MTH350

Discrete Mathematics

Module 3: Sequences, Part I

This module will cover how to define sequences using a closed formula or recursively. Also, you will learn about two types of sequences and how to find their sums. Finally, you will look at the properties and applications of the Fibonacci Sequence.

Learning Outcomes

1. Analyze sequences given as a closed formula or recursively defined.
2. Determine whether a given sequence is arithmetic or geometric.
3. Apply the Reverse and Add technique to find arithmetic sequence sums.
4. Apply the Multiply, Shift and Subtract technique to find geometric sequence sums.
5. Analyze applications of the Fibonacci Sequence in real-world contexts.

For Your Success & Readings

Module 3 will cover the first two sections of the chapter on sequences. In these sections, two ways to define sequences are presented. It will be important for you to be comfortable expressing a sequence in both ways.

You will also learn about two types of sequences: arithmetic and geometric. These sequences have specific properties that make them easy to work with. You will learn how to find sums for these sequences using methods proposed by the textbook author.

Finally, the lecture for Module 3 presents a unique sequence called the Fibonacci Sequence. You will get a chance to explore this fascinating sequence more in depth through the critical thinking assignment this week as well as in the recommended readings for this module which will give you examples of IT-related applications of the Fibonacci Sequence.

The discussion board for this week will give you an opportunity to expand your understanding of sequences and think about what types of sequences lend themselves best to each definition type. Make the most of this interactive portion of the course by posting early and by asking follow-up questions to your peers whenever possible.

Required

- Chapter 2, Sections **2.1** (http://discrete.openmathbooks.org/dmoi/sec_seq_intro.html)-**2.2** (http://discrete.openmathbooks.org/dmoi/sec_seq-arithgeom.html) in *Discrete Mathematics: An Open Introduction*
- Huang, Y., & Wen, Z. (2015). **The sequence of return words of the Fibonacci sequence** (https://www-sciencedirect-com.csuglobal.idm.oclc.org/science/article/pii/S0304397515005022?via%3Dihub). *Theoretical Computer Science, 593*, 106-116.
- Mneimneh, S. (2015). **Fibonacci in the curriculum: Not just a bad recurrence** (https://dl-acm-org.csuglobal.idm.oclc.org/citation.cfm?id=2677215). *Proceedings of the 46th ACM Technical Symposium on Computer Science Education - SIGCSE 15*. Hunter College and the Graduate Center of the City University of New York (CUNY). New York, NY: ACM.

1. Definitions

$$f(x)$$

Here, and in other areas of the course, you will find that the notion of a **function** appears in explanations of other concepts, so familiarize yourself with the following before moving on. You can learn more about functions by clicking “Learn More” at the end of this section. Even if you answer the end-of-page question correctly, it’s always a good idea to explore these concepts in greater depth!

A **function,** f**,** is a rule that assigns each **input exactly one output.** We call the output the **image** of the input.
$f: X \to Y$ is our way of saying that the function is called f, and that it goes from the set X, to the set Y.

$f(x) = y$ is read “f of x is equal to y”, means that when you input an element x from the set X, the function f gives the output y from the set Y.

For the purposes of this course, you can think of a function as a formula. For example, if the function f is defined as $f(x)=x+2$, then plugging in an input of $x=3$ gives us an output of 5 since $f(3)=3+2=5$.

Now, in this module we learn about sequences and how to define them.

04:03

Let’s begin with the most direct way of defining a sequence.

Closed formula

A **closed formula** for a sequence an where $n \in N$ is a formula for a_n using a fixed finite number of operations on n. This is what you normally think of as a formula in n, just like if you were defining a function in terms of n (because that is exactly what you are doing).

With this formula, we can think of a sequence as a **function** that assigns each index, n, a term in the sequence a_n.

Let’s look at an example. Suppose we have the sequence 0, 1, 4, 9, 16, ... In order to find the closed formula for this sequence, it might be helpful to create a diagram that aligns each index with its term. Click through the following interactive to match each index, n, with its corresponding term in the sequence.

The term a_0 is the first in the sequence and is 0.

The term a_1 is the first in the sequence and is 1.

The term a_2 is the first in the sequence and is 4.

The term a_3 is the first in the sequence and is 9.

The term a_4 is the first in the sequence and is 16.

