamental function is continuous. The main step is to show that the characteristic polynomial of the process is everywhere nonnegative.

Theorem 6.1. *If $P(\theta)$ is the characteristic function of an interpolation process of type (b, N), then any integral multiple of $2\pi/b$ which is not an integral multiple of 2π is a root of P with multiplicity at least equal to $2N$. Moreover, the first $2N-1$ derivatives of P at $\theta=0$ vanish.*

Proof. By definition, $P(\theta)=\sum_k F(k/b)\,e^{ik\theta}$. If $\theta=2\pi r/b$ (where r is a positive integer smaller than b) and if $w=e^{i2\pi r/b}$, then

$$P(\theta)=\sum_k F(k/b)w^k=\sum_{0\le s\le b-1}\sum_n F(n+s/b)w^{nb+s},$$

$$P(\theta)=\sum_{0\le s\le b-1}\sum_n F(n+s/b)w^s.$$

Since $1=\sum_n F(x-n)$ for every x (application of Lemma 4.1 and Theorem 3.2 with the polynomial equal to the constant 1),

$$P(\theta)=\sum_{0\le s\le b-1} w^s=(w^b-1)/(w-1)=0.$$

If $\theta=2\pi r/b$ (with $0\le r<b$), if $w=e^{i2\pi r/b}$, if m is a positive integer smaller than $2N$, and if $P_m=P^{(m)}(\theta)/i^m$, then

$$P_m=\sum_k k^mF(k/b)w^k=\sum_{0\le s\le b-1}\sum_n (n+s/b)^mF(n+s/b)w^{nb+s},$$

$$P_m=\sum_{0\le s\le b-1}\sum_n (n+s/b)^mF(n+s/b)w^s,$$

$$P_m=(-1)^m\sum_{0\le s\le b-1}\sum_n (n-s/b)^mF(s/b-n)w^s.$$

Since $(x-c)^m=\sum_n (n-c)^mF(x-n)$ for every x and for every c whenever $m<N$ (as shown by Lemma 4.1 and Theorem 3.2),

$$P_m=\sum_{0\le s\le b-1}(s/b-s/b)^mw^s=0. \qquad \blacksquare$$

Theorem 6.2. *The characteristic polynomial of an interpolation process of type (b, N) is everywhere nonnegative.*

Proof. The degree of the characteristic polynomial $P(\theta)$ is $bN-1$. The symmetry of $F(t)$ makes sure that $P(\theta)$ takes real values. $P(\theta)$ vanishes at the following points: $2\pi k/b$, $k=1,2,\dots,b-1$; the multiplicity of each of these roots is at least $2N$. Moreover, the first $2N-1$ successive derivatives of P' vanish at zero. Let us look at the critical points of P, the roots of the polynomial P'. $P'(\theta)$ vanishes at the following points: $2\pi k/b$, $k=0,1,\dots,b-1$; the multiplicity of each of these roots is at least $2N-1$. According to Rolle's theorem, in each interval $(2\pi k/b, 2\pi(k+1)/b)$, $k=1,2,\dots,b-2$, $P'(\theta)$ is zero for at least one point θ. The number of roots of P' is then at least $2Nb-2$, the degree of P' as a trigonometric polynomial is $bN-1$. Since the number of roots of a trigonometric polynomial cannot exceed the double of its degree, in each interval $(2\pi k/b, 2\pi(k+1)/b)$, $k=1,2,\dots,b-2$, $P'(\theta)$ is zero at exactly one point and there is no root of P' in $(0,2\pi/b)$ and in $(-2\pi/b,0)$. Since $P(0)=b$ and $P(2\pi/b)=0$, P' is negative on $(0,2\pi/b)$. The derivative $P^{(2N)}(2\pi/b)$ is positive, otherwise $P'(\theta)$ would have too many roots. When the base b is larger than 2, it can be seen that $P^{(2N)}(4\pi/b)$ is positive; first, $P^{(2N)}(4\pi/b)\neq 0$ according to the number of roots of P'; if $P^{(2N)}(4\pi/b)<0$, P would be positive around $2\pi/b$, negative around $4\pi/b$, and there would be two critical points for P in

$(2\pi/b, 4\pi/b)$, but this is impossible. This same argument can be carried through for each interval $(2\pi k/b, 2\pi(k+1)/b)$, $k=2,3,\dots,b-2$. It follows that $P(\theta)$ never take negative values. ■

Remark. As shown in the previous proof, $P^{(2N)}$ is positive at any integral multiple of $2\pi/b$ which is not an integral multiple of 2π. So any integral multiple of $2\pi/b$ which is not an integral multiple of 2π is a root of P with multiplicity *precisely* equal to $2N$.

Theorem 6.3. *For any base b, for any even positive integer N, the Fourier transform $G(\xi)$ of the interpolation process of type (b, N) is integrable.*

Proof. As already noted (Theorem 5.1), the Fourier transform $G(\xi)$ of the interpolation process is $\prod_{k\geq 1}[P(-\xi/b^k)/b]$. According to the previous theorem, $G(\xi)$ is nonnegative. If J_n is the integral of G over $(-\pi b^n, \pi b^n)$, then $J_n = \int_{(-\pi,\pi)} G(\xi)\prod_{0\leq k\leq n-1} P(-\xi b^k)\,d\xi$, since $G(b^n\xi)$ is equal to $G(\xi)\prod_{0\leq k\leq n-1}[P(-\xi b^k)/b]$. If M is the maximal value of $G(\xi)$ on $[-\pi,\pi]$, then $J_n \leq M\int_{(-\pi,\pi)}\prod_{0\leq k\leq n-1} P(-\xi b^k)\,d\xi$. We already know that the Fourier series of $\prod_{0\leq k\leq n-1} P(\xi b^k)$ is $\sum_k F(k/b^n)\,e^{ik\xi}$. So $\int_{(-\pi,\pi)}\prod_{0\leq k\leq n-1} P(-\xi b^k)\,d\xi = F(0) = 1$. Each integral J_n is bounded by M. $G(\xi)$ is then integrable. ■

Theorem 6.4. *The fundamental function $F(t)$ of an interpolation process of type (b, N) has a unique continuous extension to the real axis. The Fourier transform of that extension is $G(\xi)$.*

