Yves Meyer: restoring the role of mathematics in signal and image processing

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May 2017

The Norwegian Academy of Science and Letters has decided to award the Abel Prize for 2017 to **Yves Meyer**, École Normale Supérieure, Paris–Saclay

for his pivotal role in the development of the mathematical theory of wavelets.

Yves Meyer (1939-, Abel Prize 2017)

Outline

A biographical sketch

From Fourier to Morlet

- Fourier transform
- Gabor atoms
- Wavelet transform

First synthesis: Wavelet analysis (1984-1985)

Second synthesis: Multiresolution analysis (1986-1988)

Yves Meyer: early years

- 1939: Born in Paris.
- 1944: Family exiled to Tunisia.
- High school at Lycée Carnot de Tunis.

Lycée Carnot

University education

- 1957-1959?: École Normale Supérieure de la rue d’Ulm.
- 1960-1963: Military service (Algerian war) as teacher at Prytanée national militaire.

“Beginning a Ph.D. to avoid being drafted would be like marrying a woman for her money.”

“From teaching in high school I understood that I was more happy to share than to possess.”

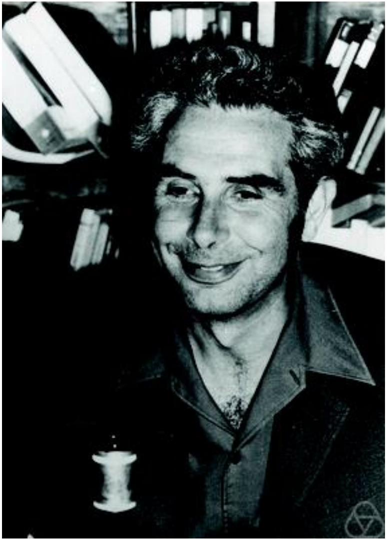
Prytanée national militaire

Doctoral degree

- 1963-1966: PhD at Strasbourg (unsupervised, formally with Jean Pierre Kahane).
- Operator theory on Hardy space H^1.
- Advice from Peter Gabriel:

  > *"Give up classical analysis. Switch to algebraic geometry (à la Grothendieck). People above 40 are completely lost now. Young people can work freely in this field. In classical analysis you are fighting against the accumulated training and experience of the old specialists."*

- Meyer's PhD thesis was soon outdone by Elias Stein.



Jean-Pierre Kahane (1926-) Elias Stein (1931-)

Meyer sets = almost lattices

- 1966-1980: Université Paris-Sud at Orsay.
- 1969: Meyer considers "almost lattices" $\Lambda \subset \mathbb{R}^n$ such that

$$\Lambda - \Lambda \subset \Lambda + F$$

  where F is a finite set.
- A Salem number is an algebraic integer $\theta > 1$ such that each Galois conjugate θ' satisfies $|\theta'| \leq 1$.
- If $\theta\Lambda \subset \Lambda$ for an almost lattice Λ then θ is a Salem number, and conversely.
- 1976: Rediscovered by Roger Penrose.
- Present in Islamic art from ca. 1200 A.D.

Madrasa portal wall in Bukhara, Uzbekistan (rotated)

Ergodic theory

- 1972: Work with Benjamin Weiss at Hebrew University, Jerusalem, to prove that Riesz products are Bernoulli shifts.

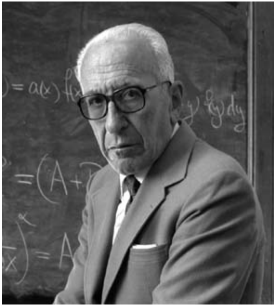
  "I was rather ignorant of ergodic theory. [...] Dive into deep waters and do not be scared to swim against the stream. But only with someone who is an expert of the field you are entering. You should never do it alone!"

Benjamin Weiss (1941-)

Calderón program

1974: Raphy Coifman at Washington University, St. Louis: “We should attack Calderón’s conjectures.”

> *“At the time I did not know that a singular integral operator could be. [...] After working day and night for two months we were able to prove the boundedness of the second commutator.”*



Alberto Calderón (1920-1998)   Ronald Raphael Coifman (1941-)

Partial differential equations

- 1980-1986: École polytechnique.
- Charles Goulaouic: Switch to PDE
- Coifman-Meyer theory of paraproducts: “road beyond pseudodifferential operators”. Led to Jean-Michel Bony’s theory of paradifferential operators
- 1980: Alan McIntosh seeks out Meyer while teaching at Orsay. Insight:

  > *“Calderón’s problem on the boundedness of the Cauchy kernel on Lipschitz curves is equivalent to Kato’s conjecture on accretive operators.”*

- After seven years, Coifman, Meyer and McIntosh prove Calderón’s conjecture.

Alan McIntosh (1942-2016)

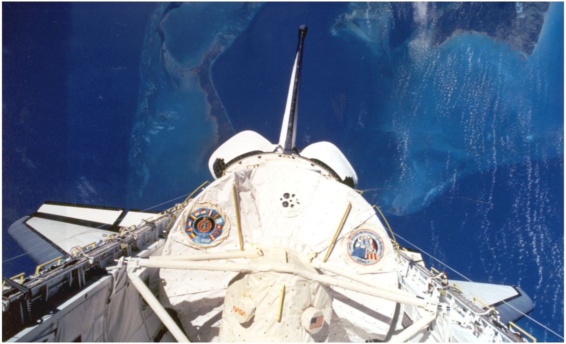
A third way

- Antoni Zygmund: “A problem should always be given its simplest and most concise formulation.”
- Nicolas Bourbaki: “A problem should be raised to its most general formulation before attacking it.”
- A third approach: “Translate a problem into the language of a completely distinct branch of mathematics.”

Antoni Zygmund (1900-1992) Nicolas Bourbaki (1934-)

Applied versus pure

- 1984: Jacques Louis Lions (head of French space agency, CNES): Can we stabilize oscillations in the Spacelab using small rockets? It reduces to “this question” in pure mathematics.
- Yves Meyer solved the problem a week later.



Spacelab module in cargo bay

Wavelets

- Autumn 1984: Jean Lascoux (while photocopying):

  “Yves, I am sure this article will mean something to you.”

- Meyer recognized Calderón’s reproducing identity in Grossmann-Morlet’s first paper on wavelets.

SIAM J. MATH. ANAL.
Vol. 15, No. 4, July 1984

009

Los moé

DECOMPOSITION OF HARDY FUNCTIONS INTO SQUARE INTEGRABLE WAVELETS OF CONSTANT SHAPE*

A. GROSSMANN† AND J. MORLET‡

Abstract. An arbitrary square integrable real-valued function (or, equivalently, the associated Hardy function) can be conveniently analyzed into a suitable family of square integrable wavelets of constant shape, (i.e. obtained by shifts and dilations from any one of them.) The resulting integral transform is isometric and self-reciprocal if the wavelets satisfy an "admissibility condition" given here. Explicit expressions are obtained in the case of a particular analyzing family that plays a role analogous to that of coherent states (Gabor wavelets) in the usual L_2-theory. They are written in terms of a modified Γ-function that is introduced and studied. From the point of view of group theory, this paper is concerned with square integrable coefficients of an irreducible representation of the nonunimodular $ax+b$-group.