What do you notice about the relationship between each index n and its corresponding term a_n? Hopefully, you noticed that a_n is just n^2. Hence, the closed formula for this example is $a_n = n^2$.

Sometimes it can be hard to find a closed formula for a sequence, and, instead, we can easily notice the pattern among the terms we are given. In this case, we might find the following way to define a sequence useful.

Recursive Definition

A **recursive definition** (sometimes called an **inductive definition**) for a sequence $a_n$where $n \in N$ consists of a **recurrence relation**: an equation relating a term of the sequence to previous terms (terms with smaller index) and an **initial condition**: a list of a few terms of the sequence (one less than the number of terms in the recurrence relation).

With a recursive definition, we need to define the first term a_0, as well as how each a_n relates to the previous term a_{n-1}.

To see how this works, consider the sequence defined recursively as $a_0 = 2$ and $a_n = a_{n-1} + 5$. This tells us that the sequence starts with the first term being 2, and then each term can be found by adding 5 to its preceding term. Note that the recursive definition must include *both* the recursion formula and where to start.

Click through the following activity to see how this recursive definition works.

Start at a_0

The first term in the sequence is a_0, which we are told is **2.**

To get the second term, add 5 to the first term.

$a_2 = a_1 + 5 = 2 + 7 = 7$. (Add 5 to get 2 + 5 = **7** as the second term).

To get the third term, add 5 to the second term.

$a_3 = a_2 + 5 = 7 + 5 = 12$. (Add 5 to get 7 + 5 = **12** as the third term.)

Continue this way to get the entire sequence.

So far, we have discovered that the sequence includes the terms **2, 7, 12, ...**. We can continue adding 5 to each term to get the next in order to obtain the entire sequence. Since this is an infinite sequence, we can just include the first few terms to show that the pattern corresponds to the recursive definition we were given.

Now, check your understanding of how sequences can be defined by answering the following question.

1.1. Functions

Recall that a function, f, is a rule that assigns each input exactly one output. We write $f: X \rightarrow Y$ to say that the function goes from the set X, to the set Y.

For example, we can define a function $f: \{1, 2, 3\} \rightarrow \{2, 4, 6\}$. In this case, we are assigning 1 to 2, 2 to 4, and 3 to 6.

Using function notation we would write $f(1) = 2$, $f(2) = 4$, and $f(3) = 6$.

In this case, we can also find a formula that relates the two sets. We notice that:

1 **multiplied by 2** is 2

2 **multiplied by 2** is 4

3 **multiplied by 2** is 6

Therefore, we can write the function as $f(x) = 2x$.

Let's look at another example. Suppose we have the function $f(x) = x^2-1$.

By plugging in values for x in the function formula $f(x)$, we can evaluate this function.

For example, for $x = 1$, $f(1) = 1^2 - 1 = 1 - 1 = \mathbf{0}$.

Check your understanding of functions with the following exercise.

1.2. Closed Formula and Recursive Definition

We have learned two ways to define a sequence. When there is a clear pattern between the index n and its corresponding term a_n, we can use the **closed formula** to define a function between n and a_n.

On the other hand, if it is easier to see a pattern among the terms themselves, we can write a **recursive definition** for the sequence that tells how to get each term based on previous ones. For this definition, it is also necessary to define the first term of the sequence.

Let's look at an example of a sequence that can be defined both with a closed formula and a recursive definition. Consider the sequence 0, 1, 3, 7, 15, 31...

To find the closed formula for this sequence, we look at each index and what term it corresponds to.

Index	**Term**
0	0
1	1
2	3
3	7
4	15
5	31

From here, we obtain the closed formula: $a_n = 2^n - 1$.

Next, looking at the relationship between the terms, we notice that each term is one more than twice the previous term. (e.g., 7 is one more than twice 3)

Thus, we have a recursive definition: $a_n = 2a_n\text{-}1 + 1$ with $a_0 = 0$.

Let's practice this once more before moving on to the next section.

2. Arithmetic and Geometric Sequences

In this section, we will learn about two particular types of sequences.

Arithmetic Sequences

If the terms of a sequence differ by a constant, we say the sequence is **arithmetic**. If the initial term (a_0) of the sequence is a and the **common difference** is d, then we have,

Recursive definition: $a_n = a_{n-1} + d$ with $a_0 = a$.

