Local Decomposition of Refinable Spaces and Wavelets

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Abstract

A convenient setting for studying *multiscale techniques* in various areas of applications is usually based on a sequence of nested closed subspaces of some function space $\mathcal{F}$ which is often referred to as *multiresolution analysis*. The concept of *wavelets* is a prominent example where the practical use of such a multiresolution analysis relies on explicit representations of the *orthogonal* difference between any two subsequent spaces. However, many applications prohibit the employment of a multiresolution analysis based on translation invariant spaces on all of $\mathbb{R}^s$, say. It is then usually difficult to compute orthogonal complements explicitly. Moreover, certain applications suggest using other types of complements, in particular, those corresponding to *biorthogonal wavelets*. The main objective of this paper is therefore to study possibly non-orthogonal but in a certain sense *stable* and even *local* decompositions of nested spaces and to develop tools which are not necessarily confined to the translation invariant setting. A convenient way of *parametrizing* such decompositions is to reformulate them in terms of matrix relations. This allows one to characterize *all* stable or local decompositions by identifying unique matrix transformations that carry one given decomposition into another one. It will be indicated how such a mechanism may help realizing several desirable features of multiscale decompositions and constructing stable multiscale bases with favorable properties. In particular, we apply these results to the identification of decompositions induced by local linear projectors. The importance of this particular application with regard to the construction of multiscale Riesz bases will be pointed out. Furthermore, we indicate possible specializations to orthogonal decompositions of spline spaces relative to non-uniform knot sequences, piecewise linear finite elements and principal shift invariant spaces. The common ground of all these examples as well as other situations of practical and theoretical interest is that *some initial* multiscale basis is available. The techniques developed here can then be used to generate from such an initial decomposition other ones with desirable properties pertaining to moment conditions or stability properties.

Key Words: Multiscale decompositions, stability, stable completions, local linear projectors, splines, finite elements, wavelets.

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1 Introduction

During the past few years *multiscale techniques* have become indispensable tools in several areas of mathematical applications. The perhaps best known representatives are *multigrid* and *multilevel methods* for the numerical solution of operator equations and the concept of *wavelet* expansions. The latter play an important role in certain areas of pure analysis [27] as well as in applications like signal processing, image and sound analysis [28, 23]. One way of viewing these concepts is to note that many tasks such as data fitting in geometric modeling, or data compression in signal and image analysis as well as the numerical treatment of operator equations can be formulated as the problem of *approximating* an (implicitly) given element f in some infinite dimensional Banach function space $\mathcal{F}$ by functions of some subspace S_j. But no matter how fine the level of discretization corresponding to S_j is chosen one such single scale of discretization will generally not be able to express the intrinsic properties of the searched element f reflecting the features of the underlying infinite dimensional space $\mathcal{F}$ which could be, for instance, a Sobolev or Besov space. Much more information is often available through studying *representations* of elements of $\mathcal{F}$ which, of course, also provide approximations as a by-product. Representations usually involve the *interaction of all* scales of discretization which often allows one to recapture asymptotic information. When $\mathcal{S}$ denotes a sequence of nested closed subspaces S_j of $\mathcal{F}$

$$S_0 \subset S_1 \subset \cdots \subset S_j \subset \cdots \subset \mathcal{F} \tag{1.1}$$

such that

$$\overline{\bigcup_{j \in \mathbb{N}_0} S_j} = \mathcal{F}, \tag{1.2}$$

such representations of f can, for instance, be obtained by

$$f = \sum_{j=0}^{\infty} (Q_j - Q_{j-1}) f, \tag{1.3}$$

where $Q_{-1} = 0$ and $\mathcal{Q} = \{Q_j\}_{j \in \mathbb{N}_0}$ is a sequence of uniformly bounded projectors Q_j from $\mathcal{F}$ onto S_j. It is clear that then the expansion (1.3) converges strongly in $\mathcal{F}$ and $(Q_j - Q_{j-1})f$ may be viewed as the *detail* of f needed to update information on f when passing to higher scales of discretization. To keep these updates possibly independent one needs at least that

$$Q_j Q_m = Q_j \quad \text{for all} \quad j \le m, \tag{1.4}$$

holds (for a more detailed discussion of related issues see [12, 13]). This condition is equivalent to the fact that the differences $Q_j - Q_{j-1}$ are projectors as well so that the detail $(Q_j - Q_{j-1})f$ sits in the particular complement

$$W_j = (Q_j - Q_{j-1})\mathcal{F} = (Q_j - Q_{j-1}) S_j \tag{1.5}$$

of S_{j-1} in S_j. One possible way of making practical use of the representation (1.3) is to determine appropriate bases for the complements W_j. Specifically, when the S_j are closed shift-invariant nested subspaces of $\mathcal{F} = L_2(\mathbb{R}^s)$ generated by the dilates and translates of a single *scaling function* one arrives at the classical notion of *multiresolution analysis* (cf. e.g. [5, 6, 23, 25, 28, 27]). Choosing then Q_j to be *orthogonal projectors*, gives rise to *wavelets* in the classical sense as basis functions for the *orthogonal* complements W_j of S_j in S_{j+1}.

The realization of such decompositions is therefore of central importance in wavelet analysis. It is this theme which will be addressed in this paper again, however, from a somewhat more general point of view in that we will neither insist on *orthogonal decompositions* nor on translation invariant spaces defined on all of $\mathbb{R}^s$. In fact, while in many applications it is of predominant importance to work with basis functions of compact support, it is known that the construction of genuinely multivariate orthonormal wavelets with compact support is a very difficult problem. This is even more so when the presence of boundary conditions relative to complicated domain geometries excludes the employment of the stationary translation invariant setting. Moreover, certain applications, for instance, the fast solution of differential, integral or, more generally, pseudodifferential equations [14, 19, 20] suggest considering certain other decompositions of nested shift-invariant spaces which are induced by local quasiinterpolants or collocation projectors like cardinal interpolation.

1.1 Background Information and Motivation

We proceed outlining briefly two such applications that have recently been studied e.g. in [14, 19, 20]. These investigations are concerned with the numerical solution of a linear operator equation

$$\mathcal{A}u = f, \tag{1.6}$$

along with appropriate side conditions. Here $\mathcal{A}$ is typically some pseudodifferential operator belonging to a sufficiently wide class of interest and is supposed to take some Sobolev space H^t into H^{t-r} where r is the order of $\mathcal{A}$. Specifically, suppose $\mathcal{A}$ is elliptic in the sense that

$$a(u,v) := (\mathcal{A}u, v), \tag{1.7}$$

where $(\cdot,\cdot)$ denotes the inner product in the underlying L_2-space, induces a norm which is equivalent to the corresponding Sobolev norm, i.e.,

$$a(u,u) \sim \|u\|_{r/2}^2, \quad u \in H^{r/2}.$$

Here and in the following '$\sim$' will always mean that the quantities on either side can be estimated by each other up to constants which are independent of the parameters these quantities may depend on. A common approach would be to use a Galerkin method based on appropriate finite dimensional spaces S_j of increasing dimension. In particular, when $\mathcal{A}$ is a partial differential operator of integral order $r = 2k$ and Φ_j is a basis of S_j consisting of compactly supported functions, this gives rise to a usually very large but *sparse* stiffness matrix $\mathcal{A}_{\Phi_j}$. To preserve sparseness one typically has to resort to iterative solvers. Unfortunately, the spectral condition number $\kappa(\mathcal{A}_{\Phi_j})$ behaves like h_j^{-2k}, where h_j is the current 'meshsize' which quickly renders classical relaxation schemes prohibitively inefficient. One preconditioning strategy that has been attracting considerable attention during the past few years is to consider trial spaces S_j which as above arise through successive refinements of coarser spaces $S_0 \subset S_1 \subset \cdots \subset S_j \subset \cdots$. Let the finite dimensional operators C_j be defined by

$$(C_j^{-1}u, v) = (Q_0 u, Q_0 v) + \sum_{l=1}^{j} 2^{jr} \left((Q_l - Q_{l-1})u, (Q_l - Q_{l-1})v\right),$$

where as above the Q_l are bounded linear projectors from L_2 onto S_l. Under mild conditions on the spaces S_j which can be expressed in terms of classical *Bernstein* and *Jackson* estimates, this gives rise to optimal preconditioners in the sense that the spectral condition number $\kappa(C_j^{1/2}\mathcal{A}_j C_j^{1/2})$ remains even uniformly bounded in j. Here $\mathcal{A}_j$ is the finite dimensional operator induced by (1.7) through restriction on S_j. This suggests studying direct sum decompositions of the form

$$S_j = \bigoplus_{l=0}^{j} W_l, \quad W_l := (Q_l - Q_{l-1})S_l, \quad Q_{-1} = 0, \tag{1.8}$$

which corresponds to the telescopic expansion

$$v = \sum_{l=0}^{j} (Q_l - Q_{l-1})v$$

whenever (1.4) holds. The explicit computation of C_j can be based on representations of the form

$$(Q_{j+1} - Q_j)v = \sum_{k \in \nabla_j} d_{j,k}(v)\psi_{j,k}, \tag{1.9}$$

where

$$\Psi^j = \{\psi_{j,k} : k \in \nabla_j\}$$

is a basis of W_{j+1} [14].

The second example concerns the analysis of a general class of Petrov-Galerkin schemes for (1.6) when $\mathcal{A}$ belongs to a certain class of pseudodifferential operators on the s-dimensional torus. The trial spaces can then be generated by periodizing appropriate shift invariant spaces generated by the shifts and dilates of a single function φ. Likewise the test functionals can be then generated by shifts and scales of a single functional η. This setting covers classical Galerkin as well as collocation schemes. The finite dimensional operator which corresponds to such discretizations has the form

$$\mathcal{A}_j := Q_j \mathcal{A} P_j,$$

where P_j denotes the orthogonal projector onto the trial space S_j, and Q_j is a linear projector induced by η onto a possibly different finite dimensional space Y_j that is to be chosen appropriately. Since for the operators under consideration the corresponding matrix representations of $\mathcal{A}_j$ are generally not sparse, an issue of crucial importance for providing efficient numerical schemes, is to approximate $\mathcal{A}_j$ by a sparse operator while still controlling consistency, stability and convergence of the perturbed schemes. Such compression techniques can be based on the decomposition

$$A_j = \sum_{l,l'=0}^{j} (Q_l - Q_{l-1}) A (P_{l'} - P_{l'-1}), \quad Q_{-1} = P_{-1} = 0,$$

(cf. [2, 21, 20]). Again this leads to studying decompositions of the type (1.8) for possibly non-orthogonal operators Q_j.

1.2 Objectives and Contents

The objective of this paper is therefore to explore possibly general classes of direct sum decompositions of nested spaces. Our reason for keeping the whole setting as general as possible is to extract relevant information for a possibly wide range of cases of practical interest covering, for instance, multiresolution on bounded domains or manifolds. The idea is then to characterize the interrelation of different such decompositions in order to develop *general mechanisms* for identifying decompositions with certain desirable features.

The paper consists of two main parts. The first one is comprised of Sections 1, 2, and 3 which are devoted to a selfcontained general study of multiresolution and multiscale decompositions. The second part of the paper is to indicate some applications and discusses various specifications in some detail.

The first part is organized as follows. In Section 2 we set up and describe a general framework of multiresolution. After collecting some preparatory facts, which are needed, for instance, to discuss issues like stability, we reformulate space decompositions of the above type in terms of *matrix relations* and list some simple consequences in the spirit of our introductory remarks. In particular, we investigate several notions of stability and especially *stability over all levels*, looking for ways of constructing multiscale bases which are actually *Riesz bases* justifying the terminology *wavelet bases*.

We then make use of this *parametrization* of decompositions in terms of matrices in order to characterize *all* decompositions which are in some sense stable or local. This is done by showing that any decomposition can be obtained from any other specific one by means of certain matrix operations (see [11] for the shift-invariant case). The usefulness of these observations lies in the following facts. In many cases of interest (see part 2 of the paper) certain initial decompositions are easy to obtain. A typical example are hierarchical bases for nodal finite elements [31]. The results from Sections 2 and 3 can then be used to construct new multiscale bases and corresponding decompositions with particular desirable properties. For instance, this might concern raising the order of vanishing moments of the basis functions $\psi_{j,k}$ in (1.9) which is important in connection with matrix compression [2, 20, 21]. Another example is to identify a basis for complements of the form (1.5) when the Q_j are given. This will be seen to be of crucial importance with regard to the above mentioned issue of constructing multiscale bases which are actually Riesz bases and therefore give rise to well conditioned multiscale transformations [12]. We comment also on the construction of *orthogonal complements* for the general setting.

The remaining part of the paper, Section 4, will be concerned with specializations of these concepts. In particular, we will point out how to obtain an *initial decomposition* in various cases and identify some concrete modifications. Specifically, Section 4.1 is concerned with decompositions induced by *quasiinterpolants* for univariate spline spaces with *arbitrary* knot sequences as well as with the construction of *orthogonal decompositions* and pre-wavelets with compact support in this case. Section 4.2 deals with decompositions of classical *finite element spaces*. Finally, in Section 4.3 we consider nested principal *shift-invariant* spaces which corresponds to the classical wavelet setting. Again decompositions induced by local projectors will be identified as specific locally finite ones.

Other applications concerning the construction of multigrid ingredients are given in [10]. Raising the order of moment conditions in connection with matrix compression and boundary integral equations has been used in [21]. In [22] the results of Section 4 will be employed to construct stable wavelet or multiscale bases on two dimensional manifolds in $\mathbb{R}^3$ which give rise to optimal convergence and compression rates for multiscale Galerkin schemes for a wide range of boundary integral equations.

2 Multiresolution

Depending on the type of application multiresolution concepts may have to be adapted to different kinds of domains or norms. To be able to extract as many essential common features as possible we choose the following general framework. As before $\mathcal{F}$ will be some fixed Banach space endowed with a norm $\|\cdot\|_{\mathcal{F}}$. Typical examples are $\mathcal{F} = L_p(\Omega)$ where (Ω, μ) is some measure space and $\|f\|_{L_p(\Omega)} = (\int_\Omega |f(x)|^p d\mu(x))^{1/p}$, $1 \leq p \leq \infty$ with the usual interpretation for $p = \infty$. In particular, one is interested in those cases where Ω is a bounded domain or a closed manifold. As before $\mathcal{S}$ will always stand for a dense nested sequence of closed subspaces of $\mathcal{F}$ (see (1.6), (1.7)). In most practical situations the spaces S_j are finite dimensional. From a conceptual point of view we find it, however, desirable to cover the model case of classical multiresolution on $\mathbb{R}^s$, i.e., to admit also infinite dimensional spaces S_j in $\mathcal{S}$. Aside from this conceptual uniformity this appears to be an appropriate framework for dealing with the *asymptotics* of the whole multiresolution sequence $\mathcal{S}$ when it comes to discussing issues like stability.

2.1 Some Prerequisites and Stability

For any (countable) collection $\Phi = \{\varphi_k : k \in \Delta\}$ we will denote by

$$S(\Phi) := \mathrm{clos}_{\mathcal{F}}\,(\mathrm{span}\,\Phi)$$

the closure of the linear span of Φ in $\mathcal{F}$. It will be convenient to abbreviate expansions in Φ as

$$\Phi^T \mathbf{c} := \sum_{k\in\Delta} c_k \varphi_k. \tag{2.1}$$

To describe quantitative properties of the generating sets Φ and to explain how the sum on the right hand side of (2.1) is to be understood we will employ a *discrete* norm

$\|\cdot\|_{\ell(\Delta)}$ which will always assumed to be *monotone*. By this we mean that

$$\|\mathbf{c}\|_{\ell(\Delta)} \leq \|\tilde{\mathbf{c}}\|_{\ell(\Delta)}, \tag{2.2}$$

whenever $|c_k| \leq |\tilde{c}_k|$, $k \in \Delta$. The space of those $\mathbf{c} = \{c_k\}_{k\in\Delta}$ for which $\|\mathbf{c}\|_{\ell(\Delta)}$ is finite is denoted by $\ell(\Delta)$. Thus in particular, $\mathbf{e}_k := \{\delta_{k,k'}\}_{k'\in\Delta} \in \ell(\Delta)$. Moreover, monotonicity implies that $\|\cdot\|_{\ell(\tilde{\Delta})}$ and hence $\ell(\tilde{\Delta})$ is well-defined for any $\tilde{\Delta} \subseteq \Delta$. Typical examples are $\ell(\Delta) = \ell_p(\Delta)$ where for $1 \leq p < \infty$

$$\|\mathbf{c}\|_{\ell_p(\Delta)} := \left(\sum_{k\in\Delta} |c_k|^p\right)^{1/p}, \quad \|\mathbf{c}\|_{\ell_\infty(\Delta)} := \sup_{k\in\Delta} |c_k|,$$

or weighted versions of these norms.

We say that Φ is $(\ell, \mathcal{F})$-*stable* if there exist positive constants γ, Γ such that

$$\gamma\|\mathbf{c}\|_{\ell(\Delta)} \leq \|\Phi^T\mathbf{c}\|_{\mathcal{F}} \leq \Gamma\|\mathbf{c}\|_{\ell(\Delta)}, \quad \mathbf{c} \in \ell(\Delta). \tag{2.3}$$

Whenever the choice of ℓ and $\mathcal{F}$ is clear from the context we will briefly say that Φ is *stable*. In this case the right hand side of (2.1) is well-defined in $\mathcal{F}$ whenever $\mathbf{c} \in \Delta$ and

$$S(\Phi) = \{\Phi^T\mathbf{c} : \mathbf{c} \in \ell(\Delta)\}. \tag{2.4}$$

We will sometimes call Φ *generator basis*. Given normed linear spaces X, Y, we denote by $[X, Y]$ the set of all bounded linear operators from X into Y. As usual we define for $A \in [X, Y]$

$$\|A\|_{[X,Y]} := \sup_{\|x\|_X \leq 1} \|Ax\|_Y.$$

When X, Y are clear from the context we will often omit the subscript from the operator norm. The adjoint of $A \in [X, Y]$ will be denoted by A^*.

Remark 2.1 *While for finite Δ (2.3) is equivalent with the linear independence of Φ in general (2.3) means that the mapping*

$$F_\Phi : \mathbf{c} \to \Phi^T\mathbf{c}$$

is an isomorphism, *i.e.,*

$$F_\Phi \in [\ell(\Delta), S(\Phi)], \quad F_\Phi^{-1} \in [S(\Phi), \ell(\Delta)],$$

and

$$\begin{aligned} \|F_\Phi\|_{[\ell(\Delta),S(\Phi)]} &= \inf\{\Gamma : \Gamma \ \text{ satisfies } \ (2.3)\}, \\ \|F_\Phi^{-1}\|^{-1}_{[S(\Phi),\ell(\Delta)]} &= \sup\{\gamma : \gamma \ \text{ satisfies } \ (2.3)\}. \end{aligned} \tag{2.5}$$

2.2 Change of Bases

A central theme in subsequent considerations is to *change bases*. The only point to be made here is that under the assumption of stability things work essentially as in the finite dimensional case. In the course of the following discussion we list some simple observations. Nevertheless, we believe that their combination will help bringing out some useful and relevant facts.

To be specific, let us consider several collections $\Phi = \{\varphi_k : k \in \Delta_\Phi\}$ and $\Theta = \{\theta_k : k \in \Delta_\theta\}$.

Remark 2.2 *If Φ and Θ are stable and $S(\Phi) \subseteq S(\Theta)$, then there exists $\mathbf{A} \in [\ell(\Delta_\Phi), \ell(\Delta_\Theta)]$ such that*

$$\Phi^T \mathbf{c} = \Theta^T \mathbf{A}\mathbf{c}, \quad \mathbf{c} \in \ell(\Delta_\Phi). \tag{2.6}$$

For any numbering of Δ_Φ and Δ_Θ $\mathbf{A}$ is represented by a matrix $(a_{m,k})_{m\in\Delta_\Theta, k\in\Delta_\Phi}$ (henceforth also denoted by $\mathbf{A}$) whose columns belong to $\ell(\Delta_\Theta)$. Moreover, the sums $\sum_{k\in\Delta_\Phi} a_{m,k}c_k$ converge absolutely for all $\mathbf{c} \in \ell(\Delta_\Phi)$. In particular, this justifies interchanging the order of summation when dealing with multiple sums in this context.

