Available online at www.sciencedirect.com
ScienceDirect
Journal of Approximation Theory 163 (2011) 413–437
Journal of Approximation Theory
www.elsevier.com/locate/jat

Full length article

Polynomial reproduction for univariate subdivision schemes of any arity

Costanza Conti[a], Kai Hormann[b,*]

[a] Dipartimento di Energetica, Università di Firenze Via C. Lombroso 6/17, 50134 Firenze, Italy
[b] Faculty of Informatics, Università della Svizzera italiana, Via Giuseppe Buffi 13, 6904 Lugano, Switzerland

Received 11 June 2010; received in revised form 13 October 2010; accepted 11 November 2010
Available online 23 November 2010

Communicated by Zuowei Shen

Abstract

In this paper, we study the ability of convergent subdivision schemes to reproduce polynomials in the sense that for initial data, which is sampled from some polynomial function, the scheme yields the same polynomial in the limit. This property is desirable because the reproduction of polynomials up to some degree d implies that a scheme has approximation order $d + 1$. We first show that any convergent, linear, uniform, and stationary subdivision scheme reproduces linear functions with respect to an appropriately chosen parameterization. We then present a simple algebraic condition for polynomial reproduction of higher order. All results are given for subdivision schemes of any arity $m \geq 2$ and we use them to derive a unified definition of general m-ary pseudo-splines. Our framework also covers non-symmetric schemes and we give an example where the smoothness of the limit functions can be increased by giving up symmetry.

Keywords: Subdivision schemes; Polynomial reproduction; Approximation order

1. Introduction

A univariate *subdivision scheme* $S_{\mathbf{a}}$ with *arity* $m \geq 2$ is based on repeated application of the refinement rule

$$f_i^{\ell+1} = \sum_{j\in\mathbb{Z}} a_{i-mj} f_j^{\ell}, \quad i \in \mathbb{Z} \tag{1}$$

* Corresponding author.
E-mail addresses: costanza.conti@unifi.it (C. Conti), kai.hormann@usi.ch (K. Hormann).

0021-9045/$ - see front matter
doi:10.1016/j.jat.2010.11.002

to generate the refined data $\boldsymbol{f}^\ell = \{f_i^\ell : i \in \mathbb{Z}\}$ for $\ell \geq 1$ from some initial data $\boldsymbol{f} = \boldsymbol{f}^0 = \{f_i^0 : i \in \mathbb{Z}\}$. The coefficients $\boldsymbol{a} = \{a_i : i \in \mathbb{Z}\}$ in (1) constitute the so-called *subdivision mask*, a compactly supported sequence of real numbers. By attaching the data f_i^ℓ to the parameter values t_i^ℓ with $t_{i+1}^\ell - t_i^\ell = m^{-\ell}$ for $i \in \mathbb{Z}$, $\ell \in \mathbb{N}$ one can establish a notion of convergence to a continuous *limit function* $g_{\boldsymbol{f}}$ by requiring that the piecewise linear functions F^ℓ which interpolate the data at level ℓ,

$$F^\ell(t_i^\ell) = f_i^\ell, \quad F^\ell|_{[t_i^\ell, t_{i+1}^\ell]} \in \pi_1, \; i \in \mathbb{Z}, \; \ell \in \mathbb{N},$$

where π_d denotes the space of polynomials of degree d, converge in the uniform norm with

$$g_{\boldsymbol{f}} = \lim_{\ell \to \infty} F^\ell. \tag{2}$$

It is clear that this limit always exists as long as the subdivision scheme applied to the initial data $\boldsymbol{\delta} = \{\delta_{i,0} : i \in \mathbb{Z}\} = \{\ldots, 0, 0, 1, 0, 0, \ldots\}$ converges in the sense of (2) to the so-called *basic limit function* $\phi_{\boldsymbol{a}} = g_{\boldsymbol{\delta}}$, because by the linearity of the refinement rule, we then have

$$g_{\boldsymbol{f}} = \sum_{j \in \mathbb{Z}} \phi_{\boldsymbol{a}}(\cdot - j) f_j^0$$

for any initial data sequence $\boldsymbol{f}$. For more background on subdivision, we refer to the seminal work of Cavaretta et al. [1] and the survey by Dyn and Levin [10].

In this paper, we only consider subdivision schemes that are convergent and *non-singular*, so that $g_{\boldsymbol{f}} = 0$ if and only if $\boldsymbol{f} = 0$. Under these assumptions we are interested in schemes that reproduce polynomials in the following sense.

Definition 1.1. A subdivision scheme $S_{\boldsymbol{a}}$ *reproduces* polynomials of degree d if it is convergent and if $g_{\boldsymbol{f}} = p$ for any polynomial $p \in \pi_d$ and initial data $f_i^0 = p(t_i^0)$, $i \in \mathbb{Z}$.

An important property of subdivision schemes is their approximation order. A subdivision scheme $S_{\boldsymbol{a}}$ is said to have approximation order d if the limit function generated from initial data, created by sampling some function $f \in C^d$ with $\|f^{(d)}\|_\infty < \infty$ uniformly with distance h, approximates f with an error of the order $O(h^d)$. Using the Taylor expansion of f and the fact that the basic limit function $\phi_{\boldsymbol{a}}$ is finitely supported, it is straightforward to show [21] that $S_{\boldsymbol{a}}$ has approximation order $d+1$ if it reproduces polynomials of degree d. A similar connection between the degree of polynomial generation and the approximation order holds only for interpolatory schemes (see Section 5.1), but not in general. For example, the approximation order of the subdivision scheme for B-splines of degree $d > 0$ is only 2, although it generates polynomials of degree d (see Section 5.3).

Despite the importance of this property, remarkably few results for systematically deriving the degree of polynomial reproduction can be found in the literature. Most papers either conclude it directly from the scheme's construction [3,4,8,19] or show it by explicitly verifying that the refinement rule (1) maps data from some polynomial of low degree to refined data from the same polynomial [11,13,15,24]. Hormann and Sabin [16] were the first, to the best of our knowledge, to derive the degree of polynomial reproduction for a family of schemes using simple algebraic considerations and the method was later generalized by Dyn et al. [9] for analysing arbitrary primal and dual binary schemes. The main contribution of this paper is to further extend their results and to derive a unified condition for polynomial reproduction that covers symmetric and non-symmetric schemes and naturally applies to m-ary subdivision schemes as well (see Section 4). Following the previous work [9,16], we use only algebraic considerations in our proof, although this condition can certainly be shown with Fourier techniques as well. As an application, we describe the construction of general m-ary pseudo-splines (see Section 5.4), which generalize the families of primal and dual binary pseudo-splines in [5,9], respectively.

Besides convergence and approximation order, two other important properties of a subdivision scheme are the support size and the smoothness of its basic limit function. Both are mutually conflicting because a higher degree of smoothness generally requires a larger support, thus leading to a more global influence of each initial data value on the limit function. Raising the arity of the subdivision scheme provides a way to overcome this dilemma to some extent. For example, the ternary and quaternary 4-point schemes discussed in [19,24], respectively, have smaller support and higher smoothness than the classical binary 4-point scheme [6], and all three schemes reproduce cubic polynomials by construction.

A simple formula for computing the support size of an m-ary subdivision scheme $S_{\boldsymbol{a}}$ can be derived as follows; compare [17]. Suppose that the mask $\boldsymbol{a}$ is supported on $[0, N]$, that is, $a_i = 0$ for $i < 0$ and $i > N$, and that the data $\boldsymbol{f}^\ell$ at level ℓ is supported on $[0, M]$. It then follows from (1) that the refined data $\boldsymbol{f}^{\ell+1}$ is supported on $[0, N + mM]$. For the initial data $\boldsymbol{\delta}$ we thus conclude by induction over ℓ that the refined data $\boldsymbol{\delta}^\ell$ is supported on $\left[0, \frac{1-m^\ell}{1-m} N\right]$, and so the support of the corresponding piecewise linear interpolating function is $\operatorname{supp}(\Delta^\ell) = \frac{1}{m^\ell}\left[-1, \frac{1-m^\ell}{1-m} N + 1\right]$. Therefore,

$$\operatorname{supp}(\phi_{\boldsymbol{a}}) = \lim_{\ell\to\infty} \operatorname{supp}(\Delta^\ell) = \left[0, \frac{N}{m-1}\right]. \tag{3}$$

Despite the advantages of schemes with arbitrary arity regarding the tradeoff between small support size and smoothness, most of the recent work in this direction [11,14,15,19,22–24,28,29] did not go beyond the investigation of quaternary schemes, because the number of mask coefficients that need to be stored and used in each subdivision step increases linearly with the arity for a fixed support size. Nevertheless, we believe that having a unified condition for polynomial reproduction of subdivision schemes with any arity $m \geq 2$ is elegant from a theoretical point of view as well as useful for the design of new schemes. Indeed, we present a novel quaternary subdivision scheme in Section 5.6 which is better than the corresponding binary scheme with the same approximation order in many aspects: it has a smaller support size, higher smoothness, and is computationally more efficient.

Another, less explored approach to increase the smoothness of the limit functions is to give up symmetry and consider subdivision schemes with non-symmetric masks and our condition for polynomial reproduction can help finding them. As an example, we derive a non-symmetric binary 3-point scheme with approximation order 3 which has C^2 limit functions, while the limit functions of its symmetric sibling are only C^1 (see Section 5.5).

1.1. Algebraic tools

Many properties of stationary subdivision schemes can be read off the subdivision mask, or equivalently, can be deduced from algebraic properties of its *symbol*

$$a(z) = \sum_{i\in\mathbb{Z}} a_i z^i, \quad z \in \mathbb{C} \setminus \{0\},$$

the Laurent polynomial associated with the mask $\boldsymbol{a}$. For example, a well-known necessary condition for an m-ary subdivision scheme $S_{\boldsymbol{a}}$ to be convergent is that the symbol $a(z)$ satisfies

$$a(1) = m \quad \text{and} \quad a(\zeta_m^j) = 0, \quad j = 1, \ldots, m-1, \tag{4}$$

where $\zeta_m^j = \exp\left(\frac{2\pi \mathrm{i}}{m} j\right)$ are the mth roots of unity. For binary schemes ($m = 2$) this was proven by Cavaretta et al. [1] and Dyn [7], and a proof for general arity can be found in [12] and [18], for example. An alternative form of condition (4) is

$$\sum_{i \in \mathbb{Z}} a_{mi+l} = 1, \quad l = 0, \ldots, m-1, \tag{5}$$

and under this condition it follows directly from (1) that constant functions are reproduced.