Closed formula: $a_n = a + dn$.

Let's look at an example of an arithmetic sequence. Consider the sequence 2, 5, 8, 11, 14, ...

Notice that there is a constant difference d between each term. In this case, that difference is 3. So we know that **d = 3**. We also notice that the first term a is 2. So **a = 2**. Hence, the closed formula for this arithmetic sequence is $\mathbf{a_n = 2 + 3n}$.

For the recursive definition, we need to know the first term and a formula for obtaining each term from previous ones. As shown above, we know that $a = 2$ and $d = 3$, so the recursive definition for this arithmetic sequence is $a_n = a_{n-1} + 3$ *with* $a_0 = 2$.

Now, let's look at the other type of sequence we will study in this section.

Geometric Sequences

A sequence is called **geometric** if the ratio between successive terms is constant. Suppose the initial term a_0 is a and the **common ratio** is r. Then we have,

Recursive definition: $a_n = ra_{n-1}$ with $a_0 = a$.

Closed formula: $a_n = a \cdot r^n$.

Let's look at an example of a geometric sequence. Consider the sequence 3, 6, 12, 24, 48, ...

Here, we notice that there is a common ratio of 2 since each term is multiplied by 2 to get to the next, and this geometric sequence has initial term 3.

Now, check your understanding of arithmetic and geometric sequences by answering the following.

2.1. Arithmetic Sequences

Recall that an arithmetic sequence is one in which the terms differ by a constant called the common difference.

If the initial term, a_0, of the sequence is a and the common difference is d, then we have,

Recursive definition: $a_n = a_n - 1 + d$ with $a_0 = a$.

Closed formula: $a_n = a + d_n$.

Let's look at an example. Consider the sequence 5, 7, 9, 11, ...

To define this function with a closed formula, we see that when $n = 0$, $a_0 = 5$, when $n = 1$, $a_1 = 7$, when $n = 2$, $a_2 = 9$, and when $n = 3$, $a_3 = 11$. So, we notice that each index n is multiplied by two and then increased by 5. In other words, we can write the formula $a_n = 2n + 5$.

To define this function recursively, we would note that $a_0 = 5$, and that there is a common difference of 2 between terms. That is, each term is two more than the previous one so we would write $a_n = a_{n-1} + 2$. Putting this all together, the recursive definition for this sequence is $a_0 = 5$ and $a_n = a_{n-1} + 2$.

Check your understanding of arithmetic sequences before moving on.

2.2. Geometric Sequences

Recall that a geometric sequence is one in which the ratio between successive terms is constant. If the initial term a_0 is a and the common ratio is r, then we have,

Recursive definition: $an = ran$-1 with $a_0 = a$.

Closed formula: $an = a{\cdot}rn$.

Let's consider an example. Consider the sequence 25, 5, 1, 1/5, ... This sequence has initial term 25 and is multiplied by 1/5 after each term. Thus, we have a recursive definition of a_n = 1/5 a_n−1 with a_0 = 25. Then, the closed formula is a_n = 5 · $(1/5)^n$.

Check your understanding of geometric sequences before moving on.

3. Sums of Arithmetic and Geometric Sequences

In this section, we will learn two methods that we can use to compute the sums of arithmetic and geometric sequences quickly. Also, we will become familiar with Σ notation for representing sums.

05:22

Summing Arithmetic Sequences: Reverse and Add:

Let's learn this strategy through an example.

Find the sum: $1 + 3 + 5 + 7 + ... + 351$.

Step 1: Determine the closed formula for the sequence.

Since this is an arithmetic sequence with initial term 1 and common difference 2, each term can be written as $a_n = 1 + 2n$.

Step 2: Determine the last term and the number of terms we want to add.

We are given the last term in the question prompt as 351.

To find the number of terms in the sequence 1, 3, 5, ..., 351, we can take the last term, 351, and set it equal to its explicit formula $a_n = 1 + 2n$. So we can set $351 = 1 + 2n$, then solve for n to get $n = 175$.

So we know that n is the index of the last term, (i.e., n goes from 0 to 175), so there are actually **176** terms in this sequence since we started at 0.

