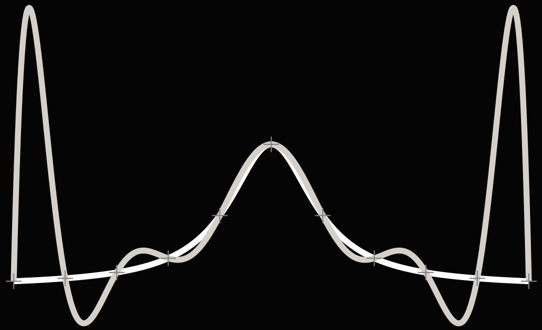
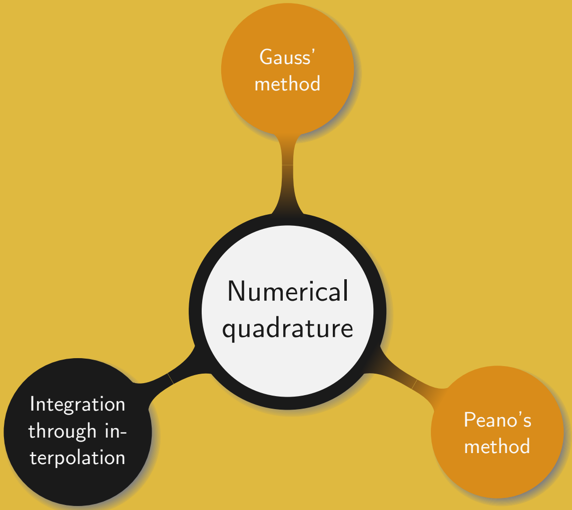


Introduction to Numerical Methods

4. Numerical quadrature

Ailin & Manuel – Summer 2022


Gauss' method
Numerical quadrature
Integration through interpolation
Peano's method

Let $[a, b]$ be a compact from $\mathbb{R}$, f be a continuous function from $[a, b]$ into $\mathbb{R}$, and w be a weight function over (a, b).

For $\alpha, \beta \in [a, b]$ we want to approximate

$$I(f) = \int_{\alpha}^{\beta} f(x)w(x)\,\mathrm{d}x,$$

by a formula of type

$$I_k(f) = \sum_{i=0}^{k} \lambda_{i,k} f(x_i),$$

where the x_i, $0 \leq i \leq n$ are nodes from $[a, b]$ (either given or to be determined), and the λ_i are to be calculated.

Remark. The maps $I_k : C[a, b] \to \mathbb{R}$, $f \mapsto I_k(f)$ are linear forms.

Definitions

Let $\left(I_k(f)\right)_{k\geq 0}$ be a sequence of approximation of $I(f)$.

1. The sequence $I_k(f)$ is *convergent* if $\lim_{k\to\infty} I_k(f) = I(f)$;
2. The *error of approximation* is given by $E_k(f) = I(f) - I_k(f)$;
3. An integration method is said to be of *order* N, $N \in \mathbb{N}$, if for any polynomial $P \in \mathbb{R}_N[x]$, $E_k(P) = 0$, $k \geq N$;
4. An integration method is said to be of *order exactly* N if it is of order N and there exists $P \in \mathbb{R}_{N+1}[x]$ such that $E_k(P) \neq 0$ for some $k \geq N$;

Remark. Since $I_k(f)$ is a linear application the order of an integration method can be easily determined by finding the largest N such that $E_k(x^N) = 0$ and $E_k(x^{N+1}) \neq 0$.
The order of a method measures how precise it is. A “good method” is expected to be convergent, precise, stable, and easily implementable.

When facing a “complex” integral the most straightforward solution would be to approximate the function by its interpolation polynomial and integrate it. Unfortunately the result quickly becomes inconsistent as the size of the interval of integration increases. To patch up this idea we divide up the initial interval $[\alpha, \beta]$ and *mesh* the domain with sub-intervals $[\alpha_i, \alpha_{i+1}]$. On each of those, the approximation of the function by a polynomial is expected to be precise enough to yield a consistent results.

In other words, for a function f continuous on $[a, b]$ and P its interpolation polynomial over $[\alpha_i, \alpha_{i+1}]$ we want to approximate

$$\int_{\alpha_i}^{\alpha_{i+1}} f(x)w(x)\,\mathrm{d}x \approx \int_{\alpha_i}^{\alpha_{i+1}} P(x)w(x)\,\mathrm{d}x.$$

Therefore this strategy will only work for “nice” weight functions.

Example. For the constant weight function $w(x) = 1$ we rewrite

$$\int_{\alpha}^{\beta} f(x)\,\mathrm{d}x = \sum_{i=0}^{k-1} \int_{\alpha_i}^{\alpha_{i+1}} f(x)\,\mathrm{d}x.$$

Assuming $f(x)$ is interpolated at $\xi_i \in [\alpha_i, \alpha_{i+1}]$ by a polynomial of degree 0 we obtain the approximation

$$\int_{\alpha_i}^{\alpha_{i+1}} f(x)\,\mathrm{d}x \approx \int_{\alpha_i}^{\alpha_{i+1}} f(\xi_i)\,\mathrm{d}x = (\alpha_{i+1} - \alpha_i)f(\xi_i). \tag{4.1}$$

We now evaluate how precise this formula is. Clearly if $f(x) = 1$ then the formula is exact. If $f(x) = x$ then we get

$$\int_{\alpha_i}^{\alpha_{i+1}} x \, \mathrm{d}x = \frac{(\alpha_{i+1} - \alpha_i)(\alpha_{i+1} + \alpha_i)}{2}.$$

From equation (4.1) we have to distinguish two cases: if $\xi_i \neq (\alpha_{i+1} + \alpha_i)/2$ then the formula is of order exactly 0; otherwise the formula is of order at least 1. But as for $f(x) = x^2$

$$\int_{\alpha_i}^{\alpha_{i+1}} x^2 \, \mathrm{d}x \neq \left(\frac{\alpha_{i+1} + \alpha_i}{2} \right)^2 (\alpha_{i+1} - \alpha_i),$$

it is of order exactly 1 and the *composed quadrature formula* over $[\alpha, \beta]$ is given by

$$\int_{\alpha}^{\beta} f(x) \approx \sum_{i=0}^{k} f(\xi_i)(\alpha_{i+1} - \alpha_i) = I_k.$$

Since this is a Riemann sum of f, $\lim_{k\to\infty} I_k = \int_\alpha^\beta f(x)\,\mathrm{d}x$. Hence the method converges.

Let f be a continuous function over $[\alpha,\beta] \subset [a,b]$. We want to derive a general formula in the case of a weight function $w(x) = 1$.

For the sake of simplicity we set $\alpha_{i+\frac{1}{2}} = (\alpha_i + \alpha_{i+1})/2$ and remap each of the $[\alpha_i, \alpha_{i+1}]$ into $[-1,1]$, a "reference interval", using the change of variable

$$\begin{aligned} \varphi_i^{-1} : [-1,1] &\longrightarrow [\alpha_i,\alpha_{i+1}] & \text{and} \quad \varphi_i : [\alpha_i,\alpha_{i+1}] &\longrightarrow [-1,1] \\ x &\longmapsto 2\frac{x-\alpha_{i+\frac{1}{2}}}{\alpha_{i+1}-\alpha_i}, & s &\longmapsto \frac{\alpha_{i+1}-\alpha_i}{2}s + \alpha_{i+\frac{1}{2}}. \end{aligned} \tag{4.2}$$

Hence, once the change of variable applied, the initial problem boils down to finding a quadrature formula for $\int_{-1}^1 \varphi(s)\,\mathrm{d}s$, when $\varphi \in C[-1,1]$.