1. Introduction.

1.1. It is well known that an arbitrary complex-valued square integrable function $\psi(t)$ admits a representation by Gaussians, shifted in direct and Fourier transformed space. If $g(t)=2^{-1/2}\pi^{-3/4}e^{-t^2/2}$ and t_0, ω_0 are arbitrary real, consider

$$g^{(t_0,\omega_0)}(t)=e^{-i\omega_0 t_0/2}e^{i\omega_0 t}g(t-t_0) \tag{1.1}$$

Grossmann-Morlet, SIAM J. MATH. ANAL., 1984)

16 precursors to wavelets, I

- Haar basis (1909)
- Franklin orthonormal system (1927)
- Littlewood-Paley theory (1930s)
- Calderón's reproducing identity (1960)
- Atomic decompositions (1972)
- Calderón-Zygmund theory
- Geometry of Banach spaces (Strömberg 1981)
- Time frequency atoms in speech signal processing (Gabor 1946)

16 precursors to wavelets, II

- Subband coding (Croisier, Esteban, Galand 1975)
- Pyramid algorithms in image processing (Burt, Adelson 1982)
- Zero-crossings in human vision (Marr 1982)
- Spline approximations
- Multipole algorithm (Rokhlin 1985)
- Refinement schemes in computer graphics
- Coherent states in quantum mechanics
- Renormalization in quantum field theory

Navier-Stokes

- 1985-1995: Ceremade, Université Paris-Dauphine.
- Proved div-curl-lemma with Pierre Louis Lions: If $B, E \in L^2(\mathbb{R}^n)$, div $E = 0$, curl $B = 0$ then $E \cdot B \in H^1(\mathbb{R}^n)$.

  "No, it cannot be true; otherwise I would have known it."

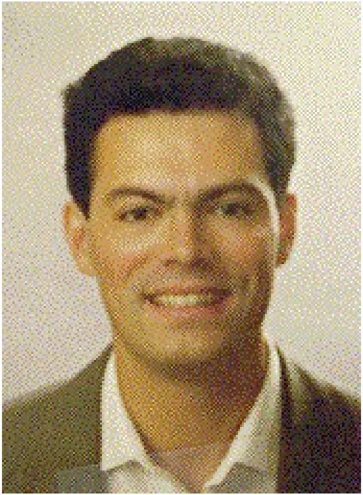
- Paul Federbush's paper "Navier and Stokes meet the wavelet." Jacques Louis Lions puzzled and irritated by title:

  "What is you opinion?"

- Yves Meyer, Marco Cannone* and Fabrice Planchon*: Prove existence of global Kato solution to Navier-Stokes when initial conditions are oscillating.

Pierre Louis Lions (1956-)   Jacques Louis Lions (1928-2001)

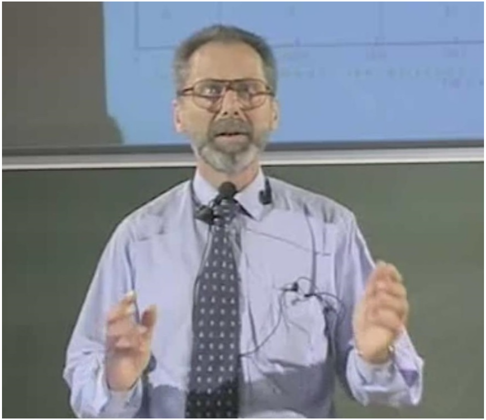
Marco Cannone* (1966-)



Fabrice Planchon* (?-)

CNRS, CNAM, Cachan

- 1995-1999: Directeur de recherche au CNRS.
- 2000: Conservatoire national des arts et métiers.
- 1999-2003: École Normale Supérieure de Cachan.



Yves Meyer at CNAM (2000)

Gauss Prize

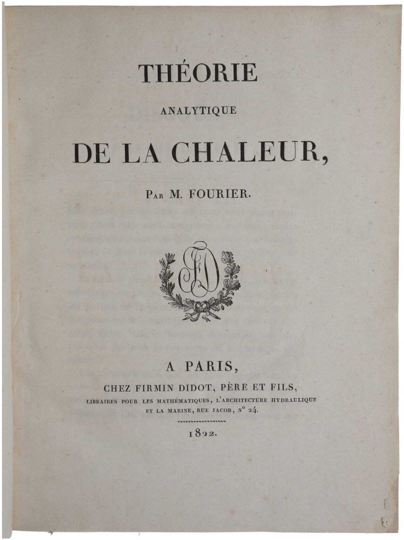
- 2003-: Professor emeritus of ENS Cachan, now ENS Paris-Saclay.
- 2010: Awarded Gauss prize at Hyderabad ICM.
- At least 51 PhD students and 138 descendants.

Yves Meyer receives Gauss prize (2010)

Outline

Outline

Joseph Fourier (1768-1830)



THÉORIE

ANALYTIQUE

DE LA CHALEUR,

PAR M. FOURIER.

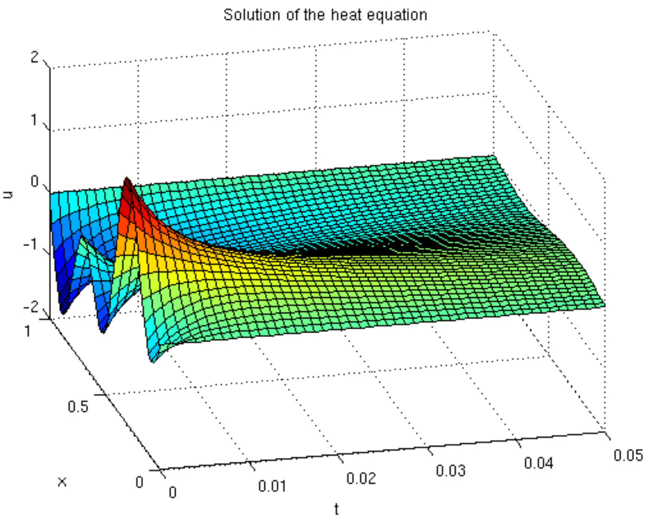
A PARIS,

CHEZ FIRMIN DIDOT, PÈRE ET FILS,

LIBRAIRES POUR LES MATHÉMATIQUES, L'ARCHITECTURE HYDRAULIQUE ET LA MARINE, RUE JACOB, N° 24.

1822.

Extended version of Fourier’s 1807 Memoir



Heat equation: $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}$

Fourier's claim

Every 2π-periodic function $f(t)$ can be represented by sum

$$a_0 + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt)$$

where

$$\begin{aligned} a_0 &= \frac{1}{2\pi} \int_0^{2\pi} f(t)\, dt \\ a_n &= \frac{1}{\pi} \int_0^{2\pi} f(t) \cos nt\, dt \\ b_n &= \frac{1}{\pi} \int_0^{2\pi} f(t) \sin nt\, dt\,. \end{aligned}$$

Fourier series

- For 2π-periodic $f(t)$ Fourier coefficients are

$$\hat{f}_n = \int_0^{2\pi} f(t) e^{-int}\, dt \qquad \text{for } n \in \mathbb{Z}$$

- Fourier series is

$$f(t) \stackrel{?}{=} \frac{1}{2\pi} \sum_n \hat{f}_n\, e^{int}$$

- Functions $\{e^{int}\}_n$ are orthonormal for inner product

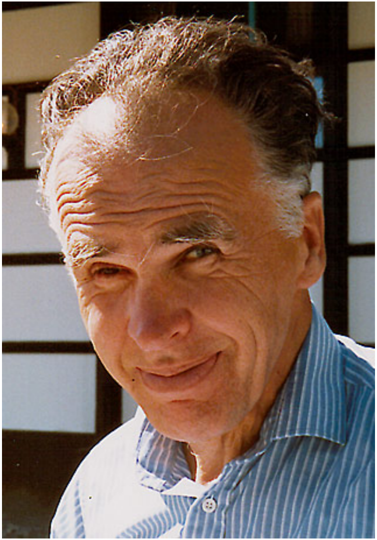
$$\langle f, g \rangle = \frac{1}{2\pi} \int_0^{2\pi} f(t) \overline{g(t)}\, dt\,.$$

1873: Paul du Bois-Reymond (1831-1889) constructed a continuous function whose Fourier series diverges at one point

1903: Lipót Fejér (1880-1959) proved Cesàro sum convergence for Fourier series of any continuous function

Henri Lebesgue (1875-1941) established convergence in norm for Fourier series of any L^2-function on $[0, 2\pi]$

1923/1926: Andrey Kolmogorov (1903-1987) constructed an L^1-function whose Fourier series diverges (almost) everywhere



1966: Lennart Carleson (1928-, Abel Prize 2006) proved that Fourier series of an L^2-function converges almost everywhere

Time-invariant linear operators

- Consider functions $g\colon u \mapsto g(u)$ for $u \in \mathbb{R}$.
- Time-invariant linear operators $L\colon g \mapsto Lg$ given by convolution products

$$Lg(u) = (f * g)(u) = \int_{-\infty}^{+\infty} f(t)g(u-t)\,dt\,.$$

- Here $f = L\delta$ is impulse response of L to the Dirac δ.

