On a C^2-nonlinear subdivision scheme avoiding Gibbs oscillations

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Abstract

This paper is devoted to the presentation and the study of a new nonlinear subdivision scheme eliminating the Gibbs oscillations close to discontinuities. Its convergence, stability and order of approximation are analyzed. It is proved that this scheme converges towards limit functions of Hölder regularity index larger than 1.192. Numerical estimates provide an Hölder regularity index of 2.438. Up to our knowledge, this subdivision scheme is the first one that achieves simultaneously the control of the Gibbs phenomenon and regularity index larger than 1 for its limit functions.

Key Words. Nonlinear subdivision scheme, limit function, regularity, stability, Gibbs phenomenon.

AMS(MOS) subject classifications. 41A05, 41A10, 65D05, 65D17

1 Introduction

Subdivision schemes are useful tools for generating smooth curves and surfaces. For convergent schemes, starting from discrete sets of control points and using basic rules of low complexity, curves or surfaces can be obtained as

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limits (called limit functions) of sequences of points generated by recursive applications of the subdivision scheme.

A simple example of subdivision scheme is the family of interpolatory subdivision schemes, based on Lagrange's interpolation that have been derived and analyzed in [11]. Other example is the family of spline subdivision schemes related to spline spaces [8].

The four-point interpolatory scheme [16], [15] is a convergent linear scheme of the first family, involving four-point stencils at each subdivision, for which the limit function is at least in the space[1] C^1. The Chaikin algorithm [7] is an example of spline subdivision scheme, with lower complexity as the previous example and converging towards C^{2-} functions[2].

For applications, for instance to computer aided geometric design or image processing, complexity and convergence/regularity are not the only quality criterion. The order of approximation, that characterizes the precision of the scheme, is an other important one. Moreover, oscillations that could occur in the limit function at the vicinity of strongly variating data (coming from the sampling of discontinuous functions) , called Gibbs oscillation, are really undesirable.

In the last decade, various attempts to improve the properties of linear subdivision schemes, have lead to nonlinear subdivision schemes. For such schemes, the subdivision rules become data dependant; in addition to the previously defined criteria, one should add a stability property that ensures that the nonlinear scheme is linearly affected by perturbations of the data (for linear schemes, the stability is a direct consequence of the convergence).

For nonlinear subdivision schemes, very few results concerning convergence or stability are available, see for instance [5], [9], [12], [21], [10] and [17].

A large family of nonlinear subdivision schemes to which belong the ENO, WENO or PPH schemes [9], [4] is made by the schemes constructed as a perturbation of the four-point linear interpolatory Lagrange scheme based on centered degree 3 polynomial interpolation. These schemes are interpolatory subdivision schemes (i.e. based on interpolation rules) and are constructed to avoid the Gibbs oscillations occurring classically for linear interpolatory schemes (see Figure 1). The schemes of this family are unfortunately characterized by a low regularity of the limit functions of type

[1] For $0 < \alpha < 1$, $f \in C^\alpha(\mathbb{R})$ iff f is bounded and $\exists C > 0$ such that $\forall x, y \in \mathbb{R}, |f(x) - f(y)| \leq C|x-y|^\alpha$
For $\alpha > 1$, $f \in C^\alpha(\mathbb{R})$ iff $f^{([\alpha])}$ is bounded and $f^{([\alpha])} \in C^{(\alpha-[\alpha])}$ where $[\alpha]$ is the integer part of α

[2] $C^{\alpha-} = \{f \in C^\beta, \forall \beta < \alpha\}$

C^{1-}. Moreover, the ENO scheme is unstable.

In [14], a new linear four-point subdivision scheme was presented. Its refinement rule is based on local cubic interpolation followed by a shift of 1/4 or, in other words an evaluation at positions 1/4 and 3/4 rather then the standard evaluation at 1/2. This new scheme was shown to be convergent towards a C^2 curve.

The aim of this paper is to analyze the scheme obtained using the same trick (shift of 1/4) for the PPH-type schemes [4] which are derived modifying the classical four point interpolatory subdivision scheme substituting the harmonic mean to the arithmetic mean. After the definition of the new scheme in section 2 we analyze successively its convergence (section 3), its stability (section 4) and its order of approximation (in section 5). Its behavior in presence of strongly variating data (Gibbs oscillations) is analyzed in section 6. The last section is devoted to concluding remarks.

2 A new nonlinear subdivision scheme

As mentioned above, the starting point of our work is the construction of N. Dyn, M.S. Floater and K. Hormann in [14]. There, a new linear four-point subdivision scheme that generates C^2 curves is presented. Its refinement rule is based on the local cubic Lagrange interpolation, followed by evaluation at positions 1/4 and 3/4 of the refined interval. For all $f \in l^{\infty}(\mathbb{Z})$, the scheme is then given by

$$\begin{aligned} (Sf)_{2n} &= -\frac{7}{128}f_{n-1} + \frac{105}{128}f_n + \frac{35}{128}f_{n+1} - \frac{5}{128}f_{n+2}, \\ (Sf)_{2n+1} &= -\frac{5}{128}f_{n-1} + \frac{35}{128}f_n + \frac{105}{128}f_{n+1} - \frac{7}{128}f_{n+2}. \end{aligned} \tag{1}$$

Following [4] where a nonlinear scheme is derived modifying the classical four-point interpolatory subdivision scheme substituting the harmonic mean to the arithmetic mean, we first obtain two new formulations of the scheme (1).

1

$$\begin{aligned} (Sf)_{2n} &= \frac{49}{64}f_n + \frac{14}{64}f_{n+1} + \frac{1}{64}f_{n+2} - \frac{7}{64}\frac{(d^2f_n + d^2f_{n+1})}{2}, \\ (Sf)_{2n+1} &= \frac{15}{64}f_n + \frac{50}{64}f_{n+1} - \frac{1}{64}f_{n+2} - \frac{5}{64}\frac{(d^2f_n + d^2f_{n+1})}{2}. \end{aligned}$$

2

$$
\begin{aligned}
(Sf)_{2n} &= -\frac{1}{64}f_{n-1} + \frac{15}{64}f_n + \frac{50}{64}f_{n+1} - \frac{5}{64}\frac{(d^2f_n + d^2f_{n+1})}{2}, \\
(Sf)_{2n+1} &= \frac{1}{64}f_{n-1} + \frac{49}{64}f_n + \frac{14}{64}f_{n+1} - \frac{7}{64}\frac{(d^2f_n + d^2f_{n+1})}{2}.
\end{aligned}
$$

where (d^2f) is defined by $d^2f_n = f_{n+1} - 2f_n + f_{n-1}$.

The two formulations differs essentially in the distribution of the points f_n contributing to the three first terms of **1** and **2**.