Proof. The Fourier transform $G(\xi)$ of the interpolation process is integrable. $G(\xi)$ is then the Fourier transform of a continuous function $\Phi(t)$: in fact, $\Phi(t) = 1/(2\pi)\int G(\xi)\,e^{it\xi}\,d\xi$. From the functional relation (5.2), $G(b\xi) = G(\xi)P(-\xi)/b$, it follows that $\Phi(t/b) = \sum_k F(k/b)\Phi(t-k)$. The interpolation process applied to the sequence $\Phi(n)$ reproduces $\Phi(t)$ for any b-adic number t. According to Theorem 3.2, for any b-adic number t, $\Phi(t) = \sum_k \Phi(k)F(t-k)$. If we introduce the sequence of trigonometric polynomials $\Pi_n(\theta) = \sum_k \Phi(k/b^n)\,e^{ik\theta}$ and if $Q(\theta)$ is the trigonometric polynomial $\sum_k \Phi(k)\,e^{ik\theta}$, then $\Pi_n(\theta) = Q(b^n\theta)P_n(\theta)$, as is seen after a short computation. The sequence of Fourier transforms $\Pi_n(\xi/b^n)/b^n$ converges to $Q(\xi)G(\xi)$. Since $\Pi_n(\xi/b^n)/b^n$ is the Fourier transform of the following measure μ_n, a linear combination of Dirac masses, $\sum_k b^{-n}\Phi(k/b^n)\delta_{k/b}n$, and since this sequence of measures converges weakly to $\Phi(t)$, we get $Q(\xi)G(\xi) = G(\xi)$. From this relation $Q(\xi)=1$, $\Phi(0)=1$, and $\Phi(n)=0$ for any integer other than zero. $\Phi(t) = F(t)$ for any b-adic number. ■

7. The Order of Regularity of the Fundamental Function

We already know that the fundamental function F of any Lagrange iterative symmetric interpolation process is continuous. Is this function differentiable? What happens for derivatives of higher order? In this section we show some results about the order of regularity of F. The two basic tools of this study are factorization of the characteristic function $P(\theta)$ and finding real values α for which $\int|\xi|^\alpha G(\xi)\, d\xi$ is finite (where G is the Fourier transform of the iterative process). We recall the definition of a *Lipschitz function of order* α; f is such a function if there is a number L such that for any two numbers x_1 and x_2, $|f(x_1)-f(x_2)|\le L|x_1-x_2|^\alpha$. The class of all Lipschitz functions of order α is customarily noted as Lip α. Some authors prefer to use the expression Hölder functions instead of Lipschitz functions. The following lemma makes a connection between the finiteness of a Fourier transform multiplied by a power and Lipschitz functions. This lemma should be known in the mathematical literature, but we were unable to find a reference about it.

Lemma 7.1. *Let $f(t)$ be a function whose Fourier transform is $g(\xi)$. We assume that $|\xi|^{m+\alpha}g(\xi)$ is integrable where m is a nonnegative integer and α is a real number of* $[0, 1]$. *If $\alpha=0$, then $f^{(m)}$ is continuous. If $\alpha\neq 0$, then $f^{(m)}$ belongs to* Lip α.

Proof. (a) That $f^{(m)}$ is continuous when $|\xi|^m g(\xi)$ is integrable is a classical result as proven in Bochner and Chandrasekharan [1].

(b) Let us assume that $\int|\xi|^\alpha|g(\xi)|<\infty$ where $\alpha\in(0,1)$. If t_1 and t_2 are two numbers such that $|t_1-t_2|<\delta$, then

$$2\pi|f(t_1)-f(t_2)|\le\int|e^{i\xi t_1}-e^{i\xi t_2}||g(\xi)|\, d\xi\le\int\min(|\xi|\delta, 2)|g(\xi)|\, d\xi.$$

If $\alpha\in[0,1]$, then, for every positive x, $\min(x,1)\le x^\alpha$. So

$$2\pi|f(t_1)-f(t_2)|\le 2\int(|\xi|\delta/2)^\alpha|g(\xi)|\, d\xi.$$

$|f(t_1)-f(t_2)|\le\int|\xi|^\alpha|g(\xi)|\, d\xi(\delta/2)^\alpha/\pi$ and then f belongs to Lip α.

(c) The general case follows rather easily from (a) and (b). ∎

Theorem 7.2. *If $P(\theta)$ is the characteristic polynomial of the interpolation process of type (b, N), then there is a trigonometric polynomial $S(\theta)$ of degree $N-1$ such that*

$$P(\theta)=S(\theta)[\sin(b\theta/2)/\sin(\theta/2)]^{2N}. \tag{7.1}$$

Proof. Let us introduce the following polynomial: $\Pi(w)=\sum_{|k|<bN}F(k/b)w^{k+2bN-1}$. From Theorem 6.1, each point $2\pi k/b$ ($k=1,2,\ldots,b-1$) is a root of multiplicity $2N$ of the equation $P(\theta)=0$, each b-root ω of unity other than 1 is a root of multiplicity $2N$ of $\Pi(w)$. $(w-\omega)^{2N}$ is a factor of $\Pi(w)$. If $\omega_1,\omega_2,\ldots,\omega_{b-1}$ are these b-roots of unity other than 1, then $\prod_{0\le k\le b}(w-\omega_k)^{2N}$ is a factor of $\Pi(w)$. $\prod_{0\le k\le b}(w-\omega_k)=(w^b-1)/(w-1)$. $\Pi(w)[(w-1)/(w^b-1)]^{2N}$ is a polynomial $\Sigma(w)$ of degree $2N-2$. The following formulas hold:

$$P(\theta)=e^{-i\theta(bN-1)}\Pi(e^{i\theta})=e^{-i\theta(bN-1)}[(e^{ib\theta}-1)/(e^{i\theta}-1)]^{2N}\Sigma(e^{i\theta}),$$

$$P(\theta)=[\sin(\theta b/2)/\sin(\theta/2)]^{2N}e^{-i\theta(N-1)}\Sigma(e^{i\theta}).$$

Since $e^{-i\theta(N-1)}\Sigma(e^{i\theta})$ is a trigonometric polynomial of degree $N-1$, $P(\theta)/[\sin(b\theta/2)/\sin(\theta/2)]^{2N}$ is a trigonometric polynomial of the same degree $N-1$. ■

Definition. This polynomial $P(\theta)/[\sin(b\theta/2)/\sin(\theta/2)]^{2N}$ is called the *secondary factor* of $P(\theta)$. This factor is denoted by $S(\theta)$.

