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Letter to the Editor

On the stability of the PPH nonlinear multiresolution ☆

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Abstract

This letter is devoted to the stability of the so-called piecewise polynomial harmonic (PPH) multiresolution transform that belongs to the class of data dependent nonlinear multiresolution algorithms. The presentation of the PPH multiresolution as some specific perturbation of a linear multiresolution allows to establish a two step contraction property that leads first to a convergence result and finally to the stability.

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1. Introduction

Recently, various attempts to improve the classical linear multiresolutions of wavelet type have led to nonlinear multiresolutions. In such frameworks, few results for convergence and stability are available [4].

In [2], in the context of image compression, a new multiresolution has been presented. Using a tensorial product, this multiresolution is based on an univariate nonlinear multiresolution called PPH multiresolution. It has been analyzed in terms of convergence and stability of an associated subdivision scheme

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following an approach for data dependent multiresolutions introduced in [4]. Edge resolution, robustness with regard to texture or noise, accuracy and compression rate have been numerically investigated. All the results seem to indicate that, opposite to other nonlinear techniques, the PPH multiresolution is stable and can be applied without specific control of error as such introduced in [1].

The aim of this letter is to establish the stability of the PPH multiresolution that, due to nonlinearity, is not a consequence of the stability of the associated subdivision scheme. The key point for that will be to present the PPH multiresolution as some perturbation of a classical linear multiresolution following [7], [11], and [5].

The letter is organized as follows: In Section 2 we present the PPH multiresolution as some perturbation of a linear interpolatory multiresolution. In Section 3 we establish a two step contraction property, deduce a convergence result and finally prove the stability of the multiresolution.

2. PPH multiresolution

The PPH multiresolution is here presented in the framework of Harten's multiresolution [9] on the whole line.

In Harten's interpolatory multiresolution, one considers a set of nested bi-infinite regular grids:

$$X^k = \{x_j^k\}_{j\in\mathbb{Z}}, \quad x_j^k = j2^{-k},$$

where k is called a scale parameter. The point-value discretization operators are defined by

$$\mathcal{D}_k : f \in C(\mathbb{R}) \mapsto f^k = \left(f_j^k\right)_{j\in\mathbb{Z}} := \left(f\left(x_j^k\right)\right)_{j\in\mathbb{Z}} \in V^k, \tag{1}$$

where V^k is the space of real sequences and $C(\mathbb{R})$ the set of continuous functions on $\mathbb{R}$. A reconstruction operator $\mathcal{R}_k$ associated to this discretization is any right inverse of $\mathcal{D}_k$ on V^k which means that

$$\left(\mathcal{R}_k f^k\right)\left(x_j^k\right) = f_j^k = f\left(x_j^k\right). \tag{2}$$

The operator defined by $\mathcal{D}_{k+1}\mathcal{R}_k$ acts between the coarse scale (k) and the fine scale ($k+1$) and is called a prediction operator. It directly provides a subdivision scheme. However, since for most function f, $\mathcal{D}_{k+1}\mathcal{R}_k f^k \neq f^{k+1}$, details, called d^k, should be added to $\mathcal{D}_{k+1}\mathcal{R}_k f^k$ to recover f^{k+1}. The multiresolution transform (see [3] for more details) of f^L is the sequence $\{f^0, d^0, \ldots, d^{L-1}\}$.

The PPH multiresolution is related to a specific prediction, S_{PPH}, adapted to the presence of discontinuities. This prediction, as it is explained in the sequel, is based on a Piecewise Polynomial interpolation of degree 3 where an arithmetic mean of local differences has been substituted by an Harmonic mean. In this letter, we emphasize the presentation of S_{PPH} as a nonlinear perturbation of a linear interpolation subdivision scheme, namely $S_{\mathcal{L}}$, the centered Lagrange interpolation of degree 1 or $S_{\mathcal{LL}}$ the centered Lagrange interpolation of degree 3.

Most of the time, for a generic k, the letters f, g will stand for f^k, g^k while $\hat{f}$, $\hat{g}$ will stand for $S_{\mathrm{PPH}}(f^k)$, $S_{\mathrm{PPH}}(g^k)$.

To be more precise, we consider the set of points f_{j-1}, f_j, f_{j+1}, f_{j+2} corresponding to subsequent values at points $x_{j-1}, x_j, x_{j+1}, x_{j+2}$ of a regular grid X and describe the prediction of the value $\hat{f}_{2j+1}$ at the mid-point $(x_j + x_{j+1})/2 = x_{j+1/2}$. As classically, $\hat{f}_{2j+1}$ is defined as the value at $x_{j+1/2}$ of a polynomial $\tilde{P}_j$ and therefore we focus on the definition of $\tilde{P}_j$.