Proof: By assumption, there exists for every $\mathbf{c} \in \ell(\Delta_\Phi)$ some $\mathbf{b} \in \ell(\Delta_\Theta)$ such that $F_\Phi \mathbf{c} = F_\Theta \mathbf{b}$, i.e., $\mathbf{b} = F_\Theta^{-1} \circ F_\Phi \mathbf{c}$. By the above remarks, stability of Φ and Θ implies that

$$\mathbf{A} := F_\Theta^{-1} \circ F_\Phi \in [\ell(\Delta_\Phi), \ell(\Delta_\Theta)] \tag{2.7}$$

which confirms (2.6). Choosing $\mathbf{c} = \mathbf{e}_k \in \ell(\Delta_\Phi)$ and setting $(a_{m,k})_{m\in\Delta_\Theta} := \mathbf{A}\mathbf{e}_k$ for $k \in \Delta_\Phi$, gives, on account of (2.6),

$$\varphi_k = \sum_{m\in\Delta_\Theta} a_{m,k}\theta_m,$$

and, again by stability, $(a_{m,k})_{m\in\Delta_\Theta} \in \ell(\Delta_\Theta)$. Moreover, considering for any $\mathbf{c} \in \ell(\Delta_\Phi)$, $m \in \Delta_\Theta$ fixed, any sequence $\tilde{\mathbf{c}}$ with $\tilde{c}_k = 0$ or $|\tilde{c}_k| = |c_k|$ such that $a_{m,k}\tilde{c}_k = |a_{m,k}\tilde{c}_k|$, monotonicity of the norm $\|\cdot\|_{\ell(\Delta_\Phi)}$ ensures that $\tilde{\mathbf{c}} \in \ell(\Delta_\Phi)$ whence the rest of the assertion follows. $\square$

Since we will make use of the following simple fact several times it is worth recording.

Remark 2.3 *If Φ is stable and $S(\Phi) = S(\Theta)$ then Θ is stable if and only if there exists an isomorphism $\mathbf{T} \in [\ell(\Delta_\Theta), \ell(\Delta_\Phi)]$ such that*

$$\Phi^T \mathbf{T}\mathbf{b} = \Theta^T \mathbf{b}, \quad \mathbf{b} \in \ell(\Delta_\Theta). \tag{2.8}$$

If Θ is stable analogous statements as in Remark 2.2 hold for the matrix representation of $\mathbf{T}$. Moreover, $\mathbf{T}$ satisfies for all $\mathbf{c} = \mathbf{T}\mathbf{b}$ the estimates

$$\|\mathbf{T}^{-1}\|^{-1}\|\mathbf{b}\|_{\ell(\Delta_\Theta)} \le \|\mathbf{c}\|_{\ell(\Delta_\Phi)} \le \|\mathbf{T}\|\|\mathbf{b}\|_{\ell(\Delta_\Theta)}, \tag{2.9}$$

and

$$\|\mathbf{T}^{-1}\|^{-1}\|F_\Phi^{-1}\|^{-1}\|\mathbf{b}\|_{\ell(\Delta_\Theta)} \le \|F_\Theta \mathbf{c}\|_{\ell(\Delta_\Phi)} \le \|\mathbf{T}\|\|F_\Phi\|\|\mathbf{b}\|_{\ell(\Delta_\Theta)}, \tag{2.10}$$

where we have omitted for notational simplicity the subscripts in the operator norms which are here clear from the context.

Proof: Since $S(\Phi) = S(\Theta)$ we can find for every $\mathbf{c} \in \ell(\Delta_\Phi)$ some $\mathbf{b} \in \ell(\Delta_\Theta)$ such that $F_\Phi \mathbf{c} = F_\Theta \mathbf{b}$ provided that Θ is stable. Thus $\mathbf{c} = F_\Phi^{-1} \circ F_\Theta \mathbf{b}$ and $\mathbf{T} := F_\Phi^{-1} \circ F_\Theta$ satisfies (2.8) and is clearly an isomorphism in $[\ell(\Delta_\Theta), \ell(\Delta_\Phi)]$. Conversely, if (2.8) holds and $\mathbf{T}$ is an isomorphism, $F_\Theta = F_\Phi \circ \mathbf{T}$ is boundedly invertible which confirms stability of Θ. The estimates (2.9), (2.10) are trivial consequences of (2.8). $\square$

2.3 Refinement Relations

The sequences $\mathcal{S}$ of nested closed subspaces S_j of $\mathcal{F}$ are supposed to have the form

$$S_j = S(\Phi_j), \quad j \in I\!N_0,$$

where the generators

$$\Phi_j = \{\varphi_{j,k} : k \in \Delta_j\}$$

will be at most countable sets. We say $\{\Phi\}$ is *uniformly stable* if there exist positive constants γ, Γ such that

$$\gamma \|\mathbf{c}\|_{\ell(\Delta_j)} \leq \|\Phi_j^T \mathbf{c}\|_{\mathcal{F}} \leq \Gamma \|\mathbf{c}\|_{\ell(\Delta_j)}, \quad \mathbf{c} \in \ell(\Delta_j),\ j \in I\!N_0, \tag{2.11}$$

where now the best uniform bounds are, in view of (2.5), given by

$$\gamma^{-1} = \sup_{j \in I\!N_0} \|F_{\Phi_j}^{-1}\|_{[S(\Phi_j), \ell(\Delta_j)]}, \quad \Gamma = \sup_{j \in I\!N_0} \|F_{\Phi_j}\|_{[\ell(\Delta_j), S(\Phi_j)]} < \infty. \tag{2.12}$$

Applying Remark 2.2 with $\Phi = \Phi_j$ and $\Theta = \Phi_{j+1}$, we infer the existence of *refinement matrices* $\mathbf{A}_{j,0} = (a^j_{m,k})_{m \in \Delta_{j+1}, k \in \Delta_j}$ such that

$$\varphi_{j,k} = \sum_{m \in \Delta_{j+1}} a^j_{m,k} \varphi_{j+1,m}, \quad k \in \Delta_j.$$

In accordance with (2.6) we can express this briefly as

$$\Phi_j^T = \Phi_{j+1}^T \mathbf{A}_{j,0}. \tag{2.13}$$

Moreover, we conclude from Remark 2.2 and (2.12) that the columns of the $\mathbf{A}_{j,0}$ are uniformly bounded in $\ell(\Delta_{j+1})$ and that

$$\|\mathbf{A}_{j,0}\|_{[\ell(\Delta_j), \ell(\Delta_{j+1})]} \leq \Gamma / \gamma. \tag{2.14}$$

2.4 Stable Completions

Without loss of generality we may assume that $\Delta_j \subset \Delta_{j+1}$ so that we can write

$$\Delta_{j+1} = \Delta_j \cup \nabla_j, \quad \Delta_j \cap \nabla_j = \emptyset.$$

Our objective is then to find *complement bases* $\Psi_j = \{\psi_{j,k} : k \in \nabla_j\}$ such that

$$S(\Phi_{j+1}) = S(\Phi_j) \oplus S(\Psi_j), \tag{2.15}$$

and that the collections $\Phi_j \cup \Psi_j$, $j \in I\!N_0$, are also uniformly stable. By the monotonicity of the discrete norms, this latter requirement implies, in particular, that the collections Ψ_j be uniformly stable. Employing Remark 2.2 with $\Theta = \Phi_{j+1}$ and $\Phi = \Psi_j$ the Ψ_j must therefore have the form

$$\Psi_j^T = \Phi_{j+1}^T \mathbf{A}_{j,1}, \tag{2.16}$$

where the matrices $\mathbf{A}_{j,1} = (a^j_{q,k})_{q \in \Delta_{j+1}, k \in \nabla_j}$ satisfy

$$(a^j_{m,k})_{m \in \Delta_{j+1}} \in \ell(\Delta_{j+1}), \quad k \in \Delta_j, \tag{2.17}$$

and

$$\mathbf{A}_{j,1} \in [\ell(\nabla_j), \ell(\Delta_{j+1})], \quad \|\mathbf{A}_{j,1}\|_{[\ell(\nabla_j),\ell(\Delta_{j+1})]} = \mathcal{O}(1), \quad j \in I\!N_0. \tag{2.18}$$

Although $\ell(\Delta_{j+1})$ can be identified with $\ell(\Delta_j \cup \nabla_j)$ we use $\binom{\mathbf{c}}{\mathbf{d}} \in \ell(\Delta_j \cup \nabla_j)$ to indicate the reference of the portions $\mathbf{c}, \mathbf{d}$ to Δ_j, ∇_j, respectively. Monotonicity of the discrete norms implies, in particular, that $\mathbf{c} \in \ell(\Delta_j)$ and $\mathbf{d} \in \ell(\nabla_j)$. Accordingly, we define the operator $\mathbf{A}_j : \ell(\Delta_j \cup \nabla_j) \to \ell(\Delta_{j+1})$ by

$$\mathbf{A}_j \begin{pmatrix} \mathbf{c} \\ \mathbf{d} \end{pmatrix} := \mathbf{A}_{j,0}\mathbf{c} + \mathbf{A}_{j,1}\mathbf{d}$$

and write briefly $\mathbf{A}_j = (\mathbf{A}_{j,0}, \mathbf{A}_{j,1})$. On account of (2.14) and (2.18), we have

$$\|\mathbf{A}_j\|_{[\ell(\Delta_j \cup \nabla_j),\ell(\Delta_{j+1})]} = \mathcal{O}(1), \quad j \in I\!N_0, \tag{2.19}$$

provided that $\{\Psi_j\}$ is uniformly stable. We may now summarize the above observations as follows.

Proposition 2.1 *Suppose that Φ_j, Φ_{j+1} and Ψ_j are interrelated by (2.13) and (2.16). The collections Φ_{j+1} and $\Phi_j \cup \Psi_j$ are stable bases for $S(\Phi_{j+1})$ if and only if $\mathbf{A}_j^{-1} \in [\ell(\Delta_{j+1}), \ell(\Delta_j \cup \nabla_j)]$, i.e., there exists $\mathbf{B}_j \in [\ell(\Delta_{j+1}), \ell(\Delta_j \cup \nabla_j)]$ such that*

$$\mathbf{A}_j\mathbf{B}_j = \mathbf{B}_j\mathbf{A}_j = \mathbf{I}, \tag{2.20}$$

and

$$\|\mathbf{B}_j\|^{-1}\|F^{-1}_{\Phi_{j+1}}\|^{-1}\|\begin{pmatrix} \mathbf{c} \\ \mathbf{d} \end{pmatrix}\|_{\ell(\Delta_j\cup\nabla_j)} \le \|\Phi_j^T\mathbf{c} + \Psi^T\mathbf{d}\|_{\mathcal{F}} \le \|\mathbf{A}_j\|\|F_{\Phi_{j+1}}\|\|\begin{pmatrix} \mathbf{c} \\ \mathbf{d} \end{pmatrix}\|_{\ell(\Delta_j\cup\nabla_j)}. \tag{2.21}$$

Here we have again omitted the subscripts of the operator norms. Moreover, $\mathbf{B}_j$ can be blocked as $\mathbf{B}_j = \binom{\mathbf{B}_{j,0}}{\mathbf{B}_{j,1}}$ where $\mathbf{B}_{j,0} \in [\ell(\Delta_{j+1}), \ell(\Delta_j)]$, $\mathbf{B}_{j,1} \in [\ell(\Delta_{j+1}), \ell(\nabla_j)]$ and

$$\Phi_{j+1}^T = \Phi_j^T\mathbf{B}_{j,0} + \Psi_j^T\mathbf{B}_{j,1}. \tag{2.22}$$

Proof: Applying Remark 2.3 with $\Phi = \Phi_{j+1}$, $\Theta = \Phi_j \cup \Psi_j$, stability of $\Phi_j \cup \Psi_j$ is seen to be equivalent to the existence of some isomorphism $\mathbf{T}_j \in [\ell(\Delta_j \cup \nabla_j), \ell(\Delta_{j+1})]$ such that

$$\Phi_{j+1}^T\mathbf{T}_j \begin{pmatrix} \mathbf{c} \\ \mathbf{d} \end{pmatrix} = \Phi_j^T\mathbf{c} + \Psi_j^T\mathbf{d}. \tag{2.23}$$

Inserting (2.13) and (2.16) on the right hand side of (2.23) and using stability of Φ_{j+1}, shows that

$$\mathbf{T}_j = \mathbf{A}_j = (\mathbf{A}_{j,0}, \mathbf{A}_{j,1}).$$

Setting $\mathbf{T}_j^{-1} := \mathbf{B}_j \in [\ell(\Delta_{j+1}), \ell(\Delta_j \cup \nabla_j)]$, the estimate (2.21) is an immediate consequence of Remark 2.3 (2.10) and we obtain

$$(b^j_{m,k})_{m\in\Delta_j\cup\nabla_j} := \mathbf{B}_j \mathbf{e}^{j+1,k} \in \ell(\Delta_j \cup \nabla_j), \quad k \in \Delta_{j+1}. \tag{2.24}$$

The submatrices $(b^j_{m,k})_{m\in\Delta_j, k\in\Delta_{j+1}}, (b^j_{m,k})_{m\in\nabla_j, k\in\Delta_{j+1}}$ correspond to the blocking $\mathbf{B}_j = \binom{\mathbf{B}_{j,0}}{\mathbf{B}_{j,1}}$, where the monotonicity of the norm $\|\cdot\|_{\ell(\Delta_j)}$ implies that $\mathbf{B}_{j,0} \in [\ell(\Delta_{j+1}), \ell(\Delta_j)]$, $\mathbf{B}_{j,1} \in [\ell(\Delta_{j+1}), \ell(\nabla_j)]$. Inserting $\tilde{\mathbf{c}} := \mathbf{T}_j\binom{\mathbf{c}}{\mathbf{d}}$, i.e., $\binom{\mathbf{c}}{\mathbf{d}} = \mathbf{B}_j\tilde{\mathbf{c}}$ into (2.23), yields (2.22) or equivalently

$$\varphi_{j+1,k} := \sum_{m\in\Delta_j} b^j_{m,k}\varphi_{j,m} + \sum_{m\in\nabla_j} b^j_{m,k}\psi_{j,m}, \quad k \in \Delta_{j+1}, \tag{2.25}$$

noting that that, in view of (2.24), the right hand side of (2.25) is well-defined. Inserting (2.13) and (2.16) into the right hand side of (2.22) and using stability of Φ_{j+1}, gives

$$\mathbf{A}_{j,0}\mathbf{B}_{j,0} + \mathbf{A}_{j,1}\mathbf{B}_{j,1} = \mathbf{I}, \tag{2.26}$$

which is $\mathbf{A}_j\mathbf{B}_j = \mathbf{I}$. Multiplying both sides of (2.22) by $\mathbf{A}_{j,0}$ and $\mathbf{A}_{j,1}$, using stability of $\Delta_j \cup \nabla_j$ as well as (2.13) and (2.16), respectively, yields

$$\mathbf{B}_{j,i}\mathbf{A}_{j,i'} = \delta_{i,i'}\mathbf{I}, \quad i, i' \in \{0, 1\}, \tag{2.27}$$

which means $\mathbf{B}_j\mathbf{A}_j = \mathbf{I}$ and hence (2.20). $\square$

Note that, by Remarks 2.2 and 2.3, the product (2.20) is well-defined for any matrix representations of $\mathbf{A}_j, \mathbf{B}_j$ relative to any fixed numbering of the elements in the index sets. Since, on account of these observations, stability permits us to treat these matrices formally as finite ones we will in the following identify always operator and its (fixed yet unspecified) matrix representation and perform corresponding operations like matrix multiplication without further comment.

Corollary 2.1 *Suppose that $\{\Phi_j\}$ is uniformly stable and that Φ_j, Ψ_j satisfy (2.13), (2.16), respectively. Then $\{\Phi_j \cup \Psi_j\}$ is uniformly stable for $\mathcal{S}$ if and only if*

$$\|\mathbf{A}_j\| = \mathcal{O}(1), \quad \|\mathbf{B}_j\| = \mathcal{O}(1), \quad j \to \infty, \tag{2.28}$$

where the $\mathbf{B}_j$ satisfy (2.20) and hence (2.22).

Given the $\mathbf{A}_{j,0}$ any $\mathbf{A}_{j,1}$ such that

$$\mathbf{A}_j \in [\ell(\Delta_j \cup \nabla_j), \ell(\Delta_{j+1})], \quad \mathbf{A}_j^{-1} \in [\ell(\Delta_{j+1}), \ell(\Delta_j \cup \nabla_j)] \tag{2.29}$$

is called **stable completion** of $\mathbf{A}_{j,0}$. A case of particular practical importance arises when the $\mathbf{A}_j$ are (uniformly) **banded** by which we mean that the rows and columns of $\mathbf{A}_j$ contain only a uniformly bounded finite number of nonzero entries and that the supports of only a uniformly bounded finite number of columns have nonempty intersection. This is typically in correspondence with the $\varphi_{j,k}$ and $\psi_{j,k}$ having compact support whose size shrinks with increasing j so that locally only a uniformly bounded finite number of functions is different from zero. More precisely, the completions $\mathbf{A}_{j,1}$ of $\mathbf{A}_{j,0}$ are called **local** if the $\mathbf{A}_j$ and $\mathbf{B}_j$ are both banded. In analogy to the discrete case we call the Φ_j *locally finite* if there exists some $N \in I\!N$ which is independent of j and x such that

$$\#\{k \in \Delta_j : \varphi_{j,k}(x) \neq 0\} \leq N. \tag{2.30}$$

This property is of predominant importance in many applications. For locally finite Φ_j it makes also sense to consider *arbitrary* linear combinations of the basis functions in Φ_j. Accordingly, with a slight abuse of terminology when Φ_j has infinite cardinality, the concept of *linear independence* is well-defined. The functions $\varphi_{j,k}$ are said to be linearly independent if the mappings F_{Φ_j} are *injective* on the space of *all* sequences $\mathbf{c} : \Delta_j \to \mathbb{C}$ (see also [1, 32]).

When dealing with local completions, the uniform stability of $\{\Phi_j \cup \Psi_j\}$ is particularly easy to check. In fact, the uniform boundedness of the operator norms of the $\mathbf{A}_j$ and $\mathbf{B}_j$ follows e.g. for standard ℓ_p-norms already from the uniform boundedness of the entries of the respective matrix representations.

As a simple example take S_j as the space of piecewise linear continuous functions on $[0,1]$ relative to some set Δ_j of knots in $[0,1]$. Let Δ_{j+1} be obtained by adding one additional knot between any two knots in Δ_j, i.e., ∇_j is comprised of the *new* knots between the old ones in Δ_j. Here the $\varphi_{j,k}, \varphi_{j+1,k}$ are the classical Courant *hat functions* on Δ_j, Δ_{j+1}, respectively. In this case one simply has $\psi_{j,k} = \varphi_{j+1,k}$ for $k \in \nabla_j$, i.e., $a^j_{m,k} = \delta_{m,k}, k \in \nabla_j, m \in \Delta_{j+1}$, $a^j_{m,k} = \varphi^j_k(m), k \in \Delta_j, m \in \Delta_{j+1}$, $b^j_{m,k} = \delta_{m,k}, k \in \nabla_j, m \in \Delta_{j+1}$, and $b^j_{m,k} = \delta_{k,m} - \varphi^j_k(m), k \in \Delta_j, m \in \Delta_{j+1}$. Here we have ignored for the moment a proper normalization of the hat functions which depends on the given norms. The bivariate analog will be discussed in more detail in Section 5 below.

Another important class of examples is encountered in connection with *principal shift-invariant spaces*. To be more specific, let S_j be the L_2-closure of the linear span of the translates $\varphi(2^j \cdot -k), k \in \mathbb{Z}^s$, where $\varphi \in L_2(I\!R^s)$ is *refinable*, i.e., there exists a *mask* $\{a_k\}_{k\in\mathbb{Z}^s}$ such that

$$\varphi(x) = \sum_{k\in\mathbb{Z}^s} a_k \varphi(2x-k) \tag{2.31}$$

holds for almost every $x \in I\!R^s$. Writing $\varphi(2^j x - k)$ as $\varphi(2^j(x - k/2^j))$ we see that here

$$\Delta_j = 2^{-j}\mathbb{Z}^s,$$

i.e.,

$$\Delta_1 \hat{=} \Delta_0 \times E, \quad E := \{0,1\}^s. \tag{2.32}$$

E is a set of representers of $\mathbb{Z}^s/2\mathbb{Z}^s$. Moreover, one has in this case

$$a^j_{k,l} = a_{k-2l}. \tag{2.33}$$

This gives rise to a Hurwitz-type matrix which was referred to as a *slanted* matrix in [17]. Section 4.3 is devoted to a more detailed discussion of this setting.

2.5 Multiscale Transformations

So far we have established simple criteria under which the collection

$$\Psi^j := \Phi_0 \cup \left(\cup_{l=0}^{j-1} \Psi_l\right) \tag{2.34}$$

forms also a basis for $S_j = S(\Phi_j)$. Ψ^j will be referred to as a *multiscale basis* for S_j. We will assume throughout the remainder of this section that $\{\Phi_j \cup \Psi_j\}$ is uniformly stable.