The next step is to study the following class of integrals: $\int |\xi|^{\alpha}G(\xi)\,d\xi$ where α is a parameter between 0 and $2N$. We start with the value $\alpha=2N$.

Lemma 7.3. *In the interpolation process of type* (b, N), *there is a constant* C_1 *such that, for any* $n\geq 0$,

$$\int_{(0,\pi b^n)} \xi^{2N}G(\xi)\,d\xi \leq C_1 b^{2Nn}\int_{(0,2\pi)}\prod_{0\leq k\leq n-1} S(\theta b^k)\,d\theta.$$

Proof.

$$I_n=\int_{(0,\pi b^n)} \xi^{2N}G(\xi)\,d\xi = b^{n(2N+1)}\int_{(0,\pi)}\xi^{2N}G(\xi b^n)\,d\xi,$$

$$I_n = b^{n(2N+1)}\int_{(0,\pi)}\xi^{2N}\prod_{0\leq k\leq n-1}[P(\xi b^k)/b]G(\xi)\,d\xi,$$

$$I_n = b^{2Nn}\int_{(0,\pi)}\xi^{2N}[\sin(b^n\xi/2)/\sin(\xi/2)]^N\left[\prod_{0\leq k\leq n-1} S(\xi b^k)\right]G(\xi)\,d\xi.$$

Since the functions of ξ, $\xi^{2N}[\sin(b^n\xi/2)/\sin(\xi/2)]^{2N}G(\xi)$, are uniformly bounded on $[0,\pi]$, there is a bound C_1 such that

$$I_n\leq C_1 b^{2Nn}\int_{(0,2\pi)}\prod_{0\leq k\leq n-1} S(\theta b^k)\,d\theta, \qquad n=0,1,2,\ldots. \quad ■$$

We need to study the behavior of another sequence of integrals. When $S(\theta)$ is a trigonometric polynomial and when b is an integer larger than 1, we set $J_n=\int_{(0,2\pi)}\prod_{0\leq k\leq n-1} S(\theta b^k)\,d\theta$. There is a close connection between the sequence J_n, an infinite matrix B which we introduce immediately, and eigenvalues of a truncated part of B.

If $S(\theta)=\sum s_k\,e^{ik\theta}$, the entries of matrix B are given as $b_{j,k}=s_{j-bk}$. Let us describe some properties of this matrix.

Lemma 7.4. *If* $S(\theta)=\sum s_k\,e^{ik\theta}$ *is a trigonometric polynomial, if* B *is the matrix* (s_{j-bk}), *and if* $\sum_k s_k^{(n)}\,e^{ik\theta}$ *is the polynomial* $\prod_{k=0,n-1} S(b^k\theta)$, *then the* (j, k)*-entry of the matrix* B^n *is the number* $s_{j-kb^n}^{(n)}$.

Proof. The action of matrix B is better understood by using formal trigonometric series. If $V=(v_k)$, then we introduce the trigonometric series $Y(\theta)=\sum_k v_k e^{ik\theta}$. We do not care about the convergence of this series. If $(w_k)=W=BV$ and if $Z(\theta)=\sum_k w_k e^{ik\theta}$, then $Z(\theta)=S(\theta)Y(b\theta)$. If $(w_k^{(n)})=W_n=B^nV$ and if $Z_n(\theta)=\sum_k w_k^{(n)} e^{ik\theta}$, then $Z_n(\theta)=\prod_{0\le k\le n-1} S(b^k\theta)Y(b^n\theta)$. The kth column of matrix B^n can be found from the trigonometric polynomial $Z_n(\theta)$ if $Y(\theta)=e^{ik\theta}$. From that formula, it is possible to see that the (j,k)-entry of matrix B^n is the number $s_{j-kb^n}^{(n)}$ if the polynomial $\prod_{k=0,n-1} S(b^k\theta)$ is equal to $\sum_k s_k^{(n)} e^{ik\theta}$. ■

Definition. We say that a matrix $B=(b_{j,k})_{-\infty<k<\infty,-\infty<j<\infty}$ is *M-concentrated* if, for every j of $[-M,M]$ and for every k outside of $[-M,M]$, $b_{j,k}=0$.

Definition. The *M-truncation* of a matrix $B=(b_{j,k})_{-\infty<k<\infty,-\infty<j<\infty}$ is the matrix $(b_{j,k})_{-M\le k\le M,-M\le j\le M}$. This matrix is denoted by $[B]_M$.

Lemma 7.5. *If $S(\theta)=\sum s_k e^{ik\theta}$ is a trigonometric polynomial of degree N, if B is the matrix (s_{j-bk}), and if M is the integral part of $N/(b-1)$, then B is an M-concentrated matrix.*

Proof. If M is the integral part of $N/(b-1)$, let us consider two indices j and k of matrix B such that $|j|\le M$ and $|k|>M$. Then $|j-bk|\ge(M+1)b-M$ and $|j-bk|\ge M(b-1)+b>N$. So $s_{j-bk}=0$. B is an M-concentrated matrix. ■

Lemma 7.6. *If B and C are two M-concentrated matrices, BC is an M-concentrated matrix and $[BC]_M=[B]_M[C]_M$.*

We leave the proof to the reader.

Corollary 7.7. *If A is an M-concentrated matrix, then for any positive integral value of n, $[A^n]_M=[A]_M^n$.*