Introducing the differences $Df_j = f_{j+1} - 2f_j + f_{j-1}$, according to [2], $\tilde{P}_j$ is the polynomial of degree 3 defined by

$$\begin{cases} \tilde{P}_j(x_l) = f_l & \text{for } j-1 \leqslant l \leqslant j+1, \\ \tilde{P}_j(x_{j+2}) = \tilde{f}_{j+2}, \end{cases} \tag{3}$$

with

$$\tilde{f}_{j+2} = f_{j+1} + f_j - f_{j-1} + 2H(Df_j, Df_{j+1}),$$

where H is defined by

$$(x, y) \in \mathbb{R}^2 \mapsto H(x, y) := \frac{xy}{x+y}\big(\operatorname{sign}(xy) + 1\big), \tag{4}$$

where $\operatorname{sign}(x) = 1$ if $x \geqslant 0$ and $\operatorname{sign}(x) = -1$ if $x < 0$.

Remarking that $f_{j+2} = f_{j+1} + f_j - f_{j-1} + 2(Df_{j+1} + Df_j)/2$, the PPH prediction operator clearly appears as some perturbation of the degree 3 centered Lagrange interpolation.

It is also useful to remark that we obtain

$$\hat{f}_{2j+1} = \frac{f_j + f_{j+1}}{2} - \frac{1}{8}H(Df_{j+1}, Df_j). \tag{5}$$

Indeed, it has to be compared with what is obtained using the degree 3 centered Lagrange interpolation [6] that writes

$$\big(S_{\mathcal{LC}}(f)\big)_{2j+1} = \frac{f_j + f_{j+1}}{2} - \frac{1}{8}\frac{D_{j+1} + D_j}{2}. \tag{6}$$

Here, both the PPH prediction and the degree 3 centered Lagrange interpolation appear as some perturbations (nonlinear for the PPH, linear for the degree 3 centered Lagrange interpolation) of the linear centered interpolation of degree 1.

A full description of the original PPH scheme is available in [2].

Remark 1. Thanks to an anonymous reviewer, we realized during the review of this letter that our PPH scheme has been introduced independently by F. Kuijt and R. van Damme [10] and M.S. Floater and C.A. Micchelli [8] in the framework of convexity preserving subdivision schemes. In [10], the PPH subdivision scheme is mentioned as an example of interpolatory subdivision scheme converging, for any initial sequence toward a limit function that is piecewise convex (concave) and continuously differentiable.

3. Stability of the PPH multiresolution

Before establishing the stability, we need the two following technical lemmas that deal with the function H defined above.

Lemma 1. *For any couples $(x, y), (x', y') \in \mathbb{R}^2$, the function H satisfies the following properties*:

(1) $H(x, y) = H(y, x)$,
(2) $H(x, y) = 0$ *if* $xy \leqslant 0$,

(3) $H(-x,-y) = -H(x,y)$,
(4) $|H(x,y)| \leqslant \max(|x|,|y|)$,
(5) $|H(x,y)| \leqslant 2\min(|x|,|y|)$,
(6) $|H(x,y) - H(x',y')| \leqslant 2\max\{|x-x'|,|y-y'|\}$.

Proof.

- The claims (1)–(5) are obvious.
- We now prove property (6) considering the following cases:
  (a) $xx' \leqslant 0$ and $yy' \leqslant 0$: We obtain

$$\begin{aligned}\left|H(x,y) - H(x',y')\right| &\leqslant \left|H(x,y)\right| + \left|H(x',y')\right| \leqslant \max\left\{|x|,|y|\right\} + \max\left\{|x'|,|y'|\right\}\\ &\leqslant 2\max\left\{|x-x'|,|y-y'|\right\}.\end{aligned}$$

