Stability of manifold-valued subdivision schemes and multiscale transformations

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Abstract

Linear subdivision schemes can be adapted in various ways so as to operate in nonlinear geometries such as Lie groups or Riemannian manifolds. It is well known that along with a linear subdivision scheme a multiscale transformation is defined. Such transformations can also be defined in a nonlinear setting. We show the stability of such nonlinear multiscale transforms. To do this we introduce a new kind of proximity condition which bounds the difference of the differential of a nonlinear subdivision scheme and a linear one. It turns out that – unlike the generic nonlinear case and modulo some minor technical assumptions – in the manifold-valued setting, convergence implies stability of the nonlinear subdivision scheme and associated nonlinear multiscale transformations.

Keywords: Lipschitz, interpolatory wavelet transform, manifold valued data, nonlinear subdivision.

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Contents

1 Introduction

There exists an ongoing interest in nonlinear subdivision schemes and their applications in data processing. This is because in many situations linear methods perform poorly and one is forced to employ nonlinear processes in order to obtain pleasing results. As examples we mention the presence of non-gaussian noise where the so-called *median interpolating scheme* has been introduced in [10], or piecewise smooth data where so-called *essentially non-oscillating (ENO) schemes* have been developed [21]. Another example is the processing of data that does not live in a vector space but in a general manifold. In this case linear methods usually cannot even be defined anymore. There has been some effort to also provide approximation tools for this situation using non linear subdivision schemes which are intrinsically defined in a manifold [29, 15, 27, 6]. These schemes are always defined as nonlinear analogue of some linear scheme. It is certainly of interest to know if these constructions share the properties of the corresponding linear ones. Among the properties that have been previously studied are

- convergence and smoothness of the limit function of manifold-valued subdivison schemes [29, 28, 18, 17, 32, 31]
- approximation order of manifold-valued subdivision schemes [12, 30], or
- smoothness characterization properties of the multiscale decomposition associated with a manifold valued subdivision scheme [19].

For each one of these properties it has turned out that essentially the nonlinear manifold-valued constructions do indeed share the same properties with the linear constructions.

In the spirit of these studies, the present work is concerned with stability properties of manifold valued subdivision schemes.

The main idea of the previous work mentioned above in this direction is to employ a *proximity condition* that bounds the difference of a nonlinear subdivision scheme to the linear scheme it is constructed from and to use perturbation results. Unfortunately, in the case of stability this path is doomed to failure. In the course of this paper we will see an example of a nonlinear scheme which is in proximity with a linear one, but which is not stable.

To overcome this difficulty, we introduce a new kind of proximity condition: a *differential proximity condition* which bounds the difference between the differential of a nonlinear scheme and a linear scheme. We manage to show that indeed stability holds for a nonlinear scheme satisfying a differential proximity condition with a stable linear scheme. We find the surprising result that the two notions of proximity – the previously employed notion of proximity and the new differential proximity condition – are equivalent if the nonlinear scheme is sufficiently smooth as an operator on $l_\infty(\mathbb{Z})$.

The main result of the present paper shows that the manifold-valued analogue of a stable subdivision scheme is itself stable and that the associated nonlinear multiscale decomposition is stable, too.

Stability is of course essential for the applicability of any multiscale transformation. This is because, for an application in data compression, de-noising or smoothing, we need to know to what extent a function is affected by the thresholding or perturbing of detail coefficients.

Aside from the practical importance of the stability property, our results also provide strong tools for the theoretical analysis of nonlinear subdivision schemes. As an example we mention the upcoming work [16], where the present results are applied to prove some new facts concerning approximation order.

Previous work There are only few general results concerning the stability of nonlinear subdivision schemes and their associated multiscale transforms. A very recent and general work is [20], where necessary and sufficient criteria for the stability of nonlinear subdivision schemes and multiscale transforms are presented. These results however cannot be applied to the manifold-valued case since they assume polynomial generation properties of the underlying subdivision scheme which do not hold in the manifold-valued case. Other results can be found in [4, 2, 24, 6, 1, 5]. All these results rely on the perturbation approach of nonlinear subdivision and multiresolution, i.e. they regard the nonlinear multiresolution as a perturbation of the linear one. This approach does not work for manifold-valued schemes either as we will see later. Our work combines the more analytical methods of [20] with the more algebraic methods in, say [1], where by algebraic we mean that no differentiation or integration takes place in the analysis. As a (rather natural) additional assumption over this previous work we shall require the underlying nonlinear subdivision operator to be a C^1-mapping.

Outline After setting up the necessary notation and reviewing some basic facts on subdivision in Section 2, in Section 3 we provide an example of an unstable subdivision scheme which is in proximity with a linear stable scheme in the sense of [29]. Motivated by this result we introduce a new differential proximity condition and, using this condition, prove the stability of a large class of nonlinear subdivision operators. Section 4 focuses on the stability of the multiresolution transform associated with a subdivision operator. After discussing some issues concerning the well-definedness of the reconstruction procedure, we show that, assuming a differential proximity condition, stability of the multiresolution transform holds. In Section 5 we apply the general results of Sections 3 and 4 to some examples of interest: the log-exp analogue in a Lie group, the log-exp analogue in a Riemannian manifold and the projection analogue.

2 Notation and preliminaries

The present section is devoted to fixing the notation and reviewing some basic facts concerning linear subdivision schemes. We shall always use boldface letters for sequences $\mathbf{p} = (p_i)_{i\in\mathbb{Z}}$.

2.1 Subdivision schemes

Let us define what we precisely mean by a subdivision scheme:

Definition 2.1. *An m-dimensional subdivision operator ($m \in \mathbb{Z}^+$) is a mapping $\mathcal{T} : l_\infty(\mathbb{Z}, \mathbb{R}^m) \to l_\infty(\mathbb{Z}, \mathbb{R}^m)$ that is* local *and has* dilation factor *2, meaning that*

$$\sigma^2 \circ \mathcal{T} = \mathcal{T} \circ \sigma,$$

where σ denotes the right-shift on $\mathbb{Z}$. Locality means that the value $(\mathcal{T}\mathbf{p})_i \in \mathbb{R}^m,\ i \in \mathbb{Z},\ \mathbf{p} \in l_\infty(\mathbb{Z}, \mathbb{R}^m)$ depends only on a finite number of points.

The dilation factor 2 in this definition for our purposes is completely arbitrary. It makes no difference if we replace it with any integer > 1 and all our results hold for this general case too, with the same proofs.

From the definition of a subdivision scheme it follows that there exists an interval $I \subset \mathbb{Z}$ of finite length L such that the value of $(\mathcal{T}\mathbf{p})_{2i+k}$ only depends on the values $\mathbf{p}\big|_{i+I}$ with $i \in \mathbb{Z},\ k \in \{0, 1\}$. We once and for all fix a norm $\|\cdot\|$ on $\mathbb{R}^m$, for instance the maximum norm. For an element $\mathbf{p} \in l_\infty(\mathbb{Z}, R^m)$ we write $\|\mathbf{p}\|_\infty := \sup_{i\in\mathbb{Z}} \|p_i\|$. For functions we use the same symbol $\|\cdot\|_\infty$ to indicate the usual *sup*-norm. For a linear operator $l_\infty(\mathbb{Z}, \mathbb{R}^m) \to l_\infty(\mathbb{Z}, \mathbb{R}^m)$ we use the symbol $\|\cdot\|$ to indicate the usual operator norm induced by $\|\cdot\|_\infty$ on $l_\infty(\mathbb{Z}, \mathbb{R}^m)$.

The purpose of a subdivision scheme is to be iterated with the interpretation that the points $\mathcal{T}^n\mathbf{p}_i$ are approximately the value of some continuous function $\mathcal{T}^\infty\mathbf{p}(t) : \mathbb{R} \to \mathbb{R}^m$ at $t = i/2^n$. The precise definition of convergence is as follows:

Definition 2.2. *A subdivision scheme $\mathcal{T}$ is called* convergent *if for any initial data $\mathbf{p} \in l_\infty(\mathbb{Z}, \mathbb{R}^m)$ there exists a continuous function $\mathcal{T}^\infty\mathbf{p}$ with*

$$\lim_{n\to\infty} \|\mathcal{T}^\infty\mathbf{p}(t) - (\mathcal{T}^n\mathbf{p})_{i_n}\| = 0, \quad \textit{whenever} \quad \lim_{n\to\infty} i_n/2^n = t.$$

In this paper we are not concerned with convergence, or smoothness properties of the limit function. If the reader is interested in these topics we refer to [18, 17, 31, 32, 29, 28]. Rather, we are interested in the following type of question:

Question 1: Is the operator $\mathcal{T}^\infty : l_\infty(\mathbb{Z}, \mathbb{R}^m) \to C(\mathbb{R}, \mathbb{R}^m)$ that maps $\mathbf{p}$ to its limit function $\mathcal{T}^\infty\mathbf{p}$ Lipschitz, i.e. does there exist a finite constant C so that

$$\|\mathcal{T}^\infty\mathbf{p}(t) - \mathcal{T}^\infty\mathbf{q}(t)\|_\infty \le C\|\mathbf{p} - \mathbf{q}\|_\infty$$

holds (at least locally)?

We will prove that the answer is “yes” for a general class of nonlinear subdivision schemes that includes for example subdivision in manifolds via the log-exp analogy [27].

Our proofs are based on comparing the nonlinear scheme $\mathcal{T}$ with a linear scheme $\mathcal{S}$ (which is stable) and using perturbation arguments to ensure that $\mathcal{T}^\infty$ is Lipschitz. First let us recall some well-known properties of linear subdivision schemes. Comprehensive introductions into the theory of linear subdivision schemes are provided e.g. by [3, 23, 13]. In the following we always write $\mathcal{S}$ for a linear subdivision scheme and $\mathcal{T}$ for a possibly nonlinear one.

Definition 2.3. *A subdivision scheme $\mathcal{S}$ is called* linear*, if there exists a sequence $(a_j)_{j\in\mathbb{Z}}$, $a_j \in \mathbb{R}$ with finite support, called the* mask *of $\mathcal{S}$, such that*

$$(\mathcal{S}\mathbf{p})_i = \sum_{j\in\mathbb{Z}} a_{i-2j} p_j, \qquad p_j \in \mathbb{R}^m.$$

It is well known and easy to see from the uniform boundedness principle that for a convergent linear scheme $\mathcal{S}$ the operator norms $\|\mathcal{S}^n\|$ are uniformly bounded by a constant $M \geq 0$ for all n. If $\mathcal{S}$ is a convergent linear subdivision scheme with $\mathcal{S}^\infty \neq 0$, it maps constant data to itself:

$$\mathcal{S}\mathbf{q} = \mathbf{q} \quad \text{for } \mathbf{q} = (\dots, q, q, q, \dots) \in l_\infty(\mathbb{Z}, \mathbb{R}) \text{ with } q \in \mathbb{R}^m. \tag{1}$$

If a (linear or nonlinear) subdivision scheme $\mathcal{S}$ satisfies (1) we say that $\mathcal{S}$ *reproduces constants*. For a nonlinear scheme $\mathcal{T}$ it might happen that it is not defined anymore for all sequences. In this case the same definition holds with the additional (and obvious) restriction that (1) needs to hold for every $q \in \mathbb{R}^m$ such that $\mathcal{T}$ is defined on $\mathbf{q} = (\dots, q, q, q, \dots)$.

A useful tool in the study of a linear (or nonlinear) subdivision scheme is the forward difference operator Δ that maps a sequence $\mathbf{p} \in l_\infty(\mathbb{Z}, \mathbb{R}^m)$ to the sequence $(p_{i+1} - p_i)_{i\in\mathbb{Z}}$. Convergence of a linear subdivision scheme can be characterized as follows (see e.g. [3]):

Theorem 2.4. *Let $\mathcal{S}$ be a linear subdivision scheme. $\mathcal{S}$ is convergent if and only if there exists a constant $C \geq 0$ and a constant $0 \leq \mu < 1$, called the* contraction factor *of $\mathcal{S}$, such that for all initial data $\mathbf{p} \in l_\infty(\mathbb{Z}, \mathbb{R}^m)$*

$$\|\Delta \mathcal{S}^j \mathbf{p}\|_\infty \leq C\mu^j \|\Delta \mathbf{p}\|_\infty.$$

It can be shown that the contraction factor is always greater or equal 1/2 unless $\mathcal{S}^\infty$ maps all data to the zero function.