The transformation $\mathbf{T}_j$ that takes the *multiscale coefficients* $\mathbf{d}^{(j)} = (\mathbf{c}^0, \mathbf{d}^0, \ldots, \mathbf{d}^{j-1})$ in the *multiscale representation*

$$f_j = \sum_{l=-1}^{j-1} \sum_{k\in\nabla_l} d_{l,k}\psi_{l,k}$$

of some $f_j \in S_j$ relative to Ψ^j into the *single scale* coefficients $\mathbf{c}^j$ in

$$f_j = \sum_{k\in\Delta_j} c_{j,k}\varphi_{j,k},$$

i.e.,

$$\mathbf{c}^j = \mathbf{T}_j \mathbf{d}^{(j)}, \tag{2.35}$$

can be realized, of course, by *pyramid schemes* as they arise in the wavelet transform (cf. [23, 28]). In fact, when

$$f_j = \Phi_j^T \mathbf{c}^j = \Phi_{j-1}^T \mathbf{c}^{j-1} + \Psi_{j-1}^T \mathbf{d}^{j-1},$$

one easily concludes from (2.13), (2.16), and (2.22) that

$$\mathbf{c}^j = \mathbf{A}_{j-1,0}\mathbf{c}^{j-1} + \mathbf{A}_{j-1,1}\mathbf{d}^{j-1} \tag{2.36}$$

$$\mathbf{c}^{j-1} = \mathbf{B}_{j-1,0}\mathbf{c}^j, \qquad \mathbf{d}^{j-1} = \mathbf{B}_{j-1,1}\mathbf{c}^j. \tag{2.37}$$

Iteration shows that the execution of $\mathbf{T}_j$ is equivalent to the scheme

$$\begin{array}{ccccccccc}
 & \mathbf{A}_{0,0} & & \mathbf{A}_{1,0} & & & & \mathbf{A}_{j-1,0} & \\
\mathbf{c}^0 & \rightarrow & \mathbf{c}^1 & \rightarrow & \mathbf{c}^2 & \rightarrow \cdots & & \rightarrow & \mathbf{c}^j \\
 & \mathbf{A}_{0,1} & & \mathbf{A}_{1,1} & & & & \mathbf{A}_{j-1,1} & \\
 & \nearrow & & \nearrow & & \nearrow \cdots & & \nearrow & \\
\mathbf{d}^0 & & \mathbf{d}^1 & & \mathbf{d}^2 & & & \mathbf{d}^{j-1} &
\end{array} \tag{2.38}$$

Similarly, the inverse procedure has the form

$$\begin{array}{ccccccccc}
 & \mathbf{B}_{j-1,0} & & \mathbf{B}_{j-2,0} & & & & \mathbf{B}_{0,0} & \\
\mathbf{c}^j & \rightarrow & \mathbf{c}^{j-1} & \rightarrow & \mathbf{c}^{j-2} & \rightarrow \cdots & & \rightarrow & \mathbf{c}^0 \\
 & \mathbf{B}_{j-1,1} & & \mathbf{B}_{j-2,1} & & & & \mathbf{B}_{0,1} & \\
 & \searrow & & \searrow & & \searrow \cdots & & \searrow & \\
 & & \mathbf{d}^{j-1} & & \mathbf{d}^{j-2} & & & & \mathbf{d}^0
\end{array} \tag{2.39}$$

Alternatively, setting

$$\hat{\mathbf{A}}_l := \begin{pmatrix} \mathbf{A}_l & 0 \\ 0 & I \end{pmatrix},$$

where I is the identity matrix corresponding to the space $W_l \oplus \cdots \oplus W_{j-1}$, we obtain

$$\mathbf{T}_j = \hat{\mathbf{A}}_{j-1} \cdots \hat{\mathbf{A}}_0. \tag{2.40}$$

Note that in typical data compression applications both transformations $\mathbf{T}_j$ and $\mathbf{T}_j^{-1}$ are needed. However, in connection with the numerical treatment of operator equations (1.6) only $\mathbf{T}_j$ is needed. In fact, denoting for instance by $\mathcal{A}_{\Phi_j}, \mathcal{A}_{\Psi^j}$ the stiffness matrices of $\mathcal{A}$ relative to the bases Φ_j, Ψ^j, respectively, the system

$$\mathcal{A}_{\Phi_j}\mathbf{c}^j = \mathbf{f}^j$$

is equivalent to

$$\mathcal{A}_{\Psi^j}\mathbf{d}^{(j)} = \mathbf{T}_j^*\mathbf{f}^j,$$

where $\mathcal{A}_{\Psi_j} = \mathbf{T}_j^*\mathcal{A}_{\Phi_j}\mathbf{T}_j$ and $\mathbf{c}^j = \mathbf{T}_j\mathbf{d}^{(j)}$.

An efficient application of multiscale techniques in any of these applications is only possible if the execution of $\mathbf{T}_j$ or $\mathbf{T}_j^{-1}$ requires only an order of floating point operations which is proportional to $\dim S_j = \#\Delta_j$. In typical applications one has $\#\Delta_j/\#\Delta_{j+1} \le \rho < 1$. In this case one infers for instance from (2.40) that $\mathbf{T}_j$ is efficient in the above sense if the $\mathbf{A}_j$ are banded.

2.6 Stability over All Levels

For the remainder of this section we will confine the discussion to the case that $\mathcal{F}$ is a *Hilbert space* with inner product $\langle\cdot,\cdot\rangle$ and that $\ell(\Delta_j) = \ell_2(\Delta_j)$. Aside from efficiency aspects it is important that the multiscale transformations $\mathbf{T}_j$ are well conditioned, by which we mean that

$$\|\mathbf{T}_j\|\ \|\mathbf{T}_j^{-1}\| = \mathcal{O}(1), \quad j \to \infty. \tag{2.41}$$

Condition (2.41) is intimately connected with norm equivalences for Hilbert spaces, in particular, for certain ranges of Sobolev spaces. It is also known to be crucial for the construction of optimal multilevel preconditioners for linear systems arising from Galerkin discretizations of elliptic operator equations [21]. A more detailed discussion of this issue can be found in [13]. We briefly recall a few facts which are relevant in this context.

Remark 2.4 *It is not difficult to see that (2.41) is equivalent to the fact that*

$$\Psi := \bigcup_{j=-1}^{\infty} \Psi_j, \quad \Psi_{-1} := \Phi_0,$$

is a Riesz-basis *and that there exists a biorthogonal Riesz basis* $\tilde{\Psi} = \{\tilde{\Psi}_{j,k} : (j,k) \in \nabla\}$, $\nabla := (\{0\} \times \Delta_0) \bigcup_{j\in\mathbb{N}_0}(\{j\} \times \nabla_j)$ *i.e.,*

$$\langle \psi_{j,k}, \tilde{\psi}_{j',k'} \rangle = \delta_{(j,k),(j',k')}, \quad (j,k),(j',k') \in \nabla, \tag{2.42}$$

and every $f \in \mathcal{F}$ admits a unique expansion

$$f = \sum_{(j,k)\in\nabla} \langle f, \tilde{\psi}_{j,k}\rangle \psi_{j,k}$$

such that

$$\|f\|_{\mathcal{F}} \sim \left(\sum_{(j,k)\in\nabla} |\langle f, \tilde{\psi}_{j,k}\rangle|^2 \right)^{1/2}, \quad f \in \mathcal{F}. \tag{2.43}$$

Remark 2.4 reveals that the choices of complement bases Ψ_j which are appropriate in the sense of (2.41) are seriously constrained. In particular, (2.41) is a significantly stronger condition than the uniform stability of $\{\Phi_j \cup \Psi_j\}_{j\in\mathbb{N}_0}$ where stability is always referred to each individual level j with uniform Riesz bounds. By contrast, loosely speaking, (2.43) means *stability over all levels*. So called *hierarchical bases* for bivariate piecewise linear finite element spaces are an important example for multiscale bases Ψ which are *not* stable over all levels but where $\{\Phi_j \cup \Psi_j\}$ is uniformly stable. A detailed discussion of this example is given in Section 4.2 below.

2.7 How to Ensure Stability over All Levels

Unfortunately, the characterization in Remark 2.4 does not tell us how to construct stable multiscale bases Ψ. It does tell us though that *biorthogonality* is a necessary ingredient. Let us briefly indicate now a principal strategy for constructing stable multiscale bases, i.e., multiscale bases which are stable over all levels. To this end, suppose first that Ψ is a Riesz basis and consider the *truncation operators*

$$Q_j f := \sum_{l=-1}^{j-1} \sum_{k\in\nabla_l} \langle f, \tilde{\psi}_{l,k}\rangle \psi_{l,k}, \quad Q_j^* f := \sum_{l=-1}^{j-1} \sum_{k\in\nabla_l} \langle f, \psi_{l,k}\rangle \tilde{\psi}_{l,k}. \tag{2.44}$$

They are, in view of (2.42) and (2.43), obviously uniformly bounded linear *projectors* which are adjoints of each other. Moreover

$$S_j = S(\Phi_j) = \operatorname{ran} Q_j, \quad \tilde{S}_j = \operatorname{ran} Q_j^*,$$

where

$$\tilde{S}_j := \bigoplus_{l=-1}^{j-1} S(\tilde{\Psi}_l).$$

The two sequences $\mathcal{S}, \tilde{\mathcal{S}}$ correspond to the two biorthogonal multiresolution sequences appearing in the context of biorthogonal wavelets for $L_2(\mathbb{R})$ [8].

Next observe that the Q_j defined by (2.44) satisfy

$$Q_l Q_j = Q_l \quad \text{for} \quad l \le j, \tag{2.45}$$

which is a way of expressing biorthogonality without specifying the complement bases (see [13] for further comments). Note also that for uniformly stable $\{\Psi_j\}$ (2.43) is equivalent to

$$\|f\|_{\mathcal{F}} \sim \left(\sum_{j=0}^{\infty} \|(Q_j - Q_{j-1})f\|_{\mathcal{F}}^2 \right)^{1/2}. \tag{2.46}$$

This suggests the following two-step procedure:

(i) Given $\{\Phi_j\}$, construct *uniformly $\mathcal{F}$-bounded* projectors $Q_j : \mathcal{F} \to S(\Phi_j)$, $j \in \mathbb{N}_0$, satisfying (2.45) such that (2.46) holds.

(ii) Identify uniformly stable bases $\{\Psi_j\}$ of the particular complement bases

$$W_j := (Q_{j+1} - Q_j)S(\Phi_{j+1}). \tag{2.47}$$

Condition (2.45) ensures that actually $W_j = (Q_{j+1} - Q_j)\mathcal{F}$ [13].

In general, (2.45) does not suffice to ensure (2.46). However, the point is that there exist criteria for the validity of (2.46) which do not require any knowledge of Ψ or $\tilde{\Psi}$ beforehand. In fact, it is shown in [13] that if $\mathcal{S}$ and $\tilde{\mathcal{S}}$ have certain (very weak) approximation and regularity properties stated in terms of direct and inverse estimates and if the Q_j satisfy (2.45) then (2.46) holds. We will therefore not address this point here any further but focus instead on (ii).

To this end, it is clear that candidates Q_j of suitable projectors, defined in terms of the Φ_j have the form

$$Q_j f = \sum_{k \in \Delta_j} \langle f, \tilde{\varphi}_{j,k} \rangle \varphi_{j,k}, \tag{2.48}$$

where the collection $\tilde{\Phi}_j = \{\tilde{\varphi}_{j,k} : k \in \Delta_j\} \subset \mathcal{F}$ is biorthogonal to Φ_j, i.e.,

$$\langle \varphi_{j,k}, \tilde{\varphi}_{j,k'} \rangle = \delta_{k,k'}, \quad k, k' \in \Delta_j. \tag{2.49}$$

Let us point out next how to interpret condition (2.45) when the projectors Q_j are given in the form (2.48).

Proposition 2.2 *Suppose that for uniformly stable $\{\Phi_j\}$ the Q_j defined by (2.48) are uniformly bounded in $\mathcal{F}$. Then the biorthogonal collections $\{\tilde{\Phi}_j\}_{j \in \mathbb{N}_0}$ are uniformly stable as well. Moreover, $\{Q_j\}_{j \in \mathbb{N}_0}$ satisfies (2.45) if and only if the $\tilde{\Phi}_j$ are also refinable, i.e., there exists $\tilde{\mathbf{A}}_{j,0} = (\tilde{a}^j_{m,k})_{m \in \Delta_{j+1}, k \in \Delta_j} \in [\ell_2(\Delta_j), \ell_2(\Delta_{j+1})]$ such that*

$$\tilde{\varphi}_{j,k} = \sum_{m \in \Delta_{j+1}} \tilde{a}^j_{m,k} \tilde{\varphi}_{j+1,m}, \quad k \in \Delta_j, \tag{2.50}$$

where

$$\tilde{\mathbf{A}}^*_{j,0} \mathbf{A}_{j,0} = \mathbf{I}, \tag{2.51}$$

and

$$\tilde{a}^j_{m,k} = \overline{\langle \varphi_{j+1,m}, \tilde{\varphi}_{j,k} \rangle}. \tag{2.52}$$

Proof: By uniform stability of $\{\Phi_j\}$ and biorthogonality (2.49), one has

$$\begin{aligned} \|\tilde{\Phi}_j^T \tilde{\mathbf{c}}\|_{\mathcal{F}} &= \sup_{f \in \mathcal{F}} \frac{\langle f, \tilde{\Phi}_j \tilde{\mathbf{c}} \rangle}{\|f\|_{\mathcal{F}}} \geq \sup_{f_j \in S_j} \frac{\langle f_j, \tilde{\Phi}_j \tilde{\mathbf{c}} \rangle}{\|f_j\|_{\mathcal{F}}} \\ &\geq \sup_{\mathbf{c} \in \ell_2(\Delta_j)} \frac{\left| \sum_{k \in \Delta_j} c_k \overline{\tilde{c}_k} \right|}{\|F_{\Phi_j}\| \|\mathbf{c}\|_{\ell_2(\Delta_j)}} = \|F_{\Phi_j}\|^{-1} \|\tilde{\mathbf{c}}\|_{\ell_2(\Delta_j)}. \end{aligned}$$

Conversely, since for $\tilde{f}_j := \tilde{\Phi}_j^T \tilde{\mathbf{c}}$ one has $\langle f, \tilde{f}_j\rangle = \langle Q_j f, \tilde{f}_j\rangle$ the uniform boundedness of the Q_j yields

$$\|\tilde{f}_j\|_{\mathcal{F}} = \sup_{f\in\mathcal{F}} \frac{|\langle Q_j f, \tilde{f}_j\rangle|}{\|f\|_{\mathcal{F}}} \lesssim \sup_{f\in\mathcal{F}} \frac{|\langle Q_j f, \tilde{f}_j\rangle|}{\|Q_j f\|_{\mathcal{F}}}.$$

Here $a \lesssim b$ means that a can be bounded by a constant multiple of b where the constant is independent of any parameters a and b may depend on. Since, on account of (2.49),

$$\langle Q_j f, \tilde{f}_j\rangle = \sum_{k\in\Delta_j} \langle f, \tilde{\varphi}_{j,k}\rangle \overline{\tilde{c}_k},$$

again uniform stability of $\{\Phi_j\}$ yields

$$\|\tilde{f}_j\| \lesssim \sup_{\mathbf{c}\in\ell_2(\Delta_j)} \frac{\left|\sum_{k\in\Delta_j} c_k \overline{\tilde{c}_k}\right|}{\|\mathbf{c}\|_{\ell_2(\Delta_j)}} = \|\tilde{\mathbf{c}}\|_{\ell_2(\Delta_j)},$$

which confirms the uniform stability of $\{\tilde{\Phi}_j\}$.

Next it is easy to see that the condition (2.45) is equivalent to the fact that the spaces

$$\tilde{S}_j := \operatorname{ran} Q_j^* = S(\tilde{\Phi}_j)$$

are also nested. Since $\{\tilde{\Phi}_j\}$ is uniformly stable we can invoke Remark 2.2 again to confirm the existence of $\tilde{\mathbf{A}}_{j,0} \in [\ell_2(\Delta_j), \ell_2(\Delta_{j+1})]$ satisfying (2.50). To identify $\tilde{\mathbf{A}}_{j,0}$ note that, since the Q_j^* are projectors and the $\tilde{S}_j$ are nested,

$$\tilde{\varphi}_{j,k} = Q_{j+1}^* \tilde{\varphi}_{j,k} = \sum_{m\in\Delta_{j+1}} \langle \tilde{\varphi}_{j,k}, \varphi_{j+1,m}\rangle \tilde{\varphi}_{j+1,m} = \sum_{m\in\Delta_{j+1}} \overline{\langle \varphi_{j+1,m}, \tilde{\varphi}_{j,k}\rangle} \tilde{\varphi}_{j+1,m},$$

which gives (2.52). Moreover, by Remark 2.2 and the fact that $\tilde{\mathbf{A}}_{j,0} \in [\ell_2(\Delta_j), \ell_2(\Delta_{j+1})]$, the product $\tilde{\mathbf{A}}_{j,0}^* \mathbf{A}_{j,0}$ is well-defined. By (2.49) and absolute convergence asserted by Remark 2.2, one has now

$$\begin{aligned}
\delta_{k,k'} &= \langle \varphi_{j,k}, \tilde{\varphi}_{j,k'}\rangle = \left\langle \sum_{m\in\Delta_{j+1}} a^j_{m,k} \varphi_{j+1,k}, \sum_{m'\in\Delta_{j+1}} \tilde{a}^j_{m',k'} \tilde{\varphi}_{j+1,m'} \right\rangle \\
&= \sum_{m,m'\in\Delta_{j+1}} a^j_{m,k} \overline{\tilde{a}^j_{m',k'}} \langle \varphi_{j,k}, \tilde{\varphi}_{j,k'}\rangle = \sum_{m\in\Delta_{j+1}} a^j_{m,k} \overline{\tilde{a}^j_{m,k'}},
\end{aligned}$$

which confirms (2.51) and finishes the proof. $\square$

A few comments on the hypotheses in Proposition 2.2 are in order. If Φ_j and $\tilde{\Phi}_j$ are biorthogonal, locally finite and if $\|\varphi_{j,k}\|_{\ell_2(\Delta_j)}, \|\tilde{\varphi}_{j,k}\|_{\ell_2(\Delta_j)} = \mathcal{O}(1),, j \in \mathbb{N}_0$ then $\{\Phi_j\}$ and $\{\tilde{\Phi}_j\}$ are uniformly stable and the Q_j (and hence the Q_j^*) are uniformly bounded. Similarly uniform stability of $\{\Phi_j\}$ and uniform boundedness of the $\|\tilde{\varphi}_{j,k}\|_{\ell_2(\Delta_j)}$ implies the uniform boundedness of the Q_j.

So far Proposition 2.48 reduces the construction of suitable projectors Q_j in step (i) above to the construction of biorthogonal refinable collections $\tilde{\Phi}_j$. When dealing with the classical case of multiresolution on $\mathbb{R}$ induced by the integer translates and dilates of a single generator such biorthogonal collections have been constructed through limit processes (see e.g. [8]). These biorthogonal systems in turn can be also adapted to intervals or more generally to bounded domains by means of suitable modifications near the boundary so that for a certain range of situations pairs of biorthogonal generator bases $\Phi_j, \tilde{\Phi}_j$ are indeed available.

But even in situations not covered by the above cases the construction of biorthogonal bases can, in principle, still be based on a limit process. A possible strategy can be outlined as follows. Given the refinement operators $\mathbf{A}_{j,0}$, one first tries to find left inverses $\tilde{\mathbf{A}}_{j,0}^*$ satisfying (2.51). Of course, the $\tilde{\mathbf{A}}_{j,0}$ are by no means unique. The central task is to find such left inverses that certain limit procedures often termed *subdivision schemes* converge. The limits of these schemes form the biorthogonal collections (see also Remark 3.3 below). Further details with regard to the present general setting can be found in [12].

Now let us suppose that pairs of uniformly stable biorthogonal collections $\{\Phi_j\}, \{\tilde{\Phi}_j\}$ are known and hence the projectors Q_j defined by (2.48). It still remains to identify stable bases Ψ_j for the complements W_j defined by (2.47). Using the same sort of arguments as above, one derives the following fact.

Remark 2.5 *Suppose that the Φ_j, $\tilde{\Phi}_j$ are stable and biorthogonal to each other (2.49) with refinement matrices $\mathbf{A}_{j,0}, \tilde{\mathbf{A}}_{j,0}$ satisfying (2.51). The additional collections $\Psi^j, \tilde{\Psi}^j$ of functions*

$$\Psi_j^T = \Phi_{j+1}^T \mathbf{A}_{j,1} \quad \tilde{\Psi}_j^T = \tilde{\Phi}_{j+1}^T \tilde{\mathbf{A}}_{j,1} \tag{2.53}$$

satisfy

$$\langle \psi_{j,k}, \tilde{\psi}_{j',k'} \rangle = \delta_{j,j'}\delta_{k,k'}, \quad (j,k),(j',k') \in \nabla, \tag{2.54}$$

if and only if the matrices $\mathbf{A}_{j,1}, \tilde{\mathbf{A}}_{j,1}$ are chosen such that $\mathbf{A}_j = (\mathbf{A}_{j,0}, \mathbf{A}_{j,1})$, $\tilde{\mathbf{A}}_j = (\tilde{\mathbf{A}}_{j,0}, \tilde{\mathbf{A}}_{j,1})$ satisfy

$$\mathbf{A}_j \tilde{\mathbf{A}}_j^* = \tilde{\mathbf{A}}_j^* \mathbf{A}_j = \mathbf{I}. \tag{2.55}$$

In analogy to (2.26), (2.27) this can be expressed in terms of the block operations

$$\mathbf{A}_{j,i}^* \tilde{\mathbf{A}}_{j,i'} = \delta_{i,i'}\mathbf{I}, \quad i, i' \in \{0,1\}, \tag{2.56}$$

and

$$\tilde{\mathbf{A}}_{j,0}\mathbf{A}_{j,0}^* + \tilde{\mathbf{A}}_{j,1}\mathbf{A}_{j,1}^* = \mathbf{I}. \tag{2.57}$$

Thus, biorthogonality is formally governed by the same type of algebraic relations as any stable completion, namely by (2.55) where, however, the inverses $\mathbf{B}_j$ now have to coincide with the adjoints $\tilde{\mathbf{A}}_j^*$ of a dual stable completion.

