Subdivision schemes in L_p spaces

Rong-Qing Jia*

Department of Mathematics, University of Alberta, Edmonton, Canada T6G 2G1

Received 24 June 1994; revised 15 November 1994
Communicated by C.A. Micchelli

Subdivision schemes play an important role in computer graphics and wavelet analysis. In this paper we are mainly concerned with convergence of subdivision schemes in L_p spaces $(1 \le p \le \infty)$. We characterize the L_p-convergence of a subdivision scheme in terms of the p-norm joint spectral radius of two matrices associated with the corresponding mask. We also discuss various properties of the limit function of a subdivision scheme, such as stability, linear independence, and smoothness.

Keywords: Subdivision schemes, refinement equations, spectral radii, stability, linear independence, smoothness.

AMS subject classification: 39B12, 41A15, 41A25, 65D99.

1. Introduction

Subdivision schemes play an important role in computer graphics and wavelet analysis. There has been an intensive study on uniform convergence of subdivision schemes. In this paper we are mainly concerned with convergence of subdivision schemes in L_p spaces $(1 \le p \le \infty)$.

As usual, for $1 \le p \le \infty$, $L_p(\mathbb{R})$ denotes the Banach space of all (complex-valued) measurable functions f on $\mathbb{R}$ such that $\|f\|_p < \infty$, where

$$\|f\|_p := \left(\int_{\mathbb{R}} |f(x)|^p \, dx \right)^{1/p} \quad \text{for } 1 \le p < \infty,$$

and $\|f\|_\infty$ is the essential supremum of f on $\mathbb{R}$. Moreover, $l_p(\mathbb{Z})$ denotes the Banach space of all (complex-valued) sequences on $\mathbb{Z}$ such that $\|a\|_p < \infty$, where

$$\|a\|_p := \left(\sum_{j \in \mathbb{Z}} |a(j)|^p \right)^{1/p} \quad \text{for } 1 \le p < \infty,$$

and $\|a\|_\infty$ is the supremum of a on $\mathbb{Z}$.

* Supported in part by NSERC Canada under Grant OGP 121336.

Let a be a finitely supported sequence on $\mathbb{Z}$. Consider the mapping T_a on $L_p(\mathbb{R})$ $(1 \le p \le \infty)$ given by

$$T_a\phi = \sum_{j\in\mathbb{Z}} a(j)\phi(2\cdot - j), \quad \phi \in L_p(\mathbb{R}). \tag{1.1}$$

Then T_a is a bounded linear operator on $L_p(\mathbb{R})$. If f is a fixed point of T_a, i.e., $T_a f = f$, then f satisfies the *refinement equation*

$$f = \sum_{j\in\mathbb{Z}} a(j)f(2\cdot - j). \tag{1.2}$$

Correspondingly, the sequence a is called the *refinement mask*. Any function satisfying a refinement equation is called a *refinable function*.

If a is a mask with $\sum_{j\in\mathbb{Z}} a(j) = 2$, then it is known (see [1] and [5]) that there is a unique compactly supported distribution f satisfying $\hat{f}(0) = 1$ and the refinement equation (1.2). This distribution is said to be *the normalized solution* to the refinement equation with mask a.

Let ϕ be a compactly supported function in $L_p(\mathbb{R})$ $(1 \le p \le \infty)$. Consider the iterates $f_n := T_a^n\phi$, $n = 0, 1, 2, \ldots$. Each f_n can be expressed as

$$f_n = \sum_{j\in\mathbb{Z}} a_n(j)\phi(2^n\cdot - j),$$

where a_n is a finitely supported sequence on $\mathbb{Z}$. Note that the sequences a_n are independent of the choice of ϕ. Indeed, a_n can also be computed by an iteration scheme. First, $a_0 = \delta$, where we use δ to denote the sequence on $\mathbb{Z}$ given by

$$\delta(j) = \begin{cases} 1 & \text{for } j = 0, \\ 0 & \text{elsewhere.} \end{cases} \tag{1.3}$$

Next, let S_a be the linear operator on $l_p(\mathbb{Z})$ defined by

$$S_a\lambda(i) := \sum_{j\in\mathbb{Z}} a(i - 2j)\lambda(j), \quad i \in \mathbb{Z}, \tag{1.4}$$

where $\lambda \in l_p(\mathbb{Z})$. Then it is easily seen that

$$a_n = S_a^n\delta, \quad n = 0, 1, 2, \ldots.$$

More generally, if $\lambda \in l_p(\mathbb{Z})$, then

$$\sum_{j\in\mathbb{Z}} \lambda(j)(T_a^n\phi)(\cdot - j) = \sum_{k\in\mathbb{Z}} (S_a^n\lambda)(k)\phi(2^n\cdot - k). \tag{1.5}$$

The operator S_a is called the *subdivision operator* associated with a.

In particular, if we choose ϕ to be the hat function:

$$\phi(x) = \begin{cases} 1 + x & \text{for } x \in [-1, 0), \\ 1 - x & \text{for } x \in [0, 1], \\ 0 & \text{for } x \in \mathbb{R}\backslash[-1, 1], \end{cases} \tag{1.6}$$

then the iteration scheme $f_n = T_a^n\phi$, $n = 0, 1, 2, \dots$, is called the *subdivision scheme* associated with mask a. This subdivision scheme is said to converge in the L_p-norm, if there is a function $f \in L_p(\mathbb{R})$ such that

$$\lim_{n\to\infty} \| f_n - f \|_p = 0.$$

If this is the case, then for any $\lambda \in l_p(\mathbb{Z})$

$$\lim_{n\to\infty} \left\| \sum_{k\in\mathbb{Z}} (S_a^n\lambda)(k)\phi(2^n\cdot - k) - \sum_{j\in\mathbb{Z}} \lambda(j)f(\cdot - j) \right\|_p = 0.$$

The preceding definitions and results can be found in [1]. Note that in [5] a subdivision scheme is referred to as a *cascade algorithm*.

Here is an outline of the paper. In section 2, we establish a basic theory for subdivision schemes in L_p spaces. In particular, we give a necessary condition and a sufficient condition for a subdivision scheme to be L_p-convergent. In order to study convergence of subdivision schemes, we introduce in section 3 the concept of the p-norm joint spectral radius of a finite collection of square matrices, and study spectral radii and joint spectral radii of various matrices. In section 4, we characterize the L_p-convergence of a subdivision scheme and discuss the Lipschitz property of the corresponding refinable function in the L_p-norm. The results of section 4 can be used to characterize smoothness of a refinable function in terms of its mask, provided this refinable function has stable shifts. In section 5, we discuss stability and another related concept of linear independence, and demonstrate that the study of smoothness of refinable functions can be reduced to those refinable functions with linearly independent shifts. Finally, in section 6, we give a rather complete discussion of smoothness of refinable functions, and compare our results with those in the literature.

We take a new approach to the study of convergence of subdivision schemes. Our approach does not require that the concerned mask be fully factorized, and so it can be applied to the multivariate case. We will report our results on multivariate refinement equations and related subdivision schemes in a forthcoming paper.

2. Subdivision schemes

The hat function ϕ defined in (1.6) enjoys two nice properties. First, the shifts of ϕ form a partition of unity, that is,

$$\sum_{j\in\mathbb{Z}} \phi(\cdot - j) = 1. \tag{2.1}$$

Second, the shifts of ϕ are L_p-stable for $1 \le p \le \infty$. We say that the shifts of a function ϕ in $L_p(\mathbb{R})$ are *stable* if there are two positive constants C_1 and C_2 such that

$$C_1\|\lambda\|_p \le \left\| \sum_{j\in\mathbb{Z}} \lambda(j)\phi(\cdot - j) \right\|_p \le C_2\|\lambda\|_p, \quad \forall\lambda \in l_p(\mathbb{Z}). \tag{2.2}$$

The above two properties can both be characterized in terms of the Fourier transform of ϕ. In this paper, the Fourier transform of a function $f \in L_1(\mathbb{R})$ is defined to be

$$\hat{f}(\xi) := \int_{\mathbb{R}} f(x)e^{-ix\xi}\, dx, \quad \xi \in \mathbb{R}.$$

By the Poisson summation formula it is easily seen that (2.1) is equivalent to saying that

$$\hat{\phi}(0) = 1 \text{ and } \hat{\phi}(2k\pi) = 0, \quad \forall k \in \mathbb{Z}\backslash\{0\}.$$

In other words,

$$\int_{\mathbb{R}} \phi(x)e^{-i2k\pi x}\, dx = \delta_{0k}, \tag{2.3}$$

where δ stands for the Kronecker sign. If a function $\phi \in L_1(\mathbb{R})$ satisfies (2.3), then we say that ϕ satisfies the *moment conditions* of order 1. The moment conditions were formulated by Schoenberg in [23] and applied to spline approximation. Furthermore, it was proved in [16] that a compactly supported function $\phi \in L_p(\mathbb{R})$ satisfies the L_p-stability condition in (2.2) if and only if, for any $\xi \in \mathbb{R}$, there exists an integer k such that

$$\hat{\phi}(\xi + 2k\pi) \neq 0.$$

Also, it was shown in [17, theorem 2.4] that if ϕ is a refinable function in $L_1(\mathbb{R})$ with compact support and $\hat{\phi}(0) = 1$, then ϕ satisfies the moment conditions of order 1.

Let a be a finitely supported sequence with $\sum_{j\in\mathbb{Z}} a(j) = 2$, and let f be the normalized solution to the refinement equation with mask a. Cavaretta et al. showed in [1] that if f is a continuous function with stable shifts, then the subdivision scheme associated with a converges uniformly, i.e., converges in the L_∞-norm. They also proved that for the subdivision scheme associated with a to converge uniformly it is necessary that

$$\sum_{j\in\mathbb{Z}} a(i-2j) = 1, \quad \forall i \in \mathbb{Z}.$$

In this section, we extend their results to the case $1 \le p \le \infty$.

We first establish a result concerning approximation by the dilated shifts of a compactly supported function. In what follows, we use $\omega(f,h)_p$ to denote the L_p-modulus of continuity of f:

$$\omega(f,h) := \sup_{|t|\le h} \|f - f(\cdot - t)\|_p, \quad h > 0.$$

Theorem 2.1

Let $1 \le p \le \infty$. Suppose $\phi \in L_p(\mathbb{R})$ is compactly supported and satisfies the moment conditions of order 1. If $f \in L_p(\mathbb{R})$ (f is a uniformly continuous function on $\mathbb{R}$ in the case $p = \infty$), then there exists a constant $C > 0$ such that

$$\left\| f - \sum_{j \in \mathbb{Z}} b_h(j)\phi(\cdot/h - j) \right\|_p \leq C\omega(f, h)_p, \quad \forall h > 0, \tag{2.4}$$

where

$$b_h(j) = \frac{1}{h}\int_{jh}^{(j+1)h} f(x)\, dx.$$

Proof
For $x \in \mathbb{R}$ and $h > 0$, let

$$f_h(x) := \frac{1}{h}\int_0^h f(x+y)\, dy = \frac{1}{h}\int_x^{x+h} f(y)\, dy.$$

Thus, f_h is absolutely continuous, and for almost every $x \in \mathbb{R}$,

$$f_h'(x) = \frac{f(x+h) - f(x)}{h}.$$

We shall show that for $1 \leq p \leq \infty$

$$\| f_h - f \|_p \leq \omega(f, h)_p. \tag{2.5}$$

The proof of (2.5) for the case $p = \infty$ is trivial, so we only deal with the case $1 \leq p < \infty$. Let q be the exponent conjugate to p, i.e., $1/p + 1/q = 1$. By Hölder's inequality, we obtain

$$\begin{aligned} |f_h(x) - f(x)| &= \left| \frac{1}{h}\int_0^h [f(x+y) - f(x)]\, dy \right| \\ &\leq \frac{1}{h}\left(\int_0^h dy\right)^{1/q}\left(\int_0^h |f(x+y) - f(x)|^p\, dy\right)^{1/p}. \end{aligned}$$

It follows that

$$\| f_h - f \|_p^p \leq \frac{1}{h}\int_0^h \int_{\mathbb{R}} |f(x+y) - f(x)|^p\, dx\, dy \leq \omega(f, h)_p^p.$$

This proves (2.5).