  (b) $xx' > 0$ and $yy' \leqslant 0$ (by symmetry, this covers also the case $xx' \leqslant 0$ and $yy' > 0$): This implies either $xy \geqslant 0$ and $x'y' \leqslant 0$ or $xy \leqslant 0$ and $x'y' \geqslant 0$. In the first subcase $H(x',y') = 0$ and

$$\left|H(x,y) - H(x',y')\right| = \left|H(x,y)\right| \leqslant 2\min\left(|x|,|y|\right) \leqslant 2|y-y'|,$$

  similarly in the second case.
  (c) $xx' > 0$ and $yy' > 0$: By property (3) we can assume $x, x' > 0$ (otherwise, change the sign of all 4 variables). If $y, y' < 0$, there is nothing to prove ($H(x,y) = H(x',y') = 0$). If also $y, y' > 0$ then the inequality follows from:

$$\begin{aligned}H(x,y) - H(x',y') = \frac{2xy(x'+y') - 2x'y'(x+y)}{(x+y)(x'+y')} &= \frac{2xx'}{(x+y)(x'+y')}(y-y')\\ &+ \frac{2yy'}{(x+y)(x'+y')}(x-x'),\end{aligned}$$

  and the observation that $xx' + yy' < (x+y)(x'+y')$. □

Remark 2. The properties (3), (5), and (6) of Lemma 1 will be fundamental for the proof of Proposition 1. Note that a stronger property (6), with a constant less than 2, not true in our case, would have been a good substitute.

Lemma 2. *The function Z defined on $\mathbb{R}^3$ by $Z(x,y,z) = x/2 - (1/8)(H(x,y) + H(x,z))$ satisfies the following properties*:

(1) $|Z(x,y,z)| \leqslant |x|/2$,
(2) $\mathrm{sign}(Z(x,y,z)) = \mathrm{sign}(x)$,
(3) $|Z(x,y,z) - Z(x',y',z')| \leqslant (1/2)|x-x'| + (1/2)\max\{|y-y'|,|z-z'|\}$.

Proof.

- Remarking that $0 \leqslant y/(x+y)(\mathrm{sign}(xy)+1) \leqslant 2$ we get (1) and (2).
- To prove (3), we set $Z = Z(x,y,z)$, $Z' = Z(x',y',z')$. First of all, we may assume $x \geqslant 0$ since $Z(-x,-y,-z) = -Z$. Then we consider the following cases:

(a) $x = 0$ or $x' \leqslant 0$: In either case,

$$|Z - Z'| \leqslant |Z| + |Z'| \leqslant \frac{|x| + |x'|}{2} = \frac{|x - x'|}{2}.$$

(b) $x, x' > 0$: We write

$$Z - Z' = \frac{x - x'}{2} - \frac{1}{8}\big(H(x, y) - H(x', y')\big) - \frac{1}{8}\big(H(x, z) - H(x', z')\big) \equiv A - B - C.$$

Now we note from the proof of property (6) of Lemma 1 that (because $x, x' > 0$) if at least one of the numbers y, y' is nonpositive, then $|B| \leqslant |y - y'|/4$ (and similarly for z, z' and C). But if $y, y' > 0$ we get from Eq. (7)

$$B = \frac{xx'}{4(x + y)(x' + y')}(y - y') + \frac{yy'}{4(x + y)(x' + y')}(x - x') \equiv B' + B'',$$

where the first term satisfies $|B'| \leqslant |y - y'|/4$ and the second has the same sign as A and satisfies $|B''| \leqslant |A|/2$ (and thus can be combined with A in a favorable way). Similarly for z, z' and C. Thus, in summary

$$|Z - Z'| \leqslant |A - B'' - C''| + \frac{|y - y'|}{4} + \frac{|z - z'|}{4} \leqslant \frac{1}{2}\big(|x - x'| + \max\big(|y - y'|, |z - z'|\big)\big),$$

where B'' and C'' are set to zero in all other cases but $y, y' > 0$ (resp. $z, z' > 0$). $\square$

We then focus on the subdivision schemes S_{PPH} associated to the PPH prediction that writes

$$f^{k-1} \mapsto S_{\mathrm{PPH}}\big(f^{k-1}\big) = \mathcal{D}_k \mathcal{R}_{k-1} f^{k-1},$$

with

$$\begin{cases} (\mathcal{D}_k \mathcal{R}_{k-1} f^{k-1})_{2j+1} = \tilde{P}_j(x^k_{j+1/2}), \\ (\mathcal{D}_k \mathcal{R}_{k-1} f^{k-1})_{2j} = f^{k-1}_j. \end{cases} \tag{7}$$

We have the following two step contraction property:

Proposition 1. *If, removing k for simplicity, $\hat{f} = S_{\mathrm{PPH}}(f)$, $\hat{g} = S_{\mathrm{PPH}}(g)$, $\bar{f} = S_{\mathrm{PPH}}(\hat{f})$ and $\bar{g} = S_{\mathrm{PPH}}(\hat{g})$ then*

$$\begin{aligned} &(1)\ \|D\hat{f}\|_\infty \leqslant \frac{1}{2}\|Df\|_\infty, \\ &(2)\ \big|D(\hat{f}_j - \hat{g}_j)\big| \leqslant \frac{1}{2}\big\|D(f - g)\big\|_\infty \quad \textit{for } j = 2n + 1, \\ &\big|D(\hat{f}_j - \hat{g}_j)\big| \leqslant \big\|D(f - g)\big\|_\infty \quad \textit{for } j = 2n, \quad \textit{and} \\ &(3)\ \big\|D(\bar{f} - \bar{g})\big\|_\infty \leqslant \frac{3}{4}\big\|D(f - g)\big\|_\infty. \end{aligned} \tag{8}$$

Proof.

- We first prove (1), i.e. for all j,

  $$|D\hat{f}_j| \leqslant \frac{1}{2}\|Df\|_\infty.$$

  Even and odd values of j should be treated separately:

  (a) $j = 2n + 1$.
  Since the prediction is interpolatory, we have

  $$\hat{f}_{j+1} - 2\hat{f}_j + \hat{f}_{j-1} = f_{n+1} - 2\hat{f}_j + f_n,$$

  with, (5), $\hat{f}_j = (f_n + f_{n+1})/2 - (1/8)H(Df_{n+1}, Df_n)$.
  Thanks to property (4) of Lemma 1, we get

  $$|f_{n+1} - 2\hat{f}_j + f_n| \leqslant \frac{1}{4}\max\{|Df_{n+1}|, |Df_n|\} \leqslant \frac{1}{4}\|Df\|_\infty.$$

  (b) $j = 2n$.
  Again, due to the interpolatory property, we get

  $$\hat{f}_{j+1} - 2\hat{f}_j + \hat{f}_{j-1} = \hat{f}_{j+1} - 2f_n + \hat{f}_{j-1}.$$

  From the definition of the PPH subdivision scheme, we get

  $$\hat{f}_{j+1} - 2f_n + \hat{f}_{j-1} = Z(Df_n, Df_{n+1}, Df_{n-1}),$$

  and property (1) of Lemma 2 gives the result.
- The proof for (2) works similarly:

  (a) $j = 2n + 1$.
  We get

  $$|f_{n+1} - 2\hat{f}_j + f_n - g_{n+1} + 2\hat{g}_j - g_n| = \frac{1}{4}\big|H(Df_{n+1}, Df_n) - H(Dg_{n+1}, Dg_n)\big|,$$

  and from property (6) of Lemma 1 we get the result.

  (b) $j = 2n$.
  It is a consequence of the property (3) of Lemma 2.
- To prove (3), thanks to (2), we only have to consider $j = 2n$. Then,

  $$\begin{aligned}\big|D(\bar{f}_j - \bar{g}_j)\big| &= \big|\bar{f}_{j+1} - 2\bar{f}_{2n} + \bar{f}_{j-1} - (\bar{g}_{j+1} - 2\bar{g}_{2n} + \bar{g}_{j-1})\big| \\ &= \big|Z(D\hat{f}_n, D\hat{f}_{n+1}, D\hat{f}_{n-1}) - Z(D\hat{g}_n, D\hat{g}_{n+1}, D\hat{g}_{n-1})\big|,\end{aligned}$$

  and the conclusion is a direct consequence of (2) and property (3) of Lemma 2 noting that either n or $n-1$ and $n+1$ are odd. □

Remark 3. The following example, $(Df_n, Df_{n+1}, Df_{n-1}) = (M+1, 0, 0)$ and $(Dg_n, Dg_{n+1}, Dg_{n-1}) = (M, 1, 1)$ with $M \to +\infty$, shows that a single step contraction property in the sense of property (2.a) ($j = 2n + 1$) of Proposition 1 is not available for $j = 2n$.

We are now able to derive directly the convergence of the subdivision scheme[2] S_{PPH}, applying Theorem 3.3 of [5]. In our context, this theorem applies as follows: If S_L is a converging linear subdivision scheme with Hölder smoothness s_L, reproducing polynomials up to degree P, if S_N is a perturbation of S_L in the sense that, calling $f^k := S_N(f^0)$ for all $f^0 \in l_\infty$,

$$\big\|S_N\big(f^k\big) - S_L\big(f^k\big)\big\|_\infty = O\big(2^{-\nu k}\big),$$

then S_N is convergent with an Hölder smoothness $s_N \geqslant \min(P, s_L, \nu) - \delta$ for all $\delta > 0$.

