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Biorthogonal Spline Wavelets on the Interval— Stability and Moment Conditions

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   This paper is concerned with the construction of biorthogonal multiresolution analyses on [0, 1] such that the corresponding wavelets realize any desired order of moment conditions throughout the interval. Our starting point is the family of biorthogonal pairs consisting of cardinal B-splines and compactly supported dual generators on $\mathbb{R}$ developed by Cohen, Daubechies, and Feauveau. In contrast to previous investigations we preserve the full degree of polynomial reproduction also for the dual multiresolution and prove in general that the corresponding modifications of dual generators near the end points of the interval still permit the biorthogonalization of the resulting bases. The subsequent construction of compactly supported biorthogonal wavelets is based on the concept of stable completions. As a first step we derive an initial decomposition of the spline spaces where the complement spaces between two successive levels are spanned by compactly supported splines which form uniformly stable bases on each level. As a second step these initial complements are then projected into the desired complements spanned by compactly supported biorthogonal wavelets. Since all generators and wavelets on the primal and the dual sides have finitely supported masks, the corresponding decomposition and reconstruction algorithms are simple and efficient. The desired number of vanishing moments is implied by the polynomial exactness of the dual multiresolution. Again due to the polynomial exactness the primal and dual spaces satisfy corresponding Jackson estimates. In addition, Bernstein inequalities can be shown to hold for a range of Sobolev norms depending on the regularity of the primal and dual wavelets. Then it follows from general principles that the wavelets form Riesz bases for $L_2([0, 1])$ and that weighted sequence norms for the coefficients of such wavelet expansions characterize Sobolev spaces and their duals on [0, 1] within a range depending on the parameters in

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the Jackson and Bernstein estimates. We conclude by addressing several issues concerning numerical implementation. In particular, we test the quantitative stability properties of corresponding multiscale transformations, indicate strategies for improving them, and present some numerical experiments.   © 1999 Academic Press

1. INTRODUCTION

The objective of this paper is the construction of biorthogonal multiresolution analyses and corresponding wavelets on the interval [0, 1] with the following properties:

(i) The primal multiresolution consists of spline spaces for any desired degree $d - 1$.

(ii) For a given degree $d - 1$ of the splines and any $\tilde{d} \in \mathbb{N}$, $\tilde{d} \geq d$ such that $d + \tilde{d}$ is even, the dual multiresolution has degree $\tilde{d} - 1$ of polynomial exactness.

(iii) As a consequence of (ii) the biorthogonal spline wavelets have the corresponding number $\tilde{d}$ of vanishing moments.

(iv) All generators and wavelets on the primal and the dual sides have finitely supported masks so that decomposition and reconstruction algorithms are simple and fast.

(v) The wavelets form Riesz bases for $L_2([0, 1])$. Moreover, discrete norms based on these wavelet expansions characterize Sobolev spaces and their duals on [0, 1] within a range depending on the regularity and degree of exactness of the involved multiresolution analyses.

1.1. Background and Motivation

The issue of constructing wavelets on the interval has been recently addressed in several papers (see, e.g., [2, 10, 14, 16, 41]). However, as far as we know none of these approaches meets the above complete list of requirements. While [10, 16] focus on orthogonal decompositions, [2] addresses biorthogonal multiresolution but fails to build in any polynomial exactness of the dual spaces. Furthermore, neither is it proved there that the central biorthogonalization of properly adjusted spanning sets is actually possible nor are the Riesz basis property and related Sobolev norm equivalences established which are of fundamental importance for many applications.

It is perhaps instructive to point out why the above requirements are important and why

we found it worthwhile to invest some further technical effort into their realization. First a few general comments. One important property of wavelets on the line is that the wavelet representation of many operators is (nearly) sparse, which is crucial for fast numerical processing. The near sparseness is an immediate consequence of a sufficiently high number of vanishing moments; see (iii). This, in turn, is implied by the corresponding polynomial exactness of the dual multiresolution throughout the respective domain as required in (ii) above.

As for (v), the stability of the multiscale transformations forming the reconstruction and decomposition procedures is known to be equivalent to the Riesz basis property of the wavelet bases. The Sobolev norm equivalences, in turn, are equivalent to the fact that for elliptic problems diagonal scalings of stiffness matrices relative to wavelet bases yield uniformly bounded condition numbers and thus facilitate fast iterative solvers [21, 28].

Wavelet schemes for the approximate solution of saddle point problems stemming, for instance, from a weak formulation of the Stokes problem or mixed formulations of second order scalar elliptic equations lead to further examples where the requirements (ii) and (v) are essential. Here it is important to construct pairs of trial spaces for pressure and velocity, say, which are compatible in the sense that the so-called *Ladyšenskaja–Babuška–Brezzi condition* is satisfied. In [22] the construction of families of such spaces for any spatial dimension and any degree of exactness was based on suitable biorthogonal multiresolution. Since the pressure is discretized there by the primal multiresolution while velocities are represented in terms of the dual multiresolution, it is important that both spaces have sufficient polynomial exactness to guarantee accurate solutions. In particular, realization of a higher degree of exactness for the velocities requires the ability to raise the exactness of the dual multiresolution independently of the degree of the primal one as in (ii) above. Again preconditioning of the resulting matrices is based on (v).

Another context where the above conditions are relevant is the numerical solution of boundary integral equations. While conventional boundary element methods usually give rise to densely populated matrices, wavelet-based discretizations often lead to nearly sparse matrices [5, 26, 27, 44]. The analysis in [28, 45] yields precise conditions on the wavelets that guarantee asymptotically optimal efficiency. By this we mean that the compressed stiffness matrices contain only an amount of nonvanishing entries of order N, N being the number of unknowns; that diagonal scalings produce uniformly bounded condition numbers; and that the solutions to the compressed systems still exhibit the *same asymptotic* accuracy as those to the unperturbed problems. When the boundary surfaces are represented by parametric mappings it is convenient to construct the wavelets on the surface by means of parametric mappings of wavelets defined on the unit square [30]. Thus, again tensor products of wavelets on the unit interval form the core of the construction. Specifically, when the integral operators have nonpositive order, asymptotic optimality requires for the primal system a higher number of moment conditions than the degree of exactness, which rules out orthogonal decompositions and stresses the importance of (ii) and (iii). Also, when the operators have negative order, preconditioning the compressed matrices requires the validity of Sobolev norm equivalences also for Sobolev spaces of negative order. Again (ii) is needed for this purpose.

These are some instances where the results available in the literature were not sufficient and thus lead us to the present investigation. One could add other examples such as

appending boundary conditions by Lagrange multipliers [37]. This leads to similar requirements on the wavelets defined on the boundary curve or surface. Furthermore, biorthogonal wavelet bases are used for the pressure computation when employing divergence free wavelets for the Stokes problem [49].

Our starting point is a family of biorthogonal multiresolution analyses on $\mathbb{R}$ developed in [15]. Specifically, we confine the discussion to the case where the primal multiresolution is generated by cardinal B-splines. Polynomial splines have numerous practical advantages over other types of scaling functions, among them explicit analytic representations and minimal support relative to their smoothness.

1.2. Biorthogonal Multiresolution in $L_2(\mathbb{R})$

A function $\theta \in L_2(\mathbb{R})$ is called *refinable* with *mask* $\mathbf{a} = \{a_k\}_{k \in \mathbb{Z}}$, $a_k \in \mathbb{R}$, if

$$\theta(x) = \sum_{k \in \mathbb{Z}} a_k \theta(2x - k), \qquad x \in \mathbb{R} \ a.e. \tag{1.2.1}$$

We say that two refinable functions θ, $\tilde{\theta}$ form a *dual pair* if

$$(\theta, \tilde{\theta}(\cdot - k))_{\mathbb{R}} = \delta_{0,k}, \qquad k \in \mathbb{Z}, \tag{1.2.2}$$

where in what follows, for any domain $\Omega \subset \mathbb{R}$,

$$(f, g)_\Omega := \int_\Omega f(x) g(x) dx.$$

It is well known that θ and $\tilde{\theta}$ can be normalized so that

$$\int_{\mathbb{R}} \theta(x) dx = \int_{\mathbb{R}} \tilde{\theta}(x) dx = 1. \tag{1.2.3}$$

Let us abbreviate for any collection $C \subset L_2(\Omega)$

$$S(C) := \mathrm{clos}_{L_2}(\mathrm{span}\ C),$$

the L_2 closure of the linear span of C. It will be convenient to write for $g \in L_2(\mathbb{R})$

$$g_{[j,k]} := 2^{j/2} g(2^j \cdot - k), \qquad j, k \in \mathbb{Z}.$$

Thus, defining

$$S_j = S(\{\theta_{[j,k]} : k \in \mathbb{Z}\}), \qquad \tilde{S}_j = S(\{\tilde{\theta}_{[j,k]} : k \in \mathbb{Z}\}), \tag{1.2.4}$$

refinability is known to imply

$$\cdots \subset S_j \subset S_{j+1} \subset \cdots, \qquad \mathrm{clos}_{L_2}\left(\bigcup_{j\in\mathbb{Z}} S_j\right) = L_2(\mathbb{R}),$$

$$\cdots \subset \tilde{S}_j \subset \tilde{S}_{j+1} \subset \cdots, \qquad \mathrm{clos}_{L_2}\left(\bigcup_{j\in\mathbb{Z}} \tilde{S}_j\right) = L_2(\mathbb{R}), \qquad (1.2.5)$$

and

$$\bigcap_{j\in\mathbb{Z}} S_j = \bigcap_{j\in\mathbb{Z}} \tilde{S}_j = \{0\}.$$

Moreover, if θ, $\tilde{\theta}$ have compact support it is easy to see that for $\mathbf{c} = \{c_k\}_{k\in\mathbb{Z}} \in l_2(\mathbb{Z})$,

$$\left\|\sum_{k\in\mathbb{Z}} c_k\theta(\,\cdot\, - k)\right\|_{L_2(\mathbb{R})} \sim \|\mathbf{c}\|_{l_2(\mathbb{Z})} \sim \left\|\sum_{k\in\mathbb{Z}} c_k\tilde{\theta}(\,\cdot\, - k)\right\|_{L_2(\mathbb{R})}, \qquad (1.2.6)$$

which due to

$$\|\theta_{[j,k]}\|_{L_2(\mathbb{R})} = \|\theta\|_{L_2(\mathbb{R})} \qquad (1.2.7)$$

implies the *uniform stability* of the scaled dilates,

$$\left\|\sum_{k\in\mathbb{Z}} c_k\theta_{[j,k]}\right\|_{L_2(\mathbb{R})} \sim \|\mathbf{c}\|_{l_2(\mathbb{Z})}, \qquad (1.2.8)$$

and likewise for $\tilde{\theta}$. Here $a \lesssim b$ means that a can be bounded by some constant multiple of b uniformly in any parameters on which a and b may depend. We write $a \sim b$ if $a \lesssim b$ and $b \lesssim a$.

θ, $\tilde{\theta}$ are called the *generators* of the multiresolution sequences $\mathcal{S} = \{S_j\}_{j\in\mathbb{Z}}$, $\tilde{\mathcal{S}} = \{\tilde{S}_j\}_{j\in\mathbb{Z}}$. Moreover, it will be convenient to refer to $\mathcal{S}$ and $\tilde{\mathcal{S}}$ as *primal* and *dual* multiresolution.

Recall that in the given circumstances the polynomial exactness of the spaces S_j determines their approximation power. We say θ is *exact of order* d if all polynomials of degree at most $d - 1$ can be written as a linear combination of the integer translates $\theta(\,\cdot\, - k)$. In fact, defining

$$\alpha_{\tilde{\theta},r}(y) := ((\,\cdot\,)^r, \tilde{\theta}(\,\cdot\, - y))_{\mathbb{R}} \qquad (1.2.9)$$

one has then, in view of (1.2.2), the explicit representation

$$x^r = \sum_{k\in\mathbb{Z}} \alpha_{\tilde{\theta},r}(k)\theta(x - k), \qquad x \in \mathbb{R} \ a.e., \ r = 0, \ldots, d - 1, \qquad (1.2.10)$$

which will be used frequently later on.

The concept of biorthogonal wavelets consists now of finding complement spaces W_j, $\tilde{W}_j$ of S_j, $\tilde{S}_j$ in S_{j+1}, $\tilde{S}_{j+1}$, respectively, satisfying

$$\tilde{W}_j \perp S_j, \qquad W_j \perp \tilde{S}_j, \tag{1.2.11}$$

so that, by (1.2.5),

$$W_j \perp \tilde{W}_r, \qquad j \neq r. \tag{1.2.12}$$

It is known [15] that by defining new masks

$$b_k := (-1)^k \tilde{a}_{1-k}, \qquad \tilde{b}_k := (-1)^k a_{1-k}, \, k \in \mathbb{Z}, \tag{1.2.13}$$

such spaces W_j, $\tilde{W}_j$ are generated by the dilates and translates of the functions

$$\psi(x) := \sum_{k \in \mathbb{Z}} b_k \theta(2x - k), \qquad \tilde{\psi}(x) := \sum_{k \in \mathbb{Z}} \tilde{b}_k \tilde{\theta}(2x - k), \tag{1.2.14}$$

which satisfy

$$(\theta, \tilde{\psi}(\cdot - k))_{\mathbb{R}} = (\tilde{\theta}, \psi(\cdot - k))_{\mathbb{R}} = 0, \qquad (\psi, \tilde{\psi}(\cdot - k))_{\mathbb{R}} = \delta_{0,k}, \, k \in \mathbb{Z}. \tag{1.2.15}$$

Note that if $\tilde{\theta}$ is exact of order $\tilde{d}$, one immediately infers from (1.2.10) and (1.2.11) that ψ has $\tilde{d}$ *vanishing moments,* i.e.,

$$\int_{\mathbb{R}} x^r \psi(x) dx = 0, \qquad r = 0, \ldots, \tilde{d} - 1. \tag{1.2.16}$$

Finally, a fact of primary importance is that the collections $\{\psi_{[j,k]} : j, \, k \in \mathbb{Z}\}$, $\{\tilde{\psi}_{[j,k]} : j, \, k \in \mathbb{Z}\}$ form (biorthogonal) *Riesz bases* of $L_2(\mathbb{R})$, which means that

$$\|v\|_{L_2(\mathbb{R})}^2 \sim \sum_{j,k \in \mathbb{Z}} |(v, \tilde{\psi}_{[j,k]})_{\mathbb{R}}|^2 \sim \sum_{j,k \in \mathbb{Z}} |(v, \psi_{[j,k]})_{\mathbb{R}}|^2, \qquad v \in L_2(\mathbb{R}). \tag{1.2.17}$$

In fact, the latter relations can be extended to norm equivalences for a certain range of Sobolev spaces.

The objective of the subsequent investigation is to construct biorthogonal wavelet bases for $L_2([0, 1])$, retaining as many properties of the above setting as possible.

1.3. The Layout of the Paper

The standard derivation of the above facts makes heavy use of Fourier techniques (see, e.g., [15]). When working on the interval such techniques are not directly applicable, which forces us to resort to alternative tools. Therefore we briefly collect in Section 2 a few general concepts which will serve that purpose and will guide later constructions.

Some of these results are known, some are implicit in various studies, and some are simply folklore. Nevertheless, we hope that the reader will benefit from our putting them briefly together since we feel that they help make several somewhat technical developments more transparent. The main tools are *stable completions* [8] and associated stability criteria as well as a mechanism for generating from some initial multiscale decomposition of a multiresolution space other complements which correspond to biorthogonal wavelets. Moreover, we recall a general criterion for establishing the Riesz basis property and Sobolev norm equivalences based on *direct* and *inverse estimates* [18]. The results presented in this section provide the guideline for the whole subsequent development.

In Section 3 we construct spline multiresolution spaces of arbitrary degree on [0, 1] along with a dual multiresolution satisfying requirements (i) and (ii) above. The basic idea is quite familiar. To construct two collections of spanning sets of multiresolution sequences that are candidates for biorthogonal bases on [0, 1], one has to form special boundary near basis functions by forming fixed linear combinations of translates of scaling functions in such a way that the resulting linear spans are still nested and contain all polynomials up to the original degree of exactness (see also [2, 16]). Since (see (3.2.5) below) the support of the dual generator $\tilde{\theta}$ is at least as large as that of θ, the degree of exactness of the dual multiresolution determines the number of summands appearing in the boundary near basis functions. Due to these modifications the resulting collections of functions have equal cardinality but are, of course, no longer biorthogonal. Apparently, the fact that a biorthogonalization is actually possible has never been established in previous investigations, not even in the case where no exactness is enforced on the dual side. After completion of this paper we became aware of a similar approach [40], in which, however, again the biorthogonalization is not rigorously justified. Therefore, we invest some effort in proving that the involved linear systems are *always nonsingular,* which is the main result in Section 3. Moreover, application of the results from Section 2 yields discrete norms which are equivalent to Sobolev norms on [0, 1] for a range depending on the regularity and the exactness of the generators.

Section 4 is devoted to the construction of biorthogonal wavelets. Here our approach differs again in an essential way from previous studies. It is divided into two steps. By adapting a factorization result for bi-infinite B-spline refinement matrices from [24] to the case at hand, we first construct in a systematic way a family of *initial* stable completions. Once this has been accomplished we can again make use of the results in Section 2 to derive next biorthogonal wavelet bases. The primal and dual wavelets all have compact support. Moreover, combining the stability of the completions with the norm equivalences from Section 3 readily confirms that these wavelet bases are Riesz bases for $L_2([0, 1])$. Furthermore, weighted coefficient norms are shown to be equivalent to Sobolev norms within certain ranges of Sobolev exponents.

Finally, in Section 5 we briefly comment on the actual computation of the various ingredients of the construction and display a list of filter coefficients as well as plots of the generators and wavelets for the example $d = 3$, $\tilde{d} = 5$. Data for several other cases can be obtained from the authors. Although the theoretical results guarantee the stability of the constructed wavelet bases, the size of the involved constants will matter in practical applications. Therefore, we have tested the condition numbers of the wavelet transformation matrices. As expected the Riesz constants grow with increasing degree of exactness.

However, we indicate strategies for stabilizing the single scale generator bases which turn out to have a significant favorable effect on the whole multiscale transformation. Corresponding numerical experiments are recorded at the end, in Section 5.3.

2. SOME GENERAL CONCEPTS

2.1. Two Scale Relations

When trying to carry over the results from Section 1.2 to the interval [0, 1] we have to give up on translation invariance. The arguments which we will employ actually hold in greater generality. Since we will use several variants, we will formulate the main facts in sufficiently general terms. For later use we record a few facts from [8]. Let $\mathcal{H}$ be some Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|_{\mathcal{H}}$. We are interested in spaces of the form $S(\Phi_j), j \geq j_0, j_0 \in \mathbb{N}$ fixed, where $\Phi_j = \{ \phi_{j,k} : k \in \Delta_j \} \subset \mathcal{H}$ are *uniformly stable* in the sense that

$$\|\mathbf{c}\|_{l_2(\Delta_j)} \sim \|\Phi_j^T \mathbf{c}\|_{\mathcal{H}}, \qquad \mathbf{c} \in l_2(\Delta_j), \tag{2.1.1}$$

holds uniformly in $j \geq j_0$. Here we have used the shorthand notation

$$\Phi_j^T \mathbf{c} := \sum_{k \in \Delta_j} c_k \phi_{j,k},$$

where Φ_j is viewed as the (column) *vector* containing the functions $\phi_{j,k}$. It is known that nestedness $S(\Phi_j) \subset S(\Phi_{j+1})$ and stability imply the existence of matrices $\mathbf{M}_{j,0} = (m_{l,k}^j)_{l \in \Delta_{j+1}, k \in \Delta_j}$ such that

$$\phi_{j,k} = \sum_{l \in \Delta_{j+1}} m_{l,k}^j \phi_{j+1,l}, \qquad k \in \Delta_j. \tag{2.1.2}$$

Of course, in the case (1.2.1) treated in Section 1.2 the refinement coefficients $m_{l,k}^j = 2^{-1/2} a_{l-2k}$ are independent of j. However, in particular, when working on the interval it will be much more convenient to regard the above refinement relation as a *matrix* relation; i.e., (2.1.2) takes the form

$$\Phi_j^T = \Phi_{j+1}^T \mathbf{M}_{j,0}. \tag{2.1.3}$$

Moreover, denoting by $[X, Y]$ the space of bounded linear operators from a normed linear space X into the normed linear space Y one has [8]

$$\mathbf{M}_{j,0} \in [l_2(\Delta_j), l_2(\Delta_{j+1})], \qquad \|\mathbf{M}_{j,0}\| = \mathbb{O}(1), j \geq j_0, \tag{2.1.4}$$

where

$$\|\mathbf{M}_{j,0}\| := \sup_{\mathbf{u} \in l_2(\Delta_j), \|\mathbf{u}\|_{l_2(\Delta_j)} = 1} \|\mathbf{M}_{j,0} \mathbf{u}\|_{l_2(\Delta_{j+1})}.$$

Although in all our applications the index sets Δ_j will be finite, we remark that the results remain valid for infinite sets Δ_j as well where the corresponding matrix–vector operations are to be understood in the sense of absolute convergence. Thus, the situation in Section 1.2 is covered as well.

2.2. Stability and Approximation

Exploiting our shorthand notation a little further we define for any two collections Φ, $\Theta \subset \mathcal{H}$ the matrix

$$\langle \Phi, \Theta \rangle := (\langle \phi, \theta \rangle)_{\phi \in \Phi, \theta \in \Theta}.$$

In particular, for $v \in \mathcal{H}$, $\langle v, \Phi_j \rangle$ denotes the (row) vectors with entries $\langle v, \phi_{j,k} \rangle$, $k \in \Delta_j$. We will make use also of the following observation whose proof will be included for the convenience of the reader.

LEMMA 2.1.   Let $\mathcal{H} = L_2(\Omega)$, where Ω is a domain in $\mathbb{R}^n$ or a manifold, and let Φ_j, $\tilde{\Phi}_j \subset \mathcal{H}$ have the following properties:

(a) Φ_j and $\tilde{\Phi}_j$ are biorthogonal, i.e.,

$$\langle \Phi_j, \tilde{\Phi}_j \rangle = \mathbf{I}. \tag{2.2.1}$$

(b) $\|\phi_{j,k}\|_{\mathcal{H}}$, $\|\tilde{\phi}_{j,k}\|_{\mathcal{H}} \lesssim 1$, $k \in \Delta_j$.
(c) Φ_j and $\tilde{\Phi}_j$ are locally finite, i.e., setting

$$\sigma_{j,k} := \operatorname{supp} \phi_{j,k}, \qquad \tilde{\sigma}_{j,k} := \operatorname{supp} \tilde{\phi}_{j,k}, \; k \in \Delta_j,$$

one has

$$\#\{k' \in \Delta_j : \sigma_{j,k'} \cap \sigma_{j,k} \neq \varnothing\}, \qquad \#\{k' \in \Delta_j : \tilde{\sigma}_{j,k'} \cap \tilde{\sigma}_{j,k} \neq \varnothing\} \lesssim 1 \qquad \textit{for all } k \in \Delta_j.$$
$$\tag{2.2.2}$$

Then

(i) $\{\Phi_j\} := \{\Phi_j\}_{j \geq j_0}$, $\{\tilde{\Phi}_j\} := \{\tilde{\Phi}_j\}_{j \geq j_0}$ are uniformly stable.
(ii) Let Ω be a domain with Lipschitz boundary and let $\Pi_l(\Omega)$ denote the space of polynomials of total degree (at most) $l - 1$ on Ω. If $\Pi_l(\Omega) \subseteq S(\Phi_j)$ then

$$\inf_{v_j \in S(\Phi_j)} \|v - v_j\|_{L_2(\Omega)} \lesssim h_j^l \|v\|_{H^l(\Omega)}, \qquad v \in H^l(\Omega),$$

where $h_j := \sup_{k \in \Delta_j} \{\operatorname{diam} \tilde{\sigma}_{j,k}, \operatorname{diam} \sigma_{j,k}\}$.
Here $H^l(\Omega)$ is the usual Sobolev space with norm $\| \cdot \|_{H^l(\Omega)}$.

Proof. By (b) and (c), one has for $k \in \Delta_j$

$$\|\Phi_j^T \mathbf{c}\|_{L_2(\sigma_{j,k})}^2 \lesssim (\sum_{k' \in \Delta_{j,k}} |c_{k'}|)^2 \lesssim \sum_{k' \in \Delta_{j,k}} |c_{k'}|^2,$$

where $\Delta_{j,k} := \{k' \in \Delta_j : \sigma_{j,k} \cap \sigma_{j,k'} \neq \varnothing\}$. Summing over $k \in \Delta_j$ and taking (2.2.2) into account provides $\|\Phi_j^T \mathbf{c}\|_{L_2(\Omega)} \lesssim \|\mathbf{c}\|_{l_2(\Delta_j)}$. Furthermore, let $v_j := \Phi_j^T \mathbf{c}$ so that by (a) and (b), $|c_k|^2 = |\langle v_j, \tilde{\phi}_{j,k} \rangle|^2 \lesssim \|v_j\|_{L_2(\tilde{\sigma}_{j,k})}^2$. Again summing over $k \in \Delta_j$ and using (2.2.2) yields $\|\mathbf{c}\|_{l_2(\Delta_j)} \lesssim \|v_j\|_{L_2(\Omega)}$, which proves (i).

