AIP | Journal of Mathematical Physics

Necessary and sufficient conditions for constructing orthonormal wavelet bases

Wayne M. Lawton

Citation: Journal of Mathematical Physics **32**, 57 (1991); doi: 10.1063/1.529093
View online: http://dx.doi.org/10.1063/1.529093
View Table of Contents: http://scitation.aip.org/content/aip/journal/jmp/32/1?ver=pdfcov
Published by the AIP Publishing

Necessary and sufficient conditions for constructing orthonormal wavelet bases

Wayne M. Lawton
AWARE, Inc., One Memorial Drive, Suite 310, 124 Mount Auburn Street, Cambridge, Massachusetts 02142

(Received 16 April 1990; accepted for publication 15 August 1990)

This paper proves a previous conjecture of the author characterizing sequences $h \in l^2(Z)$ that yield orthonormal wavelet bases of $L^2(R)$ in terms of the multiplicity of the eigenvalue 1 of an operator associated to h. The proof utilizes a result of Cohen characterizing these sequences in terms of the real zeros of their Fourier transforms. The mapping from sequences to wavelets is shown to define a continuous mapping from a subset of $l^2(Z)$ into $L^2(R)$. Related conjectures are discussed.

I. INTRODUCTION

Wavelets are families of functions $\psi_{a,b}(x) = \sqrt{a}\psi(ax-b)$; $a \neq 0$, $b \in R$ that are generated from a single function through translations and scalings. Wavelets provide means of representing functions, either by continuous or by discrete linear superpositions, similar to Fourier integrals and Fourier series. Although the term wavelet is recent,[1] previous implicit use of these functions and related scaling concepts can be found in the following areas: function theory,[2,3] multilevel numerical methods,[4–6] singular integral operators,[7] group representations,[8,9] renormalization in quantum field theory and statistical physics,[10] multirate concepts in digital signal processing,[11,12] and heirarchical techniques in image processing.[13,14] Related concepts are also discussed in more recent literature.[15–24]

Orthonormal bases of $L^2(R)$ consisting of discrete families of wavelets having infinite support were constructed by Meyer and others.[25–34] One method of constructing such bases is by applying Gram orthonormalization to the set of integer translates of a spline function to form a scaling function ϕ that retains scaling properties of spline functions but whose integer translates form an orthonormal set. The wavelet ψ is constructed from ϕ so that the discrete set formed by all normalized dyadic scalings of the integer translates of ψ form a complete orthonormal set. This construction from splines is a specific example of a more general construction based on a multiresolution analysis. Mallat[30,35–37] describes the construction of orthonormal wavelet bases from multiresolution analysis and develops several applications.

Recently Daubechies[38] constructed orthonormal bases of L^2 consisting of discrete families of compactly supported wavelets formed from normalized dyadic scalings of integer translates of a single wavelet. Unlike the construction based on a multiresolution analysis summarized above, Daubechies directly constructs compactly supported scaling functions using a recursive limiting process from a finite sequence $h \in l^2(Z)$ satisfying general constraints. The wavelet is constructed from the scaling function.

The general constraints are: h is a conjugate quadrature filter and h sums to 1. Conjugate quadrature filters were introduced by Smith and Barnwell[39] and have applications in digital signal processing, particularly in perfect reconstruction filter banks.[40,41] Daubechies derived an explicit formula that parametrizes all sequences satisfying the general constraint. Pollen has proved that these sequences may be parametrized by a subgroup of the group of paths in a unitary group.[42,43] Related parametrizations are discussed by Pollen,[44] Resnikoff,[45] and Wells.[46]

Daubechies noted that the construction above may yield wavelets that do not form orthonormal bases. To address this problem she constructed subsets of sequences satisfying special constraints and proved that scaling functions constructed from these sequences had arbitrarily high degree of regularity and that the set of their translates by integers formed an orthonormal set. These conditions imply the associated wavelets form an orthonormal basis of $L^2(R)$. These compactly supported wavelets have useful analytic and computational properties and they have been applied to diverse problems in image processing[47] and numerical analysis.[48–51]

Mallat[30] derived a condition on sequences that is sufficient to construct orthonormal wavelet bases. This condition, that the Fourier transform of h does not vanish on the interval $[-\pi/2, \pi/2]$, is less restrictive than Daubechies conditions. However, he noted that it was not known if the condition was also necessary. Cohen recently derived a condition on sequences that is both necessary and sufficient to construct orthonormal wavelet bases. This condition is expressed in terms of the set of real zeros of the Fourier transform of h.