Fourier transform

- $g(t) = e^{i\omega t}$ is an eigenfunction/eigenvector of L

$$Lg(u) = \int_{-\infty}^{+\infty} f(t) e^{i\omega(u-t)}\, dt = \hat{f}(\omega) g(u)\,.$$

- Eigenvalue

$$\hat{f}(\omega) = \int_{-\infty}^{+\infty} f(t) e^{-i\omega t}\, dt$$

  is the Fourier transform of f at $\omega \in \mathbb{R}$.

- The value $\hat{f}(\omega)$ is large when $f(t)$ is similar to $e^{i\omega t}$ for t in a set of large measure.

Inverse Fourier transform

- Fourier reconstruction formula

$$f(t) \stackrel{?}{=} \frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{f}(\omega) e^{i\omega t} \, d\omega \, .$$

- When is $f(t)$ well approximated by weighted sums or integrals of exponential functions $\{e^{i\omega t}\}_{\omega}$?
- When do few ω suffice?

Regularity and decay

A function $f(t)$ is many times differentiable if $\hat{f}(\omega)$ tends quickly to zero as $|\omega|$ grows.

Theorem

Let $r \geq 0$. If

$$\int_{-\infty}^{+\infty} |\hat{f}(\omega)|(1 + |\omega|^r)\, d\omega < \infty$$

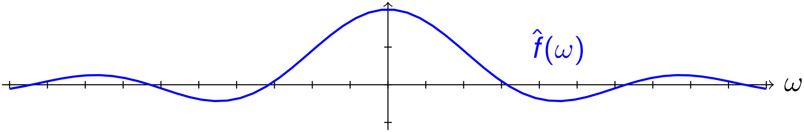
then $f(t)$ is r times continuously differentiable.

Lack of localization

- Decay of $\hat{f}$ depends on worst singular behavior of f.
- Indicator function

$$f(t) = \begin{cases} 1 & \text{for } -1 \leq t \leq 1, \\ 0 & \text{otherwise} \end{cases}$$

has Fourier transform $\hat{f}(\omega) = 2\sin(\omega)/\omega$, with slow decay.



Outline

Localization in time and frequency

- Exponential $e^{i\omega t}$ is localized in frequency, but not in time.
- Delaying $f(t)$ by t_0 is not seen by $|\hat{f}(\omega)|$.
- To study transient, time-dependent phenomena, it could be better to replace functions $e^{i\omega t}$ with functions $g(t)$ that are localized in both time and frequency.
- Can both $g(t)$ and $\hat{g}(\omega)$ have small support, or decay quickly?

Heisenberg's uncertainty principle, I

- Suppose $\|g\|^2 = \int_{-\infty}^{+\infty} |g(t)|^2 \, dt = 1$, so that $|g(t)|^2$ is a probability density on $\mathbb{R}$.
- Plancherel: $\|\hat{g}\|^2 = \int_{-\infty}^{+\infty} |\hat{g}(\omega)|^2 \, d\omega = 2\pi$.
- Mean value of t

$$\mu_t = \int_{-\infty}^{+\infty} t|g(t)|^2 \, dt \,.$$

- Variance around μ_t

$$\sigma_t^2 = \int_{-\infty}^{+\infty} (t - \mu_t)^2 |g(t)|^2 \, dt \,.$$

Heisenberg's uncertainty principle, II

- Mean value of ω

$$\mu_\omega = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \omega |\hat{g}(\omega)|^2 \, d\omega \, .$$

- Variance around μ_ω

$$\sigma_\omega^2 = \frac{1}{2\pi} \int_{-\infty}^{+\infty} (\omega - \mu_\omega)^2 |\hat{g}(\omega)|^2 \, d\omega \, .$$

Theorem (Heisenberg (1927))

$$\sigma_t \cdot \sigma_\omega \geq \frac{1}{2} \, .$$

- Theoretical limit to combined localization of time and frequency.

Werner Heisenberg (1901-1976, Nobel prize in physics 1932)

Gabor atoms

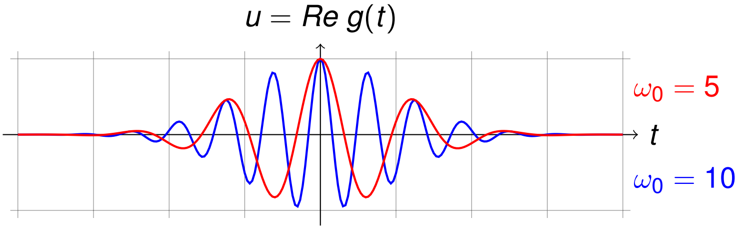
Minimum $\sigma_t \cdot \sigma_\omega = 1/2$ only for Gabor atoms

$$g(t) = ae^{-bt^2}$$

with $a, b \in \mathbb{C}$, and their twists

$$g_{t_0,\omega_0}(t) = g(t - t_0) \cdot e^{i\omega_0 t}$$

obtained by translating by t_0 in time and by ω_0 in frequency.



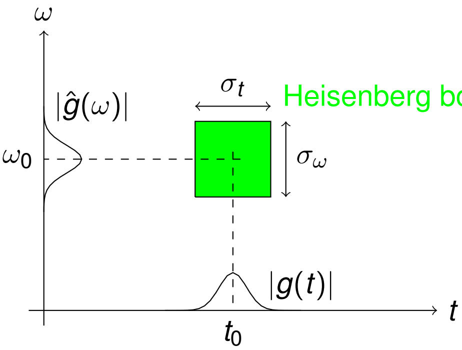
Dennis Gabor (1900-1979, Nobel prize in physics 1971)

Time-frequency support

Correlation of $f(t)$ with $g(t)$,

$$\langle f, g \rangle = \int_{-\infty}^{+\infty} f(t)\overline{g(t)}\, dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{f}(\omega)\overline{\hat{g}(\omega)}\, d\omega\,,$$

depends on f and $\hat{f}$ at (t, ω) where g and $\hat{g}$ are not neglible.