As for (ii), one has for $v \in H^l(\Omega)$ and any $P \in \Pi_l(\Omega)$

$$\|v - \langle v, \tilde{\Phi}_j \rangle \Phi_j\|_{L_2(\sigma_{j,k})} \leq \|v - P\|_{L_2(\sigma_{j,k})} + \|\langle v - P, \tilde{\Phi}_j \rangle \Phi_j\|_{L_2(\sigma_{j,k})}$$

$$\lesssim \|v - P\|_{L_2(\sigma_{j,k})} + \|v - P\|_{L_2(\hat{\sigma}_{j,k})},$$

where $\hat{\sigma}_{j,k} := \cup\{\tilde{\sigma}_{j,k'} : k' \in \Delta_{j,k}\}$. Since P was arbitrary (ii) follows from a classical Whitney-type estimate, squaring and summing over $k \in \Delta_j$, and taking again (2.2.2) into account. ∎

REMARK 2.2. *The results of Lemma 2.1 are readily extended to the case $\Phi_j \subset L_p(\Omega)$, $\tilde{\Phi}_j \subset L_q(\Omega)$, where $\frac{1}{p} + \frac{1}{q} = 1$, $1 < p, q < \infty$, replacing $H^l(\Omega)$ by $W_p^l(\Omega)$.*

The proof of Lemma 2.1 already indicates the usefulness of the projectors

$$Q_j v := \langle v, \tilde{\Phi}_j \rangle \Phi_j, \qquad Q_j^* v = \langle v, \Phi_j \rangle \tilde{\Phi}_j, \qquad (2.2.3)$$

which are obviously adjoints of each other.

Now suppose again that $\mathscr{H} = L_2(\Omega)$, where Ω is some sufficiently smooth manifold so that Sobolev spaces $H^s(\Omega)$ are well defined for the range of indices under consideration. First recall the following fact from [8].

REMARK 2.3. *Let $\{\Phi_j\}$ be uniformly stable. The Q_j defined by (2.2.3) are uniformly bounded if and only if $\{\tilde{\Phi}_j\}$ is uniformly stable as well. Moreover, the Q_j satisfy*

$$Q_l Q_j = Q_l, \qquad l \leq j, \qquad (2.2.4)$$

if and only if the $\tilde{\Phi}_j$ are also refinable, i.e., there exist matrices $\tilde{\mathbf{M}}_{j,0} = (\tilde{m}_{l,k}^j)_{l \in \Delta_{j+1}, k \in \Delta_j}$ defining uniformly bounded mappings from $l_2(\Delta_j)$ to $l_2(\Delta_{j+1})$ such that

$$\tilde{\Phi}_j^T = \tilde{\Phi}_{j+1}^T \tilde{\mathbf{M}}_{j,0}. \qquad (2.2.5)$$

Note that then (2.2.1) implies

$$\mathbf{M}_{j,0}^T \tilde{\mathbf{M}}_{j,0} = \tilde{\mathbf{M}}_{j,0}^T \mathbf{M}_{j,0} = \mathbf{I}. \qquad (2.2.6)$$

The relevance of Lemma 2.1 and Remark 2.3 is explained by the following criterion from [18] for the validity of Sobolev norm equivalences and the Riesz basis property.

THEOREM 2.4.   *Let* $\{\Phi_j\}$, $\{\tilde{\Phi}_j\}$ *be uniformly stable, refinable, biorthogonal collections and let the* Q_j *be defined by* (2.2.3). *If the Jackson-type estimate*

$$\inf_{v_j \in V_j} \|v - v_j\|_{L_2(\Omega)} \lesssim 2^{-sj} \|v\|_{H^s(\Omega)}, \qquad v \in H^s(\Omega), \ s \le \sigma, \tag{2.2.7}$$

and the Bernstein inequality

$$\|v_j\|_{H^s(\Omega)} \lesssim 2^{sj} \|v_j\|_{L_2(\Omega)}, \qquad v_j \in V_j, \ s \le \gamma, \tag{2.2.8}$$

hold for

$$V_j = \begin{Bmatrix} S(\Phi_j) \\ S(\tilde{\Phi}_j) \end{Bmatrix} \qquad with \qquad \sigma = \begin{Bmatrix} d \\ \tilde{d} \end{Bmatrix} \qquad and \qquad \gamma = \begin{Bmatrix} t \le d \\ \tilde{t} \le \tilde{d} \end{Bmatrix},$$

then

$$\|v\|_{H^s(\Omega)} \sim \Big(\sum_{j=j_0}^{\infty} 2^{2sj} \|(Q_j - Q_{j-1})v\|_{L_2(\Omega)}^2 \Big)^{1/2}, \qquad s \in (-\tilde{t}, t). \tag{2.2.9}$$

Here we have used the convention that $Q_{j_0-1} := 0$ *and that for* $s < 0$ $H^s(\Omega)$ *means the dual* $(H^{-s}(\Omega))^*$ *of* $H^{-s}(\Omega)$ *relative to the dual form induced by* $\langle \cdot, \cdot \rangle$.

The norm equivalence (2.2.9) suggests identifying stable bases Ψ_j (and $\tilde{\Psi}_j$) of the particular spaces $W_j := (Q_j - Q_{j-1})\mathcal{H}$ and $\tilde{W}_j := (Q_j^* - Q_{j-1}^*)\mathcal{H}$ which due to (2.2.4) agree with $(Q_j - Q_{j-1})S(\Phi_j)$ and $(Q_j^* - Q_{j-1}^*)S(\tilde{\Phi}_j)$, respectively. It is well known that such collections Ψ_j, $\tilde{\Psi}_j$ are actually *biorthogonal* when properly normalized. In fact, the relation $S(\tilde{\Phi}_{j-1}) \perp W_j$ can easily be confirmed as follows. For any $v \in \mathcal{H}$ one has by (2.2.1) and (2.2.5)

$$\langle (Q_j - Q_{j-1})v, \tilde{\Phi}_{j-1} \rangle = \langle \langle v, \tilde{\Phi}_j \rangle \Phi_j, \tilde{\mathbf{M}}_{j-1,0}^T \tilde{\Phi}_j \rangle - \langle \langle v, \tilde{\Phi}_{j-1} \rangle \Phi_{j-1}, \tilde{\Phi}_{j-1} \rangle$$

$$= \langle v, \tilde{\mathbf{M}}_{j-1,0}^T \tilde{\Phi}_j \rangle - \langle v, \tilde{\Phi}_{j-1} \rangle = 0.$$

2.3. Stable Completions and Biorthogonal Bases

Given two collections Φ_j, $\tilde{\Phi}_j$ of biorthogonal functions, our goal is to determine next the corresponding collections Ψ_j, $\tilde{\Psi}_j$ of biorthogonal wavelets. Our strategy is to accomplish this in two steps. In many cases it is possible to identify *some initial* complement of $S(\Phi_j)$ in $S(\Phi_{j+1})$. Then one can project this complement onto the desired complement $(Q_{j+1} - Q_j)S(\Phi_{j+1})$ while preserving stability and compact support of the basis functions. First we need a stability criterion for complement bases. We can formulate this again for the above general Hilbert space setting.

PROPOSITION 2.5 [8].   *Suppose that $\{\Phi_j\}$ is uniformly stable and (2.1.3) holds. Then $\{\Phi_j \cup \Psi_j\}$ for $\Psi_j \subset S(\Phi_{j+1})$ is uniformly stable if and only if there exists*

$$\mathbf{M}_{j,1} \in [l_2(\nabla_j), l_2(\Delta_{j+1})], \qquad \nabla_j := \Delta_{j+1}\backslash\Delta_j,$$

such that

$$\Psi_j^T = \Phi_{j+1}^T \mathbf{M}_{j,1} \tag{2.3.1}$$

and $\mathbf{M}_j = (\mathbf{M}_{j,0}, \mathbf{M}_{j,1}) \in [l_2(\Delta_j \cup \nabla_j), l_2(\Delta_{j+1})]$ is invertible and satisfies

$$\|\mathbf{M}_j\|, \|\mathbf{M}_j^{-1}\| = \mathbb{O}(1), \qquad j \geq j_0. \tag{2.3.2}$$

In fact, one has

$$c_1\|\mathbf{M}_j^{-1}\|^{-1}\left\|\begin{pmatrix}\mathbf{u}^\Delta \\ \mathbf{u}^\nabla\end{pmatrix}\right\|_{l_2(\Delta_j \cup \nabla_j)} \leq \|\Phi_j^T\mathbf{u}^\Delta + \Psi_j^T\mathbf{u}^\nabla\|_{\mathcal{H}} \leq c_2\|\mathbf{M}_j\|\left\|\begin{pmatrix}\mathbf{u}^\Delta \\ \mathbf{u}^\nabla\end{pmatrix}\right\|_{l_2(\Delta_j \cup \nabla_j)}, \tag{2.3.3}$$

where c_1, c_2 are the constants from the stability relation (2.1.1).

Writing $\mathbf{M}_j^{-1} = \mathbf{G}_j = \begin{pmatrix}\mathbf{G}_{j,0} \\ \mathbf{G}_{j,1}\end{pmatrix}$, one obtains the *reconstruction formula*

$$\Phi_{j+1}^T = \Phi_j^T\mathbf{G}_{j,0} + \Psi_j^T\mathbf{G}_{j,1}. \tag{2.3.4}$$

Given $\mathbf{M}_{j,0}$, any $\mathbf{M}_{j,1} \in [l_2(\nabla_j), l_2(\Delta_{j+1})]$ such that (2.3.2) holds is called a *stable completion* of $\mathbf{M}_{j,0}$ [8].

Clearly the pair of two scale relations (2.1.3), (2.3.1) together with (2.3.4) give rise to cascadic decomposition and reconstruction algorithms whose structures are analogous to the classical wavelet schemes on the real line. Their description in terms of the matrices $\mathbf{M}_{j,e}$, $\mathbf{G}_{j,e}$, $e \in \{0, 1\}$, can be found, e.g., in [8].

Recall that uniform stability of $\{\Phi_j \cup \Psi_j\}$ by no means implies the Riesz basis property (1.2.17) of the union of the complement bases Ψ_j for all $j \geq j_0$. This is where biorthogonality comes into play (see [17, 18]). Thus, given some stable completion $\check{\mathbf{M}}_{j,1}$ of $\mathbf{M}_{j,0}$ we need next a mechanism to generate the stable completion corresponding to the particular complements $(Q_j - Q_{j-1})S(\Phi_j)$. It can be based on the following observation from [8]. Suppose in what follows that the biorthogonal collections $\{\Phi_j\}$, $\{\tilde{\Phi}_j\}$ are both uniformly stable and refinable with refinement matrices $\mathbf{M}_{j,0}$, $\tilde{\mathbf{M}}_{j,0}$, i.e.,

$$\Phi_j^T = \Phi_{j+1}^T\mathbf{M}_{j,0}, \qquad \tilde{\Phi}_j^T = \tilde{\Phi}_{j+1}^T\tilde{\mathbf{M}}_{j,0}. \tag{2.3.5}$$

PROPOSITION 2.6 [8].   *Let $\{\Phi_j\}$, $\{\tilde{\Phi}_j\}$, $\mathbf{M}_{j,0}$, and $\tilde{\mathbf{M}}_{j,0}$ be related as above. Suppose that $\check{\mathbf{M}}_{j,1}$ is some stable completion of $\mathbf{M}_{j,0}$ and that $\check{\mathbf{G}}_j = \check{\mathbf{M}}_j^{-1} := (\mathbf{M}_{j,0}, \check{\mathbf{M}}_{j,1})^{-1}$. Then*

$$\mathbf{M}_{j,1} := (\mathbf{I} - \mathbf{M}_{j,0}\tilde{\mathbf{M}}_{j,0}^T)\check{\mathbf{M}}_{j,1} \tag{2.3.6}$$

is also a stable completion of $\mathbf{M}_{j,0}$ *and* $\mathbf{G}_j = \mathbf{M}_j^{-1}$ *has the form*

$$\mathbf{G}_j = \begin{pmatrix} \tilde{\mathbf{M}}_{j,0}^T \\ \check{\mathbf{G}}_{j,1} \end{pmatrix}. \tag{2.3.7}$$

Moreover, the collections

$$\Psi_j := \mathbf{M}_{j,1}^T \Phi_{j+1}, \qquad \tilde{\Psi}_j := \check{\mathbf{G}}_{j,1} \tilde{\Phi}_{j+1} \tag{2.3.8}$$

form biorthogonal systems,

$$\langle \Psi_j, \tilde{\Psi}_j \rangle = \mathbf{I}, \qquad \langle \Psi_j, \tilde{\Phi}_j \rangle = \langle \Phi_j, \tilde{\Psi}_j \rangle = \mathbf{0}, \tag{2.3.9}$$

so that

$$(Q_{j+1} - Q_j)S(\Phi_{j+1}) = S(\Psi_j), \qquad (Q_{j+1}^* - Q_j^*)S(\tilde{\Phi}_{j+1}) = S(\tilde{\Psi}_j). \tag{2.3.10}$$

In view of (2.3.5), (2.3.9) implies that the collections

$$\Psi = \Phi_{j_0} \cup \bigcup_{j \geq j_0} \Psi_j, \qquad \tilde{\Psi} := \tilde{\Phi}_{j_0} \cup \bigcup_{j \geq j_0} \tilde{\Psi}_j$$

are biorthogonal,

$$\langle \Psi, \tilde{\Psi} \rangle = \mathbf{I}, \tag{2.3.11}$$

where $\nabla_{j_0-1} := \Delta_{j_0}$, $\psi_{j_0-1,k} := \phi_{j_0,k}$, $\tilde{\psi}_{j_0-1,k} := \tilde{\phi}_{j_0,k}$.

Note that with

$$\mathbf{L}_j := -\tilde{\mathbf{M}}_{j,0}^T \check{\mathbf{M}}_{j,1}, \tag{2.3.12}$$

the new complement functions $\psi_{j,k}$ are obtained by updating the initial complement functions $\check{\psi}_{j,k}$ with a linear combination of the coarse level generators $\phi_{j,k}$. In fact, by (2.3.6) and (2.3.5),

$$\Psi_j^T = \Phi_{j+1}^T \mathbf{M}_{j,1} = \Phi_{j+1}^T \check{\mathbf{M}}_{j,1} + \Phi_{j+1}^T \mathbf{M}_{j,0} \mathbf{L}_j = \check{\Psi}_j^T + \Phi_j^T \mathbf{L}_j,$$

i.e.,

$$\psi_{j,k} = \check{\psi}_{j,k} + \sum_{l \in \Delta_j} (\mathbf{L}_j)_{l,k} \phi_{j,l}, \qquad k \in \nabla_j. \tag{2.3.13}$$

The following result is an immediate consequence of Theorem 2.4 and Proposition 2.6.

COROLLARY 2.7.   *Under the assumptions of Theorem 2.4 one has*

$$\|v\|_{H^s(\Omega)} \sim \Big(\sum_{j=j_0-1}^{\infty} \sum_{k\in\nabla_j} 2^{2sj}|\langle v, \tilde{\psi}_{j,k}\rangle|^2 \Big)^{1/2}, \qquad s \in (-\tilde{t}, t). \tag{2.3.14}$$

Note that, in particular, for $s = 0$ the Riesz basis property relative to $L_2(\Omega)$ of the Ψ, $\tilde{\Psi}$ is covered.

2.4. Changing Bases Continued

Clearly relations (2.3.4) and (2.3.5) describe a change of bases involving two successive scales. In addition we will also have to change bases within a given scale. It is therefore convenient to use the following simple mechanism for identifying the new refinement relations as well as corresponding stable completions. To this end, suppose that Φ'_j is refinable with refinement matrix $\mathbf{M}'_{j,0}$ and some stable completion $\mathbf{M}'_{j,1}$. Supposing that we perform a change of bases in $S(\Phi'_j)$, i.e.,

$$\Phi_j = \mathbf{C}_j\Phi'_j, \tag{2.4.1}$$

we will need the corresponding new refinement relation.

REMARK 2.8.   *For Φ'_j, Φ_j as above one has*

$$\Phi_j^T = \Phi_{j+1}^T\mathbf{M}_{j,0}, \tag{2.4.2}$$

where

$$\mathbf{M}_{j,0} = \mathbf{C}_{j+1}^{-T}\mathbf{M}'_{j,0}\mathbf{C}_j^T. \tag{2.4.3}$$

Moreover,

$$\mathbf{M}_{j,1} = \mathbf{C}_{j+1}^{-T}\mathbf{M}'_{j,1} \tag{2.4.4}$$

is the corresponding stable completion and

$$\mathbf{M}_j^{-1} = \begin{pmatrix} \mathbf{C}_j^{-T}\mathbf{G}'_{j,0}\mathbf{C}_{j+1}^T \\ \mathbf{G}'_{j,1}\mathbf{C}_{j+1}^T \end{pmatrix} =: \mathbf{G}_j. \tag{2.4.5}$$

Proof. $\Phi_j^T = (\mathbf{C}_j\Phi'_j)^T = (\Phi'_{j+1})^T\mathbf{M}'_{j,0}\mathbf{C}_j^T = \Phi_{j+1}^T\mathbf{C}_{j+1}^{-T}\mathbf{M}'_{j,0}\mathbf{C}_j^T$ confirming (2.4.2) and (2.4.3). Furthermore, one has

$$(\mathbf{C}_{j+1}^{-T}\mathbf{M}'_{j,0}\mathbf{C}_j^T, \mathbf{C}_{j+1}^{-T}\mathbf{M}'_{j,1})\begin{pmatrix} \mathbf{C}_j^{-T}\mathbf{G}'_{j,0}\mathbf{C}_{j+1}^T \\ \mathbf{G}'_{j,1}\mathbf{C}_{j+1}^T \end{pmatrix} = \mathbf{C}_{j+1}^{-T}\mathbf{M}'_j\begin{pmatrix} \mathbf{C}_j^T & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{pmatrix}\begin{pmatrix} \mathbf{C}_j^{-T} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{pmatrix}\mathbf{G}'_j\mathbf{C}_{j+1}^T$$

$$= \mathbf{C}_{j+1}^{-T}\mathbf{M}'_j\mathbf{G}'_j\mathbf{C}_{j+1}^T = \mathbf{I},$$

which proves the claim. ∎

3. BIORTHOGONAL MULTIRESOLUTION IN $L_2([0, 1])$

3.1. Boundary Functions

Suppose now that θ, $\tilde{\theta}$ form a dual pair of refinable functions as in Section 1.2 with

$$\text{supp } \theta = [l_1, l_2], \tag{3.1.1}$$

and that θ is exact of order d. Let $\mathbf{a} = \{a_k\}_{k=l_1}^{l_2}$ denote the mask of θ where $a_k := 0$ for $k < l_1$ or $k > l_2$. The essence of the following observation is well known (see, e.g., [2, 16]). However, since the precise role of the various parameters chosen here will matter and since the refinement relation below differs somewhat from the findings in [16] we will include a proof of the following fact.

LEMMA 3.1. *Suppose that*

$$l \geq -l_1 \tag{3.1.2}$$

and define

$$\theta_{j,l-d+r}^L := \sum_{m=-l_2+1}^{l-1} \alpha_{\tilde{\theta},r}(m)\theta_{[j,m]}|_{\mathbb{R}_+}, \qquad r = 0, \dots, d-1. \tag{3.1.3}$$

Then one has

$$\theta_{j,l-d+r}^L = 2^{-(r+1/2)}\left(\theta_{j+1,l-d+r}^L + \sum_{m=l}^{2l+l_1-1} \alpha_{\tilde{\theta},r}(m)\theta_{[j+1,m]} \right) + \sum_{m=2l+l_1}^{2l+l_2-2} \beta_{\tilde{\theta},r}(m)\theta_{[j+1,m]},$$

$$r = 0, \dots, d-1, \quad (3.1.4)$$

where

$$\beta_{\tilde{\theta},r}(m) := 2^{-1/2} \sum_{q=\lceil \frac{m-l_2}{2} \rceil}^{l-1} \alpha_{\tilde{\theta},r}(q)a_{m-2q} \tag{3.1.5}$$

and $\lfloor x \rfloor$ ($\lceil x \rceil$) is the largest (smallest) integer less (greater) than or equal to x.

Proof. In view of (1.2.10), one has

$$\sum_{m\in\mathbb{Z}} \alpha_{\tilde{\theta},r}(m)\theta_{[j,m]}(x) = 2^{j/2}(2^j x)^r, \qquad r = 0, \dots, d-1. \tag{3.1.6}$$

Inserting the refinement equation on $\mathbb{R}$ (1.2.1) written in the form

$$\theta_{[j,m]} = 2^{-1/2} \sum_{t=2m+l_1}^{2m+l_2} a_{t-2m}\theta_{[j+1,t]}, \qquad m \in \mathbb{Z}, \tag{3.1.7}$$

into (3.1.6) yields

$$2^{j/2}(2^j x)^r = \sum_{t\in\mathbb{Z}} 2^{-1/2} \left[\sum_{m=\lceil\frac{t-l_2}{2}\rceil}^{\lfloor\frac{t-l_1}{2}\rfloor} \alpha_{\tilde{\theta},r}(m)a_{t-2m} \right] \theta_{[j+1,t]}(x)$$

or, equivalently,

$$2^{(j+1)/2}(2^{j+1}x)^r = \sum_{t\in\mathbb{Z}} 2^{r} \left[\sum_{m=\lceil\frac{t-l_2}{2}\rceil}^{\lfloor\frac{t-l_1}{2}\rfloor} \alpha_{\tilde{\theta},r}(m)a_{t-2m} \right] \theta_{[j+1,t]}(x).$$

Comparing this with (3.1.6) for j exchanged by $j + 1$ results in the identity

$$2^{-r}\alpha_{\tilde{\theta},r}(t) = \sum_{m=\lceil\frac{t-l_2}{2}\rceil}^{\lfloor\frac{t-l_1}{2}\rfloor} \alpha_{\tilde{\theta},r}(m)a_{t-2m}. \tag{3.1.8}$$

Now by definition (3.1.3), (3.1.6) yields

$$\theta_{j,l-d+r}^{L}(x) = 2^{j/2}(2^j x)^r|_{\mathbb{R}_+} - \sum_{m=l}^{\infty} \alpha_{\tilde{\theta},r}(m)\,\theta_{[j,m]}(x)|_{\mathbb{R}_+}$$

$$= 2^{-(r+1/2)} \sum_{m=-l_2+1}^{\infty} \alpha_{\tilde{\theta},r}(m)\theta_{[j+1,m]}(x)|_{\mathbb{R}_+} - \sum_{m=l}^{\infty} \alpha_{\tilde{\theta},r}(m)\theta_{[j,m]}(x)|_{\mathbb{R}_+}.$$

Splitting the first sum and using (3.1.3) gives

$$\theta_{j,l-d+r}^{L} = 2^{-(r+1/2)}\left(\theta_{j+1,l-d+r}^{L} + \sum_{m=l}^{\infty} \alpha_{\tilde{\theta},r}(m)\theta_{[j+1,m]}|_{\mathbb{R}_+} \right) - \sum_{m=l}^{\infty} \alpha_{\tilde{\theta},r}(m)\theta_{[j,m]}|_{\mathbb{R}_+}$$

$$= 2^{-(r+1/2)}\left(\theta_{j+1,l-d+r}^{L} + \sum_{m=l}^{\infty} \alpha_{\tilde{\theta},r}(m)\theta_{[j+1,m]}|_{\mathbb{R}_+} \right)$$

$$- \sum_{m=l}^{\infty} \alpha_{\tilde{\theta},r}(m)\left(2^{-1/2} \sum_{t=2m+l_1}^{2m+l_2} a_{t-2m}\theta_{[j+1,t]}|_{\mathbb{R}_+} \right)$$

upon inserting (3.1.7). Exchanging the order of summation in the last term yields

$$\theta^L_{j,l-d+r} = 2^{-(r+1/2)}\left(\theta^L_{j+1,l-d+r} + \sum_{m=l}^{\infty} \alpha_{\tilde{\theta},r}(m)\theta_{[j+1,m]}\big|_{\mathbb{R}_+}\right) - \sum_{t=2l+l_1}^{\infty}\sum_{m=l}^{\infty} 2^{-1/2}\alpha_{\tilde{\theta},r}(m)a_{t-2m}\theta_{[j+1,t]}\big|_{\mathbb{R}_+}$$

$$= 2^{-(r+1/2)}\left(\theta^L_{j+1,l-d+r} + \sum_{m=l}^{2l+l_1-1} \alpha_{\tilde{\theta},r}(m)\,\theta_{[j+1,m]}\big|_{\mathbb{R}_+}\right)$$

$$+ \sum_{m=2l+l_1}^{\infty} \left(2^{-(r+1/2)}\alpha_{\tilde{\theta},r}(m) - 2^{-1/2}\sum_{s=l}^{\infty}\alpha_{\tilde{\theta},r}(s)a_{m-2s}\right)\theta_{[j+1,m]}\big|_{\mathbb{R}_+}.$$

Substituting (3.1.8) in the last sum, one obtains

$$\theta^L_{j,l-d+r} = 2^{-(r+1/2)}\left(\theta^L_{j+1,l-d+r} + \sum_{m=l}^{2l+l_1-1} \alpha_{\tilde{\theta},r}(m)\,\theta_{[j+1,m]}\big|_{\mathbb{R}_+}\right)$$

$$+ \sum_{m=2l+l_1}^{\infty} 2^{-1/2}\left[\sum_{q=\left\lceil\frac{m-l_2}{2}\right\rceil}^{\left\lfloor\frac{m-l_1}{2}\right\rfloor} \alpha_{\tilde{\theta},r}(q)a_{m-2q} - \sum_{q=l}^{\left\lfloor\frac{m-l_1}{2}\right\rfloor} \alpha_{\tilde{\theta},r}(q)a_{m-2q}\right]\theta_{[j+1,m]}\big|_{\mathbb{R}_+}$$

$$= 2^{-(r+1/2)}\left(\theta^L_{j+1,l-d+r} + \sum_{m=l}^{2l+l_1-1} \alpha_{\tilde{\theta},r}(m)\,\theta_{[j+1,m]}\big|_{\mathbb{R}_+}\right)$$

$$+ \sum_{m=2l+l_1}^{l_2+2l-2} 2^{-1/2}\left(\sum_{q=\left\lceil\frac{m-l_2}{2}\right\rceil}^{l-1} \alpha_{\tilde{\theta},r}(q)a_{m-2q}\right)\theta_{[j+1,m]}\big|_{\mathbb{R}_+},$$

which, in view of (3.1.5), is the asserted relation (3.1.4). ∎

Note that

$$\alpha_{\tilde{\theta},r}(y) = \int_{\mathbb{R}} (x+y)^r\tilde{\theta}(x)dx = \sum_{i=0}^{r}\binom{r}{i}y^i\int_{\mathbb{R}} x^{r-i}\tilde{\theta}(x)dx \qquad (3.1.9)$$

is a polynomial of degree r whose coefficients $\binom{r}{i}\int_{\mathbb{R}} x^{r-i}\tilde{\theta}(x)dx$ can be computed exactly with the aid of a recursion (see, e.g., [25] as well as Section 5.1).