The author[52] proved that wavelets constructed from any sequence h satisfying the general constraints form a tight frame for $L^2(R)$ and constructed a proper algebraic subset of sequences such that the tight frame of wavelets constructed from every sequence outside this subset is an orthonormal basis. This algebraic set is characterized in terms of the multiplicity of the eigenvalue 1 of an operator associated to the sequence.

This paper utilizes the results of Cohen to prove that a sequence yields wavelets that form an orthonormal basis if and only if it does not belong to the algebraic subset described above. The result is also generalized to wavelets having infinite support. It also discusses related open problems and conjectures.

II. STATEMENT OF RESULTS

A conjugate quadrature filter (CQF) is an absolutely summable complex-valued sequence h that satisfies

 0022-2488/91/010057-05$03.00

$$2\sum_n h(n)\,\overline{h(n+2m)}=\delta(m), \tag{1}$$

where $\delta(m)=1$ for $m=0$ else $\delta(m)=0$, or equivalently, whose Fourier transform defined by

$$m_0(\omega)=\sum_n h(n)\exp(-ni\omega)$$

satisfies

$$|m_0(\omega)|^2+|m_0(\omega+\pi)|^2=1, \tag{2}$$

for all real ω.

Since h is absolutely summable m_0 is continuous and Eq. (2) implies $|m_0(\omega)|\leqslant 1$ for all real ω. Furthermore, if

$$m_0(0)=1, \tag{3}$$

the function $m_1(\omega)=\exp(-i\omega)\,\overline{m_0(\omega+\pi)}$, which is the Fourier transform of the sequence $g(n)=(-1)^{1-n}\,\overline{h(1-n)}$, satisfies $m_1(0)=0$ and the pair f, g form a pair of low-pass, high-pass filters.

Let V denote the set of all absolutely summable complex sequences h satisfying Eqs. (1)–(3) and such that $m_0(\omega)$ is sufficiently regular in some neighborhood of 0 to ensure

$$\sum_{k=1}^{\infty}\left|m_0\left(\frac{\omega}{2^k}\right)-1\right|<\infty. \tag{4}$$

Daubechies proved in Ref. 38 that Eq. (4) is satisfied whenever there exists $\epsilon>0$ such that $\Sigma_n|h(n)||n|^\epsilon<\infty$. We do not know necessary and sufficient conditions on h to ensure Eq. (4).

For every $h\in V$ construct functions Φ and Ψ by

$$\Phi(\omega)=\prod_{k=1}^{\infty} m_0\left(\frac{\omega}{2^m}\right) \tag{5}$$

and

$$\Psi(\omega)=m_1(\omega/2)\Phi(\omega/2). \tag{6}$$

Since $\Phi(\omega)$ and $\Psi(\omega)$ are bounded measurable functions they are tempered distributions, therefore, they are the Fourier transforms of tempered distributions $\phi(x)$ and $\psi(x)$ that satisfy the following equations:

$$\phi(x)=2\sum_n h(n)\phi(2x-n), \tag{7}$$

$$\sum_n \phi(x-n)=1, \tag{8}$$

$$\int\phi(x)dx=1, \tag{9}$$

$$\psi(x)=2\sum_n g(n)\phi(2x-n), \tag{10}$$

$$\int\psi(x)dx=0. \tag{11}$$

For any $h\in V$, it is proved, using Fatou's lemma,[30,53] and using the weak compactness of the unit ball in L^2 in Ref. 52, that Φ and therefore Ψ, ϕ, and ψ are square integrable and that the $L^2(R)$ norms of ϕ and of ψ are equal and are $\leqslant 1$. The function ϕ is called a scaling function and the function ψ is called a wavelet. Furthermore, it is proved in these and other papers that the set $\{\psi_{j,k}(x)=2^{j/2}\psi(2^jx-k)\,|\,j,k\in Z\}$ formed from normalized dyadic scalings of integer translates of the wavelet is an orthonormal basis for $L^2(R)$ if and only if the set of integer translates of the scaling function $\{\phi(x-k)\,|\,k\in Z\}$ is orthonormal. By Poisson's summation formula, this condition is equivalent to

$$\sum_k|\Phi(x-2\pi k)|^2=1. \tag{12}$$

It follows from Eqs. (8) and (9) that this condition is also equivalent to the condition that the set $\{\phi(x-k)\,|\,k\in Z\}$ be orthogonal.