We start by approximating φ by P_φ, its Lagrange interpolation polynomial at the l distinct nodes $\tau_1, \cdots, \tau_l \in [-1,1]$. Hence

$$P_\varphi(s) = \sum_{j=0}^{l} \varphi(\tau_j)\ell_j(s) \quad \text{with } \ell_j(s) = \prod_{\substack{i=0\\ i\neq j}}^{l} \frac{s-\tau_i}{\tau_j-\tau_i}.$$

The quadrature formula over $[-1,1]$ is then given by

$$\begin{aligned}\int_{-1}^{1} \varphi(s)\,\mathrm{d}s &\approx \int_{-1}^{1} P_\varphi(s)\,\mathrm{d}s \\ &= 2\sum_{j=0}^{l} \varphi(\tau_j) w_j, \quad \text{with } w_j = \frac{1}{2}\int_{-1}^{1} \ell_j(s)\,\mathrm{d}s.\end{aligned} \tag{4.3}$$

Using the change of variable (4.2) we obtain the quadrature formula over $[\alpha_i, \alpha_{i+1}]$.

Renaming $\alpha_{i+1} - \alpha_i$ into h_i we have

$$\int_{\alpha_i}^{\alpha_{i+1}} f(x)\,\mathrm{d}x = \frac{h_i}{2}\int_{-1}^{1} \varphi_i(s)\,\mathrm{d}s \approx h_i \sum_{j=0}^{l} \varphi_i(\tau_j) w_j = h_i \sum_{j=0}^{l} w_j f(\alpha_{ij}), \tag{4.4}$$

where $\alpha_{ij} = \tau_j \frac{h_i}{2} + \alpha_{i+\frac{1}{2}}$. By construction α_{ij} is in $[\alpha_i, \alpha_{i+1}]$ since τ_j is in $[-1, 1]$.

Therefore this yields the *elementary quadrature formula*

$$\int_{\alpha_i}^{\alpha_{i+1}} f(x)\,\mathrm{d}x \approx h_i \sum_{j=0}^{l} f(\alpha_{ij}) w_j,$$

while the associated composed formula is given by

$$\int_{\alpha}^{\beta} f(x)\,\mathrm{d}x \approx \sum_{i=0}^{k-1} h_i \sum_{j=0}^{l} f(\alpha_{ij}) w_j.$$

Following the derivation of the quadrature formula we now investigate its quality, starting with the study of its convergence.

Theorem

Let f be a continuous function over $[\alpha, \beta] \subset [a, b]$. Using the previous notations, let $(I_k)_{k\in\mathbb{N}}$ be the sequence defined by

$$I_k = \sum_{i=0}^{k-1} h_i \sum_{j=0}^{l} f(\alpha_{ij}) w_j.$$

Then $(I_k)_{k\in\mathbb{N}}$ converges toward $\int_\alpha^\beta f(x)\,\mathrm{d}x$ as k tends to infinity. Moreover the method of integration through interpolation over $l+1$ nodes has order at least l.

Proof. When rearranging the terms of I_k into

$$\sum_{i=0}^{k-1} h_i \sum_{j=0}^{l} f(\alpha_{ij}) w_j = \sum_{j=0}^{l} \left[\sum_{i=0}^{k-1} h_i f(\alpha_{ij}) \right] w_j,$$

one can observe that $\sum_{i=0}^{k-1} h_i f(\alpha_{ij})$ is in fact a Riemann sum.

Thus when k tends to infinity the step of the mesh h_i tends to zero and

$$\int_{\alpha}^{\beta} f(x)\,\mathrm{d}x = \lim_{k\to\infty} \sum_{i=0}^{k-1} h_i f(\alpha_{ij}).$$

Hence

$$\lim_{k\to\infty} I_k = \sum_{j=0}^{l} \left(w_j \int_{\alpha}^{\beta} f(x)\,\mathrm{d}x \right). \tag{4.5}$$

From equation (4.5) it remains to prove that $\sum_{j=0}^{l} w_j = 1$. Recalling that $w_j = \frac{1}{2}\int_{-1}^{1} \ell_j(s)\,\mathrm{d}s$, it is clear that w_j does not depend upon the choice of the function φ.

In particular in the case where $\varphi(s) = 1$ over $[-1, 1]$, P_φ is also equal to 1 and by the quadrature formula (4.3)

$$\int_{-1}^{1} P_\varphi(s)\,\mathrm{d}s = 2\sum_{j=0}^{l} w_j = 2.$$

This proves that $\lim_{k\to\infty} I_k = \int_\alpha^\beta f(x)\,\mathrm{d}x$.

The order of the method is trivially seen as soon as one notices that $\varphi = P_\varphi$ when $\varphi \in \mathbb{R}_l[x]$. $\square$

Theorem

If w_j is strictly positive for any j in $\{0, \cdots, l\}$, then the integration method is stable with respect to the variations of the continuous function f to be integrated.

Proof. Assuming $I_k(f)$ is given by

$$I_k(f) = \sum_{i=0}^{k-1} (\alpha_{i+1} - \alpha_i) \sum_{j=0}^{l} f(\alpha_{ij}) w_j,$$

a small perturbation on f changes $f(\alpha_{ij})$ into $\widetilde{f_{ij}}$ yielding

$$I_k(\widetilde{f}) \sum_{i=0}^{k-1} (\alpha_{i+1} - \alpha_i) \sum_{j=0}^{l} \widetilde{f_{ij}} w_j.$$

Looking at the difference between $I_k(\widetilde{f})$ and $I_k(f)$ we get

$$\left|I_k(\widetilde{f}) - I_k(f)\right| \leq \sum_{i=0}^{k-1}(\alpha_{i+1} - \alpha_i)\sum_{j=0}^{l}\left|f(\alpha_{ij}) - \widetilde{f}_{ij}\right| w_j.$$

Then denoting by ε the maximum of $|f(\alpha_{ij}) - \widetilde{f}_{ij}|$, and recalling that w_j is strictly positive we see that the method is stable since

$$\left|I_k(\widetilde{f}) - I_k(f)\right| \leq \varepsilon\sum_{i=0}^{k-1}(\alpha_{i+1} - \alpha_i)\sum_{j=0}^{l} w_j = \varepsilon(\beta - \alpha).$$

$\square$

For φ in $C^{l+1}[-1,1]$, one can directly determine

$$E(\varphi) = \int_0^1 \varphi(s)\,\mathrm{d}s - \int_0^1 P_\varphi(s)\,\mathrm{d}s.$$

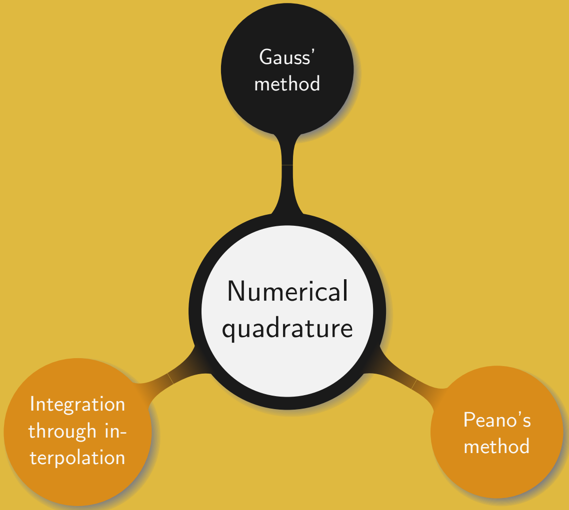
That is calculate

$$\varphi(s) - P_\varphi(s) = \frac{\varphi^{(l+1)}(\xi_x)}{(l+1)!} \prod_{j=0}^{l} (s - \tau_j),$$

and then study the result in order to find the most optimal choice of nodes and "control" $E(\varphi)$.

