EXERCISES SEMINAR 3

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11/29/2019

EXERCISE 5.1.

a) Evaluate T^2, for testing $H_0: \mu' = [7, 11]$, using the data $X = \begin{bmatrix} 2 & 12 \\ 8 & 9 \\ 6 & 9 \\ 8 & 10 \end{bmatrix}$

(i) Hypotheses:

$$H_0: \mu = \mu_0 \qquad H_1: \mu \neq \mu_0 \qquad \text{where } \mu_0 = \begin{pmatrix} 7 \\ 11 \end{pmatrix}$$

(ii) Statistic:

$$T^2 = (\bar{x} - \mu_0)' \left(\tfrac{1}{n} S\right)^{-1} (\bar{x} - \mu_0) = n(\bar{x} - \mu_0)' \cdot S^{-1} (\bar{x} - \mu_0)$$

- $n = 4$,, $\bar{x}_1 = \frac{2+8+6+8}{4} = 6$, $\bar{x}_2 = \frac{12+9+9+10}{4} = 10$ $\Big\}\ \bar{x} = \begin{bmatrix} 6 \\ 10 \end{bmatrix}$,, $1 \cdot \bar{x}' = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 6 & 10 \end{bmatrix} = \begin{bmatrix} 6 & 10 \\ 6 & 10 \\ 6 & 10 \\ 6 & 10 \end{bmatrix}$

- $X - 1\bar{x}' = \begin{bmatrix} 2-6 & 12-10 \\ 8-6 & 9-10 \\ 6-6 & 9-10 \\ 8-6 & 10-10 \end{bmatrix} = \begin{bmatrix} -4 & 2 \\ 2 & -1 \\ 0 & -1 \\ 2 & 0 \end{bmatrix}$,, $S = \frac{1}{n-1}(X - 1\bar{x}')'(X - 1\bar{x}) = \frac{1}{3} \begin{bmatrix} -4 & 2 & 0 & 2 \\ 2 & -1 & -1 & 0 \end{bmatrix} \begin{bmatrix} -4 & 2 \\ 2 & -1 \\ 0 & -1 \\ 2 & 0 \end{bmatrix} =$

$$= \frac{1}{3} \begin{bmatrix} 24 & -10 \\ -10 & 6 \end{bmatrix} = \begin{bmatrix} 8 & -10/3 \\ -10/3 & 2 \end{bmatrix}$$

- $(\bar{x} - \mu_0) = \begin{bmatrix} 6 \\ 10 \end{bmatrix} - \begin{bmatrix} 7 \\ 11 \end{bmatrix} = \begin{bmatrix} 6-7 \\ 10-11 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$

- $S^{-1} = \frac{(Adj(S))^T}{|S|} = \frac{\left(\begin{bmatrix} 2 & 10/3 \\ 10/3 & 8 \end{bmatrix}\right)^T}{8 \cdot 2 - (-\frac{10}{3})^2} = \frac{\begin{bmatrix} 2 & 10/3 \\ 10/3 & 8 \end{bmatrix}}{44/9} = \begin{bmatrix} 9/22 & 15/22 \\ 15/22 & 18/11 \end{bmatrix}$

$$\Rightarrow T^2 = 4 \cdot \begin{bmatrix} -1 & -1 \end{bmatrix} \begin{bmatrix} 9/22 & 15/22 \\ 15/22 & 18/11 \end{bmatrix} \begin{bmatrix} -1 \\ -1 \end{bmatrix} = \begin{bmatrix} -4 & -4 \end{bmatrix} \begin{bmatrix} -12/11 \\ -51/22 \end{bmatrix} = 4 \cdot \frac{12}{11} + 4 \cdot \frac{51}{22} = \frac{150}{11} = \boxed{13'\overline{63}}$$

b) Specify the distribution of T^2 for the situation in (a).

From 5-5 (page 212) $\rightarrow$ T^2 is distributed as $\frac{(n-1)p}{(n-p)} F_{p,n-p}$

So, in (a): $T^2 \sim \frac{(4-1)\,2}{4-2} F_{2,4-2} = \boxed{3\, F_{2,2}}$

c) Using (a) and (b), test H_0 at the $\alpha = 0'05$ level. What conclusion do you reach?

