Stability of Nonlinear Subdivision and Multiscale Transforms

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Abstract

Extending upon [6] and [3], we present a new general sufficient condition for the Lipschitz stability of nonlinear subdivision schemes and multiscale transforms in the univariate case. It covers the special cases (WENO, PPH) considered so far but also implies the stability in some new cases (median interpolating transform, power-p schemes, etc.). Although the investigation concentrates on multiscale transforms

$$\{v^0, d^1, \ldots, d^J\} \quad \longmapsto \quad v^J, \qquad J \geq 1,$$

in $\ell_\infty(\mathbb{Z})$ given by a stationary recursion of the form

$$v^j = Sv^{j-1} + d^j, \qquad j \geq 1,$$

involving a nonlinear subdivision operator S acting on $\ell_\infty(\mathbb{Z})$, the approach is extendable to other nonlinear multiscale transforms and norms, as well.

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1 Introduction

Subdivision methods and multiscale (or pyramid) transforms associated with a multiresolution analysis are workhorses in scientific computing, in particular, in data and geometry processing, for compression algorithms, etc.. They are well-investigated in the linear case when the components of the multiscale transform are linear, sparse maps. In this paper, we concentrate on stationary subdivision schemes

$$\mathbf{S} \; : \qquad v^0 \quad \longmapsto \quad \{v^0, v^1, \ldots, v^J, \ldots\}$$

associated with a generally nonlinear subdivision operator S acting on the sequence space $\ell_\infty(\mathbb{Z})$ given by the recursion

$$v^j = Sv^{j-1}, \qquad j \geq 1. \tag{1.1}$$

Such a subdivision scheme $\mathbf{S}$ is a special case of the synthesis part of a multiscale transform

$$\mathbf{M} \; : \qquad \{v^0, d^1, \ldots, d^J, \ldots\} \quad \longmapsto \quad \{v^0, v^1, \ldots, v^J, \ldots\}$$

given by

$$v^j = M(v^{j-1}, d^j) := Sv^{j-1} + Td^j. \tag{1.2}$$

The sequences d^j represent the detail information, and T is another bounded operator acting on $\ell_\infty(\mathbb{Z})$ (later, we will set $T = \mathrm{Id}$ for convenience). The analysis part

$$\{v^0, v^1, \ldots, v^J, \ldots\} \quad \longmapsto \quad \{v^0, d^1, \ldots, d^J, \ldots\}$$

of a complete multiscale transform is most often easy to understand, and will not be considered in this paper. Typical examples of such multiscale transforms are wavelet transforms, where S is related to the refinement equation for the associated scaling function and multiresolution analysis $\{V^j\}$, and T depends on the choice of wavelet basis in the complement spaces $\{W^j\}$, and multigrid schemes, where S models the coarse-to-fine prolongations and T the smoothers.

Considering nonlinearity in multiscale transforms is motivated by numerous case studies and applications that have appeared especially in the last decade. Many case studies are driven by the idea that efficiency in data and geometry processing can be gained by incorporating data information into multiscale transforms, i.e., into the intergrid information transfer. Almost any adaptive algorithm leads to nonlinearity. Another source is the processing of nonlinearly constrained data sets and numerical schemes for problems with manifold-valued solutions. Important applications that have emerged during the last 10 years are edge-encoding for images [1, 2], singularity capturing in numerical schemes for hyperbolic conservation laws [20, 28, 33], convexity and generally shape preserving schemes in geometry processing [27, 25, 29], normal multiresolution (an efficient geometry compression scheme based on a nonlinear multiscale transform) [17, 9, 23, 4], robust statistical methods [10, 21, 22], parametrization of manifolds and representation of data subject to nonlinear constraints [36, 35, 34], optimal models for high-dimensional data sets [7], etc. These few examples are by no means exhaustive, the references are selective, with focus on the emerging theoretical foundations for this field rather than on applications and algorithms. Some of them will be discussed in more detail in Sections 3 and 4.

Let us fix some notation. Let the integer $r > 1$ denote the so-called dilation factor. The vectors v^j given by (1.1) or (1.2) are identified with grid functions on the lattice $r^{-j}\mathbb{Z}$ or, equivalently, with a piecewise linear function f^j with knot values defined by the entries of v^j according to $f^j(ir^{-j}) = v^j_i$. Using this function setting, notions such as convergence and smoothness of the limit for $j \to \infty$ are introduced in a natural way. E.g., the subdivision scheme $\mathbf{S}$ converges if, for all initial $v^0 \in \ell_\infty(\mathbb{Z})$, the sequence f^j associated with $v^j = S^j v^0$ converges uniformly to a continuous function denoted by $f^\infty \in C(\mathbb{R})$ (this definition silently assumes that $v^0 \neq 0$ always implies $f^\infty \neq 0$). The smoothness exponent $s_\infty(\mathbf{S})$ of $\mathbf{S}$ is defined by

$$s_\infty(\mathbf{S}) := \sup\{s > 0 : \ f^\infty \in C^s(\mathbb{R}) \text{ for all } v^0 \in \ell_\infty(\mathbb{Z})\}, \tag{1.3}$$

where C^s is the space of functions of Hölder smoothness s. The quantity (1.3) describes the minimally guaranteed Hölder smoothness for the limits f^∞. Under reasonable conditions on the decay of the norms of the detail sequences d^j, the notions of convergence and smoothness for a subdivision scheme $\mathbf{S}$ carry over to the multiscale transform $\mathbf{M}$.

The basic questions of convergence and limit smoothness for $\mathbf{S}$ can be effectively answered for linear schemes if S is local and r-shift invariant, i.e., S is completely determined by a finitely supported sequence $\{a_k\}$ (the so-called mask) by setting

$$(Sv)_k = \sum_{\ell} a_{k-rl} v_l, \tag{1.4}$$

where the integer r is the dilation factor. A crucial necessary condition for such an S to lead to a convergent scheme is that it reproduces constant sequences: $S\mathbf{1} = \mathbf{1}$. The latter property implies the existence of a first derived subdivision scheme, given by another local, r-shift invariant subdivision operator $S^{[1]}$, such that $\Delta \circ S = S^{[1]} \circ \Delta$, where Δ is the forward difference operator given by by $(\Delta v)_i = v_{i+1} - v_i$, $i \in \mathbb{Z}$. Then $\mathbf{S}$ converges iff S reproduces constants and

$$\rho(S^{[1]}) := \liminf_{n\to\infty} \|(S^{[1]})^n\|_\infty^{1/n} < 1. \tag{1.5}$$

Moreover,

$$s_\infty(\mathbf{S}) \geq -\log_r(\rho(S^{[1]})) > 0.$$

More precise statements on the exact Hölder exponent are available, and involve higher-order derived schemes (e.g., $S^{[2]}$ can be defined recursively as the derived scheme of $S^{[1]}$, and so on). See, e.g., [5],[11],[13] for surveys on the basic theory of shift-invariant and local subdivision schemes.

Our main concern is the Lipschitz stability of $\mathbf{S}$ resp. $\mathbf{M}$ which (without reference to convergence) we define as follows. The multiscale transform is called Lipschitz stable (in $\ell_\infty(\mathbb{Z})$) if there is a constant C such that

$$\|v^J - \tilde{v}^J\|_\infty \leq C \left(\|v^0 - \tilde{v}^0\|_\infty + \sum_{j=1}^{J} \|d^j - \tilde{d}^j\|_\infty \right) \tag{1.6}$$

holds for all $J \geq 1$ and all data sequences $\{v^0, d^1, d^2, \ldots\}$ and $\{\tilde{v}^0, \tilde{d}^1, \tilde{d}^2, \ldots\}$, where $\tilde{v}^j = M(\tilde{v}^{j-1}, \tilde{d}^j)$, $j \geq 1$. Stability of $\mathbf{S}$ is defined in the same way, by setting $d^j = \tilde{d}^j = \mathbf{0}$ for all $j \geq 1$. Similar definitions have been given in [6, 30, 3]. We admit that requiring global Lipschitz stability in the form (1.6) is a very strong statement, especially for nonlinear scheme. However, having (1.6) provides a strong basis for ascertaining numerical stability and robustness of multiscale transforms and compression algorithms based on thresholding the detail vectors.

For linear schemes with bounded linear operators S and T convergence of $\mathbf{S}$ obviously implies stability of both $\mathbf{S}$, and $\mathbf{M}$, and (1.6) holds. However, for nonlinear schemes convergence of $\mathbf{S}$ does not necessarily imply stability (see the ENO scheme considered in [6]), and stability of $\mathbf{S}$ may not imply the stability of the associated multiscale transform $\mathbf{M}$ even for $T = \mathrm{Id}$ (an example will be given in Section 3). This and the fact that we knew only of two successful attempts to prove stability in a nonlinear situation (see [6] for the WENO subdivision scheme, and [26, 3] for the PPH multiscale transform) was the main motivation for us to consider the stability problem in more detail.

The paper is organized as follows. In Section 2, we state our main result, a new sufficient condition for stability of $\mathbf{S}$ and $\mathbf{M}$. The main difference with [6] is to formulate stability conditions in terms of spectral radii associated with the first derivative of nonlinear derived schemes $S^{[k]}$, $k \geq 1$. We highlight the difference between considering stability of $\mathbf{S}$ and $\mathbf{M}$, and argue that our sufficient conditions are close to optimal. It is also established that the stability theorem of [6] is essentially covered as a partial case. In Section 3 we demonstrate how to apply the abstract results to concrete situations. We were able to prove stability for the triadic median interpolating multiscale transform introduced in [10], the dyadic version from [31, 32] is also discussed (it provides an example where stability of $\mathbf{S}$ does not imply stability of $\mathbf{M}$). For the univariate power-p multiscale transform [33, 8] we establish stability for $1 \leq p < 8/3$, while for $p > 4$ stability cannot hold. This extends the result of [3] for the PPH scheme corresponding to the case $p = 2$ which was one of the motivating results for us. In the concluding Section 4 we briefly discuss possible extensions of our approach to normal multiresolution [24, 9] and multiscale transforms for manifold-valued functions [34, 36, 35, 15], mainly in order to point out some interesting features of nonlinear schemes that need further analysis. A major challenge ahead is the theoretical treatment of the multivariate case (first attempts [15, 8] to develop analysis tools have already been made for regular topologies and tensor-product schemes). Further progress could come from a closer interaction with the theory of dynamical systems, in particular, nonlinear iterated function systems.

2 Stability Theory

2.1 Notation and preliminary facts

In this and the following section we consider only stationary subdivision schemes $\mathbf{S}$ in $\ell_\infty(\mathbb{Z})$ given by a generally nonlinear operator $S: \ \ell_\infty(\mathbb{Z}) \to \ell_\infty(\mathbb{Z})$, and their associated multiscale transforms $\mathbf{M}$ given by $v^j := M(v^{j-1}, d^j) = Sv^{j-1} + d^j$, i.e., we set $T = \mathrm{Id}$ in (1.2). All norms used below are *infinity-norms* and hence we will simply denote them by $\|\cdot\|$. Whether the norm is on an operator, on a sequence, or on a function will be clear from the context.

Although some results are valid for more general S, we will silently assume that $S\mathbf{0} = \mathbf{0}$ (throughout the paper, $\mathbf{0}$ and $\mathbf{1}$ denote constant sequences consisting of zeros and ones, respectively), and that S is *local* and *r-shift invariant*. I.e., there are an integer $L \geq 0$ and functions $f_s: \ \mathbb{R}^{2L+1} \to \mathbb{R}$, $s = 0, 1, \ldots, r-1$, such that the action of S is given by

$$(Sv)_{ri+s} = f_s(v_{i-L}, \ldots, v_{i+L}), \qquad i \in \mathbb{Z}, \quad s = 0, \ldots, r-1. \tag{2.1}$$

Under the locality assumption, it is clear that S is well-defined on all real-valued sequences, and not just on $\ell_\infty(\mathbb{Z})$. The integer L characterizes the *support size* of S. Equivalently, r-shift invariance means that $\theta^r \circ S = S \circ \theta$, where θ is the shift operator given by $(\theta v)_i = v_{i+1}$.

Following [38] we will call S *off-set invariant* if $S(v + \alpha\mathbf{1}) = Sv + \alpha\mathbf{1}$ for all $v \in \ell_\infty(\mathbb{Z})$ and $\alpha \in \mathbb{R}$. Off-set invariance implies the reproduction of constant sequences, $S(\alpha\mathbf{1}) = \alpha\mathbf{1}$ (but not vice versa). Locality, r-shift invariance, and off-set invariance are often satisfied for practically useful nonlinear subdivision schemes.

Lemma 2.1 *If S is local, r-shift and off-set invariant, then there exists a unique local (with the same support size) and r-shift invariant operator $S^{[1]}$ (the so-called first-order derived subdivision operator associated with S) such that $\Delta \circ S = S^{[1]} \circ \Delta$.*

Proof. We give the explicit construction. For any fixed $i \in \mathbb{Z}$, write

$$v_{i+l} = v_i + w_{i+l}, \qquad w_{i+l} = \begin{cases} \sum_{m=0}^{l-1} (\Delta v)_{i+m}, & l > 0, \\ 0, & l = 0, \\ -\sum_{m=l}^{-1} (\Delta v)_{i+m}, & l < 0. \end{cases}$$

Now apply off-set invariance and (2.1):

$$(Sv)_{ri+s} = (S(w + v_i \mathbf{1}))_{ri+s} = v_i + f_s(w_{i-L}, \dots, w_{i+L}), \quad s = 0, \dots, r-1, \tag{2.2}$$

note that w depends on the arbitrarily fixed $i \in \mathbb{Z}$. From this, we see that

$$(\Delta Sv)_{ri+s} = f_{s+1}(w_{i-L}, \dots, w_{i+L}) - f_s(w_{i-L}, \dots, w_{i+L}) =: f_s^{[1]}((\Delta v)_{i-L}, \dots, (\Delta v)_{i+L-1})$$

for $s = 0, \dots, r-2$, and

$$\begin{aligned} (\Delta Sv)_{ri+r-1} &= (\Delta v)_i + f_0(-\sum_{m=L}^{-1} (\Delta v)_{i+1+m}, \dots, \sum_{m=0}^{L-1} (\Delta v)_{i+1+m}) - f_{r-1}(w_{i-L}, \dots, w_{i+L}) \\ &=: f_{r-1}^{[1]}((\Delta v)_{i-L}, \dots, (\Delta v)_{i+L}). \end{aligned}$$

This shows the claim (uniqueness is obvious).

The proof also shows that boundedness and differentiability properties of the functions f_s defining S via (2.1) automatically carry over to the functions $f_s^{[1]}$ defining $S^{[1]}$. The following statement is an immediate consequence of (2.2).