2.2 Multiscale Data Representation using Subdivision

A useful application of subdivision lies in the multiscale representation of data. The principle is as follows: Let $f : \mathbb{R} \to \mathbb{R}^m$ be a continuous function. We can analyze f by first sampling it on a grid $2^{-(j-1)}\mathbb{Z}$ to obtain data $\mathbf{p} = \mathcal{P}^{j-1} f := (f(\frac{i}{2^{j-1}}))_{i\in\mathbb{Z}}$. Then we predict the values on the grid $2^{-j}\mathbb{Z}$ using the subdivision scheme $\mathcal{T}$ and obtain an approximation $\mathcal{T}\mathbf{p}$ of the actual values $\mathcal{P}^j f = (f(\frac{i}{2^j}))_{i\in\mathbb{Z}}$. From the predicted values $\mathcal{T}\mathbf{p}$ we extract the "wavelet coefficients" as the differences

$$\lambda^j := \lambda^j f := ((\lambda^j f)_i)_{i\in\mathbb{Z}} := ((\mathcal{P}^j f)_i \ominus (\mathcal{T}\mathcal{P}^{j-1} f)_i)_{\ i\in\mathbb{Z}}$$

of the predictions and the actual samplings. The symbol $\ominus$ serves as an analogue of the conventional difference of two points in $\mathbb{R}^m$ and represents a bivariate function on $\mathbb{R}^m \times \mathbb{R}^m$ taking values in some vector space (or vector bundle). We are not concerned with the analysis of a function – this topic is studied in [19] – but with the properties of the reconstruction procedure taking data

$$(\mathcal{P}^0, \lambda^1, \lambda^2, \dots)$$

to the function f: If $\oplus$ is the inverse operation of $\ominus$ in the sense that $p \oplus (q \ominus p) = q$ for all $p, q \in \mathbb{R}^m$, then the values of f on the dyadic rationals can be reconstructed inductively from (2.2) via $(\mathcal{P}^j f)_i = (\mathcal{T}\mathcal{P}^{j-1} f)_i \oplus (\lambda^j)_i$. If f is continuous, then this procedure obviously recovers f from $(\mathcal{P}^0, \lambda^1, \dots)$. It is natural to ask what effect perturbing the coefficients λ^j has on the reconstructed function, or in other words:

Question 2: Does there exist a constant $C \geq 0$ such that for data $\mathcal{P}^0, \tilde{\mathcal{P}}^0, \lambda^1, \tilde{\lambda}^1, \dots$ and functions $f, \tilde{f}$ reconstructed from $(\mathcal{P}^0, \lambda^1, \dots)$, $(\tilde{\mathcal{P}}^0, \tilde{\lambda}^1, \dots)$ respectively, we have the inequality

$$\|f - \tilde{f}\|_\infty \leq C(\|\mathcal{P}^0 - \tilde{\mathcal{P}}^0\|_\infty + \sum_{i=1}^{\infty} \|\lambda^i - \tilde{\lambda}^i\|_\infty) \quad ?$$

Does such an inequality hold locally?

We provide an affirmative answer to this question which applies to a large class of subdivision schemes and which is true under some restrictions which cannot be avoided in our setting.

2.3 Lipschitz classes

We introduce the so-called *Lipschitz function spaces*: Let $I \subset \mathbb{R}$ be an interval. The function spaces Lip γ, where $0 < \gamma \in \mathbb{R} \setminus \mathbb{Z}^+$ are defined as follows for functions $f : I \to \mathbb{R}^d$ and $d \in \mathbb{Z}^+$: f lies in Lip γ if and only if

$$\|f\|_\infty < \infty \text{ and } \sup_{h \in \mathbb{R}\setminus\{0\}} \|h^{-(\gamma-\lfloor\gamma\rfloor)}((\frac{d}{dx})^{\lfloor\gamma\rfloor} f(x+h) - (\frac{d}{dx})^{\lfloor\gamma\rfloor} f(x))\|_\infty < \infty.$$

It can be shown [7] that this is equivalent to

$$f \in \text{Lip } \gamma \Leftrightarrow \|f\|_\infty < \infty \text{ and } |f|_{\text{Lip } \gamma} := \sup_{j\in\mathbb{Z}^+} 2^{\gamma j} \|\Delta^{\lfloor\gamma\rfloor+1}\mathcal{P}^j f\|_\infty < \infty. \tag{2}$$

For $\gamma \in \mathbb{Z}^+$ the definition of Lipschitz spaces can be obtained by interpolation methods [26]. With this definition (2) remains valid also for integers γ.

During the course of our analysis we will occasionally encounter multivariate Lipschitz function spaces containing functions $f : G \to \mathbb{R}^d$ for some domain $G \subset \mathbb{R}^s$, $s \in \mathbb{Z}^+$. In this case the definition is similar, with the differential operator $\frac{d}{dx}$ replaced by the total differential, see [26] for more information. The characterization (2) still holds in the multivariate case with the Δ operator replaced by a multivariate divided difference operator, see e.g. [19].

3 Stability from Proximity

This section is devoted to the answer to Question 1 above. We first discuss the conditions that we impose on the nonlinear scheme $\mathcal{T}$. Then we show that these conditions suffice to prove stability.

One technical issue that arises in the study of many nonlinear subdivision schemes is the fact the they may not be defined for all data $\mathbf{p} \in l_\infty(\mathbb{Z}, \mathbb{R}^m)$ anymore. We therefore introduce a family of subsets of $l_\infty(\mathbb{Z}, \mathbb{R}^m)$ convenient for our analysis:

Definition 3.1. *The set $P_{M,\delta}$ denotes all M-valued sequences such that $\|\Delta\mathbf{p}\|_\infty < \delta$. For simplicity we assume that $M \subseteq \mathbb{R}^m$ is open and convex.*

The requirement for M to be open and convex is no real restriction. In fact, if M is open, we can make M smaller to get a convex set. The convexity assumption will be needed later in some proofs in order to apply the mean value theorem for real-valued multivariate functions restricted to a line connecting two points. It is however no essential assumption, with a little more technical effort in the proofs it can just as well be replaced by path-connectedness. The typical sets M that we will encounter in the examples in Section 5 are domains of definition of charts on a differentiable manifold.

Our analysis strongly relies on so-called *proximity conditions* which are satisfied between a nonlinear and a linear subdivision scheme. They have first been used in [6] for the analysis of normal multiresolution of curves. The specific conditions that we introduce below have first been used in [29] to prove convergence and smoothness properties of nonlinear, manifold-valued schemes:

Definition 3.2. *$\mathcal{S}$ and $\mathcal{T}$ satisfy a proximity condition with exponent $\alpha > 1$ for the set $P_{M,\delta}$ if there exists a constant $C \geq 0$ such that for all $\mathbf{p} \in P_{M,\delta}$*

$$\|\mathcal{S}\mathbf{p} - \mathcal{T}\mathbf{p}\|_\infty \leq C\|\Delta\mathbf{p}\|_\infty^\alpha.$$

It is perhaps not very surprising that such an inequality is useful in the study of convergence properties, and indeed, using a perturbation result, it is possible to deduce convergence and smoothness properties of $\mathcal{T}$ from analogous properties of $\mathcal{S}$.

This kind of proximity condition does not seem well-suited for the study of stability properties. With perturbation arguments similar to [29] it seems very hard to prove stability of $\mathcal{T}$ from stability of $\mathcal{S}$. As a matter of fact it turns out that, without requiring anything else, the implication proximity $\Rightarrow$ stability is false:

Example 3.3. *Define a subdivision scheme $\mathcal{T}$ by*

$$(\mathcal{T}\mathbf{p})_{2j+k} := (\mathcal{S}\mathbf{p})_{2j+k} + \|\Delta\mathbf{p}_{j+5}\|^2|f(p_j)|, \quad k \in \{0,1\},$$

where $\mathcal{S}$ is the linear B-spline scheme defined via

$$(\mathcal{S}\mathbf{p})_{2j} = p_j \quad \textit{and} \quad (\mathcal{S}\mathbf{p})_{2j+1} = \frac{1}{2}p_j + \frac{1}{2}p_{j+1},$$

and f is some bounded function with $f(0) = 0$ and $f \notin$ Lip 1 at 0. Clearly, $\mathcal{T}$ is a subdivision scheme which is in proximity with $\mathcal{S}$ with $\alpha = 2$.

Lemma 3.4. *There exists initial data $\mathbf{p}, \mathbf{q}_n,\ n \in \mathbb{Z}^+$ such that the limit functions $\mathcal{T}^\infty\mathbf{p},\ \mathcal{T}^\infty\mathbf{q}_n$ exist and such that*

$$\limsup_{n\to\infty} \frac{\|\mathcal{T}^\infty\mathbf{p} - \mathcal{T}^\infty\mathbf{q}_n\|_\infty}{\|\mathbf{p} - \mathbf{q}_n\|_\infty} = \infty.$$

In particular, $\mathcal{T}$ is not stable.

Proof. First we note that by the general results in [29], the limit function $\mathcal{T}^\infty\mathbf{p}$ exists for all initial data $\mathbf{p}$ with $\|\Delta\mathbf{p}\|_\infty$ sufficiently small. Now set $p_5 = \delta$ and $p_j = 0$ otherwise with δ small enough for $\mathcal{T}^\infty\mathbf{p}$ to exist. By the assumption that f is not in Lip 1 at 0 it follows that there exists a sequence $(\varepsilon_n)_{n\in\mathbb{Z}^+}$ such that $\lim_{n\to\infty}\varepsilon_n = 0$, $\varepsilon_n \le \delta$ for all $n \in \mathbb{Z}^+$ and $\limsup_{n\to\infty}|f(\varepsilon_n)|/|\varepsilon_n| = \infty$. Now define $\mathbf{q}_n$ with $q_i = p_i$ for $i \neq 0$ and $q_0 = \varepsilon_n$. Again by the results in [29] it can be shown that $\mathcal{T}^\infty\mathbf{q}_n$ exists for all n. Since

$$|(\mathcal{T}^i\mathbf{q}_n)_0| \ge \|(\Delta\mathbf{q}_n)_5\|^2|f(\varepsilon_n)| = \delta^2|f(\varepsilon_n)| \quad \text{for all } i \ge 1,$$

it follows that

$$\|\mathcal{T}^\infty\mathbf{p} - \mathcal{T}^\infty\mathbf{q}_n\|_\infty/\|\mathbf{p} - \mathbf{q}_n\|_\infty \ge \delta^2|f(\varepsilon_n)|/|\varepsilon_n| \to \infty.$$

This proves the statement. □

The previous example shows that a proximity condition as defined in Definition 3.2 is not the right one to study stability properties of nonlinear subdivision schemes. It is clear that in our example the obstruction to stability comes from the fact that f is not in Lip 1, or to put it differently, that the subdivision operator $\mathcal{T}$ is not a Lipschitz-continuous operator.

We therefore restrict our attention to subdivision schemes $\mathcal{T}$ such that $\mathcal{T}$ is a C^1-mapping and introduce a more natural proximity condition for proving stability properties (which obviously are related to the differentials of the subdivision operators).
Note that the requirement on $\mathcal{T}$ to be C^1 is a very small restriction for our purposes. In most of the applications which we have in mind the scheme $\mathcal{T}$ turns out to be even analytic, see Section 5.

In the following we are going to use the differential $d\mathcal{T}$. At first sight it might be necessary to first adapt a proper definition of differentiating a subdivision operator $\mathcal{T}$ acting on an infinite dimensional Banach space. However, since by assumption $\mathcal{T}$ is a local operator we are not confronted with such issues and the differential $d\mathcal{T}$ of $\mathcal{T}$ in fact consists only of the differentials of two functions as can easily be verified from the defining assumptions of a subdivision scheme. Keeping this in mind the following definition makes perfect sense.