In section 3 we will investigate a more general approach which covers integration through interpolation and Gauss' method. In particular results on the study of the error will be proven there.

Chapter organisation



Given f a continuous function over $[a, b]$, and w a weight function over $(\alpha, \beta) \subset [a, b]$, we want to approximate the integral $\displaystyle\int_\alpha^\beta f(x)w(x)\,\mathrm{d}x$ by a quadrature formula of the form $\displaystyle\sum_{i=0}^{k} \lambda_i f(x_i)$, for some numbers λ_i, $0 \leq i \leq k$, where k is a positive integer.

For l smaller than k, Gauss method chooses $k - l + 1$ points, among the $k + 1$ nodes, outside (α, β). The goal is then to find all the nodes and λ_i such that the integration formula

$$\int_\alpha^\beta f(x)w(x)\,\mathrm{d}x \approx \sum_{i=0}^{k} \lambda_i f(x_i)$$

is of order at least $k + l$.

Theorem

Let k and l be two integers such that $k \geq l-1$ and $x_0, \cdots, x_{l-1}$ be in $[\alpha, \beta]$ while $x_l, \cdots, x_k$ are not in this interval. For x in (α, β), let $\Pi_{k,l}(x)$ denote the product $\prod_{j=l}^{k}(x - x_j)$. Then there exists a quadrature formula that is uniquely determined by $x_l, \cdots, x_k$. It is of the form

$$\sum_{i=0}^{k} \lambda_i f(x_i) \approx \int_{\alpha}^{\beta} f(x) w(x) \, \mathrm{d}x$$

and has order $k+l$. Moreover the x_i, $0 \leq i \leq l-1$, are the roots of the $(l+1)$th orthogonal polynomial P_l associated to the weight

$$\theta(x) = \operatorname{sign}\left(\Pi_{k,l}(x)\right) \Pi_{k,l}(x) w(x), \quad x \in (\alpha, \beta).$$

Proof. Before proving the existence and unicity of the quadrature formula we first need to verify that

$$\forall x \in (\alpha, \beta) \quad \theta(x) = \operatorname{sign}\left(\Pi_{k,l}(x)\right) \Pi_{k,l}(x) w(x)$$

is a weight function.

Since $x_l, \cdots, x_k \notin (\alpha, \beta)$ we see that $\Pi_{k,l}(x) = \prod_{j=l}^{k}(x - x_j)$ does not change sign over (α, β). So its sign function is either equal to 1 or -1. Either way, $\theta(x)$ is strictly positive over (α, β). Moreover

$$\int_\alpha^\beta \theta(x)\,\mathrm{d}x \leq \max_t \left|\Pi_{k,l}(t)\right| \int_\alpha^\beta w(x)\,\mathrm{d}x.$$

As $w(x)$ is a weight function over $(a, b) \supset (\alpha, \beta)$ it is clear that $\int_\alpha^\beta \theta(x)\,\mathrm{d}x < \infty$. Therefore $\theta(x)$ is a weight function over (α, β).

We now focus on the unicity of the quadrature formula, i.e. we prove that if the formula exists and is of order $k+l$, then the x_i, $0 \leq i < l$ and λ_i, $0 \leq i \leq k$ are unique.

Let $\widetilde{P}_l(x) = \prod_{j=0}^{l-1}(x - x_j)$. We want to prove that $\widetilde{P}_l = P_l$, the $(l+1)$st orthogonal polynomial associated to the weight $\theta(x)$, i.e.

$$\int_\alpha^\beta \widetilde{P}_l(x) Q(x) \theta(x)\, \mathrm{d}x = 0, \quad \forall Q \in \mathbb{R}_{l-1}[x].$$

Let Q be a polynomial of degree less than $l-1$. As the formula is of order $k+l$ and of the form $\int_\alpha^\beta f(x) w(x)\, \mathrm{d}x \approx \sum_{i=0}^k \lambda_i f(x_i)$ then

$$\int_\alpha^\beta \widetilde{P}_l(x) Q(x) \theta(x)\, \mathrm{d}x = \mathrm{sign}\left(\Pi_{k,l}(x)\right) \int_\alpha^\beta \prod_{j=0}^{k}(x - x_j) Q(x) w(x)\, \mathrm{d}x$$

$$= \sum_{j=0}^{k} \lambda_j \cdot 0 = 0.$$

This proves that for any polynomial Q of degree at most $l-1$

$$\int_{\alpha}^{\beta} \widetilde{P}_l(x)Q(x)\theta(x)\,\mathrm{d}x = 0.$$

In other words $\widetilde{P}_l$ is an orthogonal polynomial, and as it is monic it is equal to P_l (proposition 2.53). In particular this means that the roots of $\widetilde{P}_l$, $x_0, \cdots, x_{l-1}$, are uniquely defined with respect to the weight function θ.

For the unicity of the λ_i observe that we have just demonstrated that the choice of the nodes $x_l, \cdots, x_k$ fixes $x_0, \cdots, x_{l-1}$. Thus all the x_i, $0 \leq i \leq k$ are fixed and known, such that we can consider

$$\ell_i(x) = \prod_{\substack{j=0\\ j\neq i}}^{k} \frac{x - x_j}{x_i - x_j}.$$

Since ℓ_i is a polynomial of degree $k \leq k + l$, the formula is exact and we obtain

$$\begin{aligned}\int_\alpha^\beta \ell_i(x) w(x) \, \mathrm{d}x &= \sum_{j=0}^{k} \lambda_i \ell_i(x_j) \\ &= \lambda_i \ell_i(x_i) = \lambda_i.\end{aligned} \tag{4.6}$$

As a result all the λ_i are uniquely determined by the x_i, $0 \leq i \leq k$.

From the previous parts of the demonstration we know that P_l is the $(l+1)st$ orthogonal polynomial associated to the weight function $\theta(x)$ over (α, β). Furthermore P_l has degree l and its roots $x_0, \cdots, x_{l-1}$ are all distinct.

For λ_i defined as in (4.6) it is left to prove that the quadrature formula

$$\int_\alpha^\beta f(x)w(x)\,\mathrm{d}x \approx \sum_{j=0}^{k} \lambda_i f(x_j)$$

has order at least $k + l$.
By construction it is clear that the method is of order at least k, since it suffices to consider the basis composed of the Lagrange polynomials ℓ_i. So let f be a polynomial of degree strictly larger than k, but less than $k+l$. It is then possible to proceed to the euclidean division of f by $\prod_{j=0}^{k}(x - x_j)$ and there exist q and R two polynomials such that

$$f(x) = q(x)\prod_{j=0}^{k}(x - x_j) + R(x), \quad \text{with} \begin{cases} \deg R \leq k \\ \deg q \leq l - 1. \end{cases}$$

Then the quadrature formula can be rewritten in terms of q and R

$$\int_\alpha^\beta f(x)w(x)\,\mathrm{d}x = \underbrace{\int_\alpha^\beta q(x)\prod_{j=0}^{k}(x-x_j)w(x)\,\mathrm{d}x}_{A} + \int_\alpha^\beta R(x)w(x)\,\mathrm{d}x.$$

Note that R has degree less than k. As in this case we already know the quadrature formula to be exact this yields

$$\int_\alpha^\beta R(x)w(x)\,\mathrm{d}x = \sum_{i=0}^{k}\lambda_i R(x_i) = \sum_{i=0}^{k}\lambda_i f(x_i).$$

Therefore for the quadrature formula to be of order $k+l$ we need to show that A equals 0.