Not all $h\in V$ yield orthonormal bases. For instance the sequence $h(0)=h(3)=1/2$, $h(1)=h(2)=0$ yields the scaling function $\phi(x)=1/3$ on [0,3] and 0 elsewhere. The problem of determining necessary conditions and/or sufficient conditions on h that yield orthonormal wavelet bases, or equivalently, that yield scaling functions orthogonal to their translations by nonzero integers, has been the subject of several research articles.

Mallat in Ref. 30 proved, using the Lebesgue's dominated convergence theorem, that if $m_0(\omega)\neq 0$ for all ω in $[-\pi/2,\pi/2]$ then Eq. (12) is satisfied. A compact subset K of R is congruous to $[-\pi,\pi]$ if for almost all x in $[-\pi,\pi]$, there exists a unique y in K such that $x-y\in 2\pi Z$. Cohen in Refs. 54 and 55 extended this result to obtain a condition on h that is both necessary and sufficient to construct orthonormal wavelet bases. He proved the following theorem.

Theorem 1: Equation (12) is satisfied if and only if there exists a compact subset K of R congruous to $[-\pi,\pi]$ and containing 0 in it's interior such that $\Phi(\omega)\neq 0$ for all $\omega\in K$.

An elementary topological argument shows that this characterization is equivalent to the following: Eq. (12) is not satisfied if and only if there exists y in $[-\pi/2,\pi/2]$ such that $\Phi(y+2k\pi)=0$ for all $k\in Z$. Cohen's results in Ref. 54 were derived for the multidimensional case. Ausher's thesis[56] also discusses wavelet bases in one dimension with integer scaling factors greater than 2. Theorem 1 is also valid for these more general wavelet bases.

The approach developed by the author in Ref. 52 to characterizing sequences $h\in V$ that yield orthonormal wavelet bases utilizes an eigenvector relationship between a linear operator S on $l^2(Z)$ associated to h and a sequence $f\in l^2(Z)$ associated to the scaling function ϕ constructed from h. The linear operator S is defined by, for all $a\in l^2(Z)$,

$$S(a)(k)=2\sum_m\sum_n h(m)\,\overline{h(n)}\,a(2k+m-n). \tag{13}$$

Equation (1) means that the sequence $\delta(k)$ is an eigenvector of S having eigenvalue 1. The sequence f is defined by

$$f(k)=\int\phi(x)\,\overline{\phi(x-k)}dx. \tag{14}$$

Since $\phi\in L^2(R)$, f is an absolutely summable complex-valued sequence. Furthermore, Eq. (7) implies that the sequence f is also an eigenvector of S having eigenvalue 1. Let W denote the subset of all sequences h in V such that the operator S has an eigenvector having eigenvalue 1 that is not a multiple of $\delta(k)$, that is, for which the multiplicity of the eigenvalue 1 is more than 1. For every integer $N\geqslant 1$ let V_N denote the set of all sequences in V such that $h(n)=0$ for $n\leqslant 0$ and for $n\geqslant 2N-1$ and define $W_N=W\cap V_N$. The Paley–Weiner theorem implies $h \in V_N$ if and only if ϕ has compact support $[0,2N-1]$. In this case, ψ has compact support $[1-N,N]$. The main result in Ref. 52 is shown by the following theorem.

Theorem 2: Let $N \geqslant 1$, let $h \in V_N$, let ϕ and ψ be the scaling function and wavelet constructed from h, and let S be the operator constructed from h and let f be the sequence constructed from ϕ as above. Then the set of normalized dyadic scalings of integer translates of ψ forms a tight frame of $L^2(R)$, that is, for all $g \in L^2(R)$

$$g = \sum_{j,k} \langle \psi_{j,k} | g \rangle \psi_{j,k}. \tag{15}$$

Furthermore, for $N \geqslant 2 W_N$ is a nonempty proper algebraic subset of V_N and the tight frame forms an orthonormal basis unless $h \in W_N$.

