

Polynomial generation and Quasi-interpolation in stationary non-uniform subdivision

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Abstract

We study the necessary and sufficient conditions for the generation of polynomials by stationary subdivision schemes, and we show how to derive appropriate quasi-interpolation rules that have the optimal approximation order. We show that these conditions hold in the context of non-uniform subdivision as well, and we demonstrate how they can be used for the construction of stationary non-uniform subdivision schemes that have a prescribed approximation power.

Key words: Subdivision schemes, Approximation, Polynomial generation, Quasi-Interpolation.

1 Introduction

The terms polynomial preservation, or polynomial reproduction, appear frequently in approximation theory, which studies the distances between functions and function spaces. In particular, the approximation of functions that are given on grids of finer and finer resolution, is studied, and the asymptotic rate of convergence of these finer and finer approximations to the original function is called the *approximation power*.

Given an approximation operator A , we say that A reproduces polynomials up to degree m , if $Af = f$ for every $f \in \pi_m$, where π_m is the space of polynomials up to degree m . It is well known, that under certain conditions, the exactness of an approximation scheme for polynomials up to degree m , is necessary and sufficient for achieving an approximation order $m + 1$ for any function which is smooth enough.

In the recent decade, the problem of approximation by a linear combination of the integer translates of one or more basis functions (PSI and FSI spaces) has been studied extensively (see e.g. [6,8]). The particular case of a shift invariant

(PSI) space spanned by the integer translates of one *refinable* function Φ , is exactly the space of functions generated by stationary uniform subdivision.

A subdivision scheme works by applying a refinement operator to given control points, repeatedly. The control points at different refinement levels converge to a function called the limit function. If the refinement rule is the same for all levels, this is a *stationary subdivision scheme*.

Since we are interested in the approximation properties of these limit functions, we ask, for a given subdivision scheme, what polynomials can be generated as limit functions of that scheme, and how to calculate the initial control points that yield these polynomials in the limit. In our notations, we have a subdivision operator S , and we want to calculate an operator Q that maps polynomials up to degree m to sets of control points. We want to have the property $S^\infty Qf = f$, $\forall f \in \pi_m$. Given a polynomial f , Qf are the control points you choose in order to get f as the limit function of subdivision.

For interpolatory schemes, the operator Q is simply a restriction to the integer grid. Namely, the control points that yield a given polynomial in the limit are the values of that polynomial on the integer grid, provided that S generates polynomials of that degree. A more interesting question is how to calculate the operator Q for non-interpolatory schemes.

The goal of the paper is to demonstrate how the identification of the relationships between S and Q can be applied for constructing new subdivision schemes with desired approximation orders. First, we show how to find Q for a given S . Next, we demonstrate how to construct a new subdivision scheme, by fixing Q and finding S .

The main theorem in the paper states, that for stationary subdivision schemes S ,

$$S^\infty Q = id \iff SQ = Q\sigma, \quad (1)$$

over the space of polynomials up to degree m , where σ is the dilation operator $\sigma f = f\left(\frac{\cdot}{2}\right)$. This result was first presented in [17] for the case of *uniform subdivision*, i.e. when the refinement rule uses the same *stencil* or *mask* everywhere. In this work we extend the result to stationary non-uniform subdivision.

The significance of (1), is that it reduces the equation $S^\infty Q = id$, which is a complicated equation to solve, if S is the unknown, to the equation $SQ = Q\sigma$, which is linear in S . This enables us to construct new subdivision schemes, in which the weights are calculated by solving small systems of linear equations that arise from the requirement $SQ = Q\sigma$.

We also use the operator Q in a method of approximation called *quasi-interpolation*. We show that for any uniform subdivision scheme S , we can calculate Q , and then extend it to operate on non-polynomial functions. This

extension of Q is, in fact, an approximation scheme, i.e. a method of selecting control points such that limit function approaches a desired function.

The analysis of this paper serves as the basis for the work of Levin and Levin in [18]. There, we construct a bivariate piecewise-uniform subdivision scheme that operates over a grid which is half quadrilateral and half triangular.

In §2 we present the analysis in the uniform case, and demonstrate how Q can be calculated for a given S . In §3 we show that under certain conditions, (1) holds in the case of non-uniform subdivision as well, i.e. where the grid is not regular or the refinement rules vary according to the location on the grid. In §3.5 we demonstrate how this is used to construct a new non-uniform subdivision scheme.

2 Uniform Subdivision

In this section, we use results from the theory of stationary subdivision and the theory of shift-invariant spaces, and introduce the operator Q , that leads to quasi-interpolation and to a condition for polynomial generation. This was originally presented in [17]. We also demonstrate how Q can be calculated for given subdivision schemes S .

The study of the operator Q , in the uniform case, becomes useful when we proceed to construct new non-uniform (or piecewise-uniform) subdivision schemes, with prescribed approximation orders, as demonstrated in §3.5. There, we show how to generate a new subdivision scheme S when a corresponding Q is given.

2.1 Notations

Let $l = l(\mathbb{Z}^s)$ denote the collection of all sequences $P : \mathbb{Z}^s \rightarrow \mathbb{R}$. We refer to $P \in l$ as a set of *control points*. Let $C = C(\mathbb{R}^s)$ denote the space of continuous real functions on \mathbb{R}^s . Similarly, C^m stands for the spaces of m -times differentiable functions over \mathbb{R}^s whose m -th order derivatives are continuous.

We use the standard multi-index notations for \mathbb{Z}^s , $j = (j_1, \dots, j_s) \in \mathbb{Z}^s$, $j \geq 0$ if $j_1, \dots, j_s \geq 0$, $|j| = j_1 + \dots + j_s$, $x^j = x_1^{j_1} \cdot \dots \cdot x_s^{j_s}$. For $j \geq 0$ we use $j! = j_1! \cdot \dots \cdot j_s!$, $D^j = \frac{\partial^{|j|}}{\partial^{j_1} x_1 \dots \partial^{j_s} x_s}$. The entries of a multi-index sequence $P \in l$ are denoted by $P(j)$, for $j \in \mathbb{Z}^s$.

The following notations are used for arithmetic operations on subsets of \mathbb{R}^s :

Let $X, Y \subset \mathbb{R}^s$, $z \in \mathbb{R}^s$ and $\alpha \in \mathbb{R}$. Then

$$\begin{aligned} X + Y &= \{x + y \mid x \in X, y \in Y\}, \\ X + z &= \{x + z \mid x \in X\}, \\ \alpha X &= \{\alpha x \mid x \in X\}, \end{aligned}$$

For $k \geq 0$, the space of polynomials of degree at most k over \mathbb{R}^s , is denoted by $\pi_k(\mathbb{R}^s)$. Its restriction to the integer grid is denoted by $\pi_k(\mathbb{Z}^s)$.

We define a shift operator on C by

$$E_\beta f(\cdot) = f(\cdot - \beta), \quad f \in C.$$

Throughout the paper, we use the maximum norm in \mathbb{R}^s ,

$$\|x\| = \max_{i=1, \dots, s} |x_i|, \quad \forall x \in \mathbb{R}^s.$$

We denote by σ the dilation operator on C , which appears naturally in the context of stationary subdivision,

$$\sigma f = f\left(\frac{\cdot}{2}\right), \quad f \in C.$$

Uniform subdivision schemes

A uniform subdivision operator is a linear operator $S : l \rightarrow l$ which is defined by a finitely supported mask $a \in l$ through

$$(SP)(\alpha) = \sum_{\beta \in \mathbb{Z}^s} a(\alpha - 2\beta)P(\beta), \quad \forall \alpha \in \mathbb{Z}^s. \quad (2)$$

The repeated application of S to a given set of control points P , $\{S^n P\}_{n=0, \dots, \infty}$, is called a *stationary subdivision scheme*.