Note that $b_h(j) = f_h(hj)$ for all $j \in \mathbb{Z}$. Let

$$Q_h f := \sum_{j \in \mathbb{Z}} f_h(hj)\phi(\cdot/h - j).$$

Taking (2.5) into account, we see that the theorem will be established if we can show that

$$\|Q_h f - f_h\|_p \leq C\omega(f, h)_p. \tag{2.6}$$

Since ϕ satisfies the moment conditions of order 1, $\sum_{j\in\mathbb{Z}}\phi(\cdot - j) = 1$; hence we have

$$f_h(x) - Q_h f(x) = \sum_{j\in\mathbb{Z}} (f_h(x) - f_h(hj))\phi(x/h - j), \quad x \in \mathbb{R}.$$

Suppose ϕ is supported on $[-N, N]$ for some positive integer N. Fix $x \in [0, h)$ and $k \in \mathbb{Z}$ for the moment. Then

$$\begin{aligned} |f_h(x+kh) - Q_h f(x+kh)| &= \left| \sum_{|j-k|\le N} (f_h(x+kh) - f_h(hj))\phi(x/h + k - j) \right| \\ &\le \int_{(k-N)h}^{(k+N)h} |f_h'(y)|\,dy \sum_{j\in\mathbb{Z}} |\phi(x/h + k - j)|. \end{aligned}$$

Let $1 \le p < \infty$ and $1/p + 1/q = 1$. By Hölder's inequality we get

$$\int_{(k-N)h}^{(k+N)h} |f_h'(y)|\,dy \le (2Nh)^{1/q} \left(\int_{(k-N)h}^{(k+N)h} |f_h'(y)|^p\,dy \right)^{1/p}.$$

Let

$$|\phi|^0 := \sum_{j\in\mathbb{Z}} |\phi(\cdot - j)|.$$

Evidently, $|\phi|^0$ is a function lying in $L_p([0,1))$. Combining the above estimates together, we obtain

$$\begin{aligned} \|f_h - Q_h f\|_p^p &= \sum_{k\in\mathbb{Z}} \int_0^h |f_h(x+kh) - Q_h f(x+kh)|^p\,dx \\ &\le (2Nh)^{p/q} \int_0^h (|\phi|^0(x/h))^p\,dx \sum_{k\in\mathbb{Z}} \int_{(k-N)h}^{(k+N)h} |f_h'(y)|^p\,dy \\ &\le C^p \|hf_h'\|_p^p, \end{aligned}$$

where C is a positive constant. But

$$\|hf_h'\|_p = \|f(\cdot + h) - f\|_p \le \omega(f, h)_p.$$

This proves (2.6).

The case $p = \infty$ is easier and can be proved similarly. □

Theorem 2.2

Let a be a finitely supported sequence on $\mathbb{Z}$ such that $\sum_{j\in\mathbb{Z}} a(j) = 2$, and let $T = T_a$ be the linear operator given by (1.1). Suppose u is a compactly supported function in $L_p(\mathbb{R})$ $(1 \le p \le \infty)$, u satisfies the moment conditions of order 1, and the shifts of u are stable. If there exists a function $f \in L_p(\mathbb{R})$ (a continuous function f in the case

$p = \infty$) such that

$$\lim_{n\to\infty} \|T^n u - f\|_p = 0,$$

then for any compactly supported function $v \in L_p(\mathbb{R})$ satisfying the moment conditions of order 1 we also have

$$\lim_{n\to\infty} \|T^n v - f\|_p = 0.$$

Proof

For $n = 0, 1, 2, \ldots$, let

$$f_n := \sum_{j\in\mathbb{Z}} b_j(j)u(2^n \cdot - j) \quad \text{and} \quad g_n := \sum_{j\in\mathbb{Z}} b_n(j)v(2^n \cdot - j),$$

where

$$b_n(j) := \frac{1}{h}\int_{hj}^{(h+1)j} f(x)\,dx \quad \text{with } h = \frac{1}{2^n}.$$

Then by theorem 2.1 there exists a constant $C_1 > 0$ such that

$$\|f - f_n\|_p \le C_1\omega(f, 1/2^n)_p \quad \text{and} \quad \|f - g_n\|_p \le C_1\omega(f, 1/2^n)_p.$$

Observe that

$$T^n u = \sum_{j\in\mathbb{Z}} a_n(j)u(2^n \cdot - j) \quad \text{and} \quad T^n v = \sum_{j\in\mathbb{Z}} a_n(j)v(2^n \cdot - j),$$

where $a_n = S_a^n\delta$. Thus, we obtain

$$T^n u - f_n = \sum_{j\in\mathbb{Z}} [a_n(j) - b_n(j)]u(2^n \cdot - j).$$

Since the shifts of u are stable, there exists a constant $C_2 > 0$ such that

$$\|a_n - b_n\|_p \le C_2\|(f_n - T^n u)(2^{-n}\cdot)\|_p = 2^{n/p}C_2\|f_n - T^n u\|_p.$$

Furthermore, there exists a constant $C_3 > 0$ such that

$$\|(g_n - T^n v)(2^{-n}\cdot)\|_p = \left\|\sum_{j\in\mathbb{Z}} [a_n(j) - b_n(j)]v(\cdot - j)\right\|_p \le C_3\|a_n - b_n\|_p.$$

Combining the above two estimates together, we see that there exists a constant $C > 0$ such that

$$\|g_n - T^n v\|_p \le C\|f_n - T^n u\|_p.$$

Therefore we have

$$\begin{aligned}
\|f - T^n v\|_p &\le \|f - g_n\|_p + \|g_n - T^n v\|_p \\
&\le \|f - g_n\|_p + C\|f_n - T^n u\|_p \\
&\le \|f - g_n\|_p + C(\|f - f_n\|_p + \|f - T^n u\|_p).
\end{aligned}$$

But as $n \to \infty$, $\| f - g_n \|_p \to 0$, $\| f - f_n \|_p \to 0$, and $\| f - T^n u \|_p \to 0$; hence we conclude that

$$\lim_{n\to\infty} \| T^n v - f \|_p = 0. \qquad \square$$

Let f be the normalized solution to the refinement equation (1.2). Then $T_a f = f$. Suppose $f \in L_p(\mathbb{R})$ (f is continuous in the case $p = \infty$) and f satisfies the stability condition (2.2). In this case, f must satisfy the moment conditions of order 1. Thus, in theorem 2.1 we may choose u to be f and v to be the hat function. This gives the following result.

Theorem 2.3
Let a be a finitely supported sequence on $\mathbb{Z}$ with $\sum_{j\in\mathbb{Z}} a(j) = 2$, and f the normalized solution to the refinement equation

$$f = \sum_{j\in\mathbb{Z}} a(j) f(2\cdot - j).$$

If f lies in $L_p(\mathbb{R})$ $(1 \le p < \infty)$, or f is a continuous function in the case $p = \infty$, and if the shifts of f are stable, then the subdivision scheme associated with mask a converges to f in the L_p-norm.

Finally, we give a necessary condition for the L_p-convergence of a subdivision scheme.

Theorem 2.4
Let a be a finitely supported sequence on $\mathbb{Z}$ such that $\sum_{j\in\mathbb{Z}} a(j) = 2$. If the subdivision scheme associated with a converges in the L_p-norm $(1 \le p \le \infty)$, then

$$\sum_{j\in\mathbb{Z}} a(i - 2j) = 1, \quad \forall i \in \mathbb{Z}. \tag{2.7}$$

Proof
If the subdivision scheme converges in the L_p-norm and $1 \le q \le p \le \infty$, then it also converges in the L_q-norm. Thus, we only have to deal with the case $p = 1$.

Let $T = T_a$ be the linear operator given in (1.1), and let $S = S_a$ be the subdivision operator associated with a. Let ϕ be the hat function given in (1.6) and write f_n for $T^n\phi$, $n = 1, 2, \ldots$. Suppose that there exists a function $f \in L_1(\mathbb{R})$ such that $\| f_n - f \|_1 \to 0$ as $n \to \infty$. We wish to prove that (2.7) is valid.

Suppose that $a(j) = 0$ for $j \notin [-N, N]$, where N is a positive integer. Then f is supported on $[-N, N]$. Since f is a compactly supported refinable function, $\hat{f}(2k\pi) = 0$ for all $k \in \mathbb{Z}\backslash\{0\}$. Moreover, $\hat{f}(0) = \lim_{n\to\infty} \widehat{f_n}(0) = 1$. This shows that f satisfies the moment conditions of order 1, that is, $\sum_{j\in\mathbb{Z}} f(\cdot - j) = 1$. Choose an integer $m > N$, and let $g := \sum_{j=-m}^{m} f(\cdot - j)$. With $\chi_{[-m,m]}$ denoting the characteristic function of the interval $[-m, m]$ we observe that $g - \chi_{[-m,m]}$ vanishes outside $[-m - N, m + N]\backslash[-m + N, m - N]$. Therefore there exists a positive constant C_1 independent of m such that

$$\|g - \chi_{[-m,m]}\|_1 \le C_1.$$

The function g may be written as $\sum_{j\in\mathbb{Z}} \lambda(j) f(\cdot - j)$, where λ is the sequence on $\mathbb{Z}$ given by

$$\lambda(j) = \begin{cases} 1 & \text{for } |j| \le m, \\ 0 & \text{for } |j| > m. \end{cases}$$

Let $M := m + N$. Note that $S^n\lambda(j) \ne 0$ only if $j \in 2^n[-m-N, m+N] = 2^n[-M, M]$. Hence by (1.5) we get

$$\sum_{j=-m}^{m} f_n(\cdot - j) = \sum_{j=-m}^{m} (T^n\phi)(\cdot - j) = \sum_{k=-2^nM}^{2^nM} S^n\lambda(k)\phi(2^n\cdot - k).$$

Since ϕ is the hat function, there exists a constant C_2 independent of n and m such that

$$\left\| \sum_{k=-2^nM}^{2^nM} \phi(2^n\cdot - k) - \chi_{[-m,m]} \right\|_1 \le C_2.$$

Furthermore,

$$\left\| g - \sum_{j=-m}^{m} f_n(\cdot - j) \right\|_1 = \left\| \sum_{j=-m}^{m} (f - f_n)(\cdot - j) \right\|_1 \le (2m+1)\|f - f_n\|_1.$$

Combining the above estimates together, we obtain

$$\left\| \sum_{k=-2^nM}^{2^nM} [S^n\lambda(k) - 1]\phi(2^n\cdot - k) \right\|_1 \le C_1 + C_2 + (2m+1)\|f - f_n\|_1.$$

Since the shifts of ϕ are stable, we see that there exists a constant $C_3 > 0$ such that

$$\sum_{k=-2^nM}^{2^nM} |S^n\lambda(k) - 1| \le 2^nC_3(m\|f - f_n\|_1 + 1). \tag{2.8}$$

Write s_i for $\sum_{j\in\mathbb{Z}} a(i - 2j)$. Note that $s_{2i} = s_0$ and $s_{2i+1} = s_1$ for all integers i. Thus,

$$\begin{aligned} 2^nm(|s_0 - 1| + |s_1 - 1|) &\le \sum_{i=-2^{n+1}M}^{2^{n+1}M} \left| \sum_{j\in\mathbb{Z}} a(i-2j) - 1 \right| \\ &\le \sum_{i=-2^{n+1}M}^{2^{n+1}M} \left| \sum_{j\in\mathbb{Z}} a(i-2j)S^n\lambda(j) - 1 \right| \\ &\quad + \sum_{i=-2^{n+1}M}^{2^{n+1}M} \left| \sum_{j\in\mathbb{Z}} a(i-2j)(S^n\lambda(j) - 1) \right|. \end{aligned}$$

Note that $S^{n+1}\lambda(i) = \sum_{j\in\mathbb{Z}} a(i-2j)S^n\lambda(j)$ for all $i \in \mathbb{Z}$. Therefore by (2.8) we conclude that there exists a constant $C > 0$ such that

$$2^n m(|s_0 - 1| + |s_1 - 1|) \le 2^n C(m\|f - f_n\|_1 + 1).$$

It follows that

$$|s_0 - 1| + |s_1 - 1| \le C(\|f - f_n\|_1 + 1/m).$$

Letting n, $m \to \infty$ in the above inequality, we obtain $s_0 = 1$ and $s_1 = 1$, as desired. □

3. Spectral radius

There is a close connection between the spectral radius of a subdivision operator and the convergence of the corresponding subdivision scheme. Also, the spectral radius of a subdivision operator is related to the joint spectral radius of two matrices associated with the corresponding mask. This section is devoted to a study of spectral radii and joint spectral radii of various matrices.