The technique in Ref. 52 easily allows this result to be extended to noncompactly supported wavelets, with the exception that W is not an algebraic subset of V.

In the next section, we prove the following extension of this result.

Theorem 3: Let $h \in V$. Then the tight frame of wavelets constructed from h is not an orthonormal basis if and only if $h \in W$. We also prove the following theorem.

Theorem 4: The mapping from sequences to wavelets is a continuous mapping from the subset $V - W$ of $l^2(Z)$ into $L^2(R)$.

III. DERIVATIONS

The proofs of Theorems 3 and 4 require several intermediate lemmas.

Lemma 1: Let $h \in V$, let S be the operator associated to h, and let $f \in l^1(Z)$ and let F be the Fourier transform of f. Then the Fourier transform of $S(f)$ is given by

$$\left| H\left(\frac{\omega}{2}\right) \right|^2 F\left(\frac{\omega}{2}\right) + \left| H\left(\frac{\omega}{2} + \pi\right) \right|^2 F\left(\frac{\omega}{2} + \pi\right). \tag{16}$$

Proof: Follows from calculating the Fourier transform of Eq. (13).

Lemma 2: Let $h \in W$ and let S be the operator associated to h, then there exists a sequence $f \in l^1(Z)$ whose Fourier transform F is strictly positive, this means f is positive definite, such that f is not a multiple of $\delta(k)$ and such that $S(f) = f$.

Proof: If f satisfies $S(f) = f$ then Lemma 1 shows that both the real and the imaginary parts of F also satisfy Eq. (16). Therefore, if f is not a multiple of $\delta(k)$ such that $S(f) = f$, then it can be assumed without loss of generality that F is real valued. Then adding a sufficiently large positive multiple of $\delta(k)$ yields a positive definite eigenvector of S having eigenvalue 1.

Lemma 3: Let $h \in V$, let S be the operator associated to h, let ϕ be the scaling function constructed from h, and let $f \in l^1(Z)$ be a positive definite function satisfying $S(f) = f$. Let H, F, and Φ denote the Fourier transforms of h, f, and ϕ. Then the function G defined by

$$G(\omega) = H(\omega)\sqrt{F(\omega)/F(2\omega)} \tag{17}$$

is the Fourier transform of a sequence $g \in V$. Furthermore, the Fourier transform Λ of the scaling function λ constructed from g satisfies

$$\Lambda(\omega) = \Phi(\omega)/\sqrt{F(\sigma)}. \tag{18}$$

Proof: The fact g is a CQF follows from Eq. (2), Lemma 2, and the fact F has period 2π. Clearly $G(0) = 1$ therefore g sums to 1. The regularity of G at 0 and Eq. (18) are established by computing the infinite product for Λ since all but one of the terms involving F cancel.

Lemma 4: In Lemma 3, Λ satisfies the hypothesis of Theorem 1 if and only if Φ satisfies the hypothesis of Theorem 1.

Proof: Follows since the zero sets of G and H and therefore the zero sets of Λ and Φ coincide.

Proof of Theorem 3: It suffices, from Theorem 2 and the remark following it, to prove that if $h \in W$, the wavelets constructed from h do not form an orthonormal basis. Assume to the contrary that these wavelets do form an orthonormal basis. We will derive a contradiction. By Lemma 2 there exists a sequence $f \in l^1(Z)$ that is a positive definite function, satisfies $S(f) = f$, and is not a multiple of $\delta(k)$. Let Φ denote the Fourier transform of ϕ and let Λ denote the Fourier transform of the scaling function λ constructed as in Lemma 3. It follows from Theorem 1 and Lemma 4 that both ϕ and λ yield orthonormal wavelet bases. Therefore, it follows from Eq. (12), Eq. (18) of Lemma 3, and the fact F has period 2π that $\sqrt{F}$ and therefore F is constant. This contradicts the assumption that f is not a multiple of $\delta(k)$.

Lemma 5: Let ϕ_n be a sequence in $L^2(R)$ that converges as distributions to a function ϕ and such that for all n, $\|\phi_n\| = 1$ where $\|\cdot\|$ denotes the L^2 norm. Then $\|\phi_n - \phi\|$ converges to 0 if and only if $\|\phi\| = 1$.