3.2. Spline Multiresolution

We will specify now the primal multiresolution as follows. Let us denote for a sequence of knots $t_i \leq \cdots \leq t_{i+d}$ by $[t_i, \ldots, t_{i+d}]f$ the dth order divided difference of $f \in C^d(\mathbb{R})$ at $t_i, \ldots, t_{i+d}$. Setting $x^d_+ := (\max\{0, x\})^d$, the *cardinal B-spline* $_d\varphi$ of order $d \in \mathbb{N}$ is defined as

$$_d\varphi(x) := d[0, 1, \ldots, d]\left(\cdot - x - \left\lfloor \frac{d}{2} \right\rfloor\right)_+^{d-1}. \tag{3.2.1}$$

Hence, φ is *centered* around $\mu(d)/2$, i.e.,

$$_d\varphi(x + \mu(d)) = {}_d\varphi(-x), \qquad x \in \mathbb{R}, \tag{3.2.2}$$

where $\mu(d) := d \bmod 2$, and has support

$$\text{supp } {}_d\varphi = \left[\frac{1}{2}(-d + \mu(d)), \frac{1}{2}(d + \mu(d))\right] = \left[-\left\lfloor \frac{d}{2} \right\rfloor, \left\lceil \frac{d}{2} \right\rceil\right] =: [l_1, l_2]; \tag{3.2.3}$$

i.e., $d = l_2 - l_1$ and $\mu(d) = l_1 + l_2$. Thus, the B-splines of even order are centered around 0 while the ones of odd order are symmetric around $\frac{1}{2}$. The B-spline $_d\varphi$ is *refinable* with finitely supported real mask $\mathbf{a} = \{a_k\}_{k=l_1}^{l_2}$, i.e.,

$$_d\varphi(x) = \sum_{k=l_1}^{l_2} 2^{1-d}\binom{d}{k + \left\lfloor \frac{d}{2} \right\rfloor} {}_d\varphi(2x - k) =: \sum_{k=l_1}^{l_2} a_k \, {}_d\varphi(2x - k). \tag{3.2.4}$$

It has been shown in [15] that for each d and any $\tilde{d} \geq d$, $\tilde{d} \in \mathbb{N}$, so that $d + \tilde{d}$ even, there exists a function $_{d,\tilde{d}}\tilde{\varphi} \in L_2(\mathbb{R})$ with the following properties (see [15]):

(i) $_{d,\tilde{d}}\tilde{\varphi}$ has compact support,

$$\text{supp } {}_{d,\tilde{d}}\tilde{\varphi} = \left[-\frac{1}{2}d - \tilde{d} + 1 + \frac{1}{2}\mu(d), \frac{1}{2}d + \tilde{d} - 1 + \frac{1}{2}\mu(d)\right]$$

$$= [l_1 - \tilde{d} + 1, l_2 + \tilde{d} - 1] =: [\tilde{l}_1, \tilde{l}_2]. \tag{3.2.5}$$

(ii) $_{d,\tilde{d}}\tilde{\varphi}$ is refinable with finitely supported mask $\tilde{\mathbf{a}} = \{\tilde{a}_k\}_{k=\tilde{l}_1}^{\tilde{l}_2}$,

$$_{d,\tilde{d}}\tilde{\varphi}(x) = \sum_{k=\tilde{l}_1}^{\tilde{l}_2} \tilde{a}_k \, {}_{d,\tilde{d}}\tilde{\varphi}(2x - k). \tag{3.2.6}$$

(iii) $_{d,\tilde{d}}\tilde{\varphi}$ has the same symmetry properties as $_d\varphi$, i.e.,

$$_{d,\tilde{d}}\tilde{\varphi}(x + \mu(d)) = {}_{d,\tilde{d}}\tilde{\varphi}(-x), \qquad x \in \mathbb{R}. \tag{3.2.7}$$

(iv) The functions $_d\varphi$ and $_{d,\tilde{d}}\tilde{\varphi}$ form a *dual pair,* i.e.,

$$(_d\varphi, {}_{d,\tilde{d}}\tilde{\varphi}(\cdot - k))_{\mathbb{R}} = \delta_{0,k}, \qquad k \in \mathbb{Z}. \tag{3.2.8}$$

(v) $_{d,\tilde{d}}\tilde{\varphi}$ is exact of order $\tilde{d}$, i.e., all polynomials of degree less than $\tilde{d}$ can be represented as linear combinations of the translates $_{d,\tilde{d}}\tilde{\varphi}(\cdot - k)$, $k \in \mathbb{Z}$.

(vi) The regularity of $_{d,\tilde{d}}\tilde{\varphi}$ increases proportionally with $\tilde{d}$.

One easily checks that the symmetry properties (3.2.2), (3.2.7) have the discrete counterparts

$$a_k = a_{\mu(d)-k}, \qquad \tilde{a}_k = \tilde{a}_{\mu(d)-k}, \, k \in \mathbb{Z}. \tag{3.2.9}$$

In the following d, $\tilde{d}$ will be arbitrary as above but fixed so that we can suppress them as indices and write briefly φ, $\tilde{\varphi}$.

The following construction of biorthogonal multiresolution analyses on [0, 1] follows first familiar lines in that we retain translates of dilated scaling functions φ, $\tilde{\varphi}$ whose supports are fully contained in [0, 1]. Thus, the final generator bases adapted to the interval will consist of three parts, namely collections Φ_j^L, Φ_j^0, and Φ_j^R representing the basis functions adapted to the left end point of the interval, those consisting of the translates $\varphi_{[j,k]}$ supported in the interior, and those adapted to the right end of the interval, respectively. The same has to be done for the dual side. We will describe first corresponding index sets. Since by (3.2.5) the support of $\tilde{\varphi}$ is at least as large as that of φ, i.e., $\tilde{l}_2 \geq l_2$, $-\tilde{l}_1 \geq -l_1$ (even if $\tilde{d} < d$), we consider first the dual collections and fix some integer $\tilde{l}$ satisfying

$$\tilde{l} \geq \tilde{l}_2, \tag{3.2.10}$$

so that the indices

$$\tilde{\Delta}_j^0 := \{\tilde{l}, \ldots, 2^j - \tilde{l} - \mu(d)\} \tag{3.2.11}$$

correspond to translates $\tilde{\varphi}_{[j,m]}$ whose support is contained in [0, 1]. To preserve polynomial exactness of degree $\tilde{d} - 1$ we need $\tilde{d}$ additional basis functions near the left and right end of the interval each which will be constructed according to the receipe from Section 3.1. The corresponding index sets are then

$$\tilde{\Delta}_j^L := \{\tilde{l} - \tilde{d}, \ldots, \tilde{l} - 1\}, \qquad \tilde{\Delta}_j^R := \{2^j - \tilde{l} + 1 - \mu(d), \ldots, 2^j - \tilde{l} + \tilde{d} - \mu(d)\}. \tag{3.2.12}$$

We have included here a shift by $-\mu(d)$ in $\tilde{\Delta}_j^R$ to make the best possible use of *symmetry* later.

On the primal side we need bases of the same cardinality. Since the degree $d - 1$ of exactness is in general different from $\tilde{d} - 1$, the boundary index sets necessarily take the form

$$\Delta_j^L := \{\tilde{l} - \tilde{d}, \ldots, \tilde{l} - (\tilde{d} - d) - 1\},$$

$$\Delta_j^R := \{2^j - \tilde{l} + (\tilde{d} - d) + 1 - \mu(d), \ldots, 2^j - \tilde{l} + \tilde{d} - \mu(d)\}, \tag{3.2.13}$$

so that the *interior translates* $\varphi_{[j,m]}$ are determined by $m \in \Delta_j^0$, where

$$\Delta_j^0 := \{\tilde{l} - (\tilde{d} - d), \ldots, 2^j - \tilde{l} + (\tilde{d} - d) - \mu(d)\}. \tag{3.2.14}$$

Of course, it will always be assumed that j is large enough to ensure that $\tilde{l} \leq 2^j - \tilde{l} - \mu(d) + 2(\tilde{d} - d)$. By construction we have now

$$\tilde{\Delta}_j := \tilde{\Delta}_j^L \cup \tilde{\Delta}_j^0 \cup \tilde{\Delta}_j^R = \Delta_j = \Delta_j^L \cup \Delta_j^0 \cup \Delta_j^R. \tag{3.2.15}$$

Moreover, abbreviating

$$l := \tilde{l} - (\tilde{d} - d), \tag{3.2.16}$$

we observe that

$$\Delta_j^L = \{l - d, \ldots, l - 1\},$$
$$\Delta_j^0 = \{l, \ldots, 2^j - l - \mu(d)\},$$
$$\Delta_j^R = \{2^j - l + 1 - \mu(d), \ldots, 2^j - l + d - \mu(d)\}, \tag{3.2.17}$$

so that the structure of the index sets on the primal and dual side is completely analogous; see Fig. 3.2.1 for the special case $d = 3$, $\tilde{d} = 5$.

Note also that by (3.2.5) and (3.2.10), $l \geq \tilde{l}_2 - (\tilde{d} - d) = -l_1 + 2l_2 - 1$ so that the interior functions are, under the above assumption on j, indeed supported in $[0, 1]$, i.e.,

$$\text{supp } \varphi_{[j,k]} \subset [0, 1], \ k \in \Delta_j^0, \qquad \text{supp } \tilde{\varphi}_{[j,k]} \subset [0, 1], \ k \in \tilde{\Delta}_j^0. \tag{3.2.18}$$

The next step is to construct the modified basis functions near the end points of the interval for the primal and dual sides according to Section 3.1. In what follows, we will always assume that

$$j \geq \lceil \log_2(\tilde{l} + \tilde{l}_2 - 1) + 1 \rceil =: j_0 \tag{3.2.19}$$

so that the supports of the left and right end functions do not overlap (but see Remark 5.1). It will be convenient to abbreviate (recall (1.2.9))
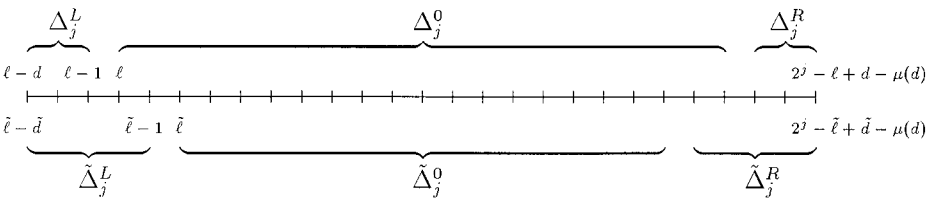
FIG. 3.2.1. Index sets for the interval for $d = 3$, $\tilde{d} = 5$, $l = 4$, $\tilde{l} = \tilde{l}_2 = 6$.

$$\tilde{\alpha}_{m,r} := \alpha_{\tilde{\varphi},r}(m), \qquad \alpha_{m,r} := \alpha_{\varphi,r}(m). \tag{3.2.20}$$

Let us see first how these quantities depend on the level j and to what extent the symmetry properties (3.2.2) and (3.2.7) are inherited. Since obviously

$$\alpha^L_{j,m,r} := 2^j \int_{\mathbb{R}} (2^j x)^r \varphi(2^j x - m)\, dx = \int_{\mathbb{R}} x^r \varphi(x - m)\, dx$$

$$\alpha^R_{j,m,r} := 2^j \int_{\mathbb{R}} (2^j(1 - x))^r \varphi(2^j x - m)\, dx = \int_{\mathbb{R}} (2^j - x)^r \varphi(x - m)\, dx, \tag{3.2.21}$$

and likewise

$$\tilde{\alpha}^L_{j,m,r} := 2^j \int_{\mathbb{R}} (2^j x)^r \tilde{\varphi}(2^j x - m)\, dx = \int_{\mathbb{R}} x^r \tilde{\varphi}(x - m)\, dx$$

$$\tilde{\alpha}^R_{j,m,r} := 2^j \int_{\mathbb{R}} (2^j(1 - x))^r \tilde{\varphi}(2^j x - m)\, dx = \int_{\mathbb{R}} (2^j - x)^r \tilde{\varphi}(x - m)\, dx, \tag{3.2.22}$$

we conclude on the one hand that

$$\sum_{m \in \mathbb{Z}} \alpha^L_{j,m,r} \tilde{\varphi}_{[j,m]}(x) = 2^{j(r+1/2)} x^r, \qquad \sum_{m \in \mathbb{Z}} \alpha^R_{j,m,r} \tilde{\varphi}_{[j,m]}(x) = 2^{j(r+1/2)}(1 - x)^r,$$

$$r = 0, \ldots, \tilde{d} - 1,$$

$$\sum_{m \in \mathbb{Z}} \tilde{\alpha}^L_{j,m,r} \varphi_{[j,m]}(x) = 2^{j(r+1/2)} x^r, \qquad \sum_{m \in \mathbb{Z}} \tilde{\alpha}^R_{j,m,r} \varphi_{[j,m]}(x) = 2^{j(r+1/2)}(1 - x)^r,$$

$$r = 0, \ldots, d - 1. \tag{3.2.23}$$

On the other hand, noting that by (3.2.2),

$$\int_{\mathbb{R}} x^r \varphi(x - m)\, dx = \int_{\mathbb{R}} (2^j - x)^r \varphi(2^j - x - m)\, dx$$

$$= \int_{\mathbb{R}} (2^j - x)^r \varphi(x - (2^j - m) + \mu(d))\, dx,$$

(3.2.21) also reveals that

$$\alpha^L_{j,m,r} = \alpha_{m,r}, \qquad \alpha^R_{j,m,r} = \alpha_{2^j - m - \mu(d),r}, \ r = 0, \ldots, \tilde{d} - 1,$$

$$\tilde{\alpha}^L_{j,m,r} = \tilde{\alpha}_{m,r}, \qquad \tilde{\alpha}^R_{j,m,r} = \tilde{\alpha}_{2^j - m - \mu(d),r}, \ r = 0, \ldots, d - 1. \tag{3.2.24}$$

We can now follow along the lines of Section 3.1 to define according to (3.1.3)

$$\phi_{j,l-d+r}^L := \sum_{m=-l_2+1}^{l-1} \tilde{\alpha}_{m,r} \varphi_{[j,m]}|_{[0,1]}, \qquad r = 0, \ldots, d-1, \qquad (3.2.25)$$

and likewise on the dual side

$$\tilde{\phi}_{j,\tilde{l}-\tilde{d}+r}^L := \sum_{m=-\tilde{l}_2+1}^{\tilde{l}-1} \alpha_{m,r} \tilde{\varphi}_{[j,m]}|_{[0,1]}, \qquad r = 0, \ldots, \tilde{d}-1. \qquad (3.2.26)$$

In view of the symmetry reflected by (3.2.23), we define the right end counterparts by

$$\phi_{j,2^j-l+d-\mu(d)-r}^R := \sum_{m=2^j-l-\mu(d)+1}^{2^j-l_1-1} \tilde{\alpha}_{j,m,r}^R \varphi_{[j,m]}|_{[0,1]}, \qquad r = 0, \ldots, d-1, \qquad (3.2.27)$$

and on the dual side

$$\tilde{\phi}_{j,2^j-\tilde{l}+\tilde{d}-\mu(d)-r}^R := \sum_{m=2^j-\tilde{l}-\mu(d)+1}^{2^j-\tilde{l}_1-1} \alpha_{j,m,r}^R \tilde{\varphi}_{[j,m]}|_{[0,1]}, \qquad r = 0, \ldots, \tilde{d}-1. \qquad (3.2.28)$$

It will be important in subsequent developments to exploit symmetry as much as possible.

REMARK 3.2. *One has for $x \in [0, 1]$*

$$\phi_{j,2^j-l+d-\mu(d)-r}^R(1-x) = \phi_{j,l-d+r}^L(x), \qquad r = 0, \ldots, d-1,$$

$$\tilde{\phi}_{j,2^j-\tilde{l}+\tilde{d}-\mu(d)-r}^R(1-x) = \tilde{\phi}_{j,\tilde{l}-\tilde{d}+r}^L(x), \qquad r = 0, \ldots, \tilde{d}-1, \qquad (3.2.29)$$

and

$$\theta_{[j,m]}(x) = \theta_{[j,2^j-m-\mu(d)]}(1-x), \qquad \theta = \varphi, \tilde{\varphi}. \qquad (3.2.30)$$

Proof. The relation (3.2.30) follows directly from (3.2.2). Moreover, by (3.2.2), (3.2.24), one has

$$\phi_{j,2^j-l+d-\mu(d)-r}^R(1-x) = \sum_{m=2^j-l+1-\mu(d)}^{2^j-l_1-1} \tilde{\alpha}_{2^j-m-\mu(d),r} 2^{j/2} \varphi(2^j(1-x)-m)|_{[0,1]}$$

$$= \sum_{m=2^j-l+1-\mu(d)}^{2^j-l_1-1} \tilde{\alpha}_{2^j-m-\mu(d),r} 2^{j/2} \varphi(2^j x - (2^j - m - \mu(d)))|_{[0,1]}$$

$$= \sum_{m=l_1+1-\mu(d)}^{l-1} \tilde{\alpha}_{m,r} 2^{j/2} \varphi(2^j x - m)|_{[0,1]}.$$

Since by (3.2.3) $l_1 - \mu(d) = -l_2$ we obtain (3.2.29). ∎

We clarify next the symmetry properties for the remaining parameters in the two-scale relations for the boundary adapted basis functions. To this end, recall (3.1.5) and set

$$\tilde{\beta}^L_{j,m,r} := \beta_{\tilde{\varphi},r}(m), \qquad \beta^L_{j,m,r} := 2^{-1/2} \sum_{q=\left\lceil \frac{m-\tilde{l}}{2} \right\rceil}^{\tilde{l}-1} \alpha_{q,r} \tilde{a}_{m-2q}. \tag{3.2.31}$$

Employing (3.2.9) and (3.2.24), one verifies that

$$\beta^L_{j,m,r} = 2^{-1/2} \sum_{q=2^{j}-\tilde{l}-\mu(d)+1}^{2^{j}-\left\lceil \frac{m-\tilde{l}_2}{2} \right\rceil - \mu(d)} \alpha^R_{j,q,r} \tilde{a}_{2^{j+1}-m-\mu(d)-2q},$$

$$\tilde{\beta}^L_{j,m,r} = 2^{-1/2} \sum_{q=2^{j}-l-\mu(d)+1}^{2^{j}-\left\lceil \frac{m-l_2}{2} \right\rceil - \mu(d)} \tilde{\alpha}^R_{j,q,r} a_{2^{j+1}-m-\mu(d)-2q}. \tag{3.2.32}$$

Thus, defining

$$\beta^R_{j,m,r} := 2^{-1/2} \sum_{q=2^{j}-l-\mu(d)+1}^{\infty} \alpha^R_{j,q,r} \tilde{a}_{m-2q}, \qquad \tilde{\beta}^R_{j,m,r} := 2^{-1/2} \sum_{q=2^{j}-\tilde{l}-\mu(d)+1}^{\infty} \tilde{\alpha}^R_{j,q,r} a_{m-2q}, \tag{3.2.33}$$

one obtains

$$\beta^R_{j,m,r} = \beta^L_{j,2^{j+1}-m-\mu(d),r}, \qquad \tilde{\beta}^R_{j,m,r} = \tilde{\beta}^L_{j,2^{j+1}-m-\mu(d),r}. \tag{3.2.34}$$

We are now prepared to construct multiresolution spaces on [0, 1]. Let

$$\Phi'_j := \{\phi^L_{j,k} : k \in \Delta^L_j\} \cup \{\varphi_{[j,k]} : k \in \Delta^0_j\} \cup \{\phi^R_{j,k} : k \in \Delta^R_j\} \tag{3.2.35}$$

and similarly

$$\tilde{\Phi}'_j := \{\tilde{\phi}^L_{j,k} : k \in \tilde{\Delta}^L_j\} \cup \{\tilde{\varphi}_{[j,k]} : k \in \tilde{\Delta}^0_j\} \cup \{\tilde{\phi}^R_{j,k} : k \in \tilde{\Delta}^R_j\}. \tag{3.2.36}$$

Finally, define

$$S_j := S(\Phi'_j), \qquad \tilde{S}_j := S(\tilde{\Phi}'_j). \tag{3.2.37}$$

PROPOSITION 3.3. (i) *The spaces S_j and $\tilde{S}_j$ are nested, i.e.,*

$$S_j \subset S_{j+1}, \qquad \tilde{S}_j \subset \tilde{S}_{j+1}, j \geq j_0. \tag{3.2.38}$$

(ii) *The spaces S_j, $\tilde{S}_j$ are exact of order d, $\tilde{d}$, respectively, i.e.,*

$$\Pi_d([0, 1]) \subset S_j, \qquad \Pi_{\tilde{d}}([0, 1]) \subset \tilde{S}_j, j \geq j_0. \tag{3.2.39}$$

Proof. As for (i), we have to show that the elements of the collections Φ'_j, $\tilde{\Phi}'_j$ are all refinable. Since by (3.2.4),

$$
\varphi(2^j x - m) = \sum_{k=l_1}^{l_2} a_k \varphi(2^{j+1} x - (2m + k)) = \sum_{k=2m+l_1}^{2m+l_2} a_{k-2m} \varphi(2^{j+1} x - k),
$$

and since $l + l_1 \geq 0$, the right-hand side involves for $m \in \Delta_j^0$ only summands for $k \in \Delta_{j+1}^0$. Thus, the interior functions are obviously refinable. On account of (3.2.10), the same holds for the dual side so that assertion (i) follows as soon as we have confirmed refinability of the functions $\phi_{j,k}^X$, $k \in \Delta_j^X$, and $\tilde{\phi}_{j,k}^X$, $\tilde{\Delta}_j^X$, $X \in \{L, R\}$.

To this end, Lemma 3.1 immediately yields the refinement relations for $\phi_{j,k}^L$, $k \in \Delta_j^L$, $\tilde{\phi}_{j,k}^L$, $k \in \tilde{\Delta}_j^L$. In fact, we infer from Lemma 3.1 and (3.2.20), (3.2.31) that

$$
\phi_{j,l-d+r}^L(x) = 2^{-(r+1/2)} \left(\phi_{j+1,l-d+r}^L(x) + \sum_{m=l}^{2l+l_1-1} \tilde{\alpha}_{m,r} \varphi_{[j+1,m]}(x) \right) + \sum_{m=2l+l_1}^{2l+l_2-2} \tilde{\beta}_{j,m,r}^L \varphi_{[j+1,m]}(x)
$$

$$(3.2.40)$$

and

$$
\tilde{\phi}_{j,\tilde{l}-\tilde{d}+r}^L(x) = 2^{-(r+1/2)} \left(\tilde{\phi}_{j+1,\tilde{l}-\tilde{d}+r}^L(x) + \sum_{m=\tilde{l}}^{2\tilde{l}+\tilde{l}_1-1} \alpha_{m,r} \tilde{\varphi}_{[j+1,m]}(x) \right) + \sum_{m=2\tilde{l}+\tilde{l}_1}^{2\tilde{l}+\tilde{l}_2-2} \beta_{j,m,r}^L \tilde{\varphi}_{[j+1,m]}(x).
$$

$$(3.2.41)$$

By Remark 3.2 and (3.2.34), one obtains

$$
\phi_{j,l-d+r}^L(x) = 2^{-(r+1/2)} \left(\phi_{j+1,2^{j+1}-l+d-\mu(d)-r}^R(1-x) + \sum_{m=2^{j+1}-2l-\mu(d)-l_1+1}^{2^{j+1}-l-\mu(d)} \tilde{\alpha}_{j+1,m,r}^R \varphi_{[j+1,m]}(1-x) \right)
$$

$$
+ \sum_{m=2^{j+1}-2l-\mu(d)-l_2+2}^{2^{j+1}-2l-\mu(d)-l_1} \tilde{\beta}_{j,m,r}^R \varphi_{[j+1,m]}(1-x). \quad (3.2.42)
$$

At the right end, we thus obtain for $r = 0, \ldots, d - 1$ the refinement relations

$$
\phi_{j,2^j-l+d-\mu(d)-r}^R(x) = 2^{-(r+1/2)} \left(\phi_{j+1,2^{j+1}-l+d-\mu(d)-r}^R(x) + \sum_{m=2^{j+1}-2l-l_1-\mu(d)+1}^{2^{j+1}-l-\mu(d)} \tilde{\alpha}_{j+1,m,r}^R \varphi_{[j+1,m]}(x) \right)
$$

$$
+ \sum_{m=2^{j+1}-2l-\mu(d)-l_2+2}^{2^{j+1}-2l-\mu(d)-l_1} \tilde{\beta}_{j,m,r}^R \varphi_{[j+1,m]}(x) \quad (3.2.43)
$$

and for $r = 0, \ldots, \tilde{d} - 1$,

$$\tilde{\phi}^R_{j,2^j-\tilde{l}+\tilde{d}-\mu(d)-r}(x) = 2^{-(r+1/2)} \Bigg(\tilde{\phi}^R_{j+1,2^{j+1}-\tilde{l}+\tilde{d}-\mu(d)-r}(x) + \sum_{m=2^{j+1}-2\tilde{l}-\tilde{l}_1-\mu(d)+1}^{2^{j+1}-\tilde{l}-\mu(d)} \alpha^R_{j+1,m,r}\tilde{\varphi}_{[j+1,m]}(x) \Bigg)$$

$$+ \sum_{m=2^{j+1}-2\tilde{l}-\mu(d)-\tilde{l}_2+2}^{2^{j+1}-2\tilde{l}-\mu(d)-\tilde{l}_1} \beta^R_{j,m,r}\tilde{\varphi}_{[j+1,m]}(x). \quad (3.2.44)$$

The proof of (ii) follows standard lines. It is short enough to be included. First note that by (3.2.21),

$$\tilde{\alpha}^R_{j,m,r} = \sum_{i=0}^{r} \binom{r}{i} 2^{j(r-i)}(-1)^i \tilde{\alpha}_{m,i}. \quad (3.2.45)$$

Thus, (3.2.23), (3.2.45), and (3.2.27) provide for any $r \in \{0, \ldots, d-1\}$ and $x \in [0, 1]$

$$2^{j(r+1/2)}(1-x)^r = \sum_{m=-l_2+1}^{2^j-l_1-1} \tilde{\alpha}^R_{j,m,r}\varphi_{[j,m]}(x)$$

$$= \sum_{m=-l_2}^{l-1} \left(\sum_{i=0}^{r} \binom{r}{i} 2^{j(r-i)}(-1)^i \tilde{\alpha}_{m,i} \right) \varphi_{[j,m]}(x) + \sum_{m=l}^{2^j-l-\mu(d)} \tilde{\alpha}^R_{j,m,r}\varphi_{[j,m]}(x)$$

$$+ \phi^R_{j,2^j-l+d-\mu(d)-r}(x)$$

$$= \sum_{i=0}^{r} \binom{r}{i} 2^{j(r-i)}(-1)^i \phi^L_{j,l-d+i}(x) + \sum_{m \in \Delta^0_j} \tilde{\alpha}^R_{j,m,r}\varphi_{[j,m]}(x) + \phi^R_{j,2^j-l+d-\mu(d)-r}(x),$$

which confirms the first part of (3.2.39). The rest is completely analogous. ∎

3.3. Biorthogonalization

By construction, the spanning sets Φ'_j and $\tilde{\Phi}'_j$ from (3.2.35) and (3.2.36) have equal cardinality. However, while the interior basis functions retain biorthogonality, this can no longer be expected to be true for the modified boundary near basis functions. Thus, it remains to verify next that these sets of functions are linearly independent and, which is a stronger but crucial property, that Φ'_j and $\tilde{\Phi}'_j$ can be *biorthogonalized*. So the question is, Can one find a change of basis for Φ'_j and $\tilde{\Phi}'_j$ so that the resulting collections are biorthogonal? Specifically, one has to determine nonsingular matrices $\mathbf{C}_j$, $\tilde{\mathbf{C}}_j$ such that

$$\Phi_j = \mathbf{C}_j \Phi'_j, \qquad \tilde{\Phi}_j = \tilde{\mathbf{C}}_j \tilde{\Phi}'_j \quad (3.3.1)$$

satisfy

$$(\Phi_j, \tilde{\Phi}_j)_{[0,1]} = \mathbf{I}. \tag{3.3.2}$$

Since

$$(\Phi_j, \tilde{\Phi}_j)_{[0,1]} = \mathbf{C}_j(\Phi'_j, \tilde{\Phi}'_j)_{[0,1]}\tilde{\mathbf{C}}_j^T, \tag{3.3.3}$$

biorthogonalization is actually possible if and only if

$$\det(\Phi'_j, \tilde{\Phi}'_j)_{[0,1]} \neq 0. \tag{3.3.4}$$

Next recall that $\#\Delta_j^X \leq \#\tilde{\Delta}_j^X$, $X \in \{L, R\}$.