Using the same strategy as in [4], we define the new nonlinear subdivision scheme S_{PPHA} associated to (1) by

If $|d^2f_n| \geq |d^2f_{n+1}|$**,**

$$
\begin{aligned}
(S_{\text{PPHA}}f)_{2n} &= \frac{49}{64}f_n + \frac{14}{64}f_{n+1} + \frac{1}{64}f_{n+2} - \frac{7}{64}\text{PPH}(d^2f_n, d^2f_{n+1}), \\
(S_{\text{PPHA}}f)_{2n+1} &= \frac{15}{64}f_n + \frac{50}{64}f_{n+1} - \frac{1}{64}f_{n+2} - \frac{5}{64}\text{PPH}(d^2f_n, d^2f_{n+1}).
\end{aligned}
$$

If $|d^2f_n| < |d^2f_{n+1}|$**,**

$$
\begin{aligned}
(S_{\text{PPHA}}f)_{2n} &= -\frac{1}{64}f_{n-1} + \frac{15}{64}f_n + \frac{50}{64}f_{n+1} - \frac{5}{64}\text{PPH}(d^2f_n, d^2f_{n+1}), \\
(S_{\text{PPHA}}f)_{2n+1} &= \frac{1}{64}f_{n-1} + \frac{49}{64}f_n + \frac{14}{64}f_{n+1} - \frac{7}{64}\text{PPH}(d^2f_n, d^2f_{n+1}),
\end{aligned}
$$

where PPH stands for the harmonic mean defined by

$$
(x, y) \in I\!R^2 \mapsto \text{PPH}(x, y) := \frac{xy}{x+y}(\text{sgn}(xy) + 1), \tag{2}
$$

with $\text{sgn}(x) = 1$ if $x \geq 0$ and $\text{sgn}(x) = -1$ if $x < 0$.

The initial motivation for the substitution of the arithmetic mean by the harmonic mean is the elimination of oscillations near strong variating data thanks to the fact that

$$
|\text{PPH}(x, y)| \leq 2\min(|x|, |y|), \tag{3}
$$

substitutes to

$$
\frac{x+y}{2} \leq max(|x|, |y|).
$$

Before analyzing in details the properties of the new scheme S_{PPHA} we summarize the most important properties of the harmonic mean in the following proposition (see [5] for more details).

Proposition 1 *For all $(x, y) \in \mathbb{R}^2$, the harmonic mean* $\textsc{pph}(x, y)$ *satisfies*

1. $\textsc{pph}(x, y) = \textsc{pph}(y, x)$.
2. $\textsc{pph}(x, y) = 0 \quad if\ xy \leq 0$.
3. $\textsc{pph}(-x, -y) = -\textsc{pph}(x, y)$.
4. $\textsc{pph}(x, y) = \frac{sign(x)+sign(y)}{2} min(|x|, |y|) \left[1 + \left|\frac{x-y}{x+y}\right|\right]$.
5. $|\textsc{pph}(x, y)| \leq \max(|x|, |y|)$.
6. $|\textsc{pph}(x, y)| \leq 2 \min(|x|, |y|)$.
7. *For* $x, y > 0$, $\min(x, y) \leq \textsc{pph}(x, y) \leq \frac{x+y}{2}$.
8. *If* $x = O(1)$, $y = O(1)$, $|y - x| = O(h)$ *and* $xy > 0$ *then*
$$|\frac{x+y}{2} - \textsc{pph}(x, y)| = O(h^2).$$
9. $|\textsc{pph}(x_1, y_1) - \textsc{pph}(x_2, y_2)| \leq 2 \max(|x_1 - x_2|, |y_1 - y_2|)$.

3 Convergence and Regularity

We recall the following definition.

Definition 1 *A subdivision scheme S is said to be convergent if*

$$\forall f \in l^\infty(\mathbb{Z}),\ \exists S^\infty f \in C^0(\mathbb{R}) \text{ such that } \lim_{j\to+\infty} sup_{n\in\mathbb{Z}} |(S^j f)_n - S^\infty f(n2^{-j})| = 0. \tag{4}$$

In order to derive the convergence, we rewrite the nonlinear subdivision scheme S_{PPHA} as a perturbation of a classical two-point linear subdivision scheme, S_{C}, introduced by G. Chaikin in [7] and defined by

$$\begin{aligned} (S_{\text{C}} f)_{2n} &= \frac{3}{4} f_n + \frac{1}{4} f_{n+1}, \\ (S_{\text{C}} f)_{2n+1} &= \frac{1}{4} f_n + \frac{3}{4} f_{n+1}. \end{aligned} \tag{5}$$

The scheme S_{C} is known to be convergent with a regularity C^{2-}.

Writing

If $|d^2 f_n| \geq |d^2 f_{n+1}|$**,**

$$\begin{aligned}(S_{\mathrm{PPHA}}f)_{2n} &= \frac{3}{4}f_n + \frac{1}{4}f_{n+1} + \frac{1}{64}d^2 f_{n+1} - \frac{7}{64}\mathrm{PPH}(d^2 f_n, d^2 f_{n+1}),\\ (S_{\mathrm{PPHA}}f)_{2n+1} &= \frac{1}{4}f_n + \frac{3}{4}f_{n+1} - \frac{1}{64}d^2 f_{n+1} - \frac{5}{64}\mathrm{PPH}(d^2 f_n, d^2 f_{n+1}),\end{aligned}$$

If $|d^2 f_n| < |d^2 f_{n+1}|$**,**

$$\begin{aligned}(S_{\mathrm{PPHA}}f)_{2n} &= \frac{3}{4}f_n + \frac{1}{4}f_{n+1} - \frac{1}{64}d^2 f_n - \frac{5}{64}\mathrm{PPH}(d^2 f_n, d^2 f_{n+1}),\\ (S_{\mathrm{PPHA}}f)_{2n+1} &= \frac{1}{4}f_n + \frac{3}{4}f_{n+1} + \frac{1}{64}d^2 f_n - \frac{7}{64}\mathrm{PPH}(d^2 f_n, d^2 f_{n+1}),\end{aligned}$$

we get that S_{PPHA} can be expressed as

$$S_{\mathrm{PPHA}}f = S_{\mathrm{C}}f + F(d^2 f),$$

with

$$F(d^2 f)_{2n} = \begin{cases} \frac{1}{64}d^2 f_{n+1} - \frac{7}{64}\mathrm{PPH}(d^2 f_n, d^2 f_{n+1}) & \text{if } |d^2 f_n| > |d^2 f_{n+1}|,\\ -\frac{1}{64}d^2 f_{n+1} - \frac{5}{64}\mathrm{PPH}(d^2 f_n, d^2 f_{n+1}) & \text{if } |d^2 f_n| < |d^2 f_{n+1}|,\end{cases} \tag{6}$$

and

$$F(d^2 f)_{2n+1} = \begin{cases} -\frac{1}{64}d^2 f_{n+1} - \frac{5}{64}\mathrm{PPH}(d^2 f_n, d^2 f_{n+1}) & \text{if } |d^2 f_n| > |d^2 f_{n+1}|,\\ \frac{1}{64}d^2 f_{n+1} - \frac{7}{64}\mathrm{PPH}(d^2 f_n, d^2 f_n) & \text{if } |d^2 f_n| < |d^2 f_{n+1}|.\end{cases} \tag{7}$$

