EXERCISES SEMINAR 2

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EXERCISE 2.2.

Given the matrices $A = \begin{bmatrix} -1 & 3 \\ 4 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 4 & -3 \\ 1 & -2 \\ -2 & 0 \end{bmatrix}$ and $c = \begin{bmatrix} 5 \\ -4 \\ 2 \end{bmatrix}$, perform the indicated multiplications.

a) $5A = 5 \cdot \begin{bmatrix} -1 & 3 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} -5 & 15 \\ 20 & 10 \end{bmatrix}$

b) $BA = \begin{bmatrix} 4 & -3 \\ 1 & -2 \\ -2 & 0 \end{bmatrix} \cdot \begin{bmatrix} -1 & 3 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} -4-12 & 12-6 \\ -1-8 & 3-4 \\ 2+0 & -6+0 \end{bmatrix} = \begin{bmatrix} -16 & 6 \\ -9 & -1 \\ 2 & -6 \end{bmatrix}$

c) $A' \cdot B' = \begin{bmatrix} -1 & 4 \\ 3 & 2 \end{bmatrix} \cdot \begin{bmatrix} 4 & 1 & -2 \\ -3 & -2 & 0 \end{bmatrix} = \begin{bmatrix} -4-12 & -1-8 & 2+0 \\ 12-6 & 3-4 & -6+0 \end{bmatrix} = \begin{bmatrix} -16 & -9 & 2 \\ 6 & -1 & -6 \end{bmatrix}$

d) $c'B = \begin{bmatrix} 5 & -4 & 2 \end{bmatrix} \begin{bmatrix} 4 & -3 \\ 1 & -2 \\ -2 & 0 \end{bmatrix} = \begin{bmatrix} 20-4-4 & -15+8+0 \end{bmatrix} = \begin{bmatrix} 12 & -7 \end{bmatrix}$

e) Is AB defined? → No, the product is undefined ⇒ A is $2 \times \underbrace{2 \text{ and B is } 3}_{\text{The middle values don't match}} \times 2$

EXERCISE 2.7.

Let $A = \begin{bmatrix} 9 & -2 \\ -2 & 6 \end{bmatrix}$

a) Determine the eigenvalues and eigenvectors of A.

(i) Eigenvalues: $\det(A - \lambda I) = 0$

· $\det\left(\begin{bmatrix} 9-\lambda & -2 \\ -2 & 6-\lambda \end{bmatrix}\right) = (9-\lambda)(6-\lambda) - (-2)(-2) = \lambda^2 - 15\lambda + 54 - 4 = \lambda^2 - 15\lambda + 50$.

· $\lambda^2 - 15\lambda + 50 = 0$; $\lambda = \frac{+15 \pm \sqrt{15^2 - 4 \cdot 1 \cdot 50}}{2 \cdot 1} = \frac{15 \pm \sqrt{225-200}}{2} = \frac{15 \pm 5}{2}$ $\begin{cases} \lambda_1 = \frac{15+5}{2} = \boxed{10} \\ \lambda_2 = \frac{15-5}{2} = \boxed{5} \end{cases}$

(ii) Eigenvectors

For $\lambda_1 = 10 \rightarrow (A - \lambda_1 I)v = 0$; $\begin{bmatrix} 9-10 & -2 \\ -2 & 6-10 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} -1 & -2 \\ -2 & -4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\left.\begin{matrix} -v_1 - 2v_2 = 0 \\ -2v_1 - 4v_2 = 0 \end{matrix}\right\}$ $v_1 = -2v_2$ so, a suitable eigenvector would be ⇒ $\boxed{(-2, 1)} = v'$

(*) Normalized eigenvector ⇒ $\frac{1}{\sqrt{v_1^2 + v_2^2}}(v_1, v_2) = \frac{1}{\sqrt{2^2+1^2}}(-2, 1) = \frac{1}{\sqrt{5}}(-2, 1) = (-2/\sqrt{5}, 1/\sqrt{5}) = e_1'$

For $\lambda_2 = 5 \rightarrow \begin{bmatrix} 9-5 & -2 \\ -2 & 6-5 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ " $\begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\left.\begin{array}{l} 4v_1 - 2v_2 = 0 \\ -2v_1 + v_2 = 0 \end{array}\right\} \quad v_1 = \frac{1}{2} v_2 \Rightarrow \boxed{v_2 = (1, 2)}$$

And the normalized eigenvector is $\rightarrow \frac{1}{\sqrt{v_1^2 + v_2^2}} e_2' = \frac{1}{\sqrt{5}}(1,2) = \left(1/\sqrt{5}, 2/\sqrt{5}\right)$

b) Write the spectral decomposition of A.