Lemma 2.2 *If S is local, r-shift and off-set invariant, with the functions f_s in (2.1) globally Lipschitz continuous then for any two sequences $v, \tilde{v} \in \ell_\infty(\mathbb{Z})$ we have*

$$\|Sv - S\tilde{v}\| \leq \|v - \tilde{v}\| + C\|\Delta(v - \tilde{v})\|.$$

By induction, one can introduce higher-order derived schemes $S^{[k]}$. More precisely, assume that S is local and r-shift invariant, and that we have already defined the derived subdivision operators $S^{[1]}, \dots, S^{[k-1]}$ for some $k \geq 2$. Then, if $S^{[k-1]}$ satisfies

$$S^{[k-1]}(w + \alpha \mathbf{1}) = S^{[k-1]} w + \frac{\alpha}{r^{k-1}} \mathbf{1} \qquad \forall w \in \ell_\infty(\mathbb{Z}),$$

using the same argument as in Lemma 2.1 we can construct a unique local and r-shift invariant operator $S^{[k]}$ such that $\Delta^k \circ S = S^{[k]} \circ \Delta^k$. We note without proof that the existence of $S^{[k]}$ is equivalent to requiring the following *off-set invariance condition for polynomials of order k* for S:

$$(S(v + p))_i = (Sv)_i + p(i/r) + q(i), \qquad i \in \mathbb{Z}, \tag{2.3}$$

holds for all $v \in \ell_\infty(\mathbb{Z})$, all polynomial sequences p generated by a polynomial $p(\cdot)$ of degree $k-1$ (i.e., given by $p_i = p(i)$, $i \in \mathbb{Z}$), with some other polynomial $q(\cdot)$ of degree less than $k-1$ (whose coefficients may depend on v and p). A less stringent condition requiring (2.3) only for $v = \mathbf{0}$ is used in [6, 30] in connection with the study of quasi-linear subdivision schemes but does not imply the existence of the nonlinear derived subdivision operator $S^{[k]}$ as introduced above. Both definitions are natural extensions of the notion of polynomial reproduction of order k for linear subdivision schemes. Unfortunately, derived subdivision operator $S^{[k]}$ rarely exist for larger values of k. In many practical cases, although true for $k = 1$, (2.3) already fails for $k = 2$. For some convexity-preserving schemes such as power-p methods, also $S^{[2]}$ exists. Despite these observations, we formulate our results for general k.

Throughout the remainder of the paper a subdivision operator $S : \ \ell_\infty(\mathbb{Z}) \to \ell_\infty(\mathbb{Z})$ will be called *k-continuous* (resp. *k-Lipschitz*, resp. *k-differentiable*) if S is local, r-shift invariant, off-set invariant for polynomials of order $\le k$, satisfies $S\mathbf{0} = \mathbf{0}$, and the functions f_s which define S via (2.1) (and thus all $f_s^{[m]}$ defining the derived subdivision operators $S^{[m]}$, $m = 1, \dots, k$) are continuous (resp. globally Lipschitz, resp. continuously differentiable with bounded partial derivatives). Some comments are in order. First, k-differentiability implies the k-Lipschitz property, and the latter implies k-continuity. Also, since $S\mathbf{0} = \mathbf{0}$, a k-Lipschitz S is automatically bounded, in the sense that $\|S^{[m]}v\| \le C\|v\|$ for all v and $m = 0, \dots, k$. Secondly, many of the used subdivision schemes in practice are k-Lipschitz, but not k-differentiable. However, since the conditions of the stability theorem below require precise estimates for Lipschitz constants, additionally assuming the existence (in some meaningful sense) of continuous first-order derivatives for $f_s^{[k]}$ would simplify this task. The technicalities triggered by the fact that the functions $f_s^{[k]}$ possess only piecewise continuous partial derivatives will be discussed in Subsection 2.3 below. Finally, an extension to only locally k-Lipschitz S and to $\ell_p(\mathbb{Z})$ norms for $1 \le p < \infty$ (though possible, see [6, 30]) is not included below, partly because the considered examples do not require this generalization but also to keep the exposition more readable.

2.2 Main stability theorem

In this section, we formulate an abstract condition for Lipschitz stability of $\mathbf{S}$ and $\mathbf{M}$ which formally does not require any additional conditions on the subdivision operator. It is the result of a thorough analysis of the stability theorem for subdivision schemes in [6], and the proof of stability for the PPH multiscale transform in [3].

Theorem 2.3 *Let $S : \ell_\infty(\mathbb{Z}) \to \ell_\infty(\mathbb{Z})$ be a given subdivision operator. Assume that for some nonnegative constants $C_0, C_1 \in \mathbb{R}$, some $\rho \in (0,1)$, and finite $k, n \in \mathbb{N}$, the inequalities*

$$\|Sv - S\tilde{v}\| \le \|v - \tilde{v}\| + C_0\|\Delta^k(v - \tilde{v})\|, \tag{2.4}$$

$$\|\Delta^k(v^n - \tilde{v}^n)\| \le \rho\|\Delta^k(v^0 - \tilde{v}^0)\| + C_1 \sum_{j=1}^{n} \|d^j - \tilde{d}^j\|, \tag{2.5}$$

hold for arbitrary $v, \tilde{v} \in \ell_\infty(\mathbb{Z})$ respectively for two arbitrary sets of multiscale data $\{v^0, d^1, \dots, d^j, \dots\}$, $\{\tilde{v}^0, \tilde{d}^1, \dots, \tilde{d}^J, \dots\}$, with their multiscale transforms defined by $v^j = Sv^{j-1} + d^j$, $\tilde{v}^j = S\tilde{v}^{j-1} + \tilde{d}^j$, $j \ge 1$. Then $\mathbf{M}$ is stable, i.e.,

$$\|v^J - \tilde{v}^J\| \le C_2\|v^0 - \tilde{v}^0\| + C_3 \sum_{j=1}^{J} \|d^j - \tilde{d}^j\| \tag{2.6}$$

holds with constants C_2, C_3 which depend on k, n, ρ, C_0, and C_1, but not on $J \geq 1$.
To obtain the stability result for the associated subdivision scheme $\mathbf{S}$*, set* $d^j = \tilde{d}^j = \mathbf{0}$ *in the above formulations.*

Proof. Fix $0 \leq j \leq J-1$. (2.4) gives rise to

$$\begin{aligned}\|v^{j+1} - \tilde{v}^{j+1}\| &= \|Sv^j + d^{j+1} - (S\tilde{v}^j + \tilde{d}^{j+1})\| \leq \|Sv^j - S\tilde{v}^j\| + \|d^{j+1} - \tilde{d}^{j+1}\| \\ &\leq \|v^j - \tilde{v}^j\| + C_0\|\Delta^k(v^j - \tilde{v}^j)\| + \|d^{j+1} - \tilde{d}^{j+1}\|.\end{aligned}$$

Applying this result iteratively J times we derive

$$\begin{aligned}\|v^J - \tilde{v}^J\| &\leq \|v^{J-1} - \tilde{v}^{J-1}\| + C_0\|\Delta^k(v^{J-1} - \tilde{v}^{J-1})\| + \|d^J - \tilde{d}^J\| \\ &\leq \|v^0 - \tilde{v}^0\| + \underbrace{C_0 \sum_{i=0}^{J-1} \|\Delta^k(v^i - \tilde{v}^i)\|}_{:=A} + \sum_{i=1}^{J} \|d^i - \tilde{d}^i\|.\end{aligned} \tag{2.7}$$

Now to prove our theorem, it suffices to show that A can be estimated by the expression in the right-hand side of (2.6). Let $s := \lfloor i/n \rfloor$. Then (2.5) implies

$$\|\Delta^k(v^i - \tilde{v}^i)\| \leq \rho^s \|\Delta^k(v^{i-sn} - \tilde{v}^{i-sn})\| + C_1 \sum_{r=0}^{s-1} \rho^r \sum_{t=rn}^{(r+1)n-1} \|d^{i-t} - \tilde{d}^{i-t}\|, \tag{2.8}$$

and after summation and using $\rho < 1$ we derive

$$A \leq C(\rho)\Big(\sum_{j=0}^{n-1} \|\Delta^k(v^j - \tilde{v}^j)\| + \sum_{j=1}^{J} \|d^j - \tilde{d}^j\|\Big).$$

To estimate $\|\Delta^k(v^j - \tilde{v}^j)\|$ for $j = 1, \dots, n-1$, we use (2.4):

$$\|Sv^{j-1} - S\tilde{v}^{j-1}\| \leq \|v^{j-1} - \tilde{v}^{j-1}\| + C_0\|\Delta^k(v^{j-1} - \tilde{v}^{j-1})\| \leq (2^k C_0 + 1)\|v^{j-1} - \tilde{v}^{j-1}\|,$$

which, applied j times, gives rise to

$$\begin{aligned}\|\Delta^k(v^j - \tilde{v}^j)\| &\leq 2^k\|Sv^{j-1} - S\tilde{v}^{j-1}\| + 2^k\|d^j - \tilde{d}^j\| \\ &\leq 2^k(2^k C_0 + 1)\|v^{j-1} - \tilde{v}^{j-1}\| + 2^k\|d^j - \tilde{d}^j\| \\ &\leq 2^k(2^k C_0 + 1)^j\|v^0 - \tilde{v}^0\| + 2^k \sum_{i=0}^{j-1} (2^k C_0 + 1)^i \|d^{j-i} - \tilde{d}^{j-i}\|.\end{aligned}$$

Thus,

$$\|\Delta^k(v^j - \tilde{v}^j)\| \leq 2^k(2^k C_0 + 1)^{n-1}\|v^0 - \tilde{v}^0\| + 2^k(2^k C_0 + 1)^{n-2} \sum_{i=1}^{n-1} \|d^i - \tilde{d}^i\|. \tag{2.9}$$

Combining (2.7), (2.8), (2.9) and

$$\sum_{i=0}^{J-1} \rho^s \leq \sum_{i=0}^{\infty} \rho^s = \sum_{i=0}^{\infty} \rho^{\lfloor i/n \rfloor} = n \sum_{i=0}^{\infty} \rho^i = \frac{n}{1-\rho},$$

we derive

$$\|v^J - \tilde v^J\| \le \|v^0 - \tilde v^0\| + C_0 \sum_{i=0}^{J-1} \rho^s \Big(2^k (2^k C_0 + 1)^{n-1} \|v^0 - \tilde v^0\| + 2^k (2^k C_0 + 1)^{n-2} \sum_{j=1}^{n-1} \|d^j - \tilde d^j\| \Big)$$
$$+ C_1 \sum_{i=1}^{J} \sum_{r=0}^{s-1} \rho^r \sum_{t=rn}^{(r+1)n-1} \|d^{i-t} - \tilde d^{i-t}\| + \sum_{i=1}^{J} \|d^i - \tilde d^i\|$$
$$\le \Big(1 + \frac{2^k C_0 (2^k C_0 + 1)^{n-1} n}{1-\rho} \Big) \|v^0 - \tilde v^0\| + C_3 \sum_{j=1}^{J} \|d^j - \tilde d^j\|.$$

The constant C_3 is finite, and does not depend on J because one can easily check that for any fixed $1 \le j \le J$ the coefficient in front of $\|d^j - \tilde d^j\|$ is of the same type as the coefficient in front of $\|v^0 - \tilde v^0\|$, i.e., a finite sum of geometric series with respect to ρ times some uniformly bounded constants. □

Let us comment on the assumptions in Theorem 2.3. First of all, the validity of the conditions of this statement automatically implies convergence of **S** and **M**. This can be seen if one sets $\tilde v_0 = \tilde d^1 = \tilde d^2 = \ldots = \mathbf{0}$, and compares with the statements in [6] or [31]. The constant $\rho < 1$ also provides a lower bound for the Hölder smoothness of the limiting functions corresponding to **S** and **M** (to speak about convergence of the multiscale transform **M**, a natural sufficient condition is to require that $\sum_{j=1}^{\infty} \|d^j\| < \infty$).

The condition (2.4) depends only on the subdivision operator S, and is usually easy to verify. Due to Lemma 2.1, this condition automatically holds with $k = 1$ for any 1-Lipschitz S. A similar statement can be obtained for $k = 2$ (assuming that S is 2-Lipschitz). However, it is easy to show that even in the linear case (2.4) cannot be fulfilled with $k \ge 3$, unless the mask of S is non-negative and S has order of polynomial reproduction at least k [19]. A reformulation of condition (2.4) so that it holds in a sensible way for all k might be possible but since in case studies we have so far always used Theorem 2.3 with either $k = 1$ or $k = 2$, we will not stress this issue further.

2.3 Nonlinear spectral radius conditions

The crucial second condition (2.5) in Theorem 2.3 is, as a rule, harder to verify than (2.4). If formulated for the case of subdivision stability ($d^j = \tilde d^j = \mathbf{0}$) it essentially represents a contraction property, similar to (1.5). More precisely, if S is k-Lipschitz, i.e., if $S^{[k]}$ exists and all the $f_s^{[k]}$, $s = 0, \ldots, r-1$ that define it are Lipschitz continuous, the condition (2.5) just states that some power $(S^{[k]})^n$ is globally strictly contracting, i.e., $(S^{[k]})^n$ possesses a Lipschitz constant strictly less than one for some finite n. In this subsection we present a convenient reformulation of the contraction property (2.5), based on nonlinear spectral radii.

Let us first consider the case when the subdivision operator S is k-differentiable. Let two sets of multiscale data as in Theorem 2.3 be given. Define absolutely continuous paths in $\ell_\infty(\mathbb{Z})$

$$\gamma^0(t) := \Delta^k (t v^0 + (1-t) \tilde v^0), \qquad \delta^j(t) := \Delta^k (t d^j + (1-t) \tilde d^j), \quad j \ge 1, \quad t \in [0,1],$$

with their k-th order differences as endpoints. Obviously, the derivatives

$$\frac{\mathrm{d}\gamma^0}{\mathrm{d}t} = \Delta^k(v^0 - \tilde{v}^0), \qquad \frac{\mathrm{d}\delta^j}{\mathrm{d}t} = \Delta^k(d^j - \tilde{d}^j), \quad , j \geq 1,$$

are constant (and thus uniformly bounded in $\ell_\infty(\mathbb{Z})$) on $[0,1]$. For each $t \in [0,1]$, consider the k-th-order derived analog of the multiscale transform $\mathbf{M}$ of $\{\gamma^0(t), \ldots, \gamma^j(t), \ldots\}$:

$$\gamma^j(t) = M^{[k]}(\gamma^{j-1}(t), \delta^j(t)) := S^{[k]}\gamma^{j-1}(t) + \delta^j(t), \qquad j \geq 1.$$

The newly created paths $\gamma^j : [0,1] \to \ell_\infty(\mathbb{Z})$ remain absolutely continuous, with derivatives belonging to L_∞ and a.e. given by the formula

$$\begin{aligned} \frac{\mathrm{d}\gamma^j}{\mathrm{d}t} &= DS^{[k]}_{\gamma^{j-1}(t)} \frac{\mathrm{d}\gamma^{j-1}}{\mathrm{d}t} + \frac{\mathrm{d}\delta^j}{\mathrm{d}t} \\ &= DS^{[k]}_{\gamma^{j-1}(t)} \frac{\mathrm{d}\gamma^{j-1}}{\mathrm{d}t} + \Delta^k(d^j - \tilde{d}^j), \end{aligned}$$

where $DS^{[k]}_\gamma$ is a bounded linear operator on $\ell_\infty(\mathbb{Z})$ representing the Frechet derivative of $S^{[k]}$ at γ.