To analyze the convergence of S_{PPHA}, we use a result proved in [3], [2] that reads:

A sufficient condition for the convergence of a nonlinear subdivision scheme $S_{NL} : l^\infty(\mathbb{Z}) \to l^\infty(\mathbb{Z})$ of the form:

$$\forall f \in l^\infty(\mathbb{Z}), \quad \forall n \in \mathbb{Z} \qquad \begin{cases} (S_{NL}f)_{2n+1} = (Sf)_{2n+1} + F(\delta f)_{2n+1},\\ (S_{NL}f)_{2n} = f_n,\end{cases} \tag{8}$$

where F is a nonlinear operator defined on $l^\infty(\mathbb{Z})$, δ is a linear and continuous operator on $l^\infty(\mathbb{Z})$ and S is a linear and convergent subdivision scheme is:

Theorem 1 *If F, S and δ given in (8) verify:*

$$\exists M > 0 \quad such\ that \quad \forall d \in l^\infty(\mathbb{Z}) \quad ||F(d)||_\infty \leq M||d||_\infty, \qquad (9)$$

$$\exists c < 1\ such\ that \qquad ||\delta S(f) + \delta F(\delta f)||_\infty \leq c||\delta f||_\infty, \qquad (10)$$

then the subdivision scheme S_{NL} is uniformly convergent. Moreover, if S is C^{α^-} convergent then, for all sequence $f \in l^\infty(\mathbb{Z})$, $S_{NL}^\infty(f) \in C^{\beta^-}$ with $\beta = \min(\alpha, -log_2(c))$.

Using theorem 1, we are going to prove the following result.

Theorem 2 *The nonlinear subdivision scheme S_{PPHA} is convergent with a regularity at least $C^{\beta-}$ with $\beta \geq -log_2(\frac{7}{16}) > 1$.*

Proof

From the properties of the harmonic mean (Proposition 1),

$$|c_1 d^2 f_{n+1} - c_2 \mathrm{PPH}(d^2 f_n, d^2 f_{n+1})| \leq \max(c_1, c_2)||d^2 f||_\infty. \qquad (11)$$

For the perturbation F defined in (6) and (7), it is then easy to see that for all $d \in l^\infty(\mathbb{Z})$,

$$||F(d)||_\infty \leq \frac{7}{64}||d||_\infty, \qquad (12)$$

that is hypothesis (9).

We now consider hypothesis (10) related, in this case, to the contraction of the second order differences $(d^2 f)$. To simplify the notations we call $f^1 = S_{\mathrm{PPHA}}(f)$.

Different cases must be considered:

Case 1: k=2n+1, study of $(d^2 f^1)_{2n+1} = f^1_{2n+2} - 2f^1_{2n+1} + f^1_{2n}$

case $1A_1$: $|d^2 f_n| \geq |d^2 f_{n+1}|$ and $|d^2 f_{n+1}| \geq |d^2 f_{n+2}|$,
case $1A_2$: $|d^2 f_n| < |d^2 f_{n+1}|$ and $|d^2 f_{n+1}| < |d^2 f_{n+2}|$,
case $1B_1$: $|d^2 f_n| \geq |d^2 f_{n+1}|$ and $|d^2 f_{n+1}| < |d^2 f_{n+2}|$,
case $1B_2$: $|d^2 f_n| < |d^2 f_{n+1}|$ and $|d^2 f_{n+1}| \geq |d^2 f_{n+2}|$.

Case 2: k=2n, study of $(d^2 f^1)_{2n} = f^1_{2n+1} - 2f^1_{2n} + f^1_{2n-1}$

case $2A_1$: $|d^2 f_n| \geq |d^2 f_{n+1}|$ and $|d^2 f_{n-1}| \geq |d^2 f_n|$,
case $2A_2$: $|d^2 f_n| < |d^2 f_{n+1}|$ and $|d^2 f_{n-1}| < |d^2 f_n|$,
case $2B_1$: $|d^2 f_n| \geq |d^2 f_{n+1}|$ and $|d^2 f_{n-1}| < |d^2 f_n|$,
case $2B_2$: $|d^2 f_n| < |d^2 f_{n+1}|$ and $|d^2 f_{n-1}| \geq |d^2 f_n|$.

The others cases follow by symmetry.

- Cases 1A: We obtain for the case 1A_1

$$\begin{aligned}
(d^2 f^1)_{2n+1} &= \frac{1}{4} f_{n+2} - \frac{2}{4} f_{n+1} + \frac{1}{4} f_n + \frac{1}{64} d^2 f_{n+2} + \frac{3}{64} d^2 f_{n+1} \\
&\quad - \frac{7}{64} \text{PPH}(d^2 f_{n+1}, d^2 f_{n+2}) + \frac{3}{64} \text{PPH}(d^2 f_n, d^2 f_{n+1}) \\
&= \frac{1}{4} d^2 f_{n+1} + \frac{1}{64} d^2 f_{n+2} + \frac{3}{64} d^2 f_{n+1} \\
&\quad - \frac{7}{64} \text{PPH}(d^2 f_{n+1}, d^2 f_{n+2}) + \frac{3}{64} \text{PPH}(d^2 f_n, d^2 f_{n+1}) \\
&= \frac{19}{64} d^2 f_{n+1} + \frac{1}{64} d^2 f_{n+2} \\
&\quad - \frac{7}{64} \text{PPH}(d^2 f_{n+1}, d^2 f_{n+2}) + \frac{3}{64} \text{PPH}(d^2 f_n, d^2 f_{n+1}).
\end{aligned}$$

Using equation (11) for $d^2 f_{n+1}$ and $\text{PPH}(d^2 f_{n+1}, d^2 f_{n+2})$ and Proposition 1, we have

$$|(d^2 f^1)_{2n+1}| \leq \frac{19+1+3}{64} ||d^2 f||_\infty \leq \frac{23}{64} ||d^2 f||_\infty. \tag{13}$$

Similarly for the case 1A_2, we have

$$\begin{aligned}
(d^2 f^1)_{2n+1} &= \frac{15}{64} d^2 f_{n+1} - \frac{3}{64} d^2 f_n - \frac{5}{64} \text{PPH}(d^2 f_{n+1}, d^2 f_{n+2}) \\
&\quad + \frac{9}{64} \text{PPH}(d^2 f_n, d^2 f_{n+1}),
\end{aligned}$$

and (13) remains valid.