The spectral decomposition of a 2x2 symmetric matrix A is given by:

$A = \lambda_1 \cdot e_1 \cdot e_1' + \lambda_2 \cdot e_2 \cdot e_2'$ where: e_1 and e_2 are the normalized eigenvectors
λ_1 and λ_2 are the eigenvalues.

So, in our case:

$$\boxed{A = 10 \cdot \begin{bmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix} \cdot \begin{bmatrix} -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix} + 5 \cdot \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix} \cdot \begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}}$$

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c) Find A^{-1}.

$$A^{-1} = \frac{(Adj(A))^T}{|A|} = \frac{\left(\begin{bmatrix} 6 & 2 \\ 2 & 9 \end{bmatrix}\right)^T}{9 \cdot 6 - (-2)(-2)} = \frac{\begin{bmatrix} 6 & 2 \\ 2 & 9 \end{bmatrix}}{50} = \begin{bmatrix} 3/25 & \frac{1}{25} \\ \frac{1}{25} & 9/50 \end{bmatrix} = \begin{bmatrix} 0'12 & 0'04 \\ 0'04 & 0'18 \end{bmatrix}$$

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d) Find eigenvalues and eigenvectors of A^{-1}.

(i) Eigenvalues: λ s.t. $|A - \lambda I| = 0$

- $\det\left(\begin{bmatrix} 0'2-\lambda & 0'04 \\ 0'04 & 0'18-\lambda \end{bmatrix}\right) = (0'2-\lambda)(0'18-\lambda) - 0'04^2 = \lambda^2 - 0'38\lambda + 0'0344$.

- $\lambda^2 - 0'38\lambda + 0'0344 = 0$ " $\lambda = \frac{+0'38 \pm \sqrt{0'38^2 - 4 \cdot 1 \cdot 0'0344}}{2 \cdot 1} = \frac{0'38 \pm 0'0825}{2} = \begin{cases} \lambda_1 = 0'23 \\ \lambda_2 = 0'15 \end{cases}$

(ii) Eigenvectors

- For $\lambda_1 = 0'2 \rightarrow v_1$ s.t. $(A - \lambda_1 I)v = 0$ " $\begin{bmatrix} 0'2-0'2 & 0'04 \\ 0'04 & 0'18-0'2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

$$\left.\begin{array}{l} 0'04 v_2 = 0 \\ 0'04 v_1 - 0'02 v_2 = 0 \end{array}\right\} \quad v_1 = \frac{1}{2} v_2 \Rightarrow \boxed{v_1' = (1, 2)}$$

Normalized eigenvector $\rightarrow \frac{1}{\sqrt{5}}(1,2) \Rightarrow e_1' = \left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)$.

- For $\lambda_2 = 0'1 \rightarrow v_2$ s.t. $(A - \lambda_2 I)v = 0$ " $\begin{bmatrix} 0'2-0'1 & 0'04 \\ 0'04 & 0'18-0'1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

$$\left.\begin{array}{l} 0'1 v_1 + 0'04 v_2 = 0 \\ 0'04 v_1 + 0'08 v_2 = 0 \end{array}\right\} \quad v_1 = 2v_2 \Rightarrow v_2' = (2, 1)$$

Normalized eigenvector $\rightarrow \frac{1}{\sqrt{5}}(2,1) \Rightarrow e_2' = \left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)$

EXERCISE 4.2

Consider a bivariate normal population with $\mu_1 = 0$, $\mu_2 = 2$, $\sigma_{11} = 2$, $\sigma_{22} = 1$, and $\rho_{12} = .5$.

a) Write out the bivariate normal density.