Theorem 7.8. *In an iterative process of type (b,N), we set M as the integral part of $(N-1)/(b-1)$ and J_n as $\int_{(0,2\pi)}\prod_{0\le k\le n-1} S(\theta b^k)\,d\theta$. If r is the spectral radius of $[B]_M$ and if s is the largest multiplicity of an eigenvalue of modulus equal to r of this matrix, then there is a constant C_2 such that, for any positive integral value n,*

$$|J_n|\le C_2 n^{s-1} r^n.$$

Proof. If the secondary factor of an iterative process of type (b,N) is $S(\theta)=P(\theta)/[\sin(b\theta/2)/\sin(\theta/2)]^{2N}=\sum s_k e^{ik\theta}$, matrix B is $(b_{j,k})=(s_{j-bk})$. The number J_n is $\int_{(0,2\pi)}\prod_{0\le k\le n-1} S(\theta b^k)\,d\theta$. J_n is the central element (entry corresponding to rank 0 and column 0) of matrix B^n. Since S is a trigonometric polynomial of degree $N-1$, matrix B is M-concentrated if M is equal to the integral part of $(N-1)/(b-1)$. According to Corollary 7.7, J_n is the central element of the truncated matrix $[B]_M^n$. If $\varphi(x)=\sum_{k\ge 0} c_k x^k$ is the minimal polynomial of $[B]_M$, then $[B]_M^n\varphi([B]_M)=\sum_{k\ge 0} c_k[B]_M^{n+k}=0$ for $n=0,1,2,\ldots$. From this matrix equation, it follows that the sequence J_n satisfies the linear recurrence $\sum_{k\geq 0} c_k J_{n+k}=0$. As can be found in Gel'fond [5] or in other classical books, the sequence J_n is a finite linear combination of the following specific sequences. These sequences are $n^j\lambda^n$ where λ is a root of the minimal polynomial $\varphi(x)$ and j is a nonnegative integer which is smaller than the multiplicity of the root λ. If r is the largest modulus of a root of φ, and if s is the largest multiplicity of a root of φ of modulus equal to r, then each admissible sequence $n^j\lambda^n$ is bounded by a suitable multiple of $n^{s-1}r^n$. There is a constant C_2 such that, for any positive integral value n, $|J_n|\leq C_2 n^{s-1}r^n$. The number r is also the spectral radius of matrix $[B]_M$. ■

Corollary 7.9. *In the process of type* (b, N), *if* r *is the spectral radius of matrix* $[B]_M$, *there is a number* C *and an integer* s *such that, for any* $n\geq 1$,

$$\int_{(0,\pi b^n)} \xi^{2N} G(\xi)\, d\xi \leq C n^s r^n b^{2Nn}.$$

Theorem 7.10. *If* r *is the spectral radius of the associated matrix* $[B]_M$ *of an interpolation process of type* (b, N) *and if* α *is a positive real value such that* $rb^\alpha<1$, *then* $\int |\xi|^\alpha G(\xi)\, d\xi<\infty$.

Proof.

$$K_n=\int_{(\pi b^{n-1},\pi b^n)} \xi^\alpha G(\xi)\, d\xi,$$

$$K_n\leq (b^{n-1})^{\alpha-2N}\int_{(\pi b^{n-1},\pi b^n)} \xi^{2N} G(\xi)\, d\xi.$$

$\sum_{n\geq 0} K_n\leq b^{2N-\alpha} C\sum_{n\geq 0}(b^\alpha r)^n n^s<\infty$ since by hypothesis $b^\alpha r<1$. So it follows that $\int |\xi|^\alpha G(\xi)\, d\xi=2(\int_{(0,\pi)} \xi^\alpha G(\xi)\, d\xi+\sum_{n\geq 0} K_n)<\infty$. ■

Definition. The number E such that $b^E=1/r$ is called the *critical exponent* of the process of interpolation. An application of Lemma 7.1 and Theorem 7.8 gives the following results.

Theorem 7.11. *If* m *is a positive integral value smaller than* E, *then the derivative of order* m *of the fundamental function is continuous. Moreover, if* m *is a nonnegative integral value and if* α *a is real number between* 0 *and* 1 *such that* $m+\alpha$ *is smaller than* E, *then* $F^{(m)}(t)$ *belongs to the Lipschitz class of order* α.

8. Finite Differences of the Fundamental Function

We found that the derivative of a suitable order of the fundamental function belongs to a Lipschitz class. We prove in this section that this result cannot be substantially improved. If $S(\theta)=\sum_k s_k e^{ik\theta}$ is the secondary factor of the characteristic polynomial $P(\theta)$, we have already defined a matrix $B=(s_{j-bk})$. We define another matrix from the fundamental function F: $A=(F(j/b-k))$. We begin with the study of eigenvalues of matrices A and B.

Lemma 8.1. *If r is an integer of $[0, 2N-1]$ and if V is the column vector $(k^r)_{-\infty<k<\infty}$, then $AV = b^{-r}V$.*

Proof. If r is an integer of $[0, 2N-1]$, it follows from Lemma 4.1 and Theorem 3.2 that $\sum_k F(j/b-k)k^r = (j/b)^r$. If V is the column vector $(k^r)_{-\infty<k<\infty}$, then $AV = b^{-r}V$. ■

Theorem 8.2. *Any eigenvalue of matrix A other than the following powers of $(1/b)$: $1, 1/b, 1/b^2, \ldots, 1/b^{2N-1}$, is an eigenvalue of matrix B. Any eigenvalue of B is an eigenvalue of A.*

Proof. Let us assume that V is column-vector $\neq 0$, with complex components, and λ is a complex number such that $AV = \lambda V$. We also assume that λ is not one of the following integral powers of $1/b$: $1, 1/b, 1/b^2, \ldots, 1/b^{2N-1}$. If $V = (v_k)$, then we introduce the trigonometric series $Y(\theta) = \sum_k v_k e^{ik\theta}$. We do not care about the convergence of this series, it is a formal trigonometric series. Since $AV = \lambda V$, then $P(\theta)Y(b\theta) = \lambda Y(\theta)$ where $P(\theta)$ is the characteristic polynomial of the interpolation process. According to Theorem 7.2, $P(\theta) = [\sin(b\theta/2)/\sin(\theta/2)]^{2N} S(\theta)$.

If $Z(\theta) = \sin^{2N}(\theta/2)Y(\theta)$, then $S(\theta)Z(b\theta) = \lambda Z(\theta)$. If $Z(\theta) = \sum_k w_k e^{ik\theta}$ and if $W = (w_k)$, then $BW = \lambda W$. Let us check that $W \neq 0$. If $W = 0$, then we would get $Z(\theta) = 0 = \sin^{2N}(\theta/2)Y(\theta)$. The meaning of the relation $\sin^{2N}(\theta/2)Y(\theta) = 0$ is that all finite differences of order $2N$ of the sequence v_k are zero. If this is the case, then there would be a polynomial $p(t)$ of degree smaller than $2N$ such that $v_k = p(k)$, $k = 0, \pm 1, \pm 2, \ldots$. If $V_r = (k^r)$, then V would be a linear combination of $\{V_r\}_{0 \le r \le 2N-1}$. As $AV_r = 1/b^r V_r$, $AV = \lambda V$ and λ is not one of the numbers $1/b_r (0 \le r \le 2N-1)$, then we would get a contradiction.