Given $\mathbf{A}_{j,0}, \tilde{\mathbf{A}}_{j,0}$ it is generally quite difficult to find the right completions $\mathbf{A}_{j,1}, \tilde{\mathbf{A}}_{j,1}$ such that (2.56), (2.57) hold. It will be one of the main applications of the subsequent results to facilitate the construction of such completions under certain circumstances.

3 A General Study of Stable Completions

We return to the general situation considered in Section 2.1, i.e., $\mathcal{F}$ is some Banach space with norm $\|\cdot\|_{\mathcal{F}}$ and $\mathcal{S}$ is a dense sequence of closed nested subspaces generated by a uniformly $(\ell, \mathcal{F})$-stable sequence $\{\Phi_j\}$. We are interested in finding suitable multiscale bases $\Psi = \cup_{j=-1}^{\infty} \Psi_j$ where 'suitable' has to be made concrete in each particular context. According to Proposition 2.1, a decomposition

$$S(\Phi_{j+1}) = S(\Phi_j) \oplus S(\Psi_j) \tag{3.1}$$

such that $\{\Phi_j \cup \Psi_j\}$ is also uniformly stable boils down to finding a stable completion of the corresponding refinement operator $\mathbf{A}_{j,0}$. For each fixed numbering of the index sets Δ_j these operators are represented by (possibly infinite) matrices

$$\mathbf{A}_{j,0} = (a^j_{m,k})_{m \in \Delta_{j+1}, k \in \Delta_j}. \tag{3.2}$$

We recall that uniform stability of $\{\Psi_j\}$ or $\{\Phi_j \cup \Psi_j\}$ expresses a quality of Ψ_j which only refers to the *single* scale j or better just to two successive scales $j, j+1$. 'Uniform' only means that the stability constants for the scale j are uniform in j. Since stability over all levels and hence asymptotics in j have been discussed above and since this property has been seen to be closely related to the fact the corresponding complements $S(\Psi_j)$ should be the image of the difference of two successive projectors satisfying (2.45) – again a property involving two successive scales – we will drop the scale index in the following subsection to simplify notation. So we will write $\mathbf{A}_0, \mathbf{A}_1$, $\mathbf{A} = (\mathbf{A}_0, \mathbf{A}_1)$ and similarly for $\mathbf{B}$. Whenever the distinction between the coarse level j and the fine level $j+1$ matters j and $j+1$ will be replaced by C and F, respectively.

We will proceed in two steps. First we will characterize for a given refinement matrix $\mathbf{A}_0$ *all* stable completions $\mathbf{A}_1$. In particular, we will show that given *some initial* stable completion of $\mathbf{A}_0$ *all* others can be parametrized by certain pairs of matrices. In a second step we will exploit this parametrization to *identify specific* stable completions which give rise to decompositions (3.1) with favorable properties. Among them will be decompositions of the form (2.47) induced by projectors.

3.1 Characterization of Stable Completions

The objective of this section is to characterize the class of stable or local completions $\mathbf{A}_1$ of a given fixed matrix $\mathbf{A}_0$ (2.55). As a first step we will identify operations which leave such classes invariant. To this end, we will always denote by $\mathbf{L}, \mathbf{K}$ matrices of the type

$$\mathbf{L} = (L_{k,m})_{k \in \nabla, m \in \Delta_C}, \quad \mathbf{K} = (K_{k,m})_{k,m \in \nabla}.$$

Such matrices will be referred to as satisfying conditions (**CL**) and (**CK**), respectively, when $\mathbf{L}$ defines a bounded map from $\ell(\nabla)$ into $\ell(\Delta_C)$ and when $\mathbf{K}$ is a bounded map from $\ell(\nabla)$ into itself possessing a bounded (right and left) inverse $\mathbf{K}^{-1}$.

Proposition 3.1 *Suppose* $\breve{\mathbf{A}}_1$ *is some stable completion of* $\mathbf{A}_0$ *and* $\breve{\mathbf{B}} = \breve{\mathbf{A}}^{-1}$, *where* $\breve{\mathbf{A}} = (\mathbf{A}_0, \breve{\mathbf{A}}_1)$. *Let* $\mathbf{L}, \mathbf{K}$ *satisfy* (**CL**), (**CK**), *respectively. Then* $\mathbf{A}_1$ *defined by*

$$\mathbf{A}_1 := \mathbf{A}_0\mathbf{L} + \breve{\mathbf{A}}_1\mathbf{K} \tag{3.3}$$

also forms a stable completion of $\mathbf{A}_0$ *and the corresponding* $\mathbf{B}$ *is given by*

$$\mathbf{B}_0 = \breve{\mathbf{B}}_0 - \mathbf{L}\mathbf{K}^{-1}\breve{\mathbf{B}}_1 \tag{3.4}$$
$$\mathbf{B}_1 = \mathbf{K}^{-1}\breve{\mathbf{B}}_1. \tag{3.5}$$

Proof: It is convenient to reinterpret (3.3), (3.4), and (3.5) as follows. Consider the mapping $\mathbf{H}_{L,K} : l(\Delta_F) \to \ell(\Delta_F)$ whose block matrix representation is given by

$$\mathbf{H}_{L,K} = \begin{pmatrix} \mathbf{I} & \mathbf{L} \\ \mathbf{0} & \mathbf{K} \end{pmatrix}, \tag{3.6}$$

where here $\mathbf{I}$ is the identity on $\ell(\Delta_C)$. Clearly $\mathbf{H}_{L,K}$ has an ℓ-inverse if and only if $\mathbf{K}$ has that property and

$$\mathbf{H}_{L,K}^{-1} = \mathbf{H}_{-LK^{-1},\,K^{-1}}. \tag{3.7}$$

In these terms (3.3), (3.4), and (3.5) may be rephrased as

$$\mathbf{A} = \breve{\mathbf{A}}\mathbf{H}_{L,K}, \qquad \mathbf{B} = \mathbf{H}_{-LK^{-1},K^{-1}}\breve{\mathbf{B}}. \tag{3.8}$$

Thus (3.7) readily assures that

$$\mathbf{BA} = \mathbf{AB} = \mathbf{I},$$

whence the assertion follows in view of the condition (**CL**), (**CK**) and Proposition 2.1. □

Our next observation states that *all* stable completions are of the above form.

Theorem 3.1 *Suppose* $\mathbf{A}_1$, $\breve{\mathbf{A}}_1$ *are any two stable completions of* $\mathbf{A}_0$ *with respective inverses* $\mathbf{B}$ *and* $\breve{\mathbf{B}}$. *Then there exist* $\mathbf{L}, \mathbf{K}$ *satisfying* (**CL**), (**CK**), *respectively, such that*

$$\mathbf{A} = \breve{\mathbf{A}}\mathbf{H}_{L,K}, \qquad \mathbf{B} = \mathbf{H}_{-LK^{-1},K^{-1}}\breve{\mathbf{B}}. \tag{3.9}$$

Proof: Let $\mathbf{H} := \breve{\mathbf{B}}\mathbf{A}$. By (2.55), one has

$$\breve{\mathbf{B}}\mathbf{A}\mathbf{B}\breve{\mathbf{A}} = \mathbf{I} = \mathbf{B}\breve{\mathbf{A}}\breve{\mathbf{B}}\mathbf{A}, \tag{3.10}$$

so that

$$\mathbf{H}^{-1} = \mathbf{B}\breve{\mathbf{A}}. \tag{3.11}$$

Furthermore, observe that, again by (2.55),

$$\breve{\mathbf{A}}\mathbf{H} = \breve{\mathbf{A}}\breve{\mathbf{B}}\mathbf{A} = \mathbf{A}, \tag{3.12}$$

and

$$\mathbf{H}^{-1}\breve{\mathbf{B}} = \mathbf{B}\breve{\mathbf{A}}\breve{\mathbf{B}} = \mathbf{B}. \tag{3.13}$$

Next note that

$$\mathbf{H} = \begin{pmatrix} \breve{\mathbf{B}}_0 \\ \breve{\mathbf{B}}_1 \end{pmatrix} (\mathbf{A}_0, \mathbf{A}_1) = \begin{pmatrix} \breve{\mathbf{B}}_0\mathbf{A}_0 & \breve{\mathbf{B}}\mathbf{A}_1 \\ \breve{\mathbf{B}}_1\mathbf{A}_0 & \breve{\mathbf{B}}_1\mathbf{A}_1 \end{pmatrix}.$$

Since $\breve{\mathbf{A}}_0 = \mathbf{A}_0$ we conclude from (2.27) that $\breve{\mathbf{B}}_0\mathbf{A}_0 = \mathbf{I}$ and $\breve{\mathbf{B}}_1\mathbf{A}_0 = \mathbf{0}$. Thus setting

$$\mathbf{L} := \breve{\mathbf{B}}\mathbf{A}_1, \quad \mathbf{K} := \breve{\mathbf{B}}_1\mathbf{A}_1, \tag{3.14}$$

and noting that, in view of an analogous representation for $\mathbf{H}^{-1}$, we have $\mathbf{K}^{-1} = \mathbf{B}\breve{\mathbf{A}}$ we readily see that $\mathbf{L}$ and $\mathbf{K}$ satisfy **(CL)** and **(CK)**, respectively. This shows that $\mathbf{H} = \mathbf{H}_{L,K}$ with $\mathbf{L}, \mathbf{K}$ defined in (3.14), finishing the proof. □

Clearly, the set $\mathcal{H}$ of all mappings $\mathbf{H}_{L,K}$, where $\mathbf{L}$ and $\mathbf{K}$ satisfy **(CL)**, **(CK)**, respectively, is a group. Thus

$$\mathcal{SC}(\mathbf{A}_0) = \{(\mathbf{AH})_1 : \mathbf{H} \in \mathcal{H}\} \tag{3.15}$$

parametrizes for any fixed stable completion $\breve{\mathbf{A}}_1$ of $\mathbf{A}_0$ the set of *all* stable completions of $\mathbf{A}_0$.

In particular, we are interested in local completions $\mathbf{A}_1$ of $\mathbf{A}_0$ where both $\mathbf{A}$ and $\mathbf{B}$ are banded.

In this context, we say that $\mathbf{L}, \mathbf{K}$ satisfy $(\mathbf{CL_0})$, $(\mathbf{CK_0})$, respectively, if $\mathbf{L}$ as well as $\mathbf{K}$ and $\mathbf{K}^{-1}$ are banded. The set $\mathcal{H}_f \subset \mathcal{H}$ consisting of those $\mathbf{H}_{L,K}$ where $\mathbf{L}, \mathbf{K}$ satisfy $(\mathbf{CL_0})$, $(\mathbf{CK_0})$ forms a subgroup of $\mathcal{H}$. Identical arguments as those used above yield

Theorem 3.2 *The set of all local completions of $\mathbf{A}_0$ is given by*

$$\{(\breve{\mathbf{A}}\mathbf{H})_1 : \mathbf{H} \in \mathcal{H}_f\},$$

where $\breve{\mathbf{A}} = (\mathbf{A}_0, \breve{\mathbf{A}}_1)$ and $\breve{\mathbf{A}}_1$ is any fixed local completion of $\mathbf{A}_0$.

Remark 3.1 *Keeping $\mathbf{K} = \mathbf{I}$ fixed and varying only $\mathbf{L}$, traces through a subclass of stable completions. According to (3.3), the new complement bases Ψ_j have the form*

$$\Psi_j^T = \Phi_{j+1}^T\mathbf{A}_0\mathbf{L} + \Phi_{j+1}^T\breve{\mathbf{A}}_1 = \Phi_j^T\mathbf{L} + \breve{\Psi}_j^T. \tag{3.16}$$

Thus, the initial complement bases $\breve{\Psi}_j$ are modified by adding a linear combination of the coarse scale generator Φ_j, i.e., for $\mathbf{L} = (\ell_{m,k})_{m\in\Delta_j, k\in\nabla_j}$ one has

$$\psi_{j,k} = \sum_{m\in\Delta_j} \ell_{m,k}\varphi_{j,m} + \breve{\psi}_{j,k}. \tag{3.17}$$

Throughout the remainder of this section we will assume that $\breve{\mathbf{A}}_{j,1}$ is a fixed stable completion of some given refinement matrix $\mathbf{A}_{j,0}$ for the jth scale and that $\breve{\mathbf{B}}_j$ is again the inverse of $\breve{\mathbf{A}}_j = (\mathbf{A}_{j,0}, \breve{\mathbf{A}}_{j,1})$ (see (2.55)). Our goal is to generate from such a completion other completions with certain desirable features. To this end, recall the basic two-scale relations (2.13) and (2.16) as

$$\Phi_j = \mathbf{A}_{j,0}^T \Phi_{j+1}, \qquad \Psi_j = \mathbf{A}_{j,1}^T \Phi_{j+1}. \tag{3.18}$$

Similarly we (formally) define for any two vectors $\Lambda \in \mathcal{F}^\Delta, \Xi \in (\mathcal{F}^*)^\nabla$ the matrix

$$\langle \Lambda, \Xi \rangle := (\langle \lambda_k, \xi_m \rangle)_{k\in\Delta, m\in\nabla},$$

where $\langle \cdot, \cdot \rangle$ denotes the dual form on $\mathcal{F} \times \mathcal{F}^*$. At this point we depart from the usual convention of viewing $\langle \cdot, \cdot \rangle$ as a *bilinear* form on $\mathcal{F} \times \mathcal{F}^*$. Here we prefer to treat $\langle \cdot, \cdot \rangle$ as a *sesquilinear form* when dealing with spaces over the complex field. The reason is to achieve as much formal consistency as possible with the important special case that $\mathcal{F}$ is a Hilbert space. Since will be switching between these different levels of generality several times our convention saves us repeatedly making corresponding distinctions with regard to complex conjugation when dealing with adjoint or conjugate operators. One easily verifies now that for any $\Delta' \times \Delta$ matrix $\mathbf{A}$ and any $\nabla' \times \nabla$ matrix $\mathbf{B}$

$$\langle \mathbf{A}\Lambda, \Xi \rangle = \mathbf{A}\langle \Lambda, \Xi \rangle, \quad \langle \Lambda, \mathbf{B}\Xi \rangle = \langle \Lambda, \Xi \rangle \overline{\mathbf{B}}^T. \tag{3.19}$$

For the matrices appearing in the present context these operations are well-defined.

3.2 Decompositions from Projectors

Assume that the collections $\Lambda_j = \{\lambda_{j,k} : k \in \Delta_j\} \subset \mathcal{F}^*$ are biorthogonal to Φ_j, i.e., in the present terms (2.49) can be rewritten as

$$\langle \Phi_j, \Lambda_j \rangle = \mathbf{I}. \tag{3.20}$$

At this point we do not require yet that Λ_j be also refinable but we will assume that

$$\|\langle f, \Lambda_j \rangle\|_{\ell(\Delta_j)} \lesssim \|f\|_{\mathcal{F}}, \quad f \in \mathcal{F}, \tag{3.21}$$

which is, on account of the stability of Φ_j, equivalent to saying that the mapping

$$Q_j(f) = \sum_{k\in\Delta_j} \langle f, \lambda_k^j \rangle \varphi_k^j = \langle f, \Lambda_j \rangle \Phi_j \tag{3.22}$$

defines a bounded projector on $\mathcal{F}$ with range S_j. An example of this type will be discussed below in Section 4.2. For our purposes it would actually suffice to require that (3.21) holds only for $f \in S(\Phi_{j+1})$.

A first important application of the results in the previous section is to identify for such given projectors the particular complement

$$W_j^Q := (Q_{j+1} - Q_j) S_{j+1} \tag{3.23}$$

for the decomposition

$$S_{j+1} = S_j \oplus W_j^Q. \tag{3.24}$$

Theorem 3.3 *Let $\{\Phi_j\}$ be refinable (2.13) and stable and let Λ_j be biorthogonal collections satisfying (3.21). Assume that $\breve{\mathbf{A}}_{j,1}$ is some stable completion of $\mathbf{A}_{j,0}$ and that $\mathbf{K}_j$ is any matrix satisfying* **(CK)**. *Define*

$$\mathbf{A}_{j,1} := (\mathbf{I} - \mathbf{A}_{j,0}\langle \Phi_{j+1}, \Lambda_j\rangle^T)\breve{\mathbf{A}}_{j,1}\mathbf{K}_j, \tag{3.25}$$

and

$$\mathbf{B}_{j,0} := \breve{\mathbf{B}}_{j,0} + \langle \Phi_{j+1}, \Lambda_j\rangle^T \breve{\mathbf{A}}_{j,1}\breve{\mathbf{B}}_{j,1} \qquad \mathbf{B}_{j,1} := \mathbf{K}_j^{-1}\breve{\mathbf{B}}_{j,1}. \tag{3.26}$$

Then $\mathbf{A}_{j,1}$ is also a stable completion of $\mathbf{A}_{j,0}$. The collection of functions

$$\Psi_j^T := \Phi_{j+1}^T \mathbf{A}_{j,1} \tag{3.27}$$

satisfies

$$\Phi_{j+1}^T = \Phi_j^T \mathbf{B}_{j,0} + \Psi_j^T \mathbf{B}_{j,1} \tag{3.28}$$

and forms a stable basis of $W_j^Q = (Q_{j+1} - Q_j)S_{j+1}$. Moreover, the functionals $\tilde{\Psi}_j$, defined by

$$\langle f, \tilde{\Psi}_j\rangle := \langle f, \overline{\mathbf{B}}_{j,1}\Lambda_{j+1}\rangle = \langle f, \Lambda_{j+1}\rangle \mathbf{B}_{j,1}^T \tag{3.29}$$

are biorthogonal to Ψ_j, i.e.,

$$\langle \Psi_j, \tilde{\Psi}_j\rangle = \mathbf{I}. \tag{3.30}$$

Thus the mapping $Q_{j+1} - Q_j$ agrees on S_{j+1} with a projector, represented there by

$$(Q_{j+1} - Q_j)f = \langle f, \tilde{\Psi}_j\rangle \Psi_j = \sum_{k\in\nabla_j} \langle f, \tilde{\psi}_{j,k}\rangle \psi_{j,k}. \tag{3.31}$$

Proof: By our assumptions on $\breve{\mathbf{A}}_{j,1}, \mathbf{K}_j, \Lambda_j$, the mapping

$$\mathbf{L} := -\langle \Phi_{j+1}, \Lambda_j\rangle^T \breve{\mathbf{A}}_{j,1}\mathbf{K}_j \tag{3.32}$$

satisfies **(CL)**. In fact, for $\mathbf{d} \in \ell(\nabla_j)$ one has $\mathbf{c} := \breve{\mathbf{A}}_{j,1}\mathbf{K}_j\mathbf{d} \in \ell(\Delta_{j+1})$ with $\|\mathbf{c}\|_{\ell(\Delta_{j+1})} \leq \|\breve{\mathbf{A}}_{j,1}\|\|\mathbf{K}_j\|\|\mathbf{d}\|_{\ell(\nabla_j)}$ and

$$\begin{aligned}\|\mathbf{Ld}\|_{\ell(\Delta_j)} &= \|\langle \mathbf{c}^T\Phi_{j+1}, \Lambda_j\rangle\|_{\ell(\Delta_j)} \lesssim \|\Phi_{j+1}^T\mathbf{c}\|_{\mathcal{F}} \\ &\lesssim \|\mathbf{c}\|_{\ell(\Delta_{j+1})} \lesssim \|\mathbf{d}\|_{\ell(\nabla_j)},\end{aligned}$$

where we have used (3.21) and the boundedness of $F_{\Phi_{j+1}}$ which in turn follows from stability. One readily infers now from (3.25) and (3.32) that

$$\mathbf{A}_j = \mathbf{H}_{L,K_j}\breve{\mathbf{A}}_{j,1}. \tag{3.33}$$

Since by (3.32),

$$\langle \Phi_{j+1}, \Lambda_j\rangle^T \breve{\mathbf{A}}_{j,1} = -\mathbf{L}\mathbf{K}_j^{-1},$$

(3.26) just means that

$$\mathbf{B}_j = \mathbf{H}_{-LK_j^{-1}, K_j^{-1}}\breve{\mathbf{B}}_j$$

so that the first part of the assertion follows from Proposition 3.1, Theorem 3.1 and Proposition 2.1.