Recalling that θ does not change sign over (α, β) (slide 4.20)

$$\begin{aligned}\prod_{j=0}^{k}(x-x_j)w(x) &= \prod_{j=0}^{l-1}(x-x_j)\prod_{j=l}^{k}(x-x_j)w(x)\\ &= P_l(x)\,\mathrm{sign}\left(\Pi_{k,l}(x)\right)\theta(x),\end{aligned}$$

such that we get

$$A = \int_{\alpha}^{\beta} q(x)P_l(x)\,\mathrm{sign}\left(\Pi_{k,l}(x)\right)\theta(x)\,\mathrm{d}x.$$

Since q is of degree less than $l-1$, P_l is orthogonal to q for the weight θ. Hence $A = 0$ and this concludes the proof. $\square$

Corollary

When $k = l - 1$, i.e. no node is taken out of (α, β), there exists a unique quadrature formula of the form

$$\int_\alpha^\beta f(x)w(x)\,\mathrm{d}x \approx \sum_{i=0}^{k} \lambda_i f(x_i),$$

and with order $2k+1$. The $x_i \in (\alpha, \beta)$, $0 \leq i \leq k$, are the roots of the $(k+2)$nd orthogonal polynomial associated to the weight w. Moreover all the λ_i are strictly positive.

Proof. The only part of the corollary that remains to be proven is that all the λ_i, $0 \leq i \leq k$, are strictly larger than 0.

From the first part of the result we know that the quadrature formula is of order at least $2k+1$, which is larger than the degree of ℓ_i^2. Thus, as the formula is exact we have

$$\int_\alpha^\beta \ell_i^2(x)w(x)\,\mathrm{d}x = \sum_{j=0}^{k} \lambda_j \ell_i^2(x_j) = \lambda_i.$$

Hence noting that $\ell_i^2(x)w(x)$ is strictly positive over (α,β), we have proven that λ_i is strictly positive. $\square$

Following the existence and unicity we now focus on the quality of Gauss' integration method and estimating its error.

Theorem

Let $x_0, \cdots, x_{l-1}$ be the nodes from (α, β) and $x_l, \cdots, x_k$ be the points of $[a, b]$ exterior to (α, β). Then for any function $f \in C^{k+l+1}[a, b]$ there exists ξ in $[a, b]$ such that

$$E(f) = \frac{f^{(k+l+1)}(\xi)}{(k+l+1)!} \int_\alpha^\beta \prod_{i=0}^{l-1} (x - x_i)^2 \prod_{j=l}^{k} (x - x_j) w(x)\, dx.$$

Proof. Let P be the Hermite interpolation polynomial of f over $[a, b]$ at the nodes $x_0, \cdots, x_{l-1}, x_l, \cdots, x_k$, with

$$P(x_i) = f(x_i), \quad \forall 0 \leq i \leq k,$$
$$P'(x_i) = f'(x_i), \quad \forall 0 \leq i \leq l-1.$$

Since Gauss' method has order at least $k+l$ we have

$$\int_\alpha^\beta P(x)w(x)\,\mathrm{d}x = \sum_{i=0}^{k} \lambda_i P(x_i) = \sum_{i=0}^{k} \lambda_i f(x_i),$$

and we can write

$$\begin{aligned} E(f) &= \int_\alpha^\beta f(x)w(x)\,\mathrm{d}x - \sum_{i=0}^{k} \lambda_i f(x_i) \\ &= \int_\alpha^\beta f(x)w(x)\,\mathrm{d}x - \int_\alpha^\beta P(x)w(x)\,\mathrm{d}x \\ &= \int_\alpha^\beta \big(f(x) - P(x)\big)\, w(x)\,\mathrm{d}x. \end{aligned}$$

From the error formula for Hermite interpolation polynomial (theorem 3.37) we know that for all $x \in [a, b]$ there exists $\xi_x \in [a, b]$ such that

$$f(x) - P(x) = \frac{f^{(k+l+1)}(\xi_x)}{(k+l+1)!}\Pi(x),$$

with $\Pi(x) = \prod_{i=0}^{k}(x - x_i)^{\alpha_i+1}$.

Note that in our case α_i is 0 for $i \in \{l, \cdots, k\}$ and 1 for $i \in \{0, \cdots, l-1\}$.

Without loss of generality we now assume $\int_\alpha^\beta \Pi(x)w(x)\,\mathrm{d}x$ to be strictly positive. This is possible since Π has a constant sign over (α, β).

Then, as $f \in C^{(k+l+1)}[a,b]$, the extreme values theorem (2.14) can be applied and

$$\frac{m}{(k+l+1)!}\int_\alpha^\beta \Pi(x)w(x)\,\mathrm{d}x \le E(f) \le \frac{M}{(k+l+1)!}\int_\alpha^\beta \Pi(x)w(x)\,\mathrm{d}x,$$

with $m = \min_{x\in[a,b]} f^{(k+l+1)}(x)$ and $M = \max_{x\in[a,b]} f^{(k+l+1)}(x)$.

Hence

$$m \le (k+l+1)!\frac{E(f)}{\int_\alpha^\beta \Pi(x)w(x)\,\mathrm{d}x} \le M. \tag{4.7}$$

Applying the intermediate values theorem (2.11) to $f^{(k+l+1)}$, for any q in $[m,M]$, there exists ξ such that $f^{(k+l+1)}(\xi) = q$.

In particular from (4.7) we know that $(k+l+1)!\frac{E(f)}{\int_\alpha^\beta \Pi(x)w(x)\,\mathrm{d}x}$ is in $[m, M]$, meaning there exists $\xi \in [a, b]$ such that

$$E(f) = \frac{f^{(k+l+1)}(\xi)}{(k+l+1)!}\int_\alpha^\beta \prod_{i=0}^{l-1}(x-x_i)^2 \prod_{j=l}^{k}(x-x_j)w(x)\,\mathrm{d}x.$$

□

In particular from (4.7) we know that $(k+l+1)!\frac{E(f)}{\int_\alpha^\beta \Pi(x)w(x)\,\mathrm{d}x}$ is in $[m, M]$, meaning there exists $\xi \in [a, b]$ such that

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□

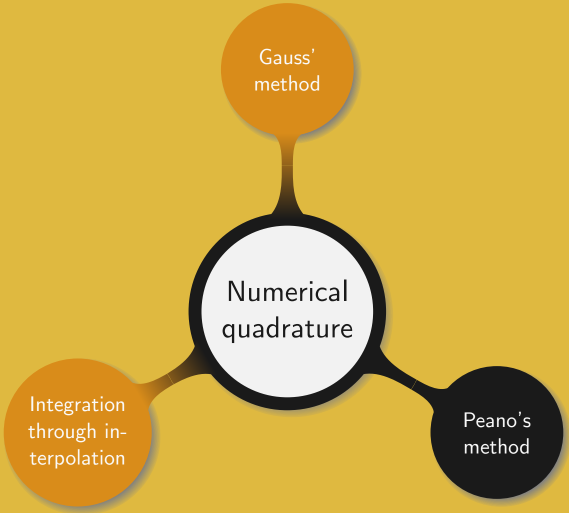
Corollary

Gauss' method has order exactly $k+l$.