- Cases 1B: We obtain for the case 1B_1

$$\begin{aligned}
(d^2 f^1)_{2n+1} &= \frac{1}{4} f_{n+2} - \frac{2}{4} f_{n+1} + \frac{1}{4} f_n - \frac{1}{64} d^2 f_{n+1} + \frac{3}{64} d^2 f_{n+1} \\
&\quad - \frac{5}{64} \text{PPH}(d^2 f_{n+1}, d^2 f_{n+2}) + \frac{3}{64} \text{PPH}(d^2 f_n, d^2 f_{n+1}) \\
&= \frac{18}{64} d^2 f_{n+1} \\
&\quad - \frac{5}{64} \text{PPH}(d^2 f_{n+1}, d^2 f_{n+2}) + \frac{3}{64} \text{PPH}(d^2 f_n, d^2 f_{n+1}).
\end{aligned}$$

Using equation (11) for $d^2 f_{n+1}$ and $\textsc{pph}(d^2 f_{n+1}, d^2 f_{n+2})$, and Proposition 1, we have

$$|(d^2 f^1)_{2n+1}| \;\; \leq \;\; \frac{18+3}{64}||d^2 f||_\infty \leq \frac{21}{64}||d^2 f||_\infty. \tag{14}$$

Similarly for the case $1\mathrm{B}_2$, we have

$$\begin{aligned}(d^2 f^1)_{2n+1} &= \frac{16}{64}d^2 f_{n+1} + \frac{1}{64}d^2 f_{n+2} - \frac{3}{64}d^2 f_n \\ &\quad -\frac{7}{64}\textsc{pph}(d^2 f_{n+1}, d^2 f_{n+2}) + \frac{9}{64}\textsc{pph}(d^2 f_n, d^2 f_{n+1}),\end{aligned}$$

and (14) remains valid.

- Cases 2A: We obtain for the case $2\mathrm{A}_1$

$$\begin{aligned}(d^2 f^1)_{2n} &= \frac{1}{4}f_{n+1} - \frac{2}{4}f_n + \frac{1}{4}f_{n-1} - \frac{3}{64}d^2 f_{n+1} - \frac{1}{64}d^2 f_n \\ &\quad +\frac{9}{64}\textsc{pph}(d^2 f_n, d^2 f_{n+1}) - \frac{5}{64}\textsc{pph}(d^2 f_{n-1}, d^2 f_n) \\ &= \frac{1}{4}d^2 f_n - \frac{3}{64}d^2 f_{n+1} - \frac{1}{64}d^2 f_n \\ &\quad +\frac{9}{64}\textsc{pph}(d^2 f_n, d^2 f_{n+1}) - \frac{5}{64}\textsc{pph}(d^2 f_{n-1}, d^2 f_n) \\ &= \frac{15}{64}d^2 f_n - \frac{3}{64}d^2 f_{n+1} \\ &\quad +\frac{9}{64}\textsc{pph}(d^2 f_n, d^2 f_{n+1}) - \frac{5}{64}\textsc{pph}(d^2 f_{n-1}, d^2 f_n).\end{aligned}$$

Using equation (11) for $d^2 f_{n+1}$ and $\textsc{pph}(d^2 f_n, d^2 f_{n+1})$, and for $d^2 f_{n-1}$ and $\textsc{pph}(d^2 f_{n-1}, d^2 f_n)$, we have

$$|(d^2 f^1)_{2n}| \;\; \leq \;\; \frac{15+9}{64}||d^2 f||_\infty \leq \frac{28}{64}||d^2 f||_\infty. \tag{15}$$

Similarly for the case $2\mathrm{A}_2$, we have

$$\begin{aligned}(d^2 f^1)_{2n} &= \frac{19}{64}d^2 f_n + \frac{1}{64}d^2 f_{n-1} \\ &\quad +\frac{3}{64}\textsc{pph}(d^2 f_n, d^2 f_{n+1}) - \frac{7}{64}\textsc{pph}(d^2 f_{n-1}, d^2 f_n),\end{aligned}$$

and (15) remains valid.

- Cases 2B: We obtain for the case $2B_1$

$$\begin{aligned}(d^2f^1)_{2n} &= \frac{1}{4}f_{n+1}-\frac{2}{4}f_n+\frac{1}{4}f_{n-1}-\frac{3}{64}d^2f_{n+1}+\frac{1}{64}d^2f_{n-1}\\ &\quad+\frac{9}{64}\text{PPH}(d^2f_n,d^2f_{n+1})-\frac{7}{64}\text{PPH}(d^2f_{n-1},d^2f_n)\\ &= \frac{1}{4}d^2f_n-\frac{3}{64}d^2f_{n+1}+\frac{1}{64}d^2f_{n-1}\\ &\quad+\frac{9}{64}\text{PPH}(d^2f_n,d^2f_{n+1})-\frac{7}{64}\text{PPH}(d^2f_{n-1},d^2f_n).\end{aligned}$$

Using equation (11) for d^2f_{n+1} and $\text{PPH}(d^2f_n,d^2f_{n+1})$, and for d^2f_n and $\text{PPH}(d^2f_{n-1},d^2f_n)$, we have

$$|(d^2f^1)_{2n}| \leq \frac{16+9+1}{64}||d^2f||_\infty \leq \frac{28}{64}||d^2f||_\infty. \tag{16}$$

Similarly for the case $2B_2$, we have

$$\begin{aligned}(d^2f^1)_{2n} &= \frac{18}{64}d^2f_n\\ &\quad+\frac{3}{64}\text{PPH}(d^2f_n,d^2f_{n+1})-\frac{5}{64}\text{PPH}(d^2f_{n-1},d^2f_n),\end{aligned}$$

and (16) remains valid.

From equations (13), (14), (15) and (16), we deduce that for all $f \in l^\infty(\mathbb{Z})$

$$||d^2S_{\text{PPHA}}f||_\infty \leq \frac{7}{16}||d^2f||_\infty. \tag{17}$$

Therefore S_{PPHA} verifies the hypothesis (10) of the Theorem 1. In particular, we obtain the convergence of S_{PPHA}.

For the regularity, we use again Theorem 1. According to the values $\alpha = 2$ and $c = \frac{7}{16}$ we obtain the regularity constant $\beta = \min\left(2, -log_2\left(\frac{7}{16}\right)\right) \approx 1.192$.