Another property that can be derived easily from the symbol is that a convergent subdivision scheme *generates* polynomials up to degree d (that is, π_d is contained in the space of all limit functions), if and only if

$$a^{(k)}(\zeta_m^j) = 0, \quad j = 1, \ldots, m-1, \; k = 0, \ldots, d. \tag{6}$$

For binary schemes, this result was first shown by Cavaretta et al. [1] and for arbitrary $m \geq 2$ it can be deduced from [21] as well as from the Strang–Fix conditions [27] by following the explanation in [26]. Clearly, condition (6) is equivalent to requiring that the symbol $a(z)$ is of the form

$$a(z) = (1 + z + \cdots + z^{m-1})^{d+1} b(z) = \left(\frac{1 - z^m}{1 - z}\right)^{d+1} b(z) \tag{7}$$

for some Laurent polynomial $b(z)$ with $b(1) = 1/m^d$. Note that both conditions are only sufficient for polynomial generation if we drop the assumption of non-singularity. For example, the binary scheme with symbol $a(z) = (3 + z^2)(1 + z^{-2})(1 + z)/8$ is convergent and generates linear functions even though it does not satisfy (7) for $d = 1$, but the scheme is singular because it yields $g_f = 0$ as limit function for any initial data $\boldsymbol{f} = \{\ldots, \alpha, -\alpha, \alpha, -\alpha, \ldots\}$, $\alpha \in \mathbb{R}$.

Summarizing the above, polynomial generation is guaranteed by the "correct" behaviour of the symbol $a(z)$ and its derivatives at all mth roots of unity ζ_m^j except $\zeta_m^0 = 1$, and if $a(z)$ behaves "correctly" at this last root of unity $z = 1$ in addition, then the scheme reproduces polynomials of degree zero. This observation led us to the idea that polynomial reproduction of higher degree might be connected to the behaviour of the derivatives of $a(z)$ at $z = 1$, and the main purpose of this paper is to report that this is indeed so.

We first noticed (see Section 2) that the conditions for polynomial generation themselves already have a strong impact on the values $a^{(k)}(1)$ and then discovered (see Section 4) that the remaining condition for polynomial reproduction of degree d is

$$a^{(k)}(1) = m \prod_{l=0}^{k-1} (\tau - l), \quad k = 0, \ldots, d, \tag{8}$$

where τ is related to the parameterization of the subdivision scheme (see Section 3).

In a nutshell, any convergent subdivision scheme reproduces constant functions. If it further generates linear functions, then it also reproduces them with respect to the appropriate parameterization, which determines τ. And if the scheme generates polynomials of degree $d > 1$ and its symbol further satisfies (8), then it also reproduces polynomials of degree d and thus has approximation order $d + 1$.

An important aspect of conditions (6) and (8) is that they are given in terms of specific values of the symbol's derivatives at certain points, which is likely to be generalized to the multivariate setting.

2. Subsymbols and their derivatives

We denote the *subsymbols* of a subdivision symbol $a(z)$ by

$$a_l(z) = \sum_{i\in\mathbb{Z}} a_{mi+l} z^{mi+l}, \quad l = 0, \ldots, m-1,\ z \in \mathbb{C} \setminus \{0\}, \tag{9}$$

and remark that the kth derivative of a subsymbol is

$$a_l^{(k)}(z) = \sum_{i\in\mathbb{Z}} q_{k,l}(i) a_{mi+l} z^{mi+l-k},$$

where $q_{k,l} \in \pi_k$ are the polynomials

$$q_{k,l}(x) = \prod_{n=0}^{k-1} (mx + l - n). \tag{10}$$

We can now establish a remarkable equivalence between the conditions for polynomial generation (6) and the behaviour of the derivatives of the symbol and its subsymbols at $z = 1$.

Lemma 2.1. *The kth derivative of a subdivision symbol $a(z)$ satisfies*

$$a^{(k)}(\zeta_m^j) = 0, \quad j = 1, \ldots, m-1,$$

if and only if the kth derivatives of all its subsymbols evaluate to the same value at $z = 1$, namely

$$a_l^{(k)}(1) = a^{(k)}(1)/m, \quad l = 0, \ldots, m-1.$$

Proof. Since the subsymbols are related to the symbol by

$$a(z) = \sum_{l=0}^{m-1} a_l(z), \tag{11}$$

we have for any $j = 0, \ldots, m-1$,

$$a^{(k)}(\zeta_m^j) = \sum_{l=0}^{m-1} a_l^{(k)}(\zeta_m^j) = \sum_{l=0}^{m-1} (\zeta_m^j)^{l-k} \sum_{i\in\mathbb{Z}} q_{k,l}(i) a_{mi+l} (\zeta_m^j)^{mi} = \sum_{l=0}^{m-1} (\zeta_m^j)^{l-k} a_l^{(k)}(1),$$

because $(\zeta_m^j)^{mi} = 1$ for all $i \in \mathbb{Z}$. This can be rewritten as the linear system

$$\begin{pmatrix} a^{(k)}(1) \\ a^{(k)}(\zeta_m^1) \\ \vdots \\ a^{(k)}(\zeta_m^{m-1}) \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \zeta_m^1 & (\zeta_m^1)^2 & \cdots & (\zeta_m^1)^{m-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \zeta_m^{m-1} & (\zeta_m^{m-1})^2 & \cdots & (\zeta_m^{m-1})^{m-1} \end{pmatrix} \begin{pmatrix} a_n^{(k)}(1) \\ \vdots \\ a_{m-1}^{(k)}(1) \\ a_0^{(k)}(1) \\ \vdots \\ a_{n-1}^{(k)}(1) \end{pmatrix}, \tag{12}$$

where $n \in \{0, \ldots, m-1\}$ such that $k \equiv n (\text{mod } m)$. Note that the system matrix V in (12) is the non-singular Vandermonde matrix associated with the distinct values ζ_m^j, $j = 0, \ldots, m-1$. Moreover, since the mth roots of unity clearly satisfy

$$\sum_{l=0}^{m-1} (\zeta_m^j)^l = \begin{cases} m, & \text{for } j = 0, \\ 0, & \text{for } j = 1, \ldots, m-1, \end{cases}$$

we have

$$\begin{pmatrix} a^{(k)}(1) \\ 0 \\ \vdots \\ 0 \end{pmatrix} = V \begin{pmatrix} a^{(k)}(1)/m \\ \vdots \\ a^{(k)}(1)/m \end{pmatrix}. \tag{13}$$

As V is non-singular, it is then clear that the vectors on the left-hand sides of (12) and (13) are identical if and only if the vectors on the right-hand sides are. $\square$

Note that the equivalence of conditions (4) and (5) follows from Lemma 2.1 by considering the special case $k = 0$.

3. Parameterization

Let us now take a look at the simplest case of polynomial reproduction, namely the reproduction of constant functions. Given some polynomial $p \in \pi_0$, $p(x) = \alpha$, with $\alpha \in \mathbb{R}$, we define the initial data by sampling p at the parameter values t_i^0, that is, $f_i^0 = p(t_i^0) = \alpha$ for $i \in \mathbb{Z}$. Now if $S_{\boldsymbol{a}}$ is a convergent subdivision scheme, its mask $\boldsymbol{a}$ satisfies condition (5) and according to the refinement rule (1) we then have $f_i^\ell = \alpha$ for all $i \in \mathbb{Z}$, $\ell \in \mathbb{N}$, and so the limit function is $g_{\boldsymbol{f}}(x) = \alpha$. By Definition 1.1, the scheme hence reproduces polynomials of degree 0.

We conclude that all convergent subdivision schemes reproduce constants, so let us raise the bar and consider the reproduction of linear polynomials. Again, we start by sampling some polynomial $p \in \pi_1$, so that $f_i^0 = p(t_i^0)$ for $i \in \mathbb{Z}$, and now the question is, under which conditions does a subdivision scheme $S_{\boldsymbol{a}}$ generate $g_{\boldsymbol{f}} = p$ as the limit function for this initial data $\boldsymbol{f}^0$. In addition to being convergent, the scheme should certainly generate linear polynomials, because we want π_1 to be among all possible limit functions. It then turns out that these two necessary conditions are also sufficient for linear reproduction, but only with respect to the appropriate parameterization.

So far, we did not assume anything special about the parameter values t_i^ℓ, except that they are uniformly spaced with distance $m^{-\ell}$, that is, $t_i^\ell = t_0^\ell + i/m^\ell$. Hence, all parameter values are uniquely determined by the value t_0^0 and the *relative shifts* $\tau_\ell = (t_0^\ell - t_0^{\ell+1})m^{\ell+1}$ between the parameterizations at level ℓ and $\ell + 1$ for $\ell \in \mathbb{N}$. In order to simplify the analysis, the value t_0^0 and all relative shifts are often set to zero, resulting in the *standard parameterization* $t_i^\ell = i/m^\ell$, because most of the properties of a subdivision scheme (for example, convergence, smoothness, support, degree of polynomial generation, reproduction of constants) do not depend on these values. They are, however, crucial for polynomial reproduction of degree $d \geq 1$.