Convention. In what follows, we will denote by Φ_j^X the set of cardinality $\tilde{d}$ obtained by adding to the functions $\phi_{j,k}^X$, $k \in \Delta_j^X$, the appropriate number of $\tilde{d} - d$ additional interior functions.

Because of the biorthogonality of the interior functions in Φ'_j, $\tilde{\Phi}'_j$, the nonsingularity of $(\Phi'_j, \tilde{\Phi}'_j)_{[0,1]}$ is equivalent to the nonsingularity of the boundary blocks

$$\mathbf{\Gamma}_{j,X} := (\Phi_j^X, \tilde{\Phi}_j^X)_{[0,1]} := ((\phi_{j,k}^X, \tilde{\phi}_{j,m}^X)_{[0,1]})_{k,m\in\tilde{\Delta}_j^X}, \qquad X \in \{L, R\}. \tag{3.3.5}$$

This problem will be further reduced by exploiting symmetry. To this end, it is convenient to denote for any matrix $\mathbf{M}$ by $\mathbf{M}^{\updownarrow}$ the matrix which is obtained by reversing the order of rows and columns of $\mathbf{M}$.

The main result of this section can now be formulated as follows.

THEOREM 3.4. *The matrices $\mathbf{\Gamma}_{j,L}$ are independent of j,*

$$\mathbf{\Gamma}_{j,L} = \mathbf{\Gamma}_L \tag{3.3.6}$$

and

$$\mathbf{\Gamma}_{j,R} = \mathbf{\Gamma}_L^{\updownarrow}, \tag{3.3.7}$$

which means $(\mathbf{\Gamma}_R)_{k,m} = (\mathbf{\Gamma}_L)_{2^j-\mu(d)-k,2^j-\mu(d)-m}$, k, $m \in \tilde{\Delta}_j^R$.

Moreover, for any dual pair φ, $\tilde{\varphi}$ satisfying (i)–(vi) at the beginning of Section 3.2, the matrix $\mathbf{\Gamma}_L$ is always nonsingular.

Proof. By (3.2.25) we have for $r = 0, \ldots, d - 1$ and $k = 0, \ldots, \tilde{d} - 1$,

$$(\phi_{j,l-d+r}^L, \tilde{\phi}_{j,\tilde{l}-\tilde{d}+k}^L)_{[0,1]} = \sum_{\nu=-l_2+1}^{l-1} \sum_{\mu=-\tilde{l}_2+1}^{\tilde{l}-1} \tilde{\alpha}_{\nu,r}\alpha_{\mu,k}(\varphi_{[j,\nu]}, \tilde{\varphi}_{[j,\mu]})_{[0,1]}$$

$$= \sum_{\nu=-l_2+1}^{l-1} \sum_{\mu=-\tilde{l}_2+1}^{\tilde{l}-1} \tilde{\alpha}_{\nu,r}\alpha_{\mu,k} \int_0^{2^j} \varphi(x - \nu)\tilde{\varphi}(x - \mu)dx. \tag{3.3.8}$$

Similarly one obtains for $r = d, \ldots, \tilde{d} - 1$, $k = 0, \ldots, \tilde{d} - 1$,

$$(\varphi_{[j,l-d+r]}, \tilde{\phi}^L_{j,\tilde{l}-\tilde{d}+k})_{[0,1]} = \sum_{\mu=-\tilde{l}_2+1}^{\tilde{l}-1} \alpha_{\mu,k} \int_0^{2^j} \varphi(x - (l - d + r))\tilde{\varphi}(x - \mu)dx.$$

Since for $j \geq j_0$ and $-l_2 + 1 \leq \nu \leq l - 1$, $-\tilde{l}_2 + 1 \leq \mu \leq \tilde{l} - 1$,

$$\int_0^{2^j} \varphi(x - \nu)\tilde{\varphi}(x - \mu)dx = \int_0^\infty \varphi(x - \nu)\tilde{\varphi}(x - \mu)dx,$$

(3.3.6) follows, while (3.3.7) is an immediate consequence of the symmetry relations (3.2.29), (3.2.30).

Now suppose first that $d \geq 2$. Note that, by (3.2.8) and the definition of $\phi^L_{j,k}$, $\tilde{\phi}^L_{j,k}$ (3.2.25), (3.2.26), and (3.2.23), we have

$$(\phi^L_{j,l-d+r}, \tilde{\phi}^L_{j,\tilde{l}-\tilde{d}+k})_{[0,1]} = (\phi^L_{j,l-d+r}, \sum_{m=-\tilde{l}_2+1}^{\infty} \alpha_{m,k}\tilde{\varphi}_{[j,m]})_{[0,1]}$$

$$= 2^{j/2}2^{kj}(\phi^L_{j,l-d+r}, (\cdot)^k)_{[0,1]}, \qquad k = 0, \ldots, \tilde{d} - 1. \qquad (3.3.9)$$

Thus

$$\mathbf{\Gamma}_L = (2^{j/2}2^{kj}(\phi^L_{j,l-d+r}, (\cdot)^k)_{[0,1]})_{r,k=0}^{\tilde{d}-1} = \mathbf{\Gamma}_L(d, \tilde{d}, l, 0),$$

where we define more generally

$$\mathbf{\Gamma}_L(d, \tilde{d}, l, \nu) := (2^{j/2}2^{kj}(\phi^L_{j,l-d+r}, (\cdot)^{\nu+k})_{[0,1]})_{r,k=0}^{\tilde{d}-1}.$$

To show that $\mathbf{\Gamma}_L$ is nonsingular, it will be useful to keep track of the dependence of the various entities on the parameters $d, \tilde{d}, l, \tilde{l}$. Therefore, we write

$$\phi^L_{j,k}(x) = \phi^L_{j,k}(x|d, \tilde{d}, l), \qquad \tilde{\alpha}_{m,r} = \tilde{\alpha}_{m,r}(d, \tilde{d}) = \int_{\mathbb{R}} x^r {}_{d,\tilde{d}}\tilde{\varphi}(x - m)dx.$$

Rewriting formula (3.4.11) in [22] in present terms (see also [30]) yields the relations

$$\frac{d}{dx} \phi^L_{j,l-d+r}(x|d, \tilde{d}, l)$$

$$= \begin{cases} -2^j {}_{d-1}\varphi_{[j,l-\mu(d-1)]}(x), & r = 0, \\ 2^j\Big(r\phi^L_{j,l-d+r-\mu(d-1)}(x|d-1, \tilde{d}+1, l - \mu(d-1)) \\ \quad - \tilde{\alpha}_{l-1,r}(d, \tilde{d}) {}_{d-1}\varphi_{[j,l-\mu(d-1)]}\Big), & r = 1, \ldots, d-1, \end{cases}$$

$$(3.3.10)$$

while

$$\frac{d}{dx}\,{}_d\varphi_{[j,k]} = 2^j\big({}_{d-1}\varphi_{[j,k+\mu(d-1)]} - {}_{d-1}\varphi_{[j,k+1-\mu(d-1)]}\big), \qquad k = l,\, l+1,\,\dots. \quad (3.3.11)$$

These relations are obtained by straightforward calculations with the aid of

$$\tilde\alpha_{m,r}(d,\tilde d) - \tilde\alpha_{m-1,r}(d,\tilde d) = r\tilde\alpha_{m-\mu(d-1),r-1}(d-1,\tilde d+1), \qquad r = 0,\dots, d-1,$$

which in turn follow from the definition and the formula

$$\frac{d}{dx}\,{}_d\varphi(x) = {}_{d-1}\varphi(x+\mu(d-1)) - {}_{d-1}\varphi(x-\mu(d)),$$

(see [30] for more details). Therefore by (3.3.10) and (3.3.11), we have for any $k = 0,\dots,\tilde d - 1$,

$$(\phi^L_{j,l-d+r},\,(\cdot)^k)_{[0,1]} = -\frac{1}{k+1}\left(\frac{d}{dx}\,\phi^L_{j,l-d+r}(\cdot\,|d,\tilde d,l),\,(\cdot)^{k+1}\right)_{[0,1]}$$

$$= \begin{cases}
\dfrac{2^j}{k+1}\big({}_{d-1}\varphi_{[j,l-\mu(d-1)]},\,(\cdot)^{k+1}\big)_{[0,1]}, & r = 0, \\[2ex]
\dfrac{2^j\tilde\alpha_{l-1,r}(d,\tilde d)}{k+1}\big({}_{d-1}\varphi_{[j,l-\mu(d-1)]},\,(\cdot)^{k+1}\big)_{[0,1]} & r = 1,\dots,d-1, \\[1ex]
\quad -\dfrac{2^j r}{k+1} \\[1ex]
\quad\times (\phi^L_{j,l-(d-1)+r-1-\mu(d-1)}(\cdot\,|d-1,\tilde d+1,l-\mu(d-1)),\,(\cdot)^{k+1})_{[0,1]}, \\[2ex]
\dfrac{2^j}{k+1}\big({}_{d-1}\varphi_{[j,l-d+r-\mu(d-1)]} - {}_{d-1}\varphi_{[j,l-d+r+1-\mu(d-1)]},\,(\cdot)^{k+1}\big)_{[0,1]}, & r = d,\dots,\tilde d - 1.
\end{cases}$$

$$\hspace{12cm}(3.3.12)$$

One readily concludes from (3.3.12) that $\Gamma_L(d,\tilde d,l,0)$ is nonsingular if and only if

$$\Gamma_L(d-1,\tilde d,l-\mu(d-1),1) = \big((2^{j(k+1/2)}\,{}_{d-1}\phi_{j,l-\mu(d-1)-(d-1)+r},\,(\cdot)^{k+1})_{[0,1]}\big)_{r,k=0}^{\tilde d-1}$$

is nonsingular. Here we have set

$${}_{d-1}\phi_{j,l-\mu(d)-(d-1)+r}$$
$$= \begin{cases}
\phi^L_{j,l-\mu(d-1)-(d-1)+r}(\cdot\,|d-1,\tilde d+1,l-\mu(d-1)), & r = 0,\dots,d-2, \\
{}_{d-1}\varphi_{[j,l-\mu(d)-(d-1)+r]}, & r = d-1,\dots,\tilde d - 1.
\end{cases}$$

Repeating this argument provides

$$\det \mathbf{\Gamma}_L \neq 0 \qquad \text{iff} \quad \det \mathbf{\Gamma}_L(1, \tilde{d}, \hat{l}, d-1) \neq 0, \tag{3.3.13}$$

where $\hat{l} := l - \mu(d-1) - \cdots - \mu(1)$. Thus we are finished as soon as we have shown that for any $l, \tilde{d} \in \mathbb{N}$, $\nu \in \mathbb{N} \cup \{0\}$ the matrix $\mathbf{\Gamma}_L(1, \tilde{d}, l, \nu)$ is nonsingular.

To this end, note that for $d = 1$, i.e., $\varphi(x) = \chi_{[0,1)}$, we have $l_1 = 0$, $l_2 = 1$. Thus (3.2.25) gives

$$\phi_{j,l-1}^L(x) = \sum_{m=0}^{l-1} \tilde{\alpha}_{m,0} \varphi_{[j,m]}(x) = \sum_{m=0}^{l-1} \varphi_{[j,m]}(x) \tag{3.3.14}$$

because, by (1.2.3), $\tilde{\alpha}_{m,0} = \int_{\mathbb{R}} \tilde{\varphi}(x-m)dx = 1$. Therefore, one has

$$2^{j(k+1/2)}(\phi_{j,l-1}^L, (\,\cdot\,)^k)_{[0,1]} = 2^{j(k+1/2)} \sum_{m=0}^{l-1} \int_0^1 2^{j/2}\chi_{[2^{-j}m, 2^{-j}(m+1))}(x)x^k dx = 2^{j(k+1)} \int_0^{2^{-j}l} x^k dx,$$

i.e.,

$$2^{j(k+1/2)}(\phi_{j,l-1}^L, (\,\cdot\,)^k)_{[0,1]} = \frac{l^{k+1}}{k+1}, \qquad k = 0, 1, 2, \ldots. \tag{3.3.15}$$

Moreover, we have

$$2^{j(k+1/2)}(\varphi_{[j,\nu]}, (\,\cdot\,)^k)_{[0,1]} = 2^{j(k+1)} \int_{2^{-j}\nu}^{2^{-j}(\nu+1)} x^k dx = \frac{1}{k+1}((\nu+1)^{k+1} - \nu^{k+1}),$$

$$\nu = l, \ldots, \tilde{l}-1.$$

Thus $\mathbf{\Gamma}_L(1, \tilde{d}, l, \nu)$ takes the form

$$\mathbf{\Gamma}_L(1, \tilde{d}, l, \nu) = \begin{bmatrix} \dfrac{l^{\nu+1}}{\nu+1} & \dfrac{l^{\nu+2}}{\nu+2} & \cdots & \dfrac{l^{\nu+\tilde{d}}}{\nu+\tilde{d}} \\[2mm] \dfrac{(l+1)^{\nu+1} - l^{\nu+1}}{\nu+1} & \dfrac{(l+1)^{\nu+2} - l^{\nu+2}}{\nu+2} & \cdots & \dfrac{(l+1)^{\nu+\tilde{d}} - l^{\nu+\tilde{d}}}{\nu+\tilde{d}} \\[2mm] \vdots & \vdots & & \vdots \\[2mm] \dfrac{\tilde{l}^{\nu+1} - (\tilde{l}-1)^{\nu+1}}{\nu+1} & \dfrac{\tilde{l}^{\nu+2} - (\tilde{l}-1)^{\nu+2}}{\nu+2} & \cdots & \dfrac{\tilde{l}^{\nu+\tilde{d}} - (\tilde{l}-1)^{\nu+\tilde{d}}}{\nu+\tilde{d}} \end{bmatrix}.$$

Adding the first row to the second one, adding the result to the third row, and so on produces the matrix

$$
\begin{bmatrix}
\dfrac{l^{\nu+1}}{\nu+1} & \dfrac{l^{\nu+2}}{\nu+2} & \cdots & \dfrac{l^{\nu+\tilde{d}}}{\nu+\tilde{d}} \\[2mm]
\dfrac{(l+1)^{\nu+1}}{\nu+1} & \dfrac{(l+1)^{\nu+2}}{\nu+2} & \cdots & \dfrac{(l+1)^{\nu+\tilde{d}}}{\nu+\tilde{d}} \\[2mm]
\vdots & \vdots & & \vdots \\[2mm]
\dfrac{\tilde{l}^{\nu+1}}{\nu+1} & \dfrac{\tilde{l}^{\nu+2}}{\nu+2} & \cdots & \dfrac{\tilde{l}^{\nu+\tilde{d}}}{\nu+\tilde{d}}
\end{bmatrix}.
$$

Dividing the ith row by $(l + i - 1)^{\nu+1}$ and multiplying then the ith column of the resulting matrix by $\nu + i$ finally produces a Vandermonde matrix which is nonsingular. Taking $\nu = 0$, this confirms the claim for $d = 1$. By (3.3.13), the case $\nu = d - 1$ (with l replaced by $\hat{l}$) verifies the assertion for any $d \geq 2$, which completes the proof. ∎

The above result ensures that there always exist matrices $\mathbf{C}_j$, $\tilde{\mathbf{C}}_j$ such that Φ_j, $\tilde{\Phi}_j$ defined by (3.3.1) are biorthogonal. Recall from (3.3.3) that, setting

$$
\Gamma_j := (\Phi_j', \tilde{\Phi}_j')_{[0,1]},
$$

the matrices $\mathbf{C}_j$, $\tilde{\mathbf{C}}_j$ have to satisfy

$$
\mathbf{C}_j \Gamma_j \tilde{\mathbf{C}}_j^T = \mathbf{I}. \tag{3.3.16}
$$

This obviously leaves a great deal of freedom in choosing the matrices $\mathbf{C}_j$, $\tilde{\mathbf{C}}_j$. Only for the sake of convenience will we fix $\mathbf{C}_j$ in Sections 3 and 4 to be the identity, which means that we keep $\Phi_j = \Phi_j'$ unchanged. Consequently, $\tilde{\mathbf{C}}_j$ is then determined by

$$
\tilde{\mathbf{C}}_j = \Gamma_j^{-T}. \tag{3.3.17}
$$

It will be seen later, in Section 5.3, that this choice is not necessarily optimal with regard to the *quantitative* stability properties of the corresponding single-scale and multiscale bases. However, since the mechanisms described in Section 2.4 show how to incorporate additional changes of bases, we can confine the subsequent theoretical investigations without loss of generality to the case (3.3.17).

Of course, by biorthogonality of the interior functions, it suffices to consider transformation matrices $\mathbf{C}_j$, $\tilde{\mathbf{C}}_j$ of the form

$$
\mathbf{C}_j = \operatorname{diag}(\mathbf{C}_L, \mathbf{I}, \mathbf{C}_R), \qquad \tilde{\mathbf{C}}_j = \operatorname{diag}(\tilde{\mathbf{C}}_L, \mathbf{I}, \tilde{\mathbf{C}}_R) \tag{3.3.18}
$$

with $\tilde{d} \times \tilde{d}$ matrices $\mathbf{C}_L$, $\mathbf{C}_R$, $\tilde{\mathbf{C}}_L$, $\tilde{\mathbf{C}}_R$. Thus let

$$
\Phi_j = \Phi_j', \qquad \tilde{\Phi}_j = \Gamma_j^{-T} \tilde{\Phi}_j', \tag{3.3.19}
$$

so that, in particular, one has

$$\tilde{\phi}_{j,k} = \sum_{l \in \tilde{\Delta}_j^L} (\tilde{\mathbf{C}}_L)_{k,l} \tilde{\phi}_{j,l}^L, \quad k \in \tilde{\Delta}_j^L, \quad \tilde{\mathbf{C}}_L := \mathbf{\Gamma}_L^{-T}. \tag{3.3.20}$$

The above findings can be summarized as follows.

COROLLARY 3.5. *The following holds:*

(i) *The collections* Φ_j, $\tilde{\Phi}_j$ *defined by (3.3.19) are biorthogonal.*

(ii) *dim* $S_j = $ *dim* $\tilde{S}_j = \#\Delta_j = 2d + 2^j - \mu(d) - 2l + 1$

$$= 2^j - 2(l - d) - \mu(d) + 1. \tag{3.3.21}$$

(iii) *The basis functions have small support, i.e.,*

$$diam(supp\ \phi_{j,k}), \ diam(supp\ \tilde{\phi}_{j,k}) \sim 2^{-j}, \quad j \geq j_0. \tag{3.3.22}$$

(iv) *The bases* $\{\Phi_j\}$, $\{\tilde{\Phi}_j\}$ *are uniformly stable.*
(v) *The projectors*

$$Q_j v = (v, \tilde{\Phi}_j)_{[0,1]} \Phi_j, \qquad Q_j^* v = (v, \Phi_j)_{[0,1]} \tilde{\Phi}_j \tag{3.3.23}$$

are uniformly bounded.

(vi) *The spaces* $\tilde{S}_j = S(\tilde{\Phi}_j)$ *are nested and exact of order* $\tilde{d}$.

Proof.   (i) follows from (3.3.16), (3.3.19), and Theorem 3.4, while (ii) is an immediate consequence of (i) and (3.2.17). (iii) results from (3.2.3), (3.2.5), and the definitions (3.2.25), (3.2.26), (3.2.35). Combining (iii) with the fact that the entries of the matrices $\mathbf{\Gamma}_{j,X}$ are independent of j, (3.3.6) yields

$$\|\phi_{j,k}\|, \|\tilde{\phi}_{j,k}\| \lesssim 1, \qquad k \in \Delta_j, j \geq j_0. \tag{3.3.24}$$

Thus (iv) follows, in view of (i), (iii), and (3.3.24), from Lemma 2.1 (i). Finally, (v) is a consequence of (iv) and Remark 2.3, and (vi) follows from Proposition 3.3. ■

Although the proof of Theorem 3.4 makes crucial use of the fact that the primal multiresolution is generated by B-splines, it is perhaps worth pointing out that, in principle, the argument can be extended to other cases as well. For instance, consider a Daubechies scaling function $_d\theta$ of sufficient regularity and order d of exactness. There is a canonical way of generating a family of dual pairs $_{d-r}\theta$, $_{d+r}\theta$ essentially by differentiation and integration described, e.g., in [11, 22, 39, 48]. One could then employ integration by parts as in the above proof but so that one ends up with a Gramian matrix whose regularity follows from the linear independence of the involved functions. Since the Daubechies scaling functions lack the above nice symmetry properties so that both ends of the interval need separate treatment and since the role of the B-splines as generators will be crucial also later for the construction of biorthogonal wavelets, we will not pursue this issue here any further.

3.4. Direct and Inverse Estimates, Norm Equivalences

Combining Corollary 3.5 with Lemma 2.1 (ii) provides the following results.

COROLLARY 3.6. *One has*

$$\inf_{v_j \in V_j} \|v - v_j\|_{L_2([0,1])} \lesssim 2^{-sj} \|v\|_{H^s([0,1])}, \qquad v \in H^s([0, 1]), \tag{3.4.1}$$

where

$$s \leq \begin{cases} d, & V_j = S_j, \\ \tilde{d}, & V_j = \tilde{S}_j. \end{cases} \tag{3.4.2}$$

As mentioned before, the Sobolev regularity of $\tilde{\varphi}$ is proportional to $\tilde{d}$. It is actually strictly positive as soon as $\tilde{\varphi} \in L_2(\mathbb{R})$ [50]. Observe that

$$\gamma := \sup\{s : \varphi \in H^s(\mathbb{R})\} = d - \frac{1}{2},$$

and let

$$\tilde{\gamma} := \sup\{s : \tilde{\varphi} \in H^s(\mathbb{R})\}. \tag{3.4.3}$$

The following fact follows from [19].