Proof. It is clear since the method is of order at least $k+l$ and

$$E(x \mapsto x^{k+l+1}) = \frac{(k+l+1)!}{(k+l+1)!}\int_\alpha^\beta \Pi(x)w(x)\,\mathrm{d}x \neq 0.$$

□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In this section we investigate a generic integration method where a minimum amount of assumptions is assumed. In particular it covers both integration through interpolation and Gauss' method.

Let $w(x)$ be a weight function over $(\alpha, \beta) \subset [a, b]$ and f be a function defined almost everywhere on $[a, b]$ and which is Lebesgue integrable on (α, β) with respect to the weight w. We are interested in the generic quadrature formula

$$\int_\alpha^\beta f(x)w(x)\,\mathrm{d}x \approx \sum_{i=0}^{k} \lambda_i f(x_i), \quad \lambda_i \in \mathbb{R}, x_i \in [a, b], \forall 0 \leq i \leq k.$$

This method is called *Peano's method*.

Definition

The application K_N, called *Peano kernel*, is defined by

$$K_N(t) = E(x \in [\alpha, \beta] \mapsto [(x-t)_+]^N),$$

where

$$[(x-t)_+]^N = \begin{cases} (x-t)^N, & \text{if } t \leq x \\ 0, & \text{if } t > x \end{cases}$$

Definition

The application K_N, called *Peano kernel*, is defined by

$$K_N(t) = E(x \in [\alpha, \beta] \mapsto [(x-t)_+]^N),$$

where

$$[(x-t)_+]^N = \begin{cases} (x-t)^N, & \text{if } t \leq x \\ 0, & \text{if } t > x \end{cases}$$

Theorem (Peano kernel theorem)

Assuming Peano's method is of positive order N, f is a $C^{N+1}[a, b]$ function, and $f^{(N)}$ is absolutely continuous then

$$E(f) = \frac{1}{N!} \int_a^b K_N(t) f^{(N+1)}(t) \, \mathrm{d}t.$$

Proof. From Taylor's theorem (2.36) f can be rewritten

$$f(x) = P_N(x) + \frac{1}{N!}\int_a^x (x-t)^N f^{(N+1)}(t)\,\mathrm{d}t,$$

with $P_N \in \mathbb{R}_N[x]$.

Replacing $(x-t)$ by $(x-t)_+$ we make the range of integration independent of x to obtain

$$f(x) = P_N(x) + \frac{1}{N!}\int_a^b [(x-t)_+]^N f^{(N+1)}(t)\,\mathrm{d}t.$$

Using the definition of f we now evaluate the error of Peano's method.

The error is given by

$$E(f) = E(P_N) + \frac{1}{N!} E\left(x \mapsto \int_a^b \left[(x-t)_+\right]^N f^{(N+1)}(t)\,\mathrm{d}t\right)$$

$$= \frac{1}{N!}\left[\int_\alpha^\beta \left(\int_a^b \left[(x-t)_+\right]^N f^{(N+1)}(t)\,\mathrm{d}t\right) w(x)\,\mathrm{d}x \right.$$

$$\left. - \sum_{i=0}^k \lambda_i \int_a^b \left[(x_i-t)_+\right]^N f^{(N+1)}(t)\,\mathrm{d}t\right].$$

By the linearity of integration and Fubini's theorem (2.33)

$$E(f) = \frac{1}{N!}\int_a^b \left[\left(\int_\alpha^\beta \left[(x-t)_+\right]^N w(x)\,\mathrm{d}x\right) f^{(N+1)}(t) \right.$$

$$\left. - \sum_{i=0}^k \lambda_i \left[(x_i-t)_+\right]^N f^{(N+1)}(t)\right] \mathrm{d}t.$$

Rearranging the terms, Peano kernel appears, yielding the result

$$E(f) = \frac{1}{N!}\int_a^b \left[\int_\alpha^\beta \left[(x-t)_+\right]^N w(x)\,\mathrm{d}x - \sum_{i=0}^k \lambda_i \left[(x_i - t)_+\right]^N\right] f^{(N+1)}(t)\,\mathrm{d}t$$

$$= \frac{1}{N!}\int_a^b K_N(t) f^{(N+1)}(t)\,\mathrm{d}t.$$

$\square$

Rearranging the terms, Peano kernel appears, yielding the result

$$E(f) = \frac{1}{N!}\int_a^b \left[\int_\alpha^\beta \left[(x-t)_+\right]^N w(x)\,\mathrm{d}x - \sum_{i=0}^{k}\lambda_i \left[(x_i-t)_+\right]^N\right] f^{(N+1)}(t)\,\mathrm{d}t$$

$$= \frac{1}{N!}\int_a^b K_N(t) f^{(N+1)}(t)\,\mathrm{d}t.$$

□

Corollary

Under the same assumptions as Peano kernel theorem we have

$$|E(f)| \leq \frac{1}{N!} \sup_{t\in[a,b]} |f^{(N+1)}(t)| \int_a^b |K_N(t)|\,\mathrm{d}t.$$

Corollary

Under the same assumptions as Peano kernel theorem and assuming Peano kernel does not change sign, there exists $\xi \in (a, b)$ such that

$$E(f) = \frac{f^{(N+1)}(\xi)}{(N+1)!} E(x \mapsto x^{N+1}).$$

Proof. Since K_N is integrable and has a constant sign over $[a, b]$, while $f^{(N+1)}$ is continuous we can apply the first mean value theorem for integrals (theorem 2.34). Then there exists $\xi \in [a, b]$ such that

$$E(f) = \frac{f^{(N+1)}(\xi)}{N!} \int_a^b K_N(t)\,\mathrm{d}t. \tag{4.8}$$

Determining (4.8) for $f(x) = x^{N+1}$ gives

$$\begin{aligned} E\left(x \mapsto x^{N+1}\right) &= \frac{(N+1)!}{N!} \int_a^b K_N(t)\,\mathrm{d}t \\ &= (N+1) \int_a^b K_N(t)\,\mathrm{d}t. \end{aligned}$$

Thus replacing $\int_a^b K_N(t)\,\mathrm{d}t$ by $E(x \mapsto x^{N+1})/(N+1)$ completes the proof.

$\square$

Example. Determine Peano kernel for the rectangle method

$$\int_\alpha^\beta f(x)\,\mathrm{d}x \approx f(\xi)(\beta - \alpha), \quad \text{for some } \xi \in [\alpha, \beta].$$

Reminding that the rectangle method has been investigated in example 4.6 we already know that it is of order 0 if ξ is not $\frac{\alpha+\beta}{2}$ and 1 otherwise.

Case $\xi \neq \frac{\alpha+\beta}{2}$:

$$\begin{aligned} K_0(t) &= E(x \mapsto [(x-t)_+]^0) \\ &= \int_\alpha^\beta [(x-t)_+]^0 \, \mathrm{d}x - [(\xi - t)_+]^0(\beta - \alpha) \\ &= \int_t^\beta \mathrm{d}x - [(\xi - t)_+]^0(\beta - \alpha) \\ &= (\beta - t) - [(\xi - t)_+]^0(\beta - \alpha) \end{aligned}$$

Now if $t \leq \xi$ then $[(\xi - t)_+]^0$ is 1, implying $K_0(t) = \alpha - t$. And when $t > \xi$ then $[(\xi - t)_+]^0$ is 0, such that $K_0(t) = \beta - t$.