We will reject H_0 in favor of H_1 if $T^2 > \frac{(n-1)p}{(n-p)} F_{p,n-p}(\alpha)$

- $T^2 = 13'\overline{63}$
- $3 \cdot F_{2,2}(0'05) = 3 \cdot 19 = 57$

$\Big\}$ $13'63 \not> 57 \Rightarrow$ We don't reject H_0: $\mu' = [7, 11]$.

$\downarrow$ Table in page 762.

EXERCISE 5.3.

a) Use expression (5-15) to evaluate T^2 for the data in Ex. 5.1.

Expression $\rightarrow$ $T^2 = \dfrac{(n-1)|\hat{\Sigma}_0|}{|\hat{\Sigma}|} - (n-1) = \dfrac{(n-1)\left|\sum_{j=1}^{n}(x_j-\mu_0)(x_j-\mu_0)'\right|}{\left|\sum_{j=1}^{n}(x_j-\bar{x})(x_j-\bar{x})'\right|} - (n-1)$

- $(n-1) = 4-1 = 3$
- $\mu_0 = \begin{bmatrix} 7 \\ 11 \end{bmatrix}$
- $X - 1\mu_0' = \begin{bmatrix} 2-7 & 12-11 \\ 8-7 & 9-11 \\ 6-7 & 9-11 \\ 8-7 & 10-11 \end{bmatrix} = \begin{bmatrix} -5 & 1 \\ 1 & -2 \\ -1 & -2 \\ 1 & -1 \end{bmatrix}$
- $X = \begin{bmatrix} 2 & 12 \\ 8 & 9 \\ 6 & 9 \\ 8 & 10 \end{bmatrix}$
- $\bar{x} = \begin{bmatrix} 6 \\ 10 \end{bmatrix}$
- $X - 1\bar{x}' = \begin{bmatrix} -4 & 2 \\ 2 & -1 \\ 0 & -1 \\ 2 & 0 \end{bmatrix}$

- $\sum_{j=1}^{n=4}(x_j-\mu_0)(x_j-\mu_0)' = \begin{bmatrix} -5 & 1 & -1 & 1 \\ 1 & -2 & -2 & -1 \end{bmatrix}\begin{bmatrix} -5 & 1 \\ 1 & -2 \\ -1 & -2 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 28 & -6 \\ -6 & 10 \end{bmatrix}$;; $\det\left(\begin{bmatrix} 26 & -6 \\ -6 & 10 \end{bmatrix}\right) = 28-(-6)^2 = 244$

- $\sum_{j=1}^{n=4}(x_j-\bar{x})\cdot(x_j-\bar{x})' = \begin{bmatrix} 24 & -10 \\ -10 & 6 \end{bmatrix}$;; $\det\left(\begin{bmatrix} 24 & -10 \\ -10 & 6 \end{bmatrix}\right) = 24\cdot 6-(-10)^2 = 44$

Evaluation $\rightarrow$ $T^2 = \dfrac{3\cdot 244}{44} - 3 = \dfrac{150}{11} = \boxed{13'6\overline{3}}$

b) Evaluate Λ in (5-13). Also, evaluate Wilk's lambda.

- $\Lambda = \left(\dfrac{|\hat{\Sigma}|}{|\hat{\Sigma}_0|}\right)^{n/2} = \left(\dfrac{44}{244}\right)^{4/2} = \left(\dfrac{11}{61}\right)^2 = \dfrac{121}{3721} = \boxed{0'0325}$

- Wilk's lambda $\rightarrow$ $\Lambda^{2/n} = \dfrac{|\hat{\Sigma}|}{|\hat{\Sigma}_0|} = \dfrac{11}{61} = \boxed{0'18033}$

EXERCISE 5.4. (theory in page 221)

Use the sweat data in Table 5.1.

a) Determine the axes of the 90% confidence ellipsoid for μ. Determine the lengths of these axes.

