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Polynomial reproduction in subdivision

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Dedicated to Professor Franz Locher on his sixtieth birthday.

We study conditions on the matrix mask of a vector subdivision scheme ensuring that certain polynomial input vectors yield polynomial output again. The conditions are in terms of a recurrence formula for the vectors which determine the structure of polynomial input with this property. From this recurrence, we obtain an algorithm to determine polynomial input of maximal degree. The algorithm can be used in the design of masks to achieve a high order of polynomial reproduction.

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1. Preliminaries and notation

We use the notation of the survey paper [11]. Let $\mathbf{\Phi} = (\phi_1, \ldots, \phi_n)^{\mathrm{T}}$ be a (column) vector of $L_2(\mathbb{R}^d)$-functions, which is refinable. The refinement mask $\mathbf{P} = (\mathbf{P}_{\boldsymbol{\alpha}})_{\boldsymbol{\alpha}\in\mathbb{Z}^d}$ consisting of a sequence of complex $(n \times n)$-matrices $\mathbf{P}_{\boldsymbol{\alpha}}$ is assumed to be finitely supported (i.e., $\mathbf{P}_{\boldsymbol{\alpha}} = \mathbf{0}$ except for finitely many $\boldsymbol{\alpha} \in \mathbb{Z}^d$), and refinement refers to a *two-scale equation*

$$\mathbf{\Phi}\left(\frac{1}{2}\mathbf{x}\right) = \sum_{\boldsymbol{\alpha}} \mathbf{P}_{\boldsymbol{\alpha}} \mathbf{\Phi}(\mathbf{x} - \boldsymbol{\alpha}). \tag{1.1}$$

In the scalar case where $\mathbf{\Phi} = \phi \in L_2(\mathbb{R}^d)$, the refinement mask is a scalar sequence $p = (p_{\boldsymbol{\alpha}})_{\boldsymbol{\alpha}\in\mathbb{Z}^d}$, and the refinement equation reads

$$\phi\left(\frac{1}{2}\mathbf{x}\right) = \sum_{\boldsymbol{\alpha}} p_{\boldsymbol{\alpha}} \phi(\mathbf{x} - \boldsymbol{\alpha}).$$

Here, and in what follows, the short-hand indexing $\sum_{\boldsymbol{\alpha}}$ of a sum indicates that the sum is extended over the lattice $\mathbb{Z}^d$.

Associated with the refinement mask are two operators [13]. The *transfer operator* $T = T_{\mathbf{P}}$ operates on $(\ell_0(\mathbb{Z}^d))^n$, the space of vector-valued sequences $\mathbf{d} = (\mathbf{d}_\alpha)_{\alpha\in\mathbb{Z}^d}$ of compact support, according to the rule

$$\mathbf{d} \mapsto T\mathbf{d} \quad \text{with } (T\mathbf{d})_\alpha = \sum_\beta \mathbf{P}_{2\alpha-\beta}\mathbf{d}_\beta,\ \alpha \in \mathbb{Z}^d.$$

The *subdivision operator* $S = S_{\mathbf{P}}$ maps the space $(\ell(\mathbb{Z}^d))^n$ of vector-valued sequences $\mathbf{c} = (\mathbf{c}_\alpha)_{\alpha\in\mathbb{Z}^d}$ into itself according to

$$\mathbf{c} \mapsto S\mathbf{c} \quad \text{with } (S\mathbf{c})_\alpha = \sum_\beta \mathbf{P}^*_{\alpha-2\beta}\mathbf{c}_\beta,\ \alpha \in \mathbb{Z}^d. \tag{1.2}$$

Here, $\mathbf{P}^*_\alpha := \overline{\mathbf{P}}^{\mathrm{T}}_\alpha$ denotes the conjugate-transpose of the matrix $\mathbf{P}_\alpha$.

In this paper we study the question under what conditions a polynomial sequence $\mathbf{c}$ is transformed into a polynomial sequence by the subdivision operator. A sequence $\mathbf{c} = (\mathbf{c}_\alpha)_{\alpha\in\mathbb{Z}^d} \in (\ell(\mathbb{Z}^d))^n$ is called *polynomial of type* Π_N*, or of type* $\Pi_{\mathbf{N}}$ *with* $\mathbf{N} = (N_1, \ldots, N_d)$, if there exists a vector-valued function $\mathbf{f} = (f_1, \ldots, f_n)^{\mathrm{T}}$ with all components being polynomials (of type Π_N, or of type $\Pi_{\mathbf{N}}$, respectively) such that

$$\mathbf{c}_\alpha = \mathbf{f}(\alpha) \quad \text{for all } \alpha \in \mathbb{Z}^d.$$

Here, we use two notions of degree for polynomials, namely, the *total degree* $N \in \mathbb{N}$,

$$f \in \Pi_N \iff f(\mathbf{z}) = \sum_{\mathbf{n}\in\mathbb{N}^d,\ |\mathbf{n}|\leqslant N} a_{\mathbf{n}}\mathbf{z}^{\mathbf{n}} \tag{1.3}$$

(with $\mathbf{n} = (n_1, \ldots, n_d)$ and $|\mathbf{n}| = n_1+\cdots+n_d$), and the *coordinate degree* $\mathbf{N} = (N_1, \ldots, N_d) \in \mathbb{N}^d$,

$$f \in \Pi_{\mathbf{N}} \iff f(\mathbf{z}) = \sum_{\mathbf{n}\in\mathbb{N}^d, n_i\leqslant N_i} a_{\mathbf{n}}\mathbf{z}^{\mathbf{n}}. \tag{1.4}$$

Thus

$$\Pi_N = \bigcup_{|\mathbf{N}|=N} \Pi_{\mathbf{N}}.$$

The question of reproduction of polynomials f by integer translates of refinable functions, or function vectors, in terms of the infinite series

$$f(\mathbf{x}) = \sum_\alpha \mathbf{c}^*_\alpha \boldsymbol{\Phi}(\mathbf{x}-\alpha),$$

plays a prominent role in the study of approximation orders from shift-invariant spaces [12,14–17]. If such a representation exists for any polynomial f of total degree less than N (for given $N \in \mathbb{N}$), the function vector is said *to have accuracy* N, following the notation coined by Heil et al., see, e.g., [1,10]. To our knowledge, accuracy order is (in general) not enough to yield approximation order. Any statement on approximation order still involves some type of (although often rather weak) assumption of 'independence' or 'stability' on the system $\mathbf{\Phi}$, let alone the assumption on the Gramian as, e.g., given in [17, theorem 3.6], and is not based on immediate properties of the mask $\mathbf{P}$ alone. The same assumption usually comes up in the context of convergence of subdivision [2,6].

This additional assumption is not needed if we study just one subdivision step. The present paper thus deals with the question under what condition a polynomial sequence $\mathbf{c}$ is transformed into another polynomial sequence in the sense that – under rescaling – we have

$$(S\mathbf{c})_{\boldsymbol{\alpha}} = \tilde{f}(\boldsymbol{\alpha}/2), \quad \boldsymbol{\alpha} \in \mathbb{Z}^d,$$

for another polynomial $\tilde{f}$. We should point to the fact that the same problem was studied by Han in [9, section 2], using a different approach. He obtained a characterization of polynomial reproduction in terms of the so-called sum rules, and by zero properties of the mask symbol. The advantage of our approach is that the characterization is constructive in the sense that we devise an algorithm which allows us – in particular in the vector-valued case – to recursively determine the order of polynomial reproduction. In addition, the results of sections 3.1 and 3.3 as well as the second example in section 4 are completely new.

In the paper we use the following notation and symbols. Vectors and matrices are usually printed boldface, and elements from the lattice $\mathbb{Z}^d$ are denoted by $\boldsymbol{\alpha}$ or $\boldsymbol{\beta}$. We heavily rely on usual multi-index notation. For $\mathbf{z} = (z_1, \ldots, z_d) \in \mathbb{C}^d$ and $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_d) \in \mathbb{Z}^d$ we put $\mathbf{z}^{\boldsymbol{\alpha}} = z_1^{\alpha_1} \cdots z_d^{\alpha_d}$, and $|\boldsymbol{\alpha}| = |\alpha_1| + \cdots + |\alpha_d|$. Restricting $\mathbf{z}$ to the torus, i.e., putting $z_k = \mathrm{e}^{\mathrm{i}\xi_k}$, $k = 1, \ldots, d$, where $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_d) \in \mathbb{R}^d$, we have $\mathbf{z}^{\boldsymbol{\alpha}} = \mathrm{e}^{\mathrm{i}\boldsymbol{\alpha}\cdot\boldsymbol{\xi}}$ with the dot standing for the usual inner product in $\mathbb{R}^d$. For a scalar λ, we put $\lambda^{\boldsymbol{\alpha}} = \lambda^{\alpha_1+\cdots+\alpha_d}$. Thus in case $\alpha_1, \ldots, \alpha_d$ are all non-negative, $\lambda^{\boldsymbol{\alpha}}$ and $\lambda^{|\boldsymbol{\alpha}|}$ denote the same number. The Fourier transform of an $L_1(\mathbb{R}^d)$-function f is given by the continuous function

$$f^\wedge(\boldsymbol{\xi}) = \int_{\mathbb{R}^d} f(\mathbf{x})\mathrm{e}^{-\mathrm{i}\boldsymbol{\xi}\cdot\mathbf{x}}\,\mathrm{d}\mathbf{x}, \tag{1.5}$$

and the notion of Fourier transform extends in the usual way to tempered distributions. In particular, the δ-distribution (evaluating a test function φ at the origin) has the Fourier transform $\delta^\wedge = 1$, and any monomial can be considered as the Fourier transform of the (distributional) derivative of δ according to the pair of formulas

$$\bigl(D^{\mathbf{n}}\delta\bigr)^\wedge(\boldsymbol{\xi}) = (\mathrm{i}\boldsymbol{\xi})^{\mathbf{n}} \quad \text{and} \quad D^{\mathbf{n}}\delta = \frac{1}{(2\pi)^d}\bigl\{(-\mathrm{i}\boldsymbol{\xi})^{\mathbf{n}}\bigr\}^\wedge. \tag{1.6}$$