Windowed Fourier transform

- Gabor (1946) showed that decomposition of audio signals as a sum or integral of twists

$$g_{t_0,\omega_0}(t) = g(t - t_0) \cdot e^{i\omega_0 t}$$

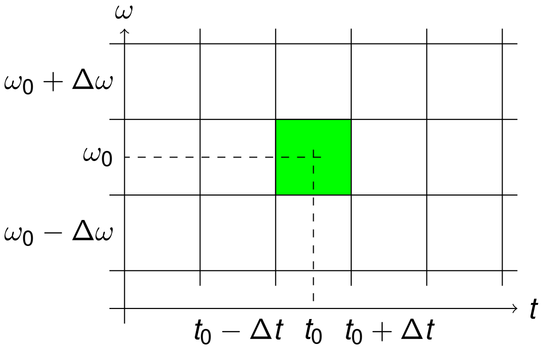
  of atoms $g(t) = ae^{-bt^2}$ (which he called logons) is closely related to our perception of sounds.
- Conjectured that a time-frequency dictionary of logons $g_{t_0,\omega_0}(t)$ for

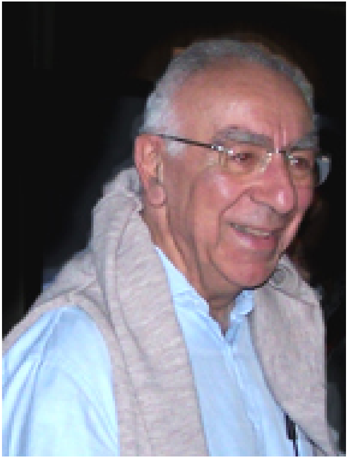
$$t_0 = k\Delta t \qquad \text{and} \qquad \omega_0 = \ell\Delta\omega \,,$$

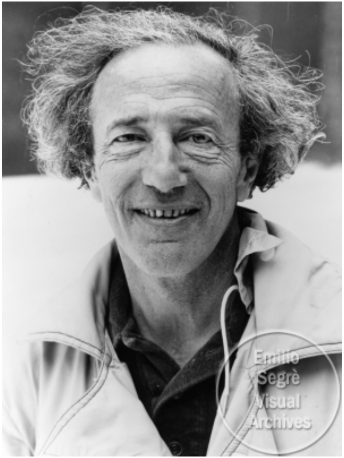
  integer multiples of fixed Δt and $\Delta\omega$, could give an orthonormal basis for $L^2(\mathbb{R})$.

Balian-Low theorem

- Such a basis would tile time-frequency plane by congruent translates of Heisenberg box of $g(t)$ by integer multiples of Δt and $\Delta\omega$.
- Roger Balian (1981) and Francis Low proved independently that an orthonormal basis cannot be obtained this way, for any smooth, localized "window" $g(t)$.







Roger Balian (1933-)   Francis Low (1921-2007)

Outline

Jean Morlet (1931-2007)

Jean Morlet, I

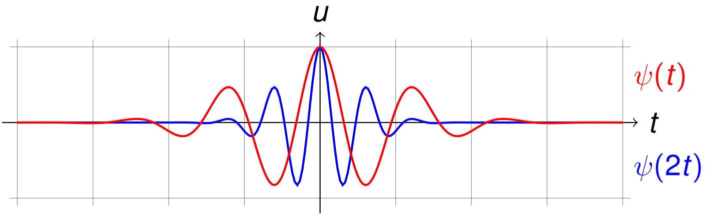
- 1970s: Worked as a geophysicist for French oil company Elf-Aquitaine.
- Processed backscattered seismic signals to obtain information about geological layers.
- Found that at high frequencies windowed Fourier transform (twisted Gabor atoms = logons) had too long duration to resolve thin borders between layers.
- 1981: Proposed to use dilations by m and translations by c

$$\psi(\frac{t-c}{m})$$

of a constant shape $\psi(t) = Re\, g(t)$.

Jean Morlet, II

- Adopted Balian's term "ondelette/wavelet" for this shape.
- Turn from time-frequency analysis to time-scale analysis.
- Elf: "If it were true [that this can work], it would be known."
- Balian directed Morlet to Alexandre Grossmann in Marseille.



Alexandre Grossmann (1930-)

Continuous wavelet transform (CWT)

- Normalize

$$\psi_{(m,c)}(t) = \frac{1}{\sqrt{m}}\psi(\frac{t-c}{m}).$$

- Inner product

$$Wf(m,c) = \langle f, \psi_{(m,c)} \rangle = \int_{-\infty}^{+\infty} f(t)\overline{\psi_{(m,c)}(t)}\, dt$$

  for $(m,c) \in (0,\infty) \times \mathbb{R}$ defines continuous wavelet transform Wf of f.

- $Wf(m,c)$ is large if f is similar to ψ at scale m near time c.

Grossmann-Morlet reconstruction formula

Theorem (Grossmann-Morlet (1984))

$$f(t) = \iint Wf(m,c)\, \psi_{(m,c)}(t)\, \frac{dm}{m}\, dc$$

for f in Hardy space $H^2(\mathbb{R}) \subset L^2(\mathbb{R})$.

- Grossmann interpreted Morlet's continuous wavelet transform as a “coherent state” for the Lie group of affine motions $t \mapsto mt + c$, $m > 0$.
- Studied for quantum mechanics by Erik W. Aslaksen and John R. Klauder (1968/1969).

Discrete wavelet series

- For a scaling factor $s > 1$, Morlet sought to approximate double integral over $(m, c) \in (0, \infty) \times \mathbb{R}$ by series such as

$$f(t) \stackrel{?}{=} \sum_{j,k} Wf_{j,k}\, s^{j/2}\psi(s^j t - k)$$

for $j, k \in \mathbb{Z}$, where

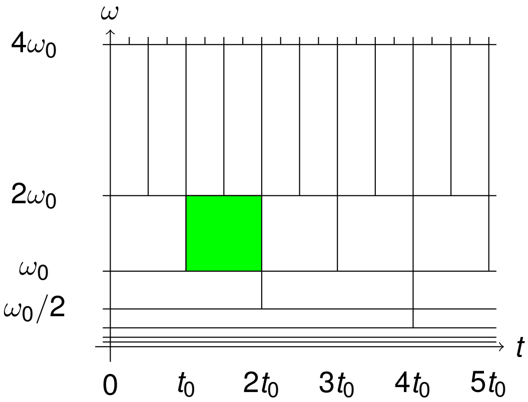
$$\psi_{(m,c)}(t) = s^{j/2}\psi(s^j t - k)$$

is dilated by $m = 1/s^j$ and translated by $c = k/s^j$.

- How to determine wavelet coefficients $Wf_{j,k}$ numerically?
- How large can s be? Is Shannon limit $s = 2$ possible?

Dyadic ($s = 2$) time-frequency tiling

- Corresponds to tiling time-frequency plane by area-preserving modifications of Heisenberg box of $\psi(t)$.
- Narrower time intervals at high frequencies.



Calderón's identity

- Yves Meyer recognized Grossmann-Morlet's formula as Alberto Calderón's reproducing identity (1964)

$$f = \int_0^\infty Q_m(Q_m^*(f)) \, \frac{dm}{m} \, ,$$

  valid for all $f \in L^2(\mathbb{R})$.
- Here $\psi(t) \in L^2(\mathbb{R})$ and we assume that

$$\int_0^\infty |\hat{\psi}(m\omega)|^2 \frac{dm}{m} = 1$$

  for almost all $\omega \in \mathbb{R}$.
- Operator $Q_m \colon f \mapsto \psi_m * f$ is convolution with $\psi_m(t) = \frac{1}{m}\psi(\frac{t}{m})$, and Q_m^* is its adjoint.

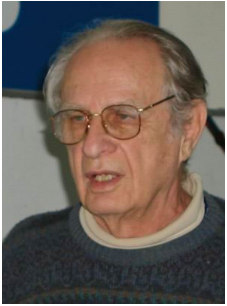
Yves Meyer:

“I recognized Calderón’s reproducing identity and I could not believe that it had something to do with signal processing.

I took the first train to Marseilles where I met Ingrid Daubechies, Alex Grossmann and Jean Morlet. It was like a fairy tale.

This happened in 1984. I fell in love with signal processing. I felt I had found my homeland, something I always wanted to do.”

Marseille group



Ingrid Daubechies Alex Grossmann Jean Morlet

A collective enterprise

Yves Meyer:

"A last advice to young mathematicians is to simply forget the torturing question of the relevance of what they are doing.

It is clear to me that the progress of mathematics is a collective enterprise.

All of us are needed."