To begin with, we review matrix norms (see [15, pp. 290–311]). Let m be a positive integer. A real-valued function $\|\cdot\|$ on the collection of all $m \times m$ complex matrices is called a *generalized matrix norm*, if the following conditions are satisfied for all $m \times m$ complex matrices A and B:

(a) $\|A\| \ge 0$, and $\|A\| = 0$ if and only if $A = 0$;
(b) $\|cA\| = |c|\|A\|$ for all complex numbers c;
(c) $\|A + B\| \le \|A\| + \|B\|$.

If, in addition,

(d) $\|AB\| \le \|A\|\,\|B\|$,

then $\|\cdot\|$ is called a *matrix norm*.

We use A^{T} to denote the transpose of a given matrix A. If $\|\cdot\|$ is a (generalized) matrix norm, then the function $A \mapsto \|A^{\mathrm{T}}\|$ also gives a (generalized) matrix norm.

Now let us introduce the p-norm joint spectral radius of a finite collection of matrices. Let $\mathcal{M}$ be a finite collection of square matrices of the same size. For a positive integer n we denote by $\mathcal{M}^n$ the nth Cartesian power of $\mathcal{M}$:

$$\mathcal{M}^n = \{(M_1, \ldots, M_n) : M_1, \ldots, M_n \in \mathcal{M}\}.$$

Let $\|\cdot\|$ be a fixed matrix norm. For $1 \le p \le \infty$, we define

$$\|\mathcal{M}^n\|_p := \left(\sum_{(M_1,\ldots,M_n)\in\mathcal{M}^n} \|M_1 \cdots M_n\|^p \right)^{1/p}. \tag{3.1}$$

It is easily seen that for positive integers j and k

$$\|\mathcal{M}^{j+k}\|_p \le \|\mathcal{M}^j\|_p \|\mathcal{M}^k\|_p.$$

The *p-norm joint spectral radius* of $\mathcal{M}$ is defined to be

$$\rho_p(\mathcal{M}) := \lim_{n\to\infty} \|\mathcal{M}^n\|_p^{1/n}.$$

It can be easily proved that this limit exists, and

$$\lim_{n\to\infty} \|\mathcal{M}^n\|_p^{1/n} = \inf_{n\geq 1} \|\mathcal{M}^n\|_p^{1/n}.$$

Indeed, if $\rho = \inf_{n\geq 1} \|\mathcal{M}^n\|_p^{1/n}$, then for any $\epsilon > 0$ there exists a positive integer k such that

$$\|\mathcal{M}^k\|_p^{1/k} < \rho + \epsilon.$$

Suppose $\|\mathcal{M}^j\|_p \leq C$ for $j = 0, 1, \ldots, k-1$, where C is a positive constant. If n is a positive integer, then n can be written as $n = ks + j$, where s is an integer and $j \in \{0, 1, \ldots, k-1\}$. Hence we have

$$\|\mathcal{M}^n\|_p \leq \|\mathcal{M}^{ks}\|_p \|\mathcal{M}^j\|_p \leq C(\rho + \epsilon)^{ks}.$$

This shows that $\limsup_{n\to\infty} \|\mathcal{M}^n\|_p^{1/n} \leq \rho + \epsilon$, and therefore our claim is verified.

We observe that $\rho_\infty(\mathcal{M})$ is the joint spectral radius introduced by Rota and Strang in [22], and $\rho_1(\mathcal{M})$ was studied by Yang in [26] as the mean spectral radius.

If $|||\cdot|||$ is a generalized matrix norm on the collection of all $m \times m$ complex matrices, then there exist two positive constants C_1 and C_2 such that for all $m \times m$ matrices A,

$$C_1\|A\| \leq |||A||| \leq C_2\|A\|.$$

It follows that

$$C_1\|\mathcal{M}^n\|_p \leq \left(\sum_{(M_1,\ldots,M_n)\in\mathcal{M}^n} |||M_1 \cdots M_n|||^p \right)^{1/p} \leq C_2\|\mathcal{M}^n\|_p,$$

where the constants C_1 and C_2 are independent of n. Therefore,

$$\lim_{n\to\infty} \left(\sum_{(M_1,\ldots,M_n)\in\mathcal{M}^n} |||M_1 \cdots M_n|||^p \right)^{1/(np)} = \rho_p(\mathcal{M}).$$

In other words, in the definition (3.1), $\|\cdot\|$ can be any generalized matrix norm.

Next, let us consider the p-norm joint spectral radius of a finite collection of linear operators on a finite-dimensional vector space V. A vector norm $\|\cdot\|$ on V induces a norm on the linear operators from V to itself. For a linear operator T on V, we define

$$\|T\| := \max_{\|x\|=1} \{\|Tx\|\}.$$

Let $\mathcal{T}$ be a finite collection of linear operators on V, the p-norm joint spectral radius of $\mathcal{T}$ is defined to be

$$\rho_p(\mathcal{T}) := \lim_{n\to\infty} \|\mathcal{T}^n\|_p^{1/n},$$

where

$$\|\mathscr{T}^n\|_p := \left(\sum_{(T_1,\ldots,T_n)\in\mathscr{T}^n} \|T_1\cdots T_n\|^p \right)^{1/p}.$$

Clearly, $\rho_p(\mathscr{T})$ is independent of the choice of the vector norm on V. Furthermore, if $\{e_1,\ldots,e_m\}$ is a basis for V and if T is a linear operator on V such that

$$Te_j = \sum_{k=1}^{m} a_{jk}e_k, \quad j = 1,\ldots,m,$$

then $M = (a_{jk})_{1\le j,k\le m}$ is the matrix representation of T with respect to the basis $\{e_1,\ldots,e_m\}$. Let $\mathscr{M}$ be the collection of the matrix representations of the operators in $\mathscr{T}$. It can be easily proved that $\rho_p(\mathscr{T}) = \rho_p(\mathscr{M})$, regardless of the choice of the basis of V.

Let $S = S_a$ be the subdivision operator associated with a as given in (1.4). We denote by $\rho_p(S_a)$ the spectral radius of the bounded linear operator S_a on $l_p(\mathbb{Z})$. Let $a_n := S_a^n\delta$ for $n = 0,1,\ldots,$ where δ is the sequence given in (1.3). It was proved in [12, corollary 2.5] that

$$\rho_p(S_a) = \lim_{n\to\infty} \|S_a^n\delta\|_p^{1/n}, \quad 1 \le p \le \infty.$$

Given a finitely supported sequence a on $\mathbb{Z}$, we use $\tilde{a}(z)$ to denote its symbol:

$$\tilde{a}(z) := \sum_{j\in\mathbb{Z}} a(j)z^j, \quad z \in \mathbb{C}\backslash\{0\}.$$

It is known (see, e.g., [1]) that

$$\tilde{a}_n(z) = \tilde{a}(z)\tilde{a}(z^2)\cdots\tilde{a}(z^{2^{n-1}}). \tag{3.2}$$

Now let a be a sequence on $\mathbb{Z}$ such that $a(j) = 0$ for $j \notin [0,N]$, where N is a positive integer. We associate to a two matrices A_0 and A_1 as follows:

$$A_0 = (a(2i-j-1))_{1\le i,j\le N} \quad \text{and} \quad A_1 = (a(2i-j))_{1\le i,j\le N}. \tag{3.3}$$

We write $\rho_p(A_0,A_1)$ for $\rho_p(\mathscr{M})$ with $\mathscr{M} := \{A_0,A_1\}$. We wish to prove

$$\rho_p(S_a) = \rho_p(A_0,A_1), \quad 1 \le p \le \infty.$$

The case $p = \infty$ was proved by Goodman et al. in [12]. Following the lines of [12], for each $i \in \mathbb{Z}$, we introduce the linear operator J_i from $l_p(\mathbb{Z})$ to $l_p(\{1,\ldots,N\})$ as follows:

$$J_i\lambda(k) = \lambda(i+1-k), \quad k = 1,\ldots,N,$$

where $\lambda \in l_p(\mathbb{Z})$. Let e_i be the ith column of the $N\times N$ identity matrix, i.e., $e_i(j) = \delta_{ij}$ for $j = 1,\ldots,N$. Then

$$J_i\delta = \begin{cases} e_{i+1} & \text{if } i \in \{0,1,\ldots,N-1\} \\ 0 & \text{otherwise.} \end{cases} \tag{3.4}$$

If $i = d_1 + 2i_1$, where $d_1 \in \{0,1\}$ and $i_1 \in \mathbb{Z}$, then for $k = 1, \dots, N$,

$$\begin{aligned} J_i S\lambda(k) &= S\lambda(i+1-k) \\ &= \sum_{j\in\mathbb{Z}} a(i+1-k-2j)\lambda(j) \\ &= \sum_{j\in\mathbb{Z}} a(d_1+2i_1+1-k-2j)\lambda(j) \\ &= \sum_{j\in\mathbb{Z}} a(d_1-1-k+2j)\lambda(i_1+1-j) \\ &= \sum_{j=1}^{N} A_{d_1}^{\mathrm{T}}(k,j) J_{i_1}\lambda(j) \\ &= A_{d_1}^{\mathrm{T}} J_{i_1}\lambda(k), \end{aligned}$$

where we have used $A(k,j)$ to denote the (k,j)-entry of a matrix A. If

$$i = d_1 + 2d_2 + \cdots + 2^{n-1}d_n + 2^n j,$$

where $d_1, d_2, \dots, d_n \in \{0,1\}$ and $j \in \mathbb{Z}$, then an inductive argument gives

$$J_i S^n = A_{d_1}^{\mathrm{T}} \cdots A_{d_n}^{\mathrm{T}} J_j$$

(see the proof of [12, theorem 2.14]). Thus, for $k \in \{1, \dots, N\}$,

$$S^n\delta(i+1-k) = J_i S^n\delta(k) = A_{d_1}^{\mathrm{T}} \cdots A_{d_n}^{\mathrm{T}} J_j\delta(k) = (A_{d_n} \cdots A_{d_1})^{\mathrm{T}} J_j\delta(k).$$

This, in connection with (3.4), shows that for $k \in \{1, \dots, N\}$

$$S^n\delta(i+1-k) = \begin{cases} 0 & \text{if } j < 0, \text{ or } j \geq N, \\ A_{d_n} \cdots A_{d_1}(j+1,k) & \text{if } 0 \leq j \leq N-1. \end{cases} \tag{3.5}$$

Theorem 3.1
Let $S = S_a$ be the subdivision operator associated with a sequence a supported on $[0, N]$, and let A_0 and A_1 be the two matrices associated with a as given in (3.3). Then

$$\rho_p(S_a) = \rho_p(A_0, A_1), \quad 1 \leq p \leq \infty.$$

Proof
In what follows, we assume that $1 \leq p < \infty$. The case $p = \infty$ can be proved similarly. Let $\mathscr{M} := \{A_0, A_1\}$. In (3.1) we choose the generalized matrix norm $\|\cdot\|$ as follows: for an $N \times N$ matrix M, let

$$\|M\| := \left(\sum_{1 \leq j,k \leq N} |M(j,k)|^p \right)^{1/p}.$$

By (3.5) we obtain

$$\begin{aligned}
\|S^n\delta\|_p^p &= \sum_{i\in\mathbb{Z}} |S^n\delta(i)|^p \\
&= \sum_{(d_1,\dots,d_n)\in\{0,1\}^n} \sum_{j=0}^{N-1} |A_{d_n}\cdots A_{d_1}(j+1,1)|^p \\
&\le \sum_{(d_1,\dots,d_n)\in\{0,1\}^n} \|A_{d_n}\cdots A_{d_1}\|^p \\
&= \|\mathscr{M}^n\|_p^p.
\end{aligned}$$