Here, $\mathbf{n} = (n_1, \ldots, n_d) \in \mathbb{N}^d$, and $D^{\mathbf{n}}$ is short for the usual mixed partial derivative. We shall make use of the formula

$$f \cdot D^{\mathbf{n}}\delta_{\mathbf{x}_0} = \sum_{\mathbf{0}\leqslant\boldsymbol{\nu}\leqslant\mathbf{n}} \binom{\mathbf{n}}{\boldsymbol{\nu}}(-1)^{\mathbf{n}-\boldsymbol{\nu}}\bigl\{D^{\mathbf{n}-\boldsymbol{\nu}}f(\mathbf{x}_0)\bigr\} \cdot D^{\boldsymbol{\nu}}\delta_{\mathbf{x}_0} \tag{1.7}$$

which applies to the product of derivatives of $\delta_{\mathbf{x}_0}$ (evaluation at $\mathbf{x}_0 \in \mathbb{R}^d$) with any C^∞-function f. Here, $\mathbf{0} = (0, \ldots, 0)$, the sum extends over all $\boldsymbol{\nu} = (\nu_1, \ldots, \nu_d) \in \mathbb{Z}^d$ with $0 \leqslant \nu_i \leqslant n_i$, and $\binom{\mathbf{n}}{\boldsymbol{\nu}} = \mathbf{n}!/(\boldsymbol{\nu}!(\mathbf{n}-\boldsymbol{\nu})!)$ is the usual multivariate binomial coefficient based on the multivariate factorial $\mathbf{n}! = n_1! \cdots n_d!$. We shall also use the formula

$$D^{\mathbf{j}}\delta(2\boldsymbol{\xi}) = 2^{-|\mathbf{j}|}\frac{1}{2^d}\sum_{\mathbf{e}\in E} D^{\mathbf{j}}\delta_{\pi\mathbf{e}}, \tag{1.8}$$

where

$$E := \{0, 1\}^d$$

denotes the set of representatives for the cosets of $\mathbb{Z}^d/2\mathbb{Z}^d$.

The z-transform of a vector-valued sequence $\mathbf{c}$ (with compact support) is the vector-valued formal Laurent series (rational function, respectively)

$$\mathbf{c}(\mathbf{z}) := \sum_{\boldsymbol{\alpha}} \mathbf{c}_{\boldsymbol{\alpha}}\mathbf{z}^{-\boldsymbol{\alpha}},$$

and its restriction to the torus $[0, 2\pi[^d$ is the Fourier transform of the sequence

$$\mathbf{c}^\wedge(\boldsymbol{\xi}) = \sum_{\boldsymbol{\alpha}} \mathbf{c}_{\boldsymbol{\alpha}}\mathrm{e}^{-\mathrm{i}\boldsymbol{\alpha}\cdot\boldsymbol{\xi}},$$

which in case of (at most polynomially) growing sequences has to be interpreted in the distributional sense. For example, the scalar-valued monomial sequence

$$m_{\mathbf{n}} = \bigl((-\mathrm{i}\boldsymbol{\alpha})^{\mathbf{n}}\bigr)_{\boldsymbol{\alpha}\in\mathbb{Z}^d} \quad \text{transforms into } m_{\mathbf{n}}^\wedge(\boldsymbol{\xi}) = D^{\mathbf{n}}\delta \tag{1.9}$$

with δ denoting here the delta distribution on the torus (evaluation at the origin of C^∞-functions which are 2π-periodic in each variable).

This notion extends to matrix-valued sequences. For the refinement mask $\mathbf{P}$ we get the Fourier transform

$$\mathbf{P}^\wedge(\boldsymbol{\xi}) := \sum_{\boldsymbol{\alpha}} \mathbf{P}_{\boldsymbol{\alpha}}\mathrm{e}^{-\mathrm{i}\boldsymbol{\alpha}\cdot\boldsymbol{\xi}}, \tag{1.10}$$

which is an $(n \times n)$-matrix of trigonometric polynomials. It is a multiple of the refinement mask symbol $\mathbf{H}$ as defined in [11], namely,

$$\mathbf{H}(\boldsymbol{\xi}) = \frac{1}{2^d}\mathbf{P}^\wedge(\boldsymbol{\xi}).$$

From definition (1.2) of the subdivision operator we get the following formula for the Fourier transforms

$$(S\mathbf{c})^\wedge(\boldsymbol{\xi}) = \bigl(\mathbf{P}^*\bigr)^\wedge(\boldsymbol{\xi})\mathbf{c}^\wedge(2\boldsymbol{\xi}) \tag{1.11}$$

which makes perfect sense in case the sequence $\mathbf{c}$ is tempered, i.e., grows at most polynomially (and then $S\mathbf{c}$ is tempered again, with the same order of growth). Since $(\mathbf{P}^*)^\wedge(\boldsymbol{\xi}) = (\mathbf{P}^\wedge)^*(-\boldsymbol{\xi})$, we have the equivalent identities

$$(S\mathbf{c})^\wedge(\boldsymbol{\xi}) = \left(\mathbf{P}^\wedge\right)^*(-\boldsymbol{\xi})\mathbf{c}^\wedge(2\boldsymbol{\xi}),$$

and

$$(S\mathbf{c})^\wedge(\boldsymbol{\xi}) = 2^d\mathbf{H}^*(-\boldsymbol{\xi})\mathbf{c}^\wedge(2\boldsymbol{\xi}).$$

The following two main sections of the paper elaborate on this identity, for polynomial sequences $\mathbf{c}$. For obvious reasons, we discuss the scalar case separately from the vector case. For the former one, we essentially recover and reprove some well-known facts in a compact and concise way, namely, we characterize polynomial reproduction of polynomial sequences in just one subdivision step by refering to zero properties of the mask symbol. We may stress that, to our knowledge, at least the last statement of proposition 2.4 (dealing with interpolatory subdivision) was not present in the literature so far.

The vector case is then addressed in section 3, and polyphase subdivision is chosen as a first introductory example in order to show some typical differences from the scalar case. Polynomial reproduction is then characterized by condition (A_0) which is actually a form of part of Jiang's condition in [17, equation (3.4)]. Based on this, we find further interesting facts concerning repeated polynomial reproduction and with respect to eigensequences, the latter refering to the proper extension of what was called interpolatory subdivision in the scalar case. In particular, theorem 3.5 is one of the main results here. We also give an algorithm, in section 3.4, how to find polynomial sequences of maximal degree which are transformed into polynomial sequences again. The closing section is devoted to two examples. In particular, a multivariate version of the univariate 'four point scheme' (due to Dubuc [5], or Dyn et al. [7]) is derived through an application of our algorithm.

2. The scalar case

We first look at the case of a scalar-valued refinement mask, and at the corresponding subdivision operator

$$S : \ell\left(\mathbb{Z}^d\right) \to \ell\left(\mathbb{Z}^d\right), \qquad c \mapsto Sc \quad \text{with} \quad (Sc)_{\boldsymbol{\alpha}} = \sum_{\boldsymbol{\beta}} \overline{p}_{\boldsymbol{\alpha}-2\boldsymbol{\beta}} c_{\boldsymbol{\beta}}. \tag{2.1}$$

In the Fourier transform domain we have

$$(Sc)^\wedge(\boldsymbol{\xi}) = \overline{p^\wedge}(-\boldsymbol{\xi})c^\wedge(2\boldsymbol{\xi}) = 2^d\overline{h}(-\boldsymbol{\xi})c^\wedge(2\boldsymbol{\xi}), \tag{2.2}$$

and we would like to describe the action of the operator on polynomial sequences. We choose $\mathbf{k} = (k_1, \dots, k_d) \in \mathbb{N}^d$ and start with a (scalar) sequence $c = (c_{\boldsymbol\alpha})_{\boldsymbol\alpha \in \mathbb{Z}^d}$ of type $\Pi_{\mathbf{k}}$ which we may write as

$$c_{\boldsymbol\alpha} = \sum_{\mathbf{0} \leqslant \mathbf{j} \leqslant \mathbf{k}} \binom{\mathbf{k}}{\mathbf{j}} v_{\mathbf{j}} (-\mathrm{i}\boldsymbol\alpha)^{\mathbf{k}-\mathbf{j}}, \quad \text{where } v_{\mathbf{0}} \neq 0, \tag{2.3}$$

for certain scalars $v_{\mathbf{j}}$. (Including the binomial coefficient in this 'Ansatz' will prove useful.) We may assume that c is not the null sequence, and that the representation is reduced, i.e., the index $\mathbf{k}$ is chosen to be minimal. This justifies the assumption $v_{\mathbf{0}} \neq 0$. Using (1.9), the sequence transforms into

$$c^\wedge = \sum_{\mathbf{0} \leqslant \mathbf{j} \leqslant \mathbf{k}} \binom{\mathbf{k}}{\mathbf{j}} v_{\mathbf{j}} D^{\mathbf{k}-\mathbf{j}} \delta, \tag{2.4}$$

and formula (2.2) yields – via (1.8) and (1.7):

$$\begin{aligned}
(Sc)^\wedge(\boldsymbol\xi) &= \sum_{\mathbf{0} \leqslant \mathbf{j} \leqslant \mathbf{k}} \binom{\mathbf{k}}{\mathbf{j}} v_{\mathbf{j}} 2^{-|\mathbf{k}-\mathbf{j}|} \frac{1}{2^d} \sum_{\mathbf{e} \in E} \overline{p^\wedge}(-\boldsymbol\xi) D^{\mathbf{k}-\mathbf{j}} \delta_{\pi \mathbf{e}} \\
&= \frac{1}{2^d} \sum_{\mathbf{0} \leqslant \mathbf{j} \leqslant \mathbf{k}} \binom{\mathbf{k}}{\mathbf{j}} v_{\mathbf{j}} 2^{-|\mathbf{k}-\mathbf{j}|} \sum_{\mathbf{e} \in E} \sum_{\mathbf{0} \leqslant \boldsymbol\ell \leqslant \mathbf{k}-\mathbf{j}} \binom{\mathbf{k}-\mathbf{j}}{\boldsymbol\ell} \bigl\{ \bigl\{ D^{\mathbf{k}-\mathbf{j}-\boldsymbol\ell} \overline{p^\wedge} \bigr\} (-\pi \mathbf{e}) \bigr\} D^{\boldsymbol\ell}\, \delta_{\pi \mathbf{e}} \\
&= \frac{1}{2^d} \sum_{\mathbf{e} \in E} \sum_{\mathbf{0} \leqslant \boldsymbol\ell \leqslant \mathbf{k}} \binom{\mathbf{k}}{\boldsymbol\ell} \Biggl\{ \sum_{\mathbf{0} \leqslant \mathbf{j} \leqslant \boldsymbol\ell} 2^{-|\mathbf{k}-\mathbf{j}|} \binom{\boldsymbol\ell}{\mathbf{j}} \bigl\{ \bigl\{ D^{\boldsymbol\ell-\mathbf{j}} \overline{p^\wedge} \bigr\} (-\pi \mathbf{e}) \bigr\} v_{\mathbf{j}} \Biggr\} D^{\mathbf{k}-\boldsymbol\ell} \delta_{\pi \mathbf{e}}.
\end{aligned}$$

Here, the terms for $\mathbf{e} = \mathbf{0}$ determine the (Fourier transform of the) polynomial part in Sc, while the other terms are non-polynomial. From this we obtain:

Lemma 2.1. Given $\mathbf{k} = (k_1, \dots, k_d) \in \mathbb{N}^d$, the following are equivalent:

(i) The polynomial sequence (2.3) of type $\Pi_{\mathbf{k}}$ is transformed into a polynomial sequence.