Theorem 3.1. *Let $S_{\boldsymbol{a}}$ be a convergent subdivision scheme that generates linear polynomials. Then $S_{\boldsymbol{a}}$ also reproduces linear polynomials if and only if the relative shifts between the parameterizations are $\tau_\ell = a'(1)/m$ for all $\ell \in \mathbb{N}$.*

Proof. According to Dyn et al. [9, Corollary 4.5], for convergent subdivision schemes, polynomial reproduction is equivalent to polynomial reproduction in each subdivision step, hence it suffices to show that $f_i^\ell = p(t_i^\ell), i \in \mathbb{Z}$ implies $f_i^{\ell+1} = p(t_i^{\ell+1}), i \in \mathbb{Z}$ for any $\ell \in \mathbb{N}$. Moreover, as any convergent subdivision scheme reproduces constants, we only need to consider the monomial $p(x) = x$. So let $\ell \in \mathbb{N}$ and $f_i^\ell = t_i^\ell, i \in \mathbb{Z}$. Then by (1) and Lemma 2.1 for $k = 1$, we have for any $l = 0, \ldots, m-1$ and $i \in \mathbb{Z}$,

$$\begin{aligned} f_{mi+l}^{\ell+1} &= \sum_{j\in\mathbb{Z}} a_{m(i-j)+l} f_j^\ell = \sum_{j\in\mathbb{Z}} a_{mj+l} f_{i-j}^\ell = \sum_{j\in\mathbb{Z}} a_{mj+l} \left(t_0^\ell + \frac{i-j}{m^\ell} \right) \\ &= \sum_{j\in\mathbb{Z}} a_{mj+l} \left(t_0^\ell + \frac{mi+l}{m^{\ell+1}} \right) - \sum_{j\in\mathbb{Z}} a_{mj+l} \frac{mj+l}{m^{\ell+1}} \\ &= a_l(1) \left(t_0^\ell + \frac{mi+l}{m^{\ell+1}} \right) - \frac{a_l'(1)}{m^{\ell+1}} \\ &= \left(t_0^\ell - \frac{a'(1)/m}{m^{\ell+1}} \right) + \frac{mi+l}{m^{\ell+1}}, \end{aligned}$$

which is equal to

$$t_{mi+l}^{\ell+1} = t_0^{\ell+1} + \frac{mi+l}{m^{\ell+1}} = \left(t_0^\ell - \frac{\tau_\ell}{m^{\ell+1}} \right) + \frac{mi+l}{m^{\ell+1}}$$

if and only if $\tau_\ell = a'(1)/m$. □

So the good news is that the reproduction of linear functions comes for free, as long as the appropriate parameterization is considered, and that the latter is stationary in the sense that it has constant relative shifts τ_ℓ at all levels $\ell \in \mathbb{N}$. Moreover, since linear reproduction is clearly necessary for polynomial reproduction of any higher degree, this motivates the following convention.

Definition 3.2. For any subdivision scheme $S_{\boldsymbol{a}}$ we denote by $\tau = a'(1)/m$ the corresponding *parametric shift* and attach the data f_i^ℓ for $i \in \mathbb{Z}, \ell \in \mathbb{N}$ to the parameter values

$$t_i^\ell = t_0^\ell + \frac{i}{m^\ell} \quad \text{with } t_0^\ell = t_0^{\ell-1} - \frac{\tau}{m^\ell}. \tag{14}$$

Note that Definition 3.2 leaves us with one degree of freedom, namely the value of t_0^0, and that the reproduction of linear functions does not depend on this choice. One common option is to set $t_0^0 = 0$, so that the initial data f_i^0 is attached to the integers $t_i^0 = i$. Another option is to attach the data $f_{m^\ell}^\ell$ to the integers in the limit. Since (14) implies

$$t_0^\ell = t_0^{\ell-1} - \frac{\tau}{m^\ell} = \cdots = t_0^0 - \tau \sum_{j=1}^{\ell} \frac{1}{m^j} = t_0^0 - \frac{\tau}{m-1} \left(1 - \frac{1}{m^\ell} \right),$$

so that

$$\lim_{\ell\to\infty} t_{m^\ell}^\ell = t_0^0 - \frac{\tau}{m-1} + i, \tag{15}$$

this second option requires to set $t_0^0 = \tau/(m-1)$.

Remark 3.3. In view of (15), the "correct" parameterization applies a shift of $\tau/(m-1)$ to the left in the limit. So, with respect to the standard parameterization $t_i^\ell = i/m^\ell$, a scheme with polynomial reproduction of degree d yields $g_f(x) = p(x + \tau/(m-1))$ as the limit function for initial data $f_i^0 = p(i), i \in \mathbb{Z}$ and any $p \in \pi_d$. Note that this does not change the leading coefficient.

4. Polynomial reproduction

Now that we have settled the issue of the correct parameterization, we are ready to attack the main goal of this paper and derive conditions on the symbol $a(z)$ of a subdivision scheme $S_{\boldsymbol{a}}$ that guarantee the reproduction of polynomials up to some degree $d > 1$. But before we can state the main theorem, we need to establish two preliminary results. The first is an immediate consequence of Lemma 2.1.

Corollary 4.1. *The kth derivative of a subdivision symbol $a(z)$ satisfies*

$$a^{(k)}(\zeta_m^j) = 0, \quad j = 1, \dots, m-1,$$

if and only if

$$\sum_{j \in \mathbb{Z}} q_{k,i}(-j) a_{i-mj} = a^{(k)}(1)/m, \quad i \in \mathbb{Z},$$

where $q_{k,i}$ are the polynomials from (10).

Proof. We first remark that by generalizing the indices of the subsymbols in (9) to all integers we get the following cyclic behaviour. For any $i \in \mathbb{Z}$ let $l \in \{0, \dots, m-1\}$ such that $i \equiv l (\text{mod } m)$. Then,

$$a_i(z) = \sum_{j \in \mathbb{Z}} a_{mj+i} z^{mj+i} = \sum_{j \in \mathbb{Z}} a_{mj+l} z^{mj+l} = a_l(z),$$

and taking the kth derivative of

$$a_i(z) = \sum_{j \in \mathbb{Z}} a_{i+mj} z^{i+mj} = \sum_{j \in \mathbb{Z}} a_{i-mj} z^{i-mj}$$

gives

$$\sum_{j \in \mathbb{Z}} q_{k,i}(-j) a_{i-mj} z^{i-mj-k} = a_i^{(k)}(z) = a_l^{(k)}(z). \tag{16}$$

The statement now follows from Lemma 2.1 by using $z = 1$ in (16). $\square$

We then use this result to derive a set of necessary and sufficient conditions for a subdivision scheme to map *monomial data* of degree $k \le d$ at level ℓ, $f_i^\ell = i^k, i \in \mathbb{Z}$ to the refined and shifted monomial data at level $\ell+1$, $f_i^{\ell+1} = \left(\frac{i-\tau}{m}\right)^k, i \in \mathbb{Z}$ in one subdivision step.

Lemma 4.2. *Let $d \in \mathbb{N}$ and $\tau \in \mathbb{R}$. Then a subdivision symbol $a(z)$ satisfies*

$$a^{(k)}(1) = m \prod_{l=0}^{k-1} (\tau - l) \quad \textit{and} \quad a^{(k)}(\zeta_m^j) = 0, \; j = 1, \dots, m-1, \; \textit{for } k = 0, \dots, d$$

if and only if

$$\sum_{j\in\mathbb{Z}} j^k a_{i-mj} = \left(\frac{i-\tau}{m}\right)^k, \quad i \in \mathbb{Z},\ \textit{for } k = 0,\ldots,d. \tag{17}$$

Proof. Note that by Corollary 4.1 the first set of conditions is equivalent to

$$\sum_{j\in\mathbb{Z}} q_{k,i}(-j) a_{i-mj} = \prod_{l=0}^{k-1}(\tau - l), \quad i \in \mathbb{Z},\ \text{for } k = 0,\ldots,d. \tag{18}$$

The proof is then by induction over d. The case $d = 0$ is trivial because both conditions reduce to

$$\sum_{j\in\mathbb{Z}} a_{i-mj} = 1, \quad i \in \mathbb{Z}.$$

So let us assume (17) and (18) to be equivalent for $k = 0,\ldots,d-1$ and prove that the equivalence then also holds for $k = d$. We start by observing that the polynomial $q_{d,i}(-x)$ is of degree d and so there certainly exist some coefficients $\gamma_0,\ldots,\gamma_d$ to express it in monomial form,

$$q_{d,i}(-x) = \sum_{n=0}^{d} \gamma_n x^n.$$

Now we use the induction hypothesis to manipulate the left-hand side of condition (18) and get for any $i \in \mathbb{Z}$

$$\begin{aligned}
\prod_{l=0}^{d-1}(\tau - l) &= \sum_{j\in\mathbb{Z}} q_{d,i}(-j) a_{i-mj} = \sum_{j\in\mathbb{Z}} \left(\sum_{n=0}^{d} \gamma_n j^n\right) a_{i-mj} \\
&= \gamma_d \sum_{j\in\mathbb{Z}} j^d a_{i-mj} + \sum_{n=0}^{d-1} \gamma_n \sum_{j\in\mathbb{Z}} j^n a_{i-mj} \\
&= \gamma_d \sum_{j\in\mathbb{Z}} j^d a_{i-mj} + \sum_{n=0}^{d-1} \gamma_n \left(\frac{i-\tau}{m}\right)^n \\
&= \gamma_d \sum_{j\in\mathbb{Z}} j^d a_{i-mj} + q_{d,i}\left(\frac{\tau-i}{m}\right) - \gamma_d \left(\frac{i-\tau}{m}\right)^d \\
&= \gamma_d \sum_{j\in\mathbb{Z}} j^d a_{i-mj} - \gamma_d \left(\frac{i-\tau}{m}\right)^d + \prod_{n=0}^{d-1}(\tau - n).
\end{aligned}$$

Since $\gamma_d \neq 0$, this is equivalent to

$$\sum_{j\in\mathbb{Z}} j^d a_{i-mj} = \left(\frac{i-\tau}{m}\right)^d,$$

which concludes the proof. □

The main result of this paper now shows that the particular mapping property for monomial data in Lemma 4.2 is equivalent to polynomial reproduction of degree d.

Theorem 4.3. *A convergent subdivision scheme $S_{\boldsymbol{a}}$ reproduces polynomials of degree d with respect to the parameterization in (14) if and only if*

$$a^{(k)}(1) = m\prod_{l=0}^{k-1}(\tau - l) \quad \textit{and} \quad a^{(k)}(\zeta_m^j) = 0, \ j = 1, \ldots, m-1$$

for $k = 0, \ldots, d$.