Case $\xi = \frac{\alpha+\beta}{2}$:

$$\begin{aligned} K_1(t) &= E(x \mapsto [(x-t)_+]^1) \\ &= \int_\alpha^\beta (x-t)_+ \,\mathrm{d}x - \left(\frac{\alpha+\beta}{2} - t\right)_+ (\beta - \alpha) \\ &= \int_t^\beta x - t \,\mathrm{d}x - \left(\frac{\alpha+\beta}{2} - t\right)_+ (\beta - \alpha) \\ &= \frac{(\beta-t)^2}{2} - \left(\frac{\alpha+\beta}{2} - t\right)_+ (\beta - \alpha) \end{aligned}$$

Now if $t \leq \xi$ then $[(\xi - t)_+]^1$ is $\left(\frac{\alpha+\beta}{2} - t\right)$, implying $K_1(t) = \frac{(t-\alpha)^2}{2}$.

And when $t > \xi$ then $[(\xi - t)_+]^1$ is 0, such that $K_1(t) = \frac{(\beta-t)^2}{2}$.

Theorem

In the case of Gauss' method, Peano kernel keeps a constant sign on $[\alpha, \beta]$.

Proof. Suppose Peano kernel changes sign on $[\alpha, \beta]$.

In that case there exists a continuous function φ over $[a, b]$ such that for all $x \in [a, b]$

$$\int_\alpha^\beta K_N(t)\varphi(t)\,\mathrm{d}t \neq \varphi(x) \int_\alpha^\beta K_N(t)\,\mathrm{d}t. \tag{4.9}$$

This is indeed true as it suffices to take $\varphi(x) = K_N(x)_+$ when $\int_\alpha^\beta K_N(t)\,\mathrm{d}t \leq 0$ and $\varphi(x) = K_N(x)_+ - K_N(x)$ otherwise.

Let $f \in C^{N+1}[a, b]$ such that $f^{(N+1)} = \varphi$ on $[a, b]$.

Thus combining Peano kernel theorem (4.36) together with equation (4.9) we see that for all $x \in [a, b]$

$$E(f) = \frac{1}{N!}\int_a^b K_N(t) f^{(N+1)}(t)\,\mathrm{d}t \neq \frac{\varphi(x)}{N!}\int_\alpha^\beta K_N(t)\,\mathrm{d}t. \tag{4.10}$$

Then from theorem 4.29, the error of Gauss' method is of the form

$$E(f) = \frac{\varphi(\xi)}{(N+1)!}\int_\alpha^\beta \Pi(x) w(x)\,\mathrm{d}x. \tag{4.11}$$

Next see that $\int_\alpha^\beta \Pi(x) w(x)\,\mathrm{d}x$ is precisely $E(x \mapsto x^{N+1})$, while $E(x \mapsto x^{N+1})$ is $(N+1)\int_a^b K_N(t)\,\mathrm{d}t$ (theorem 4.36).

Hence (4.11) can be rewritten

$$
\begin{aligned}
E(f) &= \frac{\varphi(\xi)}{N!}\int_a^b K_N(t)\,\mathrm{d}t \\
&= \frac{\varphi(\xi)}{N!}\left(\underbrace{\int_a^\alpha K_N(t)\,\mathrm{d}t}_{A} + \int_\alpha^\beta K_N(t)\,\mathrm{d}t + \underbrace{\int_\beta^b K_N(t)\,\mathrm{d}t}_{B}\right).
\end{aligned}
$$

From the definition of Peano kernel, B is 0: if $t \geq \beta$ then for all $x \leq \beta$, $(x-t)_+^N = 0$ implying $K_N(t) = 0$.

From the order of Gauss method, A is 0: if $t \leq \alpha$ then $(x-t)_+^N$ is $(x-t)^N$ and as the method has order N, the formula is exact.

As a result we have shown that

$$E(f) = \frac{1}{N!}\int_a^b K_N(t)\varphi(t)\,\mathrm{d}t = \frac{\varphi(\xi)}{N!}\int_\alpha^\beta K_N(t)\,\mathrm{d}t.$$

This contradicts (4.10) which should have been valid for all $x \in [a, b]$, including ξ.↯ $\square$

We now refocus our attention on the method of integration through interpolation, and determine the expression of the error. A particularly interesting result states that the error depends on the step of the mesh.

From the general idea behind Peano's method it is clear that integration through interpolation is a special case of this more general method. As such Peano kernel theorem (4.36) holds and we can define both the *elementary* and *composite errors*, corresponding the elementary and composite quadrature formulae.

The following discussion reuses the notations introduced in slides 4.8 to 4.10. If the method is of order $N \geq l \geq 0$, then the elementary error for $\varphi \in C^{N+1}[-1,1]$ is

$$E_{el}(\varphi) = \int_{-1}^{1} \varphi(s)\,\mathrm{d}s - 2\sum_{j=0}^{l} w_j \varphi(\tau_j) = \frac{1}{N!}\int_{-1}^{1} k_N(t)\varphi^{(N+1)}(t)\,\mathrm{d}t.$$

As for the composite quadrature formula, if the method is of order $N \geq l$, and $f \in C^{(N+1)}[\alpha,\beta]$, then the error is given by

$$E_{comp}(f) = \int_{\alpha}^{\beta} f(x)\,\mathrm{d}x - \sum_{i=0}^{k-1} h_i \sum_{j=0}^{l} w_j f(\alpha_{ij}) = \frac{1}{N!}\int_{\alpha}^{\beta} K_N(t) f^{(N+1)}(t)\,\mathrm{d}t.$$

Proposition

Let k_n and K_n denote Peano kernel in the case of the elementary error and composite error, respectively.

For all $t \in [\alpha_j, \alpha_{j+1}]$, $0 \leq \ j \leq k-1$

$$K_N(t) = \left(\frac{h_j}{2}\right) k_N\left(\frac{2}{h_j}\left(t - \alpha_{j+\frac{1}{2}}\right)\right),$$

with $h_j = \alpha_{j+1} - \alpha_j$ and $\alpha_{j+\frac{1}{2}} = \frac{\alpha_{j+1}+\alpha_j}{2}$.