A subdivision scheme S is termed *uniformly convergent*, if for every $P \in l$, there exists $F \in C$ (called the limit function) such that

$$\lim_{n \rightarrow \infty} \|S^n P - F(2^{-n} \cdot)\|_{\infty, \mathbb{Z}^s \cap 2^n D} = 0, \quad (3)$$

for any open and bounded domain $D \subset \mathbb{R}^s$. The introduction of the bounded domain D here is needed if we do not want to exclude unbounded sequences of control points and unbounded limit functions. In this paper we are specifically interested in the case where the limit function is a polynomial, and therefore, not bounded. We denote $S^\infty P = F$. We also require, as part of the definition of uniform convergence, that $S^\infty P$ is non-zero for some P .

S is called an interpolatory subdivision scheme, if $SP(2\alpha) = P(\alpha) \quad \forall \alpha \in \mathbb{Z}^s$.

We say that S belongs to the class C^m if S is uniformly convergent, and $S^\infty P \in C^m$ for every $P \in l$. For the smoothness analysis of uniform subdivision schemes the reader is referred to [3,10]. For a uniformly convergent S we define the S -refinable function $\Phi = S^\infty \delta$, where $\delta \in l$ is the sequence which is 1 at the origin and zero anywhere else. The limit function $S^\infty P$ can be expressed as a sum of integer translates of Φ

$$S^\infty P = \sum_{\alpha \in \mathbb{Z}^s} P(\alpha) \Phi(\cdot - \alpha) = \sum_{\alpha \in \mathbb{Z}^s} (S^k P)(\alpha) \Phi(2^k \cdot - \alpha), \quad \forall k \geq 0. \quad (4)$$

If S is uniformly convergent, it follows from (2) and (4) that Φ satisfies the so-called '2-scale relation'

$$\Phi(\cdot) = \sum_{\alpha \in \mathbb{Z}^s} a(\alpha) \Phi(2 \cdot - \alpha). \quad (5)$$

For convenience, it is assumed that a is supported in a cube centered at the origin,

$$a(\alpha) = 0, \quad \forall \alpha \in \mathbb{Z}^s \setminus [-\omega, \omega]^s, \quad (6)$$

for some $\omega \in \mathbb{Z}_+$. It is shown in [3] that the support of Φ is contained in the convex hull of the support of the mask a , therefore

$$\Phi(x) = 0, \quad \forall x \notin [-\omega, \omega]^s. \quad (7)$$

We call $\Omega = [-\omega, \omega]^s$ the *support* of S .

2.2 The operator Q

In this section we discuss the known fact that uniform subdivision schemes, under certain conditions, can generate polynomials as their limit functions. Our goal in the following lemmas is to establish the logical relation (1).

It is shown in [3], theorem 8.4 (see also [12]) that if $S \in C^m$ and the integer translates of Φ are linearly independent, then the space $\pi_m(\mathbb{Z}^s)$ is invariant under S . For many uniform subdivision schemes S , the space $\pi_m(\mathbb{Z}^s)$ is invariant under S even though S does not belong to C^m . Cavaretta et al. ([3], chapter 6) formulate conditions on the mask a that can be used to find the maximal m such that $\pi_m(\mathbb{Z}^s)$ is invariant under S (If the integer shifts of Φ are linearly independent, this amounts to simple algebraic conditions on a , by corollary 6.3 and theorem 6.3 in [3]). In the following results we study the mapping $S : \pi_m(\mathbb{Z}^s) \rightarrow \pi_m(\mathbb{Z}^s)$.

Lemma 2.1 *If S maps $\pi_m(\mathbb{Z}^s)$ to itself, and S is convergent, then S^∞ maps*

$\pi_m(\mathbb{Z}^s)$ to $\pi_m(\mathbb{R}^s)$, and the leading coefficients of $S^\infty p$ are the same as those of p , for any $p \in \pi_m(\mathbb{Z}^s)$.

Proof. Let $P \in \pi_m(\mathbb{Z}^s)$ and $F = S^\infty P$. In particular, we have that

$$F(z) = \lim_{n \rightarrow \infty} (S^n P)(2^n z), \quad \forall z \in \mathbb{Z}^s.$$

Let D_m denote the collection of all difference operators of order $m+1$ using only values of the given function on the integer grid. For every $\Delta \in D_m$, we have that

$$\Delta F(z) = \lim_{n \rightarrow \infty} \Delta((S^n P)(2^n \cdot))(z), \quad \forall z \in \mathbb{Z}^s.$$

But, because $S^n P$ is in $\pi_m(\mathbb{Z}^s)$, its differences of order $m+1$ are all zero. Therefore, we find that $\Delta F(z) = 0$ for all $z \in \mathbb{Z}^s$. Since this is true for all $\Delta \in D_m$, it follows that the restriction of F to \mathbb{Z}^s is a polynomial of degree m . Since $\sigma F = S^\infty S P$, we can equally deduce that the restriction of σF to \mathbb{Z}^s is a polynomial of degree m . In other words, the restriction of F to $2^{-1}\mathbb{Z}^s$ is a polynomial of degree m . We proceed by induction, and find that the restriction of F to all of the dyadic points is a polynomial. Since F is continuous, it must be in $\pi_m(\mathbb{R}^s)$.

We now show that $S^\infty P$ has the same leading coefficients as P . Since S is supported in Ω , there exists $c > 0$ such that

$$|S^\infty P(\alpha)| \leq c \|P\|_{\infty, \alpha + \Omega}, \quad \forall \alpha \in \mathbb{Z}^s, \quad \forall P \in \pi_m(\mathbb{Z}^s). \quad (8)$$

Let $P_\alpha \in \pi_0(\mathbb{Z}^s)$ be the constant polynomial $P_\alpha \equiv P(\alpha)$. Using the partition of unity property of S^∞ , we get that $S^\infty P_\alpha \equiv P(\alpha)$. We now substitute $P - P_\alpha$ in (8), and get

$$|S^\infty P(\alpha) - P(\alpha)| \leq c \|P - P(\alpha)\|_{\infty, \alpha + \Omega}, \quad \forall \alpha \in \mathbb{Z}^s.$$

Since $S^\infty P$ and P are polynomials, this necessarily implies that they have the same leading coefficients. \square

In the following, we show that S is similar to σ on $\pi_m(\mathbb{Z}^s)$. The similarity operator Q is then shown to be the inverse of S^∞ on π_m . A similar result also appeared in the context of wavelet theory in [21] (Lemma 3.2.3).

Lemma 2.2 *If S^∞ is a 1-1 map from $\pi_m(\mathbb{Z}^s)$ to $\pi_m(\mathbb{R}^s)$, then the restriction of S to $\pi_m(\mathbb{Z}^s)$ is similar to the dilation operator,*

$$SQ = Q\sigma, \quad \text{on } \pi_m(\mathbb{R}^s), \quad (9)$$

and the similarity operator $Q : \pi_m(\mathbb{R}^s) \rightarrow \pi_m(\mathbb{Z}^s)$ is the inverse of S^∞ on $\pi_m(\mathbb{R}^s)$. Moreover, the leading coefficients of Qf are the same as those of f , for all $f \in \pi_m(\mathbb{R}^s)$.

Note that the lemma only requires that S^∞ is 1-1 over a sub-space of polynomials, and not for all sequences of control points. Since this is a finite-dimensional space, this property is not difficult to check. In particular, it holds for any scheme S which satisfies the conditions of Lemma (2.1).