Proof: The only if part follows from the triangle inequality. Let $\langle .,. \rangle$ denote the inner product in $L^2(R)$. Assume $\|\phi\| = 1$. Since Schwarz functions are dense in $L^2(R)$, it follows that $\langle \phi_n, \phi \rangle$ approaches $\|\phi\|^2 = 1$. Then $\|\phi_n - \phi\|^2 = 2 - \langle \phi_n, \phi \rangle - \langle \phi, \phi_n \rangle$ which converges to 0. This completes the if part of the proof.

Lemma 6: Let $h \in V$ and define a sequence ϕ_n as follows: $\phi_0(x) = 1$ for $x \in [0,1)$ else $\phi_0(x) = 0$, and

$$\phi_{n+1}(x) = \sum_k h(k) \phi_n(2x - k)$$

scaling function ϕ constructed from h if and only if $\|\phi\| = 1$.

Proof: A Fourier transform argument shows that this sequence converges weakly to the scaling function ϕ constructed from h. Furthermore, 1 implies $\|\phi_n\| = 1$ for all n. The result follows from Lemma 5.

Lemma 7: Let $h \in V$ and let ϕ be the scaling function constructed from h. Then $h \in V - W$ if and only if $\|\phi\| = 1$.

Proof: If $h \in V - W$ then ϕ is orthogonal to its integer translates and Eqs. (8) and 9 imply $\|\phi\| = 1$. If $\|\phi\| = 1$ then Lemma 6 implies the sequence ϕ_n converges strongly to ϕ. Since for all n, ϕ_n is orthogonal to its integer translates, it follows that ϕ must be orthogonal to its integer translates and the result follows.

Proof of Theorem 4: Clearly, the scaling function ϕ constructed from $h \in V$ varies continuously in a weak sense. By Lemma 7, $\|\phi\| = 1$ for $h \in V - W$. The result follows from the argument in the proof of Lemma 5 that shows weak convergence implies strong convergence on the surface of the unit ball in $L^2(R)$.

IV. OPEN PROBLEMS

Problem 1: Clearly, for any $h(k) \in V$ and for any odd integer p, the sequence $h(pk) \in W$. The latter sequence corresponds to stretching $h(k)$ by p since the corresponding scaling functions (and wavelets) are related by stretching by a factor p along the x axis and multiplying by the factor $1/p$. A large class of nonorthonormal tight frames of wavelets can be constructed in this way. The construction generalizes to scaling functions for which the factor 2 is replaced by an integer $m > 2$ and the wavelet bases consists of dilated (by powers of m) integer translates of $m - 1$ wavelet functions. In this case the stretching factor p must be relatively prime to m.

For any $d > 0$, it is possible, using a method based on spline transition bands,[57] to construct infinite sequences $h(k) \in W$ that converge to zero as $O(k^{-d})$ and for which $h(k)$ is not a stretched version of another sequence.

Pollen has conjectured[58] that if the sequence $h(k) \in W_N$ then $h(k)$ must be a shifted version of a stretched version of a sequence in V. He has proven this conjecture[58] in one dimension for all scale factors $m \geqslant 2$ under the assumption that all nonzero terms in the sequence are equal. The scaling functions corresponding to such sequences are equal to a constant times the characteristic function of a set called a scaling tile. The method of proof however does not appear to generalize to higher dimensions nor to more general scaling functions.

Problem 2: What class of operators can be expressed in the form $\Sigma_{j,k} c_{j,k} |\psi_{j,k}\rangle \langle \psi_{j,k}|$ if the wavelets form a nonorthonormal tight frame?

Problem 3: The infinite product formula for $\Phi(\omega)$ together with the fact that the mapping $m(x) = 2x \bmod Z$ is ergodic implies that the *typical* rate of decay of $|\Phi(\omega)|$ as ω increases is $|\omega|^{\ln(H_g)/\ln(2)}$ where H_g denotes the geometric mean of $|H|$. Can this relation to be used to describe the regularity of ϕ? The multiplicative ergodic theorem of Oseledec,[59,60] may be applicable in this context.