The above recursive formula leads to the introduction of a nonlinear spectral radius associated with $\mathbf{M}$ (more precisely with the associated k-th order derived transform $M^{[k]}$) which we define as follows

$$\rho_s(\mathbf{M}, k) := \liminf_{j\to\infty} \sup_{(w^0,w^1,\ldots,w^{j-1})\in(\ell_\infty(\mathbb{Z}))^j} \|DS^{[k]}_{w^{j-1}} DS^{[k]}_{w^{j-2}} \ldots DS^{[k]}_{w^0}\|^{1/j}. \tag{2.10}$$

Note that in this definition the supremum is taken over all possible choices of w^l.

For $\mathbf{S}$ it is more appropriate to set

$$\rho_s(\mathbf{S}, k) := \liminf_{j\to\infty} \sup_{w\in\ell_\infty(\mathbb{Z})} \|DS^{[k]}_{(S^{[k]})^{j-1}w} DS^{[k]}_{(S^{[k]})^{j-2}w} \ldots DS^{[k]}_{w}\|^{1/j}. \tag{2.11}$$

To see the difference with (2.10), set $w^0 = w$, $w^l = (S^{[k]})^l w$, $l \geq 1$, and note that the w^l are now not arbitrary but depend on a single $w \in \ell_\infty(\mathbb{Z})$ (and the supremum is only taken with respect to the latter). Thus, we generally have $\rho_s(\mathbf{S}, k) \leq \rho_s(\mathbf{M}, k)$ but in many cases the inequality is strict.

Similar spectral radii (based on $S^{[k]}$ rather than on $DS^{[k]}$) have been defined in [6, 30] and [31] for the investigation of convergence and smoothness of nonlinear univariate subdivision schemes $\mathbf{S}$. In both papers, the assumption was that, for some $k \geq 1$ and all $l = 0, \ldots, k$, there exist families of data-dependent bounded linear subdivision operators $\{S^{[l]}_v\}_{v\in\ell_\infty(\mathbb{Z})}$ on $\ell_\infty(\mathbb{Z})$, such that

$$\Delta^l(Sv) = S^{[l]}_v \Delta^l v.$$

This is weaker than assuming the existence of nonlinear l-th order derived operators $S^{[l]}$ for all $l \leq k$ because despite the fact that by definition

$$\Delta^l(Sv) = S^{[l]}w = S^{[l]}_v w, \qquad w = \Delta^l v,$$

the allowed dependence of $S^{[l]}_v$ on v and not only on $w = \Delta^l v$ gives greater flexibility (but also adds ambiguity). In [6, 30], the nonlinear spectral radius

$$\rho_c(\mathbf{M}, k) := \liminf_{j\to\infty} \sup_{v^0,\ldots,v^{j-1}\in\ell_\infty(\mathbb{Z})} \|S^{[k]}_{v^{j-1}} S^{[k]}_{v^{j-2}} \ldots S^{[k]}_{v^0}\|^{1/j},$$

was introduced while in [31]

$$\rho_c(\mathbf{S},k) := \liminf_{j\to\infty} \sup_{v\in\ell_\infty(\mathbb{Z})} \|S^{[k]}_{S^{j-1}v} S^{[k]}_{S^{j-2}v} \dots S^{[k]}_{v}\|^{1/j},$$

was proposed. Assuming $\rho_c(\mathbf{M},k) < 1$ is a sufficient condition for convergence of the subdivision scheme $\mathbf{S}$, and the value of $-\log_r(\rho_c(\mathbf{M},k))$ provides a lower bound for the Hölder smoothness of the limit function (certainly, the weaker assumption $\rho_c(\mathbf{S},k) < 1$ yields the same conclusions). Following the proof in [6], it is not hard to verify that $\rho_c(\mathbf{M},k) < 1$ also ensures the convergence of the multiscale transform $\mathbf{M}$ for arbitrary data $\{v^0, d^1, d^2, \dots\}$ if $\sum_{j=1}^\infty \|d^j\| < \infty$.

Having introduced the notions of joint spectral radii, we can prove a spectral radius version of Theorem 2.3.

Theorem 2.4 *Let S be k-differentiable, and assume that (2.4) is satisfied with this $k \geq 1$. Then*

(i) $\mathbf{S}$ *is stable if* $\rho_s(\mathbf{S},k) < 1$,

(ii) $\mathbf{M}$ *is stable if* $\rho_s(\mathbf{M},k) < 1$.

Proof. Using the notation above, for any two sets of multiscale data $\{v^0, d^1, \dots, d^j, \dots\}$ and $\{\tilde v^0, \tilde d^1, \dots, \tilde d^J, \dots\}$, we can write

$$\begin{aligned}
\Delta^k(v^n - \tilde v^n) &= \gamma^n(1) - \gamma^n(0) = \int_0^1 \frac{\mathrm{d}\gamma^n}{\mathrm{d}t}\,\mathrm{d}t \\
&= \int_0^1 \left(DS^{[k]}_{\gamma^{n-1}(t)} \frac{\mathrm{d}\gamma^{n-1}}{\mathrm{d}t} + \Delta^k(d^n - \tilde d^n)\right)\mathrm{d}t \\
&= \int_0^1 \left(DS^{[k]}_{\gamma^{n-1}(t)} DS^{[k]}_{\gamma^{n-2}(t)} \frac{\mathrm{d}\gamma^{n-2}}{\mathrm{d}t} + DS^{[k]}_{\gamma^{n-1}(t)}\Delta^k(d^{n-1} - \tilde d^{n-1}) + \Delta^k(d^n - \tilde d^n)\right)\mathrm{d}t \\
&= \dots = \int_0^1 \left(\Pi_0^{n-1}(t)\Delta^k(v^0 - \tilde v^0) + \sum_{j=1}^n \Pi_j^{n-1}(t)\Delta^k(d^j - \tilde d^j)\right)\mathrm{d}t,
\end{aligned}$$

where for short $\Pi_j^{n-1}(t) = DS^{[k]}_{\gamma^{n-1}(t)} DS^{[k]}_{\gamma^{n-2}(t)} \dots DS^{[k]}_{\gamma^{j}(t)}$ for $j \leq n-1$ (for $j = n$ it is just the identity operator). Thus, we have

$$\begin{aligned}
\|\Delta^k(v^n - \tilde v^n)\| &\leq (\max_{t\in[0,1]} \|\Pi_0^{n-1}(t)\|)\|\Delta^k(v^0 - \tilde v^0)\| \\
&\qquad + 2^k \sum_{j=1}^n (\max_{t\in[0,1]} \|\Pi_j^{n-1}(t)\|)\|d^j - \tilde d^j\|.
\end{aligned}$$

Obviously, if $\rho_s(\mathbf{M},k) < 1$, then for any $\rho_s(\mathbf{M},k) < \tilde\rho < 1$, there is a constant $C = C_{\tilde\rho}$ such that

$$\|DS^{[k]}_{w^{j-1}} DS^{[k]}_{w^{j-2}} \dots DS^{[k]}_{w^0}\| \leq C\tilde\rho^j,$$

independently of the choices for w^l. In particular, $\max_{t\in[0,1]} \|\Pi_j^{n-1}(t)\| \leq C\hat\rho^{n-j}$ for all $j = 0, \dots, n$, and (2.5) follows by choosing n such that $\rho := C\tilde\rho^n < 1$.

The same consideration applies to the case $d^j = \tilde d^j = \mathbf{0}$ in (2.5), i.e., if only the stability of $\mathbf{S}$ is of concern.

Now the result follows from Theorem 2.3 $\square$

Remark 2.5 *The spectral radius conditions of Theorem 2.4 are close to optimal, i.e., when S is k-differentiable, the opposite (strict) inequality $\rho_s(\mathbf{M}, k) > 1$ implies the existence of counterexamples to the stability inequality of the multiscale transform $\mathbf{M}$ while $\rho_s(\mathbf{S}, k) > 1$ implies that stability for $\mathbf{S}$ should not be expected, i.e.,*

$$\sup_{v,\tilde{v}\in\ell_\infty(\mathbb{Z}),v\neq\tilde{v}} \frac{\|S^J v - S^J\tilde{v}\|}{\|v-\tilde{v}\|} \longrightarrow \infty, \qquad J\to\infty.$$

We sketch the argument for $\mathbf{S}$. If $\rho_s(\mathbf{S}, k) > 1$ then for each large enough J, there is a sequence $w \in \ell_\infty(\mathbb{Z})$ such that

$$\|DS^{[k]}_{w^{J-1}}\dots DS^{[k]}_{w^0}\| > \hat{\rho}^J,$$

where $w^j = (S^{[k]})^j w$, $j = 0,\dots,J-1$, and $\hat{\rho}$ is chosen such that $1 < \hat{\rho} < \rho_s(\mathbf{S}, k)$. By the definition of the operator norm this means that there is a $u \in \ell_\infty(\mathbb{Z})$ with unit norm $\|u\| = 1$ such that

$$\|u^J\| > \hat{\rho}^J, \qquad u^J := DS^{[k]}_{w^{J-1}}\dots DS^{[k]}_{w^0}u.$$

More precisely, without loss of generality we can assume that there is an index $l_J \in \mathbb{Z}$ such that $(u^J)_{l_J} > \hat{\rho}^J$. Using the locality and r-shift-invariance of S (and thus of $S^{[k]}$ and $DS^{[k]})_w$ as well, and the continuity of all $Df^{[k]}_s$ it is clear that we have a similar inequality $(u^J)_{l_J} > \frac{1}{2}\hat{\rho}^J$ for all w in a small open neighborhood W of the initial w, and that this still holds true if we change the entries w_l and u_l of any of these sequences outside a fixed interval $l_0 - L_1 \le l \le l_0 + L_1$ of indices (without loss of generality, we can assume that $\max\limits_{l_0-L_1\le l\le l_0+L_1} |u_l| = 1$). By using a trivial continuation argument, this allows us to choose the sequences $v, z \in \ell_\infty(\mathbb{Z})$ such that $(\Delta^k v - w)_l = (\Delta^k z - u)_l = 0$ for all l in this specified interval, and $\|z\| \le A\|\Delta^k z\| \le A$ holds with some absolute constant A.

Set $\tilde{v} := v + \lambda z$, and consider $v(t) := v + t(\tilde{v} - v) = v + t\lambda z$. Fix $\lambda > 0$ small enough so that $w(t) := \Delta^k v(t) = w + t\lambda u$ is completely in W. Define $v^j(t) = S^j v(t)$, $v^j = v^j(0)$, $\tilde{v}^j = v^j(1)$, and $w^j(t) = \Delta^k v^j(t) = (S^{[k]})^j w(t)$, $j \ge 0$. Then

$$\begin{aligned}
\|v^J - \tilde{v}^J\| &\ge 2^{-k}\|\Delta^k v^J(1) - \Delta^k v^J(0)\| = 2^{-k}\left\|\int_0^1 \frac{\mathrm{d}w^J(t)}{\mathrm{d}t}\mathrm{d}t\right\| \\
&\ge \lambda 2^{-k}\left|\int_0^1 (\underbrace{(DS^{[k]})_{w^{J-1}(t)}\dots(DS^{[k]})_{w^0(t)}u}_{=u^J(t)})_{l_J}\,\mathrm{d}t\right| \\
&\ge \lambda 2^{-k-1}\hat{\rho}^J \ge \lambda A^{-1}2^{-k-1}\hat{\rho}^J\|z\| \ge c\hat{\rho}^J\|v^0 - \tilde{v}^0\|.
\end{aligned}$$

For $J \to \infty$, this shows that $\mathbf{S}$ cannot be stable.

The adjustment for the case of $\mathbf{M}$ is simple. For given w^j (which are now, in contrast to the previous case, not related to each other) such that

$$\|DS^{[k]}_{w^{J-1}}\dots DS^{[k]}_{w^0}\| > \hat{\rho}^J,$$

where $1 < \hat{\rho} < \rho_s(\mathbf{M}, k)$, we choose $\delta^j := w^j - S^{[k]}w^{j-1}$ such that $w^j = M^{[k]}(w^{j-1}, \delta^j)$ for all $j \ge 1$. Then, using the same localization trick, we introduce the appropriate sequences z, $u = \Delta^k z$, v^j, and d^j such that $\Delta^k v^j = w^j$, $\Delta^k d^j = \delta^j$, $\|z\| \le A\|\Delta^k z\| \le A$. With this, considering the two sets of multiscale data

$$\{v^0, d^1, \ldots, d^J\}, \qquad \{v^0 + \lambda z, d^1, \ldots, d^J\}, \quad \lambda > 0,$$

will establish the failure of Lipschitz stability for $\mathbf{M}$.

Note that the counterexample for $\mathbf{M}$ is rather exotic, as the detail sequences d^j are determined from the w^j via the formulas $\Delta^k d^j = \delta^j = w^j - S^{[k]} w^{j-1}$. Thus, a property such as $\|d^j\| \to 0$ which is often part of the application scenario, does not necessarily hold for the above constructed counterexample. It is a topic of future research to specify the notion of stability for multiscale transforms in such a way that available a priori information on the details d^j is properly taken into account.