As for the remaining part of the claim, let $Q_j(\Psi_j) = \{Q_j(\psi_{j,k}) : k \in \nabla_j\}$. We will show next that $Q_j(\Psi_j) = 0$. From (2.16), (3.27), and (3.19) one derives

$$Q_j(\Psi_j) = \langle \Psi_j, \Lambda_j \rangle \Phi_j = \langle \mathbf{A}_{j,1}^T \Phi_{j+1}, \Lambda_j \rangle \Phi_j = \mathbf{A}_{j,1}^T \langle \Phi_{j+1}, \Lambda_j \rangle \Phi_j. \tag{3.34}$$

Now (3.18) and (3.20) yield

$$\langle \Phi_{j+1}, \Lambda_j \rangle^T \mathbf{A}_{j,0} = \left(\mathbf{A}_{j,0}^T \langle \Phi_{j+1}, \Lambda_j \rangle \right)^T = \langle \Phi_j, \Lambda_j \rangle^T = \mathbf{I}. \tag{3.35}$$

Therefore (3.25) provides

$$\begin{aligned} \langle \Phi_{j+1}, \Lambda_j \rangle^T \mathbf{A}_{j,1} &= \langle \Phi_{j+1}, \Lambda_j \rangle^T (\mathbf{I} - \mathbf{A}_{j,0} \langle \Phi_{j+1}, \Lambda_j \rangle^T) \breve{\mathbf{A}}_{j,1} \mathbf{K}_j \\ &= (\langle \Phi_{j+1}, \Lambda_j \rangle^T - \langle \Phi_{j+1}, \Lambda_j \rangle^T) \breve{\mathbf{A}}_{j,1} \mathbf{K}_j = 0. \end{aligned} \tag{3.36}$$

Thus we infer from (3.34) that $Q_j(\Psi_j) = 0$ so that

$$(Q_{j+1} - Q_j)\psi_{j,k} = \psi_{j,k}, \qquad k \in \nabla_j. \tag{3.37}$$

Hence, by Proposition 2.1, Ψ_j is a stable basis for W_j^Q. Finally, by (3.19), (3.18), and (3.20),

$$\begin{aligned} \langle \Psi_j, \tilde{\Psi}_j \rangle &= \langle \mathbf{A}_{j,1}^T \Phi_{j+1}, \overline{\mathbf{B}}_{j,1} \Lambda_{j+1} \rangle = \mathbf{A}_{j,1}^T \langle \Phi_{j+1}, \Lambda_{j+1} \rangle \mathbf{B}_{j,1}^T \\ &= \mathbf{A}_{j,1}^T \mathbf{B}_{j,1}^T = (\mathbf{B}_{j,1} \mathbf{A}_{j,1})^T = \mathbf{I}, \end{aligned}$$

where we have used (2.55) in the last step. This completes the proof. □

Again one may immediately formulate an analogous result for local decompositions and local completions.

Theorem 3.4 *In addition to the hypotheses of Theorem 3.3 Φ_j and Λ_j are locally finite and linearly independent. Assume that $\breve{\mathbf{A}}_{j,1}$ is a local completion (i.e., $\breve{\mathbf{A}}_j, \breve{\mathbf{B}}_j$ are both banded). Then the assertion of Theorem 3.3 remains valid and for $\mathbf{K}_j = \mathbf{I}$ the completion $\mathbf{A}_{j,1}$ given by (3.25) is also local. Accordingly, Ψ_j and $\tilde{\Psi}_j$ are also locally finite.*

Remark 3.2 *The set of functionals Ξ_j defined by*

$$\langle f, \Xi_j \rangle := \langle f, \overline{\mathbf{B}}_{j,0} \Lambda_{j+1} \rangle = \langle f, \Lambda_{j+1} \rangle \mathbf{B}_{j,0}^T, \tag{3.38}$$

where $\mathbf{B}_j$ is defined by (3.26), agrees with Λ_j on S_j because by (3.38), (3.18), (3.19), and (3.20),

$$\langle \Phi_j, \Xi_j \rangle = \mathbf{A}_{j,0}^T \langle \Phi_{j+1}, \Lambda_{j+1} \rangle \mathbf{B}_{j,0}^T = \mathbf{I}.$$

Replacing Λ_j by Ξ_j in the definition of Q_j would assure that the relation (1.4) $Q_j Q_{j+1} = Q_j$ for the fixed j is valid not only on S_{j+1} but on all of $\mathcal{F}$. In this case one also has, of course,

$$\langle \Phi_{j+1}, \Xi_j \rangle = \langle \Phi_{j+1}, \Lambda_{j+1} \rangle \mathbf{B}_{j,0}^T = \mathbf{B}_{j,0}^T.$$

This latter observation can be carried further. Fixing $n \in I\!N$, define for $j \leq n$

$$\Lambda_j^n := \overline{\mathbf{B}}_{j,0} \cdots \overline{\mathbf{B}}_{n,0} \Lambda_{n+1}. \tag{3.39}$$

The Λ_j^n are refinable up to the level n, i.e., by construction, one has for $n > j$

$$\Lambda_j^n = \overline{\mathbf{B}}_{j,0} \Lambda_{j+1}^n.$$

Moreover, for $j \leq n$ the Λ_j^n are still biorthogonal to the Φ_j. In fact,

$$\langle \Phi_j, \Lambda_j^n \rangle = \langle \mathbf{A}_{j,0}^T \cdots \mathbf{A}_{n,0}^T \Phi_{n+1}, \overline{\mathbf{B}}_{j,0} \cdots \overline{\mathbf{B}}_{n,0} \Lambda_{n+1} \rangle = \mathbf{I},$$

where we have used (2.27) and (3.20). As an immediate consequence we have

Remark 3.3 *If there exists a collection $\tilde{\Phi}_j$ such that for any $\mathbf{c} \in \ell(\Delta_j)$*

$$\lim_{n \to \infty} \| \mathbf{c}^T \tilde{\Phi}_j - \mathbf{c}^T \Lambda_j^n \|_{\mathcal{F}} = 0,$$

then one has

$$\langle \Phi_j, \tilde{\Phi}_j \rangle = \mathbf{I}, \quad \tilde{\Phi}_j = \overline{\mathbf{B}}_{j,0} \tilde{\Phi}_{j+1} \tag{3.40}$$

(see also Remark 2.5 and the preceding comments).

As pointed out in Section 2.7, to construct multiscale bases which are stable over all levels it is important to identify the complements in (3.23) when $\Lambda_j = \tilde{\Phi}_j$ satisfies (3.40).

Corollary 3.1 *Let Φ_j be stable and refinable with refinement matrices $\mathbf{A}_{j,0}$ (cf. (3.18)). Assume that $\breve{\mathbf{A}}_{j,1}$ is some stable completion of $\mathbf{A}_{j,0}$ and that $\breve{\mathbf{B}}_j$ is the inverse of $\breve{\mathbf{A}}_j$. Moreover, let $\tilde{\Phi}_j$ be biorthogonal to Φ_j (see (3.20) or (2.49)) and also refinable with refinement matrix $\tilde{\mathbf{A}}_{j,0}$. Then for any matrix $\mathbf{K}_j$, satisfying* **(CK)**, *there exists a stable completion $\mathbf{A}_{j,1}$ of $\mathbf{A}_{j,0}$ given by*

$$\mathbf{A}_{j,1} := (\mathbf{I} - \mathbf{A}_{j,0} \tilde{\mathbf{A}}_{j,0}^*) \breve{\mathbf{A}}_{j,1} \mathbf{K}_j, \tag{3.41}$$

such that the inverse of $\mathbf{A}_j$ is given by

$$\mathbf{B}_j = \begin{pmatrix} \tilde{\mathbf{A}}_{j,0}^* \\ \mathbf{K}_j^{-1} \breve{\mathbf{B}}_{j,1} \end{pmatrix}. \tag{3.42}$$

The collections

$$\Psi_j := \mathbf{A}_{j,1}^T \Phi_{j+1}, \quad \tilde{\Psi}_j := \overline{\mathbf{B}}_{j,1} \tilde{\Phi}_{j+1}$$

form biorthogonal systems (see (2.54))

$$\langle \Psi_j, \tilde{\Psi}_j \rangle = \mathbf{I}, \quad \langle \Psi_j, \tilde{\Phi}_j \rangle = \langle \Phi_j, \tilde{\Psi}_j \rangle = \mathbf{0},$$

so that

$$\langle \Psi_j, \tilde{\Psi}_{j'} \rangle = \delta_{j,j'} \mathbf{I}, \quad j, j' \in I\!N_0.$$

Proof: Since by refinability of $\tilde{\Phi}_j$, (3.19), and (3.20),

$$\langle \Phi_{j+1}, \tilde{\Phi}_j \rangle^T = \tilde{\mathbf{A}}_{j,0}^* \langle \Phi_{j+1}, \tilde{\Phi}_{j+1} \rangle^T = \tilde{\mathbf{A}}_{j,0}^*,$$

(3.41) follows from (3.25). Similarly, upon using (2.27) and (2.26), (3.26) gives

$$\begin{aligned} \breve{\mathbf{B}}_{j,0} + \langle \Phi_{j+1}, \tilde{\Phi}_j \rangle^T \breve{\mathbf{A}}_{j,1} \breve{\mathbf{B}}_{j,1} &= \breve{\mathbf{B}}_{j,0} + \tilde{\mathbf{A}}_{j,0}^* \breve{\mathbf{A}}_{j,1} \breve{\mathbf{B}}_{j,1} \\ &= \breve{\mathbf{B}}_{j,0} + \tilde{\mathbf{A}}_{j,0}^* (\mathbf{I} - \breve{\mathbf{A}}_{j,0} \breve{\mathbf{B}}_{j,0}) = \tilde{\mathbf{A}}_{j,0}^*. \end{aligned}$$

The rest is an immediate consequence of (3.26) and Theorem 3.3. □

The main point to be stressed here is that given

- locally finite biorthogonal generators $\Phi_j, \tilde{\Phi}_j$ which are both refinable and
- some initial banded stable completion $\breve{\mathbf{A}}_{j,1}$ of $\mathbf{A}_{j,0}$,

one can explicitly construct new stable completions

$$\mathbf{A}_{j,1} = (\mathbf{I} - \mathbf{A}_{j,0} \tilde{\mathbf{A}}_{j,0}^*) \breve{\mathbf{A}}_{j,1}$$

which are still banded and give rise to biorthogonal bases $\Psi, \tilde{\Psi}$ consisting of compactly supported function(al)s. The new complement bases Ψ_j are of the form shown in Remark 3.1 where $\mathbf{K} = \mathbf{I}$ and

$$\mathbf{L} = -\tilde{\mathbf{A}}_{j,0}^* \breve{\mathbf{A}}_{j,1}$$

is banded so that the modifications (3.17) only require the application of local masks to the coarse generators.

3.3 Orthogonal Decompositions

Throughout the remainder of this section we will assume that $\mathcal{F} = \mathcal{F}^*$. The issue of stability of multiscale decompositions (2.43) over all levels trivially reduces to studying (uniform) stability with respect to single scales when W_j is chosen as the orthogonal complement of S_j in S_{j+1}. Therefore we will comment next on the question how to generate from some stable completion $\breve{\mathbf{A}}_{j,1}$ of the refinement matrix $\mathbf{A}_{j,0}$ another completion which induces orthogonal decompositions.

One possible strategy is to apply Theorem 3.3. Since by (3.19) one has $\langle \Phi_j, \tilde{\Phi}_j \rangle = \mathbf{I}$ for $\tilde{\Phi}_j := \mathbf{G}_j \Phi_j$ if and only if

$$\mathbf{G}_j = \langle \Phi_j, \Phi_j \rangle^{-1}, \tag{3.43}$$

the orthogonal projector onto S_j has the form $\langle f, \Phi_j \rangle \mathbf{G}_j^* \Phi_j$. Combining Theorem 3.3 (3.25) with a straightforward computation shows that

$$\mathbf{A}_{j,1} := \Big(\mathbf{I} - (\mathbf{A}_{j,0} \mathbf{G}_j \mathbf{A}_{j,0}^*) \langle \Phi_{j+1}, \Phi_{j+1} \rangle \Big) \breve{\mathbf{A}}_{j,1} \mathbf{K}_j \tag{3.44}$$

is for any $\mathbf{K}_j$ satisfying **(CK)** a stable completion which gives rise to an orthogonal decomposition of S_{j+1}.

If the initial completion is even local so that, due to the compact support of the basis functions, the Gramian $\langle \Phi_{j+1}, \Phi_{j+1} \rangle$ is banded the question arises whether the completion (3.44) is also local. Since in general $\mathbf{G}_j$ will not be banded one may have to choose $\mathbf{K}_j$ judiciously. In the present generality it does not seem to be possible to decide whether such a $\mathbf{K}_j$, which also would have to be banded, exists.

Alternatively, one could employ Proposition 3.1 directly. Then the objective is to choose $\mathbf{L}, \mathbf{K}$, satisfying **(CL), (CK)**, respectively, such that

$$\Psi_j := \mathbf{A}_{j,1}^T \Phi_{j+1}$$

with

$$\mathbf{A}_{j,1} := \mathbf{A}_{j,0}\mathbf{L} + \breve{\mathbf{A}}_{j,1}\mathbf{K} \tag{3.45}$$

satisfies

$$\mathbf{0} = \langle \Phi_j, \Psi_j \rangle = \mathbf{A}_{j,0}^T \langle \Phi_{j+1}, \Phi_{j+1} \rangle \overline{\mathbf{A}_{j,1}}.$$

Observing that $\langle \Phi_j, \Phi_j \rangle = \mathbf{A}_{j,0}^T \langle \Phi_{j+1}, \Phi_{j+1} \rangle \overline{\mathbf{A}_{j,0}}$ and substituting (3.45), yields after some straightforward calculations

$$\mathbf{0} = \langle \Phi_j, \Phi_j \rangle \overline{\mathbf{L}} + \mathbf{A}_{j,0}^T \langle \Phi_{j+1}, \Phi_{j+1} \rangle \overline{\breve{\mathbf{A}}_{j,1}\mathbf{K}}.$$

Choosing

$$\mathbf{K} := \breve{\mathbf{B}}_{j,1}\mathbf{R}, \tag{3.46}$$

this finally becomes

$$\langle \Phi_j, \Phi_j \rangle (\mathbf{L} - \breve{\mathbf{B}}_{j,0}\mathbf{R}) = -\mathbf{A}_{j,0}^* \langle \Phi_{j+1}, \Phi_{j+1} \rangle \mathbf{R}. \tag{3.47}$$

Thus one could try to find $\mathbf{R}$ taking $\ell_2(\nabla_j)$ into $\ell_2(\Delta_{j+1})$ such that $\breve{\mathbf{B}}_{j,1}\mathbf{R}$ has a bounded inverse and

$$\mathbf{A}_{j,0}^* \langle \Phi_{j+1}, \Phi_{j+1} \rangle \mathbf{R} = \mathbf{0}. \tag{3.48}$$

One could then set

$$\mathbf{L} := \breve{\mathbf{B}}_{j,0}\mathbf{R} \tag{3.49}$$

to obtain with (3.45) a completion inducing orthogonal decompositions. Recalling (3.46) and (3.16), the relation (3.45) readily yields as expected

$$\mathbf{A}_{j,1} = \mathbf{R}. \tag{3.50}$$

As for locality $\mathbf{R}$ would have to be banded. Then assuming that the initial completion $\breve{\mathbf{A}}_{j,1}$ is local, $\mathbf{L}$ defined by (3.49) is automatically banded. Whether it is possible to find such a $\mathbf{R}$ will again depend on the special case at hand.

We will present below in Section 4.1 an example where this strategy can be employed successfully. In fact, we will identify such banded matrices $\mathbf{R}$ for the construction of orthogonal complements of univariate spline spaces on arbitrary knot sequences and of corresponding compactly supported basis functions.

3.4 Further Developments

Several applications of the above results have already evolved. An important property of multiscale bases are *vanishing moments*. They play an important role in connection with matrix compression and fast solvers for linear systems arising from discretizations of boundary integral equations. To be specific, the elements of Ψ should satisfy

$$\int_{\Omega} \psi_{j,k}(x)\pi(x)dx = 0, \quad \pi \in \Pi,$$

where Π is some finite dimensional linear space which has good local approximation properties. Π typically stands for a space of polynomials. More generally, when Ω is a parametrically defined surface Π contains functions of the form $\pi = P \circ \gamma^{-1}$ where P is a polynomial and Ω consists of unions of parametric patches of the form $\gamma(\Box)$, $\Box$ being some reference parameter domain. Given some initial multiscale basis Ψ corresponding to stable completions $\mathbf{A}_{j,1}$ with insufficient order of vanishing moments it was pointed out in [21] that choosing again $\mathbf{K} = \mathbf{I}$ one has to determine $\mathbf{L}_j$ in (3.3) such that

$$\mathbf{L}_j^* \mathbf{g}_{j,r} + \mathbf{A}_{j,1}^* \mathbf{g}_{j+1,r} = 0, \quad r = 1, \ldots N,$$

where

$$\mathbf{g}_{j,r} = (\mathbf{g}_{j,r,k})_{k \in \Delta_j}, \quad \mathbf{g}_{j,r,k} := \int_{\Omega} \pi_r(x)\overline{\varphi_{j,k}(x)}dx, \quad k \in \Delta_j,$$

and $\{\pi_r : r = 1, \ldots N\}$ is a basis of Π. When the $\mathbf{A}_{j,1}$ are banded one can usually find solutions $\mathbf{L}_j$ which are also banded.

In [22] Corollary 3.1 serves as the main tool for constructing a family of continuous biorthogonal multiscale bases $\Psi, \tilde{\Psi}$ on two dimensional manifolds in $\mathbb{R}^3$ which are stable over all levels and where for any $d \in \mathbb{N}$ the basis Ψ can be arranged to have vanishing moments of degree d.

4 Some Applications and Special Cases

The remainder of the paper is devoted to some applications and specializations of the above general results. We exhibit several cases where initial completions can be identified. First we will consider univariate splines on nonuniform knot sequences and construct among other things compactly supported pre-wavelets, i.e., basis functions for orthogonal complements. Then we briefly discuss piecewise linear bivariate finite elements on nested triangulations. We conclude with revisiting wavelets on $\mathbb{R}^s$. Most of the latter results on wavelets are known in essence but we find it quite instructive to revisit them from the point of view of the above concepts.

4.1 Univariate Splines and Quasiinterpolants

Our first application of the above results concerns nested univariate splines spaces on biinfinite irregular knot sequences. The need for dealing with stable decompositions of spline spaces with irregular knot sequences arises, for instance, in connection with multilevel preconditioning schemes for elliptic boundary value problems when the boundary conditions are not incorporated in the trial spaces but are appended by Lagrange multipliers [26]. This facilitates the use of shift-invariant trial spaces. However, when these trial functions are generated by translates of box splines, say, their traces on polygonal boundaries are in general univariate splines with nonuniform knot sequences.

To be more specific, let $X = \{x_i\}_{i\in\mathbb{Z}}$, $W = \{w_i\}_{i\in\mathbb{Z}}$ be strictly increasing knot sequences such that $x_i < w_i < x_{i+1}$, $i \in \mathbb{Z}$, and let $T := \{t_i\}_{i\in\mathbb{Z}} = X \cup W$, i.e., $t_{2i} = x_i$, $t_{2i+1} = w_i$, $i \in \mathbb{Z}$. Denoting by $[x_i, \ldots, x_{i+m}]f$ the $m+1$-st order divided difference of f at $x_i, \ldots, x_{i+1+m}$, we define the B-splines on the coarse and fine knot sequence

$$\begin{aligned} N_{k,m,X}(x) &:= (x_{k+m+1} - x_k)[x_k, \ldots, x_{k+1+m}](\cdot - x)_+^m \\ N_{k,m,T}(x) &:= (t_{k+1+m} - t_k)[t_k, \ldots, t_{m+k+1}](\cdot - x)_+^m, \end{aligned}$$

respectively, where $x_+^\ell = (\max\{0, x\})^\ell$. Thus defining $S_Y :=$ span Φ_Y where $\Phi_Y = \{N_{k,m,Y} : k \in \mathbb{Z}\}$, $Y \in \{X, T\}$ (Here 'span' is to be viewed as the set of any (infinite) linear combinations of the $N_{k,m,X}$, $k \in \mathbb{Z}$), we may put here $\Delta_C = X$, $\Delta_F = T$, $\varphi_{X,k} = N_{k,m,X}$, $\varphi_{T,k} = N_{k,m,T}$. Thus the case (2.32) with $E = \{0, 1\}$ applies here.

It is well-known that there exist coefficients $a_{0,i,j}$ such that

$$\varphi_{X,k}(x) = N_{k,m,X}(x) = \sum_{l\in\mathbb{Z}} a_{0,l,k} N_{l,m,T}$$

and

$$a_{0,l,k} = 0, \quad l < 2k, \quad l > 2k + m + 1,$$

so that clearly $S_X \subset S_T$. The transpose of the matrix

$$\mathbf{A}_0 = (a_{0,l,k})_{l,k\in\mathbb{Z}}$$

was called *two-slanted* in [17]. The following result was proved in [17].