Let us now look at the second part of the theorem. If λ is an eigenvalue of B, we need to prove that λ is an eigenvalue of A. The case $\lambda = 1/b^r$ (where $r = 0, 1, \ldots, 2N-1$) is dealt with in Lemma 8.1. We may assume that λ is an eigenvalue that is different from $1/b^r$ ($r = 0, 1, \ldots, 2N-1$). Then there is a vector $W = (w_k)$ such that $BW = \lambda W$. We set $Z(\theta) = \sum_k w_k e^{ik\theta}$ and the relation $S(\theta)Z(b\theta) = \lambda Z(\theta)$ is induced. If δ_{2N} is the operator of the central finite differences of order $2N$, then there is a sequence v_k such that $w_k = \delta_{2N} v_k$. If we set $Y(\theta) = \sum_k v_k e^{ik\theta}$, then $Z(\theta) = \sin^{2N}(\theta/2)Y(\theta)$. From this, it follows that $S(\theta)\sin^{2N}(b\theta/2)Y(b\theta) = \lambda \sin^{2N}(\theta/2)Y(\theta)$ and $\sin^{2N}(\theta/2)[P(\theta)Y(b\theta) - \lambda Y(\theta)] = 0$.

If $V = (v_k)$, then there is a polynomial $p(k)$ of degree smaller than $2N$ such that $AV - \lambda V = (p(k))$. It is easy to find a polynomial $q(k)$ of degree smaller than $2N$ such that $(A - \lambda I)(q(k)) = (p(k))$. The column vector $V - (q(k))$ is then an eigenvector for matrix A. ■

Lemma 8.3. *If A is the matrix $(F(j/b-k))_{-\infty<k<\infty, -\infty<j<\infty}$, then matrix A^n is $(F(j/b^n - k))$.*

Proof. We proceed by induction on n. If $A^{n+1}=(c_{j,k})$, we use the relation $A^{n+1}=AA^n$.

$$c_{j,k}=\sum_r F(j/b-r)F(r/b^n-k)=\sum_s F((j-kb^{n+1})/b-s)F(s/b^n).$$

From equation (3.1), $c_{j,k}=F(j/b^{n+1}-k)$. ■

Lemma 8.4. *Any eigenvalue of B is in the unit-disk of the complex plane.*

Proof. If λ is an eigenvalue of matrix B, there is a rank vector $W=(w_k)_{-M\le k\le M}$ which is an eigenvector for the truncated matrix $[A]_M$: $W[A]_M=\lambda W$. Since $W[A]_M^n=\lambda^n W$, it follows that

$$\sum_j w_jF(j/b^n-k)=\lambda^n w_k \qquad (\text{for } k=-M,-M+1,\dots,M-1,M).$$

Let us assume that $|\lambda|$ is larger than 1. The vector W is biorthogonal to the eigenvector $(1)_{-M\le k\le M}$ corresponding to the eigenvalue 1: $\sum_k w_k=0$. If $s_k=\sum_{j\le k} w_j$, then $\sum_j w_jF(j/b^n-k)=-\sum_j s_j[F((j+1)/b^n-k)-F(j/b^n-k)]$.

If k is an index for which $w_k\ne 0$, it is possible to find a positive number ε and a sequence j_n of integers such that $F((j_n+1)/b^n-k)-F(j_n/b^n-k)$ is larger than $\varepsilon|\lambda|^n$ in absolute value. The number ε can be chosen as $|w_k/[(2M+1)B]|$ where B is the maximum of numbers $|s_j|$. Since F is continuous, then $|\lambda|<1$. This is a contradiction. ■

Theorem 8.5. *If E is the critical exponent of the interpolation process, we set m as the integral part of E and $\alpha=E-m$. If E is not integral, $F^{(m)}(t)$ cannot belong to the Lipschitz class* Lip *s when s is larger than α. If E is integral, $F^{(m)}$ cannot be a continuous function.*

Proof. Let us first discuss the case where E is not integral. If λ is the maximal eigenvalue of matrix B, there is a rank vector $W=(w_k)_{-M\le k\le M}$ which is an eigenvector for the truncated matrix $[A]_M$: $W[A]_M=\lambda W$. Since $W[A]_M^n=\lambda^n W$, it follows that

$$\sum_j w_jF(j/b^n-k)=\lambda^n w_k \qquad (\text{for } k=-M,-M+1,\dots,M-1,M).$$

The vector W is biorthogonal to each column eigenvector corresponding to the eigenvalue $1/b^r$, $r=0,1,\dots,m$. W is then biorthogonal to any column vector $V=(p(j))_{-M\le j\le M}$ where p is a polynomial of degree smaller or equal to m. From this fact, it follows that W is a linear combination of operators of finite difference of order $m+1$ (any operator of finite difference can be viewed as a rank vector). Let us pick an index k for which $w_k\ne 0$. It is possible to find a positive number ε and a sequence D_n of operators of finite difference of order $m+1$ such that when D_n is applied to the sequence $V_n=(F(j/b^n-k))_{-M\le j\le M}$, a number D_nV_n larger than $\varepsilon\lambda^n$ in absolute value is then produced. The number ε can be chosen as $|w_k/[(2M+1)B]|$ if B is a bound of the absolute values of the coefficients used in the representation of W as a linear combination of operators of finite difference of order $m+1$.

For each value of n, there is a couple of real numbers (u_n, t_n) such that $D_n V_n = [F^{(m)}(t_n) - F^{(m)}(u_n)]b^{-nm}$ and $|u_n - t_n| \le 2M/b^n$. With these inequalities, there is no way for the function $F^{(m)}$ to belong to Lipschitz class Lip s if s is larger than α. One part of the proof is complete.

Let us discuss the case where E is integral. The maximal eigenvalue λ of matrix B is $1/b^m$.

(a) If there is a rank vector $W = (w_k)_{-M \le k \le M}$ which is at the same time an eigenvector for the truncated matrix $[A]_M$: $W[A]_M = \lambda W$ and is biorthogonal to the column eigenvector $(k^m)_{-M \le k \le M}$, then the same arguments as before show that $F^{(m)}$ cannot be continuous. A sequence of couples of real numbers (u_n, t_n) can be found such that $F^{(m)}(t_n) - F^{(m)}(u_n)$ does not converge to 0 while $|u_n - t_n| \le 2M/b^n$.