Definition 3.5. *A nonlinear subdivision scheme $\mathcal{T} \in C^1$ and a linear scheme $\mathcal{S}$ satisfy a* differential proximity condition *with exponent $\beta > 0$ for the set $P_{M,\delta}$ if there exists a constant $C \ge 0$ such that for all $\mathbf{p} \in P_{M,\delta}$*

$$\|d\mathcal{T}\big|_{\mathbf{p}} - \mathcal{S}\| \le C\|\Delta\mathbf{p}\|_\infty^\beta. \tag{3}$$

Intuitively, this definition should work better than Definition 3.2, because if $\|\Delta\mathbf{p}\|_\infty$ becomes small (which it does for a convergent scheme $\mathcal{T}$), then the difference between the differentials of $\mathcal{T}$ and $\mathcal{S}$ becomes small, too. These heuristic arguments provide some hope that, using a differential proximity condition, it should be possible to show uniform boundedness of the differentials $d\mathcal{T}^n$, and consequently stability of $\mathcal{T}$. Theorem 3.8 below turns this heuristic into a rigorous proof.

First we need the following perturbation result which will be useful in several places:

Lemma 3.6. *Let $A_i,\ i \in \mathbb{Z}^+$ be operators on a normed space with norm $\|\cdot\|$ with $\sup_{k\in\mathbb{Z}^+} \|\prod_{l=1}^k A_l\| \le M$ for a constant $M \ge 0$. If $U_i,\ i \in \mathbb{Z}^+$ are operators such that $\|U_i\| \le C\mu^i$ for some $C \ge 0,\ \mu < 1$ ($\|U_i\|$ denoting the induced operator norm), then $\|\prod_{l=1}^k (A_l + U_l)\|$ is bounded by a constant only depending on C, μ and M.*

Proof. Define the norm $\|x\|' := \sup_{k\in\mathbb{Z}^+} \|\prod_{l=1}^k A_l x\|$. Clearly, we have

$$\|x\| \le \|x\|' \le M\|x\|.$$

The operators A_l have norm ≤ 1 with respect to $\|\cdot\|'$. Therefore

$$\begin{aligned} \|\prod_{l=1}^k (A_l + U_l)x\| &\le \|\prod_{l=1}^k (A_l + U_l)x\|' \le \prod_{l=1}^k \|A_l + U_l\|'\|x\|' \\ &\le \prod_{i=1}^\infty (1 + MC\mu^i)\|x\|' \le M \prod_{i=1}^\infty (1 + MC\mu^i)\|x\|. \end{aligned}$$

This implies our statement. □

In order to be able to speak of stability we need $\mathcal{T}$ to be convergent which is usually related to the notion of *contractivity*:

Definition 3.7. *A subdivision scheme $\mathcal{T}$ is* contractive *for data in $P_{M,\delta}$ if there exists a constant $C \ge 0$ and $\mu < 1$ such that for all data $\mathbf{p} \in P_{M,\delta}$*

$$\|\Delta \mathcal{T}^j \mathbf{p}\|_\infty \le C\mu^j \|\Delta \mathbf{p}\|_\infty \quad j \in \mathbb{Z}^+. \tag{4}$$

Now we can formulate and prove our main result regarding stability of nonlinear subdivision schemes

Theorem 3.8. *Assume that $\mathcal{T}$ is a C^1-subdivision scheme satisfying a differential proximity condition with exponent $\beta > 0$ and data in $P_{M,\delta}$ with a convergent linear scheme $\mathcal{S}$. Assume further that $\mathcal{T}$ is contractive on $P_{M,\delta}$. Then there exists a constant $C \ge 0$ such that for all $\mathbf{p},\ \mathbf{q} \in P_{M,\delta}$ we have*

$$\|\mathcal{T}^\infty \mathbf{p}(t) - \mathcal{T}^\infty \mathbf{q}(t)\|_\infty \le C\|\mathbf{p} - \mathbf{q}\|_\infty.$$

Proof. We show that the operator norms $\|d\mathcal{T}^n\big|_{\mathbf{p}}\|$ are uniformly bounded in n by a constant C for all $\mathbf{p} \in P_{M,\delta}$. To do this, we apply the chain rule to $\mathcal{T}^n = \mathcal{T} \circ \mathcal{T}^{n-1}$ and obtain the following expansion, with certain operators U_j:

$$\begin{aligned} d\mathcal{T}^n\big|_{\mathbf{p}} &= d\mathcal{T}\big|_{\mathcal{T}^{n-1}\mathbf{p}} \cdot d\mathcal{T}^{n-1}\big|_{\mathcal{T}^{n-2}\mathbf{p}} \cdot \ldots \cdot d\mathcal{T}\big|_{\mathbf{p}} \\ &=: (\mathcal{S} + U_n)(\mathcal{S} + U_{n-1}) \cdot \ldots \cdot (\mathcal{S} + U_1). \end{aligned}$$

By assumption, the schemes $\mathcal{S}$ and $\mathcal{T}$ satisfy the differential proximity condition (3). Hence, thanks to the contractivity assumption (4),

$$\|U_i\| \le C\|\Delta \mathcal{T}^{i-1}\mathbf{p}\|_\infty^\beta \le C'(\mu^\beta)^i \|\Delta \mathbf{p}\|_\infty^\beta, \quad i = 1, \ldots, n.$$

Now it remains to apply Lemma 3.6 with $A_i := \mathcal{S}$ to conclude that the norms $\|d\mathcal{T}^n\big|_{\mathbf{p}}\|$ are indeed bounded.

In order to obtain a bound for $\|\mathcal{T}^\infty \mathbf{p} - \mathcal{T}^\infty \mathbf{q}\|$ we employ the definition of convergence and see that for $t \in I$,

$$\mathcal{T}^\infty \mathbf{p}(t) = \lim_{n\to\infty} (\mathcal{T}^n \mathbf{p})_{i_n},$$

with integers i_n chosen such that $i_n/2^n \to t$. An analogous formula holds for $\mathbf{q}$. Now let $\mathrm{c}(s) := \mathbf{p} + s(\mathbf{q} - \mathbf{p})$. The uniform boundedness of $\|d\mathcal{T}^n\|$ shown above implies that

$$\|(\mathcal{T}^n \mathbf{p} - \mathcal{T}^n \mathbf{q})_{i_n}\| \le \sup_{t\in(0,1)} \|d\mathcal{T}^n\big|_{\mathrm{c}(t)}\| \|\mathbf{p} - \mathbf{q}\|_\infty \le C\|\mathbf{p} - \mathbf{q}\|_\infty.$$

We let $n \to \infty$ and arrive at

$$\|\mathcal{T}^\infty \mathbf{p}(t) - \mathcal{T}^\infty \mathbf{q}(t)\|_\infty \le C\|\mathbf{p} - \mathbf{q}\|_\infty$$

for all t. □

Using this general result we go on to prove that proximity in the sense of 3.2 *does* imply stability if we impose a mild regularity assumption on the subdivision operator $\mathcal{T}$.

The proof is based on the remarkable fact that if we require the subdivision scheme $\mathcal{T}: l_\infty(\mathbb{Z}, \mathbb{R}^m) \to l_\infty(\mathbb{Z}, \mathbb{R}^m)$ to be smooth, then Definition 3.2 and Definition 3.5 almost coincide. Recall that $\mathcal{T} \in \mathrm{Lip}\ \alpha$ for $1 < \alpha < 2$ at $\mathbf{p}$ means that there exists a constant $C \geq 0$ such that for all $\mathbf{q}$ locally around $\mathbf{p}$

$$\|d\mathcal{T}\big|_{\mathbf{p}} - d\mathcal{T}\big|_{\mathbf{q}}\| \leq C\|\mathbf{p} - \mathbf{q}\|_\infty^{\alpha-1}.$$

Lemma 3.9. *Let $\mathcal{T}$ be a subdivision scheme with $\mathcal{T} \in Lip\ \alpha$, $1 < \alpha < 2$ at constant sequences and $\mathcal{S}$ a linear scheme. Then the following are equivalent:*

(i) $\mathcal{S}$ and $\mathcal{T}$ satisfy a proximity condition with exponent α for $P_{M,\delta}$,

(ii) $\mathcal{T}$ reproduces constants and $\mathcal{S}$, $\mathcal{T}$ satisfy a differential proximity condition with exponent $\alpha - 1$ for $\mathbf{p} \in P_{M,\delta}$,

(iii) $\mathcal{T}$ reproduces constants and for all constant data $\mathbf{q} = (\dots, q, q, q, \dots)$ with $q \in M$,

$$\mathcal{S} = d\mathcal{T}\big|_{\mathbf{q}}.$$

Proof. In the following we denote by $\mathcal{T}_{k,l},\ \mathcal{S}_{k,l},\ k \in \mathbb{Z},\ l \in \{1, \dots, m\}$ the real-valued functions that map initial data $\mathbf{p}$ to the l-th coordinate of $\mathcal{T}\mathbf{p}_k,\ \mathcal{S}\mathbf{p}_k$, respectively. Note that every function $\mathcal{T}_{k,l},\ \mathcal{S}_{k,l}$ depends only on $\mathbf{p}\big|_{i+I}$ with $k = 2i$ or $k = 2i + 1$ and I a fixed interval.

(ii) $\Rightarrow$ (iii) is clear, since Δ maps constant data to zero.

(iii) $\Rightarrow$ (ii): Since $\mathcal{T} \in \mathrm{Lip}\ \alpha$ at constant sequences $\mathbf{q} = (\dots, q, q, q, \dots)$ for all $q \in \mathbb{R}^m$ we have by (iii)

$$\|d\mathcal{T}_{k,l}\big|_{\mathbf{p}} - \mathcal{S}_{k,l}\| = \|d\mathcal{T}_{k,l}\big|_{\mathbf{p}} - d\mathcal{T}_{k,l}\big|_{\mathbf{q}}\| \leq C\|\mathbf{p}\big|_{i+I} - \mathbf{q}\big|_{i+I}\|_\infty^{\alpha-1}.$$

By choosing the value $q \in \mathbb{R}^m$ appropriately, we get $\|\mathbf{p}\big|_{i+I} - \mathbf{q}\big|_{i+I}\|_\infty \leq C'\|\Delta\mathbf{p}\|_\infty$ with a uniform constant C'. In summary we arrive at

$$\|d\mathcal{T}\big|_{\mathbf{p}} - \mathcal{S}\| \leq C''\|\Delta\mathbf{p}\|_\infty^{\alpha-1}$$

with another constant.

(iii) $\Rightarrow$ (i): For any constant sequence $\mathbf{q} = (\dots, q, q, q, \dots),\ q = (q_1, \dots, q_m) \in \mathbb{R}^m$ we let $\mathrm{c} : [0,1] \to l_\infty(\mathbb{Z}, \mathbb{R}^m)$ be the straight line with $\mathrm{c}(0) = \mathbf{q}$ and $\mathrm{c}(1) = \mathbf{p}$. Since $\mathcal{T} \in \mathrm{Lip}\ \alpha$,

$$\begin{aligned} \mathcal{T}_{k,l}\mathbf{p} &= \mathcal{T}_{k,l}\mathbf{q} + d\mathcal{T}_{k,l}\big|_{\mathrm{c}(\theta)}(\mathbf{p}\big|_{i+I} - \mathbf{q}\big|_{i+I}) \\ &= q_l + (d\mathcal{T}_{k,l}\big|_{\mathbf{q}} + U)(\mathbf{p}\big|_{i+I} - \mathbf{q}\big|_{i+I}) = \mathcal{S}_{k,l}\mathbf{p} + U(\mathbf{p}\big|_{i+I} - \mathbf{q}\big|_{i+I}) \end{aligned}$$

with $0 < \theta < 1$ and $\|U\| \leq C\|\mathbf{p}\big|_{i+I} - \mathbf{q}\big|_{i+I}\|_\infty^{\alpha-1}$. By choosing q appropriately we can conclude that

$$\begin{aligned} |\mathcal{T}_{k,l}\mathbf{p} - \mathcal{S}_{k,l}\mathbf{p}| &\leq \|U\|\|\mathbf{p}\big|_{i+I} - \mathbf{q}\big|_{i+I}\|_\infty \leq C\|\mathbf{p}\big|_{i+I} - \mathbf{q}\big|_{i+I}\|_\infty^{\alpha-1}\|\mathbf{p}\big|_{i+I} - \mathbf{q}\big|_{i+I}\|_\infty \\ &\leq C'\|\Delta\mathbf{p}\|_\infty^\alpha \end{aligned}$$

with another uniform constant C'. This implies (i).