(ii) For $\mathbf{e} \in E \setminus \{\mathbf{0}\}$ and $\mathbf{0} \leqslant \boldsymbol\ell \leqslant \mathbf{k}$, the scalars $v_{\mathbf{j}}$, $\mathbf{0} \leqslant \mathbf{j} \leqslant \mathbf{k}$, satisfy the condition

$$\sum_{\mathbf{0} \leqslant \mathbf{j} \leqslant \boldsymbol\ell} \frac{1}{2^{|\boldsymbol\ell-\mathbf{j}|}} \binom{\boldsymbol\ell}{\mathbf{j}} \bigl\{ \bigl\{ D^{\boldsymbol\ell-\mathbf{j}} \overline{p^\wedge} \bigr\} (-\pi \mathbf{e}) \bigr\} v_{\mathbf{j}} = 0. \tag{2.5}$$

And if these hold, then

$$(Sc)_{\boldsymbol\alpha} = \sum_{\mathbf{0} \leqslant \mathbf{j} \leqslant \mathbf{k}} \binom{\mathbf{k}}{\mathbf{j}} v_{\mathbf{j}}^{(1)} (-\mathrm{i}\boldsymbol\alpha)^{\mathbf{k}-\mathbf{j}}, \quad \boldsymbol\alpha \in \mathbb{Z}^d,$$

with

$$v_{\mathbf{j}}^{(1)} = \frac{1}{2^{d+|\mathbf{k}-\mathbf{j}|}} \sum_{\mathbf{0} \leqslant \mathbf{m} \leqslant \mathbf{j}} \binom{\mathbf{j}}{\mathbf{m}} \frac{1}{2^{|\mathbf{j}-\boldsymbol\ell|}} \bigl\{ \bigl\{ D^{\mathbf{j}-\mathbf{m}} \overline{p^\wedge} \bigr\} (\mathbf{0}) \bigr\} v_{\boldsymbol\ell} \tag{2.6}$$

for $\mathbf{0} \leqslant \mathbf{j} \leqslant \mathbf{k}$.

Of course, in (2.5) evaluation at $-\pi\mathbf{e}$ can be replaced by evaluation at $\pi\mathbf{e}$, due to the periodicity of $p^\wedge$. The following result becomes now apparent.

Proposition 2.2. For $\mathbf{k} = (k_1, \dots, k_d) \in \mathbb{N}^d$ the following are equivalent:

(i) The subdivision operator maps the space of polynomial sequences of type $\Pi_\mathbf{k}$ into itself.

(ii) Condition (ii) of lemma 2.1 holds true for any numbers $v_\mathbf{j}$, $\mathbf{0} \leqslant \mathbf{j} \leqslant \mathbf{k}$.

In particular, the following well-known result on reproduction of constant sequences is contained as the special case where $\mathbf{k} = \mathbf{0}$. Here, (2.5) reads

$$\overline{p^\wedge}(-\pi\mathbf{e})v_\mathbf{0} = 0 \quad \text{for } \mathbf{e} \in E \setminus \{\mathbf{0}\},$$

and this is equivalent to the condition on the mask symbol

$$h(\pi\mathbf{e})v_\mathbf{0} = 0 \quad \text{for } \mathbf{e} \in E \setminus \{\mathbf{0}\}.$$

Whenever this condition is satisfied, the constant sequence $c_{\boldsymbol{\alpha}} \equiv v_\mathbf{0}$, $\boldsymbol{\alpha} \in \mathbb{Z}^d$, is mapped onto

$$(Sc)_{\boldsymbol{\alpha}} = \frac{1}{2^d}\overline{p^\wedge}(\mathbf{0})v_\mathbf{0} = h(\mathbf{0})v_\mathbf{0}.$$

Thus Sc does not vanish iff $h(\mathbf{0}) \neq 0$, and $Sc = c$ iff $h(\mathbf{0}) = 1$. This observation can be generalized by looking at the leading coefficient $v_\mathbf{0}$ more closely.

Definition 2.3.

(i) The subdivision operator (2.1) is called $\Pi_\mathbf{N}$*-invariant for* $\mathbf{N} \in \mathbb{N}^d$, if any polynomial sequence $c = (c_{\boldsymbol{\alpha}})_{\boldsymbol{\alpha}\in\mathbb{Z}^d}$ of type $\Pi_\mathbf{N}$ is mapped onto another polynomial sequence of type $\Pi_\mathbf{N}$. In other words: For any polynomial $f \in \Pi_\mathbf{N}$ and $c_{\boldsymbol{\alpha}} = f(\boldsymbol{\alpha})$, $\boldsymbol{\alpha} \in \mathbb{Z}^d$, there is another polynomial $\widetilde{f} \in \Pi_\mathbf{N}$ such that

$$\widetilde{f}\left(\frac{1}{2}\boldsymbol{\alpha}\right) = (S\mathbf{c})_{\boldsymbol{\alpha}} \quad \text{for all } \boldsymbol{\alpha} \in \mathbb{Z}^d.$$

(ii) If, given $\mathbf{N}$, the subdivision operator is $\Pi_\mathbf{k}$-invariant for any $\mathbf{k} \leqslant \mathbf{N}$, then we call the subdivision operator to be *degree preserving in* $\Pi_\mathbf{N}$.

(iii) If, in addition, for any $f \in \Pi_\mathbf{N}$ the total degree of $f - \widetilde{f}$ is less than the total degree of f, then we call the subdivision operator to be *strictly degree preserving in* $\Pi_\mathbf{N}$.

(iv) Finally, if in addition $f = \widetilde{f}$, then we call the subdivision operator to be *interpolating in* $\Pi_\mathbf{N}$.

Proposition 2.4. The following are equivalent, for $\mathbf{N} \in \mathbb{N}^d$:

(i) The subdivision operator is $\Pi_\mathbf{N}$-invariant.

(ii) The subdivision operator is degree-preserving in $\Pi_{\mathbf{N}}$.

(iii) The mask symbol satisfies $D^{\boldsymbol{\ell}} h(\pi \mathbf{e}) = 0$ for any $\boldsymbol{\ell} \leqslant \mathbf{N}$ and $\mathbf{e} \in E \setminus \{\mathbf{0}\}$.

Moreover, the subdivision operator is strictly degree-preserving in $\Pi_{\mathbf{N}}$ if and only if in addition $h(\mathbf{0}) = 1$. It is interpolating in $\Pi_{\mathbf{N}}$ if and only if in addition

$$D^{\boldsymbol{\ell}} h(\mathbf{0}) = \delta_{\boldsymbol{\ell},\mathbf{0}} \quad \text{for } \boldsymbol{\ell} \leqslant \mathbf{N}.$$

Proof. The equivalence of the first three statements is clear from the above arguments and an apparent induction argument on $\mathbf{k} \leqslant \mathbf{N}$. Since the coefficients $v_{\mathbf{j}}$ of the polynomials are arbitrary, condition (2.5) of lemma 2.1 is equivalent to $\{D^{\boldsymbol{\ell}-\mathbf{j}}\overline{p^\wedge}\}(-\pi \mathbf{e}) = 0$ for $\mathbf{e} \in E \setminus \{\mathbf{0}\}$ and $\mathbf{0} \leqslant \mathbf{j} \leqslant \boldsymbol{\ell} \leqslant \mathbf{k}$. This gives the equivalence to condition (iii) in terms of the mask symbol $h = (1/2^d)p^\wedge$.

In order to see the additional statement, we rewrite (2.6) in the following form by considering the rescaling in the polynomial $\tilde{f}$:

$$\tilde{f}\left(\frac{1}{2}\boldsymbol{\alpha}\right) = (Sc)_{\boldsymbol{\alpha}} = \sum_{\mathbf{0}\leqslant\mathbf{j}\leqslant\mathbf{k}} \binom{\mathbf{k}}{\mathbf{j}} \tilde{v}_{\mathbf{j}}^{(1)} \left(-\frac{\mathrm{i}}{2}\boldsymbol{\alpha}\right)^{\mathbf{k}-\mathbf{j}}, \quad \boldsymbol{\alpha} \in \mathbb{Z}^d,$$

with

$$\tilde{v}_{\mathbf{j}}^{(1)} = \frac{1}{2^d} \sum_{\mathbf{0}\leqslant\mathbf{m}\leqslant\mathbf{j}} \binom{\mathbf{j}}{\mathbf{m}} \frac{1}{2^{|\mathbf{j}-\mathbf{m}|}} \left\{\left\{D^{\mathbf{j}-\mathbf{m}}\overline{p^\wedge}\right\}(\mathbf{0})\right\} v_{\boldsymbol{\ell}}, \quad \mathbf{0} \leqslant \mathbf{j} \leqslant \mathbf{k}.$$

The leading coefficient of $\tilde{f}$ is given by

$$\tilde{v}_{\mathbf{0}}^{(1)} := \frac{1}{2^d}\overline{p^\wedge}(\mathbf{0}) v_{\mathbf{0}} = \overline{h}(\mathbf{0}) v_{\mathbf{0}},$$

and this proves the first additional statement. The rest is verified again by an induction argument (with respect to $\mathbf{k} \leqslant \mathbf{N}$) on the equations

$$v_{\mathbf{j}} = \frac{1}{2^d} \sum_{\mathbf{0}\leqslant\mathbf{m}\leqslant\mathbf{j}} \binom{\mathbf{j}}{\mathbf{m}} \frac{1}{2^{|\mathbf{j}-\mathbf{m}|}} \left\{\left\{D^{\mathbf{j}-\mathbf{m}}\overline{p^\wedge}\right\}(\mathbf{0})\right\} v_{\boldsymbol{\ell}}, \quad \mathbf{0} \leqslant \mathbf{j} \leqslant \mathbf{k}. \qquad \square$$

Remark 2.5. In our results we have always referred to polynomial spaces $\Pi_{\mathbf{k}}$ of coordinate degree $\mathbf{k} = (k_1, \ldots, k_d) \in \mathbb{N}^d$. However, it is apparent how to modify the statements to include polynomial sequences of total degree $N \in \mathbb{N}$, namely, we have to require the conditions for all $\mathbf{k}$ with $|\mathbf{k}| \leqslant N$. For this case, condition (ii) of lemma 2.1 recovers the characterization of accuracy of degree N.