Proof.

We observe σ , S and S^∞ as operators over the space of polynomials up to degree m , in s -dimensions. Since S^∞ is invertible, we can take its inverse $Q = (S^\infty)^{-1}$. Now, we multiply the identity

$$S^\infty S = \sigma S^\infty, \quad \text{on } \pi_m(\mathbb{Z}^s),$$

from the left and from the right by Q , and we get

$$SQ = Q\sigma, \quad \text{on } \pi_m(\mathbb{Z}^s).$$

The extension of Q to an operator from $\pi_m(\mathbb{R}^s)$ to $\pi_m(\mathbb{Z}^s)$ is trivial. That S^∞ necessarily preserves the leading coefficients, follows from lemma 2.1. Since Q is its inverse, it also preserves leading coefficients. \square

Lemma 2.3 *If $Q : \pi_m(\mathbb{R}^s) \rightarrow \pi_m(\mathbb{Z}^s)$ satisfies*

$$SQ = Q\sigma, \quad \text{on } \pi_m(\mathbb{R}^s),$$

and the leading coefficients of Qf are the leading coefficients of f for all $f \in \pi_m(\mathbb{R}^s)$, then $S^\infty Q|_{\pi_m(\mathbb{R}^s)} = id$, namely

$$S^\infty Qf = f, \quad \forall f \in \pi_m(\mathbb{R}^s).$$

This lemma is proven in the more general non-uniform case (see Lemma 3.2).

By Lemmas 2.2 and 2.3:

Corollary 2.4 *If S is convergent, and $Q : \pi_m(\mathbb{R}^s) \rightarrow \pi_m(\mathbb{Z}^s)$ preserves leading coefficients then*

$$S^\infty Q = id \iff SQ = Q\sigma,$$

both identities restricted to $\pi_m(\mathbb{R}^s)$.

The significance of Corollary (2.4) is the reduction of the property $S^\infty Q = id$, which is the formal notation for polynomial generation, to the relation $SQ =$

$Q\sigma$, in which S appears as a linear term. This is useful for constructing new subdivision schemes, in which case S is the unknown. It enables us to deduce the weights of S from the solution of a system of linear equations.

2.3 Polynomial eigenvectors of S

From the relation $SQ = Q\sigma$, over the polynomials of degree up to m , we can easily derive the most significant eigenvectors of the operator S . These are the eigenvectors that yield polynomials in the limit. For $k = (k_1, \dots, k_s) \in \mathbb{Z}_+^s$, we take the monomial

$$f(x) = x^k, \quad x \in \mathbb{R}^s.$$

Then it follows that

$$SQf = Q\sigma f = Q2^{-|k|}f = 2^{-|k|}Qf.$$

Hence, Qf is an eigenvector of S with the eigenvalue $2^{-|k|}$. Moreover, we know that Qf is a polynomial of degree $|k|$, provided that Q preserves leading coefficients.

We also know the limit function that corresponds to this eigenvector. It is simply the monomial x^k , because

$$S^\infty Qf = f.$$

For example, when S is the well-known cubic B-spline subdivision scheme, its corresponding Q operator is

$$Qf(i) = f(i) - \frac{1}{6}f''(i), \quad \forall i \in \mathbb{Z}, \quad f \in \pi_3(\mathbb{R}).$$

This is shown in detail in §2.6. We can immediately write down the four eigenvectors of S with eigenvalues $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}$. These are:

$$\begin{aligned} Q(1) &= (1) = (\dots, 1, 1, 1, 1, \dots) \\ Q(x) &= (x) = (\dots, -2, -1, 0, 1, 2, \dots) \\ Q(x^2) &= (x^2 - \frac{1}{3}) = (\dots, 3\frac{2}{3}, \frac{2}{3}, -\frac{1}{3}, \frac{2}{3}, 3\frac{2}{3}, \dots) \\ Q(x^3) &= (x^3 - x) = (\dots, -7, 0, 0, 0, 7, \dots), \end{aligned}$$

using the notation $(f) = f|_{\mathbb{Z}^s}$.

2.4 Quasi-interpolation

The operator Q , introduced in section §2.2 was defined as an operator from $\pi_m(\mathbb{R}^s)$ to $\pi_m(\mathbb{Z}^s)$. In this section, we show how Q can be extended to an

operator from $C(\mathbb{R}^s)$ to $l(\mathbb{Z}^s)$ which is local and bounded (in the sense of Lemma (2.6)). We then use Q to define a quasi-interpolation operator, and we prove an approximation order result.

We assume that for a given convergent subdivision scheme S , we have that $S^\infty Qf = f$ for all $f \in \pi_m(\mathbb{R}^s)$. Since S^∞ is shift invariant over π_m , it follows that Q is shift invariant as well. The following lemma uses this fact to show that Q can be represented as a sum of differential operators, when viewing Q as an operator on the space of polynomials up to degree m .

Lemma 2.5 *Q can be expressed in the form*

$$Q = \sum_{|i| \leq r} a_i D^i \Big|_{\mathbb{Z}^s},$$

taking functions in $\pi_m(\mathbb{R}^s)$ to sequences in $\pi_m(\mathbb{Z}^s)$.

Proof. Let $a_i = \frac{1}{i!} Q(\cdot)^i(0)$, $\forall |i| \leq r$, where $Q(\cdot)^i$ stands for the application of Q to the monomial x^i . Let $\overline{Q} = \sum_{|i| \leq r} a_i D^i$. Then it follows that

$$Qp(0) = \overline{Q}p(0), \quad \forall p \in \pi_m(\mathbb{Z}^s),$$

and both \overline{Q} and Q are shift-invariant. Using that, for every $\alpha \in \mathbb{Z}^s$ we get

$$Qp(\alpha) = E_{-\alpha} Qp(0) = QE_{-\alpha} p(0) = \overline{Q}E_{-\alpha} p(0) = E_{-\alpha} \overline{Q}p(0) = \overline{Q}p(\alpha),$$

therefore $Q = \overline{Q}$. \square

Lemma 2.6 (An extension of Q) *Q can be extended to a shift invariant operator $Q : C(\mathbb{R}^s) \rightarrow l(\mathbb{Z}^s)$ which is bounded and local, namely, there exists an open and bounded domain $A \in \mathbb{R}^s$ and $c > 0$ such that*

$$|Qf(\alpha)| \leq c \|f\|_{\infty, \alpha+A}, \quad \forall \alpha \in \mathbb{Z}^s, \quad \forall f \in C(\mathbb{R}^s).$$

Proof. Any differential operator D^i defined on $\pi_m(\mathbb{R}^s)$ can be extended to a shift invariant operator which is bounded and local, by replacing it with an appropriate finite difference. The result then follows from lemma 2.5. \square

Using the extension of Q to $C(\mathbb{R}^s)$, we can approximate any continuous function f by $S^\infty Qf$. This method is known as *quasi-interpolation*. The following theorem can also be derived as a particular case from the general theory of PSI spaces, and quasi-interpolation (see e.g. [6,8]).

Theorem 2.7 (Quasi-interpolation) *Let S denote a uniformly convergent subdivision scheme, and let $Q : C(\mathbb{R}^s) \rightarrow l(\mathbb{Z}^s)$ be a local and bounded operator (in the sense of lemma 2.6), such that*

$$S^\infty Q p = p, \quad p \in \pi_m(\mathbb{R}^s).$$

Then there exists $C > 0$ such that

$$\|S^\infty Q f - f\|_\infty \leq C \|f\|_{m+1},$$

for every $f \in C^{m+1}(\mathbb{R}^s)$ with $\|f\|_{m+1} < \infty$, where $\|f\|_{m+1}$ denotes the supremum of the partial derivatives of f of order $m+1$.