Proof. The proof is again by induction over d, with the case $d = 0$ being trivial. So let us assume that the statement holds for $k = 0, \ldots, d-1$ and prove it for $k = d$. Following the same thought that we used in the proof of Theorem 3.1, it is sufficient to show that for any polynomial $p \in \pi_d$, $p(x) = x^d + q(x)$ with $q \in \pi_{d-1}$ the implication

$$f_i^\ell = p(t_i^\ell) = (t_i^\ell)^d + q(t_i^\ell), \quad i \in \mathbb{Z} \Longrightarrow f_i^{\ell+1} = p(t_i^{\ell+1}), \quad i \in \mathbb{Z}$$

holds. But this is easily verified by using the induction hypothesis, condition (17) from Lemma 4.2, and remembering from (14) that $t_0^{\ell+1} = t_0^\ell + (i-\tau)/m^{\ell+1}$, because

$$\begin{aligned}
f_i^{\ell+1} &= \sum_{j\in\mathbb{Z}} a_{i-mj} f_j^\ell = \sum_{j\in\mathbb{Z}} a_{i-mj}\left(t_0^\ell + \frac{j}{m^\ell}\right)^d + \sum_{j\in\mathbb{Z}} a_{i-mj} q(t_j^\ell) \\
&= \sum_{j\in\mathbb{Z}}\left[\sum_{n=0}^d \binom{d}{n} (t_0^\ell)^{d-n}\left(\frac{j}{m^\ell}\right)^n\right] a_{i-mj} + q(t_i^{\ell+1}) \\
&= \sum_{n=0}^d \binom{d}{n} (t_0^\ell)^{d-n}\left(\frac{1}{m^\ell}\right)^n \left(\sum_{j\in\mathbb{Z}} j^n a_{i-mj}\right) + q(t_i^{\ell+1}) \\
&= \sum_{n=0}^d \binom{d}{n} (t_0^\ell)^{d-n}\left(\frac{1}{m^\ell}\right)^n \left(\frac{i-\tau}{m}\right)^n + q(t_i^{\ell+1}) \\
&= (t_i^{\ell+1})^d + q(t_i^{\ell+1}) = p(t_i^{\ell+1}). \quad \square
\end{aligned}$$

It is clear that the degree of polynomial reproduction can never be greater than the degree of polynomial generation. But Dyn et al. [9, Corollary 4.9] made an interesting observation in the case that it is strictly smaller, and the proof carries over to subdivision schemes of any arity without changes.

Corollary 4.4. *If the degree l of polynomial reproduction of a convergent subdivision scheme $S_{\boldsymbol{a}}$ is less than the degree n of polynomial generation and $S_{\boldsymbol{a}}$ is applied to the initial data $f_i^0 = p(t_0^i)$, $i \in \mathbb{Z}$, sampled from a polynomial $p \in \pi_d$ with $l < d \leq n$, then the limit function g_f is also a polynomial of degree d and has the same $l+1$ leading coefficients as p, that is, $g_f - p \in \pi_{d-l-1}$.*

Now, suppose that we want to determine the degree of polynomial reproduction for some given scheme. Then the following proposition provides a slightly simpler way to check the necessary conditions (8).

Proposition 4.5. *Let $d \in \mathbb{N}$ and $\tau \in \mathbb{R}$. Then a subdivision symbol $a(z)$ satisfies*

$$a^{(k)}(1) = m\prod_{l=0}^{k-1}(\tau - l), \quad k = 0, \ldots, d \tag{19}$$

if and only if $b(z) = a(z^m)z^{-m\tau}$ *satisfies*

$$b(1) = m \quad \textit{and} \quad b^{(k)}(1) = 0,\ k = 1, \ldots, d, \tag{20}$$

which in turn is equivalent to require that $b(z) = (1 - z)^{d+1}c(z) + m$ *for some* $c(z)$.

Proof. We first show by induction that the identity $a(z^m)z^{-m\tau} = b(z)$ implies for any $k \in \mathbb{N}$,

$$m^k a^{(k)}(z^m) z^{m(k-\tau)} = \sum_{j=0}^{k} c_{k,j} b^{(j)}(z) z^j \tag{21a}$$

for some coefficients $c_{k,j} \in \mathbb{R}$ with

$$c_{k,0} = m^k \prod_{l=0}^{k-1} (\tau - l) \quad \text{and} \quad c_{k,k} = 1. \tag{21b}$$

The case $k = 0$ is trivial, so let us assume that (21) holds for some $k \in \mathbb{N}$. Differentiating both sides and multiplying by z then yields

$$\begin{aligned} &m^k \Big[m a^{(k+1)}(z^m) z^{m(k+1-\tau)} + m(k-\tau) a^{(k)}(z^m) z^{m(k-\tau)} \Big] \\ &\quad = \sum_{j=0}^{k} c_{k,j} \Big[b^{(j+1)}(z) z^{j+1} + j b^{(j)}(z) z^j \Big], \end{aligned}$$

and further, by using the induction hypothesis,

$$\begin{aligned} m^{k+1} a^{(k+1)}(z^m) z^{m(k+1-\tau)} &= -m(k-\tau) \sum_{j=0}^{k} c_{k,j} b^{(j)}(z) z^j \\ &\quad + \sum_{j=0}^{k} c_{k,j} b^{(j+1)}(z) z^{j+1} + \sum_{j=0}^{k} j\, c_{k,j} b^{(j)}(z) z^j \\ &= m(\tau - k) c_{k,0} b(z) + \sum_{j=1}^{k} c_{k,j} (m(\tau - k) + j) b^{(j)}(z) z^j \\ &\quad + \sum_{j=1}^{k} c_{k,j-1} b^{(j)}(z) z^j + c_{k,k} b^{(k+1)} z^{k+1} \\ &= m^{k+1} \prod_{l=0}^{k} (\tau - l)\, b(z) + \sum_{j=1}^{k} c_{k+1,j} b^{(j)}(z) z^j + b^{(k+1)} z^{k+1} \\ &= \sum_{j=0}^{k+1} c_{k+1,j} b^{(j)}(z) z^j \end{aligned}$$

with coefficients $c_{k+1,0} = m^{k+1} \prod_{l=0}^{k} (\tau - l)$, $c_{k+1,k+1} = 1$, and $c_{k+1,j} = c_{k,j-1} + c_{k,j}(m(\tau - k) + j)$ for $j = 1, \ldots, k-1$, which completes the inductive step.

Now, if (19) holds for some $d \in \mathbb{N}$, then it follows from the definition of b that

$$b(1) = a(1) = m$$

and we further conclude from (21) by induction over k that

$$b^{(k)}(1) = m^k a^{(k)}(1) - \sum_{j=0}^{k-1} c_{k,j} b^{(j)}(1) = m^{k+1} \prod_{l=0}^{k-1}(\tau - l) - m^k \prod_{l=0}^{k-1}(\tau - l) b(1) = 0$$

for any $k > 0$. On the other hand, if (20) is true for some $d \in \mathbb{N}$, then (21) yields

$$m^k a^{(k)}(1) = c_{k,0} b(1) = m^{k+1} \prod_{l=0}^{k-1}(\tau - l) \Longrightarrow a^{(k)}(1) = m \prod_{l=0}^{k-1}(\tau - l)$$

for any $k \le d$. $\square$

We conclude this section by noting that Theorem 4.3 includes the results of [9] for polynomial reproduction of binary schemes as special cases for $m = 2$. First observe that the primal and dual parameterization that were considered in [9] correspond to our general parameterization in (14) with $\tau = 0$ and $\tau = -1/2$, respectively and $t_0^0 = \tau$. It is then clear that condition (PR1) in Theorem 4.6 of [9] for polynomial reproduction with respect to the primal parameterization can be restated as

$$a(1) = 2 \quad \text{and} \quad a^{(k)}(1) = 0, \quad k = 1, \ldots, d,$$

and is therefore equivalent to the conditions on the behaviour of $a(z)$ and its derivatives at $z = 1$ in Theorem 4.3. Moreover, the equivalence of the latter conditions to condition (PR2) in Theorem 4.7 of [9] for polynomial reproduction with respect to the dual parameterization follows from Proposition 4.5.

5. Applications

A first important application of Theorem 4.3 is the analysis of “shifted” schemes. Often, when defining a subdivision scheme, the mask coefficients a_i are simply given as a sequence of numbers without further specifying the index range. For example, the binary scheme that generates cubic B-splines is usually given by the mask $\{1, 4, 6, 4, 1\}/8$, but it is not clear whether this refers to $\{a_{-2}, a_{-1}, a_0, a_1, a_2\}$, $\{a_0, a_1, a_2, a_3, a_4\}$, or any other sequence $\{a_i, \ldots, a_{i+4}\}$. The clue is that the choice of the index range has no effect on the limit curve as far as convergence, smoothness, polynomial generation, and support of the basic limit function are concerned, because it merely leads to a shift in the indices of the refined data and does not affect the data itself. The same basically holds for polynomial reproduction, too, but the specific value of the correct parametric shift τ depends on the choice of the index range.

Corollary 5.1. *If $S_{\boldsymbol{a}}$ is a subdivision scheme that reproduces polynomials up to degree d, then so does the shifted scheme $S_{\tilde{\boldsymbol{a}}}$ with symbol $\tilde{a}(z) = a(z)z^n$ for any $n \in \mathbb{Z}$.*

Proof. Let $\tau_{\boldsymbol{a}} = a'(1)/m$ be the parametric shift of the subdivision scheme $S_{\boldsymbol{a}}$. Then the parametric shift for the scheme $S_{\tilde{\boldsymbol{a}}}$ is

$$\tau_{\tilde{\boldsymbol{a}}} = \tilde{a}'(1)/m = (a'(1) + a(1)n)/m = \tau_{\boldsymbol{a}} + n$$

and the statement follows from Proposition 4.5, because

$$\tilde{b}(z) = \tilde{a}(z^m) z^{-m\tau_{\tilde{\boldsymbol{a}}}} = a(z^m) z^{mn} z^{-m(\tau_{\boldsymbol{a}}+n)} = a(z^m) z^{-m\tau_{\boldsymbol{a}}} = b(z). \quad \square$$

Going back to the previous example, this means that the correct parameterization (14) for the binary cubic B-spline scheme with mask $\{a_{-2}, a_{-1}, a_0, a_1, a_2\} = \{1, 4, 6, 4, 1\}/8$ is the standard parameterization with $\tau = 0$, while a parametric shift of $\tau = 2$ is appropriate for the equivalent scheme with mask $\{a_0, a_1, a_2, a_3, a_4\} = \{1, 4, 6, 4, 1\}/8$.

We continue by discussing other consequences of Theorem 4.3 to several kinds of univariate subdivision schemes (interpolatory, symmetric, and m-ary B-spline schemes) and use it to define a new family of general m-ary pseudo-splines.

5.1. Interpolatory schemes

An important class of subdivision schemes are those that refine the sequence $\boldsymbol{f}$ while keeping the original data in the sense that $f_{mi}^{\ell+1} = f_i^{\ell}$, $i \in \mathbb{Z}$, $\ell \in \mathbb{N}$. For obvious reasons such a scheme is called *interpolatory* and if it is convergent then the limit function is a cardinal interpolant to $\boldsymbol{f}$, that is

$$g_{\boldsymbol{f}}(i) = f_i, \quad i \in \mathbb{Z}.$$

Interpolatory schemes are characterized by the fact that the coefficients of the subdivision mask satisfy

$$a_{mi} = \delta_{i,0}, \quad i \in \mathbb{Z}, \tag{22}$$

which by (9) is equivalent to its 0th subsymbol being $a_0(z) = 1$. However, it is less known that this condition can also be stated in terms of the symbol $a(z)$.