Remark 2.6. If the refinement equation (1.1) is generalized to include arbitrary dilation matrices $\mathbf{M}$,

$$\boldsymbol{\Phi}(\mathbf{x}) = \sum_{\boldsymbol{\alpha}} \mathbf{P}_{\boldsymbol{\alpha}} \boldsymbol{\Phi}(\mathbf{M}\mathbf{x}-\boldsymbol{\alpha}),$$

then it is necessary to work a priori with total degree. We omit the details.

3. The vector case

We start with a simple case which already shows some restrictions of how polynomial reproduction should read in the vector case.

3.1. Polyphase subdivision

Let $\phi \in L_2(\mathbb{R}^d)$ satisfy the scalar-valued refinement equation

$$\phi\left(\frac{1}{2}\mathbf{x}\right) = \sum_{\boldsymbol{\alpha}} q_{\boldsymbol{\alpha}} \phi(\mathbf{x} - \boldsymbol{\alpha}) \tag{3.1}$$

with finitely supported refinement mask q. Given any integer dilation matrix $\mathbf{M} \in \mathbb{Z}^{d\times d}$, we put $\mu := |\det \mathbf{M}|$, and we let $\mathbf{e}_0, \ldots, \mathbf{e}_{\mu-1}$ denote a set of representatives for the cosets $\mathbb{Z}^d/\mathbf{M}\mathbb{Z}^d$. Equation (3.1) can then be written in the polyphase form

$$\phi\left(\frac{1}{2}\mathbf{Mx}\right) = \sum_{\boldsymbol{\alpha}} \sum_{i=0}^{\mu-1} q_{\mathbf{M}\boldsymbol{\alpha}+\mathbf{e}_i} \phi\bigl(\mathbf{M}(\mathbf{x} - \boldsymbol{\alpha}) - \mathbf{e}_i\bigr),$$

and this suggests to introduce the *polyphase vector* $\boldsymbol{\Phi} := (\phi_0, \phi_1, \ldots, \phi_{\mu-1})^{\mathrm{T}}$ with

$$\phi_j(\mathbf{x}) := \phi(\mathbf{Mx} - \mathbf{e}_j), \quad j = 0, \ldots, \mu - 1. \tag{3.2}$$

Lemma 3.1. The polyphase vector $\boldsymbol{\Phi}$ is refinable, and satisfies a refinement equation of type (1.1) with the matrix mask

$$(\mathbf{P}_{\boldsymbol{\alpha}})_{j,k} = q_{\mathbf{M}\boldsymbol{\alpha}+\mathbf{e}_k-2\mathbf{e}_j}, \quad j, k = 0, \ldots, \mu - 1, \ \boldsymbol{\alpha} \in \mathbb{Z}^d. \tag{3.3}$$

Proof. From (3.2) and (3.1) we find

$$\begin{aligned}
\phi_j\left(\frac{1}{2}\mathbf{x}\right) &= \phi\left(\frac{1}{2}(\mathbf{Mx} - 2\mathbf{e}_j)\right) = \sum_{\boldsymbol{\alpha}} q_{\boldsymbol{\alpha}} \phi(\mathbf{Mx} - 2\mathbf{e}_j - \boldsymbol{\alpha}) \\
&= \sum_{\boldsymbol{\alpha}} q_{\boldsymbol{\alpha}-2\mathbf{e}_j} \phi(\mathbf{Mx} - \boldsymbol{\alpha}) \\
&= \sum_{\boldsymbol{\alpha}} \sum_{k=0}^{\mu-1} q_{\mathbf{M}\boldsymbol{\alpha}+\mathbf{e}_k-2\mathbf{e}_j} \phi_k(\mathbf{x} - \boldsymbol{\alpha})
\end{aligned}$$

for $j = 0, \ldots, \mu - 1$. □

Now the matrix mask symbol has the (j, k)th entry

$$\mathbf{H}_{j,k}\bigl(\mathbf{M}^{\mathrm{T}}\xi\bigr) = \frac{1}{2^d} \sum_{\boldsymbol{\alpha}} q_{\mathbf{M}\boldsymbol{\alpha}+\mathbf{e}_k-2\mathbf{e}_j} \mathrm{e}^{-\mathrm{i}(\mathbf{M}\boldsymbol{\alpha})\cdot\xi}$$

and right-multiplication with the modulation vector

$$\mathbf{w}(\xi) := \bigl(\mathrm{e}^{-\mathrm{i}\mathbf{e}_0\cdot\xi}, \ldots, \mathrm{e}^{-\mathrm{i}\mathbf{e}_{\mu-1}\cdot\xi}\bigr)^{\mathrm{T}} \tag{3.4}$$

gives

$$\begin{aligned}\mathrm{e}^{+\mathrm{i}2\mathbf{e}_j\cdot\xi}\mathbf{H}_{j,k}\big(\mathbf{M}^{\mathrm{T}}\xi\big)\mathbf{w}(\xi) &= \frac{1}{2^d}\sum_{\boldsymbol{\alpha}}\sum_{k=0}^{\mu-1} q_{\mathbf{M}\boldsymbol{\alpha}+\mathbf{e}_k-2\mathbf{e}_j}\mathrm{e}^{-\mathrm{i}(\mathbf{M}\boldsymbol{\alpha}+\mathbf{e}_k-2\mathbf{e}_j)\cdot\xi}\\ &= h(\xi), \quad j = 0,\ldots,\mu-1,\end{aligned} \tag{3.5}$$

with the scalar-valued mask symbol

$$h(\xi) = \frac{1}{2^d}\sum_{\boldsymbol{\alpha}} q_{\boldsymbol{\alpha}}\mathrm{e}^{-\mathrm{i}\boldsymbol{\alpha}\cdot\xi}$$

refering to the refinement equation (3.1).

From this the property of polynomial reproduction for polyphase subdivision becomes transparent. It is a block type subdivision version of the scalar-valued subdivision scheme

$$c \mapsto S_q c: \quad (S_q c)_{\boldsymbol{\alpha}} = \sum_{\boldsymbol{\alpha}} \overline{q}_{\boldsymbol{\alpha}-2\boldsymbol{\beta}} c_{\boldsymbol{\beta}}. \tag{3.6}$$

Namely, we write the sequence $c = (c_{\boldsymbol{\alpha}})$ in terms of the vector-valued sequence $\mathbf{c} = (\mathbf{c}_{\boldsymbol{\beta}})$ where

$$\mathbf{c}_{\boldsymbol{\beta}} = (c_{\mathbf{M}\boldsymbol{\beta}+\mathbf{e}_0}\ c_{\mathbf{M}\boldsymbol{\beta}+\mathbf{e}_1}\ \ldots\ c_{\mathbf{M}\boldsymbol{\beta}+\mathbf{e}_{\mu-1}})^{\mathrm{T}}, \quad \boldsymbol{\beta} \in \mathbb{Z}^d. \tag{3.7}$$

Whence, the scalar rule (3.6) is equivalent to the vector rule

$$\mathbf{c} \mapsto S_{\mathbf{P}}\mathbf{c}: \quad (S_{\mathbf{P}}\mathbf{c})_{\boldsymbol{\alpha}} = \sum_{\boldsymbol{\alpha}} \mathbf{P}^*_{\boldsymbol{\alpha}-2\boldsymbol{\beta}}\mathbf{c}_{\boldsymbol{\beta}}, \tag{3.8}$$

with matrix mask (3.3) and

$$(S_{\mathbf{P}}\mathbf{c})_{\boldsymbol{\alpha}} = \big((S_q c)_{\mathbf{M}\boldsymbol{\alpha}+\mathbf{e}_0}\ (S_q c)_{\mathbf{M}\boldsymbol{\alpha}+\mathbf{e}_1}\ \ldots\ (S_q c)_{\mathbf{M}\boldsymbol{\alpha}+\mathbf{e}_{\mu-1}}\big)^{\mathrm{T}}. \tag{3.9}$$

In this way, we can apply the results of section 2 to polyphase subdivision via the $\mu = |\det \mathbf{M}|$ identities of equation (3.5), if we write polynomial sequences in the corresponding block form

$$\mathbf{c}_{\boldsymbol{\beta}} = \big(f(\mathbf{M}\boldsymbol{\beta}+\mathbf{e}_0)\ f(\mathbf{M}\boldsymbol{\beta}+\mathbf{e}_1)\ \ldots\ f(\mathbf{M}\boldsymbol{\beta}+\mathbf{e}_{\mu-1})\big)^{\mathrm{T}}.$$

It also shows that the components in this vector polynomial sequence are not independent of each other, a fact that will show up in the general case to follow, as well.