Theorem 4.1 *There exists a local completion $\mathbf{A}_1$ of $\mathbf{A}_0$, i.e., the matrices $\mathbf{A}_1$, and $\mathbf{B} = \binom{\mathbf{B}_0}{\mathbf{B}_1}$ are all banded. Moreover, the biinfinite matrix $\mathbf{A}$ defined by interlacing the columns of $\mathbf{A}_0$ and $\mathbf{A}_1$ is totally positive. Thus the splines*

$$\psi_k(x) := \sum_{l\in\mathbb{Z}} a_{1,l,k} N_{l,m,T}(x) \tag{4.1}$$

satisfy

$$N_{l,m,T}(x) = \sum_{k\in\mathbb{Z}} b_{0,k,l} N_{k,m,X}(x) + \sum_{k\in\mathbb{Z}} b_{1,k,l}\psi_k(x), \tag{4.2}$$

providing a local decomposition of S_T.

In [17] the matrices $\mathbf{A}_1, \mathbf{B}_0, \mathbf{B}_1$ were constructed explicitly by means of a certain factorization of $\mathbf{A}_0$. This result was applied in [17] to the construction of compactly supported biorthogonal splines in S_T. We will indicate now first that these results combined with the techniques of the previous section also lead to orthogonal decompositions spanned by compactly supported splines. To this end, we recall briefly the above mentioned factorizations. For any biinfinite matrix $\mathbf{C}$ the two-banded block-diagonal matrices $\mathbf{M}_r(\mathbf{C}) = (m_{p,q}^{(r)}(\mathbf{C}))_{p,q\in\mathbb{Z}}$ are defined for $r = 1, 2, \ldots$ by

$$m_{p,q}^{(r)} := \begin{cases} 1, & p = q; \\ -\frac{c_{i,2i+j-1}^{(r-1)}}{c_{i,2i+j}^{(r-1)}}, & (p,q) = (2i+j, 2i+j-1); \\ 0, & \text{otherwise}; \end{cases}$$

where $\mathbf{C}^{(0)} := \mathbf{C},\ \mathbf{C}^{(r)} := \mathbf{C}\mathbf{M}_1(\mathbf{C})\cdots\mathbf{M}_r(\mathbf{C})$. Note that the $\mathbf{M}_r(\mathbf{C})$ are invertible and the inverses are two-banded with the same block-diagonal structure. In addition it is convenient to introduce the elementary *slanted* matrices

$$\mathbf{I}^{(m)} := (\delta_{q,2l+m})_{l,q\in\mathbb{Z}}, \quad \mathbf{E}^{(m)}(\mathbf{C}) := (c_{l,2l+m}^{(m)}\delta_{q,2l+m})_{l,q\in\mathbb{Z}},$$

and

$$\hat{\mathbf{E}}^{(m)}(\mathbf{C}) := ((c_{l,2l+m}^{(m)})^{-1}\delta_{q,2l+m})_{l,q\in\mathbb{Z}}.$$

One easily confirms that

$$\mathbf{E}^{(m)}(\mathbf{C})(\mathbf{I}^{(m+1)})^T = 0, \quad \mathbf{E}^{(m)}(\mathbf{C})(\hat{\mathbf{E}}^{(m)}(\mathbf{C}))^T = \mathbf{I}. \tag{4.3}$$

It has been shown in [17] that

$$\mathbf{M}_{m+1}(\mathbf{A}_0)^T\cdots\mathbf{M}_1(\mathbf{A}_0)^T\mathbf{A}_0 = (\mathbf{E}^{(m+1)}(\mathbf{A}_0))^T, \tag{4.4}$$

and that a local completion is given by

$$\breve{\mathbf{A}}_1 := (\mathbf{M}_1(\mathbf{A}_0)^T)^{-1}\cdots(\mathbf{M}_{m+1}(\mathbf{A}_0)^T)^{-1}(\mathbf{I}^{(m+2)})^T, \tag{4.5}$$

while

$$\begin{aligned} \breve{\mathbf{B}}_0 &= \hat{\mathbf{E}}^{(m+1)}(\mathbf{A}_0)\mathbf{M}_{m+1}(\mathbf{A}_0)^T\cdots\mathbf{M}_1(\mathbf{A}_0)^T, \\ \breve{\mathbf{B}}_1 &= \mathbf{I}^{(m+2)}\mathbf{M}_{m+1}(\mathbf{A}_0)^T\cdots\mathbf{M}_1(\mathbf{A}_0)^T. \end{aligned} \tag{4.6}$$

We are now in a position to apply the results from Section 3.3. To this end, let

$$\mathbf{G} := \mathbf{A}_0^T\langle\Phi_T,\Phi_T\rangle = \langle\Phi_X,\Phi_T\rangle.$$

Corollary 4.1 *The matrices $M_r(\mathbf{G})$ are well-defined and*

$$\mathbf{R} := \mathbf{M}_1(\mathbf{G})\cdots\mathbf{M}_{3m+1}(\mathbf{G})(\mathbf{I}^{(3m+1)})^T \tag{4.7}$$

satisfies

$$\mathbf{A}_0^T\langle\Phi_T,\Phi_T\rangle\mathbf{R} = 0. \tag{4.8}$$

$\mathbf{A}_1 := \mathbf{R}$ is a local completion of $\mathbf{A}_0$ giving rise to a local orthogonal decomposition of S_T.

Proof: As pointed out in [17], $\mathbf{G}$ satisfies

$$\mathbf{G}\mathbf{M}_1(\mathbf{G})\cdots\mathbf{M}_{3m+1}(\mathbf{G}) = \mathbf{E}^{(3m+1)}(\mathbf{G}), \tag{4.9}$$

so that by (4.3), the matrix $\mathbf{R}$, defined by (4.7), satisfies (4.8). Since again by (4.3),

$$\mathbf{M}_{3m+1}(\mathbf{G})^{-1}\cdots\mathbf{M}_1(\mathbf{G})^{-1}\mathbf{I}^{(3m+2)}\mathbf{R} = \mathbf{I}$$

the observations in Section 4.2 imply that $\mathbf{A}_1 := \mathbf{R}$ is a completion of $\mathbf{A}_0$ giving rise to an orthogonal decomposition of S_T. By construction, $\mathbf{R}$ is banded so that the orthogonal complement of S_X in S_T is indeed spanned by compactly supported splines. □

The corresponding matrix $\mathbf{B}$ can now be determined, for instance, from (3.4), (3.5) and (3.46), (3.16).

Next we are interested in deriving from (4.1) another local decomposition of S_T which is induced by an important type of projector which plays a pivotal role in theory and applications of spline spaces, namely *quasi-interpolants*.

Defining for some fixed $\tau_k \in (x_{k+1}, x_{m+k})$

$$\langle f, \lambda_{k,X}\rangle = \lambda_{k,X}(f) := \sum_{i=0}^{m}(-1)^{m-i}\xi_{k,m}^{(m-i)}(\tau_k)f^{(i)}(\tau_k),$$

where

$$\xi_{k,m}(x) = \frac{(x-x_{k+1})\cdots(x-x_{m+k})}{m!}.$$

It is well-known that $\lambda_{k,X}(N_{l,m,X}) = \delta_{k,l}$ which gives rise to the quasi-interpolant

$$Q_{m,X}(f) := \sum_{k\in\mathbb{Z}} \lambda_{k,X}(f)N_{k,m,X}$$

(cf. [3]). Now we use (3.25) with $\mathbf{K} = \mathbf{I}$ and define

$$\hat{a}_{1,p,q} = a_{1,p,q} - \sum_{\ell\in\mathbb{Z}}\sum_{r\in\mathbb{Z}} a_{0,p,r}g_{r,\ell}a_{1,\ell,q},$$

where $a_{1,p,q}$ are the coefficients from (4.1) and $g_{k,l} = \lambda_{l,X}(N_{k,m,T})$ (cf. (3.25)).

Obviously, the functions

$$\hat{\psi}_k := \sum_{l\in\mathbb{Z}} \hat{a}^1_{1,l,k}N_{l,m,T}$$

are compactly supported and, on account of Theorem 3.3, satisfy $Q_{m,X}\hat{\psi}_k = 0$, $(Q_{m,T} - Q_{m,X})\hat{\psi}_k = \hat{\psi}_k$, $k \in \mathbb{Z}$, i.e., they span $(Q_{m,T} - Q_{m,X})S_T$. The corresponding dual functionals can be constructed as described by Theorem 3.3.

It is interesting to note that for cubic splines the above decomposition induced by quasiinterpolants actually turns out to agree with the initial decomposition described above which is based on factorizations of the refinement matrix $\mathbf{A}_0$. Details will be reported elsewhere.

4.2 Bivariate Finite Elements

Second order elliptic boundary value problems on polygonal domains $\Omega \subset I\!R^2$ are often discretized with the aid of piecewise linear continuous finite elements. More specifically, let $\mathcal{T}_0$ be a triangulation of Ω, i.e., a collection of triangles such that

(i) $\mathrm{meas}(\tau \cap \tau') = 0$, $\tau, \tau' \in \mathcal{T}_0$, $\tau \neq \tau^0$;

(ii) $\bigcup\{\tau : \tau \in \mathcal{T}_0\} = \Omega$;

(iii) $\tau \cap \tau'$ is either a common vertex, or a common edge of τ and τ', or empty.

Given a triangulation $\mathcal{T}_i$, a refinement $\mathcal{T}_{i+1}$ is obtained by subdividing each triangle $\tau \in \mathcal{T}_i$ into four congruent subtriangles. Accordingly, we denote by $\Delta_i, \mathcal{E}_i$ the set of vertices and edges in $\mathcal{T}_i$ respectively.

With each $\mathcal{T}_i$ one may canonically associate the space S_i of continuous functions on Ω whose restriction to any triangle in $\mathcal{T}_i$ is a polynomial of degree at most one. Obviously, one has $S_i \subset S_{i+1}$. Let us denote by $\varphi_{j,k}$ the unique piecewise linear continuous function in S_j satisfying $\varphi_{j,k}(k') = 2^j \delta_{k,k'}$ and set $\Phi_j = \{\varphi_{j,k} : k \in \Delta_j\}$. Thus $S_j = S(\Phi_j)$ and it is not hard to confirm that $\{\Phi\}$ is uniformly $(\ell_2(\Delta_j), L_2(\Omega))$-stable [14]. Since clearly

$$\varphi_{j,k} = \sum_{m \in \{k\} \cup \mathcal{N}_{j+1,k}} 2^{-j-1} \varphi_{j,k}(m) \varphi_{j+1,m}, \tag{4.10}$$

where $\mathcal{N}_{j,k} := \{m \in \Delta_j : \ [k,m] \in \mathcal{E}_j\}$ denotes the *star* of k in Δ_j consisting of the neighbors of k in Δ_j. Hence i.e.,

$$(\mathbf{A}_{j,0})_{m,k} = a^j_{m,k} = \begin{cases} \frac{1}{2}, & m = k, \\ \frac{1}{4}, & m \in \mathcal{N}_{j+1,k}, \\ 0, & \text{else}, \end{cases} \tag{4.11}$$

so that the refinement matrices $\mathbf{A}_{j,0}$ are obviously uniformly banded.

So called *hierarchical bases* (see [31]) span a particularly simple type of complement space. Let ∇_j denote the set of midpoints of the edges in $\mathcal{E}_j$ so that

$$\Delta_{j+1} = \Delta_j \cup \nabla_j, \quad \Delta_j \cap \nabla_j = \emptyset. \tag{4.12}$$

Defining now for $k \in \nabla_j$

$$\breve{\psi}_{j,k} := \varphi_{j+1,k}, \quad k \in \nabla_j, \tag{4.13}$$

one has

$$(\breve{\mathbf{A}}_{j,1})_{m,k} = \delta_{m,k}, \quad m \in \Delta_{j+1},\ k \in \nabla_j. \tag{4.14}$$

On the other hand, since for $m \in \Delta_j$ one has $\mathcal{N}_{j+1,m} \subseteq \nabla_j$, (4.10), (4.13) say

$$\varphi_{j+1,m} = 2\,\varphi_{j,m} - \sum_{k \in \mathcal{N}_{j+1,m}} \frac{1}{2} \breve{\psi}_{j,k}. \tag{4.15}$$

Therefore, one has in this case

$$(\breve{\mathbf{B}}_{j,0})_{k,m} = 2\delta_{k,m}, \quad m \in \Delta_{j+1},\ k \in \Delta_j, \tag{4.16}$$

while

$$(\breve{\mathbf{B}}_{j,1})_{k,m} = \begin{cases} -\frac{1}{2}, & m \in \Delta_j,\ k \in \mathcal{N}_{j+1,m}; \\ \delta_{k,m}, & k,m \in \nabla_j; \\ 0, & \text{else.} \end{cases} \tag{4.17}$$

Obviously one has

$$\|\mathbf{A}_j\|,\ \|\breve{\mathbf{B}}_j\| = \mathcal{O}(1), \quad j \in \mathbb{N}_0. \tag{4.18}$$

Thus, setting $\breve{\Psi}_j = \{\psi_{j,k} : k \in \nabla_j\}$, Corollary 2.1 ensures that $\{\Phi_j \cup \breve{\Psi}_j\}$ is also uniformly stable.

The underlying local decomposition

$$S_{j+1} = S_j \oplus W_j, \quad W_j = S(\breve{\Psi}_j) \tag{4.19}$$

of S_{j+1} corresponds to the so called *hierarchical basis* [31] which was introduced in [31] for the purpose of preconditioning stiffness matrices arising from finite element discretizations of second order elliptic boundary value problems. Obviously, one has $W_j = (I_{j+1} - I_j)S_{j+1}$ where

$$I_j f = 2^{-j} \sum_{k \in \Delta_j} f(k)\varphi_{j,k}.$$

Although $\{\Phi_j \cup \breve{\Psi}_j\}$ is uniformly stable, $\breve{\Psi}$ is not stable over all levels, i.e., does not form a Riesz basis for $L_2(\Omega)$. In particular, these interpolation projectors are, of course, not bounded in L_2. This accounts for the fact that this sort of decomposition does ultimately not lead to optimal preconditioners for second order elliptic problems.

Therefore the following type of projectors was considered in [14]. Consider first an arbitrary triangulation $\mathcal{T}$ with vertex set Δ. As before let φ_k, $k \in \Delta$ denote the piecewise linear hat functions with respect to $\mathcal{T}$. Thus for any triangle $\tau = [k,m,p] \in \mathcal{T}$ the restriction of the hat functions φ_q, $q \in \{k,m,p\}$, to τ are just the barycentric coordinates with respect to τ and therefore linearly independent affine functions. Let ζ_q^τ, $q \in \{k,m,p\}$ denote the unique affine functions satisfying

$$\int_\tau \varphi_q(x)\zeta_{q'}^\tau(x)\,\mathrm{d}x = \delta_{q,q'}, \quad q,q' \in \{k,m,p\}. \tag{4.20}$$

Let n_k denote the number of triangles sharing k as a vertex and let

$$\lambda_{\mathcal{T},k}(x) := \begin{cases} \frac{1}{n_k}\zeta_k^\tau(x), & x \in \tau \\ 0, & x \notin \operatorname{supp} \varphi_k. \end{cases} \tag{4.21}$$

Specifically, defining for $\mathcal{T}_j$ as above

$$\langle f, \lambda_{j,k} \rangle := \int_\Omega f(x)\lambda_{\mathcal{T}_j,k}(x)\,\mathrm{d}x, \quad k \in \Delta_j, \tag{4.22}$$

one obviously has

$$\langle \varphi_{j,m}, \lambda_{j,k} \rangle = \delta_{k,m}, \tag{4.23}$$

so that the corresponding mappings Q_j, defined by (3.22) relative to $\Lambda_j = \{\lambda_{j,k} : k \in \Delta_j\}$, are projectors onto S_j.

Since the functions defined by (4.21) have compact support, the matries

$$\mathbf{G}_j := \langle \Phi_{j+1}, \lambda_j \rangle = (\langle \varphi_{j+1,k}, \lambda_{j,m} \rangle)_{k \in \Delta_{j+1}, m \in \Delta_j}$$

are in this case uniformly banded. Therefore, by Theorem 3.3,

$$\mathbf{A}_{j,1} = (\mathbf{I} - \mathbf{A}_{j,0}\mathbf{G}_j^T)\breve{\mathbf{A}}_{j,1}, \quad \mathbf{B}_j = \mathbf{H}_{\mathbf{A}_{j,1}\mathbf{G}_j,\mathbf{I}}\breve{\mathbf{B}}_j \tag{4.24}$$

gives rise to the local decompositions

$$S(\Phi_{j+1}) = S(\Phi_j) \oplus S(\Psi_j),$$

with uniformly stable $\{\Phi_j \cup \Psi_j\}$ where Ψ_j are the new complement bases induced by the completions $\mathbf{A}_{j,1}$ above.

Since the biorthogonal collections Λ_j are *not* refinable the projectors Q_j do not satisfy (2.45) (see Proposition 2.2). Thus one cannot expect Ψ to be a Riesz basis for $L_2(\Omega)$. However, since $\Lambda_j \subset L_2(\Omega)$ we can consider the modifications Λ_j^n defined in (3.39). It would be interesting to see whether the limit of the Λ_j^n as $n \to \infty$ exists in the sense of Remark 3.3.

4.3 Shift Invariant Spaces

In this section we will be concerned with a setting which houses classical multiresolution and wavelets, i.e., $\mathcal{F} = L_2(\mathbb{R}^s)$. Up to the construction of local biorthogonal projectors which could again be used for an iteration of the form (3.39) in Remark 3.3 most of the material to be discussed below is essentially known. Nevertheless, it should be instructive to review these facts from the point of view taken above and, in doing so, find perhaps a new angle to attack open questions concerning the construction of concrete biorthogonal wavelets.

To this end, let M be throughout the remainder of the paper a fixed $s \times s$ matrix with integer entries whose eigenvalues have modulus strictly larger than one. Such a matrix will be called *expanding*. Moreover, $\varphi = \psi_0$ will always denote some function in $L_2(\mathbb{R}^s)$ which is $(\mathbf{a}, M)$-*refinable*, i.e., for which there exists a sequence $\mathbf{a} = \{a_\alpha\}_{\alpha \in \mathbb{Z}^s}$, called *mask*, such that

$$\psi_0(x) = \sum_{\alpha \in \mathbb{Z}^s} a_\alpha \psi_0(Mx - \alpha), \quad x \in \mathbb{R}^s, a.e. \tag{4.25}$$

Of course, the most familiar example is $M = 2I$, where here I denotes the $s \times s$ identity matrix, but other interesting cases are

$$M = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad M = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix},$$

see e.g. [7, 6, 24, 9].

Setting for $m := |\det M|$,

$$\Phi_j = \{m^{j/2}\varphi(M^j \cdot -\alpha) : \alpha \in \mathbb{Z}^s\}$$

uniform $(\ell_2(\mathbb{Z}^s), L_2(\mathbb{R}^s))$-stability is equivalent to

$$\|\mathbf{c}\|_{\ell_2(\mathbb{Z}^s)} \sim \|\sum_{\alpha\in\mathbb{Z}^s} c_\alpha \varphi(\cdot - \alpha)\|_2. \tag{4.26}$$

We will briefly say that φ is stable. Defining as usual the Fourier transform of $f \in L_1(\mathbb{R}^s)$ by

$$\hat{f}(y) = \int_{\mathbb{R}^s} f(x) e^{-ix\cdot y}\,\mathrm{d}x,$$

(4.26) is known to be equivalent to

$$\sum_{\alpha\in\mathbb{Z}^s} |\varphi(y + 2\pi\alpha)|^2 > 0, \quad x \in [-\pi, \pi]^s \tag{4.27}$$

(cf. [28, 25]) whenever

$$\varphi \in \mathcal{L}_2 := \{f \in L_2(\mathbb{R}^s) : \sum_{\alpha\in\mathbb{Z}^s} |f(\cdot - \alpha)| \in L_2([0,1]^s)\}.$$

For φ to be a nontrivial solution of the refinement equation (4.25) one must have $\hat{\varphi}(0) \neq 0$ so that we may assume without loss of generality in the sequel that

$$\hat{\varphi}(0) = 1. \tag{4.28}$$

On account of (4.25), (4.26), the $S(\Phi_j)$ are closed nested subspaces of $L_2(\mathbb{R}^s)$ and it is not hard to derive from (4.28) that, when M is expanding,

$$\overline{\bigcup_{j\in\mathbb{Z}} S(\Phi_j)} = L_2(\mathbb{R}^s)$$

(cf. e.g. [4, 25]), where the closure is taken, of course, with respect to $L_2(\mathbb{R}^s)$.