(i) • Data $\rightarrow$ $\bar{x} = \begin{bmatrix} 4'640 \\ 45'40 \\ 9'965 \end{bmatrix}$;; $S = \begin{bmatrix} 2'879 & 10'010 & -1'810 \\ 10'010 & 199'788 & -5'640 \\ -1'810 & -5'640 & 3'628 \end{bmatrix}$

- Eigenvalues $\rightarrow$ $\det(S-\lambda I) = 0$.

$$S-\lambda I = \begin{bmatrix} 2'879-\lambda & 10'010 & -1'810 \\ 10'010 & 199'788-\lambda & -5'640 \\ -1'810 & -5'640 & 3'628-\lambda \end{bmatrix}$$

$$\det(S-\lambda I) = (2'879-\lambda)(199'788-\lambda)(3'628-\lambda) + 10'010(-5'640)(-1'810) + 10'010(-5'64)(-1'61)$$
$$-(1'810)^2(199'788-\lambda) - (-5'64)^2(2'879-\lambda) - (10'010)^2(3'628-\lambda) =$$
$$= -\lambda^3 + 206'295\lambda^2 - 1310'465\lambda + 2066'786 + 284'372 + 3'2761\lambda - 654'525 + 31'8096\lambda$$
$$-91'58 + 100'2001\lambda - 363'53 = -\lambda^3 + 206'295\lambda^2 - 1175'1792\lambda + 1181'523$$

$$\det(S-\lambda I) = 0 \quad \begin{cases} \lambda_1 = 1'300359716 \approx 1'3 \\ \lambda_2 = 4'532590729 \approx 4'5 \\ \lambda_3 = 200'4620496 \approx 200'5 \end{cases}$$

• Eigenvectors $\rightarrow (S-\lambda_i I)v = 0$

$\boxed{\text{For } \lambda_1 = 1'3}$
$$\begin{bmatrix} 1'579 & 10'01 & -1'81 \\ 10'01 & 198'488 & -5'64 \\ -1'81 & -5'64 & 2'328 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow 1'579v_1 + 10'01v_2 - 1'81v_3 = 0$$
$$v_1 = \frac{1'81v_3 - 10'01v_2}{1'579} \Rightarrow v_1' = (-5'2, 1, 1)$$

$$e_1' = \frac{1}{\sqrt{v_1^2+v_2^2+v_3^2}}(v_1, v_2, v_3) = \frac{1}{\sqrt{28'97}}(-5'2, 1, 1) \Rightarrow e_1' = (-27'98, 5'38, 5'38)$$

$\boxed{\text{For } \lambda_2 = 4'5}$
$$\begin{bmatrix} -1'621 & 10'01 & -1'81 \\ 10'01 & 195'288 & -5'64 \\ -1'81 & -5'64 & -0'872 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow -1'621v_1 + 10'01v_2 - 1'81v_3 = 0$$
$$v_1 = \frac{1'81v_3 - 10'01v_2}{-1'621} \Rightarrow v_2' = (5'06, 1, 1)$$

$$e_2' = \frac{1}{\sqrt{27'6}}\cdot(5'06, 1, 1) = (26'58, 5'25, 5'25)$$
0'19

$\boxed{\text{For } \lambda_3 = 200'5}$
$$(2'879 - 200'5)v_1 + 10'01v_2 - 1'81v_3 = 0$$
$$-197'621v_1 + 10'01v_2 - 1'81v_3 = 0.$$
$$v_1 = \frac{1'81v_3 - 10'01v_2}{-197'621} \Rightarrow v_3' = (0'04, 1, 1)$$
$$e_3' = \frac{1}{\sqrt{2'001}}(0'04, 1, 1) = (0'057, 1'41, 1'41)$$

(ii) The axes of the ellipsoid are $\rightarrow \pm\sqrt{\lambda_i}\cdot\sqrt{c^2}\cdot e_i$

$$c^2 = \frac{p(n-1)}{(n-p)}F_{p,n-p}(0'1) = \frac{3(20-1)}{(20-3)}F_{3,20-3}(0'1) = \frac{57}{17}F_{3,17}(0'1) = \frac{57}{17}\cdot 5'19 = 17'40$$

$$\boxed{\text{Axes}} \begin{cases} (1) \pm\sqrt{1'3}\cdot\sqrt{17'40}\cdot[-27'98 \;\; 5'38 \;\; 5'38]' \\ (2) \pm\sqrt{4'5}\cdot\sqrt{17'40}\cdot[26'58 \;\; 5'25 \;\; 5'25]' \\ (3) \pm\sqrt{200'5}\cdot\sqrt{17'40}\cdot[0'057 \;\; 1'41 \;\; 1'41]' \end{cases}$$