Proposition 5.2. *An m-ary subdivision scheme $S_{\boldsymbol{a}}$ is interpolatory if and only if its symbol $a(z)$ satisfies*

$$\sum_{j=0}^{m-1} a(\zeta_m^j z) = m, \quad z \in \mathbb{C} \setminus \{0\},$$

where ζ_m^j are the mth roots of unity as defined in Section 1.1.

Proof. Let $a(z)$ be the symbol of an interpolatory subdivision scheme. By (11) and (22) it has the form

$$a(z) = 1 + \sum_{l=1}^{m-1} a_l(z),$$

so that

$$a(\zeta_m^j z) = 1 + \sum_{l=1}^{m-1} \left(\zeta_m^j\right)^l a_l(z), \quad j = 0, \ldots, m-1.$$

Summing up with respect to j we get

$$\sum_{j=0}^{m-1} a(\zeta_m^j z) = m + \sum_{j=0}^{m-1} \sum_{l=1}^{m-1} \left(\zeta_m^j\right)^l a_l(z) = m + \sum_{l=1}^{m-1} a_l(z) \sum_{j=0}^{m-1} \left(\zeta_m^j\right)^l = m,$$

because $\sum_{j=0}^{m-1} \left(\zeta_m^j\right)^l = 0$ for $l = 1, \ldots, m-1$. Vice versa, assuming $\sum_{j=0}^{m-1} a(\zeta_m^j z) = m$ and following the same line of reasoning as before, we end up with the relation $a_0(z) = 1$, and so $S_{\boldsymbol{a}}$ is interpolatory. $\square$

It is well known that polynomial generation and polynomial reproduction are equivalent for interpolatory schemes and our results above confirm this, as long as the standard parameterization $t_i^\ell = i/m^\ell$ is used.

Corollary 5.3. *Let $S_{\boldsymbol{a}}$ be an interpolatory subdivision scheme that generates polynomials up to degree d. Then $S_{\boldsymbol{a}}$ also reproduces polynomials up to degree d with respect to the parameterization (14) with $\tau = 0$.*

Proof. As $S_{\boldsymbol{a}}$ is an interpolatory scheme, its 0th subsymbol is $a_0(z) = 1$ and so $a_0^{(k)}(1) = 0$ for $k \geq 1$. By Lemma 2.1 we then conclude $a^{(k)}(1) = 0$ for $k = 1, \ldots, d$. In particular, this implies that the correct parametric shift is $\tau = a'(1)/m = 0$, and it follows by Theorem 4.3 that the scheme reproduces polynomials up to degree d. □

Remark 5.4. In view of the discussion above about shifted schemes, it is possible to generalize the standard definition of interpolatory schemes slightly to all schemes with $a_n(z) = z^n$ for some $n \in \mathbb{Z}$. In terms of coefficients this translates to the condition $a_{mi+n} = \delta_{i,0}$, and the original data is then kept in the sense $f_{mi+n}^{\ell+1} = f_i^\ell$, $i \in \mathbb{Z}$, $\ell \in \mathbb{N}$. Clearly, the correct parametric shift for such a scheme is $\tau = n$.

5.2. *Symmetric schemes*

Especially in a geometric context, subdivision schemes are sometimes classified into “primal” and “dual” schemes, where primal schemes are those that leave or modify the old points and create $m - 1$ new points at each old edge, while dual schemes create m new points at the old edges and “discard” the old points. For example, the binary cubic B-spline scheme is primal, while Chaikin’s scheme [2] for quadratic B-splines is dual. Mathematically, this corresponds to using different parameterizations.

Definition 5.5. The *primal* (or *standard*) *parameterization* of a subdivision scheme is based on the parameter values

$$t_i^\ell = \frac{i}{m^\ell}, \quad i \in \mathbb{Z},\ \ell \in \mathbb{N},$$

while the *dual parameterization* attaches the data f_i^ℓ to the parameter values

$$t_i^\ell = \frac{i - 1/(2(m-1))}{m^\ell}, \quad i \in \mathbb{Z},\ \ell \in \mathbb{N}.$$

Fig. 1 illustrates both concepts for binary and ternary schemes. Note that this definition is consistent with the one given in [9, Section 2] for the special case $m = 2$ and that these two parameterizations are special cases of our general parameterization in (14) for $\tau = 0$ or $\tau = -1/2$ and $t_0^0 = \tau$. The reason why no other parameterizations have been considered so far in the literature is simple: they are the only ones that provide linear reproduction if the subdivision scheme is symmetric. For the special case $m = 2$, this has already been shown in [9, Section 5], but the proof extends nicely to general arity.

Definition 5.6. A subdivision scheme $S_{\boldsymbol{a}}$ is called *odd symmetric* if

$$a_i = a_{-i}, \quad i \in \mathbb{Z},$$

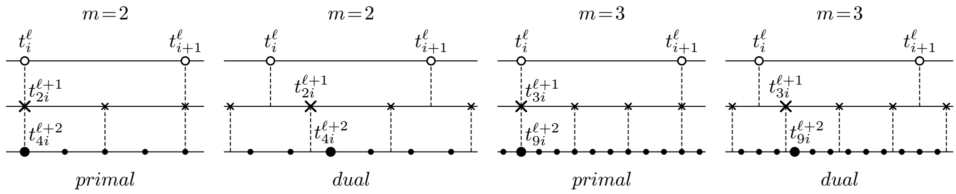

Fig. 1. Primal and dual parameterization for binary and ternary schemes.

and *even symmetric* if

$$a_{i-1} = a_{-i}, \quad i \in \mathbb{Z}.$$

In terms of Laurent polynomials, these conditions translate to $a(z) = a(z^{-1})$ and $a(z)z = a(z^{-1})$, respectively.

Corollary 5.7. *If $S_{\boldsymbol{a}}$ is an odd (even) symmetric subdivision scheme that generates linear polynomials, then the primal (dual) parameterization is the only one that yields reproduction of linear polynomials.*

Proof. If $S_{\boldsymbol{a}}$ is odd symmetric, then $a(z) = a(z^{-1})$ and by taking the first derivative of both sides,

$$a'(z) = -z^{-2}a'(z^{-1}),$$

we conclude $a'(1) = 0$. Therefore, the unique parametric shift that gives at least linear reproduction is $\tau = a'(1)/m = 0$. In case of even symmetry, the same strategy applied to $a(z)z = a(z^{-1})$ leads to $a'(1) = -a(1)/2 = -m/2$ and hence $\tau = a'(1)/m = -1/2$. □

Remark 5.8. The statement of Corollary 5.7 can be simplified by shifting the (odd or even) symmetric scheme $S_{\boldsymbol{a}}$ such that its shifted mask $\tilde{\boldsymbol{a}}$ is supported on $[0, N]$, that is, $\tilde{a}_0 \neq 0$ and $\tilde{a}_i = 0$ for $i < 0$ and $i > N$. Then the only parameterization that guarantees linear reproduction is the one with $\tau = N/2$.

5.3. Smoothing factors and B-splines

It is well known that the smoothness of an m-ary subdivision scheme $S_{\boldsymbol{a}}$ increases by one if the symbol is multiplied by the m-ary smoothing factor

$$\sigma_m(z) = \frac{1 + z + \cdots + z^{m-1}}{m} = \frac{1 - z^m}{m(1 - z)},$$

and it is clear that this also increases the degree of polynomial generation by one. However, this kind of smoothing inevitably reduces the degree of polynomial reproduction down to one.

Proposition 5.9. *Let $S_{\boldsymbol{a}}$ be a subdivision scheme that reproduces polynomials up to degree d. Then the smoothed scheme $S_{\boldsymbol{b}}$ with symbol $b(z) = \sigma_m(z)a(z)$ satisfies*

$$b(1) = m \quad \text{and} \quad b^{(k)}(\zeta_m^j) = 0, \; j = 1, \ldots, m-1, \; k = 0, \ldots, d+1,$$

and hence generates polynomials of degree $d + 1$, but it has only linear reproduction.

Proof. The statement about polynomial generation follows trivially from the fact that $b(1) = a(1)$ and by considering (7). Then, as the first derivative of $b(z)$ is

$$b'(z) = \frac{1 + z + z^2 + \cdots + z^{m-1}}{m} a'(z) + \frac{1 + 2z + 3z^2 + \cdots + (m-1)z^{m-2}}{m} a(z),$$

the correct parametric shift for $S_{\boldsymbol{b}}$ which guarantees linear reproduction is

$$\tau_{\boldsymbol{b}} = \frac{b'(1)}{m} = \frac{a'(1) + \frac{m-1}{2} a(1)}{m} = \tau_{\boldsymbol{a}} + \frac{m-1}{2}.$$

Next, computing the second derivative of $b(z)$, we have

$$\begin{aligned} b''(z) = {} & \frac{1 + z + z^2 + \cdots + z^{m-1}}{m} a''(z) + 2\frac{1 + 2z + 3z^2 + \cdots + (m-1)z^{m-2}}{m} a'(z) \\ & + \frac{2 + 6z + 12z^2 + \cdots + (m-1)(m-2)z^{m-2}}{m} a(z), \end{aligned}$$

so that

$$\begin{aligned} b''(1) &= a''(1) + (m-1)a'(1) + \frac{(m-1)(m-2)}{3} a(1) \\ &= m\tau_{\boldsymbol{a}}(\tau_{\boldsymbol{a}} - 1) + m(m-1)\tau_{\boldsymbol{a}} + \frac{m(m-1)(m-2)}{3}. \end{aligned}$$

Some straightforward simplifications then yield

$$b''(1) - m\tau_{\boldsymbol{b}}(\tau_{\boldsymbol{b}} - 1) = b''(1) - m\left(\tau_{\boldsymbol{a}} + \frac{m-1}{2}\right)\left(\tau_{\boldsymbol{a}} + \frac{m-3}{2}\right) = \frac{m(m^2-1)}{12},$$

which clearly is not equal to zero for any $m \geq 2$. And so, by Theorem 4.3, $S_{\boldsymbol{b}}$ does not reproduce polynomials of degree $d > 1$. □

As the symbol of the m-ary subdivision scheme that generates B-splines of degree n is

$$B_n(z) = m\sigma_m(z)^{n+1}, \tag{23}$$

Proposition 5.9 confirms the well-known fact that these schemes reproduce only linear polynomials and thus have approximation order 2. In consistency with Remark 5.8, the corresponding parametric shift is $\tau = B_n'(1)/m = (n+1)(m-1)/2$.