Indeed, if we choose $S_{\mathcal{L}}$, the linear interpolatory subdivision scheme for S_L (see (5)), we have $s_L = 1$ and from property (4) of Lemma 1 and Proposition 1, $\nu = 1$. With $P = 1$ we obtain the convergence of S_{PPH} with Hölder regularity $1 - \delta$.

Remark 4. It is shown in [10] that the limit curves obtained under S_{PPH} are in the worst case at most Lipschitz continuous ($s = 1$) as it can be observed considering the Dirac delta sequence $f^0 = (\ldots, 0, 0, 1, 0, 0, \ldots)$. Indeed, its limit curve is a piecewise linear hat function.

Remark 5. Using other tracks, a general approach in [4] and [2] or convexity preservation in [10] and [8], the convergence for the subdivision scheme S_{PPH} has been established. However, this property is not sufficient to ensure the stability of the associated multiresolution. The ENO-interpolatory subdivision scheme [4], constructed, as the PPH, in the Harten's framework and involving data dependent linear interpolation, is an example of a converging but unstable nonlinear subdivision scheme.

We now consider the PPH multiresolution. We first give, without proof since it is similar to the proof of Proposition 1, the following result involving the details (see Section 2) $d(f), d(g)$ and $d(\dot f), d(\dot g)$.

Proposition 2. *If, removing k for simplicity,* $\dot f = S_{\mathrm{PPH}}(f) + d(f)$, $\dot g = S_{\mathrm{PPH}}(g) + d(g)$, $\ddot f = S_{\mathrm{PPH}}(\dot f) + d(\dot f)$ *and* $\ddot g = S_{\mathrm{PPH}}(\dot g) + d(\dot g)$ *then*

$$\begin{aligned}
&(1)\ \ \|D\dot f\|_\infty \leqslant \frac{1}{2}\|Df\|_\infty + 2\big\|Dd(f)\big\|_\infty,\\
&(2)\ \ \big|D(\dot f_j - \dot g_j)\big| \leqslant \frac{1}{2}\big\|D(f-g)\big\|_\infty + 2\big\|Dd(f) - Dd(g)\big\|_\infty \quad \textit{for } j = 2n+1,\\
&\qquad \big|D(\dot f_j - \dot g_j)\big| \leqslant \big\|D(f-g)\big\|_\infty + 2\big\|Dd(f) - Dd(g)\big\|_\infty \quad \textit{for } j = 2n, \quad \textit{and}\\
&(3)\ \ \big\|D(\ddot f - \ddot g)\big\|_\infty \leqslant \frac{3}{4}\big\|D(f-g)\big\|_\infty + 2\big\|Dd(f) - Dd(g)\big\|_\infty + 2\big\|Dd(\dot f) - Dd(\dot g)\big\|_\infty.
\end{aligned} \tag{9}$$

We are then able to prove the following theorem related to the stability of the PPH reconstruction $\{f^0, d^0, \ldots, d^{L-1}\} \mapsto f^L$.

Theorem 1. *For any pair of elements* f^L, $\tilde f^L \in l_\infty(\mathbb{Z})$ *and their PPH decompositions* $\{f^0, d^0, \ldots, d^{L-1}\}$ *and* $\{\tilde f^0, \tilde d^0, \ldots, \tilde d^{L-1}\}$, *we have*

$$\big\|f^L - \tilde f^L\big\|_\infty \leqslant 9\big\|f^0 - \tilde f^0\big\|_\infty + 18\sum_{k=0}^{L-1}\big\|d^k - \tilde d^k\big\|_\infty. \tag{10}$$

[2] A subdivision scheme S is called convergent with Hölder smoothness s if, for all sequences $f^0 \in l_\infty$, the sequence of piecewise linear functions ϕ^k interpolating the points f_j^k at x_j^k converges to a function ϕ of Hölder regularity s.