Proof. We denote by $\mathbb{T}_{m,x}f$ the Taylor expansion of order m of f centered at $x \in \mathbb{R}^s$. Namely,

$$\mathbb{T}_{m,x}f(\cdot) = \sum_{|i| \leq m} \frac{1}{i!} (\cdot - x)^i D^i f(x), \quad \cdot \in \pi_m(\mathbb{R}^s).$$

From the conditions of the lemma we know that

$$S^\infty Q f(x) - f(x) = S^\infty Q (f - \mathbb{T}_{r,x}f)(x), \quad \forall f \in C^m(\mathbb{R}^s), \quad (10)$$

Since Φ is supported in Ω and bounded, it follows from (4) that there exists $c_1 > 0$ such that

$$|S^\infty Q (f - \mathbb{T}_{r,x}f)(x)| \leq c_1 \|Q (f - \mathbb{T}_{r,x}f)\|_{\infty, \mathbb{Z}^s \cap (x+\Omega)}, \quad (11)$$

for all $f \in C^m(\mathbb{R}^s)$. We also know that there exists $c_2 > 0$ such that

$$\|f - \mathbb{T}_{m,x}f\|_{\infty, (x+\Omega)} \leq c_2 \|f\|_{m+1}, \quad \forall f \in C^{m+1}(\mathbb{R}^s) \quad (12)$$

Since Q is bounded and local, it follows that there exists $c_3 > 0$ such that

$$\|Q (f - \mathbb{T}_{m,x}f)\|_{\infty, \mathbb{Z}^s \cap (x+\Omega)} \leq c_3 \|f\|_{m+1}, \quad \forall f \in C^{m+1}(\mathbb{R}^s). \quad (13)$$

The theorem follows from (10)- (13). \square

This leads to an approximation order result. We now consider finer and finer approximations of a given function f , using the quasi-interpolation operator $S^\infty Q$. The way we refine the approximation, is to apply it to the dilated version of f , σf . We then dilate the approximation back in order to compare it to the original function f . Therefore, we study the approximation error $\sigma^{-n} S^\infty Q \sigma^n f - f$. From theorem 2.7 we get

$$\begin{aligned} \|\sigma^{-n} S^\infty Q \sigma^n f - f\|_\infty &= \|S^\infty Q \sigma^n f - \sigma^n f\|_\infty \leq C \|\sigma^n f\|_{m+1} \\ &= 2^{-n(m+1)} C \|f\|_{m+1}. \end{aligned}$$

The approximation result of Theorem 2.7 can be easily extended to an estimate on the derivatives of the difference $S^\infty Qf - f$. The same proof with small changes shows that under the conditions of Theorem 2.7, for every $|j| \leq m$, there exists $C > 0$ such that

$$\|D^j S^\infty Qf - D^j f\|_\infty \leq C \|f\|_{m+1},$$

for every $f \in C^{m+1}(\mathbb{R}^s)$ such that $\|f\|_{m+1} < \infty$. Hence, the same approximant $S^\infty Qf$ that uses only values of the given function f , yields an approximation for the derivatives of f as well. However, for derivatives of order j , the approximation order is no longer $m+1$, but $m+1-|j|$, because $D^j \sigma^{-n} = 2^{n|j|} \sigma^{-n} D^j$. Namely,

$$\|D^j \sigma^{-n} S^\infty Q \sigma^n f - D^j f\|_\infty \leq 2^{-n(m+1-|j|)} C \|f\|_{m+1}.$$

2.5 How to calculate Q from S

In this section, we introduce a method to calculate the operator $Q : \pi_m(\mathbb{R}^s) \rightarrow \pi_m(\mathbb{Z}^s)$ for a given subdivision scheme S . Since we restrict our attention here to polynomials and polynomial sequences, we use the notation π_m freely, both for $\pi_m(\mathbb{R}^s)$ and $\pi_m(\mathbb{Z}^s)$.

Q is the inverse of S^∞ on π_m . In order to calculate Q , the restriction of the operator S^∞ to π_m needs to be identified. Since a polynomial is uniquely defined by its values on the integer grid, we have that

$$S^\infty p = \sum_{\alpha \in \mathbb{Z}^s} p(\alpha) \Phi(\cdot - \alpha) = \sum_{\beta \in \mathbb{Z}^s} p(\beta + \cdot) \Phi(-\beta), \quad \forall p \in \pi_m. \quad (14)$$

therefore, in order to know S^∞ on π_m , we only need to know the values of Φ at the integers, which can be calculated directly from the 2-scale relation (5). When evaluating the 2-scale relation at the integers, it becomes a system of linear equations, in the form

$$\Phi(\beta) = \sum_{\alpha} a(\alpha) \Phi(2\beta - \alpha), \quad \beta \in \text{support}(\Phi).$$

Therefore, the values of Φ at the integers are an eigenvector of a matrix whose entries are the mask coefficients, corresponding to the eigenvalue 1, and such that they sum up to 1.

Given a polynomial $p \in \pi_m$, we can rewrite any shift of it in a Taylor-expansion as follows:

$$p(\cdot + \beta) = \sum_{|i| \leq m} \frac{\beta^i}{i!} D^i p, \quad \forall p \in \pi_m(\mathbb{R}^s), \quad \forall \beta \in \mathbb{R}^s,$$

and together with (14) we get that

$$S^\infty p = \sum_{|i| \leq m} \frac{m_i}{i!} D^i p, \quad \forall p \in \pi_m, \quad (15)$$

where the *moments* m_i are given by

$$m_i = \sum_{\beta \in \mathbb{Z}^s} \Phi(-\beta) \beta^i. \quad (16)$$

After $S^\infty|_{\pi_m}$ is known in the form of (15), it is easy to calculate its inverse, Q . In the following, we demonstrate this method for the cubic B-spline scheme and for the three directional box-spline scheme.

2.6 Examples

Cubic B-spline

The cubic B-spline scheme is given by the mask $a = \frac{1}{8}[1, 4, 6, 4, 1]$, supported in $\{-2, -1, 0, 1, 2\}$. The space of all cubic polynomials $\pi_3(\mathbb{Z})$ is invariant under this scheme. In the following, we demonstrate how Q can be calculated for this specific scheme.

The first step is to identify the values of Φ at the integers:

$$\Phi(-1) = \Phi(1) = \frac{1}{6}, \quad \Phi(0) = \frac{2}{3}.$$

Then, we calculate the moments m_i defined in (16):

$$\begin{aligned} m_0 &= \Phi(1) + \Phi(0) + \Phi(-1) = 1 \\ m_1 &= -\Phi(1) + \Phi(-1) = 0 \\ m_2 &= \Phi(1) + \Phi(-1) = \frac{1}{3} \\ m_3 &= -\Phi(1) + \Phi(-1) = 0. \end{aligned} \quad (17)$$

Using (15), we can write the restriction of S^∞ to the cubic polynomials as

$$S^\infty f = f + \frac{1}{6}f'', \quad \forall f \in \pi_3.$$

The operator Q which is the inverse of S^∞ on π_3 can be written as

$$Qf = f - \frac{1}{6}f'', \quad f \in \pi_3(\mathbb{R}). \quad (18)$$

This is easy to verify:

$$S^\infty Qf = S^\infty(f - \frac{1}{6}f'') = (f - \frac{1}{6}f'' + \frac{1}{6}f'' - \frac{1}{36}f^{(4)}) = f, \quad \forall f \in \pi_3,$$

since the 4-th order derivative of a cubic polynomial is zero. The operator Q can be extended to an operator from $C(\mathbb{R})$ to $l(\mathbb{Z})$ as follows:

$$Qf(i) = -\frac{1}{6}f(i-1) + \frac{4}{3}f(i) - \frac{1}{6}f(i+1), \quad \forall i \in \mathbb{Z}.$$