Remark 2.6 *For k-differentiable S, Theorem 2.4 implies the stability result for $\mathbf{S}$ in [6].*

We will concentrate on the case $k = 1$, i.e., we will assume that S is 1-differentiable, and sketch the verification of $\rho_s(\mathbf{S}, 1) < 1$ from the assumptions of [6] (as was mentioned above, for $k = 1$ the first condition (2.4) is almost automatic). The stability theory in [6] assumes the following. Let the subdivision operator S preserve at least constant sequences, and write it in the form $Sv = S_v v$, where $\{S_v\}_{v \in \ell_\infty(\mathbb{Z})}$ is the above-mentioned family of bounded linear but data-dependent operators also preserving constants. Concerning the resulting family of first-order derived operators $\{(S_v)^{[1]}\}_{v \in \ell_\infty(\mathbb{Z})}$, two assumptions are made in [6]:

$$\|(S_v)^{[1]} - (S_{\tilde{v}})^{[1]}\| \le C\|v - \tilde{v}\|, \quad \forall\, v, \tilde{v} \in \ell_\infty(\mathbb{Z}), \tag{2.12}$$

and

$$\rho_c(\mathbf{M}, 1) < 1. \tag{2.13}$$

Obviously, (2.13) implies $\|(S_{v^{j-1}})^{[1]} \ldots (S_{v^0})^{[1]}\| \le C\tilde{\rho}^j$, with some $\tilde{\rho} < 1$, some constant C, and all $j \ge 1$. Since S is 1-differentiable, having that $DS_w^{[1]}$ is the derivative of $S^{[1]}$ at $w = \Delta v$, we can use $S^{[1]} w = S_v^{[1]} w$ and formally differentiate:

$$DS_w^{[1]} = D(S_v^{[1]})_w w + S_v^{[1]}.$$

It is not hard to see that due to the locality and r-shift invariance of S, the definition of $S_v^{[1]}$, and the differentiability assumptions on the $f_s^{[1]}$, condition (2.12) implies the boundedness of $D(S_v^{[1]})_w$ such that

$$\|D(S_v^{[1]})_w w\| \le C\|w\|$$

holds. Now apply this to estimating the operator products in the definition of $\rho_s(\mathbf{S}, 1)$. Set $v^j = S^j v$, $w^j = \Delta v^j$. Then

$$\begin{aligned}
\Pi_j \;&:=\; DS^{[1]}_{w^{j-1}} \ldots DS^{[1]}_{w^0} = ((S_{v^{j-1}})^{[1]} + E_{j-1}) \ldots ((S_{v^0})^{[1]} + E_0) \\
&=\; \sum_{l=0}^{j-1} (S_{v^{j-1}})^{[1]} \ldots (S_{v^{l+1}})^{[1]} E_l \Pi_l + (S_{v^{j-1}})^{[1]} \ldots (S_{v^0})^{[1]},
\end{aligned}$$

where $E_l = D(S^{[1]}_{v^l})_{w^l} w^l$ is a small perturbation since

$$\|E_l\| \leq C\|w^l\| = C\|\Delta v^l\| \leq C\tilde{\rho}^l$$

according to the convergence theory. Moreover,

$$\|(S_{v^{j-1}})^{[1]} \ldots (S_{v^{l+1}})^{[1]}\| \leq C\tilde{\rho}^{j-l-1},$$

and using the notation $A_l := \Pi_l$, $B_j = 1 + \sum_{l=0}^{j} A_l$, we obtain the recursions

$$A_j \leq C\tilde{\rho}^j(1 + \sum_{l=0}^{j-1} A_l) = C\tilde{\rho}^j B_{j-1}, \qquad B_j \leq (1 + C\tilde{\rho}^j)B_{j-1}, \qquad j \geq 0.$$

This gives $B_j \leq C$, and eventually $A_j = \|\Pi_j\| \leq C\tilde{\rho}^j$, independently of the choice of $v = v^0$. Thus, $\rho_s(\mathbf{S}, 1) \leq \tilde{\rho} < 1$ as well. A closer look at the argument also shows that (2.13) can be replaced by the weaker condition $\rho_c(\mathbf{S}, 1) < 1$. The weakness of the stability theory in [6] is the additional condition (2.12) which is rarely satisfied for nonlinear subdivision schemes. In particular, it is not true for the two schemes considered below. Details of the argument will be given for the power-p schemes in Section 3.2.

The main drawback of Theorem 2.4 is that it cannot be directly applied in practice. In most of the schemes, the functions f_s that define them are only piecewise continuously differentiable. In order to apply the developed machinery to k-Lipschitz schemes of this type, we need an extension of the chain rule for the superposition $\tilde{\gamma} := \phi \circ \gamma : [0,1] \to \mathbb{R}$ of a Lipschitz curve $\gamma : [0,1] \to \mathbb{R}^m$ and a piecewise C^1 Lipschitz function $\phi : \mathbb{R}^m \to \mathbb{R}$. Here m is some fixed integer which may vary from application to application. Obviously, $\tilde{\gamma}$ is again Lipschitz, and thus possesses a derivative $\tilde{\gamma}'$ a.e. on $[0,1]$. The question of concern is whether this derivative can be computed a.e. from derivatives of γ and ϕ in a meaningful way. Since the image of γ in $\mathbb{R}^m$ has measure zero this question does not have a simple answer if no additional conditions on ϕ are made.

We introduce a class C^1_{pw} of piecewise differentiable Lipschitz functions which is sufficiently broad to cover the applications of this paper (as a matter of fact, we do not know off-hand if there are interesting examples that require more than what is proved below). A continuous function $\phi : \mathbb{R}^m \to \mathbb{R}$ belongs to C^1_{pw} if there exists a locally finite polyhedral partition $\{\Omega_i\}$ of $\mathbb{R}^m$ such that the following conditions hold:

(a) Each Ω_i is a closed connected polyhedral domain in $\mathbb{R}^m$, with non-empty interior Ω_i^0. The polyhedra Ω_i may be unbounded and non-convex. The pairwise intersections $\Omega_{ij} := \Omega_i \cap \Omega_j$, $i \neq j$, are or empty, or represent d-dimensional polyhedral faces, $d = 0, 1, \ldots, m-1$, with non-empty interior Ω_{ij}^0 as subsets of the associated d-dimensional hyperplane (if $d = 0$, i.e., if Ω_{ij} is a point, then we set $\Omega_{ij}^0 = \Omega_{ij}$). Consequently, if we set $\Omega_{ii} = \Omega_i$ then $\{\Omega_{ij}^0\}$ is a partition of $\mathbb{R}^m$ into pairwise disjoint sets.

(b) The restriction of ϕ to any of the non-empty open sets $\Omega_{ii}^0 = \Omega_i^0$ and Ω_{ij}^0, $i \neq j$, is C^1, and a uniform bound on all occurring derivatives exists.

For such ϕ, we call $D\phi : \mathbb{R}^m \to \mathbb{R}^m$ an admissible derivative if $D\phi|_{\Omega_{ij}^0}$ is continuous, and for any $x \in \Omega_{ij}^0$ the directional derivative of $\phi|_{\Omega_{ij}^0}$ in a direction along the face Ω_{ij} given by a unit vector u equals $D\phi(x)u$ (if Ω_{ij} is a point ($d = 0$) then there is no restriction on the value of $D\phi$).

Lemma 2.7 *Let $\phi \in C^1_{pw}$, and let $D\phi:\ \mathbb{R}^m \to \mathbb{R}^m$ be an admissible derivative of ϕ. Then for any Lipschitz curve $\gamma:\ [0,1] \to \mathbb{R}^m$ the superposition $\tilde{\gamma} := \phi \circ \gamma:\ [0,1] \to \mathbb{R}$ satisfies*

$$\tilde{\gamma}' = (D\phi \circ \gamma)\gamma' \tag{2.14}$$

a.e. on $[0,1]$.

Proof. Since $\tilde{\gamma}$ is Lipschitz on $[0,1]$, the set $E := \{t \in (0,1):\ \tilde{\gamma}' \text{ exists}\}$ has full measure. Set $E_{ij} := \{t \in (0,1):\ \gamma(t) \in \Omega^0_{ij}\} \cap E$. Then $\cup_{i,j} E_{ij} = E$. If E_{ij} has measure zero, set $F_{ij} = E_{ij}$. In all other cases, let F_{ij} be the set of isolated points in E_{ij}, a set of measure zero. Let $t_0 \in E_{ij} \backslash F_{ij}$, and assume that Ω_{ij} is not a point ($d > 0$). Then there is a sequence $t_n \to t_0$ of points $t_n \in E_{ij} \subset \Omega^0_{ij}$, $t_n \neq t_0$, $n = 1, 2, \ldots$, for which

$$\phi(\gamma(t_n)) - \phi(\gamma(t_0)) = D\phi(\xi_n)(\gamma(t_n) - \gamma(t_0)) = D\phi(\xi_n)\gamma'(\tau_n)(t - t_0)$$

holds with certain $\xi_n \in \Omega^0_{ij}$, $\xi_n \to \gamma(t_0)$, and $\tau_n \in (0,1)$, $\tau_n \to t_0$. This follows from the admissibility of $D\phi$, the assumed C^1 property of $\phi_{\Omega^0_{ij}}$, and the differentiability of γ at t_0. Thus,

$$\tilde{\gamma}'(t_0) = \lim_{n\to\infty} \frac{\phi(\gamma(t_n)) - \phi(\gamma(t_0))}{t_n - t_0} = \lim_{n\to\infty} D\phi(\xi_n)\gamma'(\tau_n) = D\phi(\gamma(t_0))\gamma'(t_0).$$

If Ω_{ij} is a point ($d = 0$), then evidently $\tilde{\gamma}'(t_0) = 0$ and $\gamma'(t_0) = 0$ for such a t_0, and the above equality holds for any choice of $D\phi(\gamma(t_0))$. This proves (2.14) for all $t_0 \in \cup_{i,j} E_{ij} \backslash F_{ij} = E \backslash \cup_{i,j} F_{ij}$. Since the measure of $\cup_{i,j} F_{ij}$ is zero, the Lemma is proved. $\square$

We note that the proof also goes through if the underlying polyhedral partition is deformed by any non-degenerate C^1 diffeomorphism of $\mathbb{R}^m$. For the applications in our paper, the much simpler case of a partition obtained by the intersection of finitely many straight lines in $\mathbb{R}^2$ suffices. Moreover, in these applications, definitions of $D\phi$ on the subsets Ω^0_{ij}, $i \neq j$, are obtained by continuous extension from $D\phi|_{\Omega^0_i}$ resp. $D\phi|_{\Omega^0_j}$ which eases the computation of numerical estimates for the spectral radii of interest.

Let Σ_k denote the class of k-Lipschitz subdivision schemes $\mathbf{S}$, where each $f^{[k]}_s$, $s = 0, \ldots, r-1$, is the composition of finitely many functions from C^1_{pw}. Then the statement of Theorem 2.4 holds for any $S \in \Sigma_k$ if in the definition of the spectral radii we use operators $DS^{[k]}_v$ that are defined using admissable derivatives for the C^1_{pw} functions the $f^{[k]}_s$, $s = 0, \ldots, r-1$, are composed of. Applying Lemma 2.7 at each subdivision level, the proof follows line by line the proof of Theorem 2.4.

Theorem 2.8 *Let $S \in \Sigma_k$, and assume that (2.4) is satisfied with this $k \geq 1$. Then*

(i) $\mathbf{S}$ *is stable if* $\rho_s(\mathbf{S}, k) < 1$,

(ii) $\mathbf{M}$ *is stable if* $\rho_s(\mathbf{M}, k) < 1$.

We note that the statement about the sharpness of the spectral radius criterion for $\mathbf{M}$ in Remark 2.5 remains true if the required large value of $\|DS^{[k]}_{w^{J-1}} \ldots DS^{[k]}_{w^0}\|$ can be found by derivative calculations for the involved C^1_{pw} functions that only use arguments belonging to the domains Ω^0_i, and not to the lower-dimensional Ω^0_{ij}. This is obvious from the proof given for Remark 2.5. In the examples in the next section, this additional assumption can easily be verified.

As will become more evident from the examples in Section 3, due to the assumed locality and r-shift invariance of S (and consequently of $S^{[k]}$ and $(DS^{[k]})_v$), the estimation of the spectral radii in Theorem 2.8 reduces to the study of the dynamics of certain low-dimensional nonlinear iterated function systems (IFS). We use relatively crude estimates involving a few iterations of such nonlinear IFS, a thorough study of the dynamical systems aspect is left for future work.

3 Case Studies

3.1 Median Interpolating Scheme

The quadratic triadic median-interpolating scheme was introduced in [10], a dyadic version of the scheme was considered in [32]. For a real-valued continuous function f on a bounded interval I, the median of f on I is defined by

$$med(f;I) := \sup\Big\{\alpha : m(\{x : f(x) < \alpha\}) \le \frac{1}{2}m(I)\Big\},$$

where m is the Lebesgue measure. For any $v^0 \in \ell_\infty(\mathbb{Z})$ and any $i \in \mathbb{Z}$ denote by $p_i(x)$ the unique quadratic polynomial that satisfies

$$med(p_i;[i-l, i-l+1]) = v^0_{i-l}, \quad l = -1, 0, 1.$$

Then the subdivision step is given via

$$v^1_{3i+l} = med\Big(p_i;\Big[\frac{3i+l}{3}, \frac{3i+l+1}{3}\Big]\Big), \quad l = 0, 1, 2,$$

or via

$$v^1_{2i+l} = med\Big(p_i;\Big[\frac{2i+l}{2}, \frac{2i+l+1}{2}\Big]\Big), \quad l = 0, 1,$$

in the triadic, respectively the dyadic cases. We denote by $S_{med,3}$ and $S_{med,2}$ the corresponding subdivision operators.