PROPOSITION 3.7. *The inverse estimate*

$$\|v_j\|_{H^s([0,1])} \lesssim 2^{sj} \|v_j\|_{L_2([0,1])}, \qquad v_j \in V_j, \tag{3.4.4}$$

holds where

$$s < \begin{cases} \gamma, & V_j = S_j, \\ \tilde{\gamma}, & V_j = \tilde{S}_j. \end{cases} \tag{3.4.5}$$

Combining Corollaries 3.5 and 3.6 and Proposition 3.7 with Theorem 2.4 provides

COROLLARY 3.8. *Let Q_j be defined by (3.3.23). Then one has (with $Q_{j_0-1} := 0$)*

$$\left(\sum_{j=j_0}^{\infty} 2^{2sj} \|(Q_j - Q_{j-1})v\|_{L_2([0,1])}^2\right)^{1/2} \sim \begin{cases} \|v\|_{H^s([0,1])}, & s \in [0, \gamma), \\ \|v\|_{(H^{-s}([0,1]))^*}, & s \in (-\tilde{\gamma}, 0). \end{cases} \tag{3.4.6}$$

3.5. Refinement Matrices

We conclude this section with identifying the refinement matrices corresponding to Φ_j and $\tilde{\Phi}_j$. This will be of crucial importance later for the identification of stable bases for

the complements $(Q_j - Q_{j-1})S_j$. From Lemma 3.1 and (3.2.40) we infer that Φ_j satisfies (2.1.3) with

$$\mathbf{M}_{j,0} := \quad\quad (3.5.1)$$

where $\mathbf{M}_L$, $\mathbf{M}_R$ are $(d + l + l_2 - 1) \times d$ blocks of the form

$$(\mathbf{M}_L)_{m,k} = \begin{cases} 2^{-(k-l+d+1/2)}\delta_{k,m}, & m, k \in \{l - d, \dots, l - 1\} = \Delta_j^L, \\ 2^{-(k-l+d+1/2)}\tilde{\alpha}_{m,k-l+d}, & m = l, \dots, 2l + l_1 - 1, k \in \Delta_j^L, \\ \tilde{\beta}_{j,m,k-l+d}^L, & m = 2l + l_1, \dots, l_2 + 2l - 2, k \in \Delta_j^L, \end{cases} \quad (3.5.2)$$

and by (3.2.43)

$$\mathbf{M}_R = \mathbf{M}_L^{\updownarrow}, \quad\quad (3.5.3)$$

i.e.,

$$(\mathbf{M}_R)_{2^{j}-\mu(d)-m, 2^{j}-\mu(d)-k} = (\mathbf{M}_L)_{m,k}, \quad m = l - d, \dots, l_2 + 2l - 2, k \in \Delta_j^L.$$

Moreover, $\mathbf{A}_j$ has the form

$$(\mathbf{A}_j)_{m,k} = \frac{1}{\sqrt{2}} a_{m-2k}, \quad 2l + l_1 \le m \le l_2 + 2^{j+1} - 2(l + \mu(d)), k \in \Delta_j^0. \quad (3.5.4)$$

The structure of the refinement matrix $\tilde{\mathbf{M}}_{j,0}'$ corresponding to $\tilde{\Phi}_j'$ defined in (3.2.36) is completely analogous and results from replacing l, l_1, l_2, d by $\tilde{l}$, $\tilde{l}_1$, $\tilde{l}_2$, $\tilde{d}$, respectively, i.e.,

$$\tilde{\mathbf{M}}_{j,0}' = \quad\quad (3.5.5)$$

with the $(\tilde{d} + \tilde{l} + \tilde{l}_2 - 1) \times \tilde{d}$ blocks

$$(\tilde{\mathbf{M}}'_L)_{m,k} = \begin{cases} 2^{-(k-\tilde{l}+\tilde{d}+1/2)}\delta_{k,m}, & m, k \in \{\tilde{l} - \tilde{d}, \dots, \tilde{l} - 1\} = \tilde{\Delta}^L_j, \\ 2^{-(k-\tilde{l}+\tilde{d}+1/2)}\alpha_{m,k-\tilde{l}+\tilde{d}}, & m = \tilde{l}, \dots, 2\tilde{l} + \tilde{l}_1 - 1, k \in \tilde{\Delta}^L_j, \\ \beta^L_{j,m,k-\tilde{l}+\tilde{d}}, & m = 2\tilde{l} + \tilde{l}_1, \dots, \tilde{l}_2 + 2\tilde{l} - 2, k \in \tilde{\Delta}^L_j, \end{cases}$$ (3.5.6)

and

$$\tilde{\mathbf{M}}'_R = (\tilde{\mathbf{M}}'_L)^{\updownarrow}$$ (3.5.7)

as well as

$$(\tilde{\mathbf{A}}_j)_{m,k} = \frac{1}{\sqrt{2}}\,\tilde{a}_{m-2k}, \qquad 2\tilde{l} + \tilde{l}_1 \leq m \leq \tilde{l}_2 + 2^{j+1} - 2(\tilde{l} + \mu(d)), \, k \in \tilde{\Delta}^0_j. \quad (3.5.8)$$

To determine now the refinement matrices for the biorthogonalized bases $\tilde{\Phi}_j$ defined in (3.3.19), we write the biorthogonalization in the form

$$\tilde{\Phi}_j = \tilde{\mathbf{C}}_j\tilde{\Phi}'_j,$$ (3.5.9)

where

$$\tilde{\mathbf{C}}_j = \begin{pmatrix} \mathbf{\Gamma}_L^{-T} & 0 & 0 \\ 0 & \mathbf{I}^{(2^j-2\tilde{l}+1-\mu(d))} & 0 \\ 0 & 0 & \mathbf{\Gamma}_R^{-T} \end{pmatrix},$$ (3.5.10)

with $\mathbf{\Gamma}_X$ defined by (3.3.5) and $\mathbf{I}^{(r)}$ the $r \times r$ identity matrix. We readily infer now from Remark 2.8 that

$$\tilde{\mathbf{M}}_{j,0} = \tilde{\mathbf{C}}_{j+1}^{-T}\tilde{\mathbf{M}}'_{j,0}\tilde{\mathbf{C}}_j^T.$$ (3.5.11)

Keeping (3.5.5) in mind and splitting $\tilde{\mathbf{M}}'_L$ into two blocks such as

$$\tilde{\mathbf{M}}'_L = \begin{pmatrix} \mathbf{D} \\ \mathbf{K} \end{pmatrix}, \qquad \mathbf{D} = 2^{-(k-\tilde{l}+\tilde{d}+1/2)}\delta_{k,m}, \, k, m \in \tilde{\Delta}^L_j,$$

with $\mathbf{K}$ defined by (3.5.6), one easily confirms from (3.5.5) and (3.5.11) that

$$\tilde{\mathbf{M}}_{j,0} = \qquad (3.5.12)$$

where now

$$\tilde{\mathbf{M}}_L = \begin{pmatrix} \boldsymbol{\Gamma}_L \mathbf{D} \boldsymbol{\Gamma}_L^{-1} \\ \mathbf{K} \boldsymbol{\Gamma}_L^{-1} \end{pmatrix}, \qquad \tilde{\mathbf{M}}_R = \tilde{\mathbf{M}}_L^{\updownarrow}, \tag{3.5.13}$$

and $\tilde{\mathbf{A}}_j$ remains the same as in (3.5.8).

4. BIORTHOGONAL WAVELETS ON [0, 1]

4.1. An Initial Stable Completion

The most common strategy for constructing now biorthogonal wavelets for a biorthogonal multiresolution as above consists in keeping as many translates $\psi_{[j,k]}$, $\tilde{\psi}_{[j,k]}$ defined by (1.2.14) as possible whose support is sufficiently inside and complementing this set by a certain finite number of additional functions near the end points [2, 16, 40]. These additional functions are, roughly speaking, produced by projecting every second fine scale generator near the end points. Although this may in principle be a feasible approach we still feel somewhat uncomfortable with the reasoning in [2], in particular, with regard to stability of the complement bases. Therefore, we take here a completely different route suggested by the general development in Section 2.3. As a first step we will construct certain stable complement bases for the spaces S_j corresponding to a stable completion of the refinement matrices of Φ_j in the sense of Section 2.3. In a second step these initial complements will be projected into the desired ones employing again the tools from Section 2.3.

REMARK 4.1.   *We would like to stress that we do not view the following construction of an initial stable completion merely as an auxiliary ingredient of the final derivation of biorthogonal wavelets. In fact, the corresponding initial complement bases are interesting in their own right since their elements have small or even minimal support. For instance, in the case $d = 2$ the interior complement functions correspond to the* hierarchical bases *from* [51]. *Therefore, it may not be surprising that the subsequent projection into biorthogonal bases in Theorem* 4.8 *below seems to produce automatically interior wavelets which agree with those derived by* [15].

The construction of the initial stable completion $\check{\mathbf{M}}_{j,1}$ of $\mathbf{M}_{j,0}$ in (3.5.1) consists of several steps, each of which involves different matrices which are described most conveniently in a schematic block form. All these matrices will depend only *weakly* on the scale j, which means that the entries of the various blocks remain the same and only the size of the central blocks depends on j. To describe the size of the involved blocks accurately it will be convenient to abbreviate

$$p = p(j) := \#\Delta_j^0 = 2^j - 2l - \mu(d) + 1,$$

$$q = q(j) := 2p + d - 1 = 2^{j+1} - 4l - 2\mu(d) + d + 1$$

and keep in mind that (3.2.3), i.e., $l_1 = -\lfloor \frac{d}{2} \rfloor, l_2 = \lceil \frac{d}{2} \rceil$.

In these terms the interior block $\mathbf{A}_j$ in (3.5.1) is a $q \times p$ matrix of the form

$$
\mathbf{A}_j = \frac{1}{\sqrt{2}}
\begin{bmatrix}
a_{l_1} & 0 & \cdots & & & 0 \\
a_{l_1+1} & 0 & & & & \vdots \\
a_{l_1+2} & a_{l_1} & & & & \\
\vdots & \vdots & & & & \vdots \\
a_{l_2} & a_{l_2-2} & & & & \\
0 & a_{l_2-1} & & & 0 & \\
0 & a_{l_2} & & & a_{l_1} & \\
\vdots & & & & \vdots & \\
& & & & a_{l_2} & a_{l_2-2} \\
& & & & 0 & a_{l_2-1} \\
& & & & 0 & a_{l_2}
\end{bmatrix},
\tag{4.1.1}
$$

where $\mathbf{a} = \{a_k\}_{k=l_1}^{l_2}$ is the mask of $\varphi = {}_d\varphi$ in (3.2.4).

The core ingredient of our construction is a factorization of $\mathbf{A}_j$ and later of $\mathbf{M}_{j,0}$ which is inspired by similar considerations for the bi-infinite case in [24]. Employing suitable Gauss-type eliminations we will successively reduce upper and lower bands from $\mathbf{A}_j$. To this end, suppose that after i steps the resulting matrix $\mathbf{A}_j^{(i)}$ has the form

$$
\mathbf{A}_j^{(i)} =
\begin{bmatrix}
\left.\begin{matrix} 0 \\ \vdots \\ 0 \end{matrix}\right. & \left.\begin{matrix} 0 \\ \vdots \\ 0 \end{matrix}\right\}\left\lceil\frac{i}{2}\right\rceil & & 0 \\
a^{(i)}_{l_1+\left\lceil\frac{i}{2}\right\rceil} & 0 & & \\
a^{(i)}_{l_1+\left\lceil\frac{i}{2}\right\rceil+1} & 0 & & \\
\vdots & a^{(i)}_{l_1+\left\lceil\frac{i}{2}\right\rceil} & & \\
\vdots & \vdots & & \\
a^{(i)}_{l_2-\left\lfloor\frac{i}{2}\right\rfloor} & & & \\
0 & & & \\
\vdots & & & \\
& & & a^{(i)}_{l_2-\left\lfloor\frac{i}{2}\right\rfloor} \\
0 & & & \left\lfloor\frac{i}{2}\right\rfloor\left\{\begin{matrix} 0 \\ \vdots \\ 0 \end{matrix}\right.
\end{bmatrix}
\qquad \mathbf{A}_j^{(0)} := \mathbf{A}_j.
\tag{4.1.2}
$$

Defining

$$
\mathbf{U}_{i+1} := \begin{bmatrix} 1 & -\dfrac{a^{(i)}_{l_1+\left\lceil \frac{i}{2}\right\rceil}}{a^{(i)}_{l_1+\left\lceil \frac{i}{2}\right\rceil+1}} \\ 0 & 1 \end{bmatrix}, \qquad \mathbf{L}_{i+1} := \begin{bmatrix} 1 & 0 \\ -\dfrac{a^{(i)}_{l_2-\left\lfloor \frac{i}{2}\right\rfloor}}{a^{(i)}_{l_2-\left\lfloor \frac{i}{2}\right\rfloor-1}} & 1 \end{bmatrix}, \qquad (4.1.3)
$$

and setting

$$
\mathbf{H}^{(2i-1)}_j := \operatorname{diag}\left[\mathbf{I}^{(i-1)}, \underbrace{\mathbf{U}_{2i-1}, \ldots, \mathbf{U}_{2i-1}}_{p+\frac{1}{2}(d+\mu(d)-2)}, \mathbf{I}^{(2-\mu(d)-i)} \right] \in \mathbb{R}^{q\times q}
$$

$$
\mathbf{H}^{(2i)}_j := \operatorname{diag}\left[\mathbf{I}^{(2-\mu(d)-i)}, \underbrace{\mathbf{L}_{2i}, \ldots, \mathbf{L}_{2i}}_{p+\frac{1}{2}(d+\mu(d)-2)}, \mathbf{I}^{(i-1)} \right] \in \mathbb{R}^{q\times q},
$$

$$(4.1.4)$$

one easily confirms that indeed

$$
\mathbf{A}^{(i)}_j = \mathbf{H}^{(i)}_j \mathbf{A}^{(i-1)}_j, \tag{4.1.5}
$$

provided that $\mathbf{H}^{(i)}_j$ is well defined, which means that

$$
a^{(i-1)}_{l_1+\left\lceil \frac{i-1}{2}\right\rceil+1}, \qquad a^{(i-1)}_{l_2-\left\lfloor \frac{i-1}{2}\right\rfloor-1}
$$

have to be different from zero whenever

$$
a^{(i-1)}_{l_1+\left\lceil \frac{i-1}{2}\right\rceil}, \qquad a^{(i-1)}_{l_2-\left\lfloor \frac{i-1}{2}\right\rfloor}
$$

are different from zero. For $i = 1$ this is clearly the case. More generally, the following holds.

REMARK 4.2. *One has*

$$
a^{(i)}_{l_1+\left\lceil \frac{i}{2}\right\rceil}, \ldots, a^{(i)}_{l_2-\left\lfloor \frac{i}{2}\right\rfloor} \neq 0. \tag{4.1.6}
$$

Proof. This assertion has been essentially established in [24]. In fact, $\mathbf{A}_j$ is well known to be totally positive. Moreover, as pointed out in [24], the above eliminations preserve total positivity while by the results in [33, 42] the extreme entries in (4.1.6) as certain minors of $\mathbf{A}_j$ are strictly positive. Again by total positivity of $\mathbf{A}_j^{(i)}$ all the entries in between have to be strictly positive. ∎

Hence, $\mathbf{H}_j^{(i)}$ is well defined for $i = 1, \ldots, d$, and $\mathbf{A}_j^{(d)}$ has the form

$$
\mathbf{A}_j^{(d)} =
\begin{bmatrix}
\left.\begin{matrix} 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{matrix}\right\} l_2 = \left\lceil \frac{d}{2} \right\rceil \\
b \quad 0 \\
0 \quad 0 \\
0 \quad b \\
\vdots \quad 0 \quad \ddots \\
 \quad b \\
-l_1 = \left\lfloor \frac{d}{2} \right\rfloor \left\{ \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \right.
\end{bmatrix},
\tag{4.1.7}
$$

where

$$
b := a_{l_1 + \left\lceil \frac{d}{2} \right\rceil}^{(d)} = a_{l_1 + l_2}^{(d)} = a_{\mu(d)}^{(d)} \neq 0
\tag{4.1.8}
$$

and the middle nonzero block has $2p - 1$ rows. Obviously, $\mathbf{A}_j^{(d)}$ has full rank p and

$$
\mathbf{B}_j :=
\begin{bmatrix}
0 & \cdots & 0 & b^{-1} & 0 & 0 & 0 & \cdots \\
0 & \cdots & 0 & 0 & 0 & b^{-1} & 0 & \cdots \\
\underbrace{}_{\left\lceil \frac{d}{2} \right\rceil = l_2} \\
& & & & & \ddots \\
& & & & & & \underbrace{}_{\left\lfloor \frac{d}{2} \right\rfloor = -l_1} \\
& & & & b^{-1} & 0 & \cdots & 0
\end{bmatrix}
\in \mathbb{R}^{p \times q}
\tag{4.1.9}
$$

satisfies

$$
\mathbf{B}_j \mathbf{A}_j^{(d)} = \mathbf{I}^{(p)}.
\tag{4.1.10}
$$

Similarly, defining

$$
\mathbf{F}_j =
\begin{bmatrix}
\left.\begin{matrix} 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{matrix}\right\} l_2 - 1 \\
1 \quad 0 \\
0 \quad 0 \\
0 \quad 1 \\
\vdots \quad 0 \quad \ddots \\
\qquad\qquad 1 \\
-l_1 + 1 \left\{\begin{matrix} 0 \\ \vdots \\ 0 \end{matrix}\right. \\
\end{bmatrix}
\in \mathbb{R}^{p \times q}
\tag{4.1.11}
$$

essentially by shifting up each row of $\mathbf{B}_j^T$ by one, we have

$$
\mathbf{B}_j \mathbf{F}_j = \mathbf{0}.
\tag{4.1.12}
$$

After these preparations we have to pad the matrices $\mathbf{A}_j^{(d)}$, $\mathbf{B}_j$, $\mathbf{F}_j$ according to (3.5.1) to form matrices of the right size. The corresponding expanded versions will be denoted by $\hat{\mathbf{A}}_j^{(d)}$, $\hat{\mathbf{B}}_j$, $\hat{\mathbf{F}}_j$, respectively. To this end, let

$$
\left.\begin{matrix} \hat{\mathbf{A}}_j^{(d)} \\ \hat{\mathbf{B}}_j^T \end{matrix}\right\} :=
\begin{array}{c}
d\{ \\ l+l_1\{ \\ \\ \\ \\
\end{array}
\begin{bmatrix}
\mathbf{I}^{(d)} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & & \\
\mathbf{0} & \begin{Bmatrix}\mathbf{A}_j^{(d)} \\ \mathbf{B}_j^T\end{Bmatrix} & \mathbf{0} \\
& & \mathbf{0} \\
& \mathbf{0} & \mathbf{I}^{(d)}
\end{bmatrix}
\begin{array}{l}
\\ \\ \left.\right\} q = 2^{j+1} - 4l - 2\mu(d) + d + 1. \\ \left.\right\} l - l_2 + \mu(d) \\ \left.\right\} d
\end{array}
\tag{4.1.13}
$$

$$\underbrace{\qquad\qquad\qquad}_{p = 2^j - 2l - \mu(d) + 1}$$

In fact, recalling (3.2.17) and noting that

$$
d + l + l_1 = l + l_2 = l - l_2 + \mu(d) + d,
$$

one readily confirms that $\hat{\mathbf{A}}_j^{(d)}$, $\hat{\mathbf{B}}_j^T$ are $(\#\Delta_{j+1}) \times (\#\Delta_j)$ matrices.

Note that always

$$
\#\Delta_{j+1} - \#\Delta_j = 2^j
$$

is valid independent of l, $\tilde{l}$, d, $\tilde{d}$. Thus, a completion of $\hat{\mathbf{A}}_j^{(d)}$ has to be a $(\#\Delta_{j+1}) \times 2^j$ matrix. To this end, consider

$$\hat{\mathbf{F}}_j := \begin{array}{c} d\left\{\vphantom{}\right. \\ \\ \\ \\ -l_1\left\{\vphantom{}\right. \end{array} \begin{array}{|c|} \hline \mathbf{0} \\ \hline \mathbf{I}^{(l+\mu(d)-1)} \\ \hline \mathbf{F}_j \\ \hline \mathbf{I}^{(l)} \\ \hline \mathbf{0} \\ \hline \end{array} \begin{array}{c} \\ \left.\vphantom{}\right\} l_2 - 1 \\ \\ \\ \left.\vphantom{}\right\} d \end{array} \qquad (4.1.14)$$

In fact,

$$2d + l + \mu(d) - 1 + l - l_2 + 1 + l_1 + q$$
$$= 2d + 2l + \mu(d) - d + 2^{j+1} - 4l - 2\mu(d) + d + 1$$
$$= 2d - 2l - \mu(d) + 1 + 2^{j+1} = \#\Delta_{j+1},$$

while

$$l + \mu(d) - 1 + p + l = 2l - 1 + \mu(d) + 2^j - 2l - \mu(d) + 1 = 2^j,$$

so that $\hat{F}_j$ is indeed a $(\#\Delta_{j+1}) \times 2^j$ matrix.

LEMMA 4.3.   *The following relations hold:*

$$\hat{\mathbf{B}}_j \hat{\mathbf{A}}_j^{(d)} = \mathbf{I}^{(\#\Delta_j)}, \qquad \hat{\mathbf{F}}_j^T \hat{\mathbf{F}}_j = \mathbf{I}^{(2^j)}, \qquad (4.1.15)$$

and

$$\hat{\mathbf{B}}_j \hat{\mathbf{F}}_j = \mathbf{0}, \qquad \hat{\mathbf{F}}_j^T \hat{\mathbf{A}}_j^{(d)} = \mathbf{0}. \qquad (4.1.16)$$

Proof.   The relations in (4.1.15) follow from (4.1.10), (4.1.13), and (4.1.14). Next note that the lower right identity block in (4.1.14) is positioned to *miss* the lower right nonzero entry b in $\mathbf{A}_j^{(d)}$ as well as the lower right identity block $\mathbf{I}^{(d)}$ in $\hat{\mathbf{A}}_j^{(d)}$ so that the relations (4.1.16) follow from (4.1.12) and (4.1.13).   ∎

The factorization of $\mathbf{A}_j$ induced by (4.1.5) is easily carried over to a factorization of $\hat{\mathbf{A}}_j$ which is defined by (4.1.13) with $\mathbf{A}_j^{(d)}$ replaced by $\mathbf{A}_j$ given by (4.1.1). In fact, let

$$\hat{\mathbf{H}}_j^{(i)} := \mathrm{diag}(\mathbf{I}^{(l+l_2)}, \mathbf{H}_j^{(i)}, \mathbf{I}^{(l+l_2)}) \tag{4.1.17}$$

denote the corresponding expansions of the elimination matrices $\mathbf{H}_j^{(i)}$. One easily checks that the above block structure leads to the following relations.

LEMMA 4.4. *The matrix $\hat{\mathbf{A}}_j$ can be factorized as*

$$\hat{\mathbf{A}}_j = \hat{\mathbf{H}}_j^{-1}\hat{\mathbf{A}}_j^{(d)}, \tag{4.1.18}$$

where

$$\hat{\mathbf{H}}_j^{-1} = (\hat{\mathbf{H}}_j^{(1)})^{-1} \cdots (\hat{\mathbf{H}}_j^{(d)})^{-1} \tag{4.1.19}$$

and

$$(\hat{\mathbf{H}}_j^{(2i-1)})^{-1} = \mathrm{diag}\left[\mathbf{I}^{(l+l_2+i-1)}, \underbrace{\mathbf{U}_{2i-1}^{-1}, \ldots, \mathbf{U}_{2i-1}^{-1}}_{p + \frac{1}{2}(d + \mu(d) - 2)}, \mathbf{I}^{(l+l_2+2-\mu(d)-i)} \right],$$

$$(\hat{\mathbf{H}}_j^{(2i)})^{-1} = \mathrm{diag}\left[\mathbf{I}^{(l+l_2+2-\mu(d)-i)}, \underbrace{\mathbf{L}_{2i}^{-1}, \ldots, \mathbf{L}_{2i}^{-1}}_{p + \frac{1}{2}(d + \mu(d) - 2)}, \mathbf{I}^{(l+l_2+i-1)} \right]. \tag{4.1.20}$$

As in the bi-infinite case one has

REMARK 4.5. *The matrix $\hat{\mathbf{H}}_j^{-1}$ is $(d + 1)$-banded.*

Proof. This follows directly from (4.1.19), (4.1.20), and the fact that due to

$$\mathbf{U}_{i+1}^{-1} := \begin{bmatrix} 1 & \dfrac{a_{l_1+\left\lceil \frac{i}{2} \right\rceil}^{(i)}}{a_{l_1+\left\lceil \frac{i}{2} \right\rceil+1}^{(i)}} \\ 0 & 1 \end{bmatrix}, \qquad \mathbf{L}_{i+1}^{-1} := \begin{bmatrix} 1 & 0 \\ \dfrac{a_{l_2-\left\lfloor \frac{i}{2} \right\rfloor}^{(i)}}{a_{l_2-\left\lfloor \frac{i}{2} \right\rfloor-1}^{(i)}} & 1 \end{bmatrix},$$

the factors $(\hat{\mathbf{H}}_j^{(i)})^{-1}$ are still block diagonal with 2×2 blocks. ∎

The relevance of the factorization (4.1.18) relies on the observation that

$$\mathbf{M}_{j,0} = \mathbf{P}_j\hat{\mathbf{A}}_j = \mathbf{P}_j\hat{\mathbf{H}}_j^{-1}\hat{\mathbf{A}}_j^{(d)}, \tag{4.1.21}$$

where

$$\mathbf{P}_j = \begin{pmatrix} \mathbf{M}_L & & \\ & \mathbf{I}^{(\#\Delta_{j+1}-2d)} & \\ & & \mathbf{M}_R \end{pmatrix} \in \mathbb{R}^{(\#\Delta_{j+1})\times(\#\Delta_{j+1})}. \tag{4.1.22}$$

We are now in a position to state the main result of this section.