Proof. Let t be fixed in one the $[\alpha_j, \alpha_{j+1}]$. By definition

$$K_N(t) = E_{comp}(x \mapsto [(x-t)_+]^N).$$

Based on equation (4.4) this expands as

$$K_N(t) = \int_\alpha^\beta [(x-t)_+]^N \,\mathrm{d}t - \sum_{i=0}^{k-1} \frac{h_i}{2} \cdot 2 \sum_{j=0}^{l} w_j \left[\left(\tau_j \frac{h_i}{2} + \alpha_{i+\frac{1}{2}} - t \right)_+ \right]^N .$$

Then defining $f_t(x) = [(x-t)_+]^N$ and applying the bijective map φ_i (4.2) yield

$$\begin{aligned} K_N(t) &= \sum_{i=0}^{k-1} \frac{h_i}{2} \left[\int_{-1}^{1} \varphi(s)\,\mathrm{d}s - 2 \sum_{j=0}^{l} w_j \varphi_i(\tau_j) \right] \\ &= \sum_{i=0}^{k-1} \frac{h_i}{2} E_{el}(\varphi_i). \end{aligned} \tag{4.12}$$

Therefore we now need to determine $E_{el}(\varphi_i)$.
Starting from the definition of φ_i (4.2) we need to find a simpler form to express $[(s\frac{h_i}{2}+\alpha_{i+\frac{1}{2}}-t)_+]^N$. For this we note that for any positive number λ we have $[(\lambda)_+]^N=\lambda^N$. Hence we can rewrite

$$\left[\left(s\frac{h_i}{2}+\alpha_{i+\frac{1}{2}}-t\right)_+\right]^N=\left(\frac{h_i}{2}\right)^N\left[\left(s-\frac{2}{h_i}(t-\alpha_{i+\frac{1}{2}})\right)_+\right]^N.$$

Then setting $\theta_i=\frac{2}{h_i}(t-\alpha_{i+\frac{1}{2}})$ implies

$$\left[\left(s\frac{h_i}{2}+\alpha_{i+\frac{1}{2}}-t\right)_+\right]^N=\left(\frac{h_i}{2}\right)^N\left[(s-\theta_i)_+\right]^N.$$

As a result equation (4.12) can be rewritten as

$$K_N(t) = \sum_{i=0}^{k-1} \frac{h_i}{2} \left(\left(\frac{h_i}{2} \right)^N E_{el}(s \mapsto [(s-\theta_i)_+]^N \right).$$

But as k_N is by definition $E_{el}(s \mapsto [(s-t)_+]^N)$ we obtain

$$K_N(t) = \sum_{i=0}^{k-1} \left(\frac{h_i}{2} \right)^{N+1} k_N(\theta_i). \tag{4.13}$$

To conclude the proof observe that for a fixed t, there is a unique j such that $t \in [\alpha_j, \alpha_{j+1}]$. So in the sum (4.13), $K_N(\theta_i)$ is to be calculated when $i = j$, i.e. $\theta_i \in [-1, 1]$, and when $i \neq j$, i.e. $\theta_i \notin [-1, 1]$.

If $\theta_i \notin [-1,1]$ then it means it is either smaller than -1 or lager than 1.

In the case where $\theta_i < -1$ then $[(s-\theta_i)_+]^N = (s-\theta_i)^N$ is a polynomial of degree N in s. But since the method is of order N $k_N(\theta_i) = 0$. In the case where $\theta_i > 1$ then $[(s-\theta_i)_+]^N = 0$ and $k_N(\theta_i)$ is null.

Hence the only component left from sum (4.13) is when $i = j$ and we have

$$K_N(t) = \left(\frac{h_j}{2}\right) k_N\left(\frac{2}{h_j}\left(t - \alpha_{j+\frac{1}{2}}\right)\right).$$

□

This result is significant as it highlights the relation between the steps of the mesh and the expression of the error.

We now provide a more precise result on the expression of the composite error in the general case where the step of the mesh is not necessarily constant.

Proposition

If $f \in C^{N+1}[\alpha, \beta]$ and the method has order N, then

$$\left|E_{comp}(f)\right| \leq \frac{C_{N,f}}{2^{N+2}}(\beta - \alpha)h^{N+1},$$

where

$$\begin{cases} h = \max_{0 \leq i \leq k-1} h_i \\ C_{N,f} = \frac{\max_{\alpha,\beta} |f^{(N+1)}|}{N!} \int_{-1}^{1} |k_N(t)| \, \mathrm{d}t. \end{cases}$$

Proof. Following the same reasoning as the one leading to (4.12) we can easily deduce for any $f \in C^{N+1}[\alpha, \beta]$

$$E_{comp}(f) = \sum_{i=0}^{k-1} \frac{h_i}{2} E_{el}(\varphi_{i,f}), \quad \text{with } \varphi_{i,f}(s) = f\left(s\frac{h_i}{2} + \alpha_{i+\frac{1}{2}}\right). \tag{4.14}$$

Noting that $\varphi_{i,f}^{(N+1)}(s) = \left(\frac{h_i}{2}\right)^{N+1} f^{(N+1)}\left(s\frac{h_i}{2} + \alpha_{i+\frac{1}{2}}\right)$ we get

$$E_{el}(\varphi_{i,f}) = \frac{1}{N!}\int_{-1}^{1} k_N(t)\varphi_{i,f}^{(N+1)}\,\mathrm{d}t,$$

which yields

$$|E_{el}(\varphi_{i,f})| \leq \frac{\max_{[\alpha,\beta]}|f^{(N+1)}|}{N!}\left(\frac{h_i}{2}\right)^{N+1}\int_{-1}^{1}|k_N(t)|k_N\,\mathrm{d}t.$$

If we now call h the max of all the h_i, $0 \leq i \leq k-1$ and observe that their sum is exactly $\beta - \alpha$, then we obtain the expected result

$$\left|E_{comp}(f)\right| \leq \frac{\max_{[\alpha,\beta]}|f^{(N+1)}|}{N!}\left(\int_{-1}^{1}|k_N(t)|\,\mathrm{d}t\,\frac{h^{N+1}}{2^{N+2}}\sum_{i=0}^{k-1}h_i\right)$$
$$= \frac{h^{N+1}(\beta-\alpha)\max_{[\alpha,\beta]}|f^{(N+1)}|}{2^{N+2}N!}\left(\int_{-1}^{1}|k_N(t)|\,\mathrm{d}t\right)$$

□

If we now call h the max of all the h_i, $0 \leq i \leq k-1$ and observe that their sum is exactly $\beta - \alpha$, then we obtain the expected result

$$\left|E_{comp}(f)\right| \leq \frac{\max_{[\alpha,\beta]}\left|f^{(N+1)}\right|}{N!}\left(\int_{-1}^{1}\left|k_N(t)\right|\mathrm{d}t\,\frac{h^{N+1}}{2^{N+2}}\sum_{i=0}^{k-1}h_i\right)$$

$$= \frac{h^{N+1}(\beta-\alpha)\max_{[\alpha,\beta]}\left|f^{(N+1)}\right|}{2^{N+2}N!}\left(\int_{-1}^{1}\left|k_N(t)\right|\mathrm{d}t\right)$$

□

The last result we prove describes the error in the case where Peano kernel does not change sign and the step is constant. As in the previous proofs the constant sign will allow us to apply the mean value theorem to obtain a more precise result than the previous maximum.

Proposition

Let $f \in C^{N+1}[\alpha, \beta]$. If k_N has a constant sign over $[-1, 1]$, and the mesh has constant step $h = (\beta - \alpha)/k$, where k is the number of nodes, then there exists $\xi \in [\alpha, \beta]$ such that

$$E_{comp(f)} = \frac{(\beta - \alpha)h^{N+1}}{2^{N+2}N!} f^{(N+1)}(\xi) \int_{-1}^{1} k_N(t)\, \mathrm{d}t.$$

Proof. Let $f \in C^{N+1}[\alpha, \beta]$. Starting from equation (4.14) we write

$$E_{el}(\varphi_{i,f}) = \frac{1}{N!} \left(\frac{h}{2}\right)^{N+1} \int_{-1}^{1} k_N(t) f^{(N+1)}\left(t\frac{h}{2} + \alpha_{i+\frac{1}{2}}\right) \mathrm{d}t.$$

From the first mean value theorem (2.34) we know the existence of ξ_i in $[\alpha_i, \alpha_{i+1}]$ such that

$$\int_{-1}^{1} k_N(t) f^{(N+1)}\left(t\frac{h}{2} + \alpha_{i+\frac{1}{2}}\right) \mathrm{d}t = f^{(N+1)}(\xi_i) \int_{-1}^{1} k_N(t)\,\mathrm{d}t.$$