- A p-dimensional normal density has the form $\rightarrow f(x) = \frac{1}{(2\pi)^{\frac{p}{2}} |\Sigma|^{1/2}} \cdot e^{-\frac{(x-\mu)'\Sigma^{-1}(x-\mu)}{2}}$

- In our case, we have that $p = 2$; $\mu = \begin{bmatrix}\mu_1\\ \mu_2\end{bmatrix} = \begin{bmatrix}0\\ 2\end{bmatrix}$; $x = \begin{bmatrix}x_1\\ x_2\end{bmatrix}$, $(x-\mu) = \begin{bmatrix}x_1\\ x_2-2\end{bmatrix}$

- $\Sigma = \begin{bmatrix}\sigma_{11} & \sigma_{12}\\ \sigma_{12} & \sigma_{22}\end{bmatrix} = \begin{bmatrix}2 & 1/\sqrt{2}\\ \frac{1}{\sqrt{2}} & 1\end{bmatrix}$,, $\Sigma^{-1} = \frac{\begin{bmatrix}\sigma_{22} & -\sigma_{12}\\ -\sigma_{12} & \sigma_{11}\end{bmatrix}}{\sigma_{11}\cdot\sigma_{22} - \sigma_{12}^2} = \frac{\begin{bmatrix}1 & -1/\sqrt{2}\\ -\frac{1}{\sqrt{2}} & 2\end{bmatrix}}{\underbrace{2\cdot 1\cdot - \left(\frac{1}{\sqrt{2}}\right)^2}_{\frac{3}{2}}} = \begin{bmatrix}\frac{2}{3} & -\frac{\sqrt{2}}{3}\\ -\frac{\sqrt{2}}{3} & 4/3\end{bmatrix}$

$\rho_{12} = \frac{\sigma_{12}}{\sqrt{\sigma_{11}}\cdot\sqrt{\sigma_{22}}}$,, $\sigma_{12} = \rho_{12}\cdot\sqrt{\sigma_{11}}\cdot\sqrt{\sigma_{22}} = 0'5\cdot\sqrt{2}\cdot\sqrt{1} = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$ $\qquad |\Sigma| = \frac{3}{2}$

$$\boxed{f(x) = \frac{1}{(2\pi)\cdot\sqrt{3/2}} \cdot \exp\left\{-\frac{1}{2}\left[\frac{2}{3}x_1^2 - 2\cdot\frac{\sqrt{2}}{3}x_1(x_2-2) + \frac{4}{3}(x_2-2)^2\right]\right\}}$$

$$(x-\mu)'\Sigma^{-1}(x-\mu) = \begin{bmatrix}x_1 & x_2-2\end{bmatrix}\begin{bmatrix}2/3 & -\sqrt{2}/3\\ -\sqrt{2}/3 & 4/3\end{bmatrix}\begin{bmatrix}x_1\\ x_2-2\end{bmatrix} = \begin{bmatrix}\frac{2}{3}x_1 - \frac{\sqrt{2}}{3}(x_2-2) & -\frac{\sqrt{2}}{3}x_1 + \frac{4}{3}(x_2-2)\end{bmatrix}\begin{bmatrix}x_1\\ x_2-2\end{bmatrix} =$$

$$= \frac{2}{3}x_1^2 - \frac{\sqrt{2}}{3}x_1(x_2-2) + (x_2-2)\left(-\frac{\sqrt{2}}{3}x_1 + \frac{4}{3}(x_2-2)\right) = \frac{2}{3}x_1 - 2\cdot\frac{\sqrt{2}}{3}x_1(x_2-2) + \frac{4}{3}(x_2-2)^2 .$$

b) Write out the squared generalized distance expression $(x-\mu)'\Sigma^{-1}(x-\mu)$ as a function of x_1 and x_2.

$(x-\mu)'\Sigma^{-1}(x-\mu) = \frac{2}{3}x_1^2 - 2\frac{\sqrt{2}}{3}\cdot x_1(x_2-2) + \frac{4}{3}(x_2-2)^2$ $\rightarrow$ steps shown in a)

c) Determine the constant-density contour that contains 50% of the probability.