This shows that

$$\rho_p(S) = \lim_{n\to\infty} \|S^n\delta\|_p^{1/n} \le \lim_{n\to\infty} \|\mathscr{M}^n\|_p^{1/n} = \rho_p(A_0, A_1).$$

For the converse we derive from (3.5) that

$$\begin{aligned}
\|\mathscr{M}^n\|_p^p &= \sum_{(d_1,\dots,d_n)\in\{0,1\}^n} \|A_{d_n}\cdots A_{d_1}\|^p \\
&= \sum_{(d_1,\dots,d_n)\in\{0,1\}^n} \sum_{j=0}^{N-1}\sum_{k=1}^{N} |A_{d_n}\cdots A_{d_1}(j+1,k)|^p \\
&\le N\sum_{i\in\mathbb{Z}} |S^n\delta(i)|^p.
\end{aligned}$$

Therefore we have

$$\rho_p(A_0, A_1) = \lim_{n\to\infty} \|\mathscr{M}^n\|_p^{1/n} \le \rho_p(S). \qquad \square$$

Recall that e_i ($i = 1, \dots, N$) is the vector in $\mathbb{C}^N$ given by $e_i(j) = \delta_{ij}, j = 1, \dots, N$. We also view e_i as an $N \times 1$ vector. Let

$$V := \{c_1e_1 + \cdots + c_Ne_N : c_1 + \cdots + c_N = 0\}. \tag{3.6}$$

If $\sum_{j\in\mathbb{Z}} a(i-2j) = 1$ for $i = 0, 1$, then V is an invariant subspace of both A_0 and A_1. Thus, $A_0|_V$ and $A_1|_V$ are linear operators on V. Let $l(\mathbb{Z})$ be the linear space of all sequences on $\mathbb{Z}$. We use ∇ to denote the difference operator given by

$$\nabla\lambda := \lambda - \lambda(\cdot - 1) \quad \text{for } \lambda \in l(\mathbb{Z}).$$

Theorem 3.2
Let $S = S_a$ be the subdivision operator associated with a sequence a supported on $[0, N]$. Let A_0 and A_1 be the two matrices given in (3.3) and let V be the linear space given in (3.6). If $\sum_{j\in\mathbb{Z}} a(i-2j) = 1$ for $i = 0, 1$, then for $1 \le p \le \infty$,

$$\lim_{n\to\infty} \|\nabla S_a^n\delta\|_p^{1/n} = \rho_p(A_0|_V, A_1|_V).$$

Proof
Again we only deal with the case $1 \le p < \infty$. If A is an $N \times N$ matrix, then Ae_k is the kth column of A; hence

$$A(e_k - e_{k+1})(j+1) = A(j+1,k) - A(j+1,k+1).$$

Let $\mathcal{M} := \{A_0|_V, A_1|_V\}$. By (3.5) we find that for $k = 1, \ldots, N-1$,

$$\begin{aligned}\sum_{(d_1,\ldots,d_n)\in\{0,1\}^n} \sum_{j=0}^{N-1} &|A_{d_n} \cdots A_{d_1}(e_k - e_{k+1})(j+1)|^p \\ &= \sum_{i\in\mathbb{Z}} |S^n\delta(i+1-k) - S^n\delta(i-k)|^p \\ &= \sum_{i\in\mathbb{Z}} |\nabla S^n\delta(i+1-k)|^p.\end{aligned}$$

Note that $\{e_k - e_{k+1} : k = 1, \ldots, N-1\}$ forms a basis for V. Therefore we can find two positive constants C_1 and C_2 such that

$$C_1\|\mathcal{M}^n\|_p^p \le \sum_{i\in\mathbb{Z}} |\nabla S^n\delta(i)|^p \le C_2\|\mathcal{M}^n\|_p^p. \tag{3.7}$$

This shows that

$$\lim_{n\to\infty} \|\nabla S^n\delta\|_p^{1/n} = \lim_{n\to\infty} \|\mathcal{M}^n\|_p^{1/n} = \rho_p(A_0|_V, A_1|_V). \qquad \square$$

Recall that $\tilde{a}(z)$ denotes the symbol of a sequence a. If $\sum_{j\in\mathbb{Z}} a(i-2j) = 1$ for $i = 0, 1$, then $\tilde{a}(-1) = 0$. In other words,

$$\tilde{a}(z) = \tilde{b}(z)(1+z),$$

where b is another finitely supported sequence on $\mathbb{Z}$. Now assume that a is supported on $[0, N]$. Then b is supported on $[0, N-1]$. Let

$$B_0 := (b(2i-1-j))_{1\le i,j\le N-1} \quad \text{and} \quad B_1 := (b(2i-j))_{1\le i,j\le N-1}. \tag{3.8}$$

Theorem 3.3
Let a and b be two finitely supported sequences on $\mathbb{Z}$. If $\tilde{a}(z) = \tilde{b}(z)(1+z)$, then there exist two positive constants C_1 and C_2 such that for all positive integers n,

$$C_1\|S_b^n\delta\|_p \le \|\nabla S_a^n\delta\|_p \le C_2\|S_b^n\delta\|_p, \quad 1 \le p \le \infty. \tag{3.9}$$

If, in addition, a is supported on $[0, N]$, then

$$\rho_p(A_0|_V, A_1|_V) = \rho_p(B_0, B_1), \quad 1 \le p \le \infty. \tag{3.10}$$

Proof
It suffices to prove (3.9), since (3.10) follows from (3.9), by theorems 3.1 and 3.2. For $n = 1, 2, \ldots$, let $a_n := S_a^n\delta$ and $b_n := S_b^n\delta$. Then by (3.2) we have

$$\widetilde{\nabla a_n}(z) = (1-z)\tilde{a}_n(z) = (1-z^{2^n})\tilde{b}_n(z).$$

It follows that

$$\|\nabla S_a^n\delta\|_p \le 2\|S_b^n\delta\|_p, \quad 1 \le p \le \infty.$$

To prove the first inequality in (3.9), we assume that a is supported on $[0, N]$, and write

$$\tilde{b}_n(z) = Q_0(z) + Q_1(z)z^{2^n} + \cdots + Q_{N-1}(z)z^{2^n(N-1)}$$

and

$$(1-z)\tilde{a}_n(z) = R_0(z) + R_1(z)z^{2^n} + \cdots + R_{N-1}(z)z^{2^n(N-1)},$$

where Q_k and R_k are polynomials of degree less than 2^n, $k = 0, 1, \ldots, N-1$. Since $(1-z)\tilde{a}_n(z) = (1-z^{2^n})\tilde{b}_n(z)$, we have

$$R_0 = Q_0, \qquad R_1 = Q_1 - Q_0, \ldots, R_{N-1} = Q_{N-1} - Q_{N-2}.$$

It follows that

$$Q_k = R_0 + \cdots + R_k, \quad k = 0, 1, \ldots, N-1.$$

Let q_k and r_k be the sequences such that their symbols $\tilde{q}_k(z) = Q_k(z)$ and $\tilde{r}_k(z) = R_k(z)$. Then we have

$$q_k = r_0 + \cdots + r_k, \quad k = 0, 1, \ldots, N-1,$$

and therefore for $1 \le p < \infty$,

$$\|b_n\|_p^p = \sum_{k=0}^{N-1} \|q_k\|_p^p \le N^p \sum_{k=0}^{N-1} (\|r_0\|_p^p + \cdots + \|r_k\|_p^p) \le N^{p+1} \sum_{k=0}^{N-1} \|r_k\|_p^p.$$

It follows that

$$\|b_n\|_p \le N^{1+1/p} \|\nabla S_a^n\delta\|_p.$$

This is also true for $p = \infty$. The proof of (3.9) is complete. □

We remark that the L_2-spectral radius of a subdivision operator can be easily computed. Let a be a sequence supported on $[0, N]$. Suppose

$$|\tilde{a}(z)|^2 = \sum_{j \in \mathbb{Z}} q_j z^j, \quad z \in \mathbb{T},$$

where $\mathbb{T}$ denotes the unit circle $\{z \in \mathbb{C} : |z| = 1\}$. Then [12, theorem 2.10] tells us that $\rho_2(S_a) = \sqrt{\rho(Q)}$, where $\rho(Q)$ is the spectral radius of the matrix $Q := (q_{j-2k})_{-N \le j,k \le N}$. Also, see [4,11,24] for related results.

Let a be a finitely supported sequence on $\mathbb{Z}$, and let A be the matrix $(a(2i-j))_{i,j\in\mathbb{Z}}$. It was proved by Goodman et al. in [12] that for $1 \le p \le r \le \infty$,

$$2^{1/r-1/p}\rho_p(A) \le \rho_r(A) \le \rho_p(A).$$

They gave some examples showing that the above inequalities are sharp for $p = 2$ and $r = \infty$. In fact, these inequalities are sharp for all p and r with $1 \le p \le r \le \infty$. For example, if $a = \delta$, then $\rho_p(A) = 1$ for all p, $1 \le p \le \infty$.

Now consider the sequence a given by its symbol $\tilde{a}(z) = 1 + z$. For $n = 1, 2, \ldots,$ let $a_n := S^n\delta$. Then

$$\tilde{a}_n(z) = (1+z)(1+z^2)\cdots(1+z^{2^{n-1}}) = \sum_{k=0}^{2^n-1} z^k.$$

Therefore we have $\|a_n\|_p = 2^{n/p}$ and

$$\rho_p(A) = \lim_{n\to\infty} \|a_n\|_p^{1/n} = 2^{1/p}.$$

Thus, for this A and $1 \le p \le r \le \infty$,

$$2^{1/r-1/p}\rho_p(A) = \rho_r(A).$$

4. Convergence of subdivision schemes

In this section we characterize the L_p-convergence $(1 \le p \le \infty)$ of a subdivision scheme. Let a be a finitely supported sequence on $\mathbb{Z}$ with $\sum_{j\in\mathbb{Z}} a(j) = 2$, and let S_a be the subdivision operator as given in (1.4). If the subdivision scheme associated with a is L_p-convergent for some p, $1 \le p \le \infty$, then theorem 2.4 tells us that $\sum_{j\in\mathbb{Z}} a(i-2j) = 1$ for all $i \in \mathbb{Z}$. Thus, $\tilde{a}(z) = (1+z)\tilde{b}(z)$ for some sequence b. Without loss of any generality we may assume that a is supported on $[0, N]$ for some integer $N > 0$. It follows that b is supported on $[0, N-1]$. Let A_0, A_1 be defined as in (3.3), B_0, B_1 defined as in (3.8), and V the linear space given in (3.6). We showed in section 3 that

$$\lim_{n\to\infty} \|\nabla S_a^n\delta\|_p^{1/n} = \rho_p(A_0|_V, A_1|_V) = \rho_p(B_0, B_1) = \rho_p(S_b). \tag{4.1}$$

It was first proved by Dyn et al. in [10] that the subdivision scheme associated with a converges uniformly if and only if $\rho_\infty(B_0, B_1) < 1$. Also see the survey paper [9]. The following theorem characterizes the L_p-convergence of the subdivision scheme for all p, $1 \le p \le \infty$.