3.2. *The general case*

In an attempt to generalize lemma 2.1 to the vector case of polynomial sequences $\mathbf{c} = (\mathbf{c}_{\alpha})_{\alpha\in\mathbb{Z}^d}$, we replace the 'Ansatz' (2.3) by

$$\mathbf{c}_{\alpha} = \sum_{\mathbf{0}\leqslant\mathbf{j}\leqslant\mathbf{k}} \binom{\mathbf{k}}{\mathbf{j}}\mathbf{v}_{\mathbf{j}}(-\mathrm{i}\alpha)^{\mathbf{k}-\mathbf{j}}, \quad \text{where } \mathbf{v}_{\mathbf{0}} \neq \mathbf{0}, \tag{3.10}$$

with vectors $\mathbf{v_j} \in \mathbb{C}^N$ (rather than scalars). Then as before,

$$\mathbf{c}^\wedge = \sum_{\mathbf{0}\leqslant\mathbf{j}\leqslant\mathbf{k}} \binom{\mathbf{k}}{\mathbf{j}} \mathbf{v_j} D^{\mathbf{k}-\mathbf{j}}\delta$$

and thus

$$\begin{aligned}(S\mathbf{c})^\wedge(\boldsymbol{\xi}) &= \sum_{\mathbf{0}\leqslant\mathbf{j}\leqslant\mathbf{k}} \binom{\mathbf{k}}{\mathbf{j}} \mathbf{P}^{\wedge *}(-\boldsymbol{\xi})\mathbf{v_j}\frac{1}{2^d}\sum_{\mathbf{e}\in E}\frac{1}{2^{|\mathbf{k}-\mathbf{j}|}}D^{\mathbf{k}-\mathbf{j}}\delta_{\pi\mathbf{e}} \\ &= \frac{1}{2^d}\sum_{\mathbf{e}\in E}\sum_{\mathbf{0}\leqslant\boldsymbol{\ell}\leqslant\mathbf{k}}\binom{\mathbf{k}}{\boldsymbol{\ell}}\left\{\sum_{\mathbf{0}\leqslant\mathbf{j}\leqslant\boldsymbol{\ell}}\binom{\boldsymbol{\ell}}{\mathbf{j}}\frac{1}{2^{|\boldsymbol{\ell}-\mathbf{j}|}}D^{\boldsymbol{\ell}-\mathbf{j}}\big(\mathbf{P}^{\wedge *}\big)(-\pi\mathbf{e})\mathbf{v_j}\right\}\frac{1}{2^{|\mathbf{k}-\boldsymbol{\ell}|}}D^{\mathbf{k}-\boldsymbol{\ell}}\delta_{\pi\mathbf{e}}.\end{aligned}$$

Hence the vector version of lemma 2.1 reads as follows.

Lemma 3.2. For the vector subdivision operator (1.2), the following are equivalent:

(i) The polynomial sequence (3.10) is transformed into a polynomial sequence.

(ii) The family $(\mathbf{v_j})_{\mathbf{0}\leqslant\mathbf{j}\leqslant\mathbf{k}}$ satisfies *condition* (A_0): For $\mathbf{e} \in E \setminus \{\mathbf{0}\}$ and $\mathbf{0} \leqslant \boldsymbol{\ell} \leqslant \mathbf{k}$,

$$\sum_{\mathbf{0}\leqslant\mathbf{j}\leqslant\boldsymbol{\ell}}\binom{\boldsymbol{\ell}}{\mathbf{j}}\frac{1}{2^{|\boldsymbol{\ell}-\mathbf{j}|}}D^{\boldsymbol{\ell}-\mathbf{j}}\big(\mathbf{P}^{\wedge *}\big)(\pi\mathbf{e})\mathbf{v_j} = \mathbf{0}. \tag{3.11}$$

And if these hold, then

$$(S\mathbf{c})_{\boldsymbol{\alpha}} = \sum_{\mathbf{0}\leqslant\mathbf{j}\leqslant\mathbf{k}}\binom{\mathbf{k}}{\mathbf{j}}\mathbf{v}_{\mathbf{j}}^{(1)}(-\mathrm{i}\,\boldsymbol{\alpha})^{\mathbf{k}-\mathbf{j}}, \quad \boldsymbol{\alpha}\in\mathbb{Z}^d,$$

where

$$\mathbf{v}_{\mathbf{j}}^{(1)} = \frac{1}{2^{d+|\mathbf{k}-\mathbf{j}|}}\sum_{\mathbf{0}\leqslant\boldsymbol{\ell}\leqslant\mathbf{j}}\binom{\mathbf{j}}{\boldsymbol{\ell}}\frac{1}{2^{|\mathbf{j}-\boldsymbol{\ell}|}}D^{\mathbf{j}-\boldsymbol{\ell}}\big(\mathbf{P}^{\wedge *}\big)(\mathbf{0})\mathbf{v}_{\boldsymbol{\ell}} \tag{3.12}$$

for $\mathbf{0} \leqslant \mathbf{j} \leqslant \mathbf{k}$.

Remarks.

(i) Note that condition $(\mathbf{A}_0)$ is a recursive sequence of conditions which does not explicitly depend on $\mathbf{k}$. Thus, given any matrix mask $\mathbf{P}$, we can determine maximal values for $\mathbf{k}$ and corresponding polynomial sequences $\mathbf{c}$ of type (3.10) for which $S\mathbf{c}$ is polynomial as well. We just run this recurrence until it stops.

(ii) The start of the aforementioned recurrence is the condition that there exists a vector $\mathbf{v_0} \neq \mathbf{0}$ satisfying

$$\mathbf{P}^{\wedge *}(\pi\mathbf{e})\mathbf{v_0} = \mathbf{0} \quad \text{for } \mathbf{e} \in E \setminus \{\mathbf{0}\}. \tag{3.13}$$

This condition ensures that the constant sequence $\mathbf{c} = (\mathbf{v_0})_{\boldsymbol{\alpha}\in\mathbb{Z}^d}$ is mapped onto the constant sequence $S\mathbf{c} = (\mathbf{v}_\mathbf{0}^{(1)})_{\boldsymbol{\alpha}\in\mathbb{Z}^d}$, where

$$\mathbf{v}_\mathbf{0}^{(1)} = \frac{1}{2^d}\mathbf{P}^{\wedge *}(\mathbf{0})\mathbf{v_0} = \mathbf{H}^*(\mathbf{0})\mathbf{v_0}.$$

It should be noted that (3.13) is equivalent to part of what is called condition (Z_1) in [11].

(iii) We obviously have $S\mathbf{c} = \gamma\mathbf{c}$ (for some scalar γ) if and only if, in addition, $\mathbf{v_0}$ is a (right) eigenvector of $\mathbf{H}^*(\mathbf{0})$ for the eigenvalue γ, and in particular, $S\mathbf{c} = \mathbf{c}$ if and only if $\mathbf{H}^*(\mathbf{0})\mathbf{v_0} = \mathbf{v_0}$. This latter condition is the essential part of the spectral condition of order 1, as stated in [11]. It is usually formulated in terms of left eigenvectors for $\mathbf{H}(\pi\mathbf{e})$, $\mathbf{e} \in E$, and it also appears in the statements of convergence of (stationary) vector subdivision schemes.

In what follows, we make the following assumption, which will turn out to be useful.

Condition (E_0)**.** The system of linear equations (3.13) has a one-dimensional space of solutions, i.e., the vector $\mathbf{v_0} \neq \mathbf{0}$ exists, and it is unique up to a scalar factor.

Lemma 3.3. Assume that the mask $\mathbf{P}$ satisfies (E_0), and that the family $(\mathbf{v_j})_{\mathbf{0}\leqslant\mathbf{j}\leqslant\mathbf{k}}$ satisfies (A_0). Then, for $\mathbf{k}' \leqslant \mathbf{k}$ the following are equivalent:

(i) The vector sequence $(\mathbf{w_j})_{\mathbf{0}\leqslant\mathbf{j}\leqslant\mathbf{k}'}$ satisfies (A_0).

(ii) There exist scalars $a_\mathbf{j}$, $\mathbf{0} \leqslant \mathbf{j} \leqslant \mathbf{k}'$, such that

$$\mathbf{w_j} = \sum_{\mathbf{0}\leqslant\boldsymbol{\ell}\leqslant\mathbf{j}} \binom{\mathbf{j}}{\boldsymbol{\ell}} a_{\boldsymbol{\ell}}\mathbf{v}_{\mathbf{j}-\boldsymbol{\ell}} \quad \text{for } \mathbf{0} \leqslant \mathbf{j} \leqslant \mathbf{k}'. \tag{3.14}$$

Proof. Part (ii) $\Longrightarrow$ (i) is shown by a straightforward calculation. In order to show the converse, we assume that $(\mathbf{w_j})_{\mathbf{0}\leqslant\mathbf{j}\leqslant\mathbf{k}'}$ satisfies (A_0), and we show (3.14) by induction on $\mathbf{j}$. For $\mathbf{j} = \mathbf{0}$, condition (A_0) states that $\mathbf{P}^{\wedge *}(\pi\mathbf{e})\mathbf{w_0} = \mathbf{0}$, and by condition (E_0) we have $\mathbf{w_0} = a_\mathbf{0}\mathbf{v_0}$ for some scalar $a_\mathbf{0}$.

Now assume that (3.14) holds for $\mathbf{0} \leqslant \boldsymbol{\ell} \leqslant \mathbf{j}$, $\boldsymbol{\ell} \neq \mathbf{j}$. We obtain from condition (A_0) that

$$\begin{aligned}
\mathbf{P}^{\wedge *}(\pi\mathbf{e})\mathbf{w_j} &= -\sum_{\substack{\mathbf{0}\leqslant\boldsymbol{\ell}\leqslant\mathbf{j}\\ \boldsymbol{\ell}\neq\mathbf{j}}} \binom{\mathbf{j}}{\boldsymbol{\ell}} \frac{1}{2^{|\mathbf{j}-\boldsymbol{\ell}|}}\mathbf{P}^{\wedge(\mathbf{j}-\boldsymbol{\ell})*}(\pi\mathbf{e})\mathbf{w}_{\boldsymbol{\ell}} \\
&= -\sum_{\substack{\mathbf{0}\leqslant\boldsymbol{\ell}\leqslant\mathbf{j}\\ \boldsymbol{\ell}\neq\mathbf{j}}} \binom{\mathbf{j}}{\boldsymbol{\ell}} \frac{1}{2^{|\mathbf{j}-\boldsymbol{\ell}|}}\mathbf{P}^{\wedge(\mathbf{j}-\boldsymbol{\ell})*}(\pi\mathbf{e}) \sum_{\mathbf{0}\leqslant\mathbf{m}\leqslant\boldsymbol{\ell}} \binom{\boldsymbol{\ell}}{\mathbf{m}} a_\mathbf{m}\mathbf{v}_{\boldsymbol{\ell}-\mathbf{m}}
\end{aligned}$$

$$= -\sum_{\substack{\mathbf{0}\leqslant\mathbf{m}\leqslant\mathbf{j}\\ \mathbf{m}\neq\mathbf{j}}}\ \sum_{\substack{\mathbf{m}\leqslant\boldsymbol{\ell}\leqslant\mathbf{j}\\ \boldsymbol{\ell}\neq\mathbf{j}}} \frac{1}{2^{|\mathbf{j}-\boldsymbol{\ell}|}}\binom{\mathbf{j}}{\boldsymbol{\ell}}\binom{\boldsymbol{\ell}}{\mathbf{m}}\mathbf{P}^{\wedge(\mathbf{j}-\boldsymbol{\ell})*}(\pi\mathbf{e})a_{\mathbf{m}}\mathbf{v}_{\boldsymbol{\ell}-\mathbf{m}}$$

$$= -\sum_{\substack{\mathbf{0}\leqslant\mathbf{m}\leqslant\mathbf{j}\\ \mathbf{m}\neq\mathbf{j}}}\binom{\mathbf{j}}{\mathbf{m}}a_{\mathbf{m}}\sum_{\substack{\mathbf{0}\leqslant\mathbf{n}\leqslant\mathbf{j}-\mathbf{m}\\ \mathbf{n}\neq\mathbf{j}-\mathbf{m}}}\binom{\mathbf{j}-\mathbf{m}}{\mathbf{n}}\frac{1}{2^{|\mathbf{j}-\mathbf{m}-\mathbf{n}|}}\mathbf{P}^{\wedge(\mathbf{j}-\mathbf{m}-\mathbf{n})*}(\pi\mathbf{e})\mathbf{v}_{\mathbf{n}}$$

$$= -\sum_{\substack{\mathbf{0}\leqslant\mathbf{m}\leqslant\mathbf{j}\\ \mathbf{m}\neq\mathbf{j}}}\binom{\mathbf{j}}{\mathbf{m}}a_{\mathbf{m}}\big(-\mathbf{P}^{\wedge*}(\pi\mathbf{e})\mathbf{v}_{\mathbf{j}-\mathbf{m}}\big).$$