Following [32] we write the corresponding subdivision operators as functions of the centers c_i of p_i (when p_i is linear, we formally set $c_i = \pm\infty$):

$$\begin{aligned}(S_{med,3}v)_{3i} &= \frac{2}{9}v_{i-1} + \frac{8}{9}v_i - \frac{1}{9}v_{i+1} - \alpha_0(c_i)\Delta^2 v_{i-1},\\ (S_{med,3}v)_{3i+1} &= v_i - \alpha_1(c_i)\Delta^2 v_{i-1},\\ (S_{med,3}v)v_{3i+2} &= -\frac{1}{9}v_{i-1} + \frac{8}{9}v_i + \frac{2}{9}v_{i+1} - \alpha_2(c_i)\Delta^2 v_{i-1},\end{aligned} \tag{3.1}$$

and

$$(S_{med,2}v)_{2i} = \frac{5}{32}v_{i-1} + \frac{15}{16}v_i - \frac{3}{32}v_{i+1} - \tilde{\alpha}_0(c_i)\Delta^2 v_{i-1},$$
$$(S_{med,2}v)_{2i+1} = -\frac{3}{32}v_{i-1} + \frac{15}{16}v_i + \frac{5}{32}v_{i+1} - \tilde{\alpha}_1(c_i)\Delta^2 v_{i-1}, \tag{3.2}$$

where

$$\alpha_0 = \frac{8\epsilon_0 + 2\epsilon_{-2} - \epsilon_2 - \tilde{\epsilon}_{-2/3}}{9(32 - 2\epsilon_0 + \epsilon_{-2} + \epsilon_2)}, \quad \alpha_1 = \frac{9\epsilon_0 - \tilde{\epsilon}_0}{9(32 - 2\epsilon_0 + \epsilon_{-2} + \epsilon_2)}, \quad \alpha_2 = \frac{8\epsilon_0 - \epsilon_{-2} + 2\epsilon_2 - \tilde{\epsilon}_{2/3}}{9(32 - 2\epsilon_0 + \epsilon_{-2} + \epsilon_2)},$$
$$\tilde{\alpha}_0 = \frac{30\epsilon_0 + 5\epsilon_{-2} - 3\epsilon_2 - 8\tilde{\epsilon}_{-1/2}}{32(32 - 2\epsilon_0 + \epsilon_{-2} + \epsilon_2)}, \quad \tilde{\alpha}_1 = \frac{30\epsilon_0 - 3\epsilon_{-2} + 5\epsilon_2 - 8\tilde{\epsilon}_{1/2}}{32(32 - 2\epsilon_0 + \epsilon_{-2} + \epsilon_2)}, \tag{3.3}$$

with

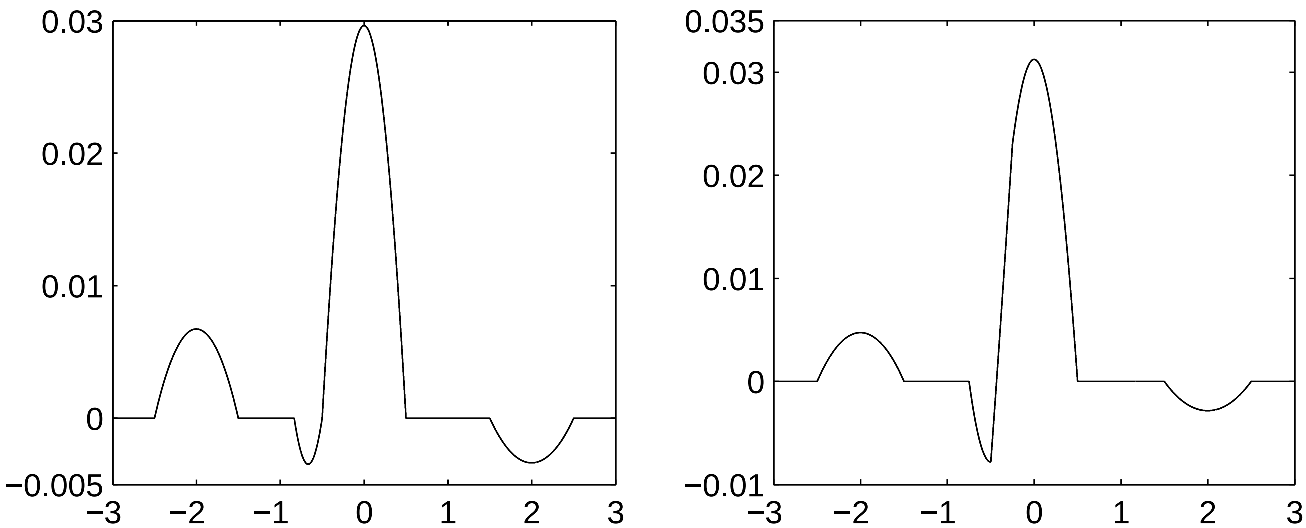

Figure 1: $\alpha_0(c)$ and $\tilde{\alpha}_0(c)$.

$$\epsilon_{-2} = (1 - 4(c+2)^2)_+; \qquad \epsilon_0 = (1 - 4c^2)_+ \qquad \epsilon_2 = (1 - 4(c-2)^2)_+$$
$$\tilde{\epsilon}_{-2/3} = (1 - 4(2+3c)^2)_+ \qquad \tilde{\epsilon}_0 = (1 - 36c^2)_+ \qquad \tilde{\epsilon}_{2/3} = (1 - 4(2-3c)^2)_+$$
$$\tilde{\epsilon}_{-1/2} = (1 - 4(-1-2c)^2)_+ \qquad \tilde{\epsilon}_{1/2} = (1 - 4(1-2c)^2)_+ \qquad .$$

For later use, we introduce some further notation. It is also not hard to prove that the quantity $\zeta_i := -(\Delta v_{i-1} + \Delta v_i)/\Delta^2 v_{i-1}$ is a continuous, piecewise differentiable, and strictly monotone function of c_i (the reader can find confirmation by going through the calculations in [32]). This function we denote by $\zeta = \zeta(c)$. Consequently, the center c_i is a continuous, piecewise differentiable function of ζ_i (and thus of the involved first-order differences as well), i.e., $c_i = c(\Delta v_{i-1}, \Delta v_i) := \zeta^{-1}(\zeta_i)$. We mention a few important symmetries that help our stability analysis:

$$\alpha_0(c) = \alpha_2(-c); \quad \alpha_1(c) = \alpha_1(-c); \quad \tilde{\alpha}_0(c) = \tilde{\alpha}_1(-c); \quad \zeta(c) = -\zeta(-c). \tag{3.4}$$

From (3.3) and (3.4) we see that $\alpha_0, \alpha_1, \alpha_2, \tilde{\alpha}_0$, and $\tilde{\alpha}_1$ are continuous and piecewise differentiable functions of c, compare Fig. 1 for the graphs of α_0 and $\tilde{\alpha}_0$.

Both schemes are known to be convergent [10, 32], with limits enjoying a certain Lipschitz regularity. In this section, we will show that the triadic median-interpolating subdivision scheme as well as its associated multiscale transform are stable, and that the dyadic median-interpolating subdivision scheme is stable, but, unless we have some restrictions on the admissible class of details, the associated multiscale transform is not. We start with

Lemma 3.1 *$S_{med,3}$ and $S_{med,2}$ both belong to Σ_1 and satisfy* (2.4) *with $k = 1$.*

Proof. We will sketch the proof for $S_{med,2} \in \Sigma_1$, in the triadic case the proof is analogous and not detailed here. Obviously $S_{med,2}$ is off-set invariant. The formulas (3.2), together with (3.4), give rise to the representation

$$\begin{aligned} (S^{[1]}_{med,2}w)_{2i} &= f_0^{[1]}(w_{i-1}, w_i) = \frac{1}{4}(w_{i-1} + w_i) + F(w_{i-1}, w_i) - F(w_i, w_{i-1}); \\ (S^{[1]}_{med,2}w)_{2i+1} &= f_1^{[1]}(w_{i-1}, w_i, w_{i+1}) = \frac{-3w_{i-1} + 22w_i - 3w_{i+1}}{32} + F(w_i, w_{i-1}) - F(w_i, w_{i+1}), \end{aligned} \tag{3.5}$$

where $F(x, y) := \tilde{\alpha}_0(c(x, y))(y - x)$, and $w = \Delta^1 v$. The formula for $c(x, y)$ is obtained via the formula $\zeta(x, y) = \frac{x+y}{x-y}$ and via the inverse function of

$$\zeta(c) = \frac{32c + \epsilon_{-2}(c) - \epsilon_2(c)}{32 + \epsilon_{-2}(c) - 2\epsilon_0(c) + \epsilon_2(c)}.$$

The elementary derivation of the last formula can be found in [19]. (3.5) implies that it is enough to show that $F \in C^1_{pw}$. Direct computations show that

$$\frac{\partial F}{\partial x} = -\tilde{\alpha}_0(c) + \frac{\tilde{\alpha}_0'(c)}{\zeta'(c)} \frac{2y}{x - y}; \qquad \frac{\partial F}{\partial y} = \tilde{\alpha}_0(c) + \frac{\tilde{\alpha}_0'(c)}{\zeta'(c)} \frac{2x}{y - x}.$$

The above gradient is not defined only on the line $x = y$ (that corresponds to $c = \infty$) or in the points of discontinuity for $\tilde{\alpha}_0'(c)/\zeta'(c)$. Fig. 2 shows that the latter happens for

$$c \in \Theta := \{-5/2, -3/2, -3/4, -1/2, -1/4, 1/2, 3/2, 5/2\},$$

which, since $\zeta(x, y)$ is homogeneous of degree 0, corresponds to 8 more lines in $\mathbb{R}^2$. But it is very easy to show that, no matter how we define $\tilde{\alpha}_0'(c)/\zeta'(c)$ for $c \in \Theta \cup \{\infty\}$ $DF := (\partial F/\partial x, \partial F/\partial y)$ is admissible. Indeed, $c = C$ gives rise to the line (t, Lt), where L is a function of C and

$$G(t) = F(t, Lt) = \tilde{\alpha}_0(C)(L - 1)t \quad \Longrightarrow \quad G'(t) = \tilde{\alpha}_0(C)(L - 1).$$

On the other hand

$$\begin{aligned} &\frac{\partial F}{\partial x}(\gamma_1(t), \gamma_2(t))\gamma_1'(t) + \frac{\partial F}{\partial y}(\gamma_1(t), \gamma_2(t))\gamma_2'(t) = \\ &\qquad \Big(-\tilde{\alpha}_0(C) + \frac{\tilde{\alpha}_0'(C)}{\zeta'(C)} \frac{2L}{1 - L}\Big)1 + \Big(\tilde{\alpha}_0(C) + \frac{\tilde{\alpha}_0'(C)}{\zeta'(C)} \frac{2}{L - 1}\Big)L = \tilde{\alpha}_0(C)(L - 1). \end{aligned}$$

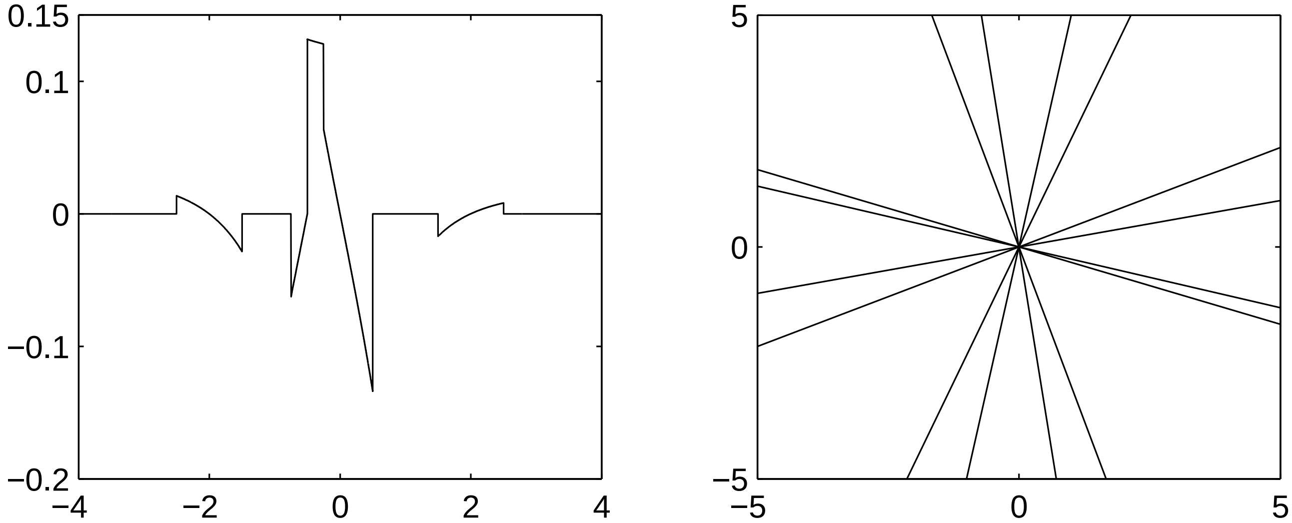

Figure 2: left: $\tilde{\alpha}_0'(c)/\zeta'(c)$; right: the lines in $\mathbb{R}^2$ that correspond to $c \in \Theta$.

There are several natural options for defining the function $\tilde{\alpha}_0'(c)/\zeta'(c)$ on Θ: we could set it to zero, or make it left- or right-continuous (the latter was done in numerical experiments). This completes the verification of $S^{[1]}_{med,2} \in \Sigma_1$. For $k = 1$, the inequality (2.4) is a consequence of Lemma 2.2. $\square$

To obtain stability results using Theorem 2.8, what remains after having established the above Lemma 3.1 is to estimate the nonlinear spectral radii $\rho_s(\mathbf{M}, 1)$ and $\rho_s(\mathbf{S}, 1)$. This will be done separately for the triadic and dyadic cases.

3.1.1 The triadic case

Using (3.1) we derive

$$\begin{aligned}
(\Delta S_{med,3} v)_{3i} &= \frac{2\Delta v_{i-1} + \Delta v_i}{9} + (\alpha_0 - \alpha_1)(c_i)\Delta^2 v_{i-1},\\
(\Delta S_{med,3} v)_{3i+1} &= \frac{\Delta v_{i-1} + 2\Delta v_i}{9} + (\alpha_1 - \alpha_2)(c_i)\Delta^2 v_{i-1},\\
(\Delta S_{med,3} v)_{3i+2} &= \frac{-\Delta v_{i-1} + 5\Delta v_i - \Delta v_{i+1}}{9} + \alpha_2(c_i)\Delta^2 v_{i-1} - \alpha_0(c_{i+1})\Delta^2 v_i.
\end{aligned}$$

Therefore, setting $w := \Delta v$, using (3.4), and doing straightforward computations, we obtain the following formulas for the nonzero entries of $DS^{[1]}_{med,3}$:

$$((DS^{[1]}_{med,3})_w)_{3i,i-1} = \frac{2}{9} - (A-C)(c_i), \qquad ((DS^{[1]}_{med,3})_w)_{3i,i} = \frac{1}{9} + (B-D)(c_i),$$
$$((DS^{[1]}_{med,3})_w)_{3i+1,i-1} = \frac{1}{9} + (B-D)(-c_i), \qquad ((DS^{[1]}_{med,3})_w)_{3i+1,i} = \frac{2}{9} - (A-C)(-c_i),$$
$$((DS^{[1]}_{med,3})_w)_{3i+2,i-1} = -\frac{1}{9} - B(-c_i), \qquad ((DS^{[1]}_{med,3})_w)_{3i+2,i+1} = -\frac{1}{9} - B(c_{i+1}),$$
$$((DS^{[1]}_{med,3})_w)_{3i+2,i} = \frac{5}{9} + A(-c_i) + A(c_{i+1}),$$

where

$$A := \alpha_0 + \frac{\alpha_0'}{\zeta'}(1-\zeta), \quad B := \alpha_0 - \frac{\alpha_0'}{\zeta'}(1+\zeta), \quad C := \alpha_1 + \frac{\alpha_1'}{\zeta'}(1-\zeta), \quad D := \alpha_1 - \frac{\alpha_1'}{\zeta'}(1+\zeta).$$

By definition these functions are piecewise continuous and bounded functions of the center parameter c.