Yves Meyer (1939-, Abel Prize 2017)

References

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- “A Wavelet Tour of Signal Processing”, Stéphane Mallat, Academic Press, 2009.
- “Interview/Essay of/by Yves Meyer”, Ulf Persson, Medlemsutskicket, Svenska Matematikersamfundet, 15 maj 2011.

Outline

A biographical sketch

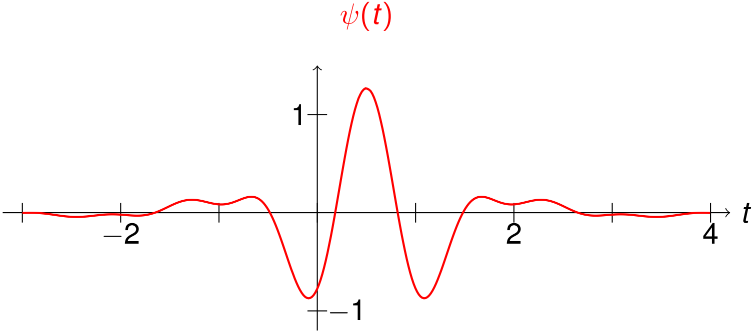
From Fourier to Morlet

- Fourier transform
- Gabor atoms
- Wavelet transform

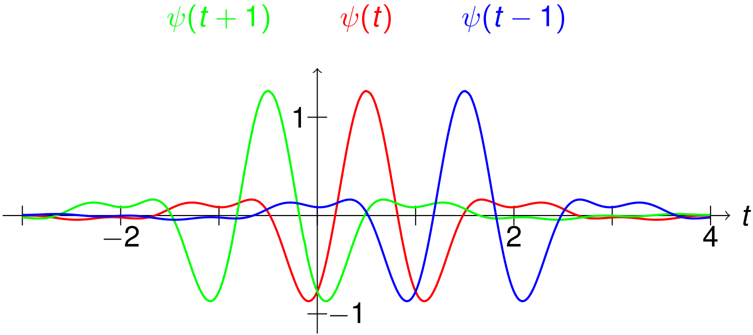
First synthesis: Wavelet analysis (1984-1985)

Second synthesis: Multiresolution analysis (1986-1988)

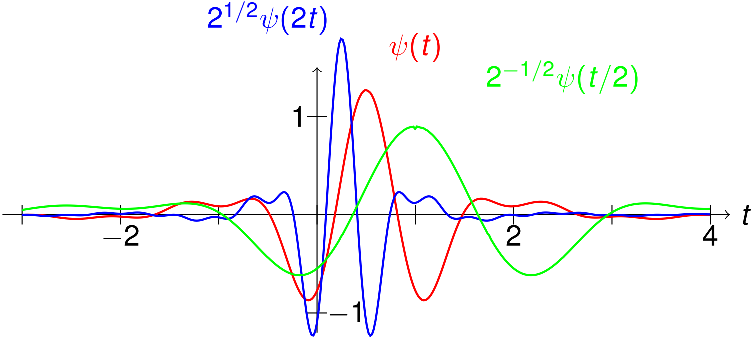
Meyer's first wavelet



Translations



Dilations



Orthonormal wavelet basis

- Let $\psi(t) \in L^2(\mathbb{R})$. Functions

  $$\psi_j(t) = 2^{j/2}\psi(2^j t)$$

  dilate $\psi(t)$ by a factor $1/2^j$, and normalize.
- Functions

  $$\psi_{j,k}(t) = \psi_j(t-k)$$

  translate ψ_j by k.
- Call $\psi(t)$ a wavelet if

  $$\psi_{j,k}(t) = 2^{j/2}\psi(2^j t - k) \qquad \text{for } j,k \in \mathbb{Z}$$

  is an orthonormal basis for $L^2(\mathbb{R})$.

Discrete wavelet transform (DWT)

- Discrete wavelet transform

$$Wf_{j,k} = \langle f, \psi_{j,k} \rangle = \int_{-\infty}^{+\infty} f(t)\overline{\psi_{j,k}(t)}\, dt \qquad \text{for } j, k \in \mathbb{Z}$$

  of $f \in L^2(\mathbb{R})$ is then given by sampling $Wf(m, c)$.

- Reconstruction formula

$$f(t) = \sum_{j,k} Wf_{j,k}\ \psi_{j,k}(t) = \sum_{j,k} \langle f, \psi_{j,k} \rangle \psi_{j,k}(t)$$

Alexandre Grossmann (1930-) Jean Morlet (1931-2007)

Scale spaces, I

- Scale $= 1/2^{j+1}$ subspace

$$W_j = \mathrm{span}\{\psi_{j,k}(t) \mid k \in \mathbb{Z}\} \subset L^2(\mathbb{R}) \qquad \text{for } j \in \mathbb{Z}$$

- Orthogonal decomposition

$$\bigoplus_{j \in \mathbb{Z}} W_j \subset L^2(\mathbb{R}) \qquad \text{dense}$$

- Scale $\geq 1/2^j$ subspace

$$V_j = \bigoplus_{i<j} W_i \qquad \text{for } j \in \mathbb{Z}$$

- Nested subspaces

$$0 \subset \cdots \subset V_j \subset V_{j+1} \subset \cdots \subset L^2(\mathbb{R})$$

Scale spaces, II

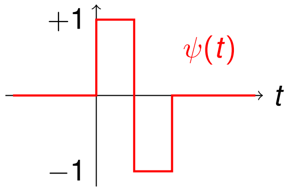
- Orthogonal sum
$$V_{j+1} = V_j \oplus W_j$$
- Scale $\geq 1/2^j$ projection
$$\mathrm{proj}_{V_j}(f) = \sum_{i<j} \sum_k \langle f, \psi_{i,k} \rangle \psi_{i,k}$$
- Scale $= 1/2^{j+1}$ details
$$\mathrm{proj}_{W_j}(f) = \sum_k \langle f, \psi_{j,k} \rangle \psi_{j,k}$$

Haar basis

- Simplest wavelet, studied by Alfréd Haar (1909):

$$\psi(t) = \begin{cases} +1 & \text{for } 0 \leq t < 1/2 \\ -1 & \text{for } 1/2 \leq t < 1 \\ 0 & \text{otherwise.} \end{cases}$$

- Discontinuous. Localized in time, but not in frequency.



Alfréd Haar (1885-1933)

Strömberg basis

- Contrary to Balian-Low prediction, orthonormal wavelet bases with more regular $\psi(t)$ can be found.
- 1981: Jan-Olov Strömberg found a continuous, piecewise-linear function $\psi(t)$ with rapid decay such that

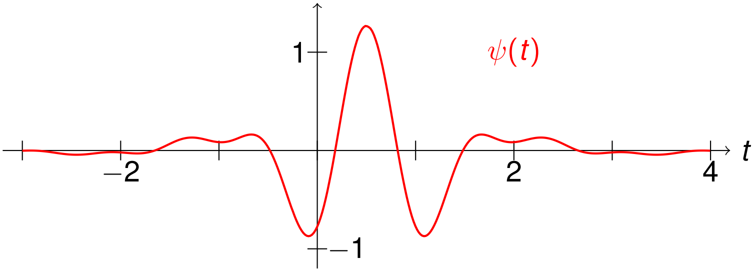
$$\psi_{j,k}(t) = 2^{j/2}\psi(2^j t - k) \qquad \text{for } j,k \in \mathbb{Z}$$

  is an orthonormal wavelet basis for $H^1(\mathbb{R})$.

Jan-Olov Strömberg (1947?-)

Meyer wavelet

- 1985: Yves Meyer found a C^∞ smooth wavelet $\psi(t)$ with rapid decay such that $\psi_{j,k}(t)$ form an orthonormal basis for $L^2(\mathbb{R})$.