Thus,

$$\mathbf{P}^{\wedge*}(\pi\mathbf{e})\left(\mathbf{w}_{\mathbf{j}} - \sum_{\substack{\mathbf{0}\leqslant\mathbf{m}\leqslant\mathbf{j}\\ \mathbf{m}\neq\mathbf{j}}}\binom{\mathbf{j}}{\mathbf{m}}a_{\mathbf{m}}\mathbf{v}_{\mathbf{j}-\mathbf{m}}\right) = \mathbf{0} \quad \text{for } \mathbf{e}\in E\setminus\{\mathbf{0}\},$$

and by condition (E_0) we find that

$$\mathbf{w}_{\mathbf{j}} - \sum_{\substack{\mathbf{0}\leqslant\mathbf{m}\leqslant\mathbf{j}\\ \mathbf{m}\neq\mathbf{j}}}\binom{\mathbf{j}}{\mathbf{m}}a_{\mathbf{m}}\mathbf{v}_{\mathbf{j}-\mathbf{m}} = a_{\mathbf{j}}\mathbf{v}_{\mathbf{0}}$$

for some scalar $a_{\mathbf{j}}$. □

Corollary 3.4. Assume that the matrix mask $\mathbf{P}$ satisfies (E_0), and assume that $\mathbf{k}$ is chosen such that the polynomial sequence (3.10) is transformed into a polynomial sequence $S\mathbf{c}$ as given by (3.12). Then for $\mathbf{k}'\leqslant\mathbf{k}$ and any polynomial sequence $\mathbf{d}$ with

$$\mathbf{d}_{\boldsymbol{\alpha}} = \sum_{\mathbf{0}\leqslant\mathbf{j}\leqslant\mathbf{k}'}\binom{\mathbf{k}'}{\mathbf{j}}\mathbf{w}_{\mathbf{j}}(-\mathrm{i}\boldsymbol{\alpha})^{\mathbf{k}'-\mathbf{j}} \tag{3.15}$$

the following are equivalent:

(i) $S\mathbf{d}$ is polynomial.

(ii) There exist scalars $a_{\mathbf{j}}$, $\mathbf{0}\leqslant\mathbf{j}\leqslant\mathbf{k}'$, such that (3.14) holds.

And if these hold, then

$$(S\mathbf{d})_{\boldsymbol{\alpha}} = \sum_{\mathbf{0}\leqslant\mathbf{j}\leqslant\mathbf{k}'}\binom{\mathbf{k}'}{\mathbf{j}}\mathbf{w}^{(1)}_{\mathbf{j}}(-\mathrm{i}\boldsymbol{\alpha})^{\mathbf{k}'-\mathbf{j}}, \quad \boldsymbol{\alpha}\in\mathbb{Z}^d,$$

where

$$\mathbf{w}^{(1)}_{\mathbf{j}} = 2^{|\mathbf{k}-\mathbf{k}'|}\sum_{\mathbf{0}\leqslant\boldsymbol{\ell}\leqslant\mathbf{j}}\binom{\mathbf{j}}{\boldsymbol{\ell}}2^{|\boldsymbol{\ell}|}a_{\boldsymbol{\ell}}\mathbf{v}^{(1)}_{\mathbf{j}-\boldsymbol{\ell}} \tag{3.16}$$

for $\mathbf{0}\leqslant\mathbf{j}\leqslant\mathbf{k}'$.

Proof. The equivalence of (i) and (ii) is immediate by lemma 3.3. The representation for the $\mathbf{w}_{\mathbf{j}}^{(1)}$ follows from a straightforward calculation using (3.14) and (3.12). $\square$

3.3. Repeated polynomial reproduction and eigensequences

Of course, it is natural to ask under which conditions repeated applications of the subdivision operator still yield polynomial output. The answer is somewhat surprising.

Theorem 3.5. Assume that the mask $\mathbf{P}$ satisfies (E_0), and that for some $\mathbf{k}$, there exists a polynomial sequence $\mathbf{c}$ as in (3.10) with $\mathbf{v_0} \neq \mathbf{0}$, such that both $S\mathbf{c}$ and $S^2\mathbf{c}$ are polynomial. Then the following hold true.

(i) $S^n\mathbf{c}$ is polynomial for all $n \in \mathbb{N}$.

(ii) $\mathbf{v_0}$ is an eigenvector of $\mathbf{P}^{\wedge *}(\mathbf{0})$, for the eigenvalue $2^d\gamma$, say.

(iii) In case $\gamma \neq 0$, for all $\mathbf{k}' \leqslant \mathbf{k}$, there exists a polynomial sequence $\mathbf{d}_{\mathbf{k}'}$ of degree $\mathbf{k}'$ satisfying

$$S\mathbf{d}_{\mathbf{k}'} = \frac{\gamma}{2^{|\mathbf{k}'|}}\mathbf{d}_{\mathbf{k}'}.$$

Proof. (i) It suffices to show that for a polynomial sequence $\mathbf{c}$, the sequences $S\mathbf{c}$ and $S^2\mathbf{c}$ being polynomial imply that $S^3\mathbf{c}$ is polynomial also, since then the claim follows by induction. By corollary 3.4, we know that $S\,(S\mathbf{c})$ being polynomial implies that there exist scalars $b_{\mathbf{j}}$, $\mathbf{0} \leqslant \mathbf{j} \leqslant \mathbf{k}$, such that

$$\mathbf{v}_{\mathbf{j}}^{(1)} = \sum_{\mathbf{0}\leqslant\boldsymbol{\ell}\leqslant\mathbf{j}} \binom{\mathbf{j}}{\boldsymbol{\ell}} b_{\boldsymbol{\ell}}\mathbf{v}_{\mathbf{j}-\boldsymbol{\ell}}, \tag{3.17}$$

and then, by (3.16),

$$\begin{aligned}\mathbf{v}_{\mathbf{j}}^{(2)} &= \sum_{\mathbf{0}\leqslant\boldsymbol{\ell}\leqslant\mathbf{j}} \binom{\mathbf{j}}{\boldsymbol{\ell}} 2^{|\boldsymbol{\ell}|} b_{\boldsymbol{\ell}}\mathbf{v}_{\mathbf{j}-\boldsymbol{\ell}}^{(1)} \\ &= \sum_{\mathbf{0}\leqslant\boldsymbol{\ell}\leqslant\mathbf{j}} \binom{\mathbf{j}}{\boldsymbol{\ell}} 2^{|\boldsymbol{\ell}|} b_{\boldsymbol{\ell}} \sum_{\mathbf{0}\leqslant\mathbf{m}\leqslant\mathbf{j}-\boldsymbol{\ell}} \binom{\mathbf{j}-\boldsymbol{\ell}}{\mathbf{m}} b_{\mathbf{m}}\mathbf{v}_{\mathbf{j}-\boldsymbol{\ell}-\mathbf{m}} \\ &= \sum_{\mathbf{0}\leqslant\mathbf{n}\leqslant\mathbf{j}} \binom{\mathbf{j}}{\mathbf{n}} \left(\sum_{\mathbf{0}\leqslant\boldsymbol{\ell}\leqslant\mathbf{n}} \binom{\mathbf{n}}{\boldsymbol{\ell}} 2^{|\boldsymbol{\ell}|} b_{\boldsymbol{\ell}} b_{\mathbf{n}-\boldsymbol{\ell}} \right) \mathbf{v}_{\mathbf{j}-\mathbf{n}},\end{aligned}$$

so again by corollary 3.4, we have that $S^3\mathbf{c}$ is polynomial as well.

(ii) For $\mathbf{j} = \mathbf{0}$, (3.17) together with (3.12) yields

$$\mathbf{v}_{\mathbf{0}}^{(1)} = b_{\mathbf{0}}\mathbf{v_0} = \frac{1}{2^{d+|\mathbf{k}|}}\mathbf{P}^{\wedge *}(\mathbf{0})\mathbf{v_0}.$$

In particular, $\gamma = 2^{|\mathbf{k}|}b_{\mathbf{0}}$.