(i) $\chi_p^2(\alpha)$ with $p=2$ and $\alpha = 0'5$ $\Rightarrow$ $c^2 = \chi_2^2(0'5) = 1'386294 \simeq 1'39$ **mirar pag 760 de int edition**
$\hookrightarrow$ R: qchisq(0'5, df=2)

(ii) The half-length of the axes are given by $\rightarrow \sqrt{\chi_2^2(0'5)}\cdot\sqrt{\lambda_i}$

- $|\Sigma - \lambda I| = 0 \Rightarrow \begin{vmatrix}2-\lambda & 1/\sqrt{2}\\ 1/\sqrt{2} & 1-\lambda\end{vmatrix} = 0 \Rightarrow (2-\lambda)(1-\lambda) - \left(\frac{1}{\sqrt{2}}\right)^2 = 0$,, $\lambda^2 - 3\lambda + \frac{3}{2} = 0$,, $\lambda = \frac{3\pm\sqrt{3}}{2}$ $\begin{cases}2'366 = \lambda_1\\ 0'634 = \lambda_2\end{cases}$

- Major axis $\rightarrow \sqrt{1'39}\cdot\sqrt{2'366} = 1'81$
- Minor axis $\rightarrow \sqrt{1'39}\cdot\sqrt{0'634} = 0'939$

(iii) The axes of the constant-density contour (ellipses) are given by $\rightarrow \pm c\sqrt{\lambda_i}\, e_i$

- Eigen vectors $e_i \rightarrow (\Sigma - \lambda_i I)\cdot v = 0$ For $\lambda_1 \rightarrow$ $\begin{cases}-0'366v_1 + \frac{1}{\sqrt{2}}v_2 = 0\\ \frac{1}{\sqrt{2}}v_1 - 1'366v_2 = 0\end{cases}$ $v_1' = (1'366, \sqrt{2}) = (0'966, 1)$; $e_1' = \frac{1}{\sqrt{0'966^2+1}}v = (1'34, 1'39)$

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For $\lambda_2 \rightarrow$ $\begin{cases}1'366v_1 + \frac{1}{\sqrt{2}}v_2 = 0\\ \frac{1}{\sqrt{2}}v_1 + 0'366v_2 = 0\end{cases}$ $v_2' = (-0'366, \sqrt{2}) = (-0'259, 1)$; $e_2' = \frac{v_2}{\sqrt{-0'259^2+1}} = (-0'268, 1'03)$

EXERCISE 4.3

Let X be $N_3(\mu, \Sigma)$ with $\mu' = [-3, 1, 4]$ and $\Sigma = \begin{bmatrix} 1 & -2 & 0 \\ -2 & 5 & 0 \\ 0 & 0 & 2 \end{bmatrix}$, which of the following are independent?

a) X_1 and X_2. → No, we can se in Σ that $\sigma_{12} = -2 \neq 0$, so X_1 and X_2 are linearly dependent.

b) X_2 and X_3 → Yes, in Σ we can se that $\sigma_{23} = \sigma_{32} = 0$, so there's no linear dependency.

c) (X_1, X_2) and (X_3) → Yes, because $cov(X_1 + X_2, X_3) = cov(X_1, X_3) + cov(X_2, X_3) = \sigma_{13} + \sigma_{23} = 0$

d) $\left(\frac{X_1+X_2}{2}\right)$ and (X_3) → Yes, → $\frac{1}{2}\sigma_{13} + \frac{1}{2}\sigma_{23} = 0$

e) X_2 and $(X_2 - \frac{5}{2}X_1 - X_3)$ → No ⇒ $Cov(X_2, X_2) - \frac{5}{2}Cov(X_2, X_1) - Cov(X_2, X_3) = \sigma_{22} - \frac{5}{2}\sigma_{21} - \sigma_{13} = 5 - \frac{5}{2}(-2) - 0 = 10 \neq 0$

EXERCISE 4.4.