The corresponding quasi-interpolation scheme applies the mask $(-\frac{1}{6}, \frac{4}{3}, -\frac{1}{6})$ to the samples of a given function at the integers. The resulting control points yield a limit function that coincides with f if f is a cubic polynomial. Otherwise, if f has 4 continuous derivatives, we get an order of approximation 4, in the sense that

$$\sigma^{-n}S^\infty Q\sigma^n f - f = o(2^{-4n}).$$

Three-directional Box-spline

Box-splines are multivariate compactly supported piecewise polynomial functions. The theory of Box-splines is presented in detail in [7]. The box-spline Φ , defined by the three directions $(1, 0), (0, 1), (1, 1)$ each taken with multiplicity 2, is known to be a compactly supported C^2 piecewise polynomial function on the regular three directional triangulation, whose polynomial pieces are of degree four. Every cubic polynomial can be generated as the sum of integer shifts of this box-spline.

Φ can be generated as the limit function of a subdivision scheme S defined by the mask

$$a = \frac{1}{16} \begin{bmatrix} 0 & 0 & 1 & 2 & 1 \\ 0 & 2 & 6 & 6 & 2 \\ 1 & 6 & 10 & 6 & 1 \\ 2 & 6 & 6 & 2 & 0 \\ 1 & 2 & 1 & 0 & 0 \end{bmatrix}. \quad (19)$$

Its support is $\Omega = [-2, 2]^2$. The Loop scheme [19] is an extension of this box-spline scheme to irregular triangular meshes.

The operator Q , which is the inverse of S^∞ in $\pi_3(\mathbb{R}^2)$, is given by

$$Qf = f - \frac{1}{6} \left(\frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_1 \partial x_2} + \frac{\partial^2 f}{\partial x_2^2} \right), \quad f \in \pi_3(\mathbb{R}^2). \quad (20)$$

In the following we derive Q , using the method of §2.5.

First, we calculate the values of the refinable function Φ on the integers, in order to identify the restriction of S^∞ to $\pi_3(\mathbb{Z}^2)$. From the 2-scale relation (5)

it can be shown that Φ is zero at all the integers except for the following:

$$\begin{aligned}\Phi(1, 0) &= \Phi(0, 1) = \Phi(1, 1) = \Phi(-1, 0) = \Phi(0, -1) = \Phi(-1, -1) = \frac{1}{12}, \\ \Phi(0, 0) &= \frac{1}{2}.\end{aligned}\tag{21}$$

Using (21) and (16), we calculate all the moments up to order 3:

$$\begin{aligned}m_{(0,0)} &= 1, & m_{(1,0)} &= m_{(0,1)} = 0, \\ m_{(2,0)} &= m_{(0,2)} = \frac{1}{3}, & m_{(1,1)} &= \frac{1}{6}, \\ m_{(3,0)} &= m_{(0,3)} = m_{(2,1)} = m_{(1,2)} = 0.\end{aligned}$$

Using (15), we get that

$$S^\infty p = p + Lp, \quad \forall p \in \pi_3(\mathbb{R}^2),\tag{22}$$

where

$$L = \frac{1}{6} \left(D^{(0,2)} + D^{(1,1)} + D^{(2,0)} \right).$$

Finally, since $(I+L)(I-L) = (I-L^2)$, and $L^2 p$ is zero for any cubic polynomial p , we conclude that $Qp = p - Lp$, as in (20).

One extension of Q , that can be used for quasi-interpolation, in the spirit of Lemma (2.6), is

$$Qf(\alpha) = \frac{3}{2}f(\alpha) - \frac{1}{12} \sum_{\delta \in \Delta} f(\alpha + \delta), \quad \alpha \in \mathbb{Z}^2,$$

where $\Delta = \{(1, 0), (1, 1), (0, 1), (-1, 0), (-1, -1), (0, -1)\}$.

3 Non-Uniform Subdivision

In §2 we restricted ourselves to subdivision schemes that are stationary and uniform. Namely, the same subdivision mask is applied everywhere on the integer grid and at every level of refinement. They were also defined as operators on the integer grid, $l(\mathbb{Z}^s)$, which is uniformly distributed over the space \mathbb{R}^s .

In this section, we extend the results of §2 to schemes that are stationary but non-uniform. The scheme is still stationary in the sense that a single operator S operates at every level of refinement. However, S is no longer restricted to the form (2). It can no longer be captured by a single mask a that operates everywhere. Also, S doesn't necessarily operate on the integer grid \mathbb{Z}^s . The grid itself can be non-uniform.

The theoretical results developed in this section describe the relationships between S and the operator Q that are necessary and sufficient for establishing the polynomial generation properties of a subdivision scheme. The practical purpose for this analysis is to introduce a method for constructing new non-uniform subdivision schemes with good approximation orders. Due to the constructive nature of our approach, we do not deal with the question of the existence of Q for given S .

In §3.1 we extend the notations of stationary subdivision to the non-uniform case. In §3.2 we discuss the kinds of schemes for which this theory is applicable. In sections 3.3 - 3.4 we extend the results of §2 to the non-uniform case. In §3.5 we demonstrate the construction of a non-uniform scheme with maximal approximation order. This is a new univariate scheme which is interpolatory on the negative side of the real line, and non-interpolatory on the positive side. We show how the relation (1) is used as the basis for constructing the scheme.

3.1 Notations

The integer grid, that was used for defining uniform subdivision, is replaced here by a set of points $X \subset \mathbb{R}^s$. A set of control points is a sequence of values on X , $P \in l(X)$. The subdivision operator S is a linear operator $S : l(X) \rightarrow l(X)$. A stationary subdivision scheme is defined as the repeated application of S to given control points $P \in l(X)$.

We say that S is *uniformly convergent*, if for every $P \in l(X)$, there exists $F \in C(\mathbb{R}^s)$ (called the limit function) such that

$$\lim_{n \rightarrow \infty} \|S^n P - F(2^{-n} \cdot)\|_{\infty, X \cap 2^n D} = 0, \quad (23)$$

for any open and bounded domain $D \subset \mathbb{R}^s$. We denote $S^\infty P = F$. We also require, as part of the definition of uniform convergence, that $S^\infty P$ is non-zero for some P . We restrict our attention to grids X such that $2X \subset X$. This enables us to define interpolatory subdivision schemes S , as schemes where $SP(2x) = P(x)$ for all $x \in X$, and for all $P \in l(X)$. We also require that the dilations of X are dense in \mathbb{R}^s , namely

$$\overline{\bigcup_{n=0}^{\infty} 2^{-n} X} = \mathbb{R}^s.$$

This guarantees that if the limit function (23) exists, it is unique.