(iii) The 'Ansatz' (3.15) for $\mathbf{d}_{\mathbf{k}'}$ with (3.14) for the $\mathbf{w}_{\mathbf{j}}$ yields (3.16) for the $\mathbf{w}_{\mathbf{j}}^{(1)}$ in $S\mathbf{d}_{\mathbf{k}'}$. Therefore, $S\mathbf{d}_{\mathbf{k}'} = (\gamma/2^{|\mathbf{k}'|})\mathbf{d}_{\mathbf{k}'}$, i.e.,

$$\mathbf{w}_{\mathbf{j}}^{(1)} = \frac{\gamma}{2^{|\mathbf{k}'|}}\mathbf{w}_{\mathbf{j}} \quad \text{for } \mathbf{0} \leqslant \mathbf{j} \leqslant \mathbf{k}',$$

is equivalent to

$$\begin{aligned}
\frac{\gamma}{2^{|\mathbf{k}'|}} \sum_{\mathbf{0}\leqslant\boldsymbol{\ell}\leqslant\mathbf{j}} \binom{\mathbf{j}}{\boldsymbol{\ell}} a_{\boldsymbol{\ell}} \mathbf{v}_{\mathbf{j}-\boldsymbol{\ell}} &= 2^{|\mathbf{k}-\mathbf{k}'|} \sum_{\mathbf{0}\leqslant\boldsymbol{\ell}\leqslant\mathbf{j}} \binom{\mathbf{j}}{\boldsymbol{\ell}} 2^{|\boldsymbol{\ell}|} a_{\boldsymbol{\ell}} \mathbf{v}_{\mathbf{j}-\boldsymbol{\ell}}^{(1)} \\
&= 2^{|\mathbf{k}-\mathbf{k}'|} \sum_{\mathbf{0}\leqslant\boldsymbol{\ell}\leqslant\mathbf{j}} \binom{\mathbf{j}}{\boldsymbol{\ell}} 2^{|\boldsymbol{\ell}|} a_{\boldsymbol{\ell}} \sum_{\mathbf{0}\leqslant\mathbf{m}\leqslant\mathbf{j}-\boldsymbol{\ell}} \binom{\mathbf{j}-\boldsymbol{\ell}}{\mathbf{m}} b_{\mathbf{m}} \mathbf{v}_{\mathbf{j}-\boldsymbol{\ell}-\mathbf{m}} \\
&= 2^{|\mathbf{k}-\mathbf{k}'|} \sum_{\mathbf{0}\leqslant\boldsymbol{\ell}\leqslant\mathbf{j}} \binom{\mathbf{j}}{\boldsymbol{\ell}} 2^{|\boldsymbol{\ell}|} a_{\boldsymbol{\ell}} \sum_{\boldsymbol{\ell}\leqslant\mathbf{n}\leqslant\mathbf{j}} \binom{\mathbf{j}-\boldsymbol{\ell}}{\mathbf{n}-\boldsymbol{\ell}} b_{\mathbf{n}-\boldsymbol{\ell}} \mathbf{v}_{\mathbf{j}-\mathbf{n}} \\
&= 2^{|\mathbf{k}-\mathbf{k}'|} \sum_{\mathbf{0}\leqslant\mathbf{n}\leqslant\mathbf{j}} \binom{\mathbf{j}}{\mathbf{n}} \biggl(\sum_{\mathbf{0}\leqslant\boldsymbol{\ell}\leqslant\mathbf{n}} \binom{\mathbf{n}}{\boldsymbol{\ell}} 2^{|\boldsymbol{\ell}|} a_{\boldsymbol{\ell}} b_{\mathbf{n}-\boldsymbol{\ell}} \biggr) \mathbf{v}_{\mathbf{j}-\mathbf{n}}
\end{aligned}$$

for $\mathbf{0} \leqslant \mathbf{j} \leqslant \mathbf{k}'$. This is again equivalent to

$$\frac{\gamma}{2^{|\mathbf{k}|}} a_{\mathbf{n}} = \sum_{\mathbf{0}\leqslant\boldsymbol{\ell}\leqslant\mathbf{n}} \binom{\mathbf{n}}{\boldsymbol{\ell}} 2^{|\boldsymbol{\ell}|} a_{\boldsymbol{\ell}} b_{\mathbf{n}-\boldsymbol{\ell}} \quad \text{for } \mathbf{0} \leqslant \mathbf{n} \leqslant \mathbf{k}'. \tag{3.18}$$

For $\mathbf{n} = \mathbf{0}$, (3.18) says

$$\frac{\gamma}{2^{|\mathbf{k}|}} a_{\mathbf{0}} = a_{\mathbf{0}} b_{\mathbf{0}}$$

which by (ii) is true for any $a_{\mathbf{0}} \in \mathbb{C}$; for $\mathbf{n} \neq \mathbf{0}$, (3.18) can be rewritten as

$$a_{\mathbf{n}} = \frac{2^{|\mathbf{k}|}}{\gamma(1-2^{|\mathbf{n}|})} \sum_{\substack{\mathbf{0}\leqslant\boldsymbol{\ell}\leqslant\mathbf{n} \\ \boldsymbol{\ell}\neq\mathbf{n}}} \binom{\mathbf{n}}{\boldsymbol{\ell}} 2^{|\boldsymbol{\ell}|} a_{\boldsymbol{\ell}} b_{\mathbf{n}-\boldsymbol{\ell}},$$

which determines the $a_{\mathbf{n}}$ recursively once $a_{\mathbf{0}}$ has been chosen. Note that any nonzero choice for $a_{\mathbf{0}}$ ensures that $\mathbf{d}_{\mathbf{k}'}$ does indeed have degree $\mathbf{k}'$. $\square$

Remarks. (i) If $\gamma = 0$, part (iii) still holds true in case $S\mathbf{c} \equiv \mathbf{0}$. Then $b_{\mathbf{j}} = 0$ for all $\mathbf{0} \leqslant \mathbf{j} \leqslant \mathbf{k}$, and thus the $a_{\mathbf{j}}$ in (3.18) can be chosen arbitrarily. If $\gamma = 0$ (and thus $b_{\mathbf{0}} = 0$), but $S\mathbf{c} \not\equiv \mathbf{0}$, consider a maximal $\widetilde{\mathbf{k}}$ such that $b_{\mathbf{j}} = 0$ for $\mathbf{0} \leqslant \mathbf{j} \leqslant \widetilde{\mathbf{k}}$. Then part (iii) holds for all $\mathbf{k}' \leqslant \widetilde{\mathbf{k}}$, namely, we have $S\mathbf{d}_{\mathbf{k}'} \equiv \mathbf{0}$ for arbitrary $a_{\mathbf{j}}$. It is worth noting that in this case, we have $S^n\mathbf{c} = \mathbf{0}$ for some (finite) n.

(ii) Given the assumptions of theorem 3.5, it is of interest to ask under what circumstances the sequence $(S^n\mathbf{c})_n$ converges in the subdivision sense. Usually, when dealing with finite sequences, this is meant to say that there exists a (nontrivial) continuous vector function $\mathbf{f}:\mathbb{R}^d \to \mathbb{C}^N$ such that

$$\lim_{n\to\infty} \left\| \mathbf{f}\left(\frac{\cdot}{2^n}\right) - S^n \mathbf{c} \right\|_\infty = 0.$$

But since we are dealing with sequences of polynomial growth, we should not expect uniform convergence. Instead, in case $\gamma = 1$, we have pointwise convergence of $S^n \mathbf{d}_{\mathbf{k}'}$ to the function $\mathbf{f}_{\mathbf{k}'}(\mathbf{x}) = (-\mathrm{i}\mathbf{x})^{\mathbf{k}'}\mathbf{w_0}$ in the sense that for any dyadic point $(\boldsymbol{\alpha}/2^m) \in \mathbb{R}^d$,

$$\lim_{n\to\infty} \left(S^n \mathbf{d}_{\mathbf{k}'}\right)_{2^n\boldsymbol{\alpha}/2^m} = \mathbf{f}_{\mathbf{k}'}\left(\frac{\boldsymbol{\alpha}}{2^m}\right),$$

and this convergence is uniform on compact subsets of $\mathbb{R}^d$. In this case, $(S^n\mathbf{c})_n$ also converges, namely, to $f\mathbf{v_0}$ for some $f \in \mathbb{P}_{\mathbf{k}}$. This can be shown by writing $\mathbf{c}$ as a linear combination of the $\mathbf{d}_{\mathbf{k}'}$, and this linear combination also can be used to determine f.

Note that this also implies that for $|\gamma| > 1$, we have $(S^n \mathbf{d}_{\mathbf{k}'})_{2^n\boldsymbol{\alpha}/2^m} \to \infty$ for all $\boldsymbol{\alpha} \neq \mathbf{0}$, $m \in \mathbb{N}$, while for $|\gamma| < 1$, we have $(S^n \mathbf{d}_{\mathbf{k}'})_{2^n\boldsymbol{\alpha}/2^m} \to 0$. (Of course, for any $\gamma \neq 0$, we can obtain convergence by simply replacing the mask $\mathbf{P}$ by $\gamma^{-1}\mathbf{P}$.)

3.4. *An explicit algorithm*

Based on the above results, we can formulate an algorithm to determine maximal values for $\mathbf{k}$. Note that in the proof of theorem 3.5(iii), equations (3.18) determining the $a_{\mathbf{n}}$ and thus the $\mathbf{w_j}$ in $\mathbf{d}_{\mathbf{k}'}$ do not depend on $\mathbf{k}'$. Therefore we can determine the sequence of coefficient vectors recursively, and whenever this recursion stops, we have found a maximal value for $\mathbf{k}$.

To this end, we once more make the 'Ansatz' (3.10) for $\mathbf{c}$. Then $S\mathbf{c} = (1/2^{|\mathbf{k}|})\mathbf{c}$ is equivalent to (3.11) and (3.12) with $\mathbf{v}_{\mathbf{j}}^{(1)} = (1/2^{|\mathbf{k}|})\mathbf{v_j}$, which we can rewrite as

$$\frac{1}{2^d} \sum_{\mathbf{0}\leqslant\boldsymbol{\ell}\leqslant\mathbf{j}} \binom{\mathbf{j}}{\boldsymbol{\ell}} \frac{1}{2^{|\mathbf{j}-\boldsymbol{\ell}|}} \mathbf{P}^{\wedge(\mathbf{j}-\boldsymbol{\ell})*}(\pi\mathbf{e})\mathbf{v}_{\boldsymbol{\ell}} = \begin{cases} \dfrac{1}{2^{|\mathbf{j}|}}\mathbf{v_j} & \text{for } \mathbf{e} = \mathbf{0}, \\ \mathbf{0} & \text{for } \mathbf{e} \in E \setminus \{\mathbf{0}\}. \end{cases}$$

Thus our recursion algorithm is the following:
For $\mathbf{j} = \mathbf{0}$:

$$\left(\frac{1}{2^d}\mathbf{P}^{\wedge*}(\pi\mathbf{e}) - \delta_{\mathbf{e},\mathbf{0}}\mathbf{I}_n\right)\mathbf{v_0} = \mathbf{0} \quad \text{for } \mathbf{e} \in E;$$

for $\mathbf{j} \neq \mathbf{0}$:

$$\left(\frac{1}{2^d}\mathbf{P}^{\wedge*}(\pi\mathbf{e}) - \delta_{\mathbf{e},\mathbf{0}}\frac{1}{2^{|\mathbf{j}|}}\mathbf{I}_n\right)\mathbf{v_j} = -\frac{1}{2^d} \sum_{\substack{\mathbf{0}\leqslant\boldsymbol{\ell}\leqslant\mathbf{j} \\ \boldsymbol{\ell}\neq\mathbf{j}}} \binom{\mathbf{j}}{\boldsymbol{\ell}} \frac{1}{2^{|\mathbf{j}-\boldsymbol{\ell}|}} \mathbf{P}^{\wedge(\mathbf{j}-\boldsymbol{\ell})*}(\pi\mathbf{e})\mathbf{v}_{\boldsymbol{\ell}} \quad \text{for } \mathbf{e} \in E.$$

Here, $\mathbf{I}_n$ denotes the $(n \times n)$ unit matrix. If (E_0) holds, then $\mathbf{v_0}$ is unique (up to a multiplicative constant), and then the $\mathbf{v_j}$ are unique also.