Theorem 4.1
The subdivision scheme associated with a mask a converges in the L_p-norm $(1 \le p \le \infty)$ if and only if

$$\rho_p(A_0|_V, A_1|_V) < 2^{1/p}. \tag{4.2}$$

Proof
Suppose that the subdivision scheme associated with a converges to f in the L_p-norm, then $f \in L_p(\mathbb{R})$ (f is continuous in the case $p = \infty$). Let T_a be the linear operator in (1.1) and let ϕ be the hat function given in (1.6). For $n = 1, 2, \ldots,$ write f_n for $T_a^n\phi$. Then

$$f_n = T_a^n\phi = \sum_{j\in\mathbb{Z}} S_a^n\delta(j)\phi(2^n\cdot - j).$$

It follows that

$$f_n(\cdot - 1/2^n) = \sum_{j\in\mathbb{Z}} S_a^n\delta(j)\phi(2^n\cdot - 1 - j) = \sum_{j\in\mathbb{Z}} S_a^n\delta(j-1)\phi(2^n\cdot - j).$$

Subtracting one from the other, we obtain

$$f_n - f_n(\cdot - 1/2^n) = \sum_{j\in\mathbb{Z}} \nabla S_a^n\delta(j)\phi(2^n\cdot - j). \tag{4.3}$$

Since the shifts of ϕ are stable, there exists a constant $C > 0$ such that

$$\|\nabla S_a^n\delta\|_p \le C_1 2^{n/p}\| f_n - f_n(\cdot - 1/2^n)\|_p.$$

But

$$\lim_{n\to\infty} \| f_n - f\|_p = 0.$$

Also, we have

$$\| f_n - f_n(\cdot - 1/2^n)\|_p \le 2\| f_n - f\|_p + \| f - f(\cdot - 1/2^n)\|_p.$$

The preceding estimates show that

$$\lim_{n\to\infty} 2^{-n/p}\|\nabla S_a^n\delta\|_p = 0. \tag{4.4}$$

Let $\mathscr{M} := \{A_0|_V, A_1|_V\}$. Then (3.7) and (4.4) together tell us that

$$\lim_{n\to\infty} 2^{-n/p}\|\mathscr{M}^n\|_p = 0.$$

If the condition in (4.2) were violated, then we would have

$$\inf_{n\ge 1} \|\mathscr{M}^n\|_p^{1/n} = \rho_p(A_0|_V, A_1|_V) \ge 2^{1/p}.$$

It follows that for all positive integers n,

$$2^{-n/p}\|\mathscr{M}^n\|_p \ge 1.$$

This contradicts the fact that $\lim_{n\to\infty} 2^{-n/p}\|\mathscr{M}^n\|_p = 0$. The necessity part of the theorem is proved.

Next, we show that (4.2) is also sufficient for the subdivision scheme to converge in the L_p-norm. In what follows we write S for S_a. Suppose

$$\rho_p(A_0|_V, A_1|_V) < 2^{1/p}.$$

Then by (4.1) we can find a positive real number $r < 2^{1/p}$ such that

$$\|\nabla S^n\delta\|_p \le C_1 r^n, \quad n = 0, 1, \ldots, \tag{4.5}$$

where C_1 is a positive constant independent of n. Recall that $f_n = T_a^n\phi$. We have

$$\begin{aligned}
f_{n+1} &= \sum_{j\in\mathbb{Z}} S^{n+1}\delta(j)\phi(2^{n+1}\cdot - j)\\
&= \sum_{j\in\mathbb{Z}} S^{n+1}\delta(2j)\phi(2^{n+1}\cdot - 2j) + \sum_{j\in\mathbb{Z}} S^{n+1}\delta(2j-1)\phi(2^{n+1}\cdot - 2j+1)\\
&= \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} a(2j-2k)S^n\delta(k)\phi(2^{n+1}\cdot - 2j)\\
&\quad + \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} a(2j-1-2k)S^n\delta(k)\phi(2^{n+1}\cdot - 2j+1).
\end{aligned}$$

Moreover, since ϕ is the hat function, we have

$$\begin{aligned}
f_n &= \sum_{j\in\mathbb{Z}} S^n\delta(j)\phi(2^n\cdot - j)\\
&= \sum_{j\in\mathbb{Z}} S^n\delta(j)[\phi(2^{n+1}\cdot - 2j+1)/2 + \phi(2^{n+1}\cdot - 2j) + \phi(2^{n+1}\cdot - 2j-1)/2]\\
&= \sum_{j\in\mathbb{Z}} S^n\delta(j)\phi(2^{n+1}\cdot - 2j) + \sum_{j\in\mathbb{Z}} [S^n\delta(j) + S^n\delta(j-1)]\phi(2^{n+1}\cdot - 2j+1)/2.
\end{aligned}$$

Subtracting one from the other and taking (2.7) into account, we get

$$\begin{aligned}
f_{n+1} - f_n &= \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} a(2j-2k)[S^n\delta(k) - S^n\delta(j)]\phi(2^{n+1}\cdot - 2j)\\
&\quad + \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} a(2j-1-2k)\\
&\quad \times [S^n\delta(k) - S^n\delta(j)/2 - S^n\delta(j-1)/2]\phi(2^{n+1}\cdot - 2j+1).
\end{aligned}$$

Suppose a is supported on $[0, N]$, where N is a positive integer. Then $a(2j-2k) \neq 0$ implies $0 \le 2j - 2k \le N$, so $j - N \le k \le j$. Also, $a(2j-1-2k) \neq 0$ implies $j - N \le k \le j$. Consequently,

$$|S^n\delta(k) - S^n\delta(j)| \le \sum_{i=j-N+1}^{j} |\nabla S^n\delta(i)|.$$

Thus, there exists a constant $C_2 > 0$ such that

$$\| f_{n+1} - f_n\|_p \le C_2 2^{-n/p}\|\nabla S^n\delta\|_p. \tag{4.6}$$

Combining this estimate with (4.5), we obtain

$$\| f_{n+1} - f_n\|_p \le C_1C_2(r2^{-1/p})^n.$$

Since $r2^{-1/p} < 1$, this shows that the series $\sum_{n=0}^{\infty} \| f_{n+1} - f_n\|_p$ converges. In other words, the subdivision scheme associated with a converges in the L_p-norm. □

For $1 \le p \le \infty$ and $0 < \mu \le 1$, we use $\mathrm{Lip}(\mu, L_p)$ to denote the linear space of all functions f in $L_p(\mathbb{R})$ for which

$$\sup_{t>0} \omega(f, t)_p / t^\mu < \infty.$$

In particular, if $p = \infty$, then $\mathrm{Lip}(\mu, L_\infty)$ is the Lipschitz space $\mathrm{Lip}\,\mu$ (see [7, p. 51]).

Theorem 4.2

Let a be a finitely supported sequence on $\mathbb{Z}$ such that $\sum_{j\in\mathbb{Z}} a(j) = 2$ and let f be a nontrivial solution to the refinement equation (1.2). If there exists a constant $C > 0$ such that

$$\|\nabla S_a^n \delta\|_p \le C 2^{(1/p-\mu)n}, \quad n = 1, 2, \ldots, \tag{4.7}$$

where $0 < \mu \le 1$ and $1 \le p \le \infty$, then $f \in \mathrm{Lip}(\mu, L_p)$. Conversely, if $f \in \mathrm{Lip}(\mu, L_p)$ has stable shifts, then (4.7) holds true.

Proof

Let T_a be the operator given in (1.1) and let $f_n = T_a^n \phi$ for $n = 1, 2, \ldots$, where ϕ is the hat function given in (1.6). From (4.3) and (4.7) we see that there exists a constant $C_1 > 0$ such that

$$\|f_n - f_n(\cdot - 1/2^n)\|_p \le C_1 2^{-\mu n}, \quad n = 1, 2, \ldots.$$

Moreover, (4.6) and (4.7) together tell us that there exists a constant $C_2 > 0$ such that

$$\|f_{n+1} - f_n\|_p \le C_2 2^{-\mu n}, \quad n = 1, 2, \ldots.$$

It follows that

$$\begin{aligned}
\|f - f(\cdot - 1/2^n)\|_p &\le \|f_n - f_n(\cdot - 1/2^n)\|_p + 2\|f - f_n\|_p \\
&\le \|f_n - f_n(\cdot - 1/2^n)\|_p + 2\sum_{m=n}^{\infty} \|f_{m+1} - f_m\|_p \\
&\le C_1 2^{-\mu n} + 2C_2 \sum_{m=n}^{\infty} 2^{-\mu m} \\
&\le C_3 2^{-\mu n},
\end{aligned}$$

where C_3 is another positive constant. Suppose

$$h = \frac{1}{2^n} + \frac{d_1}{2^{n+1}} + \frac{d_2}{2^{n+2}} + \cdots,$$

where $d_1, d_2, \ldots \in \{0, 1\}$. By what has been proved, we have

$$\|f - f(\cdot - h)\|_p \le C_3 \sum_{m=n}^{\infty} 2^{-m\mu} \le C h^\mu,$$

where C is a positive constant independent of h. This shows that $f \in \mathrm{Lip}(\mu, L_p)$.

Conversely, suppose that $f \in \mathrm{Lip}(\mu, L_p)$ has stable shifts. The desired inequality (4.7) follows immediately from the following identity:

$$f - f(\cdot - 1/2^n) = \sum_{j\in\mathbb{Z}} \nabla S_a^n \delta(j) f(2^n \cdot - j).$$

The latter is true because of (1.2). This completes the proof of the theorem. □

If $\tilde{a}(z) = \tilde{b}(z)(1+z)$, then theorem 3.3 tells us that (4.7) is equivalent to saying that there exists a constant $C > 0$ such that

$$\|S_b^n \delta\|_p \le C 2^{(1/p-\mu)n}, \quad n = 1, 2, \ldots.$$

As a corollary of theorem 4.2, we see that

$$\tilde{b}(1) = 1 \Rightarrow \rho_p(S_b) \ge 2^{1/p-1}.$$

In the case $p = \infty$, (3.7) shows that (4.7) is equivalent to saying that there exists a constant $C > 0$ such that

$$\max_{d_1,\ldots,d_n\in\{0,1\}} \|A_{d_1} \cdots A_{d_n}|_V\|_\infty \le C 2^{-\mu n}, \quad n = 1, 2, \ldots. \tag{4.8}$$

Using a different method, Daubechies and Lagarias showed in [6] that f lies in Lip μ provided (4.8) is satisfied.

Now we give an example to illustrate the theorems obtained in this section.

Example 4.3
Let a be the mask given by

$$\tilde{a}(z) = 2^{1-k}(1+z)^k(z^6 - z^3 + 1),$$

where k is a positive integer. It was shown in [1, p. 163] that the subdivision scheme associated with a converges uniformly if and only if $k \ge 2$. Let us investigate its convergence in the L_p-norm $(1 \le p \le \infty)$.

First, consider the case $k = 1$. Write $\tilde{a}(z) = \tilde{b}(z)(1+z)$, where

$$\tilde{b}(z) := z^6 - z^3 + 1 = (z^9+1)/(z^3+1).$$

For $n = 1, 2, \ldots$, let $b_n := S_b^n \delta$. Then

$$\begin{aligned}\tilde{b}_n(z) &= \prod_{k=1}^{n} \tilde{b}(z^{2^{k-1}}) = \prod_{k=1}^{n} \frac{1+z^{9\cdot 2^{k-1}}}{1+z^{3\cdot 2^{k-1}}} \\ &= \frac{1-z^{9\cdot 2^n}}{1-z^{3\cdot 2^n}} \frac{1-z^3}{1-z^9} = \frac{(1+z^{3\cdot 2^n}+z^{3\cdot 2^{n+1}})(1-z^3)}{1-z^9} \\ &= \frac{1+z^{3\cdot 2^n}+z^{3\cdot 2^{n+1}}-z^3-z^{3(2^n+1)}-z^{3(2^{n+1}+1)}}{1-z^9}.\end{aligned}$$

Suppose n is an odd integer. Then $2^n \equiv 2 (\mathrm{mod}\, 3)$. In this case, we may write

$$\tilde{b}_n(z) = \tilde{c}_1(z) + \tilde{c}_2(z) + \tilde{c}_3(z),$$

where

$$\tilde{c}_1(z) = \frac{1 - z^{3(2^n+1)}}{1 - z^9} = 1 + z^9 + \cdots + z^{9(2^n-2)/3},$$

$$\tilde{c}_2(z) = \frac{-z^3(1 - z^{3(2^{n+1}-1)})}{1 - z^9} = -z^3(1 + z^9 + \cdots + z^{9(2^{n+1}-4)/3}),$$

$$\tilde{c}_3(z) = \frac{z^{3\cdot 2^n} - z^{3(2^{n+1}+1)}}{1 - z^9} = z^{3\cdot 2^n}(1 + z^9 + \cdots + z^{9(2^n-2)/3}).$$

We observe that the supports of the sequences c_1, c_2, and c_3 are disjoint from each other. Moreover,

$$\|c_1\|_\infty = \|c_2\|_\infty = \|c_3\|_\infty = 1.$$

This shows that $\|b_n\|_\infty = 1$ for all odd integers n. Similarly, we can show that $\|b_n\|_\infty = 1$ for all even integers n. Therefore, we have $\rho_\infty(S_b) = 1$. For $1 \le p < \infty$ and odd integers n, we have

$$\|b_n\|_p^p = \|c_1\|_p^p + \|c_2\|_p^p + \|c_3\|_p^p = (2^{n+2} + 1)/3.$$

Similarly, we can show that $\|b_n\|_p^p = (2^{n+2} - 1)/3$ for all even integers n. Therefore, we have

$$\rho_p(S_b) = \lim_{n\to\infty} \|b_n\|_p^{1/n} = 2^{1/p}.$$

By theorem 4.1, we conclude that the subdivision scheme associated with a diverges in the L_p-norm for any p, $1 \le p \le \infty$. However, any solution to the refinement equation with mask a lies in L_∞ (see [1, p. 163] and example 5.4).