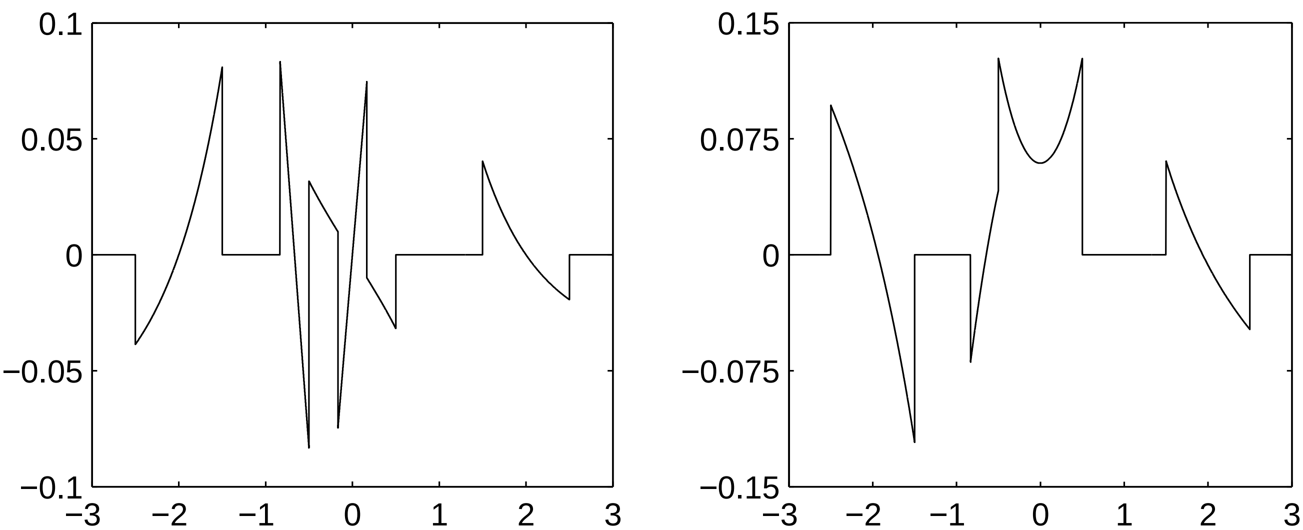

Figure 3: Graphs of $(B-A+C-D)(c)$ and $(A+B)(c)$.

For further use, let us denote the entries of $(DS^{[1]}_{med,3})_w$ by $t_{i,j} := ((DS^{[1]}_{med,3})_w)_{i,j}$. Direct computations show that the elements $t_{3i,i-1}$, $t_{3i,i}$, $t_{3i+1,i-1}$, $t_{3i+1,i}$, $t_{3i+2,i}$ are positive, and that the elements $t_{3i+2,i-1}$, $t_{3i+2,i+1}$ are negative. By shift invariance, this gives complete knowledge about the sign pattern of the non-zero entries in $(DS^{[1]}_{med,3})_w$. More precisely, the ranges for each one of these entries have been numerically computed as follows (since all functions involved are piecewise rational and explicitly available, this can be backed by tedious analytic estimates as well):

$$\begin{pmatrix} t_{3i,i-1} & t_{3i,i} & 0 \\ t_{3i+1,i-1} & t_{3i+1,i} & 0 \\ t_{3i+2,i-1} & t_{3i+2,i} & t_{3i+2,i+1} \end{pmatrix} = \begin{pmatrix} (0.1546, 0.3232) & (0.0771, 0.1616) & 0 \\ (0.0771, 0.1616) & (0.1546, 0.3232) & 0 \\ (-0.3016, -0.0476) & (0.3536, 0.9366) & (-0.3016, -0.0476) \end{pmatrix}.$$

We will use these bounds throughout the remainder of this subsection.

To prepare for norm estimates for the operators $(DS^{[1]}_{med,3})_w$ and their products, observe that

$$\begin{aligned}\sum_j |t_{3i,j}| &= t_{3i,i-1} + t_{3i,i} = \frac{1}{3} + (B-A+C-D)(c_i),\\ \sum_j |t_{3i+1,j}| &= t_{3i+1,i-1} + t_{3i+1,i} = \frac{1}{3} + (B-A+C-D)(-c_i),\\ \sum_j |t_{3i+2,j}| &= -t_{3i+2,i-1} + t_{3i+2,i} - t_{3i+2,i+1}\\ &= \frac{7}{9} + (A+B)(-c_i) + (A+B)(c_{i+1}),\end{aligned}$$

and, thus (compare Fig. 3)

$$\sum_j |((DS^{[1]}_{med,3})_w)_{3i+l,j}| \leq 0.4166, \qquad l=0,1, \qquad\qquad \sum_j |((DS^{[1]}_{med,3})_w)_{3i+2,j}| \leq 1.0318.$$

In other words, $\|(DS^{[1]}_{med,3})_w\| \leq 1.0318$, which is not sufficient for establishing stability. However, we have

Lemma 3.2 *For any $u, w \in \ell_\infty(\mathbb{Z})$*

$$\|(DS^{[1]}_{med,3})_u (DS^{[1]}_{med,3})_w\| \leq 0.9706.$$

Thus, the spectral radius $\rho_s(\mathbf{M}, 1)$ associated with the multiscale transform $\mathbf{M}$ defined by $S_{med,3}$ is less than one.

Proof. Let us denote by $t_{i,j}$ as before the elements of both $(DS^{[1]}_{med,3})_w$ and $(DS^{[1]}_{med,3})_u$ (by the indices of an element it will be clear, exactly to which matrix it belongs), and by $t^2_{i,j}$ - the elements of the product $(DS^{[1]}_{med,3})_u(DS^{[1]}_{med,3})_w$. By standard techniques for $l = 0,1,3,4,6,7$ we have:

$$\sum_j |t^2_{9i+l,j}| \leq \sum_{j,s} |t_{9i+l,s} t_{s,j}| \leq \sum_s |t_{9i+l,s}| (\sum_j |t_{s,j}|) \leq 0.4166 \cdot 1.0318 = 0.4298.$$

Consider $l = 2$. Since $(DS^{[1]}_{med,3})_u$ and $(DS^{[1]}_{med,3})_w$ are local and shift-invariant, we can perform our analysis on a finite-dimensional subspace of $\ell_\infty(\mathbb{Z})$. Therefore, to determine the elements $\{t^2_{9i+2,j}\}$ we need to consider only the following product of a 1×3 vector and a 3×3 matrix

$$\begin{pmatrix} t_{9i+2,3i-1} & t_{9i+2,3i} & t_{9i+2,3i+1} \end{pmatrix} \begin{pmatrix} t_{3i-1,i-2} & t_{3i-1,i-1} & t_{3i-1,i} \\ 0 & t_{3i,i-1} & t_{3i,i} \\ 0 & t_{3i+1,i-1} & t_{3i+1,i} \end{pmatrix} = \begin{pmatrix} - & + & - \end{pmatrix} \begin{pmatrix} - & + & - \\ 0 & + & + \\ 0 & + & + \end{pmatrix}.$$

Using again the ranges of the entries $t_{i,j}$ given above, we obtain the following bounds for the absolute values of $t^2_{9i+2,\cdot}$:

$$|t^2_{9i+2,i-2}| \leq 0.0910; \quad |t^2_{9i+2,i-1}| \leq 0.2782; \quad |t^2_{9i+2,i}| \leq 0.2350,$$

and, thus

$$\sum_j |t^2_{9i+2,j}| \le 0.6042.$$

Due to symmetries, the case $l = 5$ is absolutely analogous to the case $l = 2$.

It remains to look at the case $l = 8$, where we have the following situation:

$$\begin{pmatrix} t_{9i+8,3i+1} & t_{9i+8,3i+2} & t_{9i+8,3i+3} \end{pmatrix} \begin{pmatrix} t_{3i+1,i-1} & t_{3i+1,i} & 0 \\ t_{3i+2,i-1} & t_{3i+2,i} & t_{3i+2,i+1} \\ 0 & t_{3i+3,i} & t_{3i+3,i+1} \end{pmatrix} = \begin{pmatrix} - & + & - \end{pmatrix} \begin{pmatrix} + & + & 0 \\ - & + & - \\ 0 & + & + \end{pmatrix}.$$

According to the sign pattern of the $t_{i,j}$, both $t^2_{9i+8,i-1}$ and $t^2_{9i+8,i+1}$ are non-positive. If we assume that $t^2_{9i+8,i}$ is non-positive as well, and usubstitute the bounds for $t_{i,j}$ from the previous page, we derive that

$$\sum_j |t^2_{9i+8,j}| = -\sum_{j=i-1}^{i+1} t^2_{9i+8,j} \le 0.6836.$$

Otherwise, if $t^2_{9i+8,i}$ is positive then we write

$$\begin{aligned}\sum_j |t^2_{9i+8,j}| &= -t_{9i+8,3i+1}(t_{3i+1,i-1} - t_{3i+1,i}) + t_{9i+8,3i+2}(-t_{3i+2,i-1} + t_{3i+2,i} - t_{3i+2,i+1}) \\ &\quad - t_{9i+8,3i+3}(-t_{3i+3,i} + t_{3i+3,i+1}).\end{aligned}$$

While for the first and third term we can simply use the bounds for $t_{i,j}$, e.g.,

$$-t_{9i+8,3i+1}(t_{3i+1,i-1} - t_{3i+1,i}) \le 0.3016(0.1616 - 0.1546) = 0.0021,$$

for the second term we use the previously established inequalities

$$t_{9i+8,3i+2} \le 0.9366, \qquad -t_{3i+2,i-1} + t_{3i+2,i} - t_{3i+2,i+1} \le 1.0318.$$

This leads to the desired estimate

$$\sum_j |t^2_{9i+8,j}| \le 2 \cdot 0.0021 + 0.9366 \cdot 1.0318 = 0.9706.$$

□

Now, combining Lemma 3.1 and Lemma 3.2 with Theorem 2.8 gives

Theorem 3.3 *The triadic, quadratic, median-interpolating subdivision scheme* **S** *and its associated multiscale transform* **M** *are Lipschitz stable.*

3.1.2 The dyadic case

Applying the same analysis as for the triadic case, we obtain

$$
\begin{aligned}
(\Delta S_{med,2}v)_{2i} &= \frac{1}{4}(\Delta v_{i-1} + \Delta v_i) + \tilde{\alpha}_2(c_i)\Delta^2 v_{i-1},\\
(\Delta S_{med,2}v)_{2i+1} &= \frac{-3\Delta v_{i-1} + 22\Delta v_i - 3\Delta v_{i+1}}{32} + \tilde{\alpha}_1(c_i)\Delta^2 v_{i-1} - \tilde{\alpha}_0(c_{i+1})\Delta^2 v_i,
\end{aligned}
$$

where $\tilde{\alpha}_2 := \tilde{\alpha}_0 - \tilde{\alpha}_1$. Therefore, setting as before $w := \Delta v$, writing S instead of $S_{med,2}$, and denoting $t_{i,j} := ((DS^{[1]}_{med,2})_w)_{i,j}$ for the entries of $(DS^{[1]}_{med,2})_w$ gives rise to

$$
\begin{aligned}
t_{2i,i-1} &= \frac{1}{4} - \tilde{A}(c_i) + \tilde{B}(-c_i), & t_{2i,i} &= \frac{1}{4} + \tilde{B}(c_i) - \tilde{A}(-c_i),\\
t_{2i+1,i-1} &= -\frac{3}{32} - \tilde{B}(-c_i), & t_{2i+1,i+1} &= -\frac{3}{32} - \tilde{B}(c_{i+1}),\\
t_{2i+1,i} &= \frac{11}{16} + \tilde{A}(-c_i) + \tilde{A}(c_{i+1}),
\end{aligned}
$$

where

$$
\tilde{A} := \tilde{\alpha}_0 + \frac{\tilde{\alpha}_0'}{\zeta'}(1-\zeta), \quad \tilde{B} := \tilde{\alpha}_0 - \frac{\tilde{\alpha}_0'}{\zeta'}(1+\zeta).
$$

Again, as in the triadic case we have a sign pattern for the entries, namely $t_{2i,i-1}$, $t_{2i,i}$, $t_{2i+1,i}$ are positive while $t_{2i+1,i-1}$, $t_{2i+1,i+1}$ are negative. The numerically computed ranges of the relevant $t_{i,j}$ are as follows:

$$
\begin{pmatrix} t_{2i,i-1} & t_{2i,i} & 0 \\ t_{2i+1,i-1} & t_{2i+1,i} & t_{2i+1,i+1} \end{pmatrix} = \begin{pmatrix} (0.1684, 0.3636) & (0.1684, 0.3636) & 0 \\ (-0.2946, -0.02) & (0.4687, 1.0669) & (-0.2946, -0.02) \end{pmatrix}.
$$

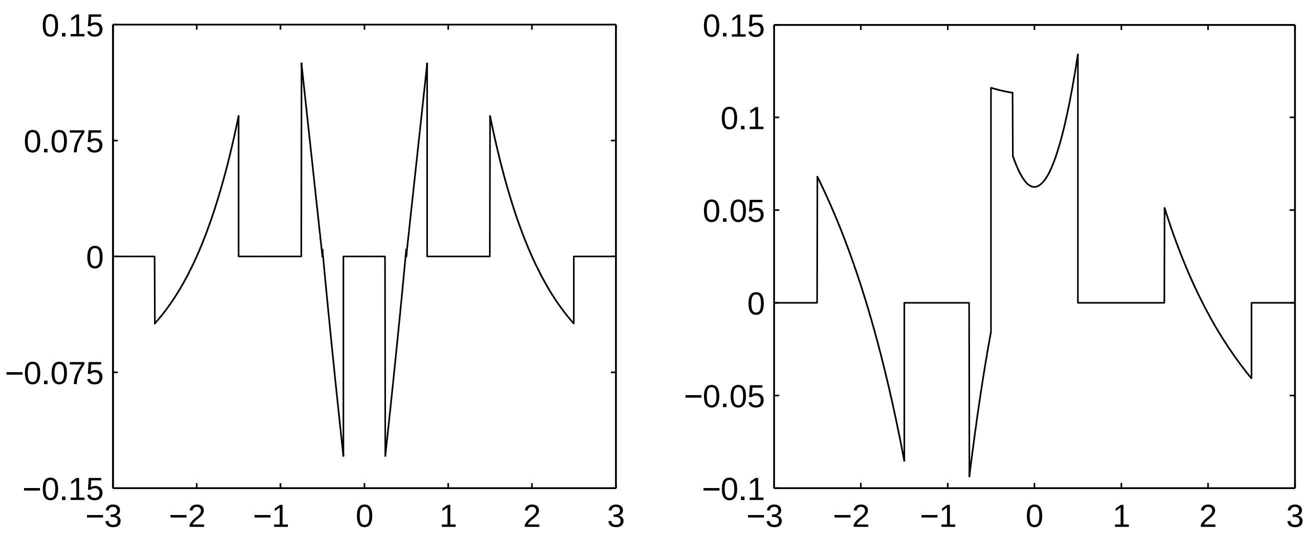

Figure 4: $(\tilde{B} - \tilde{A})(c) + (\tilde{B} - \tilde{A})(-c)$ and $\tilde{A}(c) + \tilde{B}(c)$.