Then the composite error (4.14) can be expressed as

$$E_{comp}(f) = \frac{h^{N+1}}{2^{N+2}N!}\frac{\beta-\alpha}{k}\left[\sum_{i=0}^{k-1} f^{(N+1)}(\xi_i)\right]\int_{-1}^{1} k_N(t)\,\mathrm{d}t. \tag{4.15}$$

In order to rewrite the sum observe that

$$\sum_{i=0}^{k-1} \min_{[\alpha,\beta]} f^{(N+1)} \le \sum_{i=0}^{k-1} f^{(N+1)}(\xi_i) \le \sum_{i=0}^{k-1} \max_{[\alpha,\beta]} f^{(N+1)}.$$

Looking at the left and right sums we directly see that

$$\min_{[\alpha,\beta]} f^{(N+1)} \leq \frac{1}{k}\sum_{i=0}^{k-1} f^{(N+1)}(\xi_i) \leq \max_{[\alpha,\beta]} f^{(N+1)}.$$

Therefore from the intermediate values theorem (2.11) there exists $\xi \in [\alpha, \beta]$ such that $\frac{1}{k}\sum_{i=0}^{k-1} f^{(N+1)}(\xi_i) = f^{(N+1)}(\xi)$.

Finally (4.15) can be rewritten

$$E_{comp}(f) = \frac{h^{N+1} f^{(N+1)}(\xi)}{2^{N+2} N!}(\beta - \alpha)\int_{-1}^{1} k_N(t)\,\mathrm{d}t.$$

□

Let $(x_{jk})_{0\leq j\leq k}$ be a sequence of points in an interval $[a,b]\subset\mathbb{R}$. Given $(\alpha,\beta)\subset[a,b]$, and a weight function $w:(\alpha,\beta)\to\mathbb{R}^+$ we consider the quadrature formula, for a function $f\in C[a,b]$,

$$\int_\alpha^\beta f(x)w(x)\,\mathrm{d}x \approx \sum_{j=0}^{k}\lambda_{jk}f(x_{jk}),$$

and define its error

$$E_k(f)=\int_\alpha^\beta f(t)w(t)\,\mathrm{d}t-\sum_{j=0}^{k}\lambda_{jk}f(x_{jk}).$$

We now introduce a result which provides necessary and sufficient conditions for the error to converge, i.e. $\lim_{k\to\infty}E_k(f)=0$.

Theorem (Polya's convergence)

Using the previous notations, the error $E_k(f)$ converges to 0 as k tends to infinity if and only if the following two conditions are met

1. For any positive integer n, $\lim_{k\to\infty} E_k(x \mapsto x^n) = 0$;
2. There exists M, strictly positive, such that for all $k \in \mathbb{N}$ $\sum_{j=0}^{k} |\lambda_{jk}| \leq M$;

Proof. Let f be a continuous function over $[a, b]$.

We first prove that if the two conditions hold then $\lim\limits_{k\to\infty} E_k(f) = 0$.

By Stone-Weierstrass theorem (2.22) there exists a sequence of polynomials $(P_n)_{n\in\mathbb{N}}$ such that

$$\lim_{n\to\infty} \|f - P_n\|_\infty = \lim_{n\to\infty} \max_{x\in[a,b]} |f(x) - P_n(x)| = 0.$$

As $C[a,b]$ and $\mathbb{R}$ are vector spaces while E_k is a linear application we would like to apply proposition 1.24. We therefore only need to prove that $\|E_k\|$ is finite.

This is straightforward. Since $\int_\alpha^\beta f(t)w(t)\,\mathrm{d}t$ is finite, all the $f(x_{jk})$ are finite, and from (ii), there exists $M > 0$ such that for all $k \in \mathbb{N}$, $\sum_{j=0}^{k} |\lambda_{jk}| \leq M$. Thus $E_k(P_n)$ converges to $E_k(f)$.

Now from our first condition we know that $E_k(P_n)$ tends to 0 as k tends to infinity. This yields the first implication.

For the converse, (i) is trivial since it suffices to take $f(x) = x^n$. Therefore we only need to prove (ii). For (ii) first observe that E_k is continuous (theorem 1.28)

$$|E_k(f)| \leq \max_{[a,b]} |f| \left(\int_\alpha^\beta w(x)\,\mathrm{d}x + \sum_{j=0}^{k} |\lambda_{jk}| \right).$$

Hence $(E_k)_{k\geq 0}$ is a sequence of continuous linear applications from the Banach space $C[a,b]$ into the Banach space $\mathbb{R}$. Thus we can apply Banach-Steinhaus theorem (2.20) to obtain the existence of some m such that $\sup_k \|E_k\|$ is less than m.

Let g_k be a continuous function over $[a,b]$, with $\max_{[a,b]} |g_k| = 1$ and $g_k(\lambda_{jk}) = \operatorname{sign}(\lambda_{jk})$. Then for all k we see that

$$\begin{aligned}\sum_{j=0}^{k} |\lambda_{jk}| &= \int_\alpha^\beta g_k(x)w(x)\,\mathrm{d}x - E_k(g_k)\\ &\leq \int_\alpha^\beta w(x)\,\mathrm{d}x + m.\end{aligned}$$

Choosing $M = \int_\alpha^\beta w(x)\,\mathrm{d}x + m$ completes the proof. □

Example. Lets consider the Gauss-Chebyshev integration method, i.e. Gauss method together with Chebyshev nodes.

The weight function associated to Chebyshev polynomials is defined on $(-1,1)$ by $w(x)=\frac{1}{\sqrt{1-x^2}}$, and for $0\leq j\leq k$, $k\in\mathbb{N}$,

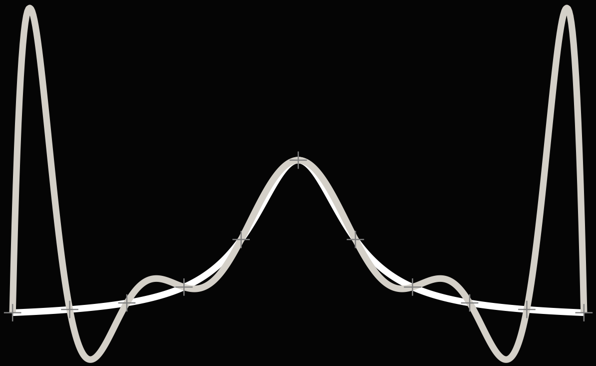
$$x_{jk}=\cos\frac{(2j+1)\pi}{2(k+1)},\qquad \lambda_{jk}=\frac{\pi}{k+1}.$$

All the λ_{jk} being positive, $\sum_{j=0}^{k}|\lambda_{jk}|=\sum_{j=0}^{k}\lambda_{jk}=\pi$. So the second condition of Polya's convergence theorem is met. The first one is also verified since Gauss method is of order $2k+1$ (corollary 4.27): for all $1\leq n\leq 2k+1$ the formula is exact, and when k tends to infinity $E_k(x\mapsto x^n)$ tends to 0.

By Polya's convergence theorem (4.61), the Gauss-Chebyshev method is convergent.

Key points

- What is the order of an integration method?
- Is interpolation useful for numerical integration?
- Describe Peano's method
- Why does Peano's method covers other methods?
- What does it mean for a method to be convergent?



Thank you!