(b) Otherwise, there is a row eigenvector W and a row vector W' such that $W[A]_M = \lambda W$, $W'[A]_M = \lambda W' + W$, and W' is biorthogonal to the column eigenvector $(k^m)_{-M \le k \le M}$. If $W = (w_j)$ and $W' = (w'_j)$, then $W[A]^n_M = \lambda^n W$ and $W'[A]^n_M = \lambda^n W' + n\lambda^{n-1} W$.

$$\sum_k w'_j F(j/b^n - k) = \lambda^n w'_k + n\lambda^{n-1} w_k \qquad (\text{for } k = -M, -M+1, \ldots, M-1, M).$$

Similar arguments show that $F^{(m)}$ cannot be continuous. There is a positive number ε and a sequence of couples of real numbers (u_n, t_n) can be found such that $|F^{(m)}(t_n) - F^{(m)}(u_n)| > \varepsilon n$ while $|u_n - t_n| \le 2M/b^n$. ■

9. Examples of Secondary Factors

To illustrate previous theorems, we give examples of secondary factors. We bring back the polynomial

$$\Pi(w) = \sum_{k=-bN+1, bN-1} F(k/b) w^{k+2Nb-1}.$$

We have shown that $\Pi(w)$ is divisible by $[(w^b - 1)/(w-1)]^{2N}$, the quotient is a polynomial $\Sigma(w)$ of degree $2N-2$. We compute the first two coefficients σ_0 and σ_1 of $\Sigma(w)$.

Lemma 9.1. *In the interpolation process of type* (b, N),

$$\sigma_0 = (1/b)^{2N-1}\left[\prod_{k=1, N-1} (1 - k^2 b^2)\right]/(2N-1)!,$$

$$\sigma_1 + 2N\sigma_0 = 2(1/b)^{2N-1}\left[\prod_{k=1, N-1} (4 - k^2 b^2)\right]/(2N-1)!.$$

Proof. We used the fact that the first two coefficients of $\Pi(w)$ are $F(-bN + 1/b)$ and $F(-bN + 2/b)$. According to formula (4.2),

$$F(-bN + r/b) = \left[\prod_{k=-N+1, N-1} (r/b - k)\right] \Big/ (2N-1)!.$$

The first two coefficients of $[(w^b-1)/(w-1)]^{2N}$ are 1 and $2N$. Since $(\sigma_0+\sigma_1 w+\cdots)(1+2Nw+\cdots)=\Pi(w)$, a simple computation concludes the proof. ■

Lemma 9.2. *For an interpolation process of type* $(b, 2)$, *the secondary factor is* $S(y)=[(b^2+2)-(b^2-1)\cos y]/(3b^3)$.

Lemma 9.3. *For an interpolation process of type* $(b, 3)$, $S(y)$ *is* $[3(4b^4+5b^2+11)-2(b^2-1)(8b^2+13)\cos y+(b^2-1)(4b^2-1)\cos(2y)]/(60b^5)$.

Last two lemmas can be proved through Lemma 9.1 and with a little algebra. We give the first secondary factors S (when $N=2$) for $b=2$, 3, 4. If $b=2$, $S(y)=1/4-\cos y/8$. If $b=3$, $S(y)=(11-8\cos y)/81$. If $b=4$, $S(y)=(6-5\cos y)/64$.

Let us look at eigenvalues of matrix $[B]_M$ as previously described. If $b=2$, then $[B]_M$ is a matrix of order 3 whose eigenvalues are 1/4 and $-1/16$. The critical exponent is in this case $E=2$. This means that the fundamental function is differentiable and F' belongs to Lipschitz class Lip s for any $s\in(0,1)$.

If $b>2$, then the matrix $[B]_M$ is of order 1. The unique eigenvalue λ is the constant coefficient in the Fourier expansion of $S(y)$, $\lambda=(b^2+2)/(3b^3)$. The critical exponent E is the solution to equation $b^E=3b^3/(b^2+2)$. E is between 1 and 2. The function F always has a continuous derivative. For example, if $b=3$, then $E=1.817$ and if $b=4$, then $E=1.707$.

We give the first secondary factors S (when $N=3$) for $b=2$, 3, 4:

$$S(y)=[19-18\cos y+3\cos(2y)]/128 \quad \text{if} \quad b=2.$$

$$S(y)=[57-68\cos y+14\cos(2y)]/729 \quad \text{if} \quad b=3.$$

$$S(y)=[223-282\cos y+63\cos(2y)]/4096 \quad \text{if} \quad b=4.$$

Let us look at eigenvalues of matrix $[B]_M$ as previously described. If $b=2$, then $[B]_M$ is a matrix of order 5 whose eigenvalues are 3/256, $-9/128$, 9/64, and $-1/16$. The largest eigenvalue of matrix $[B]_M$ is 9/64. The critical exponent in this case is $E=\log_2(64/9)=2.830$. This means that the fundamental function is twice differentiable and F'' belongs to Lipschitz class Lip s for any $s\in(0, 0.830)$.

If $b=3$, then $[B]_M$ is a matrix of order 3 whose eigenvalues are 7/729 and 57/729. The critical exponent is then 2.319. F'' belongs to Lipschitz class Lip s for any $s\in(0, 0.319)$. If $b>3$, than $[B]_M$ is of order 1. The unique eigenvalue λ is the constant coefficient in the Fourier expansion of $S(y)$: $(4b^4+5b^2+11)/20b^5)$. The critical exponent E is the root of the equation $b^E=1/\lambda$. If $b=4$, then $E=2.099$, F'' belongs to Lip s for any $s\in(0, 0.099)$. If $b\geq 5$, then $1<E<2$.