In contrast to uniform subdivision, $S^\infty P$ can no longer be expressed as a sum of integer translates of one compactly supported refinable basis function, as in

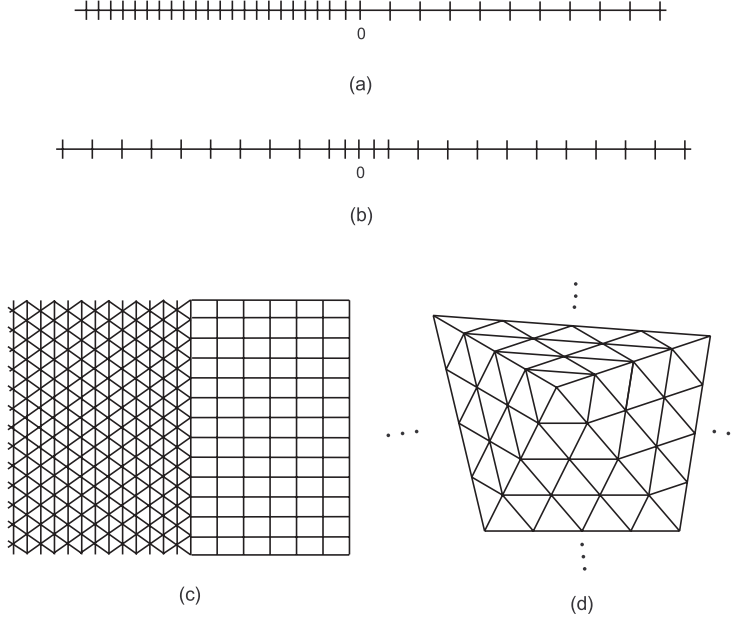


Fig. 1. Non-uniform grids. (a) Different grid spacing from both sides of the origin. (b) Higher density near the origin. (c) A grid that combines quadrilaterals on one side of the axis with triangles on the other. (d) A k -regular grid. The only vertex that has valency other than 6 is at the origin.

(4). However, we are still interested in the notion of locality of the subdivision scheme. We say that S is local with support $\Omega \subset \mathbb{R}^s$, or that S^∞ is supported on Ω , if for all $x \in \mathbb{R}^s$, and for all $P \in l(X)$,

$$P|_{(x+\Omega) \cap X} = 0 \Rightarrow S^\infty P(x) = 0. \quad (24)$$

3.2 Kinds of non-uniform schemes

A non-uniform scheme can be defined on a non-uniform grid X . Several examples of non-uniform grids in \mathbb{R}^1 and in \mathbb{R}^2 are shown in Fig. 1. They all satisfy the relation $2X \subset X$, and $2^{-n}X$ converges to a set which is dense in \mathbb{R}^1 and \mathbb{R}^2 respectively. A particular extension of the four-point scheme to a semi-uniform grid such as in Fig. 1(a) was analyzed in [17]. Subdivision schemes for the grids in Fig. 1(b) and 1(c) have not yet been studied. A grid such as the k -regular grid in Fig. 1(d) appears naturally around extraordinary vertices in subdivision meshes [2,9], but their analysis takes a different direction than this paper, in that the smoothness around extraordinary vertices is analyzed with respect to a special parameterization called the *characteristic map* (see e.g [20]).

On the other hand, a subdivision operator defined on a uniform grid, or even the integer grid, may be a non-uniform subdivision operator, if it operates dif-

ferently on different areas of the grid. A class of univariate schemes operating differently on either side of the origin was analyzed in [12]. In the bivariate case, such non-uniformity is used in crease-rules for subdivision surfaces (see e.g. [13,22]). These are similar to boundary rules that operate near boundaries of meshes [1]. The setting where the grid is uniform, but the rules change near the boundary is the one used in *combined subdivision schemes*, [14–17], which are subdivision schemes that satisfy boundary conditions.

The kinds of non-uniformity that we consider in this paper do not include schemes that have no regular structure in them, such as the univariate schemes described in [4,5], that can add samples not only at the middle of intervals, and operate differently at each refinement level.

3.3 The operator Q

In this section we show that (1) is valid in the non-uniform case, under certain assumptions.

We observe a significant difference between the uniform and the non-uniform case, as far as polynomial generation is concerned. In both cases, we ask how to choose the control points so that we get a polynomial in the limit. The answer comes in the form of an operator Q , because $S^\infty Q$ is the identity on a subspace of polynomials. But, in the uniform case, in order to get a polynomial in the limit, we had to take control points that lie on a polynomial, whereas in the non-uniform case this is no longer true.

The operator Q is no longer an operator from $\pi_m(\mathbb{R}^s)$ to $\pi_m(X)$, and we consider a more general

$$Q : \pi_m(\mathbb{R}^s) \rightarrow l(X).$$

However, we will still need a notion parallel to preservation of leading coefficients, which is one of the conditions of Lemma 2.3. We say that $Q : \pi_m(\mathbb{R}^s) \rightarrow l(X)$ preserves leading coefficients, if

$$f \in \pi_k(\mathbb{R}^s) \Rightarrow |Qf(x) - f(x)| = o(\|x\|^k), \quad \text{as } \|x\| \rightarrow \infty, \quad (25)$$

for all $k \leq m$.

In the following we establish (1) for the non-uniform case.

Lemma 3.1 *Let $S : l(X) \rightarrow l(X)$ be a convergent subdivision operator. If S^∞ is an injection, namely, $S^\infty P = 0 \Rightarrow P = 0$, and $Q : \pi_m(\mathbb{R}^s) \rightarrow l(X)$ is such that*

$$S^\infty Qf = f, \quad \forall f \in \pi_m(\mathbb{R}^s), \quad (26)$$

then

$$SQf = Q\sigma f, \quad \forall f \in \pi_m(\mathbb{R}^s). \quad (27)$$

Proof. From the definition of S^∞ it follows, similarly to the case of uniform subdivision, that $S^\infty S = \sigma S^\infty$. Using (26), we get

$$S^\infty SQf = \sigma S^\infty Qf = \sigma f,$$

for any $f \in \pi_m(\mathbb{R}^s)$. On the other hand,

$$S^\infty Q\sigma f = \sigma f.$$

Therefore,

$$S^\infty SQf = S^\infty Q\sigma f, \quad \forall f \in \pi_m(\mathbb{R}^s),$$

and (27) follows from the injectivity of S^∞ . \square

Lemma 3.2 *Let $S : l(X) \rightarrow l(X)$ be a convergent subdivision operator, and let Q denote a linear operator $Q : \pi_m(\mathbb{R}^s) \rightarrow l(X)$. If*

$$SQf = Q\sigma f, \quad \forall f \in \pi_m(\mathbb{R}^s), \quad (28)$$

and Q preserves leading coefficients, in the sense of (25), then

$$S^\infty Qf = f, \quad \forall f \in \pi_m(\mathbb{R}^s), \quad (29)$$

Proof. It is sufficient to show that $S^\infty Qf = f$ for all homogeneous polynomials f in $\pi_m(\mathbb{R}^s)$, i.e. polynomials such that $\sigma f = 2^{-k}f$ where $k \leq m$ is the degree of the polynomial. We prove the lemma directly from the definition of the limit function (23). We aim to show that

$$\lim_{n \rightarrow \infty} \|S^n Qf - f(2^{-n} \cdot)\|_{\infty, X \cap 2^n D} = 0, \quad (30)$$

for all open and bounded domains $D \subset \mathbb{R}^s$.

From (28) it follows that $S^n Qf = Q\sigma^n f$, and because f is a homogeneous polynomial,

$$S^n Qf = Q2^{-nk} f = 2^{-nk} Qf.$$

Thus,

$$\|S^n Qf - f(2^{-n} \cdot)\|_{\infty, X \cap 2^n D} = 2^{-nk} \|Qf - f\|_{\infty, X \cap 2^n D}.$$

But, because Q satisfies (25), we have that $\|Qf - f\|_{\infty, X \cap 2^n D} = o(2^{nk})$, and (30) follows. \square

Corollary 3.3 *If S is convergent, S^∞ is an injection, and $Q : \pi_m(\mathbb{R}^s) \rightarrow l(X)$ preserves leading coefficients in the sense of (25), then*

$$S^\infty Q = id \iff SQ = Q\sigma,$$

both identities restricted to $\pi_m(\mathbb{R}^s)$.