To see how this fits into the general setting considered before, let

$$E := \mathbb{Z}^s \cap \{Mx : x \in [0,1]^s\}, \quad E_* = E \setminus \{0\}, \tag{4.29}$$

and recall (cf. [16]), that

$$m := |\det M| = \#E, \tag{4.30}$$

as well as

$$\mathbb{Z}^s = \bigcup_{e\in E} (e + M\mathbb{Z}^s), \tag{4.31}$$

i.e., E is a set of representers for $\mathbb{Z}^s/M\mathbb{Z}^s$. Thus we may put here

$$\Delta_j = M^{-j}\mathbb{Z}^s, \quad \nabla_j = \bigcup_{e\in E_*} M^{-j}(M^{-1}e + \mathbb{Z}^s). \tag{4.32}$$

In our previous language we would have then $k = (e, \alpha)$. However, since clearly, Δ_j and ∇_j are isomorphic to $\mathbb{Z}^s$ or $\{0\} \times \mathbb{Z}^s$ and $E \times \mathbb{Z}^s$ which will mostly employ a stationary and in this case more convenient indexing. Therefore the refinement matrices now take the stationary form

$$\mathbf{A}_{j,0} = \mathbf{A}_0 = (a_{\alpha - M\beta})_{\alpha,\beta\in\mathbb{Z}^s}.$$

4.3.1 Initial Stable Completions

We proceed collecting a few quantitative versions of the general facts stated in Section 2.3. The same arguments as those used in Theorem 3.4 of [25] yield (see also [6])

Remark 4.1 *When ψ_0 is stable, $(\mathbf{a}, M)$-refinable and decays exponentially, then the mask coefficients a_α must also decay exponentially as $\|\alpha\|$ increases where $\|\cdot\|$ is any norm on $\mathbb{R}^s$.*

When φ has compact support, we may work with the concept of linear independence. Recall that the shifts $\varphi(\cdot - \alpha)$, $\alpha \in \mathbb{Z}^s$, are called (algebraically) *linearly independent*, if the mapping F_{Φ_0}

$$\mathbf{c} \mapsto \sum_{\alpha \in \mathbb{Z}^s} c_\alpha \varphi(\cdot - \alpha)$$

is injective for *all* sequences $\mathbf{c}$ on $\mathbb{Z}^s$ (cf. Section 2.4).

Remark 4.2 *Let φ have compact support so that its Fourier Transform extends to an entire function on all of $\mathbb{C}^s$. Then linear independence of the shifts $\varphi(\cdot - \alpha)$ is known to be equivalent to the fact that there exists no $z \in \mathbb{C}^s$ such that $\hat{\varphi}(z + 2\pi\alpha) = 0$ for all $\alpha \in \mathbb{Z}^s$ (cf. [15, 30]). In particular, linear independence implies therefore stability. Moreover, linear independence implies the existence of compactly supported biorthogonal functionals [1, 32]. This, in turn, can be used to show that the corresponding mask must thus be finitely supported.*

We are now ready to formulate simple conditions that ensure the existence of stable or even local decompositions of the spaces $S(\Phi_j)$.

Theorem 4.2 *Suppose $\psi_0 := \varphi \in L_2(\mathbb{R}^s)$ decays exponentially, is $(\mathbf{a}^0, M)$-refinable and stable. Then there exist exponentially decaying masks $\mathbf{a}^e, \mathbf{b}^e$, $e \in E$, such that the functions*

$$\psi_e(x) := \sum_{\alpha \in \mathbb{Z}^s} a^e_\alpha \psi_0(Mx - \alpha), \quad e \in E, \tag{4.33}$$

satisfy

$$\psi_0(Mx - e) = \sum_{e' \in E} \sum_{\alpha \in \mathbb{Z}^s} b^{e'}_{e - M\alpha} \psi_{e'}(x - \alpha), \quad e \in E. \tag{4.34}$$

Moreover, setting $\Psi_j := \cup_{e \in E} \Psi_{e,j}$, $\Psi_{0,j} = \Phi_j$, $\Psi_{e,j} = \{m^{j/2} \psi_e(M^j \cdot - \alpha) : \alpha \in \mathbb{Z}^s\}$, the sequence $\{\cup_{e \in E} \Psi_{e,j}\}$ is uniformly stable and

$$S(\Phi_{j+1}) = \bigoplus_{e \in E} S(\Psi_{e,j}) \tag{4.35}$$

form stable decompositions. If in addition $\varphi = \psi_0$ has compact support and linearly independent shifts the masks $\mathbf{a}^e, \mathbf{b}^e$ can be chosen to be all finitely supported so that the decomposition (4.35) is even local.

Although these facts are known in essence (see e.g [6, 25]) we will sketch some steps of the proof since their ingredients will be needed anyway. To this end, it is convenient to identify with any sequence $\mathbf{c} \in \ell_1(\mathbb{Z}^s)$ the (formal) Laurent series

$$\mathbf{c}(z) = \sum_{\alpha \in \mathbb{Z}^s} c_\alpha z^\alpha .$$

Furthermore, let $\mathbf{c}_e(z) = \sum_{\alpha \in \mathbb{Z}^s} c_{e+M\alpha} z^\alpha$ so that

$$\mathbf{c}(z) = \sum_{e \in E} z^e c_e(z^M), \tag{4.36}$$

where $z^m = (z^{M_1}, \dots, z^{M_i})^T$ and M_i denotes the i-th column of M.

A key role is played by the following observation.

Lemma 4.1 *Let $\psi_0 \in L_2(\mathbb{R}^s)$ be $(\mathbf{a}^0, M)$ refinable. Suppose ψ_0 has compact support and stable integer translates, then the Laurent polynomials $\mathbf{a}_e^0(z)$, $e \in E$, have no common zero on the torus $\mathcal{T}^s := \{z \in \mathbb{C}^s : |z_i| = 1, i = 1, \dots, s\}$. If ψ_0 has linearly independent integer translates, the $\mathbf{a}_e^0(z)$, $e \in E$, have no common zero in $(\mathbb{C} \setminus \{0\})^s$.*

The above assumptions may be weakened. For instance, in the first part of the assertion it would suffice to assume $\psi_0 \in \mathcal{L}_2$. The idea of the proof is the same as in the case $M = 2I$ (see e.g. [25]). The result may also be viewed as a special case of a corresponding result in [18].

The proof of Theorem 4.2 relies now on the following consequence of Lemma 4.1.

Proposition 4.1 *Let ψ_0 satisfy the hypotheses of Lemma 4.1 with finitely supported $\mathbf{a}^0$. Then there exist additional finitely supported masks $\mathbf{a}^e$, $e \in E_*$ such that the matrix*

$$\mathbf{A}(z) := (\mathbf{a}_e^{e'}(z))_{e,e' \in E} \tag{4.37}$$

is invertible on $\mathcal{T}^s$ when ψ_0 has stable integer translates and satisfies $\det \mathbf{A}(z) = 1$, *$z \in (\mathbb{C} \setminus \{0\})^s$, when ψ_0 has linearly independent integer translates. Thus the matrix*

$$\mathbf{B}(z) := (\mathbf{b}_e^{e'}(z))_{e',e \in E}, \tag{4.38}$$

defined by

$$\mathbf{B}(z^{-1}) = \mathbf{A}(z)^{-1} \tag{4.39}$$

defines exponentially decaying masks $\mathbf{b}^{e'}$, $e' \in E$, when ψ_0 has stable integer translates and even finitely supported masks, when ψ_0 has linearly independent integer translates.

In view of Lemma 4.1, the proof follows exactly the lines of [25] ensuring *extensibility* of $\mathbf{a}^0$ in the case of stability and using the Quillen-Suslin Theorem in the case of linear independence.

It only remains now to reinterpret these facts in terms of the setting considered in Sections 2, 3. To this end, let

$$\mathbf{A}_e := (a^e_{\alpha - M\beta})_{\alpha,\beta \in \mathbb{Z}^s}, \quad \mathbf{B}^e := (b^e_{\beta - M\alpha})_{\alpha,\beta \in \mathbb{Z}^s}, \tag{4.40}$$

which may be viewed as mappings from $\ell_2(\mathbb{Z}^s)$ into $\ell_2(M^{-1}\mathbb{Z}^s)$ and $\ell_2(M^{-1}\mathbb{Z}^s)$ into $\ell_2(\mathbb{Z}^s)$, respectively. Again one may assemble such matrices to a single one by defining the operators $\mathbf{A} : \ell_2(\cup_{e\in E}(M^{-1}e + \mathbb{Z}^s)) \to \ell_2(M^{-1}\mathbb{Z}^s)$ and $\mathbf{B} : \ell_2(M^{-1}\mathbb{Z}^s) \to \ell_2(\cup_{e\in E}(M^{-1}e + \mathbb{Z}^s))$ by

$$\mathbf{A}\mathbf{c} := \sum_{e\in E} \mathbf{A}_e \mathbf{c}_e \qquad (\mathbf{B}\mathbf{c})_{e+M\alpha} = (\mathbf{B}^e\mathbf{c})_\alpha, \quad e \in E,\ \alpha \in \mathbb{Z}^s, \tag{4.41}$$

where $\mathbf{c}_e := \{c_{e+M\alpha}\}_{\alpha\in\mathbb{Z}^s}$. It is now easy to verify that (2.20) or in the present terms

$$\mathbf{AB} = \mathbf{BA} = \mathbf{I} \tag{4.42}$$

is equivalent to

$$\mathbf{A}(z)\mathbf{B}(z^{-1}) = \mathbf{I}. \tag{4.43}$$

In fact, the validity of the relations

$$\sum_{e'\in E}\sum_{\beta\in\mathbb{Z}^s} a^{e'}_{e''+M(\mu-\beta)} b^{e'}_{e-M\beta} = \delta_{0,\mu}\delta_{e,e''}, \quad \mu \in \mathbb{Z}^s, e, e'' \in E, \tag{4.44}$$

which are, in view of Proposition 4.1, well-defined, is equivalent to

$$\sum_{e'\in E} a^{e'}_{e''}(z) b^{e'}_e(z^{-1}) = \delta_{e,e''}, \quad e, e'' \in E, z \in \mathcal{T}^s, \tag{4.45}$$

This can be seen by multiplying both sides of (4.44) by z^μ and summing over $\mu \in \mathbb{Z}^s$. Now (4.44) corresponds to (4.42) while (4.45) is just (4.43). But since $\mathbf{A}(z)$ is a finite matrix, (4.39) is equivalent to (4.43).

Taking Lemma 4.1 into account, the above observations combined with Proposition 2.1 may be summarized as follows.

Proposition 4.2 *Let $\psi_0 \in L_2(\mathbb{R}^s)$ be $(\mathbf{a}^0, M)$-refinable and suppose ψ_0 decays exponentially and has stable shifts (has compact support and linearly independent shifts). Then the following statements are equivalent:*

- *(i) The functions ψ_e, $e \in E_*$, defined by (4.33) relative to exponentially decaying (finitely supported) masks $\mathbf{a}^e, e \in E_*$, satisfy (4.34) relative to masks $\mathbf{b}^e$, $e \in E$, with the same properties and are stable (linearly independent).*
- *(ii) $\mathbf{A}(z)\mathbf{B}(z^{-1}) = \mathbf{I}$, $z \in \mathcal{T}^s$ ($z \in (\mathbb{C} \setminus \{0\})^s$), where $\mathbf{A}(z)$, $\mathbf{B}(z)$ are defined by (4.37), (4.38).*

The proof of Theorem 4.2 is now an immediate consequence of Proposition 4.2.

We wish to mention another way of generating some initial completion which is analogous to the situation considered in Section 5.2 above.

To this end, suppose that $\psi_0 \in C(\mathbb{R}^s)$ has compact support, is $(\mathbf{a}^0, M)$-refinable and satisfies

$$\sigma(x) = \sum_{\alpha\in\mathbb{Z}^s} \psi_0(\alpha) e^{i\alpha\cdot x} \neq 0, \quad x \in \mathbb{R}^s, \tag{4.46}$$

or, equivalently,

$$\sigma(x) = \sum_{\alpha \in I\!\!R^s} \hat{\psi}_0(x + 2\pi\alpha) \neq 0, \quad x \in I\!\!R^s. \tag{4.47}$$

Thus ϕ_0 defined by

$$\hat{\phi}_0(x) = \hat{\psi}_0(x)/\sigma(x) \tag{4.48}$$

has the form

$$\phi_0(x) = \sum_{\alpha \in \mathbb{Z}^s} g_\alpha \psi_0(x - \alpha), \tag{4.49}$$

where

$$\frac{1}{\sigma(x)} = \sum_{\alpha \in \mathbb{Z}^s} g_\alpha e^{-ix \cdot \alpha},$$

i.e., by Wiener's Lemma, the coefficients g_α decay exponentially, and

$$\phi_0(\alpha) = \delta_{0,\alpha}, \quad \alpha \in \mathbb{Z}^s. \tag{4.50}$$

Clearly, ϕ_0 is $(\hat{\mathbf{a}}^0, M)$-refinable with

$$\hat{\mathbf{a}}^0(e^{-iu}) = \mathbf{a}^0(e^{-iu})\sigma(M^{-T}u)/\sigma(u)$$

and

$$S(\Phi_j) = S(\{\phi_0(M^j \cdot -\alpha)\}_{\alpha \in \mathbb{Z}^s}). \tag{4.51}$$

From (4.48) we see that

$$\sum_{\alpha \in \mathbb{Z}^s} |\phi_0(u + 2\pi\alpha)|^2 = |\sigma(u)|^{-2} \sum_{\alpha \in \mathbb{Z}^s} \left|\hat{\psi}_0(u + 2\pi\alpha)\right|^2$$

so that the shifts $\phi_0(\cdot - \alpha)$, $\alpha \in \mathbb{Z}^s$ are, on account of (4.27), also stable.

Now define

$$\phi_e(x) := \phi_0(Mx - e), \quad e \in E_*, \tag{4.52}$$

so that

$$\phi_e(\alpha) = \phi_0(M\alpha - e) = 0, \quad \alpha \in \mathbb{Z}^s. \tag{4.53}$$

Moreover,

$$\phi_0(Mx - e) = \begin{cases} \phi_e(x), & e \in E_* \\ \phi_0(x) - \sum_{e' \in E_*} \sum_{\alpha \in \mathbb{Z}^s} \phi_0(\alpha + M^{-1}e')\phi_{e'}(x - \alpha), & e = 0. \end{cases} \tag{4.54}$$

In fact, let

$$h(x) = \phi_0(x) - \sum_{e' \in E_*} \sum_{\alpha \in \mathbb{Z}^s} \phi_0(\alpha + M^{-1}e')\phi_{e'}(x - \alpha)$$

and note that, in view of (4.53),

$$h(0) = \phi_0(0) - \sum_{e' \in E_*} \sum_{\alpha \in \mathbb{Z}^s} \phi_0(\alpha + M^{-1}e')\phi_{e'}(-\alpha) = \phi_0(0) = 1. \tag{4.55}$$

Now suppose that $\nu \in \mathbb{Z}^s$, $\nu \neq 0$, and write $\nu = M\beta + e''$ for some $\beta \in \mathbb{Z}^s$, $\beta'' \in E$. Then by (4.52),

$$
\begin{aligned}
h(M^{-1}\nu) &= h(\beta + M^{-1}e'') \\
&= \phi_0(\beta + M^{-1}e'') - \sum_{e' \in E_*} \sum_{\alpha \in \mathbb{Z}^s} \phi_0(\alpha + M^{-1}e')\phi_{e'}(\beta + M^{-1}e'' - \alpha) \\
&= \phi_0(\beta + M^{-1}e'') - \sum_{e' \in E_*} \sum_{\alpha \in \mathbb{Z}^s} \phi_0(\alpha + M^{-1}e')\phi_0(M(\beta - \alpha) + e'' - e').
\end{aligned}
$$

Since $M\beta + e'' = M\alpha + e$ if and only if $\alpha = \beta$, $e = e''$, (4.50) yields

$$
h(M^{-1}\nu) = \phi_0(\beta + M^{-1}e'') - \phi_0(\beta + M^{-1}e'') = 0. \tag{4.56}
$$

Since the $\phi_0(\alpha+M^{-1}e')$ decay exponentially, it is clear that $h \in S(\{\phi_0(M\cdot-\alpha)\}_{\alpha\in\mathbb{Z}^s}) = S(\Phi_1)$. From (4.55) and (4.56) we infer that $h(x) = \phi_0(Mx)$ which confirms (4.54).

Thus the functions ϕ_e, $e \in E$, satisfy (4.33) and (4.34) with

$$
\hat{a}^e_\alpha = \delta_{\alpha,e}, \quad e \in E_*, \tag{4.57}
$$

and, by (4.54),

$$
\hat{b}^{e'}_{e-M\alpha} = \begin{cases} \delta_{e,e'}\delta_{0,\alpha}, & e \in E_*,\ e' \in E,\ \alpha \in \mathbb{Z}^s, \\ (2\delta_{0,e'} - 1)\phi_0(\alpha + M^{-1}e'). & \end{cases} \tag{4.58}
$$

Since the sequences $\hat{a}^{e'}_\alpha$, $\hat{b}^e_\alpha$ are either finitely supported or exponentially decaying, this defines a stable decomposition. In fact, one has for instance

$$
\hat{A}_e = (\delta_{\beta,M\alpha+e})_{\alpha,\beta\in\mathbb{Z}^s}, \quad e \in E_*. \tag{4.59}
$$

We wish to see next what this decomposition means in terms of the original generator ψ_0. Firstly, by (4.49),

$$
\phi_e(x) = \sum_{\alpha\in\mathbb{Z}^s} g_\alpha \psi_0(Mx - e - \alpha) = \sum_{\alpha\in\mathbb{Z}^s} g_{\alpha-e}\psi_0(Mx - \alpha), \quad e \in E_*, \tag{4.60}
$$

i.e.

$$
a^e_\alpha = g_{\alpha-e}, \quad \alpha \in \mathbb{Z}^s, e \in E_*. \tag{4.61}
$$

Moreover,

$$
\begin{aligned}
\psi_0(Mx - e) &= \sum_{\alpha\in\mathbb{Z}^s} \psi_0(M(M^{-1}\alpha) - e)\psi_0(Mx - \alpha) \\
&= \sum_{e'\in E} \sum_{\nu\in\mathbb{Z}^s} \psi_0(M(M^{-1}(M\nu + e')) - e)\phi_0(Mx - M\nu - e') \\
&= \sum_{e'\in E} \sum_{\nu\in\mathbb{Z}^s} \psi_0(M(\nu + M^{-1}e') - e)\phi_0(Mx - e' - M\nu) \\
&= \sum_{e'\in E} \sum_{\nu\in\mathbb{Z}^s} \psi_0(M\nu + e' - e)\phi_{e'}(x - \nu).
\end{aligned}
$$

Since

$$\sum_{\nu\in\mathbb{Z}^s}\psi_0(M\nu-e)\phi_0(x-\nu)=\sum_{\alpha\in\mathbb{Z}^s}\left(\sum_{\beta\in\mathbb{Z}^s}\psi_0(M\alpha-e-M\beta)g_\beta\right)\psi_0(x-\alpha),$$

we obtain

$$b^0_{-\alpha}=\sum_{\beta\in\mathbb{Z}^s}\psi_0(\alpha-M\beta)g_\beta,\quad \alpha\in\mathbb{Z}^s, \tag{4.62}$$

and

$$b^{e'}_{-\alpha}=\psi_0(\alpha+e'),\quad e'\in E_*,\alpha\in\mathbb{Z}^s. \tag{4.63}$$

Thus the rows and columns of the matrices $\mathbf{A}_e$, $\mathbf{B}^e$, $e\in E$, defined by (4.61), (4.62) and (4.63) are either finitely supported or exponentially decaying. We immediately infer from Proposition 4.2 that this establishes a stable decomposition.

4.3.2 Decompositions Induced by Local L_2-Bounded Projectors

Given such initial decompositions one may now apply the results from Section 4. In this context it is convenient to reinterpret the relevant matrix identities in terms of symbols. In particular, the matrices $\mathbf{L},\mathbf{K}$ should be blocked appropriately, i.e., for sequences $\mathbf{l}$, $\mathbf{k}^e, e\in E_*$, we set

$$\mathbf{L}_e:=(l_{e+M(\alpha-\beta)})_{\alpha,\beta\in\mathbb{Z}^s},\quad \mathbf{K}_{e,e'}:=(k^{e'}_{e+M(\alpha-\beta)})_{\alpha,\beta\in\mathbb{Z}^s},\quad e,e'\in E_*.$$

Moreover set,

$$(\mathbf{K}^{-1})_{e,e'}:=\mathbf{N}_{e,e'},\quad e,e'\in E_*,$$

and note that, for instance, (3.3) reads now

$$\hat{\mathbf{A}}_e=\mathbf{A}_0\mathbf{L}_e+\sum_{e'\in E_*}\mathbf{A}_{e'}\mathbf{K}_{e',e},\quad e\in E_*.$$

Since

$$(\mathbf{K}^{-1}\mathbf{B}_1)_e(z)=\sum_{e'\in E_*}\mathbf{n}^{e'}_e(z^{-M})\mathbf{b}^{e'}(z),$$

one easily verifies that the relations (3.3), (3.4), and (3.5) are equivalent to

$$\begin{aligned}
\hat{\mathbf{a}}^e(z) &= \mathbf{a}^0(z)\mathbf{l}_e(z^M)+\sum_{e'\in E_*}\mathbf{a}^{e'}(z)\mathbf{k}^e_{e'}(z^M)\\
\hat{\mathbf{b}}^0(z) &= \mathbf{b}^0(z)-\sum_{e,e'\in E_*}\mathbf{b}^{e'}(z)\mathbf{n}^{e'}_e(z^M)\mathbf{l}_e(z^M),\quad z\in\mathcal{T}^s\\
\hat{\mathbf{b}}^e(z) &= \sum_{e'\in E_*}\mathbf{b}^{e'}(z)\mathbf{n}^{e'}_e(z^M),\quad e\in E_*,\ z\in\mathcal{T}^s.
\end{aligned} \tag{4.64}$$

Again possible applications concern *local* L_2-bounded projectors. We will construct next such projectors with range $S(\Phi_j)$ and biorthogonal systems in $S(\Phi_{j+1})$.