5.4. *Pseudo-splines*

Theorem 4.3 also allows us to generalize the family of binary pseudo-splines both to general arity and to arbitrary parameterizations. Primal pseudo-splines with odd symmetry were first presented by Dong and Shen [5], while even symmetric dual pseudo-splines were later discovered by Dyn et al. [9].

For any $\tau \in \mathbb{R}$ and $n, l \in \mathbb{N}$ the m-ary pseudo-spline is defined to be the scheme with minimal support that generates polynomials of degree n and whose symbol satisfies the necessary conditions

$$a^{(k)}(1) = m\prod_{i=0}^{k-1}(\tau - i), \quad k = 0, \ldots, l \tag{24}$$

for reproduction of polynomials up to degree l. Its actual degree of polynomial reproduction is $\min(n, l)$ and its symbol can be written as

$$a(z) = B_n(z)b(z), \tag{25}$$

where $B_n(z)$ is the symbol of the m-ary degree n B-spline scheme in (23) and $b(z)$ is the polynomial of lowest possible degree such that $a(z)$ satisfies (24).

Using the Leibniz rule, we see that this set of conditions is equivalent to

$$\sum_{i=0}^{k} \binom{k}{i} b^{(i)}(1) B_n^{(k-i)}(1) = m \prod_{i=0}^{k-1} (\tau - i) =: c_k, \quad k = 0, \ldots, l,$$

which can be rewritten as the linear system

$$A\boldsymbol{d} = \boldsymbol{c}, \tag{26}$$

where $\boldsymbol{d} = \big(b(1), b'(1), \ldots, b^{(l)}(1)\big)^T$, $\boldsymbol{c} = (c_0, c_1, \ldots, c_l)^T$, and A is the lower triangular $(l+1) \times (l+1)$ matrix with coefficients

$$a_{k,i} = \binom{k}{i} B_n^{(k-i)}(1), \quad k = 0, \ldots, l,\ i = 0, \ldots, k.$$

Note that the lower diagonal elements of A have the recursive structure

$$a_{k,i} = \frac{k}{i} a_{k-1,i-1} = \frac{k(k-1)}{i(i-1)} a_{k-2,i-2} = \cdots = \binom{k}{i} a_{k-i,0},$$

a fact that will be very useful in what follows. In particular, $a_{k,k} = a_{0,0}$ for all $k = 0, \ldots, l$.

Letting $(d_0, d_1, \ldots, d_l)^T = A^{-1}\boldsymbol{c}$, it is clear that $d_k = b^{(k)}(1)$ for $k = 0, \ldots, l$ if and only if

$$b(z) = \sum_{k=0}^{l} \frac{(z-1)^k}{k!} d_k + (z-1)^{l+1} r(z) \tag{27}$$

for some polynomial $r(z)$ and that $b(z)$ has the lowest possible degree if $r(z) = 0$. Hence, the symbol of the general pseudo-spline is

$$a(z) = B_n(z) \sum_{k=0}^{l} \frac{(z-1)^k}{k!} d_k,$$

which is a polynomial of degree $(n+1)(m-1)+l$ in general and occasionally one less. For example, in the special case $l = 1$ the linear system (26) is simply

$$\begin{pmatrix} m & 0 \\ B_n'(1) & m \end{pmatrix} \begin{pmatrix} d_0 \\ d_1 \end{pmatrix} = \begin{pmatrix} m \\ \tau m \end{pmatrix},$$

and for $\tau = B_n'(1)/m$ the solution vector of this system is $(1, 0)$, so that the pseudo-splines reduce to the m-ary B-splines.

We continue with an important result concerning the structure of the matrix A^{-1} for general $l \in \mathbb{N}$.

Lemma 5.10. *Let $A = (a_{k,i})_{0\le k,i\le l}$ be a lower triangular $(l+1)\times(l+1)$ matrix with $a_{0,0} \ne 0$, any values $a_{k,0}$, $1 \le k \le l$ in the first column, and*

$$a_{k,i} = \binom{k}{i} a_{k-i,0}, \quad 1 \le i \le k \le l, \tag{28}$$

for the remaining elements. Then $A^{-1} = (\tilde{a}_{k,i})_{0\le k,i\le l}$ is also a lower triangular matrix with elements

$$\tilde{a}_{0,0} = \frac{1}{a_{0,0}}, \qquad \tilde{a}_{k,0} = -\frac{1}{a_{0,0}} \sum_{j=0}^{k-1} \binom{k}{j} a_{k-j,0}\tilde{a}_{j,0}, \quad 1 \le k \le l \tag{29}$$

in the first column and

$$\tilde{a}_{k,i} = \binom{k}{i} \tilde{a}_{k-i,0}, \quad 1 \le i \le k \le l \tag{30}$$

otherwise.

Proof. The formulas for the elements $\tilde{a}_{k,i}$ of A^{-1} can be verified by using the fact that the product of the kth row of A and the ith column of A^{-1} must satisfy

$$\sum_{j=0}^{l} a_{k,j}\tilde{a}_{j,i} = \sum_{j=i}^{k} a_{k,j}\tilde{a}_{j,i} = \delta_{k,i}, \quad 0 \le k, i \le l.$$

This is clearly true if $i > k$, because the sum is empty, and if $i = k$ it reduces to $a_{k,k}\tilde{a}_{k,k} = a_{0,0}\tilde{a}_{0,0} = 1$. Finally, if $i < k$ then by (28) and (30), we have

$$\begin{aligned}
\sum_{j=i}^{k} a_{k,j}\tilde{a}_{j,i} &= \sum_{j=0}^{k-i} a_{k,i+j}\tilde{a}_{i+j,i} = \sum_{j=0}^{k-i} \binom{k}{i+j} a_{k-i-j,0} \binom{i+j}{i} \tilde{a}_{j,0} \\
&= \binom{k}{i} \sum_{j=0}^{k-i} \binom{k-i}{j} a_{k-i-j,0}\tilde{a}_{j,0} \\
&= \binom{k}{i} \left(\sum_{j=0}^{k-i-1} \binom{k-i}{j} a_{k-i-j,0}\tilde{a}_{j,0} + a_{0,0}\tilde{a}_{k-i,0} \right),
\end{aligned}$$

which reduces to zero by considering recursion (29) for $\tilde{a}_{k-i,0}$. □

Note that the matrices that appear in the construction of the pseudo-splines are exactly of this kind, with elements

$$a_{k,0} = B_n^{(k)}(1), \quad 0 \le k \le l$$

in the first column.

5.4.1. Binary pseudo-splines

For binary pseudo-splines ($m = 2$) the formulas above can be simplified considerably by using a remarkable binomial identity that we could not find in the literature.

Lemma 5.11. *For any* $k, n \in \mathbb{N}$*,*

$$\alpha_{k,n} = \sum_{j=0}^{k} (-1)^j \binom{n}{j} \binom{n+k-j}{n} = 1.$$

Proof. For $k = 0$ and $n \in \mathbb{N}$ as well as for $k \in \mathbb{N}$ and $n = 0$, the identity is easily verified, and the rest follows by induction through the recursion

$$\alpha_{k,n} = \alpha_{k-1,n} + \alpha_{k,n-1} - \alpha_{k-1,n-1},$$

which we get by using

$$\begin{aligned}\binom{n}{j}\binom{n+k-j}{n} &= \binom{n}{j}\binom{n+k-j-1}{n} + \binom{n}{j}\binom{n+k-j-1}{n-1} \\ &= \binom{n}{j}\binom{n+(k-1)-j}{n} + \binom{n-1}{j}\binom{(n-1)+k-j}{n-1} \\ &\quad + \binom{n-1}{j-1}\binom{(n-1)+(k-1)-(j-1)}{n-1}. \quad \square\end{aligned}$$

We can now derive a simple closed form for the elements of the inverse of A in (26).

Corollary 5.12. *In the particular case of binary pseudo-splines* $(m = 2)$*, we have*

$$a_{k,0} = 2\binom{n+1}{k}\frac{k!}{2^k}, \quad 0 \le k \le l,$$

and the elements in the first column of the inverse matrix are

$$\tilde{a}_{k,0} = \frac{(-1)^k}{2}\binom{n+k}{n}\frac{k!}{2^k}, \quad 0 \le k \le l. \tag{31}$$

Proof. The first statement can be easily seen by considering

$$\frac{\partial^k}{\partial z^k}(1+z)^n = \prod_{i=0}^{k-1}(n-i)(1+z)^{n-k} = \binom{n}{k} k!(1+z)^{n-k}$$

and remembering that $B_n(z) = (1+z)^{n+1}/2^n$. The second statement follows by induction over k. For $k = 0$, we clearly have

$$\tilde{a}_{0,0} = \frac{1}{2} = \frac{1}{a_{0,0}},$$

so let us assume that (31) is true for some $k \ge 0$. Then using the recursion in (29) we get

$$\begin{aligned}\tilde{a}_{k+1,0} &= -\frac{1}{2}\sum_{j=0}^{k}\binom{k+1}{j} a_{k+1-j,0}\tilde{a}_{j,0} \\ &= -\frac{1}{2}\sum_{j=0}^{k}\binom{k+1}{j}\binom{n+1}{k+1-j}\frac{(k+1-j)!}{2^{k+1-j}}(-1)^j\binom{n+j}{n}\frac{j!}{2^j}\end{aligned}$$

$$= -\frac{1}{2}\sum_{j=0}^{k}\binom{k+1}{k-j}\binom{n+1}{j+1}\frac{(j+1)!}{2^{j+1}}(-1)^{k-j}\binom{n+k-j}{n}\frac{(k-j)!}{2^{k-j}}$$
$$= \frac{(-1)^{k+1}}{2}\frac{(k+1)!}{2^{k+1}}\sum_{j=0}^{k}(-1)^j\binom{n+1}{j+1}\binom{n+k-j}{n}$$
$$= \frac{(-1)^{k+1}}{2}\binom{n+k+1}{n}\frac{(k+1)!}{2^{k+1}},$$

where we used Lemma 5.11 to conclude the last identity,

$$\begin{aligned}
&\sum_{j=0}^{k}(-1)^j\binom{n+1}{j+1}\binom{n+k-j}{n}\\
&\quad= \sum_{j=0}^{k}(-1)^j\binom{n}{j}\binom{n+k-j}{n} + \sum_{j=0}^{k}(-1)^j\binom{n}{j+1}\binom{n+k-j}{n}\\
&\quad= 1 + \sum_{j=1}^{k+1}(-1)^{j-1}\binom{n}{j}\binom{n+k-(j-1)}{n}\\
&\quad= 1 - \sum_{j=0}^{k+1}(-1)^j\binom{n}{j}\binom{n+(k+1)-j}{n} + \binom{n+k+1}{n}\\
&\quad= 1 - 1 + \binom{n+k+1}{n}. \quad \square
\end{aligned}$$