Let X be $N_3(\mu, \Sigma)$ with $\mu' = [2, -3, 1]$ and $\Sigma = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 2 \\ 1 & 2 & 2 \end{bmatrix}$

a) Find the distribution of $3X_1 - 2X_2 + X_3$.

$$E[3X_1 - 2X_2 + X_3] = 3E[X_1] - 2E[X_2] + E[X_3] = 3\mu_1 - 2\mu_2 + \mu_3 = 3\cdot 2 + 2\cdot 3 + 1 = 13.$$

$$V[3X_1 - 2X_2 + X_3] = 3^2\sigma_{11} + (-2)^2\sigma_{22} + 1^2\sigma_{33} + 2\cdot 3(-2)\sigma_{12} + 2\cdot 3\cdot 1\,\sigma_{13} + 2(-2)\cdot 1\,\sigma_{23} = 9.$$

So, $3X_1 - 2X_2 + X_3 \sim N(13, 9)$

b) Relabel the variables if necessary, and find a 2×1 vector a such that X_2 and $X_2 - a'\begin{bmatrix} X_1 \\ X_3 \end{bmatrix}$ are independent.

- $X_2 - a'\begin{bmatrix} X_1 \\ X_3 \end{bmatrix} = X_2 - [a_1 \; a_3]\begin{bmatrix} X_1 \\ X_3 \end{bmatrix} = X_2 - a_1X_1 - a_3X_3$

- For X_2 and $X_2 - a'\begin{bmatrix} X_1 \\ X_3 \end{bmatrix}$ to be independent ⇒ $Cov(X_2, X_2 - a_1X_1 - a_3X_3) = 0$

$$Cov(X_2, X_2 - a_1X_1 - a_3X_3) = Cov(X_2, X_2) - a_1 Cov(X_2, X_1) - a_3 Cov(X_2, X_3) =$$
$$= \sigma_{22} - a_1\sigma_{21} - a_3\sigma_{23} = 3 - a_1 - 2a_3.$$

$3 - a_1 - 2a_3 = 0 \;\; \therefore \; +a_1 = +3 - 2a_3 \;\; \Rightarrow \; a' = [a_1 \; a_3] = [3 - 2a_3 \;\; a_3]$

- We can relabel $a_3 = 1$ and $a_1 = 3 - 2\cdot 1 = 1$ so $a' = [1 \; 1]$ and $Cov(X_2, X_2 - X_1 - X_3) = 0$.

EXERCISE 4.5.

Specify each of the following.

a) The conditional distribution of X_1 given that $X_2 = x_2$ for the joint distribution in Exercise 4.2.

$$X_1 | X_2 = x_2 \sim N\left(\mu_1 + \frac{\sigma_{12}}{\sigma_{22}}(x_2 - \mu_2),\; \sigma_{11} - \frac{\sigma_{12}^2}{\sigma_{22}}\right) = \left(0 + \frac{1/\sqrt{2}}{1}(x_2 - 2),\; 2 - \frac{\left(\frac{1}{\sqrt{2}}\right)^2}{1}\right)$$

$$\boxed{X_1 | X_2 = x_2 \sim N\left(\frac{1}{\sqrt{2}}(x_2 - 2),\; \frac{3}{2}\right)}$$

b) The conditional distribution of X_2, given that $X_1 = x_1$ and $X_3 = x_3$ for $f(x_1, x_2 x_3)$ in Exercise 4.3.

- $E(X_2 | X_1, X_3) = \mu_2 + \frac{\sigma_{21}}{\sigma_{11}}(x_1 - \mu_1) + \frac{\sigma_{23}}{\sigma_{33}}(x_3 - \mu_3) = 1 + \frac{(-2)}{1}(x_1 + 3) + \cancel{\frac{0}{2}(x_3 - 4)} = 1 - 2x_1 - 6 = -2x_1 - 5$

- $Var(X_2 | X_1, X_3) = \sigma_{22} - \frac{\sigma_{21}^2}{\sigma_{11}} - \frac{\sigma_{23}^2}{\sigma_{33}} = 5 - \frac{(-2)^2}{1} - 0 = 5 - 4 = 1.$

$\boxed{X_2 | X_1, X_3 \sim N(-2x_1 - 5, 1)}$

c) The conditional distribution of X_3, given that $X_1 = x_1$ and $X_2 = x_2$ for the joint distribution in Exercise 4.4.