$\square$

Numerical Regularity

Following [20], the regularity of a limit function can be evaluated numerically. Using S_1 and S_2 the subdivision schemes for the differences of order 1 and 2 associated to S_{PPHA} (that can be derived due to the specific definition of S_{PPHA}), the following quantities are estimated for $k = 1, 2$,

$$-log_2\left(2^k\frac{||(S_k^{j+1}f)_{n+1}-(S_k^{j+1}f)_n||_\infty}{||(S_k^jf)_{n+1}-(S_k^jf)_n||_\infty}\right).$$

They provide an estimate for β_1 and β_2 such that the limit function belongs to $C^{1+\beta_1-}$ and $C^{2+\beta_2-}$. From table 1, the numerical estimate of the regularity is $C^{2.438-}$. We recall that the corresponding estimate for the linear scheme [14] is $C^{2.67-}$.

j	5	6	7	8	9	10
β_1	0.9999	0.9999	1	1	1	1
β_2	0.4395	0.7738	1.2615	0.6541	0.4387	0.4388

Table 1: Numerical estimates of the limit function regularity $C^{1+\beta_1-}$ and $C^{2+\beta_2-}$ for S_{PPHA}.

4 Stability

For simplicity in notations we call for any initial sequence f^0, and any $j \in \mathbb{N}$, $f^{j+1} = S(f^j)$. We recall the following definition.

Definition 2 *A convergent subdivision scheme is stable if*

$$\exists C < +\infty \text{ such that } \forall f^0, g^0 \in l^\infty(\mathbb{Z}) \quad ||S^\infty f - S^\infty g||_{L^\infty} \leq C||f^0 - g^0||_{l^\infty}. \tag{18}$$

As for the convergence, to derive the stability of S_{PPHA} we use the following theorem of [2].

Theorem 3 *If F, S and δ given in (8) verify: $\exists M > 0, c < 1$ such that $\forall f, g, d_1, d_2$,*

$$||F(d_1) - F(d_2)||_\infty \leq M||d_1 - d_2||_\infty, \tag{19}$$

$$\|\delta(S_{NL}f - S_{NL}g)\|_\infty \leq c\|\delta(f - g)\|_\infty, \tag{20}$$

then the nonlinear subdivision scheme S_{NL} is stable.

In order to check the hypotheses of Theorem 3 for $S_{NL} = S_{\text{PPHA}}$ we first prove the following lemma.

Lemma 1 *Let be f, $g \in l^\infty(\mathbb{Z})$, if $|d^2 f_n| \geq |d^2 f_{n+1}|$ and $|d^2 g_{n+1}| \geq |d^2 g_n|$ then*

$$|d^2 f_{n+1} + d^2 g_n - 2\text{PPH}(d^2 g_n, d^2 g_{n+1})| \leq 3||d^2 f - d^2 g||_\infty.$$

Proof
We consider different cases.

• If $d^2 g_{n+1} d^2 g_n < 0$, we have

$$d^2 f_{n+1} + d^2 g_n - 2\textsc{pph}(d^2 g_n, d^2 g_{n+1}) = d^2 f_{n+1} + d^2 g_n.$$

$*$ if $d^2 f_{n+1} d^2 g_n < 0$, using that $|d^2 f_n| \geq |d^2 f_{n+1}|$

$$|d^2 f_{n+1} + d^2 g_n| \leq |d^2 f_{n+1} - d^2 g_n| \leq |d^2 f_n - d^2 g_n|.$$

$*$ if $d^2 f_{n+1} d^2 g_n \geq 0$, we have $d^2 f_{n+1} d^2 g_{n+1} < 0$ and with $|d^2 g_{n+1}| \geq |d^2 g_n|$, we obtain

$$|d^2 f_{n+1} + d^2 g_n| \leq |d^2 f_{n+1} - d^2 g_{n+1}|.$$

• If $d^2 g_{n+1} d^2 g_n \geq 0$, we recall (Proposition 1) that if $x, y > 0$, $\min(x, y) \leq \textsc{pph}(x, y) \leq max(x, y)$.
Without loss of generality, we suppose that $d^2 g_n \geq 0$. We denote

$$H = d^2 f_{n+1} + d^2 g_n - 2\textsc{pph}(d^2 g_n, d^2 g_{n+1}).$$

$*$ if $H > 0$,

$$\begin{aligned} H &\leq d^2 f_{n+1} + d^2 g_n - 2\min(|d^2 g_n|, |d^2 g_{n+1}|) \\ &\leq d^2 f_{n+1} - d^2 g_n \\ &\leq |d^2 f_n - d^2 g_n|. \end{aligned}$$

$*$ if $H < 0$,

$$\begin{aligned} H &\geq d^2 f_{n+1} + d^2 g_n - 2\max(|d^2 g_n|, |d^2 g_{n+1}|) \\ &\geq d^2 f_{n+1} + d^2 g_n - 2d^2 g_{n+1} \\ &\geq d^2 f_{n+1} - d^2 g_{n+1} + d^2 g_n - d^2 g_{n+1}. \end{aligned}$$

We have again to consider different cases according to the sign of $d^2 f_n$.

– $d^2 f_n \geq 0$, we have $d^2 f_n - d^2 f_{n+1} \geq 0$

$$\begin{aligned} H &\geq (d^2 f_{n+1} - d^2 g_{n+1}) + (d^2 g_n - d^2 g_{n+1}) - (d^2 f_n - d^2 f_{n+1}) \\ &\geq 2(d^2 f_{n+1} - d^2 g_{n+1}) + (d^2 g_n - d^2 f_n) \geq -3||d^2 f - d^2 g||_\infty. \end{aligned}$$

– $d^2f_n < 0$ and $d^2f_{n+1}d^2f_n \geq 0$, then $d^2f_{n+1} < 0$

$$\begin{aligned} H &\geq (d^2f_{n+1} - d^2g_{n+1}) + (d^2f_{n+1} - d^2g_{n+1}) \\ &\geq 2(d^2f_{n+1} - d^2g_{n+1}) \geq -2||d^2f - d^2g||_\infty. \end{aligned}$$

– $d^2f_n < 0$ and $d^2f_{n+1}d^2f_n < 0$, we have from hypothesis that $d^2f_{n+1} + d^2f_n < 0$,

$$\begin{aligned} H &\geq (d^2f_{n+1} - d^2g_{n+1}) + (d^g_n - d^2g_{n+1}) + d^2f_{n+1} + d^2f_n \\ &\geq 2(d^2f_{n+1} - d^2g_{n+1}) + d^2f_n + d^2g_n \\ &\geq 2(d^2f_{n+1} - d^2g_{n+1}) + d^2f_n - d^2g_n \geq -3||d^2f - d^2g||_\infty. \end{aligned}$$

□

We are now ready to prove the stability of S_{PPHA}.

Theorem 4 *The scheme S_{PPHA} is stable.*

Proof

We check the hypotheses of Theorem 3.

Firstly, we start with the hypothesis (19) for F.
Using the expressions of perturbation F, (6) and (7), and Proposition 1, we obtain for all $d_1, d_2 \in l^\infty(\mathbb{Z})$ that

$$||F(d_1) - F(d_2)||_\infty \leq \frac{1 + 7 \cdot 2}{64}||d_1 - d_2||_\infty.$$

Secondly, we have to verify the contraction hypothesis (20).