Overall, it turns out that the general binary pseudo-spline has the symbol

$$\begin{aligned}
a(z) &= \frac{(1+z)^{n+1}}{2^n}\sum_{k=0}^{l}\left(\frac{1-z}{2}\right)^k d_k\\
&= \frac{(1+z)^{n+1}}{2^n}\sum_{k=0}^{l}\left(\frac{1-z}{2}\right)^k \sum_{i=0}^{k}\binom{k}{i}\tilde{a}_{k-i,0}c_i\\
&= \frac{(1+z)^{n+1}}{2^n}\sum_{k=0}^{l}\left(\frac{1-z}{2}\right)^k \sum_{i=0}^{k}\binom{k}{i}\tilde{a}_{k-i,0}\,2\binom{\tau}{i}i!\\
&= \frac{(1+z)^{n+1}}{2^n}\sum_{k=0}^{l}\left(\frac{1-z}{2}\right)^k \sum_{i=0}^{k}(-2)^i\binom{n+k-i}{n}\binom{\tau}{i}, \qquad (32)
\end{aligned}$$

which is a polynomial of degree $n+l+1$ in general and $n+l$ in some special cases.

Remark 5.13. One of these special cases occurs for odd l and $\tau = (n+l)/2$, when the general binary pseudo-splines reduce to the primal pseudo-splines if n is odd and to the dual pseudo-splines if n is even.

5.4.2. Pseudo-splines of arity $m > 2$

For general $m > 2$ we were not able to find a simple closed form for the elements $\tilde{a}_{k,i}$ of A^{-1}, but at least the coefficients $a_{k,0}$ of A can be found with the help of the multinomial theorem and then $\tilde{a}_{k,i}$ can be computed using Lemma 5.10.

Theorem 5.14 (*Multinomial Theorem*)**.** *For $\mathbf{z} \in \mathbb{R}^{m-1}$, $\boldsymbol{\alpha} \in \mathbb{N}^{m-1}$ with $\mathbf{z} = (z_1, z_2, \ldots, z_{m-1})$, and $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_{m-1})$,*

$$(z_1 + z_2 + \cdots + z_{m-1})^n = \sum_{|\boldsymbol{\alpha}|=n} \frac{n!}{\boldsymbol{\alpha}!} \prod_{i=1}^{m-1} \mathbf{z}^{\boldsymbol{\alpha}} = \sum_{|\boldsymbol{\alpha}|=n} \frac{n!}{\boldsymbol{\alpha}!} \prod_{i=1}^{m-1} z_i^{\alpha_i},$$

where $|\boldsymbol{\alpha}| = \alpha_1 + \cdots + \alpha_{m-1}$.

Letting $\ell(\boldsymbol{\alpha}) := \sum_{i=1}^{m-1} i\alpha_i$ and applying the multinomial theorem for $z_i = z^i$, we have

$$\frac{\partial^k}{\partial z^k}(z + \cdots + z^{m-1})^j = \frac{\partial^k}{\partial z^k} \sum_{|\boldsymbol{\alpha}|=j} \frac{j!}{\boldsymbol{\alpha}!} z^{\ell(\boldsymbol{\alpha})} = \sum_{|\boldsymbol{\alpha}|=j} \frac{j!}{\boldsymbol{\alpha}!} \binom{\ell(\boldsymbol{\alpha})}{k} k! z^{\ell(\boldsymbol{\alpha})-k},$$

and since

$$\begin{aligned}\frac{\partial^k}{\partial z^k}(1 + z + \cdots + z^{m-1})^n &= \frac{\partial^k}{\partial z^k}\big(1 + (z + \cdots + z^{m-1})\big)^n \\ &= \sum_{j=0}^{n} \binom{n}{j} \frac{\partial^k}{\partial z^k}(z + \cdots + z^{m-1})^j,\end{aligned}$$

it follows that

$$\frac{\partial^k}{\partial z^k}(1 + z + \cdots + z^{m-1})^n = \sum_{j=0}^{n} \binom{n}{j} \sum_{|\boldsymbol{\alpha}|=j} \frac{j!}{\boldsymbol{\alpha}!} \binom{\ell(\boldsymbol{\alpha})}{k} k! z^{\ell(\boldsymbol{\alpha})-k}.$$

Hence, the elements in the first column of A are

$$a_{k,0} = \frac{\partial^k}{\partial z^k} \frac{(1 + z + \cdots + z^{m-1})^{n+1}}{m^n} = \frac{k!}{m^n} \sum_{j=0}^{n+1} \binom{n+1}{j} \sum_{|\boldsymbol{\alpha}|=j} \frac{j!}{\boldsymbol{\alpha}!} \binom{\ell(\boldsymbol{\alpha})}{k}.$$

5.5. Non-symmetric binary 3-point schemes

As an example of the discussion above, let us consider the simplest case beyond B-splines, namely binary pseudo-splines that reproduce quadratic polynomials ($m = n = l = 2$). According to (32), their symbol is

$$a(z) = \frac{(1 + z)^3}{8}(b_0 + b_1 z + b_2 z^2)$$

with

$$b_0 = (\tau - 2)(\tau - 4), \qquad b_1 = 2 - b_0 - b_2, \qquad b_2 = (\tau - 1)(\tau - 3).$$

In general, these schemes have approximation order 3 and their basic limit functions are supported on $[-\tau, 5 - \tau]$. Fig. 2 shows some plots for several values of τ that we consider in the following.

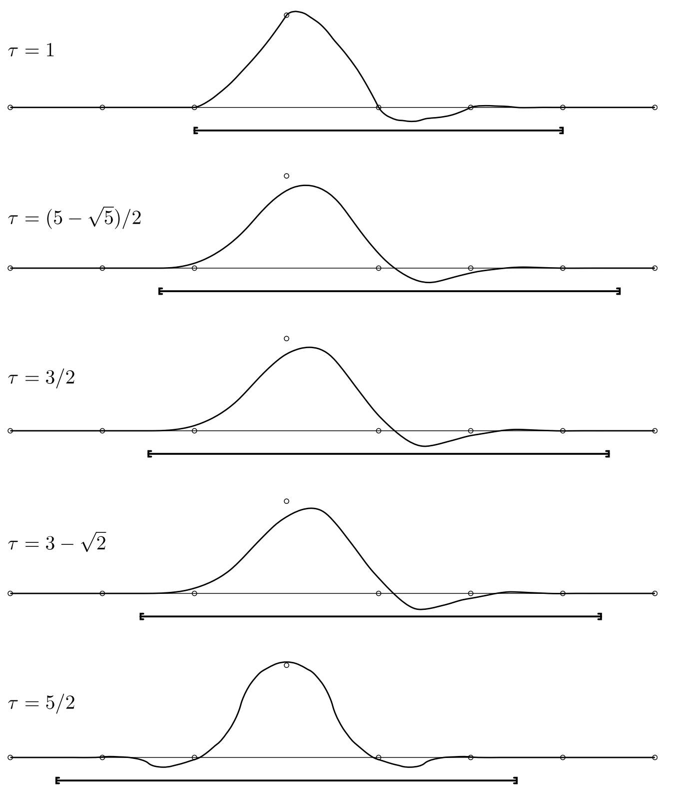

Fig. 2. Basic limit functions of binary pseudo-splines and their support for different parametric shifts.

Clearly, $a(z)$ is symmetric if and only if $\tau = 5/2$, yielding the mask $\{-3, 5, 30, 30, 5, -3\}/32$. This scheme is known as the *dual* 3-*point scheme* and its limit functions are C^1-continuous [16]. However, our general approach allows us to give up symmetry and trade it in for other desirable properties.

For example, letting $\tau = 1$ or $\tau = 3$ reduces the support of the basic limit function by one and gives the interpolatory schemes with masks

$$\{3, 8, 6, 0, -1\}/8 \quad \text{and} \quad \{-1, 0, 6, 8, 3\}/8,$$

which are non-symmetric but symmetric to each other. Another interpretation of these two schemes is that the new data $f_{2i+1}^{\ell+1}$ is computed by sampling the unique quadratic polynomial that interpolates f_i^ℓ, f_{i+1}^ℓ and either f_{i-1}^ℓ or f_{i+2}^ℓ at the midpoint between f_i^ℓ and f_{i+1}^ℓ. From this point of view it is also clear that the scheme reproduces quadratic polynomials by construction.

Another interesting aspect is that τ can be chosen so that the limit functions are C^2. Following Rioul [25], the Hölder regularity of the limit function is $3 - \log_2(\mu)$, where μ is the *joint spectral radius* of the matrices

$$B_0 = \begin{pmatrix} b_1 & 0 \\ b_2 & b_0 \end{pmatrix} \quad \text{and} \quad B_1 = \begin{pmatrix} b_2 & b_0 \\ 0 & b_1 \end{pmatrix}.$$

For $3-\sqrt{2} \le \tau \le 2+\sqrt{2}$ it is easy to see that the spectral radii of B_0 and B_1 as well as their maximum norms are

$$\rho(B_0) = \rho(B_1) = \|B_0\|_\infty = \|B_1\|_\infty = b_1.$$

Since they bound μ from below and above, as shown in [20], we conclude that the Hölder regularity for this range of τ is

$$3 - \log_2(b_1)$$

and that the limit functions are C^2-continuous for

$$\frac{\sqrt{3}}{2} < \left|\tau - \frac{5}{2}\right| \le \sqrt{2} - \frac{1}{2}.$$

Within this range, the maximal smoothness ($\approx$ 2.1294) is obtained for $\tau = 3-\sqrt{2}$ and $\tau = 2+\sqrt{2}$, giving the C^2 schemes with masks

$$\{1, 2+2\sqrt{2}, 2+4\sqrt{2}, 4, 5-4\sqrt{2}, 2-2\sqrt{2}\}/8 \quad \text{and}$$
$$\{2-2\sqrt{2}, 5-4\sqrt{2}, 4, 2+4\sqrt{2}, 2+2\sqrt{2}, 1\}/8,$$

which again are non-symmetric but symmetric to each other. In addition, we find that the interpolatory schemes above have Hölder regularity ≈ 1.415.