PROPOSITION 4.6.   *The matrices*

$$\check{\mathbf{M}}_{j,1} := \mathbf{P}_j \hat{\mathbf{H}}_j^{-1} \hat{\mathbf{F}}_j \tag{4.1.23}$$

are uniformly stable completions of the refinement matrices $\mathbf{M}_{j,0}$ *(3.5.1) for the bases* Φ_j. *Moreover, the inverse*

$$\check{\mathbf{G}}_j = \begin{pmatrix} \check{\mathbf{G}}_{j,0} \\ \check{\mathbf{G}}_{j,1} \end{pmatrix}$$

of $\check{\mathbf{M}}_j = (\mathbf{M}_{j,0}, \check{\mathbf{M}}_{j,1})$ *is given by*

$$\check{\mathbf{G}}_{j,0} = \hat{\mathbf{B}}_j \hat{\mathbf{H}}_j \mathbf{P}_j^{-1}, \qquad \check{\mathbf{G}}_{j,1} = \hat{\mathbf{F}}_j^T \hat{\mathbf{H}}_j \mathbf{P}_j^{-1}. \tag{4.1.24}$$

Proof.   By Lemmas 4.3 and 4.4 and (4.1.15), (4.1.21), we have

$$\check{\mathbf{G}}_{j,0}\mathbf{M}_{j,0} = \hat{\mathbf{B}}_j \hat{\mathbf{H}}_j \mathbf{P}_j^{-1} \mathbf{P}_j \hat{\mathbf{A}}_j = \hat{\mathbf{B}}_j \hat{\mathbf{H}}_j \hat{\mathbf{H}}_j^{-1} \hat{\mathbf{A}}_j^{(d)} = \hat{\mathbf{B}}_j \hat{\mathbf{A}}_j^{(d)} = \mathbf{I}^{(\#\Delta_j)} \tag{4.1.25}$$

and

$$\check{\mathbf{G}}_{j,1}\check{\mathbf{M}}_{j,1} = \hat{\mathbf{F}}_j^T \hat{\mathbf{F}}_j = \mathbf{I}^{(2^j)},$$

while similarly by (4.1.23), (4.1.16),

$$\check{\mathbf{G}}_{j,0}\check{\mathbf{M}}_{j,1} = \hat{\mathbf{B}}_j \hat{\mathbf{F}}_j = \mathbf{0}, \qquad \check{\mathbf{G}}_{j,1}\mathbf{M}_{j,0} = \hat{\mathbf{F}}_j^T \hat{\mathbf{A}}_j^{(d)} = \mathbf{0}. \tag{4.1.26}$$

This shows that

$$\check{\mathbf{G}}_j \check{\mathbf{M}}_j = \mathbf{I}^{(\#\Delta_{j+1})}.$$

Next note that $\mathbf{P}_j^{-1}$ has a similar block structure as $\mathbf{P}_j$ with the size of the upper left and lower right blocks independent of j. Thus, the only dependence of $\mathbf{P}_j$ and $\mathbf{P}_j^{-1}$ on j lies

in the size of the central identity block. A completely analogous statement is obviously true for all the other involved matrices in that all the upper left and lower right blocks are independent of j while the entries of the central blocks are also stationary and only the size of the central blocks depends on j. Thus, by Remark 4.5, $\check{\mathbf{M}}_j$ and $\check{\mathbf{G}}_j$ are both uniformly banded with entries independent of j in the above sense. Hence one trivially has that

$$\|\check{\mathbf{M}}_j\|, \|\check{\mathbf{G}}_j\| = \mathcal{O}(1), \qquad j \geq j_0,$$

where $\|\cdot\|$ denotes the spectral norm. The claimed uniform stability of the completions is therefore an immediate consequence of Proposition 2.5. ∎

REMARK 4.7.   *Note that when d is odd, one has $l_2 - 1 \neq -l_1 + 1$ so that the upper and lower zero blocks in the matrix $\mathbf{F}_j$ in (4.1.11) have different size. This asymmetry is inherited by $\check{\mathbf{M}}_{j,1}$, $\check{\mathbf{G}}_{j,1}$ in (4.1.23), (4.1.24) and, consequently, by the primal and dual wavelets in (4.2.2) below. However, it has been shown in [29] how to "symmetrize" the wavelets while preserving locality. By this we mean that the left end wavelets are related to the right end ones by relations like those in Remark 3.2. In particular, the corresponding completions satisfy $\mathbf{M}_{j,1} = \mathbf{M}_{j,1}^{\updownarrow}$, $\tilde{\mathbf{M}}_{j,1} = \hat{\mathbf{M}}_{j,1}^{\updownarrow}$.*

4.2. Biorthogonal Wavelet Bases

It merely remains to put all the collected ingredients together to formulate the following main result of this paper.

THEOREM 4.8.   *Adhering to the above notation for $\mathbf{M}_{j,0}$, $\tilde{\mathbf{M}}_{j,0}$, $\check{\mathbf{M}}_{j,1}$ defined by (3.5.1), (3.5.12), and (4.1.23), respectively, let*

$$\mathbf{M}_{j,1} := (\mathbf{I}^{(\#\Delta_{j+1})} - \mathbf{M}_{j,0}\tilde{\mathbf{M}}_{j,0}^T)\check{\mathbf{M}}_{j,1}. \tag{4.2.1}$$

Then the following statements hold:

   (i) The $\mathbf{M}_{j,1}$ are uniformly stable completions of the $\mathbf{M}_{j,0}$. The inverse $\mathbf{G}_j$ of $\mathbf{M}_j = (\mathbf{M}_{j,0}, \mathbf{M}_{j,1})$ is given by

$$\mathbf{G}_j = \begin{pmatrix} \tilde{\mathbf{M}}_{j,0}^T \\ \check{\mathbf{G}}_{j,1} \end{pmatrix},$$

where $\check{\mathbf{G}}_{j,1}$ is defined by (4.1.24) and $\mathbf{M}_j$ and $\mathbf{G}_j$ are uniformly banded.

   (ii) Setting

$$\Psi_j^T := \Phi_{j+1}^T \mathbf{M}_{j,1}, \qquad \tilde{\Psi}_j^T = \tilde{\Phi}_{j+1}^T \check{\mathbf{G}}_{j,1}^T \tag{4.2.2}$$

and

$$\Psi := \Phi_{j_0} \cup \bigcup_{j=j_0}^{\infty} \Psi_j, \qquad \tilde{\Psi} := \tilde{\Phi}_{j_0} \cup \bigcup_{j=j_0}^{\infty} \tilde{\Psi}_j \tag{4.2.3}$$

then Ψ, $\tilde{\Psi}$ are biorthogonal Riesz bases for $L_2([0, 1])$, i.e., for $\Psi_{j_0-1} := \Phi_{j_0}$, $\tilde{\Psi}_{j_0-1}$ $:= \tilde{\Phi}_{j_0}$

$$(\Psi, \tilde{\Psi})_{[0,1]} = \mathbf{I}, \tag{4.2.4}$$

and

$$\mathrm{diam}(\mathrm{supp}\ \psi_{j,k}),\ \mathrm{diam}(\mathrm{supp}\ \tilde{\psi}_{j,k}) \sim 2^{-j}, \qquad j \geq j_0. \tag{4.2.5}$$

(iii) For $\tilde{\gamma}$ defined in (3.4.3) one has

$$(\|(v, \tilde{\Phi}_{j_0})_{[0,1]}\|^2_{l_2(\Delta_{j_0})} + \sum_{j=j_0}^{\infty} 2^{2sj}\|(v, \tilde{\Psi}_j)_{[0,1]}\|^2_{l_2(\nabla_j)})^{1/2} \sim \begin{cases} \|v\|_{H^s([0,1])}, & s \in [0, \gamma), \\ \|v\|_{(H^{-s}([0,1]))^*}, & s \in (-\tilde{\gamma}, 0). \end{cases}$$

Proof. (i) and part of (ii) are immediate consequences of Propositions 2.6 and 4.6 as well as the fact that the columns in $\tilde{\mathbf{M}}_{j,0}$ have uniformly bounded lengths. The Riesz basis property and (iii) follow from Corollary 3.8 and the uniform stability of the Ψ_j asserted by (i). ■

5. COMPUTATIONAL ISSUES AND EXAMPLE

5.1. Some Ingredients of the Construction

In this section we wish to complement the above theoretical developments by some comments on the concrete computation of its ingredients which are

(i) The coefficients $\alpha_{m,r}$, $\tilde{\alpha}_{m,r}$ in (3.2.20) and $\beta^L_{j,m,r}$, $\tilde{\beta}^L_{j,m,r}$ in (3.2.31);
(ii) the matrices Γ_L from (3.3.5), (3.3.6), and their inverses;
(iii) the matrices $\mathbf{M}_L$, $\tilde{\mathbf{M}}'_L$, $\tilde{\mathbf{M}}_L$ in (3.5.2), (3.5.6), (3.5.13).

All these quantities have been shown to be independent of j. Moreover, by symmetry, the corresponding right end counterparts, Γ_R, $\mathbf{M}_R$, $\tilde{\mathbf{M}}'_R$, and $\tilde{\mathbf{M}}_R$ are simply obtained by reversing the order of rows and columns, i.e., by forming, e.g., $\mathbf{M}_R = \mathbf{M}_L^{\updownarrow}$, so that it suffices to determine the left end quantities.

A glance at the formulae quoted above reveals that the coefficients $\alpha_{m,r}$, $\tilde{\alpha}_{m,r}$ serve as main building blocks in all of the quantities in (ii) and (iii). We will therefore describe first the *exact* (up to round off) computation of the $\alpha_{m,r}$, $\tilde{\alpha}_{m,r}$.

Ad (i). It immediately follows from (3.1.9) and the normalization

$$\int_{\mathbb{R}} \varphi(x)dx = \int_{\mathbb{R}} \tilde{\varphi}(x)dx = 1$$

that

$$\alpha_{m,0} = \tilde{\alpha}_{m,0} = 1, \qquad m \in \mathbb{Z}, \tag{5.1.1}$$

as well as

$$\alpha_{m,r} = \sum_{i=0}^{r} \binom{r}{i} m^i \alpha_{0,r-i}, \qquad r = 1, \ldots, \tilde{d} - 1, \tag{5.1.2}$$

and analogously for $\tilde{\alpha}_{m,r}$. The coefficients $\alpha_{0,s}$, $\tilde{\alpha}_{0,s}$, in turn, can be determined with the aid of the following familiar recursion (see, e.g., [25, 47]),

$$\alpha_{0,r} = (2^{r+1} - 2)^{-1} \sum_{k=l_1}^{l_2} a_k \sum_{s=0}^{r-1} \binom{r}{s} k^{r-s} \alpha_{0,s}, \qquad r = 1, \ldots, \tilde{d} - 1, \tag{5.1.3}$$

and analogously for $\tilde{\alpha}_{0,r}$, $r = 1, \ldots, d - 1$. In fact, (5.1.3) holds for all $r \in \mathbb{N}$.

Consequently, $\beta^L_{j,m,r}$, $\tilde{\beta}^L_{j,m,r}$ can be computed exactly from (3.2.31) and (5.1.1)–(5.1.3).

Ad (ii). Recall from (3.3.8) that for $r = 0, \ldots, d - 1$ and $k = 0, \ldots, \tilde{d} - 1$,

$$(\phi^L_{j,l-d+r}, \tilde{\phi}^L_{j,\tilde{l}-\tilde{d}+k})_{[0,1]} = \sum_{\nu=-l_2+1}^{l-1} \sum_{\mu=-\tilde{l}_2+1}^{\tilde{l}-1} \tilde{\alpha}_{\nu,r} \alpha_{\mu,k} \int_0^{2^j} \varphi(x - \nu) \tilde{\varphi}(x - \mu) dx, \tag{5.1.4}$$

while for $r = d, \ldots, \tilde{d} - 1$ and $k = 0, \ldots, \tilde{d} - 1$,

$$(\varphi_{[j,l-d+r]}, \tilde{\phi}^L_{j,\tilde{l}-\tilde{d}+k})_{[0,1]} = \sum_{\mu=-\tilde{l}_2+1}^{\tilde{l}-1} \alpha_{\mu,k} \int_0^{2^j} \varphi(x - (l - d + r)) \tilde{\varphi}(x - \mu) dx. \tag{5.1.5}$$

Thus, it remains to compute the expressions

$$I(\nu, \mu) := \int_0^{2^j} \varphi(x - \nu) \tilde{\varphi}(x - \mu) dx = \int_0^{\infty} \varphi(x - \nu) \tilde{\varphi}(x - \mu) dx,$$

for $-l_2 + 1 \leq \nu \leq l - d + \tilde{d} - 1$, $-\tilde{l}_2 + 1 \leq \mu \leq \tilde{l} - 1$, where we have used again that for $j \geq j_0$ the integrals do not depend on the upper limit 2^j in the given range of ν and μ. First note that by (3.2.8),

$$I(\nu, \mu) = \delta_{\nu,\mu}, \tag{5.1.6}$$

for

$$\nu = -l_1, \ldots, l - d + \tilde{d} - 1, \qquad \mu = -\tilde{l}_2 + 1, \ldots, \tilde{l} - 1,$$

and

$$\nu = -l_2 + 1, \ldots, l - 1, \qquad \mu = -\tilde{l}_1, \ldots, \tilde{l} - 1.$$

Consequently, it suffices to consider $I(\nu, \mu)$ for the remaining range $-l_2 + 1 \leq \nu < -l_1$, $-\tilde{l}_2 + 1 \leq \mu < -\tilde{l}_1$. To this end, note that in view of (3.2.3),

$$I(\nu, \mu) = \sum_{i=0}^{\nu+l_2-1} \int_{\mathbb{R}} \chi_{[i,i+1)}(x)\varphi(x - \nu)\tilde{\varphi}(x - \mu)dx$$

$$= \sum_{i=0}^{\nu+l_2-1} \int_{\mathbb{R}} \chi_{[0,1)}(x)\varphi(x - (\nu - i))\tilde{\varphi}(x - (\mu - i))dx$$

$$= \sum_{s=-l_2+1}^{\nu} z(s, s + \mu - \nu), \qquad (5.1.7)$$

where

$$z(s, t) := \int_{\mathbb{R}} \chi_{[0,1)}(x)\varphi(x - s)\tilde{\varphi}(x - t)dx.$$

The quantities $z(s, t)$, in turn, can be computed (up to round off) precisely as solution coefficients of an eigenvector/moment problem [25]. A documentation of the corresponding software is given in [3, 38].

By (3.3.20) it remains to invert the $(\tilde{d} \times \tilde{d})$-matrix Γ_L, e.g., with the aid of a QR factorization.

Ad (iii). Assembling the matrices $\mathbf{M}_L$, $\tilde{\mathbf{M}}'_L$, $\tilde{\mathbf{M}}_L$ requires only the information collected under (i) and (ii).

It is also clear from the construction that the final stable completions (4.2.1) leading to the biorthogonal wavelets (4.2.2) have to be computed only once for some fixed j (e.g., $j = j_0$). The corresponding quantities for arbitrary $j \geq j_0$ are then simply obtained by properly stretching the stationary interior blocks as indicated in (3.5.1), (3.5.12), (4.1.23), (4.1.24), and (4.2.1).

REMARK 5.1. *In some applications it might be important to start the multilevel decomposition with a very coarse initial level $j < j_0$. One should keep in mind that the concept of biorthogonality is primarily asymptotic in nature. For instance, it affects the validity of norm equivalences as well as moment conditions which become relevant when j becomes large. Thus, for finitely many low levels one could always resort to simple decompositions of the primal spline spaces only. For example,* hierarchical bases [51] *would provide simple splittings for levels $j < j_0$ in the case $d = 2$.*

REMARK 5.2. *One should keep in mind that in (4.2.2) the wavelets are written in terms of the generator bases on the interval. Their representation as linear combinations of scaled translates of the scaling functions φ, $\tilde{\varphi}$ from Section 3.2 restricted to the interval can be derived from this representation in a straightforward manner. Such explicit formulae have indeed been used for plotting the functions in Sections 5.4.*

5.2. Basis Transformations

A first implementation of the above results revealed the following problems concerning the quantitative properties of the above bases. Recall that, whenever a collection Θ is a Riesz basis of the closure of its span, one has

$$c\|\mathbf{c}\|_{l_2} \le \|\sum_{\theta \in \Theta} c_\theta \theta\|_{L_2} \le C\|\mathbf{c}\|_{l_2},$$

and the supremum of all c and the infimum of all C for which the above relation holds are called *Riesz constants* of Θ. The ratio of the upper and lower constant is called the *condition* of Θ.

We have noticed the following:

(a) Although Theorem 3.4 establishes that $\mathbf{\Gamma}_X$ is nonsingular, the example in Section 5.4 shows that in this case cond($\mathbf{\Gamma}_X$) $\gg 1$. Consequently, by (3.5.11) and (4.2.1) the entries of the mask matrices $\tilde{\mathbf{M}}_{j,0}$, $\mathbf{M}_{j,1}$ lose relative accuracy.

(b) The plots of the example in Section 5.4 indicate that the Riesz constants of Φ_j, $\tilde{\Phi}_j$, although uniformly bounded, are considerably large. A similar behavior has to be expected for the corresponding wavelet bases.

However, one should note that once a pair of biorthogonal generator bases Φ_j, $\tilde{\Phi}_j$ have been determined, corresponding modifications are conveniently facilitated with the aid of the mechanism from Section 2.4.

We propose the following strategy. First we wish to improve the condition of the single-scale generator bases Φ_j, $\tilde{\Phi}_j$. Since the monomial bases are increasingly ill conditioned, one expects that the collections $\Phi_j^X := \{\phi_{j,k}^X : k \in \Delta_j^X\}$, $X \in \{L, R\}$, and their dual analogs $\tilde{\Phi}_j^X$ of boundary functions from (3.2.35), (3.2.36), and (3.3.5) inherit this property. This suggests starting from well-conditioned polynomial bases $\mathcal{P} = \{p_i\}_{i=0}^{d-1}$, $\tilde{\mathcal{P}} = \{\tilde{p}_i\}_{i=0}^{\tilde{d}-1}$ of Π_d and $\Pi_{\tilde{d}}$, respectively, given by

$$p_i(x) = \sum_{r=0}^{d-1} z_{i,r} x^r, \qquad \tilde{p}_i(x) = \sum_{r=0}^{\tilde{d}-1} \tilde{z}_{i,r} x^r, \ x \in [0, 1]. \tag{5.2.1}$$

Defining $\mathbf{Z}_L = (z_{r,i})_{r,i=0,\ldots,d-1}$, $\mathbf{Z}_R = \mathbf{Z}_L^{\updownarrow}$, and correspondingly $\tilde{\mathbf{Z}}_L = (\tilde{z}_{r,i})_{r,i=0,\ldots,\tilde{d}-1}$, $\tilde{\mathbf{Z}}_R = \tilde{\mathbf{Z}}_L^{\updownarrow}$, we make the following ansatz for the new collections of boundary generators:

$$\Phi_j^{X,\text{new}} = \mathbf{Z}_X \Phi_j^X, \qquad \tilde{\Phi}_j^{X,\text{new}} = \tilde{\mathbf{Z}}_X \tilde{\Phi}_j^X, \ X \in \{L, R\}. \tag{5.2.2}$$

The new generators have yet to be biorthogonalized. It is easy to see that the matrix $\mathbf{\Gamma}_X$ from (3.3.5), (3.3.6) has to be replaced by $\mathbf{Z}_X \mathbf{\Gamma}_X \tilde{\mathbf{Z}}_X^T$. Hence, the matrix $\tilde{\mathbf{C}}_X$ which defines the biorthogonalized dual generators in (3.3.20) is now determined by the relation

$$\mathbf{Z}_X \mathbf{\Gamma}_X \tilde{\mathbf{Z}}_X^T \tilde{\mathbf{C}}_X^T = \mathbf{I}, \qquad X \in \{L, R\}. \tag{5.2.3}$$

For good choices of $\mathcal{P}$, $\tilde{\mathcal{P}}$ one expects this to alleviate problems (a) and (b).

Two possibilities for $\mathcal{P}$, $\tilde{\mathcal{P}}$ suggest themselves:

- Orthonormal polynomials have optimal L_2 condition numbers.
- Bernstein polynomials are known from computer-aided geometric design to be well conditioned relative to the supremum norm [32].

Note that the biorthogonalization as formulated above affects only the dual generators. Thus, employing Bernstein generators defined by using Z_X from (5.3.4) below preserves homogeneous boundary conditions of all but one boundary generator on the primal side, which is an advantage with regard to incorporating boundary conditions.

Alternatively, one can combine the change of basis directly with biorthogonalization according to our initial ansatz (3.3.1). Consider for Φ'_j, $\tilde{\Phi}'_j$ given in (3.2.35), (3.2.36) the new bases

$$\Phi_j^{\text{new}} = \mathbf{C}_j \Phi'_j, \qquad \tilde{\Phi}_j^{\text{new}} = \tilde{\mathbf{C}}_j \tilde{\Phi}'_j, \tag{5.2.4}$$

where, on account of the remarks preceding Theorem 3.4, the matrices $\mathbf{C}_j$, $\tilde{\mathbf{C}}_j$ must have the form

$$\mathbf{C}_j = \begin{pmatrix} \mathbf{C}_L & & \mathbf{0} \\ & \mathbf{I}^{(2^j - 2\tilde{l} + 1 - \mu(d))} & \\ \mathbf{0} & & \mathbf{C}_L^{\updownarrow} \end{pmatrix}, \qquad \tilde{\mathbf{C}}_j = \begin{pmatrix} \tilde{\mathbf{C}}_L & & \mathbf{0} \\ & \mathbf{I}^{(2^j - 2\tilde{l} + 1 - \mu(d))} & \\ \mathbf{0} & & \tilde{\mathbf{C}}_L^{\updownarrow} \end{pmatrix}. \tag{5.2.5}$$

Thus $\mathbf{C}_L$, $\tilde{\mathbf{C}}_L$ have to be chosen such that Φ_j^{new}, $\tilde{\Phi}_j^{\text{new}}$ are biorthogonal,

$$(\Phi_j^{\text{new}}, \tilde{\Phi}_j^{\text{new}})_{[0,1]} = \mathbf{I}, \tag{5.2.6}$$

which is equivalent to

$$\mathbf{C}_L \mathbf{\Gamma}_L \tilde{\mathbf{C}}_L^T = \mathbf{I}. \tag{5.2.7}$$

Again, one could base the choice of $\mathbf{C}_L$, $\tilde{\mathbf{C}}_L$ on transformations $\mathbf{Z}_L$, $\tilde{\mathbf{Z}}_L$ from (5.2.2). Instead, we mention here the following simple version which mainly addresses problem (a). Let

$$\mathbf{\Gamma}_L = \mathbf{U} \mathbf{\Sigma} \mathbf{V}^T \tag{5.2.8}$$

be the singular value decomposition of $\mathbf{\Gamma}_L$ given in (3.3.5), (3.3.6); that is, $\mathbf{\Sigma}$ is a diagonal and $\mathbf{U}$, $\mathbf{V}$ are orthogonal matrices. Defining

$$\mathbf{C}_L = \mathbf{\Sigma}^{-1/2} \mathbf{U}^T \in \mathbb{R}^{\tilde{d} \times \tilde{d}}, \qquad \mathbf{C}_R = \mathbf{C}_L^{\updownarrow},$$

$$\tilde{\mathbf{C}}_L = \mathbf{\Sigma}^{-1/2} \mathbf{V}^T \in \mathbb{R}^{\tilde{d} \times \tilde{d}}, \qquad \tilde{\mathbf{C}}_R = \tilde{\mathbf{C}}_L^{\updownarrow}, \tag{5.2.9}$$

implies (5.2.7), which, in turn, confirms (5.2.6). The mask matrices of the primal and dual generators (5.2.4) and corresponding wavelets for the transformations (5.2.5) are then constructed as follows:

1. compute $\mathbf{M}_{j,0}$ from (3.5.1) and the initial stable completion $\check{\mathbf{M}}_{j,1}$ and inverse $\check{\mathbf{G}}_j$ as in (4.1.23), (4.1.24);

2. apply the basis transformation $\mathbf{C}_j$ from (5.2.5) to $\check{\mathbf{M}}_j = (\mathbf{M}_{j,0}, \check{\mathbf{M}}_{j,1})$ and $\check{\mathbf{G}}_j$ described in Remark 2.8;

3. compute according to (2.4.3)

$$\tilde{\mathbf{M}}_{j,0} := \tilde{\mathbf{C}}_{j+1}^{-T} \tilde{\mathbf{M}}'_{j,0} \tilde{\mathbf{C}}_j^T \qquad (5.2.10)$$

with $\tilde{\mathbf{C}}_j$ given in (5.2.5) and $\tilde{\mathbf{M}}'_{j,0}$ from (3.5.5);

4. apply Theorem 4.8 to obtain the final biorthogonal system.

Recall that the situation in Section 3.5 corresponds to $\mathbf{C}_j = \mathbf{I}$ and $\tilde{\mathbf{C}}_j$ from (3.3.17) or (3.5.10). Obviously, there are many more possibilities of constructing $\mathbf{C}_L$, $\tilde{\mathbf{C}}_L$ satisfying (5.2.7) that take additional issues into account, such as preservation of boundary conditions or Riesz constants of the wavelets.

A detailed discussion of all these issues would go beyond the scope of the present paper and has therefore been deferred to [23]. Both of the options indicated above, namely the use of Bernstein basis polynomials and the singular value decomposition, have been thoroughly explored in [23]. The results may be roughly summarized as follows.

• The Riesz constants for the complement bases Ψ_j are independent of the choice of the single-scale generator bases.

• In almost all cases the Bernstein basis polynomials give rise to the best improvement of the condition of Φ_j, $\tilde{\Phi}_j$. This improvement grows significantly with increasing degree of polynomial exactness.

It remains to investigate the condition of the whole multiscale wavelet bases or, equivalently, of the corresponding multiscale transformations.

5.3. Condition Numbers of the Wavelet Transform

In Section 5.2 and [23] we have investigated how an appropriate change of basis of the boundary near functions affects the Riesz constants of Φ_j, $\tilde{\Phi}_j$. In addition, we have discussed in [23] attempts at balancing the numerical stability of the biorthogonalization process depending on the condition of Γ_X against the (possibly increasing) condition of the bases Φ_j, $\tilde{\Phi}_j$.