Step 3: Call the sum we are looking for S. Then reverse S and add it to itself.

$$
\begin{array}{rcl}
S & = & 1 + 3 + 5 + 7 + \dots + 349 + 351 \\
S & = & 351 + 349 + 347 + 345 + \dots + 3 + 1 \\
\hline
2S & = & 352 + 352 + 352 + 352 + \dots + 352 + 352
\end{array}
$$

Notice that by reversing and adding S to itself, we obtain an equation involving S that we can solve. From Step 2 we know that there are 176 of the 352s on the right hand side, so we can solve for S in Step 4.

Step 4: Solve the equation for S.

Now we have $2S = 352 \times 176$ and can solve for S to get, S = **30,976** as the desired sum.

Summing Geometric Sequences: Multiply, Shift and Subtract

Here is an example to see how this strategy works.

Find the sum: 3 + 6 + 12 + 24 + ... + 12,288.

Step 1: Determine the closed formula for the sequence.

This is a geometric sequence with initial term 3, and common ratio 2 since you multiply by 2 to get to each next term.

Step 2: Multiply the sum, S, by the common ratio.

The common ratio is 2 in this case, so we have:

$2S = 2\times3 + 2\times6 + \ldots + 2\times12{,}288 = 6 + 12 + \ldots + 24{,}576$.

Step 3: Subtract $2S - S$ to get the answer.

When subtracting $2S - S$, everything will cancel except for the first term and the last term multiplied by the common ratio.

$$
\begin{array}{rrl}
 & 2S = & \phantom{3 +{}} 6 + 12 + \ldots + 12{,}288 + 24{,}576 \\
- & S = & 3 + 6 + 12 + \ldots + 12{,}288 \\
\hline
 & S = & -3 + 0 + 0 + \ldots + 0 \qquad + 24{,}576
\end{array}
$$

Hence, $2S - S = -3 + 24{,}576 =$ **24,573** is the desired sum.

Note: This strategy only works when the common ratio is 2, but it is nonetheless worth learning as it is a useful strategy in other contexts (e.g., converting a repeating decimal to a fraction).

∑ notation

To simplify writing out sums, we will use notation as follows:

This means add up the as k goes from 0 to n. This is called sigma notation.

For example, we can write the sum 0 + 2 + 4 + 6 + ... + 200 as.

Notice that starting again from the sigma notation,

we can expand the sum by plugging in values from 0 to 100 for k: $2\times0 + 2\times1 + 2\times2 + .. + 2\times100 = 0 + 2 + 4 + \ldots + 200$.

To find this sum, we notice that the sequence 0, 2, 4, 6, ..., 200 is an arithmetic sequence so we can use the Reverse and Add method:

Step 1: Determine the closed formula for the sequence.

This is an arithmetic sequence with initial term 0 and common difference 2. So we can write the closed formula as $a_n = 0 + 2n$ or just $a_n = 2n$.

Step 2: Determine the last term and the number of terms we want to add.

We are given the last term in the question prompt as 200.

To find the number of terms in the sequence 0, 2, 4, ..., 200, we can take the last term, 200, and set it equal to its explicit formula $a_n = 2n$. So we can set $200 = 2n$, then solve for n to get $n = 100$.

So, we know that n is the index of the last term, (i.e., n goes from 0 to 100), so there are actually **101** terms in this sequence since we started at 0.

Step 3: Call the sum we are looking for S. Then reverse S and add it to itself.

$$\begin{array}{lcl} S & = & 0 + 2 + \ldots + 198 + 200 \\ S & = & 200 + 198 + \ldots + 2 + 0 \\ \hline 2S & = & 200 + 200 + \ldots + 200 + 200 \end{array}$$

Step 4: Solve the equation for S.

Now we have $2S = 200 \times 101$ and can solve for S to get, S **= 10,100** as the desired sum.

Before moving on, check your understanding of this section with the following problem.

3.1. Summing Arithmetic Sequences: Reverse and Add

Let’s look at another worked example of how to sum arithmetic sequences using the **Reverse and Add** method.

Find the sum of the first 100 terms of the sequence a_n *which starts as 8, 14, 20, 26, … . In other words, find:*

Step 1: Determine the closed formula for the sequence.

The closed formula is $a_n=8+6n$ since it starts at 8 and increases by 6 after each term.

Step 2: Determine the last term and the number of terms we want to add.

This time, we are told that there are 100 terms in the desired sum.