The observations in §2.3 are also easily extended to the non-uniform case. Provided that Q satisfies $SQ = Q\sigma$ over the polynomials up to degree m , then Qf is an eigenvector of S with eigenvalue $2^{-|k|}$, if f is the monomial $f(x) = x^k$, for $|k| \leq m$. The difference from the uniform case is, that now the values of the eigenvector Qf do not necessarily lie on a polynomial. The corresponding limit function is still a polynomial, however, because $S^\infty Qf = f$.

3.4 Quasi-interpolation

In §2.4 we have shown that if $Q : \pi_m(\mathbb{R}^s) \rightarrow \pi_m(\mathbb{Z}^s)$ satisfies

$$S^\infty Qf = f, \quad \forall f \in \pi_m(\mathbb{R}^s),$$

then there is a bounded and local extension to Q as an operator $Q : C(\mathbb{R}^s) \rightarrow l(\mathbb{Z}^s)$, and this extension is used in an approximation scheme that has approximation order $m + 1$ (Theorem 2.7).

For the extension of Q to non-polynomial functions in §2.4, we used the fact that Q was shift-invariant. In the non-uniform case, Q is not necessarily shift-invariant. Therefore, we will not describe how Q can be extended. Instead, we will assume here that Q can be extended to an operator $Q : C(\mathbb{R}^s) \rightarrow l(X)$, which is bounded and local, in the sense that there exists an open and bounded domain $A \in \mathbb{R}^s$ and $c > 0$ such that

$$|Qf(\alpha)| \leq c \|f\|_{\infty, \alpha+A}, \quad \forall \alpha \in X, \forall f \in C(\mathbb{R}^s). \quad (31)$$

Theorem 2.7 extends easily to the non-uniform case, with a similar proof, provided that we assume that S^∞ is local and bounded, in the following sense: We say that S is local, if (24) is satisfied with a support Ω which is bounded. We require, in addition, that there exists $c > 0$ such that

$$|S^\infty P(x)| \leq c \|P\|_{(x+\Omega) \cap X}, \quad \forall P \in l(X), \quad \forall x \in \mathbb{R}^s.$$

In the uniform case, this bound on $S^\infty P$ follows easily from the convergence of S . In the non-uniform case, this is not necessarily true. However, since this in itself is a fundamental property that is desirable to have in S , it doesn't add any new restrictions.

The approximation order $m + 1$ for the function values, and $m + 1 - |j|$ for derivatives of order j , follows easily, just as in the uniform case.

3.5 Example

In this section, we use the above theory to construct a non-uniform univariate scheme, that is interpolatory on the left side of the real line, and non-interpolatory on the right side.

The four point interpolatory scheme [11] is a family of interpolatory schemes, given by the mask $a = [-w, 0, \frac{1}{2} + w, 1, \frac{1}{2} + w, 0, -w]$, supported on $\{-3, \dots, 3\}$, where w is a tension parameter. It is shown in [10] (see also [11]) that the limit function Φ belongs to C^1 when w is in the range $0 < w < \frac{-1+\sqrt{5}}{8}$. Moreover, when w is in this range, Φ has Hölder continuous first derivatives, of order $1 - \epsilon$, for all $0 < \epsilon < 1$. In this section we restrict our attention to the case $w = \frac{1}{16}$ where the four-point scheme reproduces cubic polynomials. Since the scheme is interpolatory, the corresponding operator Q is the identity.

We now proceed to combine the four-point scheme with the cubic B-spline scheme, described in §2.6. Using the notations of §3.1, the grid X is the set of integers $X = \mathbb{Z}$, and our goal is to construct $S : l(\mathbb{Z}) \rightarrow l(\mathbb{Z})$ that generates all cubic polynomials $\pi_3(\mathbb{R})$, such that it is interpolatory on one half and non-interpolatory on the other half of the real line.

From Corollary 3.3 we know that it is sufficient to show that for some $Q : \pi_3(\mathbb{R}) \rightarrow l(\mathbb{Z})$,

$$SQ = Q\sigma. \quad (32)$$

So, in order to generate S we also need to determine the appropriate Q . Clearly, there is no unique way to satisfy this equation. We will look for the scheme S that differs from the cubic B-spline scheme or from the four-point scheme at the minimal number of points, and whose support is minimal.

With these guidelines, we first decide on Q , and then solve the equation for S . We define $Q : \pi_3(\mathbb{R}) \rightarrow l(\mathbb{Z})$ as follows:

$$Qf(i) = \begin{cases} f(i) & i \leq 0 \\ f(i) - \frac{1}{6}f''(i) & i > 0 \end{cases}, \quad \forall f \in \pi_3(\mathbb{R}), \quad \forall i \in \mathbb{Z}.$$

The left side of Q is the identity, which coincides with the Q operator for the interpolatory four-point scheme. The right side of Q coincides with the Q operator for the cubic B-spline scheme (18).

For given $P \in l(\mathbb{Z})$ we define $SP(i)$ for $i = 3, 4, \dots$ by the cubic B-spline

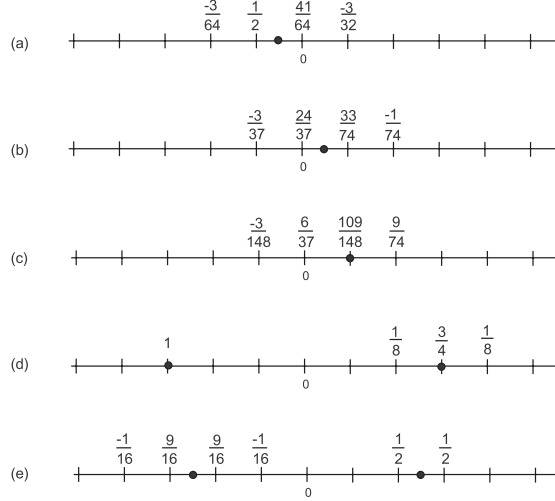


Fig. 2. A non-uniform subdivision scheme combining the four-point scheme on the left side of the real line with the cubic B-spline scheme on the right side. (a) The stencil for $SP(-1)$. (b) The stencil for $SP(1)$. (c) The stencil for $SP(2)$. (d-e) The regular four-point and cubic B-spline stencils.

scheme, and for $i = 0, -2, -3, -4, \dots$ by the four point scheme with tension parameter $w = \frac{1}{16}$. It is then easy to see that $SQf(i) = Q\sigma f(i)$ for all cubics f and for $i \in \mathbb{Z}$ except for $i = -1, 1, 2$. We now need to define $SP(-1)$, $SP(1)$ and $SP(2)$ for arbitrary P , such that S satisfies (32).