For a couple f, $g \in l^\infty(\mathbb{Z})$, we study $(d^2f^1 - d^2g^1)_k$ for $k = 2n+1$ (case 1) or $k = 2n$ (case 2).

We consider different cases, according to the proof of Theorem 2 for f or g.

For k=2n+1, we have 7 cases to study, see table 2, the others cases being deduced by symmetry.

• Case $1A_1 - 1A_1$: with equation (13) we can obtain directly

$$\begin{aligned} |d^2f^1_{2n+1} - d^2g^1_{2n+1}| &\leq \frac{19 + 1 + 7 \cdot 2 + 3 \cdot 2}{64}||d^2f - d^2g||_\infty \\ &\leq \frac{5}{8}||d^2f - d^2g||_\infty. \end{aligned}$$

	f verifies	g verifies	notation		f verifies	g verifies	notation
Case 1A	1A$_1$	1A$_1$	$1A_1-1A_1$	Case 1B	1B$_1$	1A$_2$	$1B_1-1A_2$
	1A$_1$	1A$_2$	$1A_1-1A_2$		1B$_1$	1B$_1$	$1B_1-1B_1$
	1A$_1$	1B$_1$	$1A_1-1B_1$		1B$_1$	1B$_2$	$1B_1-1B_2$
	1A$_1$	1B$_2$	$1A_1-1B_2$		1B$_2$	1A$_2$	$1B_2-1A_2$

Table 2: Cases to consider for $k=2n+1$ in the proof of the stability of S_{PPHA}.

- Case $1A_1-1A_2$: from equations (13) and (14), we obtain

$$\begin{aligned}
d^2f^1_{2n+1}-d^2g^1_{2n+1} &= \frac{19}{64}d^2f_{n+1}-\frac{15}{64}d^2g_{n+1}+\frac{1}{64}d^2f_{n+2}+\frac{3}{64}d^2g_n-\frac{7}{64}\text{PPH}(d^2f_{n+1},d^2f_{n+2})\\
&\quad+\frac{5}{64}\text{PPH}(d^2g_{n+1},d^2g_{n+2})+\frac{3}{64}\text{PPH}(d^2f_n,d^2f_{n+1})+\frac{9}{64}\text{PPH}(d^2g_n,d^2g_{n+1})\\
&= \frac{16}{64}(d^2f_{n+1}-d^2g_{n+1})-\frac{7}{64}(\text{PPH}(d^2f_{n+1},d^2f_{n+2})-\text{PPH}(d^2g_{n+1},d^2g_{n+2}))\\
&\quad+\frac{1}{64}(d^2f_{n+2}+d^2g_{n+1}-2\text{PPH}(d^2g_{n+1},d^2g_{n+2}))\\
&\quad+\frac{3}{64}(\text{PPH}(d^2f_n,d^2f_{n+1})-\text{PPH}(d^2g_{n+1},d^2g_{n+2}))\\
&\quad+\frac{3}{64}(d^2g_n+d^2f_{n+1}-2\text{PPH}(d^2g_n,d^2g_{n+1})).
\end{aligned}$$

Applying Lemma 1 and Proposition 1, we obtain

$$\begin{aligned}
|d^2f^1_{2n+1}-d^2g^1_{2n+1}| &\leq \frac{16+7\cdot 2+2+1+3\cdot 2+3\cdot 3}{64}||d^2f-d^2g||_\infty\\
&\leq \frac{45}{64}||d^2f-d^2g||_\infty.
\end{aligned}$$

Similarly

- Case $1A_1-1B_1$: from equations (13) and (14), we obtain applying the Lemma 1 and the Proposition 1

$$|d^2f^1_{2n+1}-d^2g^1_{2n+1}| \leq \frac{21}{32}||d^2f-d^2g||_\infty.$$

- Case $1A_1-1B_2$: from equations (13) and (15), we obtain applying the Lemma 1 and the Proposition 1

$$|d^2f^1_{2n+1}-d^2g^1_{2n+1}| \leq \frac{23}{32}||d^2f-d^2g||_\infty.$$

• Case $1B_1 - 1A_2$: from equations (14) and (14), we obtain applying the Lemma 1 and the Proposition 1

$$|d^2 f^1_{2n+1} - d^2 g^1_{2n+1}| \leq \frac{5}{8}||d^2 f - d^2 g||_\infty.$$

• Case $1B_1 - 1B_1$: from equation (14) we conclude

$$\begin{aligned} |d^2 f^1_{2n+1} - d^2 g^1_{2n+1}| &\leq \frac{18 + 5\cdot 2 + 3\cdot 2}{64}||d^2 f - d^2 g||_\infty \\ &\leq \frac{17}{32}||d^2 f - d^2 g||_\infty. \end{aligned}$$

• Case $1B_1 - 1B_2$: from equations (14) and (15), we obtain applying the Lemma 1 and the Proposition 1

$$|d^2 f^1_{2n+1} - d^2 g^1_{2n+1}| \leq \frac{3}{8}||d^2 f - d^2 g||_\infty.$$

• Case $1B_2 - 1A_2$: from equations (15) and (14), we obtain applying the Lemma 1 and the Proposition 1

$$|d^2 f^1_{2n+1} - d^2 g^1_{2n+1}| \leq \frac{11}{16}||d^2 f - d^2 g||_\infty.$$

For k=2n, we have 7 other cases to study, see table 3. The others cases are deduced by symmetry.
We notice that in equations (14) and (15) the same coefficients, but the cases are not completely equivalent.

	f verifies	g verifies	notation		f verifies	g verifies	notation
Case 2A	2A$_1$	2A$_1$	$2A_1 - 2A_1$	Case 2B	2B$_1$	2A$_2$	$2B_1 - 2A_2$
	2A$_1$	2A$_2$	$2A_1 - 2A_2$		2B$_1$	2B$_1$	$2B_1 - 2B_1$
	2A$_1$	2B$_1$	$2A_1 - 2B_1$		2B$_1$	2B$_2$	$2B_1 - 2B_2$
	2A$_1$	2B$_2$	$2A_1 - 1B_2$		2B$_2$	2A$_2$	$2B_2 - 2A_2$

Table 3: Cases to consider for $k = 2n$ in the proof of the stability of S_{PPHA}.