To find the last term, we just need to plug in n = 99 to get a_{99} = 8 + 6×99 = **602**.

Step 3: Call the sum we are looking for S. Then reverse S and add it to itself.

$$
\begin{array}{lcccccccccc}
S & = & 8 & + & 14 & + & \ldots. & + & 596 & + & 602 \\
S & = & 602 & + & 596 & + & \ldots. & + & 14 & + & 8 \\
\hline
2S & = & 610 & + & 610 & + & \ldots.. & + & 610 & + & 610
\end{array}
$$

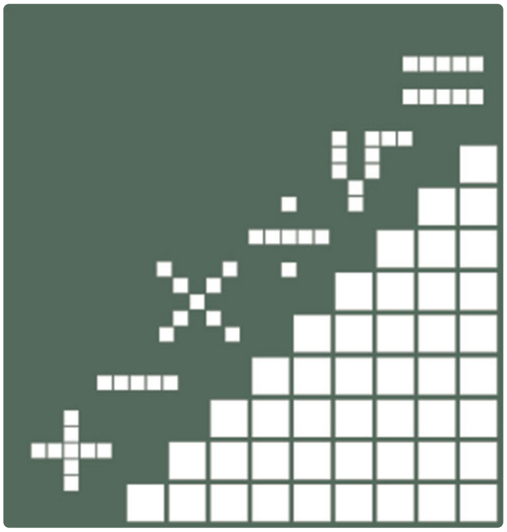
We know from Step 2 that there are 100 terms in the sequence we are adding, so there are 100 of the 610’s on the right hand side of the equation.

Step 4: Solve the equation for S.

Now we have $2S$ = 610 × 100 and can solve for S to get, S **= 30,500** as the desired sum.

Now, try the following problem to check your understanding.

3.2. Summing Geometric Sequences: Multiply, Shift and Subtract



Let's look at another worked example of how to sum geometric sequences using the **Multiply, Shift, and Subtract** method.

Find the sum: 2 + 4 + 8 + ... + 2,048.

Step 1: Determine the closed formula for the sequence.

This is a geometric sequence with initial term 2, and common ratio 2 since you multiply by 2 to get to each next term.

Step 2: Multiply the sum, S, by the common ratio.

The common ratio is 2 in this case, so we have:

$2S = 2\times2 + 2\times4 + \ldots + 2\times2{,}048 = 4 + 8 + \ldots + 4{,}096.$

Step 3: Subtract $2S - S$ to get the answer.

When subtracting $2S - S$, everything will cancel except for the first term and the last term multiplied by the common ratio.

Hence, $2S - S = -2 + 4{,}096 =$ **4,094** is the desired sum.

Now, try the following question on your own to check your understanding.

3.3. ∑ notation

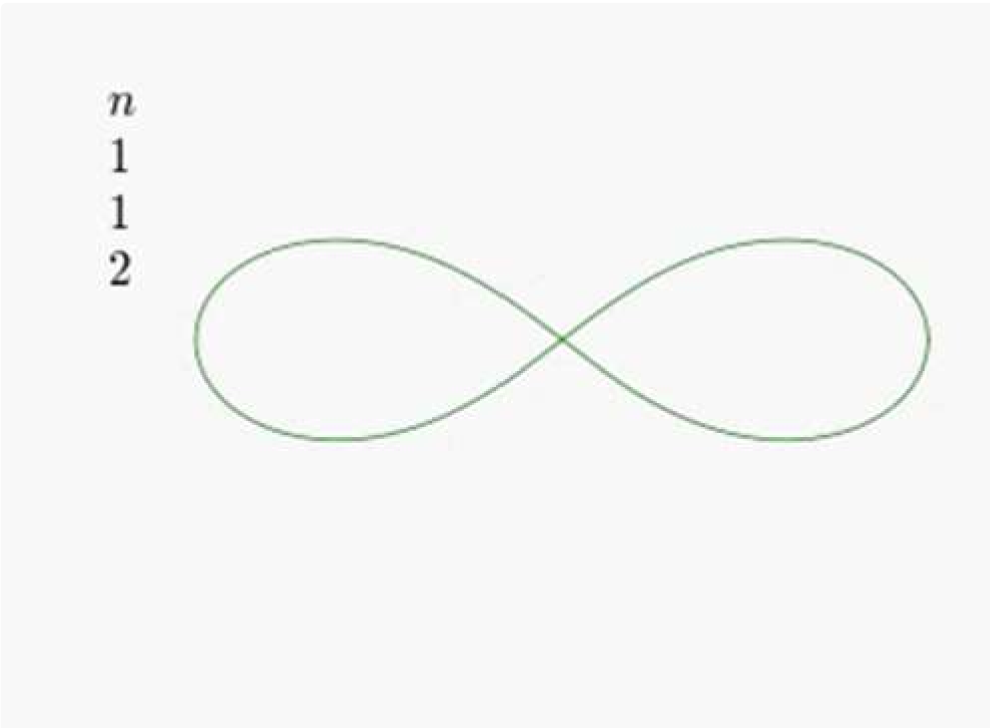
To learn more about sigma, ∑, notation, take a look at the following video which provides a more in-depth look at how it can be useful to represent the sum of a sequence.