Let for $f,g\in L_2(\mathbb{R}^s)$

$$(f,g)=\int_{\mathbb{R}^s}f(x)\overline{g(x)}\,\mathrm{d}x.$$

Theorem 4.3 *Suppose $\psi_0 \in L_2(\mathbb{R}^s)$ is $(\mathbf{a}^0, M)$-refinable, has compact support and linearly independent shifts. Then there exist functions*

$$\psi_e(x) = \sum_{\alpha\in\mathbb{Z}^s} a_\alpha^e \psi_0(Mx-\alpha), \quad e \in E_*, \tag{4.65}$$

and

$$\eta_e(x) = \sum_{\alpha\in\mathbb{Z}^s} d_\alpha^e \psi_0(Mx-\alpha), \quad e \in E, \tag{4.66}$$

such that

$$(\psi_e, \eta_{e'}(\cdot-\alpha)) = \delta_{e,e'}\delta_{0,\alpha}, \quad e,e' \in E, \alpha \in \mathbb{Z}^s. \tag{4.67}$$

Moreover, the masks $\mathbf{a}^e$, $e \in E_*$, $\mathbf{d}^e$, $e \in E$, can be chosen in such a way that either the masks $\mathbf{a}^e$, $e \in E_*$, decay exponentially and the masks $\mathbf{d}^e$, $e \in E$, are all finitely supported or the $\mathbf{a}^e$, $e \in E_*$, are finitely supported and the $\mathbf{d}^e$, $e \in E$, decay exponentially.

Proof: Note that the Gramian $\langle \Phi_1, \Phi_1 \rangle$ corresponds now to the Toeplitz matrix $m^{-1}((\psi_0(\cdot-\alpha), \psi_0(\cdot-\beta)))_{\alpha,\beta\in\mathbb{Z}^s}$. Let

$$g_\alpha := (\psi_0, \psi_0(\cdot+\alpha)), \quad \mathbf{G} := (m^{-1}g_{\alpha-\beta})_{\alpha,\beta\in\mathbb{Z}^s}, \tag{4.68}$$

and note that by (4.66) and (4.25),

$$\begin{aligned}(\psi_e, \eta_{e'}(\cdot-\alpha)) &= m^{-1}\sum_{\beta\in\mathbb{Z}^s}\sum_{\nu\in\mathbb{Z}^s} a_\beta^e \overline{d}_\nu^{e'} g_{\beta-\nu-M\alpha}, \quad e,e' \in E,\\ &= \sum_{\nu\in\mathbb{Z}^s}\left(m^{-1}\sum_{\beta\in\mathbb{Z}^s} a_\beta^e g_{\beta-\nu}\right)\overline{d}_{\nu-M\alpha}^{e'}, \quad e,e' \in E.\end{aligned} \tag{4.69}$$

Thus

$$(\psi_e, \eta_{e'}(\cdot-\alpha)) = \delta_{0,\alpha}\delta_{e,e'}, \quad e,e' \in E, \alpha \in \mathbb{Z}^s,$$

if and only if

$$\mathbf{D}^{e'}\mathbf{G}\mathbf{A}_e = \delta_{e,e'}\mathbf{I}, \quad e,e' \in E, \tag{4.70}$$

where as earlier $\mathbf{D}^{e'} = (\overline{d}_{\alpha-M\beta}^{e'})_{\beta,\alpha\in\mathbb{Z}^s}$ (cf. (4.39)). Defining $\hat{\mathbf{A}} := \mathbf{G}\mathbf{A}$ and $\hat{\mathbf{D}} := \mathbf{D}\mathbf{G}$, where $\mathbf{A}$ and $\mathbf{D}$ are defined as in (4.41), this is equivalent to

$$\hat{\mathbf{D}}\mathbf{A} = \mathbf{D}\hat{\mathbf{A}} = \mathbf{I}. \tag{4.71}$$

By Proposition 4.1, there exist additional finitely supported masks $\mathbf{a}^e$, $e \in E_*$ and finitely supported masks $\mathbf{b}^e$, $e \in E$, such that $\mathbf{A}\mathbf{B}^* = \mathbf{I}$. Now note that

$$\sum_{\alpha\in\mathbb{Z}^s} g_\alpha e^{-ix\cdot\alpha} = \sum_{\alpha\in\mathbb{Z}^s} |\psi_0(x+2\pi\alpha)|^2 > 0, \quad x \in \mathbb{R}^s,$$

by stability of the shifts $\psi_0(\cdot-\alpha)$, so that $\mathbf{G}$ possesses an inverse $\mathbf{G}^{-1}$ whose entries decay exponentially. Thus

$$\mathbf{D} := \mathbf{B}\mathbf{G}^{-1} \tag{4.72}$$

satisfies (4.66), (4.67) and has exponentially decaying entries. Since the masks $\mathbf{a}^e$, $e \in E$, are finitely supported, the second part of the assertion follows.

Concerning the rest of the claim, we observe that the matrix $\hat{\mathbf{A}}_0$ is again a Hurwitz-type matrix

$$\hat{\mathbf{A}}_0 = (\hat{a}^0_{\alpha - M\beta})_{\alpha,\beta\in\mathbb{Z}^s}, \quad \hat{a}^0_\alpha = m^{-1} \sum_{\nu\in\mathbb{Z}^s} a^0_{\nu+\alpha} g_\nu.$$

Moreover, since ψ_0 has compact support and $\mathbf{a}^0$ has finite support the new mask $\hat{\mathbf{a}}^0$ has still finite support. To finish the proof of Theorem 4.3 we need the following lemma.

Lemma 4.2 *The Laurent polynomials $\hat{\mathbf{a}}^0_e(z)$, $e \in E$, have no common zero in $(\mathbb{C} \setminus \{0\})^s$.*

Proof: Suppose that for some $z_0 \in (\mathbb{C} \setminus \{0\})^s$ $\hat{\mathbf{a}}^0_e(z_0) = 0$, $e \in \mathbb{Z}^s$. Then for $\beta = e + M\beta'$, $e \in E$,

$$\begin{aligned}\left(\psi_0(M\cdot - \beta), \sum_{\alpha\in\mathbb{Z}^s} \bar{z}_0^{-\alpha}\psi_0(\cdot - \alpha)\right) &= m^{-1}\sum_{\alpha\in\mathbb{Z}^s} z_0^{-\alpha} \sum_{\mu\in\mathbb{Z}^s} g_{\beta-\mu} a^0_{\mu - M\alpha} \\ &= \sum_{\alpha\in\mathbb{Z}^s} z_0^{-\alpha}\hat{a}^0_{\beta - M\alpha} = z_0^{-\beta'}\hat{a}^0_e(z_0) = 0,\end{aligned}$$

contradicting linear independence of the $\psi_0(\cdot - \alpha)$, $\alpha \in \mathbb{Z}^s$. $\square$

We are now ready to complete the proof of Theorem 4.3. We argue as in [25] and infer form the Quillen-Suslin Theorem that there exist additional finitely supported masks $\hat{\mathbf{a}}^e$, $e \in E_*$, such that $\hat{\mathbf{A}}(z) = (\hat{a}^e_{e'}(z))_{e,e'\in E}$ has determinant one on $(\mathbb{C} \setminus \{0\})^s$.

In this case we define the masks $\mathbf{d}^e$, $e \in E$, by

$$\mathbf{D}(z^{-1}) = \hat{\mathbf{A}}(z)^{-1},$$

which therefore are also finitely supported. Moreover, by (4.71) we have to set now

$$\mathbf{A}_e = \mathbf{G}^{-1}\hat{\mathbf{A}}_e, \quad e \in E_*, \tag{4.73}$$

i.e., the masks $\mathbf{a}^e$, $e \in E_*$ are merely exponentially decaying. This completes the proof of Theorem 4.3. $\square$

For $f \in L_2(\mathbb{R}^s)$ let

$$f^j_\alpha := m^{j/2} f(M^j \cdot - \alpha)$$

and define for η_0 from Theorem 4.3

$$Q_j f := \sum_{\alpha\in\mathbb{Z}^s} (f, \eta^j_{0,\alpha})\psi^j_{0,\alpha}, \quad j \in \mathbb{Z}.$$

Clearly the Q_j are projectors onto $S(\Phi_j)$ with uniformly bounded L_2-norm. Specifically, let us assume in the sequel that the η_e have been chosen to have compact support. Clearly, one has

$$(Q_1 - Q_0)\psi^0_{0,\alpha} = 0, \quad (Q_1 - Q_0)\psi^0_{e,\alpha} = \psi_{e,\alpha}, \quad e \in E_*, \alpha \in \mathbb{Z}^s,$$

i.e., the spaces

$$W_j = \bigoplus_{e \in E_*} S(\Psi_{e,j}) \tag{4.74}$$

are of the form (3.23). But according to Theorem 4.3, either the functions ψ_e or the dual system η_e, $e \in E$, will lack compact support.

One way to remedy this is to make use of Theorem 3.3. To this end, Theorem 4.3 asserts the existence of a finitely supported mask $\mathbf{d}^0$ such that

$$\eta_0(x) = \sum_{\alpha \in \mathbb{Z}^s} d^0_\alpha \psi_0(Mx - \alpha) \tag{4.75}$$

is dual to ψ_0, i.e.,

$$(\psi_0, \eta_0(\cdot - \alpha)) = \delta_{0,\alpha}, \quad \alpha \in \mathbb{Z}^s. \tag{4.76}$$

Furthermore, according to Theorem 4.2, let $\hat{\mathbf{a}}^e$, $e \in E_*$, $\hat{\mathbf{b}}^e$, $e \in E$, a collection of finitely supported masks that correspond to a local completion $\hat{A}_1$ of A_0, i.e., the matrices $\mathbf{A}, \mathbf{B}$ are both banded.

Theorem 4.4 *For ψ_0, η_0 as in Theorem 4.3 and $\hat{\mathbf{A}}_1, \hat{\mathbf{B}}$ as above let*

$$\mathbf{a}^e(z) := \hat{\mathbf{a}}^e(z) - \mathbf{a}^0(z) \sum_{e' \in E} \hat{\mathbf{a}}^e_{e'}(z^M)\mathbf{c}_{e'}(z^{-M}), \tag{4.77}$$

where

$$c_\alpha := m^{-1/2} \sum_{\beta \in \mathbb{Z}^s} \overline{d^0_\beta} g_{\beta - \alpha}, \tag{4.78}$$

and

$$\mathbf{b}^0(z) = \mathbf{c}(z^{-1}), \qquad \mathbf{b}^e(z) = \hat{\mathbf{b}}^e(z), \quad e \in E_*. \tag{4.79}$$

Then the functions

$$\psi_e(x) = \sum_{\alpha \in \mathbb{Z}^s} a^e_\alpha \psi_0(Mx - \alpha), \quad e \in E, \tag{4.80}$$

$$\eta_e(x) = \sum_{\nu \in \mathbb{Z}^s} \left(m^{1/2} \sum_{\alpha \in \mathbb{Z}^s} b^e_\alpha d^0_{\nu - M\alpha} \right) \psi_0(M^2 x - \nu) \tag{4.81}$$

satisfy

$$(\psi_e, \eta_{e'}(\cdot - \alpha)) = \delta_{e,e'}\delta_{0,\alpha}, \quad e, e' \in E, \alpha \in \mathbb{Z}^s. \tag{4.82}$$

Proof: The matrix $\mathbf{C} = (c_{\alpha - M\beta})_{\alpha,\beta \in \mathbb{Z}^s} = \langle \Phi_{j+1}, \Lambda_j \rangle$ in (3.25) takes here the form

$$m^{1/2}(\psi_0(M \cdot - \alpha), \eta_0(\cdot - \beta)) = m^{-1/2} \sum_{\nu \in \mathbb{Z}^s} \overline{d^0_\nu} g_{\nu + M\beta - \alpha} = c_{\alpha - M\beta}. \tag{4.83}$$

Due to the finite support of the sequences $\mathbf{d}^0, \mathbf{h}$, the sequence $\mathbf{c}$ is also finitely supported. Thus by Theorem 3.3 with $\mathbf{K} = \mathbf{I}$ the matrices

$$\mathbf{A}_e := (I - \mathbf{A}_0 \mathbf{C}^T)\hat{\mathbf{A}}_e, \qquad e \in E_*, \tag{4.84}$$

give rise to another local completion $\mathbf{A}_1$ of $\mathbf{A}_0$. Moreover, by (3.26), one has

$$\mathbf{B}^0 = \hat{\mathbf{B}}^0 + \sum_{e\in E_*} \mathbf{C}^T \hat{\mathbf{A}}_e \hat{\mathbf{B}}^e = \hat{\mathbf{B}}^0 - \mathbf{C}^T \mathbf{A}_0 \hat{\mathbf{B}}^0 + \mathbf{C}^T, \quad \mathbf{B}^e = \hat{\mathbf{B}}^e, \quad e \in E_*. \tag{4.85}$$

where we have also used (2.57). However, since by (2.13), in general $\langle \Phi_{j+1}, \Lambda_j\rangle^T \mathbf{A}_0 = \mathbf{I}$, whenever Φ_j and Λ_j are dual, our assumptions on ψ_0 and η_0 provide $\mathbf{C}^T \mathbf{A}_0 = \mathbf{I}$ so that

$$\mathbf{B}_0 = \mathbf{C}^T, \qquad \mathbf{B}_e = \hat{\mathbf{B}}_e, e \in E_*.$$

Now (4.64) combined with (4.84) yields (4.77). The rest of the assertion follows form (3.29) for $\eta^e_\alpha(f) = (f, \eta_e(\cdot - \alpha))$, $\hat{b}^e_{\alpha,\beta} = b^e_{\beta - M\alpha}$. □

Theorem 4.4 shows that by resorting to $S(\Phi_1)$ and $S(\Phi_2)$ one can construct a dual system where all the involved functions have compact support. If one does not insist on the dual system $\eta_e, e \in E$, to belong to $S(\Phi_1)$ or $S(\Phi_2)$ but instead wants η_0 to be also $(\mathbf{b}^0, M)$-refinable for some finitely supported mask $\mathbf{b}^0$ one would obtain *biorthogonal wavelets* (see Proposition 2.2). By Remark 2.5 the corresponding additional masks $\mathbf{a}^e$, $e \in E_*$, that establish a decomposition of $S(\psi_0)$ and the masks $\mathbf{b}^e, e \in E$, of the dual system are again determined by the relations (3.42). The above local biorthogonal systems may be used now as starting point for the limit processes in (3.39).

Moreover, Theorem 3.1 and Theorem 3.3 offer ways of modifying $\mathbf{A}_1$ and $\mathbf{B}$ so as to increase, for instance, the regularity or degree of exactness of the dual system.

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References

[1] A. Ben-Artzi, A. Ron, On the integer translates of a compactly supported function: dual bases and linear projectors, SIAM J. Math. Anal., 21(1990), 1550–1562.

[2] G. Beylkin, R. Coifman, V. Rokhlin, Fast wavelet transforms and numerical algorithms I, Comm. Pure and Appl. Math., Vol. XLIV, (1991), 141-183.

[3] C. de Boor, A practical Guide to Splines, Springer, 1978.

[4] C. de Boor, R. A. de Vore, A. Ron, On the construction of multivariate (pre–) wavelets, Constr. Approx. 9(1993), 123-166.

[5] C.K. Chui, An Introduction to Wavelets, Academic Press, 1992.

[6] C.K. Chui, C. Li, A general framework of multivariate wavelets with duals, ACHA, 1(1994), 368-390.

[7] A. Cohen and I. Daubechies, Non–Separable Bidimensional Wavelet Bases, Revista Mat. Iberoamericana, 9(1993), 51-137.

[8] A. Cohen, I. Daubechies and J.–C. Feauveau, *Biorthogonal bases of compactly supported wavelets*, Comm. Pure Appl. Math., 45(1992), 485-560.

[9] S. Dahlke, W. Dahmen, V. Latour, Smoothness of refinable functions obtained by convolution products, Applied and Computational Harmonic Analysis, 2(1995), 68-84.

[10] S. Dahlke, A. Kunoth, Biorthogonal wavelets and multigrid, Proceedings of the 9th GammSeminar (W. Hackbusch, G. Wittum eds.), NNFM series Vieweg, Kiel, 1993.

[11] W. Dahmen, Decomposition of refinable spaces and applications to operator equations, Numerical Algorithms, 5(1993), 229-245.

[12] W. Dahmen, Some remarks on multiscale transformations, stability and biorthogonality, in *Curves and Surfaces II*, P.J. Laurent, A. LeMéhauté, L.L. Schumaker (eds.), Academic Press, 1994, 157-188.

[13] W. Dahmen, Stability of multiscale transformations, IGPM Report 109, Nov. 1994, RWTH Aachen, to appear in Journal of Fourier Analysis and Applications.

[14] W. Dahmen, A. Kunoth, Multilevel preconditioning, Numer. Math., 63(1992), 315–344.

[15] W. Dahmen, C.A. Micchelli, Translates of multivariate splines, Linear Algebra Appl., 52/53(1983), 217–234.

[16] W. Dahmen, C.A. Micchelli, On the solution of certain systems of partial difference equations, Trans. Amer. Math. Soc., 292(1985), 305–320.

[17] W. Dahmen, C.A. Micchelli, Banded matrices with banded inverses II: Locally finite decompositions of spline spaces, Constructive Approximation, 9(1993), 263–281.

[18] W. Dahmen, C.A. Micchelli, Biorthogonal wavelet expansions, IGPM Report # 114, RWTH Aachen, May 1995.

[19] W. Dahmen, R. Schneider, S. Prössdorf, Wavelet approximation methods for pseudodifferential equations I: Stability and convergence, Math. Z., 215(1994), 583-620.

[20] W. Dahmen, R. Schneider, S. Prössdorf, Wavelet approximation methods for pseudodifferential equations II: Matrix compression and fast solution, Advances in Computational Mathematics, 1(1993), 259-335.

[21] W. Dahmen, S. Prössdorf, R. Schneider, Multiscale methods for pseudo-differential equations on smooth closed manifolds, in: Proceedings of the International Conference on Wavelets: Theory, Algorithms, and Applications, C.K. Chui, L. Montefusco, L. Puccio (eds.), Academic Press, 385-424, 1994.

[22] W. Dahmen, R. Schneider, Biorthogonal wavelets on two-dimensional manifolds in $\mathbb{R}^3$ and adaptive quadrature, in preparation.

[23] I. Daubechies, Orthonormal bases of compactly supported wavelets, Communications on Pure and Applied Mathematics, 41(1988), 909-996.

[24] K.H. Gröchenig, W.R. Madych, Haar bases and self-affine tilings of $\mathbb{R}^n$, IEEE Trans. Inform. Th. 38(2), 1992, 556–568

[25] R.Q. Jia, C.A. Micchelli, Using the refinement equation for the construction of pre-wavelets II: Powers of two, in *Curves and Surfaces*, P.J. Laurent, A. LeMéhauté, L.L. Schumaker, eds., Academic Press, New York, 1991, 209–246.

[26] A. Kunoth, Multilevel preconditioning - appending boundary conditions by Lagrange multipliers, Advances in Computational Mathematics **4**, No. 1,2 (1995), 145–170.

[27] Y. Meyer, *Ondelettes et Opérateurs 2 : Opérateur de Caldéron-Zygmund*, Hermann, Paris 1990.

[28] S.G. Mallat, Multiresolution approximation and wavelet orthonormal basis of $L_2(\mathbb{R})$, Trans. Amer. Math. Soc., 315(1989), 69–87.

[29] C.A. Micchelli, Using the refinement equation for the construction of pre-wavelets, Numerical Algorithms, 1(1991), 75–116.

[30] A. Ron, A necessary and sufficient condition for the linear independence of the integer translates of compactly supported distribution, Constructive Approximation, 5(1989), 297–308.

[31] H. Yserentant, On the multilevel splitting of finite element spaces, Numer. Math., 49 (1986), 379–412.

[32] K. Zhao, Global linear independence and finitely supported dual basis, SIAM J. Math. Anal., 23(1992), 1352-1355.

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