Of course, the issue of perhaps most practical importance is the condition of the corresponding multiscale bases or, equivalently, of the corresponding multiscale transformations. Denote by $J \geq j_0$ some fixed highest level of resolution. Then the change of coefficients from multiscale to single-scale representation

$$\mathbf{T}_J: \mathbf{d} \to \mathbf{c}, \qquad (5.3.1)$$

TABLE 5.3.1
Condition Numbers of $\mathbf{T}_J$, Standard Situation

		finest level J						
d	$\tilde{d}$	4	5	6	7	8	9	10
2	2	$5.4e+00$	$6.5e+00$	$7.3e+00$	$8.1e+00$	$8.7e+00$	$9.2e+00$	$9.5e+00$
2	4	–	$4.7e+00$	$6.2e+00$	$8.2e+00$	$1.0e+01$	$1.0e+01$	$1.1e+01$
2	6	–	–	$4.4e+00$	$6.2e+00$	$8.9e+00$	$1.1e+01$	$1.2e+01$
2	8	–	–	$4.4e+00$	$6.5e+00$	$9.7e+00$	$1.3e+01$	$1.4e+01$
2	10	–	–	–	$5.9e+00$	$8.2e+00$	$1.2e+01$	$1.6e+01$
3	3	–	$8.2e+02$	$1.4e+03$	$1.4e+03$	$1.5e+03$	$1.6e+03$	$1.7e+03$
4	4	–	–	$3.1e+05$	$5.3e+05$	$8.8e+05$	$8.8e+05$	$9.6e+05$
5	5	–	–	$4.1e+07$	$2.6e+08$	$3.2e+08$	$3.4e+08$	$3.4e+08$
6	6	–	–	$4.6e+09$	$2.8e+10$	$2.7e+10$	$5.6e+10$	$5.7e+10$

commonly referred to as the *fast wavelet transform,* has the representation

$$\mathbf{T}_J = \mathbf{T}_{J,J-1} \cdots \mathbf{T}_{J,j_0}, \tag{5.3.2}$$

where

$$\mathbf{T}_{J,j} := \begin{pmatrix} \mathbf{M}_j & \mathbf{0} \\ \mathbf{0} & \mathbf{I}^{(\#\Delta_J - \#\Delta_j)} \end{pmatrix} \in \mathbb{R}^{(\#\Delta_J) \times (\#\Delta_J)}. \tag{5.3.3}$$

Schematically $\mathbf{T}_J$ can be visualized as a pyramid scheme. Correspondingly, the inverse transform $\mathbf{T}_J^{-1}$ can be written also in product structure (5.3.2) involving the matrices $\mathbf{G}_j$.

Although the above theoretical development guarantees that Ψ, $\tilde{\Psi}$ are Riesz bases, the fact that one dispenses with orthonormality opens the risk that, in principle, the condition of the $\Psi^J := \Phi_{j_0} \cup \Psi_{j_0} \cup \cdots \cup \Psi_{J-1}$, $\tilde{\Psi}^J := \tilde{\Phi}_{j_0} \cup \tilde{\Psi}_{j_0} \cup \cdots \cup \tilde{\Psi}_{J-1}$ may become prohibitively large. In fact, corresponding investigations for bases defined on all of $\mathbb{R}$ [35] indicate that the condition number of the wavelet transform grows with increasing degrees of exactness d, $\tilde{d}$. Comparing our results reported below with recent calculations [36], again concerning biorthogonal wavelets defined on all of $\mathbb{R}$, reveals that the difference between the condition numbers is not as high as expected. For a wide range of d, $\tilde{d}$, one observes factors between 1 and 10. For instance, for d, $\tilde{d} = 6$, the factor is only 1.5. Since the generator basis is always part of the whole wavelet basis, its condition must be expected to affect also the condition of the multiscale basis and, hence, also of $\mathbf{T}_J$. In fact, one can observe that changing the basis while keeping the multiresolution sequence fixed, leaves the wavelet basis unchanged [23]. In principle, one could consider also changing the wavelet bases Ψ_j within W_j. Here we have not used this option but have confined our stabilization to the generator bases since corresponding wavelet bases Ψ_j were observed in [23] to have an already moderate condition.

TABLE 5.3.2

Condition Numbers of T_J, Transformation with SVD

		finest level J						
d	$\tilde{d}$	4	5	6	7	8	9	10
2	2	$3.6e+00$	$4.2e+00$	$5.1e+00$	$5.7e+00$	$6.2e+00$	$6.6e+00$	$6.8e+00$
2	4	—	$4.6e+01$	$4.2e+01$	$3.8e+01$	$3.8e+01$	$3.5e+01$	$3.2e+01$
2	6	—	—	$5.1e+03$	$6.3e+03$	$3.5e+03$	$3.4e+03$	$3.4e+03$
2	8	—	—	$2.0e+06$	$1.9e+06$	$1.7e+06$	$8.7e+05$	$9.3e+05$
2	10	—	—	—	$1.5e+09$	$1.4e+09$	$8.9e+08$	$4.7e+08$
3	3	—	$1.6e+02$	$2.0e+02$	$2.4e+02$	$2.7e+02$	$2.9e+02$	$3.0e+02$
3	11	—	—	—	$1.3e+10$	$5.4e+09$	$5.6e+09$	$4.6e+09$
4	4	—	—	$7.4e+01$	$1.1e+02$	$1.5e+02$	$2.0e+02$	$2.0e+02$
5	5	—	—	$4.2e+02$	$7.0e+02$	$1.1e+03$	$1.8e+03$	$2.7e+03$
6	6	—	—	$1.4e+03$	$2.8e+03$	$5.6e+03$	$1.3e+04$	$2.7e+04$
7	7	—	—	—	$9.4e+03$	$2.4e+04$	$6.0e+04$	$1.8e+05$

We have tested the following variants:

(i) the standard situation $\mathbf{C}_j = \mathbf{I}$, $\tilde{\mathbf{C}}_j = \Gamma_j^{-T}$ from (3.5.10) (Table 5.3.1);

(ii) biorthogonalization via singular value decomposition as in (5.2.9) (Table 5.3.2);

(iii) definition of Φ_j, $\tilde{\Phi}_j$ with the aid of Bernstein basis polynomials

$$P_r(x) := b^{-d+1} \binom{d-1}{r} x^r (b-x)^{d-1-r}, \qquad r = 0, \ldots, d-1,$$

which corresponds to (5.2.2) with a matrix $\mathbf{Z}_L$ given by

$$(\mathbf{Z}_L)_{l,r} = \begin{cases} (-1)^{r-l} \binom{d-1}{r} \binom{r}{l} b^{-r}, & r \geq l, \\ 0 & \text{otherwise,} \end{cases} \quad , l, r = 0, \ldots, d-1, \quad (5.3.4)$$

and correspondingly for the dual side, see [23] for more details (Table 5.3.3). Note that the definition of Bernstein basis polynomials refers to a specific interval $[0, b]$ for some $b > 0$ on which they form a partition of unity. We have tested in [23] several choices of b which turned out to have a significant effect on the resulting Riesz constants. The star in the column for b in Table 5.3.3 means that this value has *not* been optimized, which leaves further chances for improvement.

Tables 5.3.1–5.3.3 show the spectral condition number of $\mathbf{T}_J$ for different values of d, $\tilde{d}$ and different levels J. As expected, the condition numbers of the multiscale transformations exhibit a strong growth in d and $\tilde{d}$. They also appear to increase when d is fixed and only $\tilde{d}$ grows. The stabilization of the generator basis is seen to result in a significant

TABLE 5.3.3

Condition Numbers of T_J, Transformation to Bernstein Basis Form

			finest level J						
d	$\tilde{d}$	b	4	5	6	7	8	9	10
2	2	0.7	$5.2e+00$	$5.7e+00$	$6.3e+00$	$7.0e+00$	$7.6e+00$	$8.0e+00$	$8.4e+00$
2	4	0.7	—	$5.1e+00$	$5.3e+00$	$6.0e+00$	$6.9e+00$	$7.5e+00$	$7.8e+00$
2	6	0.8	—	—	$3.9e+00$	$4.7e+00$	$5.0e+00$	$6.2e+00$	$7.1e+00$
2	8	0.8	—	—	$4.4e+00$	$5.7e+00$	$5.9e+00$	$7.5e+00$	$8.7e+00$
2	10	0.8*	—	—	—	$6.9e+00$	$9.0e+00$	$9.5e+00$	$1.2e+01$
3	3	3.6	—	$8.4e+00$	$2.1e+01$	$2.6e+01$	$2.9e+01$	$3.1e+01$	$3.3e+01$
3	5	3.7	—	—	$7.3e+00$	$1.8e+01$	$2.2e+01$	$2.6e+01$	$2.8e+01$
3	7	3.5	—	—	$1.2e+01$	$3.0e+01$	$4.2e+01$	$4.9e+01$	$5.6e+01$
3	9	3.5*	—	—	—	$2.2e+01$	$6.1e+01$	$9.5e+01$	$1.2e+02$
3	11	3.5*	—	—	—	$2.0e+01$	$5.5e+01$	$8.7e+01$	$1.0e+02$
4	4	3.5	—	—	$3.3e+01$	$1.2e+02$	$1.7e+02$	$2.2e+02$	$2.8e+02$
4	6	3.6	—	—	$2.6e+01$	$7.8e+01$	$1.1e+02$	$1.2e+02$	$1.3e+02$
4	8	4.1	—	—	—	$1.6e+01$	$7.5e+01$	$1.2e+02$	$1.4e+02$
4	10	4.4	—	—	—	$1.5e+01$	$9.2e+01$	$1.8e+02$	$2.2e+02$
4	12	4.5*	—	—	—	$1.4e+01$	$1.0e+02$	$2.6e+02$	$4.7e+02$
5	5	5.7	—	—	$1.4e+02$	$9.1e+02$	$1.4e+03$	$2.2e+03$	$3.5e+03$
5	7	5.9	—	—	—	$8.4e+01$	$4.5e+02$	$5.9e+02$	$7.1e+02$
5	9	6.0*	—	—	—	$8.4e+01$	$4.9e+02$	$7.7e+02$	$1.0e+03$
5	11	6.0*	—	—	—	$9.1e+01$	$1.1e+03$	$1.3e+04$	$1.5e+05$
6	6	6.0*	—	—	$3.4e+02$	$4.1e+03$	$8.4e+03$	$1.8e+04$	$3.8e+04$
6	8	6.0*	—	—	—	$2.8e+02$	$2.1e+03$	$3.4e+03$	$4.7e+03$
6	10	6.0*	—	—	—	$2.8e+02$	$2.4e+03$	$4.2e+03$	$8.6e+03$
6	12	6.0*	—	—	—	$2.7e+04$	$9.4e+07$	$3.3e+11$	-1
7	7	6.0*	—	—	—	$1.5e+05$	$1.9e+06$	$2.3e+06$	$2.8e+06$
7	9	6.0*	—	—	—	$1.5e+05$	$2.1e+06$	$3.4e+06$	$3.5e+06$
7	11	6.0*	—	—	—	$1.6e+05$	$2.5e+07$	$4.2e+09$	$7.0e+11$
8	8	6.0*	—	—	—	$2.2e+06$	$1.2e+07$	$4.1e+07$	$6.9e+07$
8	10	6.0*	—	—	—	$2.2e+06$	$4.7e+07$	$1.8e+08$	$7.2e+08$
8	12	6.0*	—	—	—	$2.7e+06$	$6.5e+09$	$1.2e+13$	-1
9	9	6.0*	—	—	—	$6.6e+11$	-1	-1	-1
10	10	6.0*	—	—	—	$1.0e+13$	-1	-1	-1

Note. The star in the column for b means that this value has not been optimized.

improvement of the condition numbers of T_J, for instance for $d = \tilde{d} = 5$ by five orders of magnitude; see Tables 5.3.1 and 5.3.3. However, in principle, a growth of condition numbers of T_J cannot be avoided since the condition of the B-spline basis is known to grow rapidly with d.

We have confined the calculations to $J \leq 10$. In order to obtain safe information, we

have employed the singular value decomposition for determining the condition number which requires to assemble $\mathbf{T}_J$ explicitly. (Of course, in any application one would *not* assemble $\mathbf{T}_J$ but would successively apply the factors $\mathbf{T}_{J,j}$.) For $J > 10$ one would run into storage problems. (All computations were made in double precision on a Silicon Graphics Indy Workstation with 64 MB memory.) One could have used power iterations without assembling the transformations $\mathbf{T}_J$, $\mathbf{T}_J^{-1}$ to cover larger values of J.

The sign -1 in the tables indicates that the condition number could no longer be numerically determined.

Since the condition numbers are by far best for the variant (iii), we display an extensive record for this case and only a few sample values for (i), (ii) to illustrate the difference.

5.4. Example $d = 3$, $\tilde{d} = 5$

The example of biorthogonal bases for $d = 3$, $\tilde{d} = 5$ is computed with respect to

$$j = j_0 = 5. \tag{5.4.1}$$

First we list a table of the parameters j, d, $\tilde{d}$, l_1, l_2, $\tilde{l}_1$, $\tilde{l}_2$, $\mu(d)$, l, $\tilde{l}$, $\#\Delta_j$, $\#\Delta_{j+1}$ and the mask coefficients $\mathbf{a}$, $\tilde{\mathbf{a}}$ [15] which are scaled here such that they sum up to 2. The *first realization* follows exactly the derivation in the paper and computes the filter coefficients as derived in Sections 3 and 4. This corresponds to the case $\mathbf{C}_j = \mathbf{I}$, $\tilde{\mathbf{C}}_j$ from (3.3.19). The coefficients for *any* stabilized version can be obtained from those by the techniques from Section 2.4.

We display the data and plots in the following order:

1. the nonzero pattern of the refinement matrices $\mathbf{M}_{j,0}$, $\tilde{\mathbf{M}}_{j,0}$, $\mathbf{M}_{j,1}$, and $\check{\mathbf{G}}_{j,1}^T$ from (3.5.1), (3.5.12), (4.2.1), and (4.2.2), respectively. In each of the figures nz denotes the number of non zero entries of the particular matrix;

2. the stationary matrices $\mathbf{M}_L$, $\tilde{\mathbf{M}}_L$ from (3.5.2), (3.5.13), respectively, and the corresponding ones for the wavelets denoted by $\mathbf{W}_L$, $\tilde{\mathbf{W}}_L$. Here $(\mathbf{C})_{\text{col } i-k}$ always means that columns i through k of $\mathbf{C}$ are displayed. Due to the lack of symmetry the matrices $\mathbf{W}_R$, $\tilde{\mathbf{W}}_R$ are also displayed, see Remark 4.7;

3. the nonzero entries of one column of the interior parts $\mathbf{A}_j$, $\tilde{\mathbf{A}}_j$ given in (3.5.4) and (3.5.8) which will be denoted by $[\mathbf{A}_j]$, $[\tilde{\mathbf{A}}_j]$, and the nonzero entries of one corresponding column from the interior of the refinement matrices of the wavelets denoted by $[\mathbf{W}_j]$, $[\tilde{\mathbf{W}}_j]$;

4. plots of the primal generators $\phi_{j,k}^L$ for $k \in \Delta_j^L$ and $k = l$ (as representative for the interior ones) defined in (3.2.35). Recall that the functions at the right end of the interval follow by symmetry (3.2.29);

5. plots of the dual generators $\tilde{\phi}_{j,k}^L$ for $k \in \tilde{\Delta}_j^L$, before and after biorthogonalization given in (3.2.36) and (3.3.19), and $\tilde{\phi}_{j,k}$ for $k = \tilde{l}$ as representative for the interior ones defined in (3.2.18),

6. plots of the primal and dual wavelets $\psi_{j,k}$, $\tilde{\psi}_{j,k}$ defined in (4.2.2). We plot all the boundary-adapted functions at the left boundary plus the first interior one and all the boundary near right functions. Recall that for d and $\tilde{d}$ odd, the primal and dual wavelets

at the left and right boundary ends cannot be reproduced by symmetry arguments due to the form of the core matrix $\hat{\mathbf{F}}_j$ in (4.1.14), see Remark 4.7.

The *second realization* concerns the stabilized generator basis for the above example. Here we have used (5.2.4) based on the Bernstein basis transformation (5.3.4). We only display the nonzero pattern of the refinement matrices according to 1 as well as plots of the primal and dual generators $\phi_{j,k}$, $\tilde{\phi}_{j,k}$ for $k \in \tilde{\Delta}_j^L$ from (5.2.4). The corresponding new refinement matrices can be generated with the aid of the techniques from Section 2.4 from the previous listings. We also dispense with displaying the wavelets since it has been shown in [23] that the primal and dual wavelets $\psi_{j,k}$, $\tilde{\psi}_{j,k}$ remain the same. Note that the stabilized dual generators have significantly smaller L_∞-norm; compare Figs. 5.4.4 and 5.4.9.

The primal and dual wavelets corresponding to the interior stationary part can be shown to coincide with those constructed in [15] although they have been generated in a completely different way.

Before finally listing the data and plots, we conclude by mentioning that all computations have been made in **C++** with double precision and plots generated by using TECPLOT. Most recent versions of various mask matrices can be found on our homepages.

$j = 5$			
$d = 3$	$\ell_1 = -1$	$\ell_2 = 2$	$\ell = 4$
$\tilde{d} = 5$	$\tilde{\ell}_1 = -5$	$\tilde{\ell}_2 = 6$	$\tilde{\ell} = 6$
$\mu(d) = 1$	$\#\Delta_5 = 30$	$\#\Delta_6 = 62$	

$a_k =$

```
a[-1]= 2.500000000000e-01
a[ 0]= 7.500000000000e-01
a[ 1]= 7.500000000000e-01
a[ 2]= 2.500000000000e-01
-------------------------
```

$\tilde{a}_k =$

```
at[-5]=-1.953125000000e-02
at[-4]= 5.859375000000e-02
at[-3]= 7.421875000000e-02
at[-2]=-3.789062500000e-01
at[-1]=-1.015625000000e-01
at[ 0]= 1.367187500000e+00
at[ 1]= 1.367187500000e+00
at[ 2]=-1.015625000000e-01
at[ 3]=-3.789062500000e-01
at[ 4]= 7.421875000000e-02
at[ 5]= 5.859375000000e-02
at[ 6]=-1.953125000000e-02
-------------------------------
```

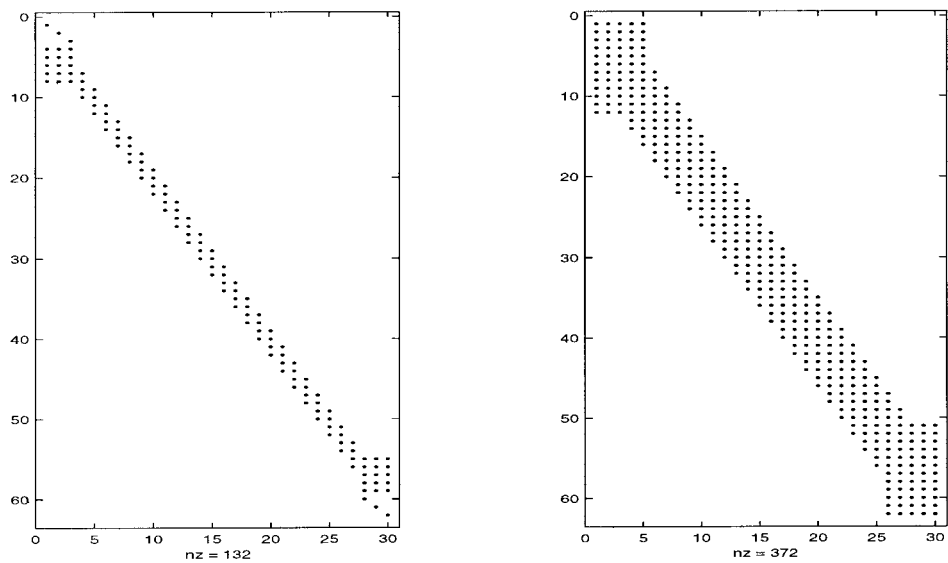*First Realization: The Case* $\mathbf{C}_j = \mathbf{I}$, $\tilde{\mathbf{C}}_j$ *from (3.5.10)*

FIG. 5.4.1. Nonzero pattern of refinement matrices $\mathbf{M}_{5,0}$, $\tilde{\mathbf{M}}_{5,0}$.
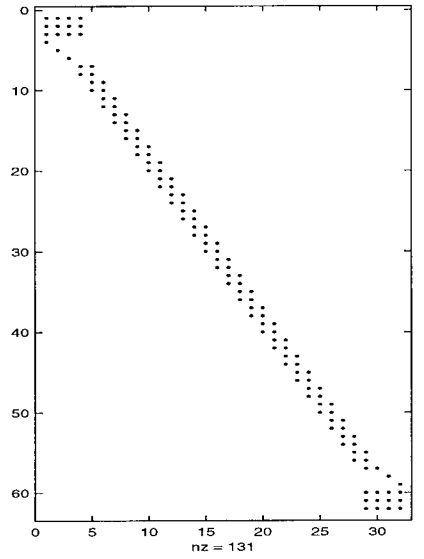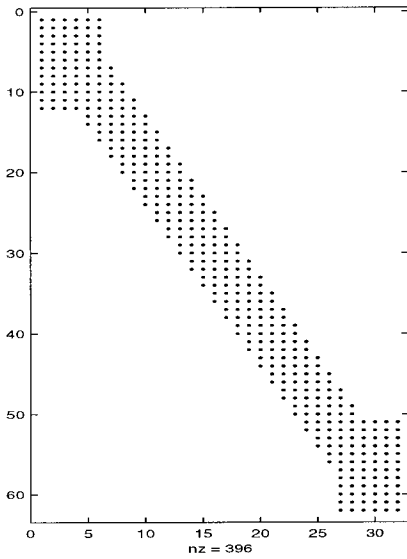
FIG. 5.4.2. Nonzero pattern of refinement matrices $\mathbf{M}_{5,1}$, $\check{\mathbf{G}}_{5,1}^T$.

$\mathbf{M}_L =$

7.071067811865475e-01	0	0
0	3.535533905932737e-01	0
0	0	1.767766952966369e-01
7.071067811865475e-01	1.590990257669732e+00	3.535533905932737e+00
7.071067811865475e-01	1.944543648263005e+00	5.303300858899106e+00
7.071067811865475e-01	2.298097038856280e+00	7.424621202458749e+00
5.303300858899106e-01	1.856155300614687e+00	6.363961030678928e+00
1.767766952966369e-01	6.187184335382290e-01	2.121320343559642e+00

$[\mathbf{A}_j] =$ $[\tilde{\mathbf{A}}_j] =$

1.767766952966369e-01	-1.381067932004976e-02
5.303300858899106e-01	4.143203796014926e-02
5.303300858899106e-01	5.248058141618906e-02
1.767766952966369e-01	-2.679271788089653e-01
---------------------	-7.181553246425873e-02
	9.667475524034830e-01
	9.667475524034830e-01
	-7.181553246425873e-02
	-2.679271788089653e-01
	5.248058141618906e-02
	4.143203796014926e-02
	-1.381067932004976e-02

$(\tilde{\mathbf{M}}_L)_{\mathrm{col}_{1-3}} =$

1.464195068483706e+00	7.279261453870778e-02	-1.914543858304005e-01
-1.066710564626374e+00	4.253770027747176e+00	-1.408332951705722e+00
-2.730568596993177e+00	1.367065594278123e+01	-4.189177028711525e+00
-1.744091502703539e-01	4.064623849190092e-01	-8.080312618538944e-02
-3.761765986223176e-02	1.487795358588295e-02	4.834320402381143e-02
7.181553246426509e-02	-2.453565828916364e-01	1.226815170693305e-01
1.244276441597032e-01	-3.403590245816694e-01	1.377640434863369e-01
-1.236384624842302e-02	2.597159313765296e-02	-8.360966768574229e-03
-3.143573673802357e-02	9.422641304253075e-02	-4.025025555082884e-02
6.971104799632144e-03	-2.023499395880890e-02	8.069623471882894e-03
3.945908377147589e-03	-1.257053668723529e-02	6.015519425702109e-03
-1.315302792386230e-03	4.190178895743024e-03	-2.005173141900407e-03

$(\tilde{\mathbf{M}}_L)_{\mathrm{col}_{4-5}} =$

2.869602962004055e-01	-5.643454266760350e-02
1.003627628636403e+00	-1.924798000629059e-01
2.134154835569323e+00	-3.730091347550064e-01
-2.105969775732293e-01	4.744753501675202e-02
-2.776589281996133e-01	5.182829860282134e-02

$$
\begin{array}{rr}
-1.242272170675189\text{e-}01 & 1.240142632820795\text{e-}02 \\
1.200677583957113\text{e-}01 & 1.400394830178797\text{e-}02 \\
9.418880313727929\text{e-}01 & -2.629827140261618\text{e-}01 \\
9.243494677892521\text{e-}01 & -6.327143289657755\text{e-}02 \\
-6.192600638942736\text{e-}02 & 9.647584924664194\text{e-}01 \\
-2.631536139127917\text{e-}01 & 9.657812074947856\text{e-}01 \\
5.088939311746454\text{e-}02 & -7.149341749469484\text{e-}02 \\
4.143203796014205\text{e-}02 & -2.679271788089644\text{e-}01 \\
-1.381067932004987\text{e-}02 & 5.248058141618883\text{e-}02 \\
0 & 4.143203796014937\text{e-}02 \\
0 & -1.381067932004970\text{e-}02
\end{array}
$$

$(\mathbf{W}_L)_{\text{col}_{1-3}} =$

$$
\begin{array}{rrr}
1.233258928572\text{e-}01 & 2.659970238083\text{e-}02 & -5.078125000008\text{e-}02 \\
-1.437061543368\text{e-}01 & -5.260150935352\text{e-}03 & 8.674665178574\text{e-}02 \\
1.428410961670\text{e-}02 & -8.545951847383\text{e-}03 & -2.168723316150\text{e-}02 \\
7.623303906756\text{e-}01 & -1.679900137759\text{e-}01 & -9.416598019417\text{e-}02 \\
-2.385346674943\text{e-}01 & 7.412903168149\text{e-}01 & -2.242916600234\text{e-}01 \\
-2.108315064307\text{e-}01 & -3.665212562890\text{e-}01 & 6.022081938244\text{e-}01 \\
-1.105063066893\text{e-}01 & -2.662366543840\text{e-}01 & -3.414459325397\text{e-}01 \\
6.244093172983\text{e-}02 & 4.214412252999\text{e-}02 & -5.525403911564\text{e-}02 \\
1.032982947643\text{e-}01 & 1.380888478904\text{e-}01 & 6.368914753402\text{e-}02 \\
1.206578241415\text{e-}02 & 2.159752169717\text{e-}02 & 1.538362740930\text{e-}02 \\
-2.516285532070\text{e-}02 & -2.748610604957\text{e-}02 & -6.576849489798\text{e-}03 \\
-8.387618440233\text{e-}03 & -9.162035349857\text{e-}03 & -2.192283163266\text{e-}03 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{array}
$$