(i) $\Rightarrow$ (iii): We let $\mathbf{e}$ be the sequence with $\mathbf{e}_j = e_r$ and zero elsewhere, $e_r \in \mathbb{R}^m$ is the r-th canonical basis vector and $r \in \{1, \dots, m\}$, $j \in \mathbb{Z}$. For $t > 0$ and $\mathbf{q} = (\dots, q, q, q, \dots)$, $q = (q_1, \dots, q_m) \in \mathbb{R}^m$ we define $\mathbf{p} := \mathbf{q} + t\mathbf{e}$. Then $\Delta\mathbf{p} = t\Delta\mathbf{e}$. We have

$$\mathcal{T}_{k,l}\mathbf{p} = \mathcal{T}_{k,l}\mathbf{q} + d\mathcal{T}_{k,l}\big|_{\mathbf{q}+t'\mathbf{e}}(t\mathbf{e}) = q_l + d\mathcal{T}_{k,l}\big|_{\mathbf{q}}(t\mathbf{e}) + (d\mathcal{T}_{k,l}\big|_{\mathbf{q}+t'\mathbf{e}} - d\mathcal{T}_{k,l}\big|_{\mathbf{q}})(t\mathbf{e})$$

with $0 < t' < t$. We let $U := (d\mathcal{T}_{k,l}\big|_{\mathbf{q}+t'\mathbf{e}} - d\mathcal{T}_{k,l}\big|_{\mathbf{q}})$ and note that

$$\|U\| \leq Ct'^{\alpha-1}\|\mathbf{e}\|_\infty^{\alpha-1} < Ct^{\alpha-1}\|\mathbf{e}\|_\infty^{\alpha-1} \tag{5}$$

with a uniform constant C.

Since $\mathcal{S}\mathbf{p} = \mathbf{q} + \mathcal{S}t\mathbf{e}$ by (1),

$$\mathcal{T}_{k,l}\mathbf{p} - \mathcal{S}_{k,l}\mathbf{p} - Ut\mathbf{e} = (d\mathcal{T}_{k,l}\big|_{\mathbf{q}} - \mathcal{S}_{k,l})t\mathbf{e}.$$

By (i) we have that $|\mathcal{T}_{k,l}\mathbf{p} - \mathcal{S}_{k,l}\mathbf{p}| \leq C'\|\Delta\mathbf{p}\|_\infty^\alpha \leq C't^\alpha\|\Delta\mathbf{e}\|_\infty^\alpha$ with some constant C'. Since by (5) also $|Ut\mathbf{e}| \leq Ct^\alpha\|\mathbf{e}\|_\infty$, the triangle inequality implies that

$$C''t^\alpha \geq t|(d\mathcal{T}_{k,l}\big|_{\mathbf{q}} - \mathcal{S}_{k,l})\mathbf{e}|$$

with some uniform constant C''. Dividing by t and letting $t \to 0$ yields

$$(d\mathcal{T}\big|_{\mathbf{q}} - \mathcal{S})\mathbf{e} = 0.$$

This implies (iii). □

The equivalence of the two notions of proximity (Definitions 3.2 and 3.5) in particular implies that the schemes $\mathcal{S}$ and $\mathcal{T}$ are in proximity if and only if they agree up to first order on constant data. This insight might open up room for new ideas and we plan to study it in more depth in the future.

As already mentioned above, a proximity condition already implies convergence and contractivity of $\mathcal{T}$. The following theorem has been proven in [15]

Theorem 3.10. *Assume that $\mathcal{T}$ and $\mathcal{S}$ satisfy a proximity condition with exponent $\alpha > 1$ for data $\mathbf{p} \in P_{M,\delta}$. If $\mathcal{S}$ is a convergent linear scheme, then there exists $M' \subseteq M,\ \delta' > 0,\ 0 \leq \mu < 1$ such that for all $\mathbf{p} \in P_{M',\delta'}$*

$$\|\Delta\mathcal{T}^j\mathbf{p}\|_\infty \leq C\mu^j\|\Delta\mathbf{p}\|_\infty$$

and $\mathcal{T}^j\mathbf{p} \in P_{M,\delta}$.

With the help of these results we are now able to prove that any smooth $\mathcal{T}$ is stable, provided it satisfies a proximity condition with a convergent linear scheme.

Theorem 3.11. *Assume that $\mathcal{T} \in C^1$ is a subdivision scheme, Lip α at constant sequences, satisfying a proximity condition with exponent $\alpha > 1$ and data in $P_{M,\delta}$ with a convergent linear scheme $\mathcal{S}$. Then there exists a constant $C \geq 0$ such that for all $\mathbf{p},\ \mathbf{q} \in P_{M',\delta'}$ and $M',\ \delta'$ from Theorem 3.10:*

$$\|\mathcal{T}^\infty\mathbf{p}(t) - \mathcal{T}^\infty\mathbf{q}(t)\|_\infty \leq C\|\mathbf{p} - \mathbf{q}\|_\infty.$$

Proof. This is a direct consequence of Lemma 3.9 and Theorems 3.8 and 3.10. □

4 Stability of the Multiscale Decomposition associated with a Subdivision operator

4.1 Definition of the multiscale decomposition associated with a nonlinear subdivision scheme

We proceed by answering Question 2: Does there exist a constant $C \geq 0$ such that for data $\mathcal{P}^0, \tilde{\mathcal{P}}^0, \lambda^1, \tilde{\lambda}^1, \ldots$ we have the inequality

$$\|f - \tilde{f}\|_\infty \leq C(\|\mathcal{P}^0 - \tilde{\mathcal{P}}^0\|_\infty + \sum_{i=1}^{\infty}\|\lambda^i - \tilde{\lambda}^i\|_\infty)$$

for the functions $f, \tilde{f}$ reconstructed from data $(\mathcal{P}^0, \lambda^1, \ldots)$, $(\tilde{\mathcal{P}}^0, \tilde{\lambda}^1, \ldots)$ respectively? Does such an inequality hold locally?

Recall that we have defined $\mathcal{P}^j$ to be the *sampling operator* that maps a bounded, continuous function $f : \mathbb{R} \to \mathbb{R}^m$ to its samples $(f(i/2^j))_{i\in\mathbb{Z}} \in l_\infty(\mathbb{Z}, \mathbb{R}^m)$. For $\mathcal{S}$ the linear B-spline scheme, Faber [14] introduced a multiscale decomposition built from the samples of a function $f \in$ Lip γ already in 1908. The idea is that one stores not the sampling data $\mathcal{P}^j f$, but instead the "prediction error" that one makes in predicting the values of $\mathcal{P}^j f$ from the coarser values $\mathcal{P}^{j-1} f$ via a subdivision scheme. Hence, a function f gets mapped to its initial samples $\mathcal{P}^0 f$ together with the prediction errors

$$\lambda^j = \mathcal{S}\mathcal{P}^{j-1} - \mathcal{P}^j. \tag{6}$$

The intuitive reason for doing this is that for a continuous function f and for a convergent scheme $\mathcal{S}$, the values λ^j should decrease rather quickly. In fact, it is possible to characterize the smoothness order of a continuous function f by the decay rate of the coefficient sequences λ^j [9, 19]. One of the aims of our approach is to make such decompositions available for manifold-valued data and to prove that the properties of the linear decomposition carry over to the manifold setting. The key property a manifold-valued transformation should satisfy is to be *intrinsically defined.* While most quantitative results can be shown just by considering a chart and making all computations in a Euclidean setting, we still need an intrinsic way to define our multiscale transformation. We come back to this issue later in Section 5. For now we simply remark that the definition of λ^j via (6) has no intrinsic meaning if applied to local coordinate representations of a manifold. We therefore need a little more generality in our definition. Our setup is that we have an m-dimensional image space $\mathbb{R}^m$ for the functions f and a k-dimensional vector space $\mathbb{R}^k$ of wavelet coefficients.

Definition 4.1. *The multiscale transformation maps a function $f \in Lip\ \gamma$ with $\gamma > 0$ to a coarse sample $\mathcal{P}^0 f$ and a countable collection of wavelet coefficients $\lambda^j \in l_\infty(\mathbb{Z}, \mathbb{R}^k)$ according to*

$$\lambda^j_i := (\lambda^j f)_i := (\mathcal{P}^j f)_i \ominus (\mathcal{T}\mathcal{P}^{j-1} f)_i, \quad j \in \mathbb{Z}^+,\ i \in \mathbb{Z}.$$

*Here, the operation $\ominus$ is a nonlinear analogue of the operation "*minus*" and represents a bivariate function $\phi : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^k$ mapping two points to their difference vector. In order to be able to reconstruct our function f from data $(\mathcal{P}^0, \lambda^1, \dots)$ we need an inverse operation $\oplus$ serving as an analogue of the euclidean point-vector addition $+$. It must satisfy the relation*

$$p \oplus (q \ominus p) = q \quad \text{for all } p, q \in \mathbb{R}^m.$$

We write $\psi(p, v)$ for $p \oplus v$ and $p \in \mathbb{R}^m, v \in \mathbb{R}^k$ for some $k \in \mathbb{Z}^+$. For technical reasons we need the following not too restrictive properties to hold for the function ψ:

- *ψ together with its inverse $\phi : (p, q) \mapsto p \ominus q$ are C^1 mappings,*
- *$d_1\psi \in Lip\ \nu$ for some $\nu > 0$ where d_1 means differentiation with respect to the variable p,*
- *$\psi(p, 0) = p \oplus 0 = p$ for all $p \in \mathbb{R}^m$.*

We now define the reconstruction procedure.

Definition 4.2. *The* reconstruction procedure *consists of the functions*

$$\mathcal{P}^n : l_\infty(\mathbb{Z}, \mathbb{R}^m) \times l_\infty(\mathbb{Z}, \mathbb{R}^k)^n \to l_\infty(\mathbb{Z}, \mathbb{R}^m),$$

which are inductively defined by

$$\mathcal{P}^n(\mathcal{P}^0, \lambda^1, \dots, \lambda^n) := \Psi(\mathcal{T}\mathcal{P}^{n-1}(\mathcal{P}^0, \lambda^1, \dots, \lambda^{n-1}), \lambda^n) \quad \mathcal{P}^0 := \mathcal{P}^0,$$

and $\Psi(\mathcal{T}\mathcal{P}^{n-1}, \lambda^n)_i := \psi((\mathcal{T}\mathcal{P}^{n-1})_i, (\lambda^n)_i)$, $i \in \mathbb{Z}$.

Again, by locality, although $\mathcal{P}^n$ is a function acting on an infinite dimensional space, it makes sense to speak of the derivatives $\frac{d}{d\lambda^l}\mathcal{P}^n$ and $\frac{d}{d\mathcal{P}^0}\mathcal{P}^n$.

Although we abuse notation by calling the reconstruction functions $\mathcal{P}^n$ it will be possible to distinguish them from the sampling operator $\mathcal{P}^n$ which acts on continuous functions.

So far we have not looked at the question for which initial data $(\mathcal{P}^0, \lambda^1, \dots)$ the reconstruction procedure is *well-defined.* This means that there exists a continuous function f with

$$\mathcal{P}^n f = \mathcal{P}^n(P^0, \dots, \lambda^n) \quad \text{for all } n \in \mathbb{Z}^+.$$

Conditions for well-definedness are studied in the next section.

4.2 Well-definedness of the reconstruction procedure

We denote by B the open unit ball in $\mathbb{R}^m$ and use the notation $U + V$ for the *Minkowski sum* between two subsets $U, V \subset \mathbb{R}^m$.

Lemma 4.3. *Let $\mathcal{T}$ be in proximity with a linear convergent scheme $\mathcal{S}$ for all initial data $\mathbf{p} \in P_{M,\delta}$ and assume that $\mathcal{S}$ has contraction factor μ_0. Assume that $0 < \mu_0 < \mu < 1$ and let $M' \subset M$ be any subset of M that stays away from the boundary of M, i.e. there exists $\rho > 0$ with*

$$M' + \rho \cdot B \subset M. \tag{7}$$

Then with $\nu := -\log_2 \mu$ and $R \in \mathbb{Z}$ there exist constants δ', C such that all data

$$(\mathcal{P}^0, \lambda^1, \lambda^2, \dots)$$

satisfying

$$\mathcal{P}^0 \in P_{M',\delta'} \quad \text{and } \|\lambda^j\|_\infty \le C\mu^j \tag{8}$$

belong to a function $f \in Lip\ \nu$ with $|f|_{Lip\ \nu}$ depending only on δ', C.