• Case $2A_1 - 2A_1$: with equation (15) we can conclude

$$\begin{aligned}
|d^2 f^1_{2n} - d^2 g^1_{2n}| &\leq \frac{15+3+9\cdot 2+5\cdot 2}{64}||d^2 f - d^2 g||_\infty \\
&\leq \frac{23}{32}||d^2 f - d^2 g||_\infty.
\end{aligned}$$

• Case $2A_1 - 2A_2$: from equations (15) and (16), we obtain

$$\begin{aligned}
d^2 f^1_{2n} - d^2 g^1_{2n} &= \frac{15}{64}d^2 f_n - \frac{19}{64}d^2 g_n - \frac{3}{64}d^2 f_{n+1} + \frac{9}{64}\text{PPH}(d^2 f_n, d^2 f_{n+1}) \\
&\quad -\frac{3}{64}\text{PPH}(d^2 g_n, d^2 g_{n+1}) - \frac{5}{64}\text{PPH}(d^2 f_{n-1}, d^2 f_n) + \frac{7}{64}\text{PPH}(d^2 g_{n-1}, d^2 g_n)) \\
&= \frac{16}{64}(d^2 f_n - d^2 g_n) + \frac{9}{64}(\text{PPH}(d^2 f_n, d^2 f_{n+1}) - \text{PPH}(d^2 g_n, d^2 g_{n-1})) \\
&\quad -\frac{3}{64}(d^2 f_{n+1} + d^2 g_n - 2\text{PPH}(d^2 g_n, d^2 g_{n+1})) \\
&\quad -\frac{5}{64}(\text{PPH}(d^2 f_{n-1}, d^2 f_n) - \text{PPH}(d^2 g_{n-1}, d^2 g_n)) \\
&\quad -\frac{1}{64}(d^2 f_n + d^2 g_{n-1} - 2\text{PPH}(d^2 g_{n-1}, d^2 g_n)).
\end{aligned}$$

Applying the Lemma 1 and the Proposition 1, we obtain

$$\begin{aligned}
|d^2 f^1_{2n} - d^2 g^1_{2n}| &\leq \frac{16+9\cdot 2+3\cdot 3+5\cdot 2+3}{64}||d^2 f - d^2 g||_\infty \\
&\leq \frac{28}{32}||d^2 f - d^2 g||_\infty.
\end{aligned}$$

Similarly,

• Case $2A_1 - 2B_1$: from equations (15) and (16), we obtain applying the Lemma 1 and the Proposition 1

$$|d^2 f^1_{2n} - d^2 g^1_{2n}| \leq \frac{25}{32}||d^2 f - d^2 g||_\infty.$$

• Case $2A_1 - 2B_2$: from equations (15) and (17), we obtain applying the Lemma 1 and the Proposition 1

$$|d^2 f^1_{2n} - d^2 g^1_{2n}| \leq \frac{13}{16}||d^2 f - d^2 g||_\infty.$$

• Case $2B_1-2A_2$: from equations (16) and (16), we obtain applying the Lemma 1 and the Proposition 1

$$|d^2 f^1_{2n} - d^2 g^1_{2n}| \leq \frac{57}{64}||d^2 f - d^2 g||_\infty.$$

• Case $2B_1-2B_1$: from equation (16) we conclude

$$\begin{aligned} |d^2 f^1_{2n} - d^2 g^1_{2n}| &\leq \frac{16+3+1+9\cdot 2+7\cdot 2}{64}||d^2 f - d^2 g||_\infty \\ &\leq \frac{21}{32}||d^2 f - d^2 g||_\infty. \end{aligned}$$

• Case $2B_1-2B_2$: from equations (16) and (17), we obtain applying the Lemma 1 and the Proposition 1

$$|d^2 f^1_{2n} - d^2 g^1_{2n}| \leq \frac{15}{16}||d^2 f - d^2 g||_\infty.$$

• Case $2B_2-2A_2$: from equations (17) and (16), we obtain applying the lemma 1 and the proposition 1

$$|d^2 f^1_{2n} - d^2 g^1_{2n}| \leq \frac{21}{32}||d^2 f - d^2 g||_\infty.$$

Finally, the hypotheses of theorem (20) are verified and stability can be established.

□

5 Order of approximation

In this section we consider the reproduction of polynomials and the order of approximation of S_{PPHA}.

We recall the following definitions.

Definition 3 *A subdivision scheme S is said to reproduce polynomials of degree k if for all polynomial P of degree k:*

if $\forall n \in \mathbb{Z}, f_n = P(n)$, then $\exists \tilde{P}$ a polynomial of degree k such that $(Sf)_n = \tilde{P}(2^{-1}n)$.

Definition 4 *A subdivision scheme S is said to have an order k of approximation if for all function $g \in C^k$ and all $h > 0$,*

$$if\ f = g(h.),\ then\ |Sf - g(2^{-1}h.)| \leq Ch^k.$$

We then have the following property.

Proposition 2 *S_{PPHA} reproduces the polynomials of degree 2 with translation of $\frac{1}{4}$.*

Proof

We remark that for any P, polynomial of degree 2, and $p = (P(n))_{n\in\mathbb{Z}}$, we have

$$\mathrm{PPH}(d^2p_n, d^2p_{n+1}) = \frac{p_n + p_{n+1}}{2}.$$

Therefore, for the initial sequence $p = (p_n)_{n\in\mathbb{Z}}$, $S_{\mathrm{PPHA}}(p)$ coincides with the application to p of the linear scheme [14]. In particular, the results of N. Dyn, M.S. Floater and K. Hormann [14] can be applied and the property of definition 3 is satisfied with $\tilde{P}(.) = P(. - 1/4)$.

□

Concerning the order of approximation the following proposition holds.

Proposition 3 *For all function $g \in C^4([0,1])$ and $h > 0$, if*

$$f = (g((n - \frac{1}{2})h))_{n\in\mathbb{Z}},$$

then
if $d^2f_nd^2f_{n+1} > 0$ for all $n \in \mathbb{Z}$, then

$$||(S_{\mathrm{PPHA}}f)_n - g(2^{-1}h(n - \frac{1}{2}))||_\infty = O(h^4),$$

otherwise

$$||(S_{\mathrm{PPHA}}f)_n - g(2^{-1}h(n - \frac{1}{2}))||_\infty = O(h^3).$$

Proof

According to Proposition 1, we have that if $d^2f_nd^2f_{n+1} > 0$ for all $n \in \mathbb{N}$ then

$$|\mathrm{PPH}(d^2f_n, d^2f_{n+1}) - \frac{d^2f_n + d^2f_{n+1}}{2}| = O(h^4).$$

Therefore, is S stands for the linear scheme defined in [14], according to the definition of the S_{PPHA},

$$||S_{\mathrm{PPHA}}f - Sf||_\infty = O(h^4).$$

Since (see [14]) the scheme S is of order of approximation 4 we get the result when $d^2f_nd^2f_{n+1} > 0$. In other case, the reproduction of polynomials leads to

$$||(S_{\rm PPHA}f)_n - g(2^{-1}h(n-\frac{1}{2}))||_\infty = O(h^3).$$

$\square$

Remark 1 *Following [20] one can also establish, using the stability of* $S_{\rm PPHA}$ *that* $||S^\infty_{\rm PPHA}f - g||_\infty = O(h^3)$.

6 Elimination of the Gibbs phenomenon

In this section we focus on the behavior of the scheme in presence of strongly variating data. The reference behavior deals with a step function as shown on Figure 1. As it is visible on Figure 1 left, high order linear schemes suffer from an oscillating behavior named as Gibbs phenomenon.