We look for subdivision rules near the origin that have as small support as possible. The weights of the stencils will be determined by the linear equation (32). We look for S of the form:

$$\begin{aligned} SP(-1) &= a_0P(-2) + a_1P(-1) + a_2P(0) + a_3P(1) \\ SP(1) &= b_0P(-1) + b_1P(0) + b_2P(1) + b_3P(2) \\ SP(2) &= c_0P(-1) + c_1P(0) + c_2P(1) + c_3P(2) \end{aligned} \quad (33)$$

The four variables a_0, \dots, a_3 are determined by the condition $SQf(-1) = Q\sigma f(-1)$, $\forall f \in \pi_3(\mathbb{R})$. By substituting in f , the monomials up to degree 3, we get four equations:

$$\begin{aligned} 1 : & \quad a_0 + a_1 + a_2 + a_3 = 1 \\ x : & \quad -2a_0 - a_1 + a_3 = -\frac{1}{2} \\ x^2 : & \quad 4a_0 + a_1 + \frac{2}{3}a_3 = \frac{1}{4} \\ x^3 : & \quad -8a_0 - a_1 = -\frac{1}{8} \end{aligned}$$

Similarly, from the conditions $SQf(1) = Q\sigma f(1)$ and $SQf(2) = Q\sigma f(2)$ for all cubics f , we get the following equations for the rest of the subdivision

mask:

$$\begin{array}{lll}
1 : & b_0 + b_1 + b_2 + b_3 = 1 & c_0 + c_1 + c_2 + c_3 = 1 \\
x : & -b_0 + b_2 + 2b_3 = \frac{1}{2} & -c_0 + c_2 + 2c_3 = 1 \\
x^2 : & b_0 + \frac{2}{3}b_2 + \frac{11}{3}b_3 = \frac{1}{6} & c_0 + \frac{2}{3}c_2 + \frac{11}{3}c_3 = \frac{11}{12} \\
x^3 : & -b_0 + 6b_3 = 0 & -c_0 + 6c_3 = \frac{3}{4}
\end{array}$$

All of these systems have unique solutions, given by $a = (-\frac{3}{64}, \frac{1}{2}, \frac{41}{64}, -\frac{3}{32})$, $b = (-\frac{3}{37}, \frac{24}{37}, \frac{33}{74}, -\frac{1}{74})$, $c = (-\frac{3}{148}, \frac{6}{37}, \frac{109}{148}, \frac{9}{74})$. The rules (33), together with the four-point scheme on the left and the cubic B-spline scheme on the right, form the new scheme S . All of masks used in S are depicted in Fig. 2.

We can easily show that the limit functions of S are C^1 -continuous. Clearly, they are C^1 -continuous away from the origin, because both cubic B-splines and the limit function of the four-point scheme are C^1 -continuous. For the smoothness analysis near the origin it is sufficient to show that the 5×5 local subdivision matrix mapping $P(-2), \dots, P(2)$ to $SP(-2), \dots, SP(2)$ has eigenvalues $1, \frac{1}{2}, \lambda_1, \lambda_2, \lambda_3$, with $|\lambda_i| < \frac{1}{2}$. In fact, we found that $\lambda_1 = \frac{1}{4}$, $\lambda_2 = \frac{57}{296} = 0.1926\dots$, $\lambda_3 = \frac{1}{8}$. Since smoothness analysis is out of the scope of this paper, we refer the reader to [12,17] for more details.

Next, we demonstrate how one can approximate using this new scheme, by applying the results of §2.4. Since S generates all cubics, we know that in order to get approximation order 4, all we need is to extend Q to an operator from $C(\mathbb{R})$ to $l(\mathbb{Z}^s)$ such that it is bounded and local in the sense of (31). We suggest here one such extension:

$$Qf(i) = \begin{cases} f(i) & i \leq 0 \\ f(i) - \frac{f(i+1) - 2f(i) + f(i-1)}{6} & i > 0 \end{cases}, \quad \forall f \in C(\mathbb{R}), \quad \forall i \in \mathbb{Z}.$$

For any continuous function f , we will use Qf as control points, and approximate f by the limit function $S^\infty Qf$. We know that the approximation is exact, namely $f = S^\infty Qf$, whenever f is a cubic polynomial. For example, in order to get the function $f(x) = x^2$ in the limit, the control points that we need to choose are $Qf = (\dots, 9, 4, 1, 0, 1 - \frac{1}{3}, 4 - \frac{1}{3}, 9 - \frac{1}{3}, \dots)$ (see Fig. 3).

The application of Q to f requires sampling f at the integers. In order to refine the approximation, we can take samples of f at denser grids, such as the half-integers. This is equivalent to an approximation of f by $\sigma^{-1}S^\infty Q\sigma f$. We tested this scheme, by measuring $(\sigma^{-n}S^\infty Q\sigma^n f(x) - f(x))$ for $f(x) = \cos(x)$. The error graphs, and the maximal errors are displayed in Fig. 4. The maximal errors for $n = 0, 1, 2, 3$ are 0.026234, 0.0018346, 0.00011706, $7.2429 \cdot 10^{-6}$, with decrease ratios 14.3, 15.67, 16.16 (The last approximation used samples of $\cos(x)$ at intervals of $\frac{1}{8}$). This is in accordance with the theory that guaran-

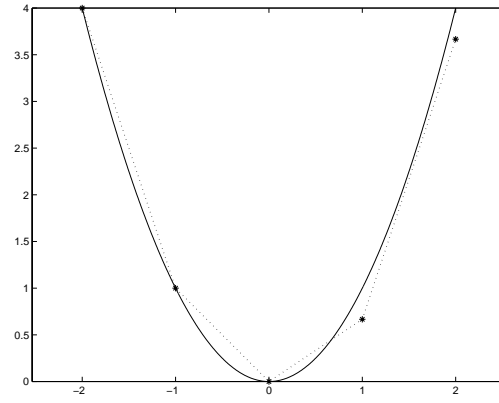


Fig. 3. The control points that yield the limit function $f(x) = x^2$

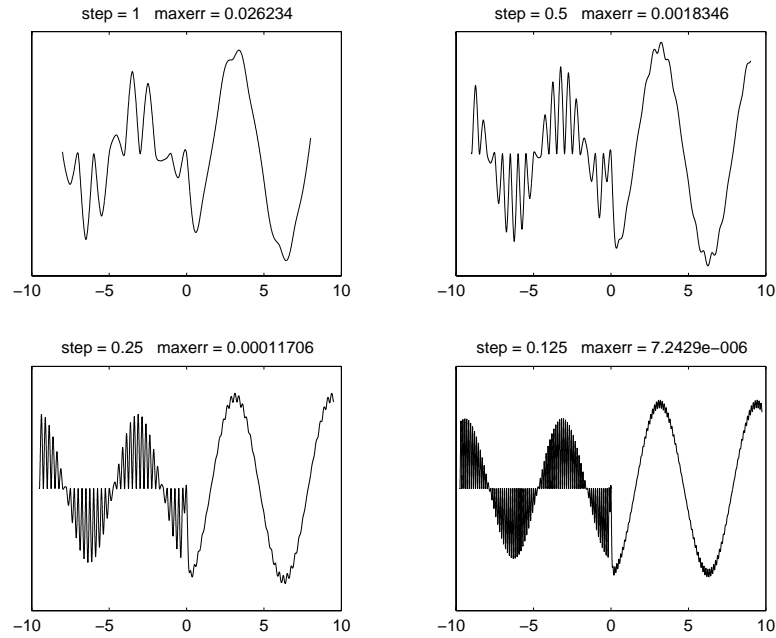


Fig. 4. The errors in approximation of $\cos(x)$ using $S^\infty Q \cos(x)$ with sampling steps from 1 to $\frac{1}{8}$.

tees the optimal order of approximation 4, namely the error bound decreases by 16 as n increases by 1. Although the shape of the error graph near the origin shows a special behavior, it is not of higher magnitude than the errors away from the origin. This can only be achieved by a careful choice of the subdivision rules near the origin.

4 Conclusion

The introduction of the operator Q , together with the identity (1) provide a method for designing new schemes with desired polynomial generation properties. As in the example given in §3.5, the relation $SQ = Q\sigma$ is viewed as a linear equation where the unknown is the subdivision scheme S . The solution of this equation yields a subdivision scheme S with the property that $S^\infty Q$ reproduces polynomials. The same method can be used to construct various non-uniform subdivision schemes.

In particular, our method was used in [18] to construct a bivariate piecewise-uniform subdivision scheme that operates over a grid which is half quadrilateral and half triangular.

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