$(\mathbf{W}_L)_{\text{col}_{4-6}} =$

$$
\begin{array}{rrr}
-9.570312500010\text{e-}02 & -2.058531746037\text{e-}02 & 2.480158730291\text{e-}03 \\
1.319335937500\text{e-}01 & 3.092935090702\text{e-}02 & -3.950538548930\text{e-}03 \\
-2.684339735243\text{e-}02 & -6.639800037903\text{e-}03 & 9.452476840872\text{e-}04 \\
-3.886990017364\text{e-}02 & -1.419923913682\text{e-}02 & 3.607688941848\text{e-}03 \\
-1.753702799479\text{e-}01 & -4.966788860883\text{e-}02 & 9.109627233790\text{e-}03 \\
-3.655574544271\text{e-}01 & -9.841613815664\text{e-}02 & 1.650206089391\text{e-}02 \\
6.931944444444\text{e-}01 & 6.766172209626\text{e-}02 & -2.837203168293\text{e-}02 \\
8.854166666697\text{e-}04 & 4.485656921499\text{e-}01 & -1.255126504967\text{e-}01 \\
-2.443033854167\text{e-}01 & -4.750727175791\text{e-}01 & 3.595260028609\text{e-}02 \\
-4.237196180556\text{e-}02 & -3.658684042400\text{e-}02 & 4.560237206655\text{e-}01 \\
4.394531250000\text{e-}02 & 1.304816569485\text{e-}01 & -4.561847060253\text{e-}01 \\
1.464843750000\text{e-}02 & 2.613277453839\text{e-}02 & -3.400601311953\text{e-}02 \\
0 & -1.953125000000\text{e-}02 & 1.263020833333\text{e-}01 \\
0 & -6.510416666667\text{e-}03 & 2.473958333333\text{e-}02 \\
0 & 0 & -1.953125000000\text{e-}02 \\
0 & 0 & -6.510416666667\text{e-}03
\end{array}
$$

$[\mathbf{W}_j] =$ $\qquad\qquad\qquad [\tilde{\mathbf{W}}_j] =$

```
    6.510416666666617e-03          -3.750000000000000e-01
    1.953124999999999e-02           1.125000000000000e+00
   -2.473958333333332e-02          -1.125000000000000e+00
   -1.263020833333333e-01           3.750000000000000e-01
    3.385416666666671e-02          -----------------------
    4.557291666666668e-01
   -4.557291666666665e-01
   -3.385416666666664e-02
    1.263020833333333e-01
    2.473958333333333e-02
   -1.953125000000000e-02
   -6.510416666666665e-03
-------------------------
```

$(\mathbf{W}_R)_{\mathrm{col}_{27-29}} =$

```
    6.510416666666665e-03                      0                              0
    1.953125000000001e-02                      0                              0
   -2.473958333333332e-02          6.510416666666665e-03                      0
   -1.263020833333333e-01          1.953125000000001e-02                      0
    3.400601311953458e-02         -2.613277453838641e-02          1.302083333333344e-02
    4.561847060252704e-01         -1.304816569484926e-01          3.906250000000031e-02
   -4.560237206655115e-01          3.658684042399879e-02         -3.766396604938099e-02
   -3.595260028614439e-02          4.750727175790878e-01         -2.171585648148105e-01
    1.255126504967086e-01         -4.485656921499052e-01          7.870370370373082e-04
    2.837203168304741e-02         -6.766172209631334e-02          6.161728395061624e-01
   -1.650206089379612e-02          9.841613815657325e-02         -3.249399594907542e-01
   -9.109627233822000e-03          4.966788860875476e-02         -1.558846932870466e-01
   -3.607688941969410e-03          1.419923913674576e-02         -3.455102237655011e-02
   -9.452476840607657e-04          6.639800037904760e-03         -2.386079764660564e-02
    3.950538548755067e-03         -3.092935090703859e-02          1.172743055555599e-01
   -2.480158730151896e-03          2.058531746032423e-02         -8.506944444445691e-02
----------------------------------------------------------------------------
```

$(\mathbf{W}_R)_{\mathrm{col}_{30-32}} =$

```
                   0                  0                  0
                   0                  0                  0
                   0                  0                  0
                   0                  0                  0
    1.245615433673e-02  -9.162035349857e-03  -8.387618440233e-03
    3.736846301020e-02  -2.748610604957e-02  -2.516285532070e-02
   -2.698833439626e-02   2.159752169717e-02   1.206578241415e-02
   -1.806142378826e-01   1.380888478904e-01   1.032982947643e-01
   -5.436862244898e-02   4.214412252999e-02   6.244093172983e-02
    3.517485119047e-01  -2.662366543840e-01  -1.105063066893e-01
    2.366507393973e-01  -3.665212562890e-01  -2.108315064307e-01
```

```
-3.996619399713e-01  7.412903168149e-01 -2.385346674943e-01
-1.330358803678e-01 -1.679900137759e-01  7.623303906756e-01
-4.853063051393e-02 -8.545951847382e-03  1.428410961670e-02
 2.186802455358e-01 -5.260150935353e-03 -1.437061543368e-01
-1.464843750002e-01  2.659970238083e-02  1.233258928572e-01
```

$(\tilde{\mathbf{W}}_L)_{\mathrm{col}_{1-3}} =$

```
   -1.000000000000000e+00    -1.000000000000000e+00    -1.000000000000000e+00
   -4.500000000000000e+00    -5.500000000000000e+00    -6.500000000000000e+00
   -2.000000000000000e+01    -3.000000000000000e+01    -4.200000000000000e+01
    1.000000000000000e+00                         0                         0
                        0     1.000000000000000e+00                         0
                        0                         0     1.000000000000000e+00
```

$(\tilde{\mathbf{W}}_L)_{\mathrm{col}_4} =$

```
   -6.666666666666665e-01
   -4.666666666666667e+00
   -3.200000000000000e+01
                        0
                        0
                        0
    1.000000000000000e+00
   -3.333333333333333e-01
                        0
                        0
```

$(\tilde{\mathbf{W}}_R)_{\mathrm{col}_{29}} =$

```
   -3.750000000000000e-01
    1.125000000000000e+00
   -1.125000000000000e+00
                        0
                        0
    1.125000000000000e+01
    2.062500000000000e+00
    3.750000000000000e-01
```

$(\tilde{\mathbf{W}}_R)_{\mathrm{col}_{30-32}} =$

```
                        0                         0                         0
                        0                         0                         0
    1.000000000000000e+00                         0                         0
                        0     1.000000000000000e+00                         0
                        0                         0     1.000000000000000e+00
   -4.200000000000000e+01    -3.000000000000000e+01    -2.000000000000000e+01
   -6.500000000000000e+00    -5.500000000000000e+00    -4.500000000000000e+00
   -1.000000000000000e+00    -1.000000000000000e+00    -1.000000000000000e+00
```
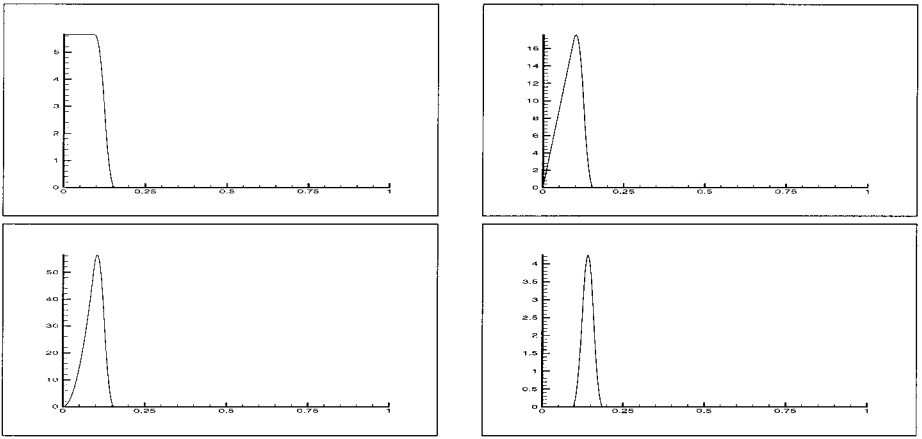--

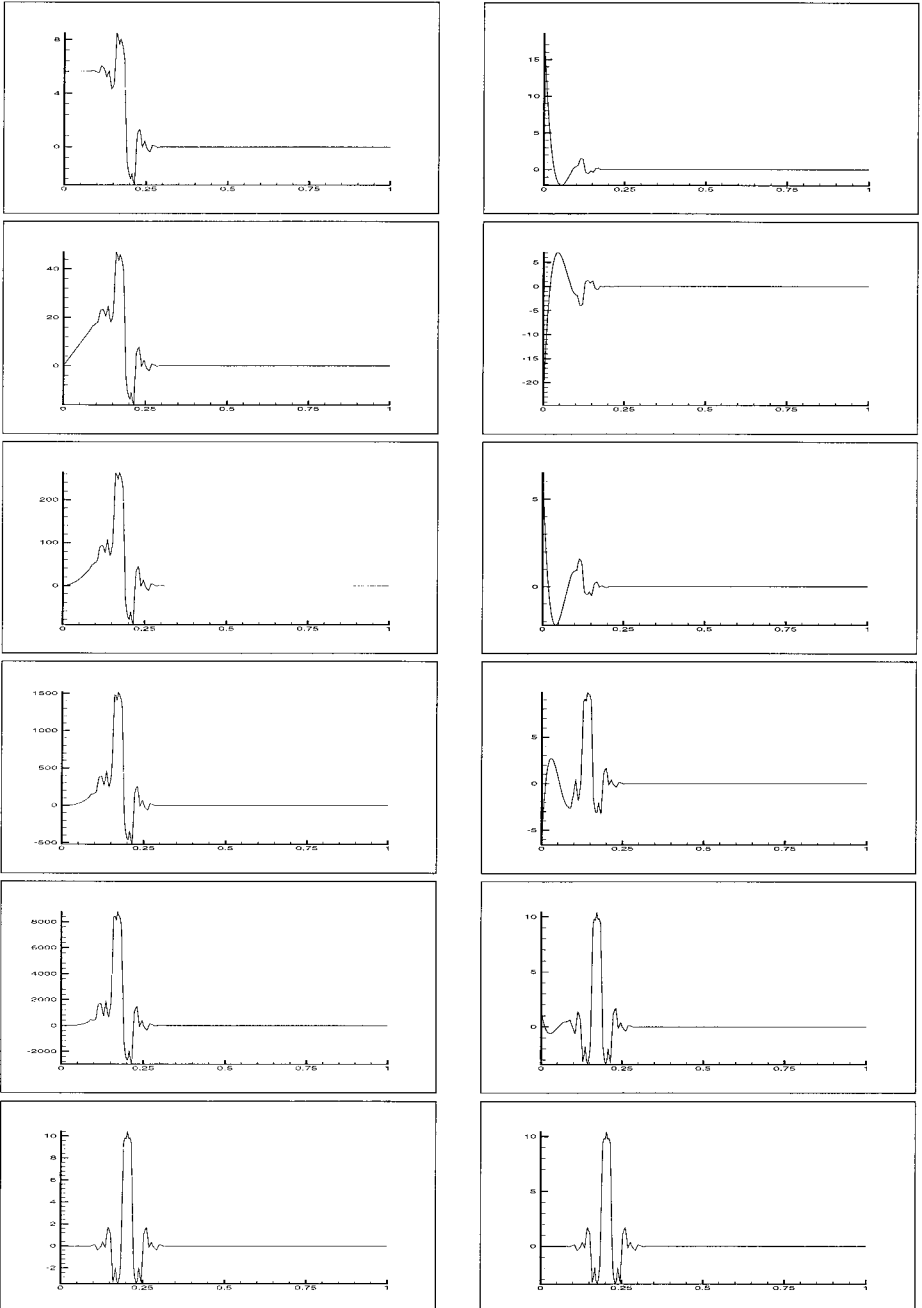**FIG. 5.4.3.** Primal generators $\phi_{5,k}$.

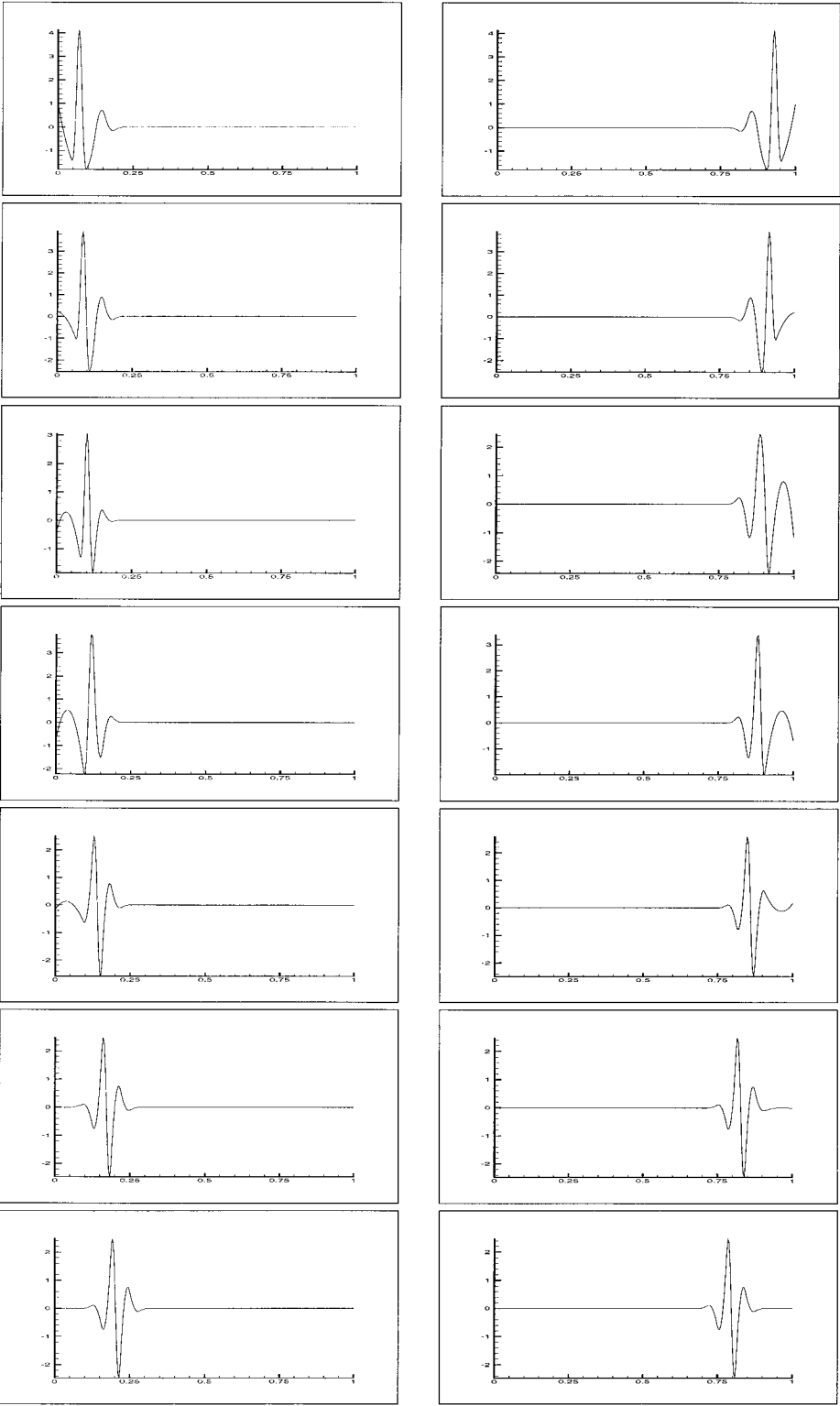**FIG. 5.4.4.** Dual generators $\tilde{\phi}_{5,k}$ before and after biorthogonalization.

FIG. 5.4.5. Primal wavelets $\psi_{5,k}$ at left and right boundaries.
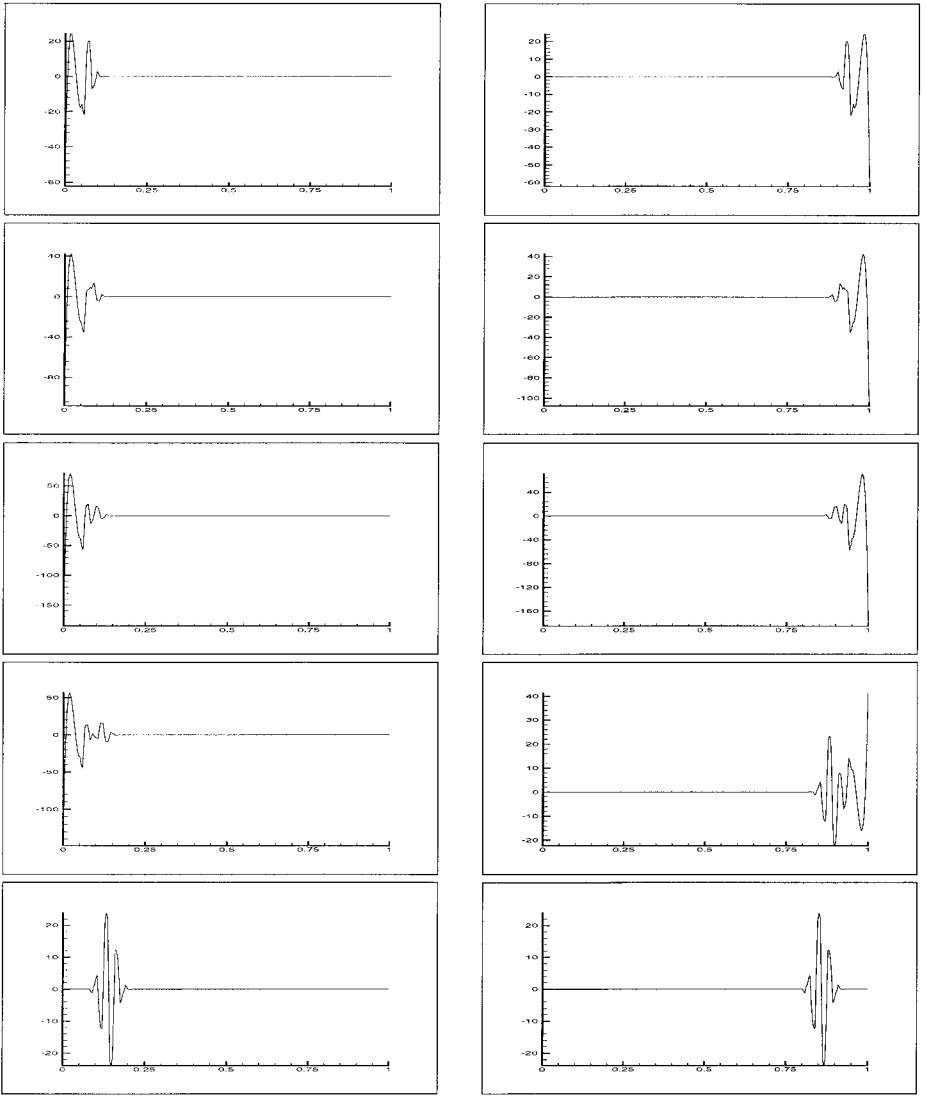
FIG. 5.4.6. Dual wavelets $\tilde{\psi}_{5,k}$ at left and right boundaries.

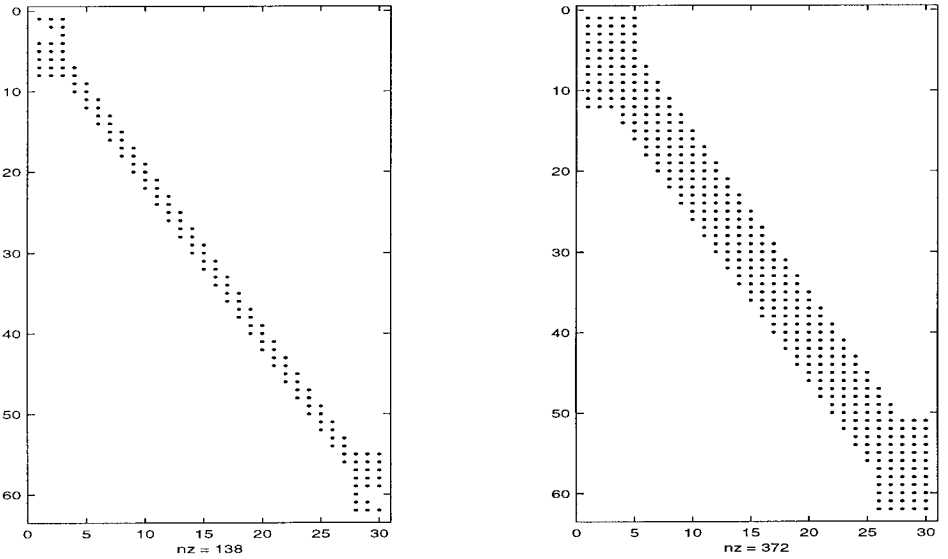*Second Realization: Bernstein Stabilization from (5.3.4)*

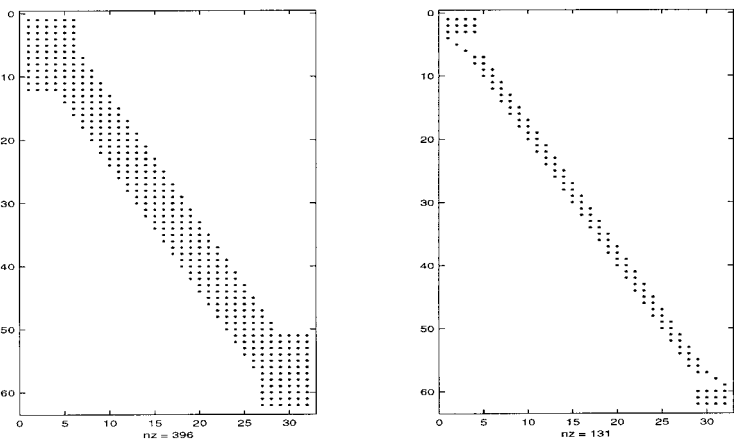**FIG. 5.4.7.** Nonzero pattern of refinement matrices $\mathbf{M}_{5,0}$, $\tilde{\mathbf{M}}_{5,0}$.

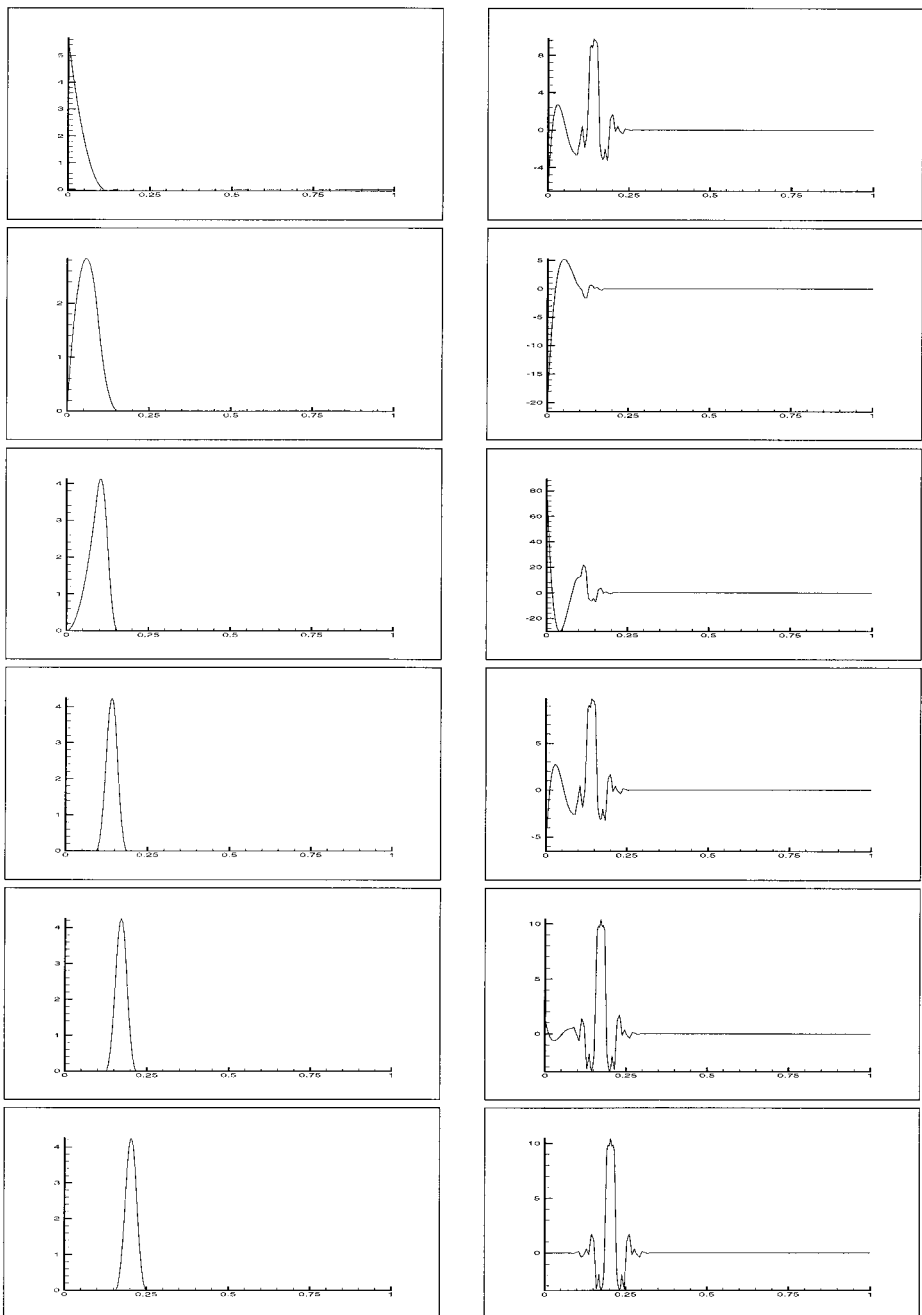**FIG. 5.4.8.** Nonzero pattern of refinement matrices $\mathbf{M}_{5,1}$, $\check{\mathbf{G}}_{5,1}^{T}$.

FIG. 5.4.9. Primal and dual generators $\phi_{5,k}$ for the Bernstein case. Only primal and dual generators are shown because the wavelets are the same as those in Figs. 5.4.5 and 5.4.6.

ACKNOWLEDGMENTS

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