Yet another special case occurs for $\tau = (5 \pm \sqrt{5})/2$, which are the unique values that give an additional factor of $1+z$, hence these particular schemes generate even cubic polynomials. Their symbols are

$$a(z) = \frac{(1+z)^4}{16}\big((1 \mp \sqrt{5}) + (1 \pm \sqrt{5})z\big),$$

and of course they can also be derived by considering the binary pseudo-splines with $n=3$ and $l=2$ and finding that the leading coefficient is zero for these values of τ. The joint spectral radius analysis becomes trivial for these schemes and reveals that the Hölder regularity of the limit functions is

$$4 - \log_2\big(\max\big(|1-\sqrt{5}|, |1+\sqrt{5}|\big)\big) \approx 2.3058.$$

Finally, we would like to remark that a more refined joint spectral radius analysis shows that also the non-symmetric dual schemes for $\tau = 3/2$ and $\tau = 7/2$ with masks

$$\{5, 21, 30, 14, -3, -3\}/32 \quad \text{and} \quad \{-3, -3, 14, 30, 21, 5\}/32$$

have C^2 limit functions.

5.6. A quaternary 4-point scheme

To further stress the usefulness of our approach and the advantages that subdivision schemes of higher arity offer, let us consider the following construction of a symmetric quaternary scheme which reproduces quadratic polynomials and is better than the symmetric binary scheme from the previous section in many aspects.

According to (25) and (27), the symbol of any quaternary scheme $S_{\boldsymbol{a}}$ with quadratic reproduction (i.e., $m=4$ and $n=l=2$) is

$$a(z) = \frac{(1+z+z^2+z^3)^3}{16}b(z), \quad b(z) = \big(b_0 + b_1 z + b_2 z^2 + (z-1)^3 r(z)\big)$$

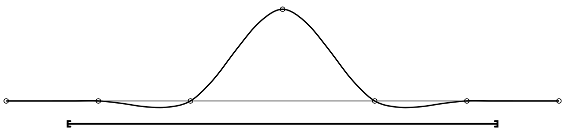

Fig. 3. Basic limit function of the quaternary 4-point scheme and its support.

with

$$b_0 = (\tau - 4)(\tau - 8)/2, \qquad b_1 = 1 - b_0 - b_2, \qquad b_2 = (\tau - 3)(\tau - 7)/2$$

and some polynomial $r(z)$. By (3), the support size of the scheme is less than 5 if r is at most a quadratic polynomial, hence $b \in \pi_5$. Moreover, $S_{\boldsymbol{a}}$ is symmetric if and only if the six coefficients of $b(z)$ are symmetric, and that turns out to be the case only if $\tau = 7$ and $r(z) = w - 3/2 - z - wz^2$ for some $w \in \mathbb{R}$. By construction, $S_{\boldsymbol{b}}$ is the subdivision scheme for the differences of the second divided differences of $S_{\boldsymbol{a}}$. Hence we would like $S_{\boldsymbol{b}}$ to be contractive, so that the limit functions of $S_{\boldsymbol{a}}$ are C^2-continuous. This can be achieved, for example, by choosing $w = 5/12$ which gives $\|S_{\boldsymbol{b}}\|_\infty = 2/3$.

The resulting quaternary scheme $S_{\boldsymbol{a}}$ has the mask

$$\{-5, -12, -13, 0, 45, 108, 165, 192, 165, 108, 45, 0, -13, -12, -5\}/192, \tag{33}$$

and approximation order 3 because it reproduces quadratic polynomials by construction. Its basic limit function is C^2 with Hölder regularity at least 2.2924 and the size of its support is 14/3; see Fig. 3. So, if we compare it to the symmetric binary scheme from the previous section, it has the same approximation order, higher smoothness, and smaller support. Moreover, it is clear from (33) that $S_{\boldsymbol{a}}$ is interpolatory and that one subdivision step for a sequence of n points requires $12n$ multiplications and $9n$ additions. In contrast, computing the same number of $4n$ refined points with the binary 3-point scheme requires two subdivision steps with a total cost of $18n$ multiplications and $9n$ additions, so the quaternary scheme is also more efficient. To conclude, we should point out that $S_{\boldsymbol{a}}$ is a true quaternary scheme and not an iterated, two-step binary scheme: if the symbol $\tilde{a}(z)$ of a binary scheme has degree d, then the degree of the iterated symbol $\tilde{a}(z)\tilde{a}(z^2)$ is $3d$, but the symbol of $S_{\boldsymbol{a}}$ has degree 14, which is clearly not a multiple of 3.

References

[1] A.S. Cavaretta, W. Dahmen, C.A. Micchelli, Stationary subdivision, Mem. Amer. Math. Soc. 93 (453) (1991) vi+186.

[2] G.M. Chaikin, An algorithm for high speed curve generation, Comput. Graph. Image Process. 3 (4) (1974) 346–349.

[3] S.W. Choi, B.-G. Lee, Y.J. Lee, J. Yoon, Stationary subdivision schemes reproducing polynomials, Comput. Aided Geom. Design 23 (4) (2006) 351–360.

[4] G. Deslauriers, S. Dubuc, Symmetric iterative interpolation processes, Constr. Approx. 5 (1) (1989) 49–68.

[5] B. Dong, Z. Shen, Pseudo-splines, wavelets and framelets, Appl. Comput. Harmon. Anal. 22 (1) (2007) 78–104.

[6] S. Dubuc, Interpolation through an iterative scheme, J. Math. Anal. Appl. 114 (1) (1986) 185–204.

[7] N. Dyn, Subdivision schemes in computer-aided geometric design, in: W. Light (Ed.), Advances in Numerical Analysis, vol. II, Oxford University Press, New York, 1992, pp. 36–104.

[8] N. Dyn, M.S. Floater, K. Hormann, A C^2 four-point subdivision scheme with fourth order accuracy and its extensions, in: M. Dæhlen, K. Mørken, L.L. Schumaker (Eds.), Mathematical Methods for Curves and Surfaces, Tromsø 2004, in: Modern Methods in Mathematics, Nashboro Press, Brentwood, TN, 2005, pp. 145–156.

[9] N. Dyn, K. Hormann, M.A. Sabin, Z. Shen, Polynomial reproduction by symmetric subdivision schemes, J. Approx. Theory 155 (1) (2008) 28–42.

[10] N. Dyn, D. Levin, Subdivision schemes in geometric modelling, Acta Numer. 11 (2002) 73–144.

[11] K. Faheem, G. Mustafa, Ternary six-point interpolating subdivision scheme, Lobachevskii J. Math. 29 (3) (2008) 153–163.

[12] B. Han, R.-Q. Jia, Multivariate refinement equations and convergence of subdivision schemes, SIAM J. Math. Anal. 29 (5) (1998) 1177–1199.

[13] M.F. Hassan, Further analysis of ternary and 3-point univariate subdivision schemes, Technical Report UCAM-CL-TR-599, University of Cambridge, Computer Laboratory, August 2004.

[14] M.F. Hassan, N.A. Dodgson, Ternary and three-point univariate subdivision schemes, in: A. Cohen, J.-L. Merrien, L.L. Schumaker (Eds.), Curve and Surface Fitting, Saint-Malo 2002, in: Modern Methods in Mathematics, Nashboro Press, Brentwood, TN, 2003, pp. 199–208.

[15] M.F. Hassan, I.P. Ivrissimtzis, N.A. Dodgson, M.A. Sabin, An interpolating 4-point C^2 ternary stationary subdivision scheme, Comput. Aided Geom. Design 19 (1) (2002) 1–18.

[16] K. Hormann, M.A. Sabin, A family of subdivision schemes with cubic precision, Comput. Aided Geom. Design 25 (1) (2008) 41–52.

[17] I. Ivrissimtzis, M.A. Sabin, N.A. Dodgson, On the support of recursive subdivision, ACM Trans. Graph. 23 (4) (2004) 1043–1060.

[18] K.P. Ko, B.-G. Lee, G.J. Yoon, A study on the mask of interpolatory symmetric subdivision schemes, Appl. Math. Comput. 187 (2) (2007) 609–621.

[19] K.P. Ko, B.-G. Lee, G.J. Yoon, A ternary 4-point approximating subdivision scheme, Appl. Math. Comput. 190 (2) (2007) 1563–1573.

[20] J.C. Lagarias, Y. Wang, The finiteness conjecture for the generalized spectral radius of a set of matrices, Linear Algebra Appl. 214 (1995) 17–42.

[21] A. Levin, Polynomial generation and quasi-interpolation in stationary non-uniform subdivision, Comput. Aided Geom. Design 20 (1) (2003) 41–60.

[22] J. Lian, On a-ary subdivision for curve design: I. 4-point and 6-point interpolatory schemes, Appl. Appl. Math. 3 (1) (2008) 18–29.

[23] J. Lian, On a-ary subdivision for curve design: II. 3-point and 5-point interpolatory schemes, Appl. Appl. Math. 3 (2) (2008) 176–187.

[24] G. Mustafa, F. Khan, A new 4-point C^3 quaternary approximating subdivision scheme, Abstr. Appl. Anal. (2009) Article ID 301967, 14 pages.

[25] O. Rioul, Simple regularity criteria for subdivision schemes, SIAM J. Math. Anal. 23 (6) (1992) 1544–1576.

[26] T. Sauer, Multivariate refinable functions, differences and ideals—a simple tutorial, J. Comput. Appl. Math. 221 (2) (2008) 447–459.

[27] G. Strang, G.J. Fix, A Fourier analysis of the finite element variational method, in: G. Geymonat (Ed.), Constructive Aspects of Functional Analysis, in: Centro Internazionale Matematico Estivo, vol. 2, Edizioni Cremonese, Roma, 1973, pp. 795–840.

[28] H. Wang, K. Qin, Improved ternary subdivision interpolation scheme, Tsinghua Sci. Technol. 10 (1) (2005) 128–132.

[29] H. Zheng, H. Zhao, Z. Ye, M. Zhou, Differentiability of a 4-point ternary subdivision scheme and its application, IAENG Int. J. Appl. Math. 36 (1) (2007) Article 4.