Proof. We first note that we can without loss of generality assume that the function $\ominus$ is given by ordinary subtraction $(p, q) \mapsto p - q$. The general case can be easily obtained by noting that, because $\ominus$ is a C^1 function, locally there exist Lipschitz constants C_1, C_2 with

$$C_1\|p \ominus q\| \le \|p - q\| \le C_2\|p \ominus q\|.$$

By assumption there exists a constant C_S with

$$\|\Delta\mathcal{S}\mathbf{p}\|_\infty \le C_S\mu_0^j\|\Delta\mathbf{p}\|_\infty \quad \text{for all } \mathbf{p} \in l_\infty(\mathbb{Z}, \mathbb{R}^m),\ j \in \mathbb{Z}^+. \tag{9}$$

Since $\mathcal{S}$ and $\mathcal{T}$ are in proximity, there exists a constant C_P such that

$$\|\mathcal{S}\mathbf{p} - \mathcal{T}\mathbf{p}\|_\infty \le C_P\|\Delta\mathbf{p}\|_\infty^\alpha \quad \text{for all } \mathbf{p} \in P_{M,\delta}.$$

Further it is well-known [3] that there exists a constant C_L with

$$\sup_{i\in\mathbb{Z}} \inf_{k\in\mathbb{Z}} \|(\mathcal{S}\mathbf{p})_i - p_k\| \le C_L\|\Delta\mathbf{p}\|_\infty \quad \text{for all } \mathbf{p} \in l_\infty(\mathbb{Z}, \mathbb{R}^m).$$

Without loss of generality, we may assume that $C_S, C_L \ge 1$. Write $\tau := \mu_0/\mu < 1$ and find constants $C, \delta' > 0$ such that

$$C_P\left(\frac{8C_S}{1-\tau}\right)^\alpha C^{\alpha-1}\mu^{-1} < 1, \tag{10}$$

$$\delta' \le \inf\left(\left(\frac{C\mu}{C_P}\left(\frac{2C_S}{1-\tau} - 1\right)\right)^{1/\alpha}, \frac{4C}{1-\tau}\right), \tag{11}$$

$$\frac{4CC_S}{1-\tau} + C_S\delta' < \delta, \tag{12}$$

and

$$\frac{C}{1-\mu} + C_P\left(\frac{8CC_S}{1-\tau}\right)^\alpha \frac{1}{1-\mu^\alpha} + C_L\frac{8CC_S}{1-\tau}\frac{1}{1-\mu} < \rho. \tag{13}$$

The above inequalities assure that the following estimates go through. After our choice of C and δ', we now show inductively that for data satisfying (8), the reconstructed data $\mathcal{P}^j$ lies in the class $P_{\delta,M}$ and

$$\|\Delta\mathcal{P}^j\|_\infty \le \frac{4CC_S}{1-\tau}\mu^j + C_S\mu_0^j\|\Delta\mathcal{P}^0\|_\infty. \tag{14}$$

for all $j \in \mathbb{Z}$. Let us start with the case $j = 1$: We have that

$$\begin{aligned}
\|\Delta\mathcal{P}^1\|_\infty &\le \|\Delta(\mathcal{P}^1 - \mathcal{T}\mathcal{P}^0)\|_\infty + \|\Delta(\mathcal{T}\mathcal{P}^0 - \mathcal{S}\mathcal{P}^0)\|_\infty + \|\Delta\mathcal{S}\mathcal{P}^0\|_\infty \\
&\le 2C\mu + 2C_P\|\Delta\mathcal{P}^0\|_\infty^\alpha + C_S\mu_0\|\Delta\mathcal{P}^0\|_\infty \\
&\le (2C + 2C_P\delta'^\alpha/\mu)\mu + C_S\mu_0\|\Delta\mathcal{P}^0\|_\infty \le \frac{4CC_S}{1-\tau}\mu + C_S\mu_0\|\Delta\mathcal{P}^0\|_\infty
\end{aligned}$$

because of (11). In particular, because of (12), we have $\|\Delta\mathcal{P}^1\|_\infty \leq \delta$. We show that $\mathcal{P}^1$ takes values in M:

$$\begin{aligned}
\sup_{i\in\mathbb{Z}} \inf_{x\in M'} \|(\mathcal{P}^1)_i - x\| &\leq \sup_{i\in\mathbb{Z}} \inf_{k\in\mathbb{Z}} \|(\mathcal{P}^1)_i - (\mathcal{P}^0)_k\| \\
&\leq \sup_{i\in\mathbb{Z}} \inf_{k\in\mathbb{Z}} \big(\|(\mathcal{P}^1)_i - (\mathcal{T}\mathcal{P}^0)_i\| + \|(\mathcal{T}\mathcal{P}^0)_i - (\mathcal{S}\mathcal{P}^0)_i\| + \|(\mathcal{S}\mathcal{P}^0)_i - (\mathcal{P}^0)_k\|\big) \\
&\leq C\mu + C_P(\delta')^\alpha + C_S\delta' < \rho
\end{aligned}$$

because of (11), (13) and the assumption that $C_S, C_L \geq 1$. With (7) we can conclude that $\mathcal{P}^1$ takes values in M and hence $\mathcal{P}^1 \in P_{M,\delta}$. We go on to prove the induction step $j-1 \mapsto j$ and estimate

$$\begin{aligned}
\|\Delta\mathcal{P}^j\|_\infty &\leq \sum_{k=1}^{j} \left(\|\Delta(\mathcal{S}^{j-k}\mathcal{P}^k - \mathcal{S}^{j-k}\mathcal{T}\mathcal{P}^{k-1})\|_\infty + \|\Delta(\mathcal{S}^{j-k}\mathcal{T}\mathcal{P}^{k-1} - \mathcal{S}^{j-k}\mathcal{S}\mathcal{P}^{k-1})\|_\infty \right) + \|\Delta\mathcal{S}^j\mathcal{P}^0\|_\infty \\
&\leq \sum_{k=1}^{j} \left(C_S\mu_0^{j-k}\|\Delta(\mathcal{P}^k - \mathcal{T}\mathcal{P}^{k-1})\|_\infty + C_S\mu_0^{j-k}\|\Delta(\mathcal{T}\mathcal{P}^{k-1} - \mathcal{S}\mathcal{P}^{k-1})\|_\infty \right) + C_S\mu_0^j\|\Delta\mathcal{P}^0\|_\infty \\
&\leq 2C_SC\sum_{k=1}^{j}\mu_0^{j-k}\mu^k + 2C_SC_P\sum_{k=1}^{j}\mu_0^{j-k}\|\Delta\mathcal{P}^{k-1}\|_\infty^\alpha + C_S\mu_0^j\|\Delta\mathcal{P}^0\|_\infty \\
&\leq \frac{2C_SC}{1-\tau}\mu^j + 2C_SC_P\sum_{k=1}^{j}\mu_0^{j-k}\Big(\frac{4CC_S}{1-\tau}\mu^{k-1} + C_S\mu_0^{k-1}\|\Delta\mathcal{P}^0\|_\infty\Big)^\alpha + C_S\mu_0^j\|\Delta\mathcal{P}^0\|_\infty \\
&\leq \frac{2C_SC}{1-\tau}\mu^j + 2C_SC_P\sum_{k=1}^{j}\mu_0^{j-k}\Big(\frac{8CC_S}{1-\tau}\mu^{k-1}\Big)^\alpha + C_S\mu_0^j\|\Delta\mathcal{P}^0\|_\infty \\
&\leq \frac{2C_SC}{1-\tau}\mu^j\left(1 + C_P\Big(\frac{8C_S}{1-\tau}\Big)^\alpha C^{\alpha-1}\mu^{-1}\right) + C_S\mu_0^j\|\Delta\mathcal{P}^0\|_\infty \\
&\leq \frac{4CC_S}{1-\tau}\mu^j + C_S\mu_0^j\|\Delta\mathcal{P}^0\|_\infty.
\end{aligned}$$

We have used (11) and (10). In particular, because of (12), we get that $\|\Delta\mathcal{P}^j\|_\infty < \delta$. Now all that needs to be shown is that $\mathcal{P}^j$ still takes values in M:

$$\begin{aligned}
\sup_{i\in\mathbb{Z}} \inf_{x\in M'} \|(\mathcal{P}^j)_i - x\| &\leq \sup_{i\in\mathbb{Z}} \inf_{k\in\mathbb{Z}} \|(\mathcal{P}^j)_i - (\mathcal{P}^0)_k\| \leq \sum_{l=1}^{j} \sup_{i\in\mathbb{Z}} \inf_{k\in\mathbb{Z}} \|(\mathcal{P}^l)_i - (\mathcal{P}^{l-1})_k\| \\
&\leq \sum_{l=1}^{j} \sup_{i\in\mathbb{Z}} \inf_{k\in\mathbb{Z}} \big(\|(\mathcal{P}^l)_i - (\mathcal{T}\mathcal{P}^{l-1})_i\| + \|(\mathcal{T}\mathcal{P}^{l-1})_i - (\mathcal{S}\mathcal{P}^{l-1})_i\| \\
&\quad + \|(\mathcal{S}\mathcal{P}^{l-1})_i - (\mathcal{P}^{l-1})_k\|\big) \\
&\leq C\sum_{l=1}^{j}\mu^j + C_P\sum_{l=1}^{j}\|\Delta\mathcal{P}^{l-1}\|_\infty^\alpha + C_L\|\Delta\mathcal{P}^{l-1}\|_\infty \\
&\leq \frac{C}{1-\mu} + C_P\sum_{l=1}^{j}\Big(\frac{4CC_S}{1-\tau}\mu^{l-1} + \delta'\mu_0^{l-1}\Big)^\alpha + C_L\sum_{l=1}^{j}\Big(\frac{4CC_S}{1-\tau}\mu^{l-1} + \delta'\mu_0^{l-1}\Big) \\
&\leq \frac{C}{1-\mu'} + C_P\sum_{l=1}^{j}\Big(\frac{8CC_S}{1-\tau}\mu^{l-1}\Big)^\alpha + C_L\sum_{l=1}^{j}\frac{8CC_S}{1-\tau}\mu^{l-1} \\
&\leq \frac{C}{1-\mu} + C_P\left(\frac{8CC_S}{1-\tau}\right)^\alpha\frac{1}{1-\mu^\alpha} + C_L\frac{8CC_S}{1-\tau}\frac{1}{1-\mu} \\
&< \rho.
\end{aligned}$$

With (7) we can conclude that $\mathcal{P}^j$ takes values in M. We have now shown that for data satisfying (8) the reconstruction procedure is well-defined and the reconstructed data satisfies (14). Now all assertions follow from (2). □

Remark 4.4. *It is well-known that the contractivity factor of a convergent linear scheme $\mathcal{S}$ must satisfy*

$\frac{1}{2} \le \mu_0$ *as shown e.g. in [28].*

4.3 Stability of the reconstruction procedure

Similar to the proof of the stability of the subdivision scheme, the key to stability is to show that the differentials of the reconstruction procedure are uniformly bounded. This is done by the next lemma.

Lemma 4.5. *Let $f \in Lip\ \gamma,\ 1 > \gamma > 0$ with wavelet decomposition $(\mathcal{P}^0 f, \lambda^1 f, \dots)$ with respect to the subdivision scheme $\mathcal{T} \in Lip\ \alpha,\ \alpha > 1$, and the operations $\ominus,\ \oplus$. Assume that $\mathcal{T}$ satisfies a proximity condition with a linear, convergent subdivision scheme $\mathcal{S}$ with exponent α for data in $P_{M,\delta}$. Then, if $\mathcal{P}^n f \in P_{M,\delta}$ for all $n \in \mathbb{Z}^+$, the operator norms*

$$\Big\|\frac{d}{d\lambda^l}\mathcal{P}^n\big|_{(\mathcal{P}^0 f,\lambda^1 f,\dots)}\Big\| \quad \text{and} \quad \Big\|\frac{d}{d\mathcal{P}^0}\mathcal{P}^n\big|_{(\mathcal{P}^0 f,\lambda^1 f,\dots)}\Big\|$$

are bounded by a constant $C \ge 0$ independent of l and n and depending only on $\|\Delta\mathcal{P}^0 f\|_\infty$ and the seminorm $|f|_{Lip\ \gamma}$.