ACKNOWLEDGMENTS

The author thanks Ingrid Daubechies for explaining the results of Albert Cohen and my colleagues at AWARE, Inc. and Rice University for helpful discussions. Theorems 3 and 4 were motivated by the conjecture of David Pollen and by insightful and persistent questions by Professor Sidney Burrus and Ramesh Gopinath concerning the relationships between Fourier analysis and the construction of scaling functions and wavelets.

This research was supported in part by the Advanced Research Projects Agency of the Department of Defense and was monitored by the Air Force Office of Scientific Research under Contract No. F49620-89-C-0125. The United States Government is authorized to reproduce and distribute reprints for governmental purposes notwithstanding any copyright notation hereon.

[1] A. Grossman and J. Morlet, SIAM J. Math. Anal. **15**, 723 (1984).
[2] A. Haar, Ph.D. dissertation; Math. Ann. **69**, 336 (1989).
[3] J. O. Stromberg, "A modified franklin system and higher order spline systems on r^n as unconditional bases for hardy spaces," Conference in Honor of Antoni Zygmund—Wadsworth Math Series, vol. 2, pp. 475–493.
[4] R. P. Fedorenko, USSR Comput. Math. Phys. **1**, 1092 (1990).
[5] R. P. Fedorenko, USSR Comput. Math. Phys. **4**, 227 (1964).
[6] A. Brandt, Math. Comput. **31**, 333 (1977).
[7] A. P. Calderón, Studia Math. **24**, 113 (1964).
[8] E. W. Aslaksen and J. R. Klauder, J. Math. Phys. **9**, 206 (1968).
[9] E. W. Aslaksen and J. R. Klauder, J. Math. Phys. **10**, 2267 (1969).
[10] K. G. Wilson and J. Kogut, Phys. Rep. C **12**, 75 (1974).
[11] R. E. Crochiere and L. R. Rabiner, *Multirate Digital Signal Processing* (Prentice-Hall, Englewood Cliffs, NJ, 1983).
[12] A. Witkin, "Scale space filtering," in *Proceedings of the International Joint Conference on Artificial Intelligence*, Karsruhe, West Germany, 1983 (Morgan-Kaufmann, Palo Alto, CA, 1983).
[13] P. Burt and E. Adelson, *IEEE* Commun. Commun. **31**, 482 (1983).
[14] P. Burt and E. Adelson, ACM Trans. Graph. **2**, 217 (1983).
[15] W. L. Briggs, *A Multigrid Tutorial* (SIAM, Philadelphia, PA, 1987).
[16] P. Goupillaud, A. Grossman, and J. Morlet. SIAM J. Math. Anal. **15**, 723 (1984).
[17] A. Grossman, J. Morlet, and T. Paul. J. Math. Phys. **26**, 2473 (1985).
[18] A. Grossman, J. Morlet, and T. Paul. Ann Inst. H. Poincaré **45**, 293 (1986).
[19] I. Daubechies, J. Math Phys. **27**, 1271 (1986).
[20] I. Daubechies, *IEEE* Trans. Inf. Theory, **36**, 961 (1990).
[21] T. Paul, J. Math. Phys. **25**, 3252 (1985).
[22] T. Paul, "Affine coherent states and the radial schrodinger equation. I. Radial harminic oscillator and hydrogen atom," preprint (1990).
[23] P. Tchamitchian, C. R. Acad. Sci. Paris **303**, 215 (1986).
[24] F. Argoul, A. Arneodo, J. Elezgaray, and G. Grasseau. Phys. Lett. A **135**, 327 (1989).
[25] Y. Meyer, Séminaire Bourbaki, p. 662, 1985–1986.
[26] Y. Meyer, Cahiers Mathematiques de la Decision, No. 8703, Ceremade, Universite de Paris Dauphine, 1987.
[27] Y. Meyer, *Ondelletes et Operateurs* (Hermann, Paris, 1988).
[28] P. G. Lemarié and Y. Meyer. Rev. Mat. Iberoamericana **2**, 1 (1986).
[29] P. G. Lemarié, J. Math. Pures Appl. **67**, 227 (1988).
[30] S. Mallat, Trans. Am. Math. Soc. XX, XXX (1989).
[31] S. Jaffard, C. R. Acad. Sci. Paris **308**, 79 (1989).
[32] S. Jaffard and Y. Meyer, J. Math. Pures Appl. **68**, 95 (1989).
[33] G. Battle, Commun. Math. Phys. **110**, 601 (1987).
[34] G. Battle and P. Federbrush, Commun. Math. Phys. **109**, 417 (1987).
[35] S. Mallat, *IEEE* Trans. Pattern Recogn. Machine Int. **11** (7), 674 (1989).
[36] S. Mallat, *IEEE* Trans. Acoust. Speech Signal Proc. **ASSP-37**, 2091 (1989).
[37] S. Mallat and S. Zhong, New York University, Courant Institute, Tech. Rep. No. 483, Robotics Rep. No. 219, December 1989.
[38] I. Daubechies, Commun. Pure Appl. Math. **41**, 909 (1985).
[39] M. J. Smith and T. P. Barnwell, *IEEE* Trans. Acoust. Speech Signal Proc. **ASSP-34**, 434 (1986).
[40] P. P. Vaidyanathan, T. Q. Nguyen, Z. Doganata, and T. Saramaki, *IEEE* Trans. Acoust. Speech Signal Proc. **ASSP-37**, 1042 (1989).
[41] M. Vetterli and D. Le Gall, *IEEE* Trans. Acoust. Speech Signal Proc. **ASSP-37**, 1057 (1989).
[42] D. Pollen, J. Am. Math. Soc. **3**, 611 (1990).
[43] H. L. Resnikoff, "Foundations of arithmeticum analysis: Compactly supported wavelets and the wavelet group," AWARE Inc. Tech. Rep. AD890507 (1989).
[44] D. Pollen, "Daubechies' scaling function on [0,3]," to appear in J. Am. Math. Soc.
[45] H. L. Resnikoff, "Weierstrass functions and compactly supported wavelets," AWARE Inc. Tech. Rep. AD900810 (1989).
[46] Jr. R. O. Wells, "Parameterizing smooth compactly supported wavelets," to appear in Trans. Am. Math. Soc.
[47] W. Zettler, J. Huffman, and D. Linden. "Application of compacty supported wavelets to image compression," Proceedings of 1990 SPIE/SPSE Symposium on Electronic Imaging, Santa Clara, CA, March 1990.
[48] R. Glowinski, W. Lawton, M. Ravachol, and E. Tennenbaum, "Wavelet solution of linear and nonlinear elliptic, parabolic, and hyperbolic problems in one space variable," AWARE, Inc. Report, May 27, 1989, *Proceedings of the Ninth International Conference on Computing Methods in*