Now consider the case $k \ge 2$. Let

$$\tilde{b}(z) := ((1 + z)/2)^{k-1}(1 - z^3 + z^6).$$

Using a similar method, we can show that there exists a constant $C > 0$ such that for all positive integers n,

$$\|b_n\|_p \le C 2^{(1/p-1)n}, \quad 1 \le p \le \infty.$$

By theorems 4.2 and 3.3, we conclude that the subdivision scheme associated with a converges in the L_p-norm for all p, $1 \le p \le \infty$. Moreover, any solution to the refinement equation with mask a lies in Lip 1.

Theorem 4.1 characterizes the L_p-convergence of a subdivision scheme, and so it also gives a characterization for the existence of L_p-solutions (continuous solutions in the case $p = \infty$) to the corresponding refinement equation when the shifts of the solution are stable. If the shifts of the solution are not stable, then (4.2) is sufficient but not necessary for the existence of L_p-solutions, as was illustrated by example 4.3. To tackle this problem, Micchelli and Prautzsch in [20] proposed to use a suitable common invariant subspace W of A_0 and A_1 instead of V in (4.2). In this way, they characterized refinement equations that have continuous solutions. Their idea was further developed by Collela and Heil in [2], and by Wang in [25]. In the same spirit, Wang in [26] investigated integrable refinable functions, and Lau and Wang in [19] gave a characterization of L_p-solutions of refinement equations for $1 \leq p < \infty$.

There is another approach adopted by Villemoes in [24]. It was proved there that, for a refinable function f, there exists another refinable function g having stable shifts such that f and g have the same smoothness. In the next section we shall improve his result by considering linear independence of the shifts of a compactly supported distribution.

5. Stability and linear independence

As was seen in the previous sections, stability plays an important role in our investigation of subdivision schemes. This section is devoted to a study of stability and another related concept of linear independence.

Let ϕ be a compactly supported distribution on $\mathbb{R}$. If

$$\sum_{j\in\mathbb{Z}} \lambda(j)\phi(\cdot - j) = 0 \Rightarrow \lambda = 0,$$

then the shifts of ϕ are said to be *linearly independent*. See [3,17,21] for various properties concerning linear independence of the shifts of one or more compactly supported distributions. In particular, it was proved by Ron in [21] that the shifts of ϕ are linearly independent if and only if for any $\zeta \in \mathbb{C}$ there exists some $k \in \mathbb{Z}$ such that

$$\hat{\phi}(\zeta + 2\pi k) \neq 0.$$

Thus, if $\phi \in L_p(\mathbb{R})$ $(1 \leq p \leq \infty)$ has linearly independent shifts, then the shifts of ϕ are stable. In other words, linear independence implies stability. This motivates us to extend the concept of stability to compactly supported distributions. We say that the shifts of a compactly supported distribution ϕ are stable if for any $\xi \in \mathbb{R}$ there exists some $k \in \mathbb{Z}$ such that

$$\hat{\phi}(\xi + 2\pi k) \neq 0.$$

Let a be a finitely supported sequence with $\sum_{j\in\mathbb{Z}} a(j) = 2$, and let ϕ be the normalized solution to the refinement equation with mask a. It is desirable to characterize stability and linear independence of the shifts of ϕ in terms of the corresponding mask. The following theorem of Jia and Wang (see [18]) gives such a characterization.

Theorem 5.1
The shifts of ϕ are linearly independent if and only if the symbol $\tilde{a}(z)$ satisfies the following two conditions:

(a) $\tilde{a}(z)$ does not have any symmetric zeros in $\mathbb{C}\backslash\{0\}$;
(b) for any odd integer $m > 1$ and a primitive mth root ω of unity, there exists an integer $d \geq 0$ such that $\tilde{a}(-\omega^{2^d}) \neq 0$.

Moreover, the shifts of ϕ are stable if and only if $\tilde{a}(z)$ satisfies (b) and $\tilde{a}(z)$ does not have any symmetric zeros on the unit circle.

Example 5.2
Suppose

$$\tilde{a}(z) = a_0 + a_1 z + \cdots + a_5 z^5,$$

where $a_0 \neq 0$ and all the coefficients $a_0, \ldots, a_5$ are real. Let us check the conditions (a) and (b) of theorem 5.1 for $\tilde{a}(z)$. First, $\tilde{a}(z)$ have symmetric zeros if and only if $\tilde{a}(z)$ and $\tilde{a}(-z)$ have common zeros. The latter is equivalent to saying that $a_0 + a_2 z^2 + a_4 z^4$ and $a_1 + a_3 z^2 + a_5 z^4$ have common zeros. By using the resultant (see [13, p. 33]) of these two polynomials we have the following conclusion. The polynomial $\tilde{a}(z)$ satisfies (a) if and only if either $a_4 = a_5 = 0$ and $a_0 a_3 - a_1 a_2 \neq 0$, or

$$\begin{vmatrix} a_0 & a_2 & a_4 & 0 \\ 0 & a_0 & a_2 & a_4 \\ a_1 & a_3 & a_5 & 0 \\ 0 & a_1 & a_3 & a_5 \end{vmatrix} \neq 0.$$

Second, let us consider condition (b). For an odd integer $m > 1$, let d be the smallest positive integer such that

$$2^d \equiv 1 \pmod{m}.$$

Let ω be a primitive mth root of unity. Then $\tilde{a}(z)$ satisfies condition (b) for m and ω if and only if $\tilde{a}(z)$ is not divisible by

$$(z+\omega)(z+\omega^2)\cdots(z+\omega^{2^{d-1}}). \tag{5.1}$$

In our case we only have to check those m for which m divides $2^r - 1$ for some r, $2 \leq r \leq 5$, that is, $m = 3$, 5, 7, 15 or 31. For $m = 3$ we have $d = 2$. If ω is a cubic root of unity, then

$$(z+\omega)(z+\omega^2) = (z^3+1)/(z+1) = z^2 - z + 1.$$

For $m = 5$ we have $d = 4$. If ω is a fifth root of unity, then

$$(z+\omega)(z+\omega^2)(z+\omega^4)(z+\omega^8) = (z^5+1)/(z+1) = z^4 - z^3 + z^2 - z + 1.$$

Since the coefficients $a_0, \ldots, a_5$ are real, $\tilde{a}(z)$ is not divisible by the polynomial in (5.1) for $m = 7$, 15, or 31. Thus, we conclude that $\tilde{a}(z)$ satisfies (b) if and only if it is not divisible by either $(z^3+1)/(z+1)$ or $(z^5+1)/(z+1)$.

Theorem 5.3
Suppose a is a finitely supported sequence on $\mathbb{Z}$ such that $\sum_{j\in\mathbb{Z}} a(j) = 2$. Let ϕ be a nontrivial distribution such that

$$\phi = \sum_{j\in\mathbb{Z}} a(j)\phi(2\cdot -j). \tag{5.2}$$

Then there exists a finitely supported sequence b with $\tilde{b}(1) = 2$ such that any nontrivial solution ψ of the refinement equation

$$\psi = \sum_{j\in\mathbb{Z}} b(j)\psi(2\cdot -j)$$

has linearly independent shifts and

$$\phi = \sum_{k\in\mathbb{Z}} c(k)\psi(\cdot - k) \tag{5.3}$$

for some finitely supported sequence c. Consequently, ϕ and ψ have the same smoothness.

Proof
Without loss of generality we may assume that $\hat{\phi}(0) = 1$ and $\hat{\psi}(0) = 1$. We use the semi-convolution notation $\psi *' c$ to denote the sum in (5.3). Suppose c is a finitely supported sequence such that $\tilde{c}(1) = 1$ and

$$\tilde{b}(z) = \tilde{a}(z)\tilde{c}(z)/\tilde{c}(z^2).$$

Then $\phi = \psi *' c$. Indeed, for almost every $\xi \in \mathbb{R}$,

$$\frac{\widehat{\psi *' c}(2\xi)}{\widehat{\psi *' c}(\xi)} = \frac{\hat{\psi}(2\xi)}{\hat{\psi}(\xi)}\,\frac{\tilde{c}(e^{-2i\xi})}{\tilde{c}(e^{-i\xi})} = \frac{\tilde{b}(e^{-i\xi})}{2}\,\frac{\tilde{c}(e^{-2i\xi})}{\tilde{c}(e^{-i\xi})} = \frac{\tilde{a}(e^{-i\xi})}{2} = \frac{\hat{\phi}(2\xi)}{\hat{\phi}(\xi)}.$$

This shows that $\psi *' c$ also satisfies the refinement equation in (5.2). But $(\psi *' c)\hat{}(0) = 1 = \hat{\phi}(0)$. Therefore $\psi *' c = \phi$.

Now suppose that the shifts of ϕ are not linearly independent. We shall show that there exists a finitely supported sequence b such that $\tilde{b}(z)$ satisfies the conditions (a) and (b) of theorem 5.1, and that $\tilde{b}(z) = \tilde{a}(z)\tilde{c}(z)/\tilde{c}(z^2)$ for some finitely supported sequence c. Indeed, if $\tilde{a}(z)$ has symmetric zeros θ and $-\theta$ ($\theta \notin \{0, 1\}$), then set

$$\tilde{b}(z) := \tilde{a}(z)\frac{z-\theta^2}{z^2-\theta^2}.$$

Repeating this process until $\tilde{b}(z)$ satisfies condition (a) of theorem 5.1. If $\tilde{a}(z)$ satisfies condition (a) but not condition (b) of theorem 5.1, then there exists an odd integer $m > 1$ and an mth primitive root ω of unity such that $-\omega$, $-\omega^2, \ldots, -\omega^{2^{d-1}}$ are zeros of $\tilde{a}(z)$, where d is the smallest positive integer such that $2^d \equiv 1 (\mathrm{mod}\ m)$. Set

$$\tilde{b}(z) := \tilde{a}(z) \prod_{k=0}^{d-1} \frac{z - \omega^{2^k}}{z^2 - \omega^{2^k}}.$$

Continuing in this way, we can find a finitely supported sequence b which satisfies both conditions (a) and (b) of theorem 5.1. If ψ is the normalized solution to the refinement equation with mask b, then ψ meets the requirement of theorem 5.3.