Proof. We break up the proof into several parts. In order to keep the notation simple, we shall denote several different constants by C and contraction factors which are greater than 0 and less than 1 by μ. Recall that we write $\psi(p,v)$ for $p \oplus v$ and $\phi(p,q)$ for $p \ominus q$, $p,q \in \mathbb{R}^m,\ v \in \mathbb{R}^k$. We require that ψ and ϕ are continuously differentiable mappings with $d_1\psi \in \mathrm{Lip}\ \nu$ for some $\nu > 0$.

(i): We first show that for $f \in \mathrm{Lip}\ \gamma$ the wavelet coefficients decay geometrically: Note that for points $p,q \in \mathbb{R}^m$ and some $0 < \theta < 1$

$$\|(p \ominus q)_l\| = \|\phi(q,p)_l\| = \|\phi(q,q)_l + d_2\phi\big|_{\mathrm{c}(\theta)}(p-q)_l\| \le \|d_2\phi\|\|p-q\|$$

with $\mathrm{c}(t) := u + t(v-u)$, d_2 meaning differentiation with respect to the second variable and the subscript l the l-th coordinate in $\mathbb{R}^k$, $l \in \{1,\dots,k\}$. We estimate the wavelet coefficients as follows:

$$\begin{aligned} \|\lambda^j f\|_\infty &= \|(\mathcal{T}\mathcal{P}^{j-1}f \ominus \mathcal{P}^j f\|_\infty \\ &\le \sup_{0<t<1}\|d_2\Phi\big|_{\mathrm{c}(t)}\|\|\mathcal{T}\mathcal{P}^{j-1}f - \mathcal{P}^j f\|_\infty \\ &\le \sup_{0<t<1}\|d_2\Phi\big|_{\mathrm{c}(t)}\|(\|\mathcal{T}\mathcal{P}^{j-1}f - \mathcal{S}\mathcal{P}^{j-1}f\|_\infty \\ &\quad +\|\mathcal{S}\mathcal{P}^{j-1}f - \mathcal{P}^j f\|_\infty). \end{aligned}$$

Since $\mathcal{S},\ \mathcal{T}$ satisfy a proximity condition, we can estimate the first term in the brackets by a constant times $\|\Delta\mathcal{P}^j f\|_\infty^\alpha$. Since $f \in \mathrm{Lip}\ \gamma$, there exist $C \ge 0,\ \mu < 1$ with $\|\Delta\mathcal{P}^j f\|_\infty \le C\mu^j$ with $C = |f|_{\mathrm{Lip}\ \gamma}$ and $\mu = 2^{-\gamma}$. Therefore the first term is bounded by a constant times $\mu^{\alpha j}$. Now to the second term: Recall that for $i \in \mathbb{Z},\ k \in \{0,1\}$ the value of $\mathcal{S}\mathbf{p}_{2i+k}$ depends only on $\mathbf{p}$ restricted to $i + I$ where I is a fixed interval $\subset \mathbb{Z}$ of length L. By (1) we have for every $q \in \mathbb{R}^m$, $\mathbf{q} = (\dots,q,q,q,\dots)$,

$$\begin{aligned} \|(\mathcal{S}\mathcal{P}^{j-1}f)_{2i+k} - (\mathcal{P}^j f)_{2i+k}\| &= \|\mathcal{S}(\mathcal{P}^{j-1}f - \mathbf{q})_{2i+k} - ((\mathcal{P}^j f)_{2i+k} - q)\| \\ &\le \|\mathcal{S}\|\max_{l\in i+I}\|(\mathcal{P}^{j-1}f)_l - q\| + \|(\mathcal{P}^j f)_{2i+k} - q\| \end{aligned}$$

By choosing q appropriately, we obtain

$$\|\mathcal{S}\mathcal{P}^{j-1}f - \mathcal{P}^j f\|_\infty \quad \le \quad C\|\Delta\mathcal{P}^j f\|_\infty \le C'\mu^j$$

for another constant C'. Combining these two estimates we arrive at

$$\|\lambda^j f\|_\infty \le C2^{-j\gamma}$$

for yet another uniform constant C.

(ii): Now we want to obtain bounds on the differentials of the reconstruction procedure. In what follows we write $\lambda^j := \lambda^j f$. We begin by computing for $l < n$

$$\begin{aligned} \frac{d}{d\lambda^l}\mathcal{P}^n &= \frac{d}{d\lambda^l}\Psi(\mathcal{T}\mathcal{P}^{n-1},\lambda^n) = d_1\Psi\big|_{(\mathcal{T}\mathcal{P}^{n-1},\lambda^n)} d\mathcal{T}\big|_{\mathcal{P}^{n-1}} \frac{d}{d\lambda^l}\mathcal{P}^{n-1} \\ &= (d_1\Psi\big|_{(\mathcal{T}\mathcal{P}^{n-1},0)} + U_n) d\mathcal{T}\big|_{\mathcal{P}^{n-1}} \frac{d}{d\lambda^l}\mathcal{P}^{n-1} \\ &= (I + U_n)(S + \tilde{U}_n)\frac{d}{d\lambda^l}\mathcal{P}^{n-1}, \end{aligned} \tag{15}$$

with I the identity mapping. Let us first obtain a bound for $\|U_n\|$: Since $d_1\Psi \in \text{Lip }\nu$, we obtain

$$\|U_n\| = \|d_1\Psi\big|_{(\mathcal{T}\mathcal{P}^{n-1},0)} - d_1\Psi\big|_{(\mathcal{T}\mathcal{P}^{n-1},\lambda^n)}\| \le C\|\lambda^n\|_\infty^\nu \le C'(\mu^\nu)^n,$$

because by (i) the wavelet coefficients decay geometrically. The Matrix $\tilde{U}_n$ is bounded by a constant times $\|\Delta\mathcal{P}^{n-1}\|_\infty^{\alpha-1}$ since we assumed that a proximity condition with exponent $\alpha > 1$ holds. By Lemma 3.9 (i) $\Rightarrow$ (ii), this implies that a differential proximity condition holds and further, $\|\tilde{U}_n\| \le C\|\Delta\mathcal{P}^{n-1}\|_\infty^{\alpha-1}$. By the above estimate and the fact that $f \in \text{Lip }\gamma$ we finally get

$$\|U_n\|,\ \|\tilde{U}_n\| \le C\mu^n$$

for another constant $C \ge 0$ and some $0 < \mu < 1$. Since also $\|S\|$ is uniformly bounded, this implies

$$\frac{d}{d\lambda^l}\mathcal{P}^n = (S + V_n)\frac{d}{d\lambda^l}\mathcal{P}^{n-1}, \tag{16}$$

where $\|V_n\| \le C\mu^n$ with a uniform constant $C \ge 0$.

For $l > n$ we have $\frac{d}{d\lambda^l}\mathcal{P}^n = 0$. If $l = n$, then $\frac{d}{d\lambda^l}\mathcal{P}^n = d_2\Psi\big|_{(\mathcal{T}\mathcal{P}^{n-1},\lambda^n)}$. Now we apply (16) inductively and obtain

$$\frac{d}{d\lambda^l}\mathcal{P}^n = \prod_{j=l+1}^n (\mathcal{S} + V_j)\frac{d}{d\lambda^l}\mathcal{P}^l = \prod_{j=l+1}^n (\mathcal{S} + V_j) d_2\Psi\big|_{(\mathcal{T}\mathcal{P}^{l-1},\lambda^l)}$$

with $\|V_j\| \le C\mu^j$. Applying Lemma 3.6 yields a uniform bound for $\|\frac{d}{d\lambda^l}\mathcal{P}^n\|$.

We still need to estimate $\|\frac{d}{d\mathcal{P}^0}\mathcal{P}^n\|$. Similar to the derivation of (15), (16), we obtain

$$\frac{d}{d\mathcal{P}^0}\mathcal{P}^n = \prod_{j=1}^n (\mathcal{S} + W_j)$$

with matrices $\|W_j\| \le C\mu^j$. Using Lemma 3.6 again, we arrive at boundedness of $\|\frac{d}{d\mathcal{P}^0}\mathcal{P}^n\|$. We have now proven that the norms $\|\frac{d}{d\lambda^l}\mathcal{P}^n\|$, $\|\frac{d}{d\mathcal{P}^0}\mathcal{P}^n\|$ are bounded by a constant $C \ge 0$ independent of l, n and depending only on γ and C.

□

We state and prove our main stability theorem for Euclidean data.

Theorem 4.6. *With the assumptions of Lemma 4.5, let $f \in$ Lip γ with $0 < \gamma < -\log_2\mu_0 \le 1$ (μ_0 being the contraction factor of $\mathcal{S}$) and $\mathcal{P}^n f \in P_{M'',\delta}$ for all $n \in \mathbb{Z}^+$ where M'' is a set that stays away from the boundary of M' of Lemma 4.3. Then there exist $\delta_0, \varepsilon_0 > 0$ and a constant $C \ge 0$ depending only (in a monotonely decreasing way) on $\|\Delta\mathcal{P}^0\|$ and $|f|_{\text{Lip }\gamma}$ such that for all initial data*

$$(\tilde{\mathcal{P}}^0, \tilde{\lambda}^1, \tilde{\lambda}^2, \dots)$$

with

$$\|\tilde{\mathcal{P}}^0 - \mathcal{P}^0 f\|_\infty \le \delta_0 \text{ and } \|\tilde{\lambda}^j - \lambda^j f\|_\infty \le \varepsilon_0/2^{\gamma j}, \quad j \in \mathbb{Z}^+, \tag{17}$$

the reconstruction from (4.6) yields a well-defined continuous function $\tilde{f}$ with

$$\|\mathcal{P}^n(f - \tilde{f})\|_\infty \le C\big(\|\mathcal{P}^0(f - \tilde{f})\|_\infty + \sum_{j=1}^n \|\lambda^j - \tilde{\lambda}^j\|_\infty\big).$$

Proof. From the first part of the proof of Lemma 4.5 we know that, since $f \in \text{Lip}\,\gamma$, the norms of the wavelet coefficients decay according to $\|\lambda^j\|_\infty = O(\mu^j)$ with the hidden constant depending only on γ and $|f|_{\text{Lip}\,\gamma}$ and $\mu = 2^{-\gamma}$. It is also clear by definition that $\|\Delta \mathcal{P}^j f\|_\infty \leq |f|_{\text{Lip}\,\gamma} 2^{-\gamma j}$. Hence there exists j_0 such that $\|\Delta \mathcal{P}^{j_0}\|_\infty < \delta'$ and $\|\lambda^{j_0+j}\| < C'/2^{\gamma j}$ for all $j \geq 0$ with δ', C' from Lemma 4.3. Denoting $\tilde{\mathcal{P}}^j := \mathcal{P}^j(\tilde{\mathcal{P}}^0, \tilde{\lambda}^1, \dots)$, we can now pick $\varepsilon_0,\ \delta_0 > 0$ such that for all data satisfying (17) we have $\|\Delta \tilde{\mathcal{P}}^{j_0}\| < \delta'$, $\|\tilde{\lambda}^{j_0+j}\| < C'/2^{\gamma j}$, and $\tilde{\mathcal{P}}^{j_0}$ assumes values in M'. Any such data, by Lemma 4.3, defines a continuous function $g \in \text{Lip}\,\gamma$ with $|g|_{\text{Lip}\,\gamma}$ only depending on δ', C'. For all data which obey (17), by Lemma 4.5, the norms $\|\frac{d}{d\lambda^{j_0+k}} \mathcal{P}^{j_0+j}\|$, $\|\frac{d}{d\mathcal{P}^{j_0}} \mathcal{P}^{j_0+j}\|$, $k, j \geq 0$ are uniformly bounded by a constant $C_1 > 0$. Define

$$C_2 := \max_{j,k=1,\dots,j_0} \sup_{(\tilde{\mathcal{P}}^0, \tilde{\lambda}^1, \dots, \tilde{\lambda}^{j_0}) \text{ with } (17)} \|\frac{d}{d\lambda^k} \mathcal{P}^j\| < \infty.$$

We now show that the theorem holds with $C := \max(C_1, C_1 \cdot C_2, C_2)$:

For data $(\tilde{\mathcal{P}}^0, \tilde{\lambda}^1, \dots)$ with (17), the estimate

$$\begin{aligned}
\|\mathcal{P}^n(\mathcal{P}^0, \lambda^1, \dots, \lambda^n) \; &- \; \mathcal{P}^n(\tilde{\mathcal{P}}^0, \tilde{\lambda}^1, \dots, \tilde{\lambda}^n)\|_\infty \\
&\leq \; \|\mathcal{P}^n(\mathcal{P}^0, \lambda^1, \dots, \lambda^n) - \mathcal{P}^n(\tilde{\mathcal{P}}^0, \lambda^1, \dots, \lambda^n)\|_\infty + \\
&\quad\; \|\mathcal{P}^n(\tilde{\mathcal{P}}^0, \lambda^1, \dots, \lambda^n) - \mathcal{P}^n(\tilde{\mathcal{P}}^0, \tilde{\lambda}^1, \lambda^2, \dots, \lambda^n)\|_\infty + \dots + \\
&\quad\; \|\mathcal{P}^n(\tilde{\mathcal{P}}^0, \tilde{\lambda}^1, \dots, \tilde{\lambda}^{n-1}, \lambda^n) - \mathcal{P}^n(\tilde{\mathcal{P}}^0, \tilde{\lambda}^1, \dots, \tilde{\lambda}^{n-1}, \tilde{\lambda}^n)\|_\infty \\
&\leq \; \sup_{t \in (0,1)} \|\frac{d}{d\mathcal{P}^0} \mathcal{P}^n|_{(t\mathcal{P}^0 + (1-t)\tilde{\mathcal{P}}^0, \lambda^1, \dots, \lambda^n)}\| \|\mathcal{P}^0 - \tilde{\mathcal{P}}^0\|_\infty + \\
&\quad\; \sup_{t \in (0,1)} \|\frac{d}{d\lambda^1} \mathcal{P}^n|_{(\tilde{\mathcal{P}}^0, t\lambda^1 + (1-t)\tilde{\lambda}^1, \dots, \lambda^n)}\| \|\lambda^1 - \tilde{\lambda}^1\|_\infty + \dots + \\
&\quad\; \sup_{t \in (0,1)} \|\frac{d}{d\lambda^n} \mathcal{P}^n|_{(\tilde{\mathcal{P}}^0, \tilde{\lambda}^1, \dots, \tilde{\lambda}^{n-1}, t\lambda^n + (1-t)\tilde{\lambda}^n)}\| \|\lambda^n - \tilde{\lambda}^n\|_\infty
\end{aligned}$$

holds. Now we distinguish three cases to show that the norms $\|\frac{d}{d\lambda^i} \mathcal{P}^j\|$, $i, j \in \mathbb{Z}^+$ are uniformly bounded by C:

Case 1: $i, j > j_0$. For this case, boundedness has been shown with the constant C_1.

Case 2: $i, j \leq j_0$. In this case the expression is bounded by C_2.

Case 3: $i > j_0, j \leq j_0$. Here we apply the chain rule to obtain

$$\|\frac{d}{d\lambda^i} \mathcal{P}^j\| \leq \|\frac{d}{d\mathcal{P}^0} \mathcal{P}^j\| \|\frac{d}{d\lambda^j} \mathcal{P}^{j_0}\| \leq C_1 \cdot C_2.$$

The proof is complete. □

Comparison with the results in [20]: We would like to say a few words on how our results compare to the recent work [20]. The idea of both our work as well as [20] is to obtain global estimates on the differentials of the iterates of the subdivision and reconstruction operators. However, other than that, the approaches are quite different. The fundamental assumption in [20] is that the nonlinear subdivision scheme admits some algebraic factorization properties which allow for the construction of so-called *derived schemes* which are intertwining maps between a difference operator and the subdivision operator. Based on the differentials of these derived schemes a spectral quantity is introduced which turns out to accurately describe the stability properties. It also turns out that convergence of the nonlinear scheme, stability of the nonlinear scheme and stability of the reconstruction procedure are in general independent of each other.

The algebraic factorization properties assumed in [20] do not hold for the applications we have in mind, namely manifold-valued subdivision, so the results of [20] cannot be applied at all. The nonlinearity of manifold-valued schemes has no algebraic structure; it stems from the rather arbitrary curvature properties of some manifold – this is the reason why algebraic factorization properties do not hold in this case. We overcome this difficulty by using a variant of the perturbation approach which has been successfully used e.g. in [29, 6] together with the above mentioned idea to bound the differentials of the iterates of the nonlinear subdivision operator.

A remarkable difference between our stability results and the results in [20] is that with our assumptions (i.e. a proximity condition), stability of the subdivision scheme and the reconstruction procedure essentially follow from convergence of the scheme. Another difference lies in the assumptions made on the initial data sequences for the reconstruction procedure: we assume exponential decay of the detail coefficients while in [20] no such assumption is made. In our framework this assumption is natural and poses almost no restriction: as can be seen from the proofs in this section, the detail coefficients coming from any reasonable function satisfy it.

5 Examples

In the previous sections we have shown some general results concerning the stability of nonlinear subdivision schemes. In that we have always required our data to lie in some Euclidean space. In the present section we discuss some examples of nonlinear multiscale decompositions which operate on manifold-valued data. While again quantitative results can easily be obtained by using a chart, we still need to state an analogue of Theorem 4.6 in an intrinsic setting. In particular, it is desirable to have an intrinsic way of comparing two wavelet coefficients which may be, for example, two tangent vectors at different points, i.e., two elements of different vector spaces. If a metric is available for our manifold, then perhaps the most natural way of comparing two such vectors is to parallel translate the first vector into the tangent space of the second one along a geodesic curve. We discuss this below, but first we treat the simpler case of a Lie group, where the tangent bundle is trivial.

5.1 The log-exp analogue in Lie groups

The log-exp analogue in Lie groups and the corresponding multi-scale decomposition has first been studied in [27]. Given a Lie group $(G,\cdot)$ with Lie-algebra $\mathfrak{g}$, the log-exp analogue $\mathcal{T}$ of a linear subdivision scheme $\mathcal{S}$ is defined on G-valued sequences $\mathbf{p}=(p_i)_{i\in\mathbb{Z}}$ via

$$(\mathcal{T}\mathbf{p})_{2i+k} := p_i \cdot \exp(\sum_{j\in\mathbb{Z}} a_{2i+k-2j}\log(p_i^{-1}\cdot p_j)) =: p_i \oplus \big(\sum_{j\in\mathbb{Z}} a_{2i+k-j}p_j \ominus p_i\big).$$

Here exp is the exponential mapping of G that diffeomorphically maps a neighborhood of $0\in\mathfrak{g}$ to a neighborhood of the unit element in G, log is its inverse, and (a_i) is the mask of $\mathcal{S}$. The detail coefficients of a continuous G-valued function f are consequently defined via

$$(\lambda^j)_i := \log\big((\mathcal{P}^j f)_i^{-1}\cdot(\mathcal{T}\mathcal{P}^{j-1})_i\big) = (\mathcal{T}\mathcal{P}^{j-1})_i \ominus (\mathcal{P}^j f)_i.$$

It is convenient that in the case of a Lie group all detail coefficients lie in the same vector space $\mathfrak{g}$. This makes it possible to directly apply our stability results from the previous sections using local charts. We fix a norm $\|\cdot\|$ in $\mathfrak{g}$ and define for two points $p,q\in G$ with $p^{-1}\cdot q$ small enough for $q\ominus p$ to be defined

$$d(p,q) := \|q\ominus p\|.$$

For G-valued sequences $\mathbf{p},\mathbf{q}$ we write $d(\mathbf{p},\mathbf{q})_\infty := \sup_{i\in\mathbb{Z}} d(p_i,q_i)$. Similarly, for a sequence $\lambda=(\lambda_i)_{i\in\mathbb{Z}}$ we write $\|\lambda\|_\infty := \sup_{i\in\mathbb{Z}}\|\lambda_i\|$. It is easy to see and has been shown e.g. in [15] that locally a proximity condition of order $\alpha=2$ is satisfied between $\mathcal{S}$ and $\mathcal{T}$. Now we can state our main stability theorem for the log-exp analogue in Lie groups.

Theorem 5.1. *For all G-valued functions $f\in Lip\,\gamma$ there exist $\delta_0,\varepsilon_0>0$ and a constant $C\geq 0$ such that for all initial data*

$$(\tilde{\mathcal{P}}^0,\tilde\lambda^1,\tilde\lambda^2,\dots)$$

with

$$d(\tilde{\mathcal{P}}^0,\mathcal{P}^0 f)_\infty \leq \delta_0 \quad \text{and} \quad \|\tilde\lambda^j-\lambda^j f\|_\infty \leq \varepsilon_0/2^{\gamma j},\quad j\in\mathbb{Z}^+, \tag{18}$$

the reconstruction yields a well-defined continuous function $\tilde f$ with

$$d(\mathcal{P}^n f,\mathcal{P}^n\tilde f)_\infty \quad \leq \quad C\Big(d(\mathcal{P}^0 f,\tilde{\mathcal{P}}^0)_\infty + \sum_{j=1}^n \|\lambda^j f-\tilde\lambda^j\|_\infty\Big).$$

The constants δ_0,ε_0,C are uniform for data values in a compact set.

Proof. We only sketch the argument. The first simple remark is that the requirement $0 < \gamma < -\log_2 \mu_0 \leq 1$, where μ_0 is the contraction factor of $\mathcal{S}$ poses no loss of generality since all our assumptions hold also for any $0 < \gamma' < \gamma$ if they hold for $\gamma > 0$. The proof is then just a simple application of Theorem 4.6 after pulling back the subdivision and reconstruction procedure to Euclidean space using a local chart such that the domain of definition of the chart is open and convex. For each chart we get different constants $\delta_0, \varepsilon_0, C$, therefore uniform constants can in general only be achieved for data values in a compact set. $\square$

Remark 5.2. *Actually, more can be said regarding the smoothness of the reconstructed function $\tilde{f}$. The decay rate of wavelet coefficients of $\tilde{f}$ is clearly the same as the decay rate of f. This implies by the results in [19] that, provided the underlying linear scheme $\mathcal{S}$ is interpolatory and sufficiently smooth, $\tilde{f}$ is as smooth as f. The same is true if $\mathcal{S}$ is not interpolatory if we define $\mathcal{T}$ in a different way [31]. This remark also applies to the log-exp analogue in a Riemannian manifold which is studied below.*

5.2 The log-exp analogue in Riemannian manifolds

Before we proceed further, we introduce some basic facts of Riemannian geometry. For more information on this topic, the interested reader is referred to [8].

Some basics of Riemannian geometry An m-dimensional *Riemannian manifold* is by definition an m-dimensional smooth differentiable manifold $\mathcal{M}$ together with a smooth $(0,2)$-tensor field

$$p \in \mathcal{M} \mapsto \langle \cdot, \cdot \rangle_p$$

such that for each $p \in \mathcal{M}$, the bilinear form $(u, v) \mapsto \langle u, v \rangle_p \in T_p\mathcal{M}$ is a positive definite inner product on $T_p\mathcal{M}$. With the help of the inner product we define the *length* of a smooth curve $\mathrm{c} : I \to \mathcal{M}$, where I is some interval $\subseteq \mathbb{R}$, by

$$l(\mathrm{c}) := \int_I \|\dot{\mathrm{c}}(t)\| dt := \int_I \langle \dot{\mathrm{c}}(t), \dot{\mathrm{c}}(t) \rangle_{\mathrm{c}(t)}^{1/2} dt.$$

A *vector field* V along a curve $\mathrm{c} : I \to \mathcal{M}$ is a mapping that smoothly assigns to each parameter value $t \in I$ a tangent vector $V(t) \in T_{\mathrm{c}(t)}\mathcal{M}$. A Riemannian manifold possesses a canonical way of differentiating a vector field along a curve via the *Riemannian connection.* This derivative is called the *covariant derivative.* With respect to a chart $\varphi : U \subset \mathbb{R}^m \to \mathcal{M}$ it can be written as

$$\frac{D}{dt} V(t) := \left(\frac{dv^l}{dt} + v^k x^j \Gamma^l_{jk}\right) \partial_l,$$

where $V(t)$ is a vector field along $\gamma(t)$. This means that $V(t) \in T_{\mathrm{c}(t)}\mathcal{M}$ with coordinate representation $v^j \partial_j$. The coefficients x^k represent the coefficients of the tangent vector field $\dot{\mathrm{c}}$, i.e. $\dot{\mathrm{c}} = x^k \partial_k$ and ∂_k are the basis vector fields induced by the parametrization $(\varphi, d\varphi)$. The quantities Γ^l_{jk} are the *Christoffel symbols of the first kind.* Note that we have used the Einstein sum convention.