Yves Meyer

Meyer basis

- Meyer wavelets provide a universal unconditional basis for almost all classical Banach spaces (L^p for $1 < p < \infty$, Hardy, BMO, Hölder, Sobolev and Besov spaces).
- Extended to functions on $\mathbb{R}^n$ for $n \geq 2$ by Pierre-Gilles Lemarié* and Yves Meyer (December 1985).

Pierre Gilles Lemarié-Rieusset*

Multifractal analysis

- Hölder regularity of e.g. Brownian motion can be detected by decay rate of wavelet coefficients $Wf_{j,k}$.
- Stéphane Jaffard* used wavelet analysis to determine Hölder exponents, varying with position, of multifractal functions.

- Used by Marie Farge as a tool for studying turbulence.
- Energy is transferred from large to small scales, suggesting time-scale analysis is appropriate.

Stéphane Jaffard* (1962-) Marie Farge (1953-)

Wavelets and operators

- Fourier analysis is adapted to time-invariant operators, which diagonalize functions $g(t) = e^{i\omega t}$.

- Wavelet analysis is adapted to operators that are (nearly) diagonalized by a wavelet basis $\psi_{j,k}(t)$.
- Generate algebra of “Calderón-Zygmund operators”, including pseudo-differential operators and singular integral operators.
- Need criteria for L^2-continuity, to replace Fourier transform.
- Theorem $T(1)$ of Guy David* and Jean-Lin Journé* provides one such tool.


Guy David*(1957-) Jean-Lin Journé* (1957-2016)

Outline

Wavelets and multiresolution approximations

- 1985: Coifman and Meyer axiomatized approximation of $L^2(\mathbb{R})$ by nested subspaces of functions at varying resolutions = inverse scales.
- Aim: Fill gap between few wavelets then at hand, and a general formalism satisfied by all orthonormal wavelet bases.

$$\text{Multi-resolution approximations} \Longleftarrow \text{Orthonormal wavelet bases}$$

Ronald Raphael Coifman (1941-)

Multi-resolution approximation

- Let
$$V_0 \subset L^2(\mathbb{R})$$
be a space of "scale $\geq 2^0 = 1$ approximations".
- Define space V_j of "scale $\geq 1/2^j$ approximations" by
$$f(t) \in V_0 \iff f(2^j t) \in V_j$$
- Assume $V_0 \subset V_1$, so that
$$\cdots \subset V_j \subset V_{j+1} \subset \ldots$$
- Assume $\bigcap_j V_j = 0$ and $\bigcup_j V_j$ is dense in $L^2(\mathbb{R})$.

Scaling function

- A function $\phi(t)$, such that integer translates

$$\phi_n(t) = \phi(t-n) \qquad \text{for } n \in \mathbb{Z}$$

  form an orthonormal basis for V_0, is called a scaling function for the multiresolution approximation.
- The functions

$$\phi_{j,n}(t) = 2^{j/2}\phi(2^j t - n) \qquad \text{for } n \in \mathbb{Z}$$

  then form an orthonormal basis for V_j.
- Fourier transform $\hat{\phi}(\omega)$ satisfies

$$\sum_k |\hat{\phi}(\omega + 2\pi k)|^2 = 1\,.$$

Haar approximations, I

- Haar's wavelet basis fits into this framework.
- $V_0 \subset L^2(\mathbb{R})$ is space of functions that are constant on each interval $[n, n+1)$, for $n \in \mathbb{Z}$.
- Scaling function

$$\phi(t) = \begin{cases} +1 & \text{for } 0 \le t < 1 \\ 0 & \text{otherwise.} \end{cases}$$

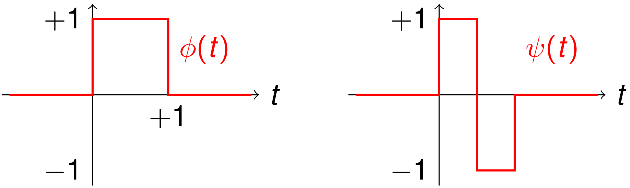
- V_j is space of functions that are constant on each interval $[n/2^j, (n+1)/2^j)$.

Haar approximations, II

- Note that $V_0 \subset V_1$.
- Translates $\psi_n(t) = \psi(t - n)$ of Haar wavelet

$$\psi(t) = \begin{cases} +1 & \text{for } 0 \le t < 1/2 \\ -1 & \text{for } 1/2 \le t < 1 \\ 0 & \text{otherwise} \end{cases}$$

form orthonormal basis for $W_0 = V_1 - V_0$.



Wavelets and conjugate mirror filters

- 1986: Stéphane Mallat and Yves Meyer connected multiresolution approximations to conjugate mirror filters.
- Parallel between multiresolution approximations and Laplacian pyramid scheme for numerical image processing, introduced by Peter J. Burt and Edward H. Adelson (1983).
- Wavelet coefficients can be calculated by a generalization of quadrature mirror filters, invented by D. Esteban and C. Galand for compression and transmission algorithms for digital telephones (1977).

Stéphane Mallat (1962-)

Orthogonal projections

$$0 \subset \dots \subset V_j \subset V_{j+1} \subset \dots \subset L^2(\mathbb{R})$$

$$\begin{tikzcd}
0 & \dots \arrow[l] & V_j \arrow[l] \arrow[d] & V_{j+1} \arrow[l, "a"'] \arrow[d, "d"] & \dots \arrow[l] & L^2(\mathbb{R}) \arrow[l] \\
 & \dots & W_{j-1} & W_j & \dots &
\end{tikzcd}$$

Conjugate mirror filters $\Longleftarrow$ Multi-resolution approximations $\Longleftarrow$ Orthonormal wavelet bases

Haar approximations and details

- Filter input $(x, y) \in \mathbb{R}^2$.
- Approximation and details:

$$a = \frac{1}{2}x + \frac{1}{2}y \qquad \text{and} \qquad d = \frac{1}{2}x - \frac{1}{2}y$$

-

$$\begin{tikzcd} x \arrow[d] \arrow[dr] & y \arrow[d] \arrow[dl] \\ a & d \end{tikzcd}$$

- Recovery: $x = a + d$ and $y = a - d$.

Repeat

- Signal $(f_k)_k$ for $1 \leq k \leq N$
- Details $(d_k)_k$ for $1 \leq k \leq N-1$ and final approximation a_{N-1}.

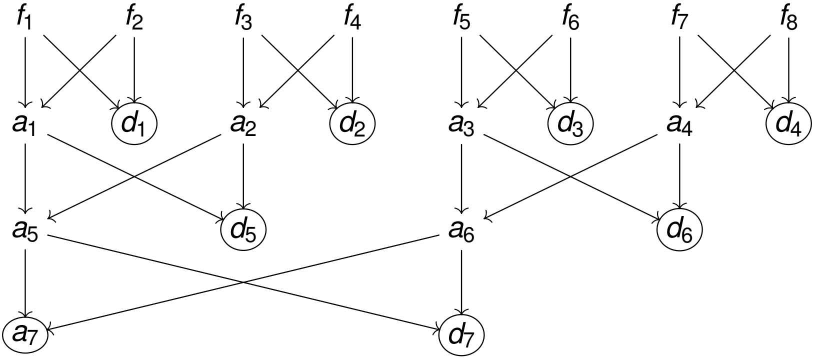
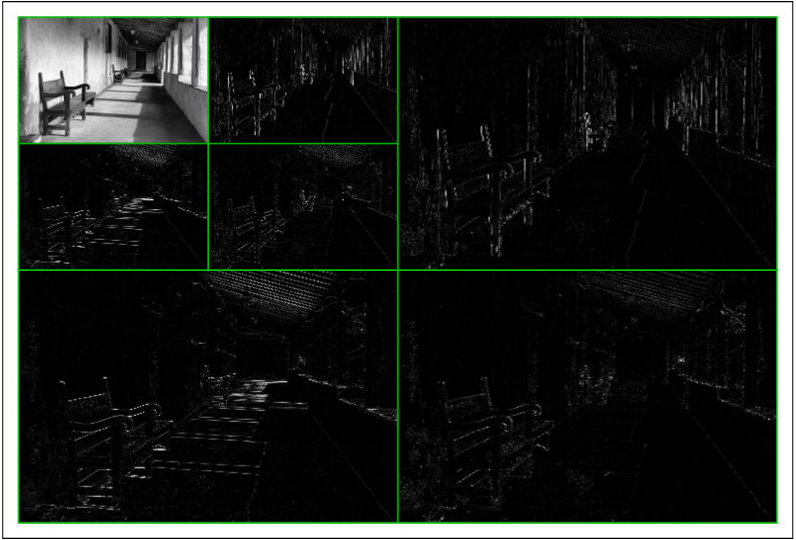