10. The Derivatives of the Fundamental Function

If $F(t)$ is the fundamental function and if r is a positive integer smaller than the critical exponent E, then we will indicate how to compute the rth derivative of F, $F^{(r)}(t)$, when t is a b-adic rational number. As we show immediately, the main point is the computation of $F^{(r)}(t)$ at integral values of t. This is a consequence of formula (3.1). If we differentiate both sides of this formula r times with respect to t, then we get

$$F^{(r)}(k/b^m)/b^{rm} = \sum_n F(n/b^m)F^{(r)}(k-n). \tag{10.1}$$

We consider the case of an interpolation process of type (b, N). $F(t)$ is an even function and $F'(t)$ is an odd function. In particular, $F'(0)=0$. $F^{(r)}$ is an even function if r is even and $F^{(r)}$ is odd when r is odd. We find the relationship between $F^{(r)}(t)$ at integral values of t and row eigenvectors of the matrix $A=(F(j/b-k))$. We compute $F'(n)$ for every integer n, for every base b, for $N=2$ and 3. We compute $F''(n)$ for every integer n, for every base b, for $N=3$.

Theorem 10.1 *If r is a nonnegative integer smaller than the critical exponent of an interpolation process of type (b, N), then the row vector $(F^{(r)}(n))_{-\infty<n<\infty}$ is an eigenvector of the matrix $A=(a_{j,k})=(F(j/b-k))$ for the eigenvalue $1/b^r$. Moreover, $\sum_n n^r F^{(r)}(n)=(-1)^r r!$.*

Proof. If $V=(F^{(r)}(j))_{-\infty<j<\infty}$ and $(w_j)=VA$, then

$$w_k = \sum_j F^{(r)}(j)F(j/b-k) = \sum_n F^{(r)}(n+kb)F(n/b).$$

If, in the last member, n is changed to $-n$ and if the symmetry of F is used, then $w_k=\sum_n F^{(r)}(kb-n)F(n/b)$. From formula (10.1),

$$w_k = F^{(r)}(k/b)/b^r = v_k/b^r.$$

W is an eigenvector and $1/b^r$ is the eigenvalue. In order to prove the last part of the theorem, we use Lemma 4.1 with the polynomial $P(t)=t^r$ and Theorem 3.2: $t^r=\sum_n n^r F(t-n)$. If this identity is differentiated r times with respect to t, then

$$r! = \sum_n n^r F^{(r)}(t-n) = (-1)^r \sum_n n^r F^{(r)}(t+n).$$

If t is set to zero, then $\sum_n n^r F^{(r)}(n)=(-1)^r r!$. ∎

Corollary 10.2. *If r is a nonnegative integer smaller than the critical exponent of an interpolation process of type (b, N), if M is the largest integer smaller than $N+(n-1)/(b-1)$, if $V=(v_j)_{-M\le j\le M}$ is a row eigenvector of the truncated matrix $[A]_M=[(F(j/b-k))]_M$ for the eigenvalue $1/b^r$, and if $\sum_n n^r v_n=(-1)^r r!$, then $V=(F^{(r)}(n))$.*

This follows from the fact that $1/b^r$ is a simple eigenvalue of matrix $[A]_M$.

Theorem 10.3. *In the interpolation process of type $(b, 2)$, $F'(1)=-2/3$ and $F'(2)=1/12$ whatever the base b.*

Proof. From Lemma 3.1, $F(t)$ vanishes on $[2+1/(b-1),\infty)$. $F(t)=0$ on $[3,\infty)$ and $F'(3)=0$. In order to compute $F'(2)$, Theorem 3.2 and Lemma 4.1 are invoked; if $p(t)=t(t^2-1)$, the following identity holds when $|t|<1$:

$$-p(t)=p(2)[F(t+2)-F(t-2)]+p(3)[F(t+3)-F(t-3)].$$

If we differentiate both sides with respect to t and if we set t to 0, $F'(2)=-p'(0)/(2p(2))=1/12$.

Computation of $F'(1)$ is performed using a similar argument. If $q(t)=t(t^2-4)$ and if $|t|<1$, then $-q(t)=q(1)[F(t+1)-F(t-1)]+q(3)[F(t+3)-F(t-3)]$. So $F'(1)=-q'(0)/(2q(1))=-2/3$. ∎

Theorem 10.4. *In the interpolation process of type* $(b,3)$ *with a base* $b\geq 3$, $F'(1)=-3/4$, $F'(2)=3/20$, and $F'(3)=-1/60$.

Proof. We argue as in the previous theorem. From Lemma 3.1, $F(t)$ vanishes on $[3+2/(b-1),\infty)$. Since $b\geq 3$, then $F(t)=0$ on $[4,\infty)$. $F'(4)=0$. In order to compute $F'(3)$, Theorem 3.2 and Lemma 4.1 are again invoked; if $p(t)=t(t^2-1)(t^2-4)$, the following identity holds when $|t|<1$:

$$\begin{aligned}-p(t)=p(2)[F(t+2)-F(t-2)]+p(3)[F(t+3)-F(t-3)]\\ +p(4)[F(t+4)-F(t-4)].\end{aligned}$$

Hence $F'(3)=-p'(0)/(2p(3))=-1/60$.

Computation of $F'(2)$ is performed using the polynomial $q(t)=t(t^2-1)(t^2-9)$. $F'(2)=-q'(0)/(2q(2))=3/20$. Computation of $F'(1)$ is done by using the polynomial $r(t)=t(t^2-4)(t^2-9)$. $F'(1)=-r'(0)/(2r(1))=-3/4$. ∎

For computation of the derivatives of F at integral points for six nodes with base 2, there is a small complication because $F'(4)\neq 0$. Using the same polynomials p, q, and r as in the previous proof, we get three equations relating $F'(1)$, $F'(2)$, $F'(3)$, and $F'(4)$:

$$F'(1)+r(4)F'(4)/r(1)=-r'(0)/(2r(1)),$$

$$F'(2)+q(4)F'(4)/q(2)=-q'(0)/(2q(2)),$$

and

$$F'(3)+p(4)F'(4)/p(3)=-p'(0)/(2p(3)).$$

Another equation can be found by using the fact that $F(t+4)=-3F(2t+3)/256$ when t is positive. So $F'(4)=-3F'(3)/128$. The solution of these linear equations is the content of next theorem.

Theorem 10.5. *In the interpolation process of type* $(2,3)$, $F'(1)=-272/365$, $F'(2)=53/365$, $F'(3)=-16/1095$, *and* $F'(4)=-1/2920$.

Interpolation processes of type (b, N) for which F is twice differentiable occur in the following cases: $N=3$ and $b=2$, 3, or 4. After some computation, the following results about $F''(n)$ can be proved.