The vector field V is called *parallel* along c, if $\frac{D}{dt} V \equiv 0$. By the linearity of the equation $\frac{DV}{dt} = 0$, for curves c with $\mathrm{c}(0) = p$ and $v \in T_p\mathcal{M}$, there exists a unique vector field V along c with $V(0) = v$ and V is parallel along c. If $\mathrm{c}(1) = q$, then the vector $V(1)$ is called the result of *parallel transport* of v from p to q along c. We denote the linear mapping that maps the vector v to the vector $V(1)$ by $\mathrm{Pt}_0^1(\mathrm{c}) : T_p\mathcal{M} \to T_q\mathcal{M}$.

Obviously, the tangent vector field $t \mapsto \dot{\mathrm{c}}(t)$ is a vector field along c. The curve c is called a *geodesic*, if $\dot{\mathrm{c}}$ is parallel along c. It is well known that geodesics locally minimize arc length. Moreover, locally any two points can be joined by a unique shortest curve which then is a geodesic. Conversely, given a point $p \in \mathcal{M}$ and a vector $v \in T_p\mathcal{M}$ of sufficiently small norm, then there exists a unique geodesic $\mathrm{c} : [-2, 2] \to \mathcal{M}$ with $\mathrm{c}(0) = p$ and $\dot{\mathrm{c}}(0) = v$. The point $\mathrm{c}(1)$ is called $\exp_p(v)$. This mapping $(p, v) \mapsto \exp_p(v)$ is a smooth map in both variables and it also possesses an inverse $\log_p$ with $\log_p(\exp_p(v)) = v$ which is also smooth in both variables. As the notation already suggests, the function exp is called *exponential mapping.* The notion of arc length naturally induces a metric on $\mathcal{M}$: For $p, q \in \mathcal{M}$ we define

$$\mathrm{d}(p, q) := \inf\{l(\mathrm{c}) : \mathrm{c}(0) = p \text{ and } \mathrm{c}(1) = q\}.$$

The multiscale decomposition in Riemannian manifolds We now explain the multiscale transformation associated with the so-called *log-exp analogue* of a linear subdivision scheme $\mathcal{S}$: It is defined by

$$(\mathcal{T}\mathbf{p})_{2i+k} := \exp_{p_i}\Big(\sum_{j\in\mathbb{Z}} a_{2i+k-2j}\log_{p_i}(p_j)\Big) =: p_i \oplus \big(\sum_{j\in\mathbb{Z}} a_{2i+k-j}p_j \ominus p_i\big),$$

where (a_i) is the mask of $\mathcal{S}$ and $i\in\mathbb{Z}$, $k\in\{0,1\}$. The multiscale transformation is set up as follows:

$$(\lambda^j)_i := \log_{(\mathcal{P}^j f)_i}((\mathcal{T}\mathcal{P}^{j-1})_i) = (\mathcal{T}\mathcal{P}^{j-1})_i \ominus (\mathcal{P}^j f)_i.$$

The reconstruction of a continuous function from data $(\mathcal{P}^0,\lambda^1,\lambda^2,\dots)$ is defined inductively by $\mathcal{P}^j = \mathcal{T}\mathcal{P}^{j-1}\oplus\lambda^j$. Observe that now each wavelet coefficient lies in a different vector space, namely $(\lambda^j)_i \in T_{(\mathcal{T}\mathcal{P}^{j-1})_i}\mathcal{M}$. The question now is how to compare two wavelet coefficients which lie in different tangent spaces. If a subdivision scheme $\mathcal{T}$ operates in $\mathcal{M}$, we can by the locality of the reconstruction procedure and the fact that our data becomes arbitrarily dense, pull the data back into Euclidean space via a chart φ. This chart induces a chart of $T\mathcal{M}$ via $(\varphi, d\varphi)$ and we can compare two wavelet coefficients in the Euclidean setting. The problem in using Theorem 4.6 directly with respect to this chart is that the quantities $\|\lambda^i - \tilde\lambda^i\|$ in Theorem 4.6 have no intrinsic meaning.

Therefore we need to find an intrinsic way of comparing wavelet coefficients living in different tangent spaces. Probably the most natural way is the following:

Definition 5.3. *Assume that $p,q\in\mathcal{M}$ can be connected by a unique geodesic c with $c(0)=p$ and $c(1)=q$. The distance $\mathbf{d}(v,w)$ between two tangent vectors $v\in T_p\mathcal{M}$ and $w\in T_q\mathcal{M}$ is defined as $\|Pt_0^1(c)v - w\|_q$, where $\|\cdot\|_q$ is the norm induced by the Riemannian metric in $T_q\mathcal{M}$.*

Remark 5.4. *The distance $\mathbf{d}$ is symmetric with respect to v and w. This follows from the fact that the parallel transport is an isometry.*

Now we can state our main stability theorem for the Riemannian log-exp analogue:

Theorem 5.5. *For all $\mathcal{M}$-valued functions $f\in Lip\ \gamma$ there exist $\delta_0,\varepsilon_0>0$ and a constant $C\geq 0$ such that for all initial data*

$$(\tilde{\mathcal{P}}^0,\tilde\lambda^1,\tilde\lambda^2,\dots)$$

with

$$d(\tilde{\mathcal{P}}^0,\mathcal{P}^0 f)_\infty \leq \delta_0 \quad \textit{and} \quad \mathbf{d}\big(\tilde\lambda^j,\lambda^j f\big)_\infty \leq \varepsilon_0/2^{\gamma j}, \quad j\in\mathbb{Z}^+, \tag{19}$$

the reconstruction yields a well-defined continuous function $\tilde f$ with

$$d(\mathcal{P}^n f,\mathcal{P}^n\tilde f)_\infty \;\leq\; C\bigg(d(\mathcal{P}^0 f,\tilde{\mathcal{P}}^0)_\infty + \sum_{j=1}^n \mathbf{d}\big(\lambda^j f,\tilde\lambda^j\big)_\infty\bigg). \tag{20}$$

The constants δ_0,ε_0,C are uniform for data values in a compact set.

Proof. Unlike the Lie group case, here we cannot simply invoke Theorem 4.6 to establish our result. This is because, by pulling back with respect to a chart and using 4.6 we lose the geometric meaning of the distance $\mathbf{d}$. The solution to this problem is as follows: Instead of assuming that all wavelet coefficients lie in different tangent spaces we regard the wavelet coefficients $(\tilde\lambda^j)_i$ as elements of $T_{(\mathcal{T}\mathcal{P}^{j-1}f)_i}\mathcal{M}$ instead of elements of $T_{(\mathcal{T}\tilde{\mathcal{P}}^{j-1})_i}\mathcal{M}$, the symbol $\tilde{\mathcal{P}}^j$ as usual denoting the j-th reconstruction step from the data $(\tilde{\mathcal{P}}^0,\tilde\lambda^1,\tilde\lambda^2,\dots)$. According to this interpretation we need to redefine the functions

$$\mathcal{P}^j\big(\tilde{\mathcal{P}}^0,\tilde\lambda^1,\dots,\tilde\lambda^j\big)_i := (\mathcal{P}^{j-1})_i \,\tilde\oplus\, (\tilde\lambda^j)_i,$$

where $i\in\mathbb{Z}$ and

$$p\,\tilde\oplus\, v := p\oplus \mathrm{Pt}_q^p(v) \quad \text{for} \quad v\in T_q\mathcal{M}.$$

Here Pt_q^p denotes the parallel transport along the unique shortest geodesic connecting q and p. Using this interpretation of the multiscale transformation the detail coefficients live in fixed vector spaces determined by the function f. This makes it possible to compare two vectors $(\lambda^j)_i$ and $(\tilde\lambda^j)_i$ (which now live in the same vector space) using the Riemannian metric in $(\mathcal{T}\mathcal{P}^{j-1})_i$. This difference is clearly the same as $\mathbf{d}((\lambda^j)_i,(\tilde\lambda^j)_i)$ where now $(\tilde\lambda^j)_i$ is interpreted as the usual detail coefficient living in $T_{(\mathcal{T}\tilde{\mathcal{P}}^{j-1})_i}\mathcal{M}$. We can thus directly apply our results from the previous section and repeat the proof of Theorem 5.1 to prove Theorem 5.5. $\square$

5.3 The projection analogue

We also want to say something about the so-called projection analogue [17]. It is defined by first applying a linear subdivision scheme $\mathcal{S}$ to data living in $\mathbb{R}^m$ and then applying a retraction mapping P, i.e. the projection analogue of $\mathcal{S}$ is defined via

$$\mathcal{T}\mathbf{p} := P \circ \mathcal{S}\mathbf{p}.$$

We consider the case where a (hyper-) surface $\mathcal{M}$ in $\mathbb{R}^m$ is given, and P is some smooth projection onto $\mathcal{M}$ defined locally around $\mathcal{M}$, for example the closest point-projection. The results of the previous sections cannot be directly applied to this particular nonlinear scheme, since $\mathcal{T}$ is only defined on $\mathcal{M}$-valued data. In Definition 3.1 we have explicitly asked for an open and convex domain of definition for $\mathcal{T}$. Neither of these requirements is satisfied by the projection analogue of a linear scheme. The proof that $\mathcal{T}$ is stable is however still almost the same: We can just repeat the arguments in Section 3 and replace every occurrence of a line $\mathrm{c}(t)$ connecting two points by the unique geodesic connecting the points. Since the norms of tangent vectors of a geodesic remain constant along the geodesic, all the arguments still go through. This implies that the subdivision operator $\mathcal{T} = P \circ \mathcal{S}$ is stable.

Another problem exists: The multiscale transformation associated with $\mathcal{T}$ is defined via

$$(\lambda^j f)_i = (\mathcal{P}^j f)_i - (\mathcal{T}\mathcal{P}^{j-1} f)_i. \tag{21}$$

Observe that the values for $(\lambda^j)_i$ cannot be chosen arbitrarily, but must be chosen such that $(\mathcal{T}\mathcal{P}^{j-1} f)_i + (\lambda^j)_i$ is in $\mathcal{M}$. The solution for this problem is easy: We replace (21) with

$$(\lambda^j f)_i = \log_{(\mathcal{T}\mathcal{P}^{j-1})_i}((\mathcal{P}^j f)_i).$$

It is well-known and easy to see that the two definitions are equivalent in that the norms of both definitions are proportional. Now the reconstruction procedure fits into the framework of the log-exp analogue for Riemannian manifolds and we can apply Theorem 5.5.

6 Concluding remarks

We conclude our work with a few remarks. First, we would like to mention that all our results remain valid in the following more general framework: Instead of defining $\mathcal{P}^n f := (f(i/2^j))_{i\in\mathbb{Z}}$, we can define

$$\bar{\mathcal{P}}^n f := \mathcal{U} \circ \mathcal{P}^n f,$$

where $\mathcal{U}$ is a local smoothing operator and

$$(\bar{\lambda}^j)_i := (\bar{\mathcal{P}}^j f)_i \ominus (\mathcal{T}\bar{\mathcal{P}}^{j-1} f)_i.$$

The purpose of defining such a decomposition is to reduce *aliasing* effects and to make the methods better suited for applications in noise-removal. As an example we mention [27], where $\mathcal{U}$ is a nonlinear average imputing operator. For the linear case compare also [11, 25].

As already mentioned in the introduction, our results also remain valid for any dilation factor > 1 with the same proofs. Also the extension of our results to the multivariate case is straightforward using the methods developed in [15]. A very natural question for future research is if the regularity conditions on $\mathcal{T}$ are really necessary. Example 3.3 shows that $\mathcal{T} \in \mathrm{Lip}\ 1$ is definitely necessary. Is this also sufficient? Does proximity and $\mathcal{T} \in \mathrm{Lip}\ 1$ imply stability? We do not know the answer.

As another possible direction for future research we would like to mention the recent work [22], where directionality is encompassed into the (linear) multiscale decomposition. It seems that these constructions can also be defined for manifold-valued data. It will be an interesting task to study the properties of these nonlinear decompositions.

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