- $E(X_3 | X_1, X_2) = \mu_3 + \frac{\sigma_{31}}{\sigma_{11}}(x_1 - \mu_1) + \frac{\sigma_{32}}{\sigma_{22}}(x_2 - \mu_2) = 1 + \frac{1}{1}(x_1 - 2) + \frac{2}{3}(x_2 - 3) = x_1 + \frac{2}{3}x_2 + 1$
- $V(X_3 | X_1, X_2) = \sigma_{33} - \frac{\sigma_{31}^2}{\sigma_{11}} - \frac{\sigma_{32}^2}{\sigma_{33}} = 2 - \frac{1^2}{1} - \frac{4}{2} = 2 - 1 - 2 = -1.$

$\boxed{X_3 | X_1 = x_1, X_3 = x_3 \sim N(x_1 + \frac{2}{3}x_2 + 1, -1)}$

EXERCISE 4.21

Let $X_1, \dots, X_{60}$ be a random sample of size 60 from a 4-variate normal distribution having mean μ and covariance Σ. Specify each of the following completely.

a) The distribution of $\bar{X}$

- $E[\bar{X}] = E\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right] = \frac{1}{n}\sum_{i=1}^{n} E[X_i] = \frac{1}{n} \cdot n \cdot \mu = \mu.$
- $V[\bar{X}] = V\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right] = \frac{1}{n^2} \cdot \sum_{i=1}^{n} V[X_i] = \frac{1}{n^2} \cdot n \cdot \sigma = \frac{1}{n} \cdot \underset{\Sigma}{\overset{\shortparallel}{\sigma}}$

$\boxed{\bar{X}_{60} \sim N_4(\mu, \frac{1}{n}\Sigma)}$

b) The distribution of $(X_1 - \mu)' \Sigma^{-1} (X_1 - \mu)$

From a) we know that $X_1 \sim N_4(\mu, \Sigma)$ so:

- $E[X_1 - \mu] = E[X_1] - \mu = \mu - \mu = 0$
- $V[X_1 - \mu] = V[X_1] - 0 = \Sigma$

Then $(X_1 - \mu) \sim N_4(0, \Sigma)$ and assuming that $|\Sigma| > 0$, from result 4.7 (page 163), then

$$(X_1 - \mu)' \Sigma^{-1} (X_1 - \mu) \sim \chi_p^2$$

c) The distribution of $n(\bar{X} - \mu)' \Sigma^{-1} (\bar{X} - \mu)$.

From a) we know that $\bar{X} \sim N_4(\mu, \frac{1}{n}\Sigma)$.

- $E[n(\bar{X} - \mu)] = n[E(\bar{X}) - \mu] = n(\mu - \mu) = 0$
- $V[n\bar{X} - n\mu] = nV[\bar{X}] - 0 = n \cdot \frac{1}{n}\Sigma = \Sigma$

Then $n(\bar{X} - \mu) \sim N_4(0, \Sigma)$ so, again $\rightarrow$ $n(\bar{X} - \mu)' \Sigma^{-1} (\bar{X} - \mu) \sim \chi_p^2$.

d) The approximate distribution of $n(\bar{x}-\mu)' S^{-1}(\bar{x}-\mu)$.

When the sample size is large as in our case ($n=60$) it is possible to use S intead of Σ, so the distribution would approximately be same as c) $\rightarrow \chi^2_p$.

EXERCISE 4.22.

Let $X_1, X_2, \_, X_{75}$ be a random sample from a population distribution with mean μ and covariance matrix Σ. What is the appropiate distribution of each of the following?

a) $\bar{X} \sim N_p(\mu, \frac{1}{n}\Sigma)$ $\rightarrow$ steps shown in exercise 4.21.

b) $n(\bar{x}-\mu)' S^{-1}(\bar{x}-\mu)$

From 4-28 we see that for $n-p$ large, as in this case,

$$n(\bar{x}-\mu)' S^{-1}(\bar{x}-\mu) \sim \chi^2_p. \text{ , approximately.}$$