Finally, if $\tilde{c}(z) = 1 - \theta z$, where $\theta \in \mathbb{C}$, then $\phi = \psi - \theta\psi(\cdot - 1)$ and

$$\psi = \phi + \theta\phi(\cdot - 1) + \theta^2\phi(\cdot - 2) + \cdots,$$

and therefore ϕ and ψ have the same smoothness. In general, this can be proved by induction on the length of the support of c. □

Example 5.4

Let a be the sequence given by its symbol

$$\tilde{a}(z) = 2^{1-k}(1+z)^k(z^6 - z^3 + 1),$$

where k is a positive integer, and let ϕ be the normalized solution to the refinement equation with mask a. We find that the smallest positive integer d such that $2^d \equiv 1 (\mathrm{mod}\ 9)$ is $d = 6$. With $\omega = \exp(i2\pi/9)$ we have

$$\prod_{j=1}^{6} (z + \omega^{2^{j-1}}) = (z^9 + 1)/(z^3 + 1) = z^6 - z^3 + 1.$$

Therefore, by theorem 5.1, the shifts of ϕ are not linearly independent. To find a distribution ψ satisfying the conditions in theorem 5.3, we follow the procedure described in its proof. Let

$$\tilde{c}(z) := \tfrac{1}{3} \prod_{j=1}^{6} (z - \omega^{2^{j-1}}) = (z^6 + z^3 + 1)/3.$$

Then

$$\tilde{b}(z) := \tilde{a}(z) \frac{\tilde{c}(z)}{\tilde{c}(z^2)} = 2^{1-k}(1+z)^k.$$

The normalized solution ψ to the refinement equation with mask b is the B-spline of order k. Evidently, the shifts of ψ are linearly independent. Moreover, theorem 5.3 tells us that

$$\phi = \psi *' c = [\psi + \psi(\cdot - 3) + \psi(\cdot - 6)]/3.$$

6. Smoothness

In this section we demonstrate that theorem 4.2 can be used to give a satisfactory analysis of the smoothness of a refinable function. We begin with a lemma which relates a refinable function with its derivatives.

Lemma 6.1

Suppose a and b are two finitely supported sequences on $\mathbb{Z}$ such that $\tilde{b}(1) = 2$ and

$$\tilde{a}(z) = \tilde{b}(z)\left(\frac{1+z}{2}\right)^m,$$

where m is a positive integer. If f and g are the normalized solutions to the refinement equations with mask a and b respectively, then

$$\nabla^m g = D^m f.$$

Proof

We only have to deal with the case $m = 1$, since the general case can be easily treated by induction on m. We observe that

$$\widehat{\nabla g}(\xi) = (1 - e^{-i\xi})\hat{g}(\xi), \quad \forall \xi \in \mathbb{R}.$$

Let $h(\xi) := \widehat{\nabla g}(\xi)/(i\xi)$ for $\xi \in \mathbb{R}$. Then $\hat{h}(0) = 1$, and for all $\xi \in \mathbb{R}$,

$$\frac{h(2\xi)}{h(\xi)} = \frac{1}{2}\frac{\widehat{\nabla g}(2\xi)}{\widehat{\nabla g}(\xi)} = \frac{\tilde{b}(e^{-i\xi})}{2}\frac{1+e^{-i\xi}}{2} = \frac{\tilde{a}(e^{-i\xi})}{2} = \frac{\hat{f}(2\xi)}{\hat{f}(\xi)}.$$

Hence $h = \hat{f}$. This shows that

$$\widehat{\nabla g}(\xi) = i\xi\hat{f}(\xi) = \widehat{f'}(\xi), \quad \forall \xi \in \mathbb{R}.$$

Therefore $\nabla g = f'$. □

We are in a position to investigate smoothness of refinable functions. Suppose a is a finitely supported sequence and there is a nontrivial solution $f \in L_1(\mathbb{R})$ to the refinement equation with mask a. Then $\tilde{a}(1) = 2^m$ for some positive integer m. Moreover, there exists a solution $g \in L_1(\mathbb{R})$ to the refinement equation with mask $a/2^{m-1}$ such that $g^{(m-1)} = f$ (see [5, theorem 3.1]). Thus, without loss of generality, we may assume that $\tilde{a}(1) = 2$. Furthermore, by theorem 5.3 we see that the study of smoothness of refinable functions can be reduced to those refinable functions with linearly independent shifts.

Theorem 6.2

Suppose a is a finitely supported sequence on $\mathbb{Z}$ with $\tilde{a}(1) = 2$. Let f be the normalized solution to the refinement equation with mask a. Assume that the shifts of f are stable. Then for $1 \le p \le \infty$ and $0 < \mu \le 1$, the mth derivative of f lies in $\mathrm{Lip}(\mu, L_p)$ if and only if the following two conditions are satisfied.

(a) $\tilde{a}(z)$ can be written as $\tilde{a}(z) = \tilde{b}(z)(1+z)^{m+1}/2^m$, where b is another finitely supported sequence.

(b) There exists a constant $C > 0$ such that $\|S_b^n\delta\|_p \le C2^{(1/p-\mu)n}$ for all positive integers n.

Proof
Suppose both conditions (a) and (b) are satisfied. Let c be the sequence with its symbol $\tilde{c}(z) = \tilde{b}(z)(1+z)$, and let g be the normalized solution to the refinement equation with mask c. By lemma 6.1, we have $\nabla^m g = f^{(m)}$. But g and $\nabla^m g$ have the same smoothness. Thus, $f^{(m)} \in \mathrm{Lip}(\mu, L_p)$ is equivalent to saying that $g \in \mathrm{Lip}(\mu, L_p)$. The latter is equivalent to condition (b), by theorems 4.2 and 3.3. This proves the sufficiency part of the theorem.

To prove the necessity part we only have to show that $\tilde{a}(z)$ is divisible by $(1+z)^{m+1}$ provided $f^{(m)} \in L_p(\mathbb{R})$. This was proved in [24, corollary 7.2]. For completeness we give a proof based on induction. That (a) is true for $m = 0$ comes directly from theorem 2.4. Suppose our assertion is true for m. If $f^{(m+1)} \in L_p(\mathbb{R})$, then by induction hypothesis, $\tilde{a}(z)$ can be written as $\tilde{a}(z) = \tilde{b}(z)(1+z)^{m+1}/2^{m+1}$. Let g be the normalized solution to the refinement equation with mask b. Then $\nabla^{m+1} g = f^{(m+1)}$. But the shifts of g are stable, and so the subdivision scheme associated with b is L_p-convergent. Therefore $\tilde{b}(z)$ is divisible by $1 + z$, by theorem 2.4. This shows that $\tilde{a}(z)$ is divisible by $(1+z)^{m+2}$, thereby completing the induction procedure. □

Let us recall the definition of the Lipschitz spaces (see [7, pp. 51–54]). For a positive integer r, we use $\omega_r(f, \cdot)_p$ to denote the rth modulus of smoothness of a function $f \in L_p(\mathbb{R})$ (a continuous function in the case $p = \infty$). For $\alpha > 0$ we write $\alpha = m + \mu$, where m is an integer and $0 < \mu \leq 1$. The Lipschitz space $\mathrm{Lip}(\alpha, L_p)$ consists of all functions $f \in L_p(\mathbb{R})$ for which the mth derivative of f lies in $\mathrm{Lip}(\mu, L_p)$. Thus, theorem 6.2 gives a characterization of refinable functions in Lipschitz spaces.

Eirola in [11] gave a characterization of refinable functions in Sobolev spaces. Furthermore, Villemoes in [24] provided a characterization of refinable functions in Besov spaces. In the rest of this paper we compare theorem 6.2 with the results in [24].

Let us recall the definition of Besov spaces from [7, pp. 54–57]. Let $\alpha > 0$ be given and let $r := \lfloor \alpha \rfloor + 1$, where $\lfloor \alpha \rfloor$ stands for the largest integer $\leq \alpha$. For $0 < p, q \leq \infty$, the Besov space $B_q^\alpha(L_p)$ is the collection of all functions $f \in L_p(\mathbb{R})$ such that

$$|f|_{B_q^\alpha(L_p)} := \begin{cases} \left(\sum_{k \in \mathbb{Z}} [2^{k\alpha} \omega_r(f, 2^{-k})_p]^q \right)^{1/q}, & 0 < q < \infty, \\ \sup_{k \in \mathbb{Z}} \{2^{k\alpha} \omega_r(f, 2^{-k})_p\}, & q = \infty, \end{cases}$$

is finite. The (quasi)norm for $B_q^\alpha(L_p)$ is

$$\|f\|_{B_q^\alpha(L_p)} := \|f\|_p + |f|_{B_q^\alpha(L_p)}.$$

If $0 < \beta < \alpha$, then for all $0 < q, q_1 \leq \infty$,

$$B_q^\alpha(L_p) \subseteq B_{q_1}^\beta(L_p). \tag{6.1}$$

For $q = \infty$, $B^\alpha_\infty(L_p)$ is the generalized Lipschitz space $\mathrm{Lip}^*(\alpha, L_p)$, which consists of all functions f in $L_p(\mathbb{R})$ with

$$\sup_{t>0}\{t^{-\alpha}\omega_r(f,t)_p\} < \infty.$$

It is known that the space $\mathrm{Lip}^*(\alpha, L_p)$ contains $\mathrm{Lip}(\alpha, L_p)$, and for $0 < \alpha < 1$, the two spaces coincide.

Motivated by [24, definition 3.1], we define the *critical exponent* of a function f in L_p to be

$$\nu_p(f) := \sup\{\alpha : f \in B^\alpha_q(L_p)\}.$$

By (6.1) we see that $\nu_p(f)$ is independent of the choice of q. Indeed, we have

$$\nu_p(f) = \sup\{\alpha : f \in \mathrm{Lip}(\alpha, L_p)\} = \sup\{\alpha : f \in \mathrm{Lip}^*(\alpha, L_p)\}.$$

Now let a be a finitely supported sequence on $\mathbb{Z}$ with $\tilde{a}(1) = 2$, and let f be the normalized solution to the refinement equation with mask a. Suppose that the shifts of f are stable. To determine the critical exponent $\nu_p(f)$, we find the largest integer m such that in the factorization

$$\tilde{a}(z) = \tilde{b}(z)(1+z)^{m+1}/2^m$$

the spectral radius $\rho_p(S_b) < 2^{1/p}$. Suppose $\rho_p(S_b) = 2^{1/p-\mu}$, where $0 < \mu \le 1$. Then by theorem 6.2 we obtain

$$\nu_p(f) = m + \mu.$$

In [24], the symbol $\tilde{a}(z)$ is factored as

$$\tilde{a}(z) = \tilde{c}(z)(1+z)^M/2^M,$$

where m is a positive integer and $\tilde{c}(-1) \neq 0$. Suppose $\rho_p(S_c) = 2^{1/p+\beta}$ and $\beta < M$. Then [24, theorem 3.2] tells us that

$$\nu_p(f) = M - \beta.$$

It follows that $m + \mu = M - \beta$. This motivates us to establish the following lemma, which is of independent interest.

Lemma 6.3

Let a and b be two finitely supported sequences such that

$$\tilde{a}(z) = \tilde{b}(z)(1+z)/2.$$

Then for $1 \le p \le \infty$,

$$\rho_p(S_b)/2 \le \rho_p(S_a) \le \rho_p(S_b). \tag{6.2}$$

Moreover, if $\tilde{a}(1) = 1$ and $\tilde{a}(z)$ satisfies the conditions (a) and (b) of theorem 5.1, then $\rho_p(S_a) = \rho_p(S_b)/2$ if and only if $\rho_p(S_b) \ge 2^{1/p}$.

Proof
Without loss of any generality, we may assume that b is supported on $[0, N-1]$ for some positive integer N. Let c be the sequence given by its symbol $\tilde{c}(z) = (1+z)/2$. Thus, $\tilde{a}(z) = \tilde{b}(z)\tilde{c}(z)$. For $n = 1, 2, \ldots$, let a_n, b_n and c_n denote the sequences $S_a^n\delta$, $S_b^n\delta$, $S_c^n\delta$, respectively. Then $\tilde{a}_n(z) = \tilde{b}_n(z)\tilde{c}_n(z)$, so $a_n = b_n * c_n$, where $a * b$ denotes the (discrete) convolution of two sequences a and b. By Young's inequality we have

$$\|a_n\|_p \leq \|b_n\|_p \|c_n\|_1.$$

But $\|c_n\|_1 = 1$ for all n. Therefore, we have

$$\rho_p(S_a) = \lim_{n\to\infty} \|a_n\|_p^{1/n} \leq \lim_{n\to\infty} \|b_n\|_p^{1/n} = \rho_p(S_b).$$

To prove the second inequality, we apply theorem 3.3 to $\tilde{a}(z)$ and $\tilde{b}(z)/2$ and obtain

$$2^{-n}\|b_n\|_p \leq C\|\nabla a_n\|_p \leq 2C\|a_n\|_p, \quad 1 \leq p \leq \infty,$$

where C is a constant independent of n. It follows that

$$\rho_p(S_b) = \lim_{n\to\infty} \|b_n\|_p^{1/n} \leq 2\rho_p(S_a).$$

Let us now prove the last statement. Since $\tilde{a}(z) = 1$, we must have $\rho_p(S_a) \geq 2^{1/p-1}$. Hence, $\rho_p(S_a) = \rho_p(S_b)/2$ implies $\rho_p(S_b) \geq 2^{1/p}$.