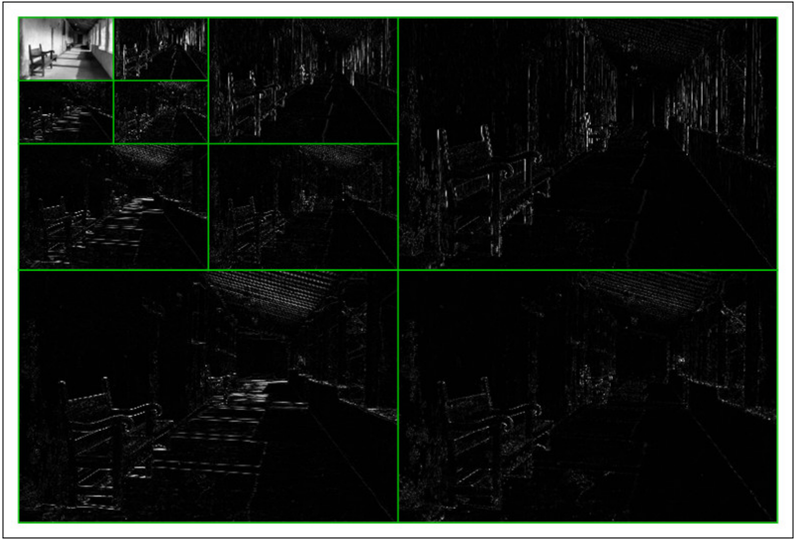
Photo by Radka Tezaur

Vertical Haar approximation and detail

Vertical and horizontal Haar approximation and detail



Twofold vertical and horizontal Haar approximation and detail



Threefold vertical and horizontal Haar approximation and detail

Threefold Haar approximation (rescaled)

Scaling equation

- Since $V_0 \subset V_1$, we can write

$$\phi(t) = \sum_n h_n \cdot \sqrt{2}\phi(2t - n)$$

  for a sequence $(h_n)_n \in \ell^2(\mathbb{Z})$
- Fourier transform

$$\hat{h}(\omega) = \sum_n h_n e^{-in\omega}$$

  satisfies

$$\hat{\phi}(\omega) = \frac{1}{\sqrt{2}}\hat{h}(\frac{\omega}{2})\hat{\phi}(\frac{\omega}{2})$$

Quadrature condition

Proposition (Mallat, Meyer)

Fourier transform $\hat{h}(\omega) = \sum_n h_n e^{-in\omega}$ *satisfies*

$$|\hat{h}(\omega)|^2 + |\hat{h}(\omega + \pi)|^2 = 2 \tag{1}$$

for all $\omega \in \mathbb{R}$*, and*

$$\hat{h}(0) = \sqrt{2}\,. \tag{2}$$

Proof.

(1) Use $\hat{\phi}(\omega) = \frac{1}{\sqrt{2}}\hat{h}(\omega/2)\hat{\phi}(\omega/2)$ and $\sum_k |\hat{\phi}(\omega + 2\pi k)|^2 = 1$.

(2) Completeness of multiresolution approximation implies $|\hat{\phi}(0)| = 1 \neq 0$. $\square$

Approximation as convolution

- Orthogonal projection $V_1 \to V_0$ takes a signal

$$f(t) = \sum_n f_n \cdot \sqrt{2}\phi(2t - n)$$

to $\sum_m a_m \cdot \phi(t - m)$, where

$$a_m = \langle f(t), \phi(t - m) \rangle = \sum_n f_n h_{n-2m} \,.$$

- Mallat recognized $(f_n)_n \mapsto (a_m)_m$ as approximation (low pass) half of a conjugate mirror filter.
- $(a_m)_m$ is given by convolving $(f_n)_n$ with $(h_{-n})_n$ and only keeping even terms (downsampling).

Wavelet construction

Theorem (Mallat, Meyer (1986))

Let $(g_n)_n \in \ell^2(\mathbb{Z})$ and $\psi(t) \in L^2(\mathbb{R})$ have Fourier transforms

$$\hat{g}(\omega) = \sum_n g_n e^{-i\omega n} = e^{-i\omega} \hat{h}^*(\omega + \pi)$$

$$\hat{\psi}(\omega) = \frac{1}{\sqrt{2}} \hat{g}(\frac{\omega}{2}) \hat{\phi}(\frac{\omega}{2}) \, .$$

Then the translates

$$\psi_n(t) = \psi(t - n) \qquad \text{for } n \in \mathbb{Z}$$

form an orthonormal basis for $W_0 \subset V_1$, and the $\psi_{j,k}(t)$ form an orthonormal wavelet basis for $L^2(\mathbb{R})$.

Detail as convolution

- Orthogonal projection $V_1 \to W_0$ takes a signal

$$f(t) = \sum_n f_n \cdot \sqrt{2}\phi(2t - n)$$

to $\sum_m d_m \cdot \psi(t - m)$, where

$$d_m = \langle f(t), \psi(t - m)\rangle = \sum_n f_n g_{n-2m} \, .$$

- Hence $(f_n)_n \mapsto (d_m)_m$ is detail (high pass) half of a conjugate mirror filter, given by convolving with $(g_{-n})_n$ and downsampling.
- Detail filter is conjugate mirror of approximation filter:

$$g_n = (-1)^{n-1} h_{1-n} \, .$$

Cascade

Fast wavelet transformation (FWT)

- Start with a signal in V_H, at high resolution 2^H

$$f(t) = \sum_n f_n \cdot \phi_{H,n}(t)$$

- Convolve $(f_n)_n$ with $(g_n)_n$ to get details in W_{H-1}

$$\sum_m d_m \cdot \psi_{H,m}(t)$$

- Convolve $(f_n)_n$ with $(h_n)_n$ to get approximation at next lower resolution, and repeat. Store details at each level.
- Finish with details in $W_{H-1}, \ldots, W_L$, and approximation in V_L, at a low resolution 2^L.

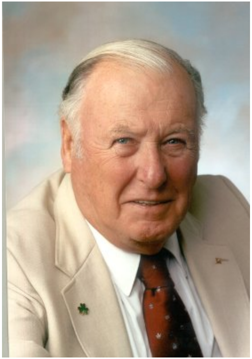
Performance

- Many coefficients $Wf_{j,k} = \langle f(t), \psi_{j,k}(t) \rangle$ may be neglible, e.g., if f is regular near k at scale $1/2^j$, where the needed regularity depends on $\psi(t)$.
- These can then be ignored, allowing for compression.
- Remaining coefficients can be analyzed to detect transient features.
- FWT requires $O(N)$ multiplications for input of size N.
- Proportionality constant given by length (= taps) of filter $(h_n)_n$, or by length of support of $\psi(t)$.