Therefore

$$\sum_j |t_{2i,j}| = t_{2i,i-1} + t_{2i,i} = \frac{1}{2} + (\tilde{B} - \tilde{A})(c_i) + (\tilde{B} - \tilde{A})(-c_i);$$

$$\sum_j |t_{2i+1,j}| = -t_{2i+1,i-1} + t_{2i+1,i} - t_{2i+1,i+1} = \frac{7}{8} + (\tilde{A} + \tilde{B})(-c_i) + (\tilde{A} + \tilde{B})(c_{i+1}).$$

Direct computations (see Fig. 4) show that for any $w \in \ell_\infty(\mathbb{Z})$

$$\sum_j \left| ((DS^{[1]})_w)_{2i,j} \right| \le C_{even} := 0.6250, \quad \sum_j \left| ((DS^{[1]})_w)_{2i+1,j} \right| \le C_{odd} := 1.1430.$$

Let w_1, w_2, w_3 be three arbitrarily fixed sequences in $\ell_\infty(\mathbb{Z})$. We will denote the elements of $(DS^{[1]})_{w_i}$, $i = 1, 2, 3$ by $t_{i,j}$ as before, and the elements of $(DS^{[1]})_{w_3}(DS^{[1]})_{w_2}(DS^{[1]})_{w_1}$ by $t^3_{i,j}$.

Lemma 3.4 $\sum_j |t^3_{8i+l,j}| \le C_{even} C^2_{odd} = 0.8165, \qquad l = 0, 1, \ldots, 6.$

The proof is similar to that of Lemma 3.2 and can be found in [19].

Such kind of contractivity cannot be obtained for t^3_{8i+7}, because of the existence of an $w \in \ell_\infty(\mathbb{Z})$, such that $\rho((DS^{[1]})_w) > 1$. For example, take $\bar{w} = \Delta \bar{v}$, where $\bar{v} := \{v_{i-1}, v_i, v_{i+1}, v_{i+2}\}$ with $c_i = -c_{i+1} = 1/2 - \epsilon$ and small $\epsilon > 0$. Then, according to our formulas, the limit as $\epsilon \to 0$ of the submatrix of $(DS^{[1]})_{\bar{w}}$ corresponding to the index pairs $\{2i, 2i+1, 2i+2\} \times \{i-1, i, i+1\}$ is computed as

$$\lim_{\epsilon \to 0} (DS^{[1]})_{\bar{w}}\big|_{\{2i,2i+1,2i+2\}\times\{i-1,i,i+1\}} = \frac{1}{448} \begin{pmatrix} 109 & 117 & 0 \\ -9 & 478 & -9 \\ 0 & 117 & 109 \end{pmatrix}.$$

Since this 3×3 matrix has a numerically computed spectral radius of $1.0540 > 1$, this implies that $\rho_s(\mathbf{M}, 1) > 1$, where $\mathbf{M}$ is the associate multiscale transform for $S_{med,2}$. For the subdivision case, when $w_2 = S^{[1]} w_1$, and $w_3 = S^{[1]} w_2$ the following inequality

$$\sum_j |t^3_{8i+7,j}| \le 0.81 \tag{3.6}$$

has been verified by numerical methods only ([19]), and needs further scrutiny. To summarize, up to the formal mathematical proof of the inequality (3.6), we showed that *the subdivision scheme* $\mathbf{S}$ *associated with the dyadic quadratic median-interpolating subdivision operator* $S_{med,2}$ *is Lipschitz stable, but its associated multiscale transform* $\mathbf{M}$ *is not.*

3.2 Power-p Scheme

For $p \in [1, \infty)$, the power-p subdivision operator S is defined by the formula

$$(Sv)_{2i} = v_i, \qquad (Sv)_{2i+1} = \frac{v_i + v_{i+1}}{2} - \frac{1}{8} H_p(\Delta^2 v_{i-1}, \Delta^2 v_i), \qquad i \in \mathbb{Z}, \tag{3.7}$$

where the so-called limiter H_p is given by

$$H_p(x,y) = \begin{cases} \frac{x+y}{2}\left(1 - \left|\frac{x-y}{x+y}\right|^p\right), & xy > 0, \\ 0, & xy \le 0. \end{cases}$$

We do not show the dependence on p in the notation for S, as this could lead to confusion with notation used in previous sections. Power-p schemes have been introduced in the context of generalized ENO-methods for hyperbolic problems [33], and can be useful for compressing piecewise smooth data and functions. A straightforward calculation shows that if $v|_{\{i_0,\dots,i_1\}}$ is a convex (concave, linear) segment of v, then $(Sv)|_{\{2i_0,\dots,2i_1\}}$ preserves this property if $p \in [1,2]$. The formula is constructed such that for $\Delta^2 v_{i-1} = \Delta^2 v_i$ the obtained value $(Sv)_{2i+1}$ is the same as for the linear interpolating scheme based on symmetric cubic Lagrange interpolation. It has been shown that the power-p subdivision scheme is convergent, with Hölder exponent 1 for all p. For $p = 2$, it coincides with the so-called PPH scheme [14, 25] whose stability in the sense of both $\mathbf{S}$ and $\mathbf{M}$ was settled in [26, 3]. We apply our stability theorems to study the general case $p \ge 1$, and obtain

Theorem 3.5 *a) The power-p subdivision operator S belongs to Σ_2 for all $p \ge 1$.*
b) The multiscale transform $\mathbf{M}$ associated with S is stable if $1 \le p < 8/3$ since $\rho(\mathbf{M}, 2) < 1$.
c) If $p > 4$ then $\rho(\mathbf{M}, 2) > 1$, i.e., $\mathbf{M}$ is unstable.

Proof. From (3.7), by straightforward calculation, we obtain that

$$(\Delta S v)_{2i} = \frac{\Delta v_i}{2} - \frac{1}{8} H_p(\Delta^2 v_{i-1}, \Delta^2 v_i), \qquad (\Delta S v)_{2i+1} = \frac{\Delta v_i}{2} + \frac{1}{8} H_p(\Delta^2 v_{i-1}, \Delta^2 v_i), \qquad i \in \mathbb{Z},$$

and

$$\begin{aligned} (\Delta^2 S v)_{2i} &= \tfrac{1}{4} H_p(\Delta^2 v_{i-1}, \Delta^2 v_i), \\ (\Delta^2 S v)_{2i+1} &= \tfrac{\Delta^2 v_i}{2} - \tfrac{1}{8}(H_p(\Delta^2 v_{i-1}, \Delta^2 v_i) + H_p(\Delta^2 v_i, \Delta^2 v_{i+1})), \end{aligned} \qquad i \in \mathbb{Z}. \tag{3.8}$$

Rewrite (3.8) in terms of $S^{[2]}$ by setting $w := \Delta^2 v$

$$(S^{[2]} w)_{2i} = \frac{1}{4} H_p(w_{i-1}, w_i), \qquad (S^{[2]} w)_{2i+1} = \frac{w_i}{2} - \frac{1}{8}(H_p(w_{i-1}, w_i) + H_p(w_i, w_{i+1})), \qquad i \in \mathbb{Z}.$$

Hence, as in the median case, it suffices to show that $H_p \in C^1_{pw}$.

Lemma 3.6 *For $xy > 0$, H_p is a C^∞ function, with bounded first partial derivatives*

$$\begin{aligned} \frac{\partial H_p}{\partial x}(x,y) &= \phi(-t) = \frac{1}{2}(1 + (p-1)|t|^p - pt|t|^{p-2}) \in (0,p), \\ \frac{\partial H_p}{\partial y}(x,y) &= \phi(t) := \frac{1}{2}(1 + (p-1)|t|^p + pt|t|^{p-2}) \in (0,p), \end{aligned}$$

where $t := (x-y)/(x+y) \in (-1,1)$.

Therefore, setting $\frac{\partial H_p}{\partial x}(x,y) = \frac{\partial H_p}{\partial y}(x,y) = 0$ if $xy \le 0$ gives $H_p \in C^1_{pw}$, and thus $S \in \Sigma_2$.

Property (2.4) holds with $k = 2$ by direct inspection, see the definition (3.7) of S. According to Theorem 2.8, to prove stability of $\mathbf{M}$, we need to estimate the spectral radius $\rho(\mathbf{M}, 2)$. The matrix representation of $DS^{[2]}$ is given entry-by-entry as follows:

$$
\begin{aligned}
((DS^{[2]})_w)_{2i,i-1} &= \frac{1}{4}\frac{\partial H_p}{\partial x}(w_{i-1}, w_i), & ((DS^{[2]})_w)_{2i,i} &= \frac{1}{4}\frac{\partial H_p}{\partial y}(w_{i-1}, w_i),\\
((DS^{[2]})_w)_{2i+1,i-1} &= -\frac{1}{8}\frac{\partial H_p}{\partial x}(w_{i-1}, w_i), & ((DS^{[2]})_w)_{2i+1,i+1} &= -\frac{1}{8}\frac{\partial H_p}{\partial y}(w_i, w_{i+1}),\\
((DS^{[2]})_w)_{2i+1,i} &= \frac{1}{2} - \frac{1}{8}\frac{\partial H_p}{\partial y}(w_{i-1}, w_i) - \frac{1}{8}\frac{\partial H_p}{\partial x}(w_i, w_{i+1}), & & i \in \mathbb{Z},
\end{aligned}
$$

all other entries of $(DS^{[2]})_w$ are zero.

We continue with the lower estimate for $\rho(\mathbf{M}, 2)$ from c). Consider a sequence $w^0(\epsilon)$ such that $w^0|_{i-1,i,i+1} = (\epsilon, 1, \epsilon)$, with small $\epsilon > 0$. According to the formulas for the entries of $DS^{[2]}_w$ and using Lemma 3.6, we can write

$$((DS^{[2]})_{w^0(\epsilon)} z)|_{2i,2i+1,2i+2} = T_\epsilon(z|_{i-1,i,i+1}), \qquad z \in \ell_\infty(\mathbb{Z}),$$

where the corresponding 3×3 submatrix T_ϵ of $(DS^{[2]})_{w^0(\epsilon)}$ satisfies the limit property

$$\lim_{\epsilon \to 0} T_\epsilon = T_0 := \begin{pmatrix} p/4 & 0 & 0 \\ -p/8 & 1/2 & -p/8 \\ 0 & 0 & p/4 \end{pmatrix}.$$

Now choosing a similar w^1 such that $w^1|_{2i,2i+1,2i+2} = (\epsilon, 1, \epsilon)$, and so on, and letting $\epsilon \to 0$, it is obvious that

$$\rho(\mathbf{M}, 2) \ge \rho(T_0) = p/4,$$

This proves c).

For b), we establish first an auxiliary result. Assume that $0 < A \le B$ are two positive numbers, and that $z \in \ell_\infty(\mathbb{Z})$ satisfies $\|z\|_e := \sup |z_{2i}| \le A$, and $\|z\|_o := \sup |z_{2i+1}| \le B$. Then, with $a = p/4$, we have

$$\|(DS^{[2]})_w z\|_e \le aB, \qquad \|(DS^{[2]})_w z\|_o \le \max(A/2 + aB, aA + B/2) \tag{3.9}$$

for arbitrary w. Indeed, the estimate for even indices is obvious, because

$$
\begin{aligned}
|((DS^{[2]})_w z)_{2i}| &\le \frac{1}{4}\left(\frac{\partial H_p}{\partial x}(w_{i-1}, w_i)|z_{i-1}| + \frac{\partial H_p}{\partial y}(w_{i-1}, w_i)|z_i|\right)\\
&\le \frac{B}{4}\left(\frac{\partial H_p}{\partial x}(w_{i-1}, w_i) + \frac{\partial H_p}{\partial y}(w_{i-1}, w_i)\right) \le aB
\end{aligned}
$$

holds according to the formulas in Lemma 3.6. For odd indices, let us denote $t = (w_{i-1} - w_i)/(w_{i-1} + w_i)$ if $w_{i-1}w_i > 0$ and (for symmetry reasons) $t' = (w_{i+1} - w_i)/(w_{i+1} + w_i)$ if $w_{i+1}w_i > 0$. Then, if i is odd, we have

$$|((DS^{[2]})_w z)_{2i+1}| \le \frac{A}{8}(\phi(-t) + \phi(-t')) + \frac{B}{8}|4 - (\phi(t) + \phi(t'))|.$$

In this and the following formulas, we silently set $\phi(\pm t) = \phi(\pm t') = 0$ if the arguments t or t' are not defined. If $\phi(t) + \phi(t') \leq 4$ then we continue with

$$|((DS^{[2]})_w z)_{2i+1}| \leq \frac{A}{8}(\phi(-t) + \phi(-t')) + \frac{B}{2} \leq aA + \frac{B}{2},$$

if $\phi(t) + \phi(t') > 4$ (this can only happen if $p > 2$) then

$$|((DS^{[2]})_w z)_{2i+1}| \leq \frac{B}{8}(\phi(-t) + \phi(t) + \phi(-t') + \phi(t')) - \frac{B}{2} \leq (a - \frac{1}{2})B.$$

If i is even, the roles of A and B change:

$$|((DS^{[2]})_w z)_{2i+1}| \leq \frac{B}{8}(\phi(-t) + \phi(-t')) + \frac{A}{8}|4 - (\phi(t) + \phi(t'))|.$$

Again, we have two cases. If $\phi(t) + \phi(t') \leq 4$ then

$$|((DS^{[2]})_w z)_{2i+1}| \leq \frac{B}{8}(\phi(-t) + \phi(-t')) + \frac{A}{2} \leq \frac{A}{2} + aB,$$

otherwise

$$|((DS^{[2]})_w z)_{2i+1}| \leq \frac{B}{8}(\phi(-t) + \phi(t) + \phi(-t') + \phi(t')) - \frac{A}{2} \leq aB.$$

Thus, we have proved the claim.