Now suppose $\rho_p(S_b) \geq 2^{1/p}$. We wish to prove $\rho_p(S_a) = \rho_p(S_b)/2$. Let

$$\tilde{c}(z) := 2\tilde{a}(z)(1+z)^m/2^m,$$

where m is a nonnegative integer, and let f be the normalized solution to the refinement equation with mask c. Since $\tilde{a}(z)$ satisfies the conditions (a) and (b) of theorem 5.1, the shifts of f are stable. We choose m to be the smallest integer such that $f \in L_p(\mathbb{R})$. Then $m \geq 1$, for otherwise $\tilde{c}(z) = (1+z)\tilde{b}(z)$ and $f \in L_p(\mathbb{R})$ imply $\rho_p(S_b) < 2^{1/p}$, contradicting the assumption $\rho_p(S_b) \geq 2^{1/p}$. Suppose the critical exponent $\nu_p(f) = \mu$. Then $0 < \mu \leq 1$. Write $\tilde{c}(z) = (1+z)\tilde{d}(z)$, where $\tilde{d}(z) = \tilde{a}(z)(1+z)^{m-1}/2^{m-1}$. By theorem 4.2, we have $\rho_p(S_d) = 2^{1/p-\mu}$, and therefore by (6.2) we obtain

$$\rho_p(S_a) \leq 2^{m-1}\rho_p(S_d) = 2^{1/p-\mu+m-1}.$$

Furthermore, write

$$\tilde{a}(z) = \tilde{q}(z)(1+z)^k/2^k,$$

where $\tilde{q}(-1) \neq 0$ and $k \geq 1$. Then

$$\tilde{c}(z) = 2\tilde{q}(z)(1+z)^{m+k}/2^{m+k}.$$

Suppose $\rho_p(S_q) = 2^{1/p+\beta}$. Then [24, theorem 3.2] tells us that $\mu = \nu_p(f) = k + m - \beta - 1$. Moreover,

$$\tilde{b}(z) = \tilde{q}(z)(1+z)^{k-1}/2^{k-1}.$$

Therefore we have

$$\rho_p(S_b) \geq \rho_p(S_q)/2^{k-1} = 2^{1/p+\beta-k+1} = 2^{1/p+m-\mu}.$$

This shows that $\rho_p(S_b) \geq 2\rho_p(S_a)$. On the other hand, $\rho_p(S_b) \leq 2\rho_p(S_a)$. We conclude that $\rho_p(S_b) = 2\rho_p(S_a)$, as desired. □

The following example illustrates lemma 6.3.

Example 6.4
Let

$$\tilde{b}(z) = \lambda + \mu z \quad \text{and} \quad \tilde{a}(z) = \tilde{b}(z)(1+z)/2,$$

where λ and μ are real numbers and $\lambda + \mu = 1$. To compute $\rho_2(S_b)$, we consider

$$|\tilde{b}(z)|^2 = \tilde{b}(z)\tilde{b}(z^{-1}) = \lambda\mu z^{-1} + (\lambda^2 + \mu^2) + \lambda\mu z, \quad z \in \mathbb{T}.$$

Thus, $\rho_2(S_b)^2$ is the spectral radius of the matrix

$$\begin{pmatrix} \lambda\mu & \lambda\mu & 0 \\ 0 & \lambda^2+\mu^2 & 0 \\ 0 & \lambda\mu & \lambda\mu \end{pmatrix}.$$

Since $|\lambda\mu| \leq (\lambda^2 + \mu^2)/2$, the spectral radius of this matrix is $\lambda^2 + \mu^2$. Therefore we obtain

$$\rho_2(S_b) = \sqrt{\lambda^2 + \mu^2} = \sqrt{\lambda^2 + (1-\lambda)^2} = \sqrt{2\lambda^2 - 2\lambda + 1}.$$

Similarly, we have

$$4|\tilde{a}(z)|^2 = \lambda\mu z^{-2} + z^{-1} + (1 + \lambda^2\mu^2) + z + \lambda\mu z^2, \quad z \in \mathbb{T}.$$

Thus, $4\rho_2(S_a)^2$ is the spectral radius of the matrix

$$\begin{pmatrix} \lambda\mu & 1+\lambda^2+\mu^2 & \lambda\mu & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & \lambda\mu & 1+\lambda^2+\mu^2 & \lambda\mu & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & \lambda\mu & 1+\lambda^2+\mu^2 & \lambda\mu \end{pmatrix}. \tag{6.3}$$

By computing the eigenvalues of this matrix we get

$$\rho_2(S_a) = \begin{cases} \sqrt{2\lambda^2 - 2\lambda + 1}/2 & \text{if } 2\lambda^2 - 2\lambda - 1 \geq 0, \\ 2^{-1/2} & \text{if } 2\lambda^2 - 2\lambda - 1 < 0. \end{cases}$$

It follows that

$$\frac{\rho_2(S_a)}{\rho_2(S_b)} = \begin{cases} 1/2 & \text{if } 2\lambda^2 - 2\lambda - 1 \geq 0, \\ 1/\sqrt{2(2\lambda^2 - 2\lambda + 1)} & \text{if } 2\lambda^2 - 2\lambda - 1 < 0. \end{cases}$$

This shows that $\rho_2(S_a)/\rho_2(S_b) = 1/2$ if and only if $\rho_2(S_b) \geq 2^{1/2} = \sqrt{2}$. Moreover, $\rho_2(S_a)/\rho_2(S_b)$ can take any value between 1/2 and 1.

Choose $\lambda = (1+\sqrt{3})/2$ in $\tilde{a}(z)$ and $\tilde{b}(z)$. Let $\tilde{c}(z) = \tilde{a}(z)(1+z)$, and let f be the normalized solution to the refinement equation with mask c. By the preceding computation we have $\rho_2(S_a) = 2^{-1/2}$. Moreover, since the largest eigenvalue of the matrix in (6.3) is simple, we can find a constant $C > 0$ such that for all positive integers n,

$$\|S_a^n \delta\|_2 \leq C 2^{-n/2}.$$

Thus, by theorem 6.2, $f \in \mathrm{Lip}(1, L_2)$. But [24, §5] only tells us $f \in \mathrm{Lip}^*(1, L_2)$. In this regard, theorem 6.2 gives a sharper result than [24] does.

Finally, we want to mention an example considered in [8]. Let a be the sequence given by its symbol

$$\tilde{a}(z) = (-1 + 4z - z^2)(1+z)^4/2^4,$$

and let f be the normalized solution to the refinement equation with mask a. Then [24, corollary 4.3] tells us that $f' \in \mathrm{Lip}^*(1, L_\infty)$. But Heil and Colella in [14] pointed out that $f' \notin \mathrm{Lip}\ 1$. Dubuc in [8] proved directly that f' is almost Lipschitz in the sense that there exists a constant $C > 0$ such that

$$\left|f'(x+h) - f'(x)\right| \leq C|h|\left|\log|h|\right|, \quad \forall x, h \in \mathbb{R}.$$

The latter conclusion can also be derived from [6, theorem 3.1].

References

[1] A.S. Cavaretta, W. Dahmen and C.A. Micchelli, *Stationary Subdivision*, Memoirs of Amer. Math. Soc., Vol. 93 (1991).

[2] D. Colella and C. Heil, Characterizations of scaling functions: continuous solutions, SIAM J. Matrix Anal. Appl. 15 (1994) 496–518.

[3] W. Dahmen and C.A. Micchelli, Translates of multivariate splines, Lin. Alg. Appl. 52 (1983) 217–234.

[4] G. Deslauriers and S. Dubuc, Symmetric iterative interpolation process, Constr. Approx. 5 (1989) 49–68.

[5] I. Daubechies and J.C. Lagarias, Two-scale difference equations: I. Existence and global regularity of solutions, SIAM J. Math. Anal. 22 (1991) 1388–1410.

[6] I. Daubechies and J.C. Lagarias, Two-scale difference equations: II. Local regularity, infinite products of matrices and fractals, SIAM J. Math. Anal. 23 (1992) 1031–1079.

[7] R.A. DeVore and G.G. Lorentz, *Constructive Approximation* (Springer, Berlin, 1993).

[8] S. Dubuc, Interpolation through an iterative scheme, J. Math. Anal. Appl. 114 (1986) 185–205.

[9] N. Dyn, Subdivision schemes in computer aided geometric design, in: *Advances in Numerical Analysis II – Wavelets, Subdivision Algorithms and Radius Functions*, ed. W.A. Light (Clarendon Press, Oxford, 1991) pp. 36–104.

[10] N. Dyn, J.A. Gregory and D. Levin, Analysis of uniform binary subdivision schemes for curve design, Constr. Approx. 7 (1991) 127–147.

[11] T. Eirola, Sobolev characterization of solutions of dilation equations, SIAM J. Math. Anal. 23 (1992) 1015–1030.

[12] T.N.T. Goodman, C.A. Micchelli and J.D. Ward, Spectral radius formulas for subdivision operators, in: *Recent Advances in Wavelet Analysis*, eds. L.L. Schumaker and G. Webb (Academic Press, 1994) pp. 335–360.

[13] J. Harris, *Algebraic Geometry* (Springer, New York, 1992).

[14] C. Heil and D. Colella, Dilation equations and the smoothness of compactly supported wavelets, in: *Wavelets: Mathematics and Applications*, eds. J.L. Benedetto and M.W. Frazier (CRC Press, Boca Raton, 1994) pp. 163–201.

[15] R.A. Horn and C.R. Johnson, *Matrix Analysis* (Cambridge University Press, Cambridge, 1985).

[16] R.Q. Jia and C.A. Micchelli, On linear independence of integer translates of a finite number of functions, Research Report CS-90-10 (1990), University of Waterloo. A revised version appeared in Proc. Edinburgh Math. Soc. 36 (1992) 69–85.

[17] R.Q. Jia and C.A. Micchelli, Using the refinement equation for the construction of pre-wavelets II: Powers of two, in: *Curves and Surfaces*, eds. P.J. Laurent, A. Le Méhauté and L.L. Schumaker (Academic Press, New York, 1991) pp. 209–246.

[18] R.Q. Jia and J.Z. Wang, Stability and linear independence associated with wavelet decompositions, Proc. Amer. Math. Soc. 117 (1993) 1115–1124.

[19] K.S. Lau and J.R. Wang, Characterization of L^p-solutions for the two-scale dilation equations, preprint.

[20] C.A. Micchelli and H. Prautzsch, Uniform refinement of curves, Lin. Alg. Appl. 114/115 (1989) 841–870.

[21] A. Ron, A necessary and sufficient condition for the linear independence of the integer translates of a compactly supported distribution, Constr. Approx. 5 (1989) 297–308.

[22] G.-C. Rota and W.G. Strang, A note on the joint spectral radius, Indag. Math. 22 (1960) 379–381.

[23] I.J. Schoenberg, Contributions to the problem of approximation of equidistant data by analytic functions, Quart. Appl. Math. 4 (1946) 45–99, 112–141.

[24] L.F. Villemoes, Wavelet analysis of refinement equations, SIAM J. Math. Anal. 25 (1994) 1433–1460.

[25] Y. Wang, Two-scale dilation equations and the cascade algorithm, Random Comp. Dynamics, to appear.

[26] Y. Wang, Two-scale dilation equations and the mean spectral radius, Random Comp. Dynamics, to appear.