Proof. Using the definition of the PPH multiresolution $f^{k+1} = S_{\mathrm{PPH}}(f^k) + d^k$ or, more precisely,

$$f_{2j}^{k+1} = f_j^k, \qquad f_{2j+1}^{k+1} = \frac{f_j^k + f_{j+1}^k}{2} - \frac{1}{8} H\big(Df_{j+1}^k, Df_j^k\big) + d_{2j+1}^k,$$

we have

$$\|f^{k+1} - \tilde{f}^{k+1}\|_\infty \leqslant \|f^k - \tilde{f}^k\|_\infty + \frac{1}{4}\|D\big(f^k - \tilde{f}^k\big)\|_\infty + \|d^k - \tilde{d}^k\|_\infty,$$

by subtracting the similar expression for $\tilde{f}$, and using norm inequalities and property (6) of Lemma 1 correspondingly.

Thus,

$$\|f^L - \tilde{f}^L\|_\infty \leqslant \|f^0 - \tilde{f}^0\|_\infty + \frac{1}{4}\sum_{k=0}^{L-1}\|D\big(f^k - \tilde{f}^k\big)\|_\infty + \sum_{k=0}^{L-1}\|d^k - \tilde{d}^k\|_\infty. \tag{11}$$

Using property (3) of Proposition 2 we can derive, according to the parity of k, the following inequalities:

(a) If $k = 2l$, $l \geqslant 1$ then

$$\begin{aligned}\|D\big(f^k - \tilde{f}^k\big)\|_\infty \leqslant \left(\frac{3}{4}\right)^l \|D\big(f^0 - \tilde{f}^0\big)\|_\infty + 2\sum_{m=1}^{l}\left(\frac{3}{4}\right)^{m-1}\big(\|D\big(d^{2l-2m} - \tilde{d}^{2l-2m}\big)\|_\infty \\ + \|D\big(d^{2l-2m+1} - \tilde{d}^{2l-2m+1}\big)\|_\infty\big).\end{aligned}$$

(b) If $k = 2l+1$, $l \geqslant 0$ then

$$\begin{aligned}\|D\big(f^k - \tilde{f}^k\big)\|_\infty \leqslant \left(\frac{3}{4}\right)^l \|D\big(f^0 - \tilde{f}^0\big)\|_\infty + 2\Bigg(\sum_{m=1}^{l}\left(\frac{3}{4}\right)^{m-1}\big(\|D\big(d^{2l+1-2m} - \tilde{d}^{2l+1-2m}\big)\|_\infty \\ + \|D\big(d^{2l-2m+2} - \tilde{d}^{2l-2m+2}\big)\|_\infty\big) + \left(\frac{3}{4}\right)^l \|D\big(d^0 - \tilde{d}^0\big)\|_\infty\Bigg).\end{aligned}$$

Summing over k, plugging into Eq. (11) and using

$$\sum_{n=0}^{\infty}\left(\frac{3}{4}\right)^n = 4,$$

and

$$\|D(g - \tilde{g})\|_\infty \leqslant 4\|g - \tilde{g}\|_\infty,$$

we get the announced estimate. □

Finally, we have a last theorem related to the stability of the PPH decomposition $f^L \mapsto \{f^0, d^0, \dots, d^{L-1}\}$.

Theorem 2. *Given two PPH decompositions* $\{f^0, d^0, \ldots, d^{L-1}\}$ *and* $\{\tilde{f}^0, \tilde{d}^0, \ldots, \tilde{d}^{L-1}\}$, *corresponding to* $f^L, \tilde{f}^L \in l_\infty(\mathbb{Z})$, *then*

$$\|f^0 - \tilde{f}^0\|_\infty \leqslant \|f^L - \tilde{f}^L\|_\infty, \qquad \|d^k - \tilde{d}^k\|_\infty \leqslant 3\|f^L - \tilde{f}^L\|_\infty \quad \forall 0 \leqslant k \leqslant L-1.$$

Proof. The first inequality comes directly from the interpolatory property. For the second we write:

$$\begin{aligned}
|d_j^k - \tilde{d}_j^k| &\leqslant \|f^{k+1} - \tilde{f}^{k+1}\|_\infty + \|S_{\mathrm{PPH}}(f^k) - S_{\mathrm{PPH}}(\tilde{f}^k)\|_\infty \\
&\leqslant \|f^{k+1} - \tilde{f}^{k+1}\|_\infty + \|f^k - \tilde{f}^k\|_\infty + \frac{1}{4}\|D(f^k - \tilde{f}^k)\|_\infty \\
&\leqslant \|f^{k+1} - \tilde{f}^{k+1}\|_\infty + 2\|f^k - \tilde{f}^k\|_\infty \leqslant 3\|f^L - \tilde{f}^L\|_\infty. \qquad \square
\end{aligned}$$

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