For $a = p/4 < 1/2$, the estimate (3.9) automatically implies $\|(D^{[2]}S)_w\| \leq a + 1/2 < 1$, and the desired spectral radius estimate follows. If $a \geq 1/2$ then it simplifies to

$$\|(DS^{[2]})_w z\|_e \leq aB, \qquad \|(DS^{[2]})_w z\|_o \leq A/2 + aB.$$

Thus, if we take an arbitrary $z \in \ell_\infty(\mathbb{Z})$ with $\|z\| \leq 1$ and denote $z^j := (DS^{[2]})_{w^{j-1}} \ldots (DS^{[2]})_{w^0} z$, $j \geq 1$, then

$$\|z^j\| = \max(\|z^j\|_e, \|z^j\|_o) \leq \|M_a^j\|, \qquad M_a := \begin{pmatrix} 0 & a \\ 1/2 & a \end{pmatrix}.$$

Thus, $\rho(\mathbf{M}, 2) \leq \rho(M_a)$ and the desired spectral radius estimate follows if

$$\rho(M_a) = (a + \sqrt{a(a+2)})/2 < 1.$$

But this is equivalent to $p/4 = a < 2/3$, i.e., $p < 8/3$ as was stated in b). This concludes the proof of Theorem 3.5.

For $\mathbf{M}$, the case $8/3 \leq p \leq 4$ is still open, while stability for $\mathbf{S}$ is not settled for $p \geq 8/3$.

Remark 3.7 *The stability theorem from [6] is not applicable to the power-p subdivision.*

Suppose we were able to write

$$Sv = S_v v$$

with a Lipschitz continuous family of linear subdivision operators S_v, which are local 2-shift invariant and have order of polynomial reproduction 2. Then

$$S_w^{[2]} = (S_v)^{[2]} w \qquad w = \Delta^2 v,$$

where $(S_v)^{[2]}$ is another well-defined family of Lipschitz continuous, local, 2-shift invariant subdivision operators.

Then we must have

$$\left(S_w^{[2]}\right)_0 = \frac{1}{4} H_p(w_{-1}, w_0) = \sum_{|k|\le L} a_k(v|_{[-L,L]}) w_{-k}$$

where the coefficient functions a_k represent the entries $\left((S_v)^{[2]}\right)_{0,-k}$ and are thus Lipschitz continuous.

Let us now specialize to sequences of the form

$$v_i = \begin{cases} y(i-1) & , \ i \ge 1 \\ -xi & , \ i \le 0 \end{cases} .$$

Obviously $w_{-1} = x$, $w_0 = y$, $w_i = 0$ otherwise, and

$$H_p(x, y) = \tilde{a}_{-1}(x, y)x + \tilde{a}_0(x, y)y$$

with Lipschitz continuous $\tilde{a}_{-1}$ and $\tilde{a}_0$.
Now set $x = (1 + \alpha)y$ with fixed but arbitrary $\alpha > 0$, and let $y \to 0$:

$$\tilde{a}_{-1}(0, 0)(1 + \alpha) + \tilde{a}_0(0, 0) = \lim_{y\to 0} \frac{x + y}{2y}\Big(1 - (\frac{x - y}{x + y})^p\Big) = \big(1 + \frac{\alpha}{2}\big)\Big(1 - (\frac{\alpha}{2 + \alpha})^p\Big).$$

Such an identity can not hold for any $p \ge 1$. Indeed, for a non-integer $p > 1$ this is obvious, while for an integer $p \ge 2$ the right-hand side is a rational function of α with a non-trivial denominator, and thus can not coincide with the linear function from the left-hand side. For $p = 1$ we get

$$\tilde{a}_{-1}(0, 0)(1 + \alpha) + \tilde{a}_0(0, 0) = 1,$$

i.e., $\tilde{a}_{-1}(0, 0) = 0$, $\tilde{a}_0(0, 0) = 1$, which leads to a contradiction if we repeat the same exercise with $y = (1 + \alpha)x$.

4 Discussion

In the previous section, we have demonstrated the applications of our stability theorems to two examples. As was mentioned in Section 2.3, the stability results for quasi-linear schemes [6] are covered as well. In particular, stability of WENO subdivision **S** and multiresolution **M** can be established using Theorem 2.8. The case study [18] relates the results of [26] on **S** stability of a large class of monotonicity- and convexity-preserving schemes to our above framework. This list could be continued, however, it is also worthwhile to look into other extensions, in particular, into other types of nonlinear multiresolutions that are not covered by the formula (1.2).

4.1 Normal Multiresolution

Normal multiresolution for surfaces [17, 23, 24] is an example of a multiscale transform where nonlinearity is introduced not via a nonlinear subdivision operator S but via a nonlinear transformation T for the detail part. The univariate case (i.e., normal multiresolution of curves in the plane) is investigated in [9]. For illustration purposes, we consider the simplest case, where $r = 2$, and S corresponds to a cubic interpolatory scheme. To describe the transformation step

$$v^j = M(v^{j-1}, d^j) = Sv^{j-1} + T(v^{j-1})d^j, \qquad j \geq 1,$$

we need to use vector-valued sequences $v^j = \{v_i^j = (x_i^j, y_i^j)^T\} \in \ell_\infty(\mathbb{Z})^2$ and scalar-valued detail sequences $d^j = \{d_i^j\} \in \ell_\infty(\mathbb{Z})$. Then the definition of M in this case is

$$v_{2i}^j = v_i^{j-1}, \qquad v_{2i+1}^j = \underbrace{\frac{-v_i^{j-1} + 9v_i^{j-1} + 9v_{i+1}^{j-1} - v_{i+2}^{j-1}}{16}}_{=:(Sv^{j-1})_{2i+1}} + \underbrace{\frac{((\Delta v^{j-1})_i)^\perp}{\|(\Delta v^{j-1})_i\|}}_{=:n_i^{j-1}} d_i^j, \qquad i \in \mathbb{Z},$$

where $((x, y)^T)^\perp = (-y, x)^T$ for any vector. Thus, this scheme is interpolating, and new points are inserted along the normal direction n_i^{j-1} to the segment Δv_i^{j-1} between two consecutive points v_i^{j-1} and v_{i+1}^{j-1} by marching a distance d_i^j from the point $(Sv^{j-1})_{2i+1}$ predicted by the linear scheme. For a smooth curve, in the analysis step of this transform one starts from a coarse set of points on the curve represented by a finite section of v^0, and determines the d^j such that the newly inserted points v_{2i+1}^j are given by the closest intersection point of the above described normal with the curve. This explains the name normal multiresolution. A particular advantage of the scheme compared to a traditional transform $\mathbf{M}$ as considered before is that only scalar detail sequences need to be stored, and that in addition the entries d_i^j are much smaller due to choosing the geometry-adapted local coordinate frame given by Δv_i^{j-1} and n_i^{j-1}. We refer to [9] for more details, and, in particular, for the exact formulation and the technically cumbersome proof of the following property of the above described normal multiscale transform: If the given curve is in the Lipschitz class $C^{1+\alpha}$ for some $\alpha > 0$, and v^0 is not too "coarse", then

$$\|d^j\| = \mathrm{O}(2^{-j(1+\beta)}), \qquad j \to \infty, \tag{4.1}$$

for any $\beta < \min(\alpha, 1)$ (the bound $\beta \leq 1$ comes from our choice for S which produces limit curves of Hölder class $C^{2-\epsilon}$ for any $\epsilon > 0$).

Let us sketch the proof of a stability result using our approach (see [9] for a different treatment). Due to the structure of the formula for the above $\mathbf{M}$, the definition (1.6) of stability seems natural. Obviously, the first derived multiscale transform $M^{[1]}$ exists, and can be written as follows:

$$\begin{aligned} w_{2i}^j &= \frac{w_{i-1}^{j-1}}{16} + \frac{w_i^{j-1}}{2} - \frac{w_{i+1}^{j-1}}{16} + \frac{(w_i^{j-1})^\perp}{\|w_i^{j-1}\|} d_i^j, \\ w_{2i+1}^j &= -\frac{w_{i-1}^{j-1}}{16} + \frac{w_i^{j-1}}{2} + \frac{w_{i+1}^{j-1}}{16} - \frac{(w_i^{j-1})^\perp}{\|w_i^{j-1}\|} d_i^j, \end{aligned}$$

where Δv^j is replaced by w^j. Following our approach, for proving stability for $\mathbf{M}$, Lipschitz contractivity of $M^{[1]}$ after several iterations would be desirable. As in the previous sections, let us rely on formally differentiating the above formulas using the short-hand notation $w_i^j = (X_i^j, Y_i^j)^T$, $\hat{X}_i^j := X_i^j/\|w_i^{j-1}\|$, $\hat{Y}_i^j := Y_i^j/\|w_i^{j-1}\|$, and $s_i^j := d_i^j/\|w_i^{j-1}\|$:

$$\begin{aligned}
\mathrm{d}X_{2i}^j &= \mathrm{d}\Big(\frac{X_{i-1}^{j-1}}{16} + \frac{X_i^{j-1}}{2} - \frac{X_{i+1}^{j-1}}{16} - \hat{Y}_i^{j-1} d_i^j\Big) \\
&= \frac{1}{16}\mathrm{d}X_{i-1}^{j-1} + \big(\frac{1}{2} + s_i^j \hat{X}_i^{j-1}\hat{Y}_i^{j-1}\big)\mathrm{d}X_i^{j-1} - \frac{1}{16}\mathrm{d}X_{i+1}^{j-1} - s_i^j(\hat{X}_i^{j-1})^2\mathrm{d}Y_i^{j-1} - \hat{Y}_i^{j-1}\mathrm{d}d_i^j \\
\mathrm{d}Y_{2i}^j &= \mathrm{d}\Big(\frac{Y_{i-1}^{j-1}}{16} + \frac{Y_i^{j-1}}{2} - \frac{Y_{i+1}^{j-1}}{16} + \hat{X}_i^{j-1} d_i^j\Big) \\
&= s_i^j(\hat{Y}_i^{j-1})^2\mathrm{d}X_i^{j-1} + \frac{1}{16}\mathrm{d}Y_{i-1}^{j-1} + \big(\frac{1}{2} - s_i^j \hat{X}_i^{j-1}\hat{Y}_i^{j-1}\big)\mathrm{d}Y_i^{j-1} - \frac{1}{16}\mathrm{d}Y_{i+1}^{j-1} + \hat{X}_i^{j-1}\mathrm{d}d_i^j,
\end{aligned}$$

similarly for X_{2i+1}^j, Y_{2i+1}^j. This allows us to identify the entries of the matrices A_j and B_j that characterize $DM_{w^{j-1},d^j}^{[1]}$ via the recursion $\mathrm{d}w^j = A_j \mathrm{d}w^{j-1} + B_j \mathrm{d}d^j$.

Thus, since $|\hat{X}_i^{j-1}|, |\hat{Y}_i^{j-1}| \le 1$ by definition, and $\|s^j\| \to 0$ at a geometric rate of $\mathrm{O}(2^{-j\beta})$ if (4.1) holds (this needs a little extra argument), we see that the infinite matrices A_j, B_j satisfy

$$\|A_j\| = \frac{5}{8} + \mathrm{O}(2^{-j\beta}), \qquad \|B_j\| = \mathrm{O}(1), \qquad j \to \infty.$$

Note that formally A_j converges to the matrix representation of the first derived scheme $S^{[1]}$ of the underlying linear subdivision scheme S. This crude estimate establishes stability, in particular, $\rho_s(\mathbf{M}, 1) \le 5/8$ would follow but this time only for the class of multiscale data v^0 and d^j for which $\|s^j\|$ decays at a geometric rate. A similar argument will go through for an arbitrary interpolating S whose limit Hölder smoothness is > 1.

We admit that this sketch of a proof needs to be backed by a rigorous argument. Since multivariate normal multiresolution, with its applications to surface and image compression, deserves a more thorough case study we will not pursue this here. The example is instructive in that important applications require the study of stability of multiscale transforms under various natural assumptions on the coarse representation v^0 and the detail sequences d^j, and not for *arbitrary* multiscale data, as was done in the previous sections.

4.2 Manifold-valued subdivision and multiscale transforms

Subdivision and multiscale transforms for manifold-valued functions has become an area of intensive research [34, 36, 35, 15, 37, 12, 39]. The departure point for the vast majority of the proposed schemes is a suitable linear subdivision operator S acting in the linear space the manifold is embedded in. Work concentrates on settling convergence, smoothness, and approximation orders by so-called proximity conditions while the issue of stability has been ignored so far. Unfortunately, it cannot be directly tackled within our approach either (since nonlinear derived schemes do not necessarily exist). Without entering into a detailed discussion, we will only look at two simple examples for the standard (matrix) Lie-group-valued case, and indicate possible solutions.

A very elegant and simple answer is available for the exp-log schemes for a given matrix Lie group G embedded in the linear matrix space X_G. Consider an arbitrary linear subdivision operator

S acting componentwise on X_G-valued sequences. Denote by g the associated Lie algebra which is a linear subspace in X_G. Suppose that the G-valued sequence v^0 is mapped to the g-valued sequence z^0 such that $v_i^0 = \exp(z_i^0)$ for all $i \in \mathbb{Z}$ (due to the fact that $\exp(\cdot) : g \to G$ is only a local diffeomorphism between g and G, the construction of z^0 is not unique which might cause some technical problems). Then, admitting g-valued detail sequences d^j, one can naturally define the synthesis part of a *nonlinear* multiscale transform for G-valued sequences (called exp-log scheme) from the *linear* multiscale transform of g-valued sequence given by S as follows:

$$v^j = \exp(Sz^{j-1} + d^j).$$

See [34, 37] for various concrete versions of this construction. Using this definition and using the smoothness of the exp map, stability of $\mathbf{M}$ is obvious. Similarly to normal multiresolution, the problem is shifted to the analysis part of the multiscale transform which needs a careful selection of the elements $z^j = \log(v_i^j) \in g$ that are used to determine the details according to $d^j = z^j - Sz^{j-1}$. In a fine-to-coarse approach, this problem is present especially for small j, and disappears for larger j if the limit curve is sufficiently smooth. To avoid these difficulties, a typical assumption is that v^0 itself corresponds to a sufficiently fine sampling of the limit object (see, e.g., [37] for a discussion). Removing such assumptions is one of the challenges in this area.

Another natural scheme is the projection analogue [35, 15] of linear schemes where S is applied directly to the G-valued v^{j-1}, modified by a X_G-valued correction d^j, and then projected back onto G by a suitable retraction $P : X_G \to G$. This works for general manifold-valued v^j as well. I.e., the multiscale transform is given by the formula

$$v^j = M(v^{j-1}, d^j) := P(Sv^{j-1} + d^j), \qquad j \geq 1.$$

Again, due to the appearance of the nonlinear map P, such a multiscale transform does not admit a derived scheme $M^{[1]}$, and our analysis does not apply directly. However, it is obvious that following the approach in Section 2.3, stability would follow if one establishes

$$\|DP_{x^j}S \dots DP_{x^1}S\| \leq C, \qquad j \geq 1 \qquad (x^j := Sv^{j-1} + d^j),$$

under reasonable conditions on v^0, d^j (and consequently x^j). We refer to possibly related work [16], where proximity conditions are studied using Taylor expansions of the commutator $PS - SP$.

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