According to D. Gottlieb and C.W. Shu [18], given a punctually discontinuous function f and its sampling f^h defined by $f^h_n = f(nh)$, the Gibbs phenomenon deals with the convergence of $S^\infty(f^h)$ towards f. It can be characterized by two features ([18] p. 244):

1. Away from the discontinuity the convergence is rather slow and for any point x,
$$|f(x) - S^\infty(f^h)(x)| = O(h).$$
2. There is an overshoot, close to the discontinuity, that does not diminish with reducing h; thus
$$\max|f(x) - S^\infty(f^h)(x)| \text{ does not tend to zero with } h.$$

We are now going to prove that the nonlinear schemes $S_{\rm PPHA}$ does not suffer from the Gibbs phenomenon oscillations, as it can be guessed from Figure 1. We have indeed the following

Proposition 4 *Given* $0 \le \xi \le h$, *for any function* f *defined by:*

$$\forall x \le \xi, f(x) = f_-(x)\, with\, f_- \in C^\infty(]-\infty,\xi]\,,$$
$$\forall x > \xi, f(x) = f_+(x)\, with\, f_+ \in C^\infty([\xi,+\infty[\,,$$

and discontinuous in ξ, *we have:*

- *if* $|x| \ge \frac{9}{2}h, |f(x-\frac{1}{2}) - S^\infty_{\rm PPHA}(f^h)(x)| = O(h^3)$,

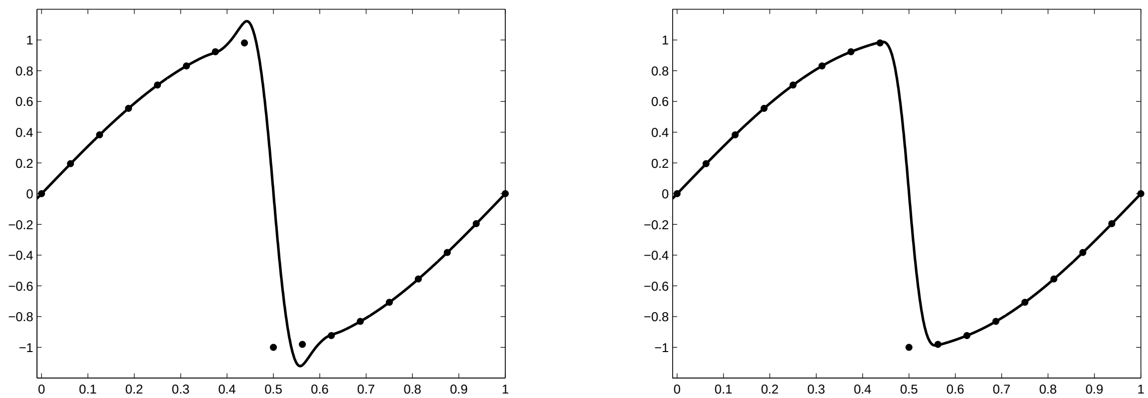

Figure 1: Comparison of limit functions for the same initial sequence (sampling of function (21)). Left, linear scheme (1), right nonlinear scheme S_{PPHA}

- *if* $|x| \le \frac{9}{2}h, f_-(0) + O(h) \ge S^{\infty}_{\text{PPHA}}(f^h)(x) \le f_-(h) + O(h)$.

Proof

Without loss of generality, we focus on $[0, +\infty[$ and suppose that $f_-(\xi) > f_+(\xi)$.

We first consider a single application of S_{PPHA}. Using Proposition 3 we get:

- for $n \ge 2$ and $n_1 \in \{2n, 2n+1\}$, $|S_{\text{PPHA}}(f^h)_{n_1} - f_+(2^{-1}h(n_1 - \frac{1}{2})| = O(h^3)$
- for $n = 1$ since f is discontinuous in ξ, $d^2 f_n = O(1)$ and $d^2 f_{n+1} = O(h^2)$. Then, from Proposition 1, $\text{PPH}(d^2 f_n, d^2 f_{n+1}) = O(h^2)$. Moreover, according to the definition of S_{PPHA} as a perturbation of the Chaikin scheme S_C we get that $|S_{\text{PPHA}}(f^h)_{n_1} - S_C(f^h)_{n_1}| = O(h^2)$. Since S_C is a second order scheme we get that $|S_{\text{PPHA}}(f^h)_{n_1} - f_+(2^{-1}h(n_1 - \frac{1}{2})| = O(h^2)$ for $n_1 \in \{2n, 2n+1\}$.
- for $n = 0$, $d^2 f_n d^2 f_{n+1} \le 0$ and therefore, according to the Proposition 1, $\text{PPH}(d^2 f_n, d^2 f_{n+1}) = 0$. It is then easy to check, from the definition of S_{PPHA} that $f_-(0) \le S_{\text{PPHA}} f_{2n} \le S_{\text{PPHA}} f_{2n+1} \le f_+(h)$. However, writing S_{PPHA} as a perturbation of S_C we get that $|S_{\text{PPHA}}(f^h)_{n_1} - S_C(f^h)_{n_1}| = O(d^2 f^h)$.

Iterating, according to the stability of S_{PPHA} we get:

- for $x \geq \frac{9}{2}h$, $|S^{\infty}_{\text{PPHA}} f^h(x) - f_+(x-1/2)| = O(h^3)$.
- for $0 \leq x \leq \frac{9}{2}h$, the contraction of the second order differences (equation (17)) and the fact that the Chaikin scheme S_C does not produce Gibbs oscillations allow to conclude.

$\square$

Before concluding this work, we come back to Figure 1 and to the comparison between the limit functions obtained with S_{PPHA} and the limit function obtained with linear subdivision schemes starting from the sampling f^h of the discontinuous function:

$$f(x) = \begin{cases} sin(\pi x) & \text{for } x \in [0, 0.5] \\ -sin(\pi x) & \text{for } x \in]0.5, 1] \, . \end{cases} \tag{21}$$

It appears from Figure 1 that the nonlinear scheme S_{PPHA} exhibits a much better behavior close to the discontinuity than the linear scheme of comparable complexity. From Proposition 3 we know moreover that the scheme S_{PPHA} is of higher order than the Chaikin scheme.

7 Conclusions

In this paper, a new nonlinear subdivision scheme has been defined. It has many desirable properties. It is convergent with a regularity proved to be at least $C^{1.192-}$ and numerically estimated at $C^{2.438-}$. By construction, it is adapted to the presence of isolated discontinuities and the Gibbs phenomenon is eliminated. The scheme is also stable, that due the nonlinear nature is not a consequence of the convergence. Moreover, its order of convergence is 3. Recalling that it is constructed from a four-point centered stencil, all these properties make of this scheme a very good candidate for various applications.

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