4. Examples

Example 4.1 (Splines of order four with double knots [4]). Here, $d = 1$. The space $S_4^1(\mathbb{Z})$ of polynomial splines of order 4 (degree 3) with double knots at the integers is spanned by the integer translates of the two splines

$$\phi_1(x) = \phi_1(-x)$$

with

$$\phi_1(x) = \begin{cases} (2x+1)(x-1)^2 & \text{for } 0 \leqslant x \leqslant 1, \\ 0 & \text{for } x > 1 \end{cases}$$

and

$$\phi_2(x) = -\phi_2(-x)$$

with

$$\phi_2(x) = \begin{cases} x(x-1)^2 & \text{for } 0 \leqslant x \leqslant 1, \\ 0 & \text{for } x > 1. \end{cases}$$

The two generators are normalized so as to satisfy the interpolating condition

$$D^j \phi_\nu(\alpha) = \delta_{0,\alpha}\delta_{j,\nu-1}, \quad \text{for } \alpha \in \mathbb{Z} \text{ and } j, \nu-1 = 0, 1.$$

Hence, any spline $s \in S_4^1(\mathbb{Z})$ can be written in terms of the interpolation formula

$$s(x) = \sum_\alpha \left\{ s(\alpha)\phi_1(x-\alpha) + s'(\alpha)\phi_2(x-\alpha) \right\},$$

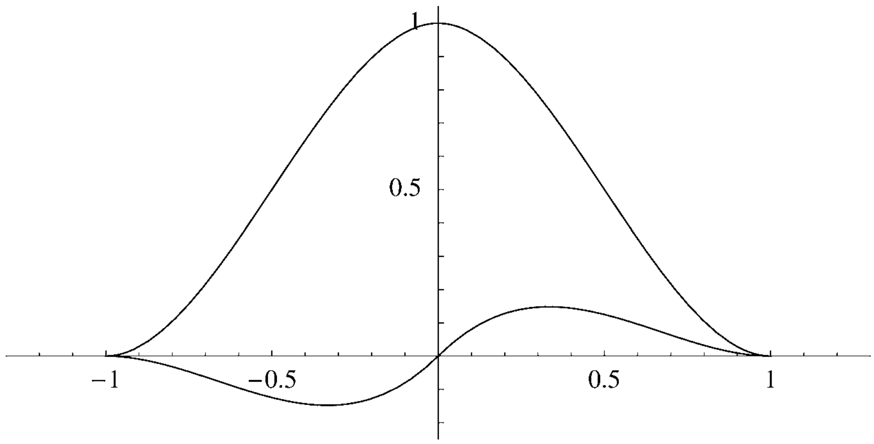

Figure 1. Generators ϕ_1 and ϕ_2 for splines of order four with double knots.

and in particular, any polynomial of degree 3 can be written this way. The system $\mathbf{\Phi} = \binom{\phi_1}{\phi_2}$ is refinable with matrix mask $\mathbf{P} = (\mathbf{P}_\alpha)_{\alpha\in\mathbb{Z}}$ given by

$$\cdots \quad \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \underbrace{\begin{pmatrix} \frac{1}{2} & \frac{3}{4} \\ -\frac{1}{8} & -\frac{1}{8} \end{pmatrix}}_{\mathbf{P}_{-1}} \quad \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}}_{\mathbf{P}_0} \quad \underbrace{\begin{pmatrix} \frac{1}{2} & -\frac{3}{4} \\ \frac{1}{8} & -\frac{1}{8} \end{pmatrix}}_{\mathbf{P}_{+1}} \quad \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \cdots$$

whence,

$$\mathbf{P}^\wedge(\xi) = \begin{pmatrix} \frac{1}{2}\mathrm{e}^{+\mathrm{i}\xi} + 1 + \frac{1}{2}\mathrm{e}^{-\mathrm{i}\xi} & \frac{3}{4}\mathrm{e}^{+\mathrm{i}\xi} + 0 - \frac{3}{4}\mathrm{e}^{-\mathrm{i}\xi} \\ -\frac{1}{8}\mathrm{e}^{+\mathrm{i}\xi} + 0 + \frac{1}{8}\mathrm{e}^{-\mathrm{i}\xi} & -\frac{1}{8}\mathrm{e}^{+\mathrm{i}\xi} + \frac{1}{2} - \frac{1}{8}\mathrm{e}^{-\mathrm{i}\xi} \end{pmatrix}.$$

Checking with our algorithm, we find the vectors

$$\mathbf{v}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \mathbf{v}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad \mathbf{v}_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \text{and} \quad \mathbf{v}_3 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Example 4.2 (Modified pyramidal scheme). Our second example is bivariate ($d = 2$). We start with piecewise linear splines on the four-directional mesh (as studied by Conti and Jetter [3], and others, see, e.g., Goodman [8]) with corresponding refinement mask

$$\begin{array}{cc}
\left(\begin{array}{ccccccc}
 & \vdots & \vdots & \vdots & \vdots & \vdots & \\
\cdots & 0 & 0 & 0 & 0 & 0 & \cdots \\
\cdots & 0 & 0 & \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 1 & \frac{1}{2} \end{pmatrix} & 0 & \cdots \\
\cdots & 0 & \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 1 & \frac{1}{2} \\ 0 & \frac{1}{2} \end{pmatrix} & \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} & 0 & \cdots \\
\cdots & 0 & \begin{pmatrix} 0 & \frac{1}{2} \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 0 \end{pmatrix} & 0 & 0 & \cdots \\
\cdots & 0 & 0 & 0 & 0 & 0 & \cdots \\
 & \vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}\right) & \leftarrow \alpha_2 = 0 \\
\begin{array}{c} \uparrow \\ \alpha_1 = 0 \end{array} &
\end{array}$$

where the position of the (2×2)-matrices $\mathbf{P}_{\boldsymbol{\alpha}}$ within the bi-infinite matrix $\mathbf{P}$ refers to the index $\boldsymbol{\alpha} = (\alpha_1, \alpha_2) \in \mathbb{Z}^2$, with α_1 increasing from left to right and α_2 from bottom up. The subdivision scheme reproduces linear polynomials.

In an attempt to increase the order of polynomial reproduction, we introduced several parameters in the scheme and applied our algorithm. As a particular result, we have found the matrix mask

$$
\begin{array}{cc}
\left(\begin{array}{ccccccc}
0 & 0 & 0 & \begin{pmatrix} -\frac{1}{16} & 0 \\ 0 & 0 \end{pmatrix} & 0 & 0 & 0 \\
0 & 0 & \begin{pmatrix} 0 & 0 \\ 0 & -\frac{1}{16} \end{pmatrix} & 0 & 0 & \begin{pmatrix} 0 & 0 \\ 0 & -\frac{1}{16} \end{pmatrix} & 0 \\
0 & \begin{pmatrix} 0 & -\frac{1}{16} \\ 0 & 0 \end{pmatrix} & 0 & \begin{pmatrix} \frac{9}{16} & 0 \\ 0 & \frac{9}{16} \end{pmatrix} & \begin{pmatrix} 0 & -\frac{1}{16} \\ 1 & \frac{9}{16} \end{pmatrix} & 0 & 0 \\
\begin{pmatrix} -\frac{1}{16} & 0 \\ 0 & 0 \end{pmatrix} & 0 & \begin{pmatrix} \frac{9}{16} & \frac{9}{16} \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 1 & \frac{9}{16} \\ 0 & \frac{9}{16} \end{pmatrix} & \begin{pmatrix} \frac{9}{16} & 0 \\ 0 & \frac{9}{16} \end{pmatrix} & 0 & \begin{pmatrix} -\frac{1}{16} & 0 \\ 0 & 0 \end{pmatrix} \\
0 & 0 & \begin{pmatrix} 0 & \frac{9}{16} \\ 0 & -\frac{1}{16} \end{pmatrix} & \begin{pmatrix} \frac{9}{16} & \frac{9}{16} \\ 0 & 0 \end{pmatrix} & 0 & \begin{pmatrix} 0 & 0 \\ 0 & -\frac{1}{16} \end{pmatrix} & 0 \\
0 & \begin{pmatrix} 0 & -\frac{1}{16} \\ 0 & 0 \end{pmatrix} & 0 & 0 & \begin{pmatrix} 0 & -\frac{1}{16} \\ 0 & 0 \end{pmatrix} & 0 & 0 \\
0 & 0 & 0 & \begin{pmatrix} -\frac{1}{16} & 0 \\ 0 & 0 \end{pmatrix} & 0 & 0 & 0
\end{array}\right)
& \begin{array}{c} \\ \\ \\ \\ \\ \leftarrow \\ \\ \\ \\ \\ \\ \end{array} \\
\begin{array}{ccccccc} & & & \uparrow & & & \\ & & & \alpha_1=0. & & & \end{array} &
\end{array}
$$

where $\leftarrow \alpha_2 = 0$.

It is interesting to see that this scheme originates from the former one by replacing all entries $\frac{1}{2}$ by $\frac{1}{2}+\omega$ with $\omega = \frac{1}{16}$, and by replacing some other entries 0 by $-\omega$. In this sense, this extends the construction of the "four point scheme" as described by Dubuc [5] and Dyn et al. [7].

It is surprising to see that the order of the modified scheme is four (i.e., the subdivision scheme reproduces polynomials up to total degree three). Here, the vectors $\mathbf{v}_{\boldsymbol{\ell}}$ are given by

$$
\mathbf{v}_{\mathbf{0}} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{v}_{\boldsymbol{\ell}} = \begin{pmatrix} 0 \\ 2^{-|\boldsymbol{\ell}|} \end{pmatrix}, \quad 1 \leqslant |\boldsymbol{\ell}| \leqslant 3.
$$

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