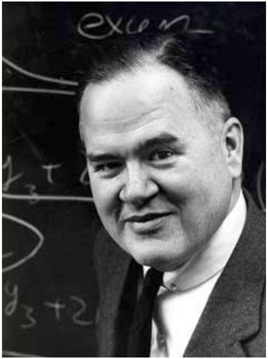
Fast Fourier transform (FFT)

Algorithmic structure of orthonormal wavelet basis makes the FWT about as fast as the Gauss/Cooley-Tukey FFT.

C.F. Gauss (1777-1855)



J. Cooley (1926-2016)



J. Tukey (1915-2000)

Vanishing moments

Definition

$\psi(t)$ has m vanishing moments if

$$\int_{-\infty}^{\infty} t^r \psi(t)\, dt = 0$$

for $0 \leq r < m$.

- Haar wavelet has only one vanishing moment.
- Meyer wavelet has infinitely many vanishing moments.
- If $f(t)$ is C^{m-1} smooth near t_0, then

  $$Wf_{j,k} = \langle f(t), \psi_{j,k}(t) \rangle$$

  is small for $2^j t_0$ near k.

Regularity

If $\psi(t)$ has m vanishing moments, $\hat{h}(\omega) = \sum_n h_n e^{-in\omega}$ has a zero of order m at $\omega = \pi$, and we can factor

$$\hat{h}(\omega) = \sqrt{2}\Big(\frac{1 + e^{-i\omega}}{2}\Big)^m \cdot R(e^{-i\omega})\,.$$

Theorem (Tchamitchian* (1987))

Let $B = \sup_\omega |R(e^{-i\omega})|$. Functions $\phi(t)$ and $\psi(t)$ are uniformly C^α Hölder/Lipschitz regular for

$$\alpha < m - \log_2(B) - 1\,.$$

Philippe Tchamitchian* (1957-)

Compactly supported wavelets

- Consider wavelets $\psi(t)$ with $m \geq 1$ vanishing moments.
- Filter $(h_n)_n$ of minimal length corresponds to polynomial $R(z)$ of minimal degree in $z = e^{-i\omega}$.
- For $(h_n)_n$ real
$$|R(z)|^2 = P(y)$$
is polynomial in $y = \sin^2(\omega/2)$.
- Quadrature condition (1) equivalent to
$$(1-y)^m P(y) + y^m P(1-y) = 1 \tag{3}$$
with $P(y) \geq 0$ for $y \in [0,1]$.

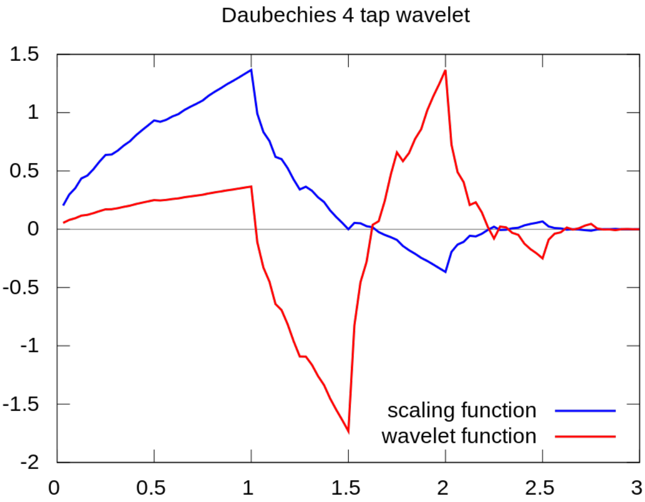
Daubechies wavelets

Theorem (Daubechies (1988))

Real conjugate mirror filters $(h_n)_n$, such that $\psi(t)$ has m vanishing moments, have length $\geq 2m$. Equality is achieved for

$$P(y) = \sum_{k=0}^{m-1} \binom{m+k-1}{k} y^k \, .$$

Associated Daubechies wavelets have compact support, of minimal length for given number m of vanishing moments.



Daubechies scaling function $\phi(t)$ and wavelet $\psi(t)$, for $m = 2$

Ingrid Daubechies (1954-)

Wavelet bases are multiresolution approximations

1988: Pierre Gilles Lemarié* proved that all orthonormal wavelet bases (for wavelets with sufficiently fast decay) arise from multiresolution approximations.

$$\text{Conjugate mirror filters} \Longleftarrow \text{Multi-resolution approximations} \Longleftrightarrow \text{Orthonormal wavelet bases}$$

Which conjugate mirror filters come from multiresolution approximations?

Theorem (Mallat, Meyer (1986))

If $\hat{h}(\omega) = \sum_n h_n e^{-i\omega n}$ *with* $|\hat{h}(\omega)|^2 + |\hat{h}(\omega + \pi)|^2 = 2$ *and* $\hat{h}(0) = \sqrt{2}$ *satisfies*

- $\hat{h}(\omega)$ *is* C^1 *near* 0
- $\hat{h}(\omega)$ *is bounded away from* 0 *on* $[-\pi/2, \pi/2]$

then

$$\hat{\phi}(\omega) = \prod_{j=1}^{\infty} \frac{\hat{h}(\omega/2^j)}{\sqrt{2}}$$

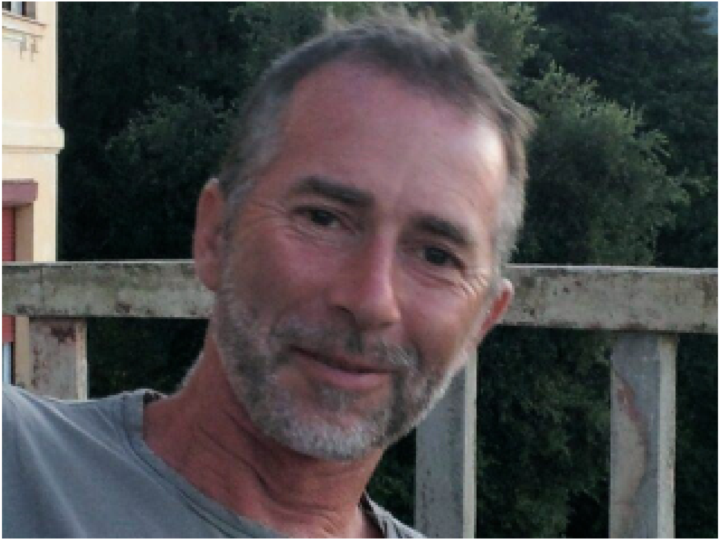
is the Fourier transform of a scaling function $\phi(t)$.

Stable QMFs are FWTs

1989: Albert Cohen* found a necessary and sufficient condition for a conjugate mirror filter $(h_n)_n$ to come from a multiresolution analysis, hence also from an an orthonormal wavelet basis.

Stable conjugate mirror filters	$\Longleftrightarrow$	Multi-resolution approximations	$\Longleftrightarrow$	Orthonormal wavelet bases

Cohen's condition identifies numerically stable filters among all possible conjugate mirror filters. Previously, tuning coefficients of a quadrature mirror filter was done empirically.



Albert Cohen* (1965-)

Later developments

- 1992: Cohen*-Daubechies-Feauveau biorthogonal wavelets (used in JPEG2000)
- 1994: Donoho-Johnstone wavelet shrinkage (noise reduction)
- 1997: F.G. Meyer-Coifman: Brushlets (texture analysis)
- 2000: Candés-Donoho: Curvelets (curve singularities)
- 2003: Donoho-Elad sparse representation via ℓ^1-minimization
- 2006: Donoho, Candès-Romberg-Tao compressive sensing (superresolution)

Yves Meyer (ca. 2013)

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- “Image Compression”, Patrick J. Van Fleet, www.whydomath.org, 2011.