(iii) Lengths of these axes $\rightarrow 2\cdot\sqrt{\lambda_i}\cdot\sqrt{c^2}$.

$$\boxed{\text{Lengths}} \begin{cases} (1) \sqrt{1'3}\cdot\sqrt{17'40}\cdot 2 = 9'51 \\ (2) \sqrt{4'5}\cdot\sqrt{17'40}\cdot 2 = 17'697 \\ (3) \sqrt{200'5}\cdot\sqrt{17'40}\cdot 2 = 118'13 \end{cases}$$

EXERCISE 5.7.

Find simultaneous 95% T^2 confidence intervals for μ_1, μ_2 and μ_3 using Result 5.3. Construct the 95% Bonferroni intervals using (5-29). Compare the two sets of intervals.

(i) $\boxed{T^2 \text{ confidence intervals}}$ $\rightarrow$ $a'\bar{x} - c\sqrt{\frac{a'Sa}{n}} \leq a'\mu \leq a'\bar{x} + c\sqrt{\frac{a'Sa}{n}}$

- $c = \sqrt{\frac{p(n-1)}{(n-p)} F_{p,n-p}(\alpha)} = \sqrt{\frac{3(20-1)}{(20-3)} F_{3,17}(0'05)} = \sqrt{\frac{57}{17} \cdot 3'20} = 3'2756$

- Interval for μ_1 $\quad \bar{x}_1 \pm c \cdot \sqrt{\frac{S_{11}}{n}} \Rightarrow 4'640 \pm 3'2756 \cdot \sqrt{\frac{2'879}{20}} \Rightarrow 4'640 \pm 1'2428$.

  $\Rightarrow \boxed{(3'397, 5'883)}$

- Interval for μ_2 $\quad \bar{x}_2 \pm c \cdot \sqrt{\frac{S_{22}}{n}} \Rightarrow 45'40 \pm 3'2756 \cdot \sqrt{\frac{199'788}{20}} \Rightarrow 45'40 \pm 10'3529$

  $\Rightarrow \boxed{(35'047, 55'753)}$

- Interval for μ_3 $\quad \bar{x}_3 \pm c\sqrt{\frac{S_{33}}{n}} \Rightarrow 9'965 \pm 3'2756 \cdot \sqrt{\frac{3'628}{20}} \Rightarrow 9'965 \pm 1'395$

  $\Rightarrow \boxed{(8'57, 11'36)}$

(ii) $\boxed{\text{Bonferroni intervals}}$ $\rightarrow$ $\bar{x}_p \pm t_{n-1}\left(\frac{\alpha}{2p}\right)\sqrt{\frac{S_{pp}}{n}}$

- $n = 20$, $p = 3$ $\Rightarrow t_{20-1}\left(\frac{0'05}{2 \cdot 3}\right) = t_{19}(0'008\overline{3}) = 2'625$ (table page 759)

- Interval for μ_1 $\rightarrow$ $4'640 \pm 2'625 \cdot \sqrt{\frac{2'879}{20}} \Rightarrow 4'640 \pm 0'996 \Rightarrow \boxed{(3'644, 5'636)}$
- Interval for μ_2 $\rightarrow$ $45'40 \pm 2'625 \cdot \sqrt{\frac{199'788}{20}} \Rightarrow 45'40 \pm 8'297 \Rightarrow \boxed{(37'103, 53'697)}$
- Interval for μ_3 $\rightarrow$ $9'965 \pm 2'625 \cdot \sqrt{\frac{3'628}{20}} \Rightarrow 9'965 \pm 1'12 \Rightarrow \boxed{(8'845, 11'085)}$

(iii) $\boxed{\text{Comparison}}$

- For each component mean (μ_i), the Bonferroni interval falls within the T^2 interval, so the Bonferroni intervals provide more precise estimates than the T^2 intervals.

  This always happens because: $t_{n-1}\left(\frac{\alpha}{2p}\right) < \frac{p(n-1)}{(n-p)} \cdot F_{p,n-p}(\alpha)$