Applied Sciences and Engineering, sponsored by INRIA, (SIAM, Philadelphia, PA, 1990).
[49] W. M. Lawton, "Wavelet discretization methods for surface estimation and reconstruction," Procedings of 1990 SPIE/SPSE Symposium on Electronic Imaging, Santa Clara, CA., 12–14 February 1990, SPIE-1251 (SPIE, Bellingham, Washington, 1990).
[50] G. Beylkin, R. Coifman, and V. Rokhlin, Res. Rep. YALEU/DCS/RR-696, August, revised December 1989.
[51] A. Latto and E. Tenenbaum, "The wavelet-galerkin method and burgers' equation," to appear in Compte Rendus (1990).
[52] W. M. Lawton, J. Math. Phys. **31**, 1898 (1990).
[53] A. Cohen, to appear in Revista Mathematica Ibero Americano.
[54] A. Cohen, Ann. Inst. H. Poincaré (1990).
[55] A. Cohen, "Wavelets, quadrature mirror filters and digital picture processing," Proceedings of the Seventh Army Annual Conference on Applied Mathematics and Computing, West Point, 1990.
[56] P. Auscher, "Ondelettes," Ph.D. thesis, Université de Paris-Dauphine (1988).
[57] C. S. Burrus, A. W. Soweto, and R. A. Gopinath, to appear in *IEEE* Trans. Acoust. Signal Speech Proc.
[58] D. Pollen, "Scaling tiles," AWARE Inc. Tech. Rep. AD890715.1.1 (1989).
[59] V. I. Oseledec, Trans. Moscow Math. Soc. **19**, 197 (1968).
[60] U. Krengel, *Ergodic Theorems* (de Gruyter, New York, 1985).