Theorem 10.6. *In the interpolation process of type* $(2,3)$ *(the base is* 2*),* $F''(0)=-295/56$, $F''(1)=356/105$, $F''(2)=-92/105$, $F''(3)=4/35$, *and* $F''(4)=3/560$.

Theorem 10.7. *In the interpolation process of type* $(b,3)$ *with a base* $b=3$ *or* 4,

$$F''(0)=(25/18)(4b^4+36b^3-17b^2-23)/(4b^4-20b^3+5b^2+11),$$

$F''(1)=-13/24-3F''(0)/4$, $F''(2)=2/3+3F''(0)/10$, $F''(3)=-1/8-F''(0)/20$ *and, for* $n\geq 4$, $F''(n)=0$.

11. Error Bounds in Interpolation

Let $f(t)$ be a real function defined on the real axis. We use the following sequence: $y(n)=f(n)$ for relative integers n. $y(t)$ is the interpolation of the sequence $y(n)$ according to scheme (4.1) for a given base b and with $2N$ nodes. The error in the interpolation is the function $e(t)=f(t)-y(t)$. We would like to get simple bounds for $e(t)$.

Theorem 11.1. *Let* $y(t)$ *be the interpolating function for a function* f *according to scheme* (4.1) *with a base* b *and an even number* $2N$ *of nodes. If* $t\in[0,1]$ *and if* $p(t)$ *is the polynomial of degree* $2N-1$ *such that* $p(n)=f(n)$, $n=-N+1, -N+2,\ldots,N-1$, *and* N, *then*

$$|f(t)-y(t)|\leq|f(t)-p(t)|+\sum_{k>0}(|f(N+k)-p(N+k)|$$
$$+|f(-N+1-k)-p(-N+1-k)|)\varepsilon_k.$$

The number ε_k *is the maximal value of* $|F(t)|$ *on* $[k-1,k]$.

Proof. We use the triangular inequality $|f(t)-y(t)|=|f(t)-p(t)+p(t)-y(t)|\leq|f(t)-p(t)|+|p(t)-y(t)|$. If $t\in[0,1]$, then Theorem 3.2 tells us that $p(t)-y(t)=\sum_k[p(k)-f(k)]F(t-k)$. Since $p(k)=f(k)$ for $k=-N+1,-n+2,\ldots,N-1$, N and since F is an even function, then

$$p(t)-y(t)=\sum_{k>N}\{[p(k)-f(k)]F(k-t)+[p(-k+1)-f(-k+1)]F(k-1+t)\},$$

$$|p(t)-y(t)|\leq\sum_{k>N}(|p(k)-f(k)|+|p(-k+1)-f(-k+1)|)\varepsilon_k.$$

The required inequality is then proved. ∎

A similar argument can be used to obtain bounds about the rth derivative of $y(t)-p(t)$. If $\varepsilon_k^{(r)}$ is the maximal value of $|F^{(r)}(t)|$ on $[k-1,k]$, then

$$|f^{(r)}(t)-y^{(r)}(t)|\leq|p^{(r)}(t)-y^{(r)}(t)|$$
$$+\sum_{k>N}(|p(k)-f(k)|+|p(-k+1)-f(-k+1)|)\varepsilon_k^{(r)}.$$

We give a table of numbers $\varepsilon_k^{(r)}$ for $b=2$, 3, and 4 and for $N=2$ and 3. These numbers were found after numerical experiments.

Case $N=2$

b	ε_3	$\varepsilon_3^{(1)}$
2	0.00459	0.08333
3	0.00316	0.08333
4	0.00251	0.08333

Case $N=3$

b	ε_4	ε_5	$\varepsilon_4^{(1)}$	$\varepsilon_5^{(1)}$	$\varepsilon_4^{(2)}$	$\varepsilon_5^{(2)}$
2	0.00122	0.00001	0.01461	0.00034	0.11429	0.00536
3	0.00083	0	0.16667	0	0.36111	0
4	0.00063	0	0.16667	0	1.15152	0

Remark. When $N=2$, it seems that the exact value of $\varepsilon_3^{(1)}$ is $1/12$ which is the value of $F'(2)$. When $N=3$, the exact values of $\varepsilon_4^{(1)}$, $\varepsilon_5^{(1)}$, $\varepsilon_4^{(2)}$, and $\varepsilon_5^{(2)}$ are probably $|F'(3)|$, $|F'(4)|$, $|F''(3)|$, and $|F''(4)|$, respectively.

12. Conclusion

The main purpose of this paper was to study a process of interpolation which could be useful for numerical analysis. This paper is a sequel to three other works, [4], [2], and [3]. In [4] the second author introduced the interpolation process of type (2, 2), the base is 2, the number of nodes is four. The fundamental function F of that case was then studied; in [2] we indicated that some properties of F could be derived more easily by using Fourier transform. In [2] and in [3] we defined what we call the model of iterative interpolation with a base b. As has been noted, this model embodies all von Koch–Mandelbrot curves as described in Mandelbrot's book [6]. We say that an iterative interpolation process is in L^2 if the Fourier transform of the process is in L^2. In [3] we found all von Koch–Mandelbrot curves which are in L^2.

If there is a choice to be done amongst a base b, it seems better to take the base b equal to 2. We can expect that the interpolation which will be produced will be smoother than with another base. If the complexity of the computations is increased when a large number of moving nodes is used, however, the accuracy of the interpolation will probably be improved with "analytic" data. It should be interesting to know the behavior of the interpolation process of type $(2, N)$ for other values of N such as $N=4$, 5, and 6.

We now present some open questions. Is it true that any interpolation process of type (b, N), $N \geq 2$, gives rise to a continuously differentiable fundamental function? If E is the critical exponent of an interpolation process of type (b, N) and if m is the integral part of E, is it possible to find the Hausdorff dimension of the graph of $F^{(m)}$? The last question is the following. How can we handle the study of the general iterative interpolation process when it is not related to polynomial interpolation? In (2.1) we may take parameters p_k as reals or even as complex numbers. The main question is to find conditions on the parameters p_k such that the interpolating function $y(t)$ defined on b-adic numbers has a continuous extension to the real axis. It does not seem easy to complete this task satisfactorily. We conjecture that the fundamental function has a continuous extension if the value 1 is a simple eigenvalue of matrix $A = (F(j/b - k))$ and if the modulus of every other eigenvalue is smaller than 1.

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