- **Summation Notation** (https://www.khanacademy.org/math/algebra2/sequences-and-series/alg2-sigma-notation/v/sigma-notation-sum)

Now, check your understanding of how to use this notation with the following question:

4. The Fibonacci Sequence

One of the most interesting sequences that has applications in several real-world contexts is the **Fibonacci Sequence**: 0, 1, 1, 2, 3, 5, 8, 13, 21, ...



Picknick, 2014, CC-BY-SA-4.0

This sequence can be recursively defined with initial conditions $F_0=0$ *and* $F_1=1$ $F_n=F_{n-1}+F_{n-2}$. In other words, after the first two initial terms, 0 and 1, each term can be found by adding together the two previous ones.

To hear more about the fascinating properties of this sequence, take a look at this video.

- **The magic of Fibonacci numbers | Arthur Benjamin** (https://www.ted.com/talks/arthur_benjamin_the_magic_of_fibonacci_numbers)

One of the most widely known applications of the Fibonacci Sequence is how it appears in nature. The following video gives a nice overview of some examples where this occurs.

- **Golden Ratio Fibonacci Sequence TEDxEast: Matthew Cross** (https://ed.ted.com/on/AGPGLmOi#review)

Check your understanding of the Fibonacci Sequence with the following exercise.

4.1. Applications of the Fibonacci Sequence

One important aspect of the Fibonacci sequence is that the ratio between the numbers gets closer and closer to the irrational number

= 1.618033988... . This ratio is also called the **golden ratio** and is discussed in the following video.

The Golden Ratio (https://www.khanacademy.org/math/geometry-home/geometry-lines/the-golden-ratio/v/the-golden-ratio)

Now, take a look at this introduction to the sequence and how to view it as a function that can be programmed.

Exercise - Write a Fibonacci Function (https://www.youtube.com/watch?v=Bdbc1ZC-vhw)

Finally, check your understanding of one example of how the Fibonacci Sequence can be applied to the real world.

5. Summary

Click through the following interactives to view worked examples that cover key points for each module outcome.

Module Outcome #1: Analyze sequences given as a closed formula or recursively defined.

Consider the sequence -3, 3, 9, ...

What is the initial term of this sequence?

-3 is the first term of the sequence.

What is the common difference?

6 is the common difference since you add 6 to get to each subsequent term.

What is a closed formula for this sequence?

$a_n = -3 + 6n$ is a closed formula since it allows you to determine any term based on its index.

What is a recursive definition for this sequence?

$a_n = a_{n-1} + 6$ and $a_0 = -3$ is a recursive definition since, given the first term, we can build all other terms based on it.

Module Outcome #2: Determine whether a given sequence is arithmetic or geometric.

Determine whether the following sequences are arithmetic, geometric, or neither.

5, 9, 13, 17, ...

This is an **arithmetic** sequence with initial term 5 and common difference 4.

5, 15, 45, ...

This is a **geometric** sequence with initial term 5 and common ratio 3.

5, 7, 11, 15, ...

This is **neither** arithmetic nor geometric since there is no common difference or common ratio between the numbers.

Module Outcome #3: Apply the Reverse and Add technique to find arithmetic sequence sums.

Find the sum 2+3 + ... + 103.

Step 1: Determine the closed formula for the sequence.

Since this is an arithmetic sequence with initial term 2 and common difference 1, each term can be written as $a_n = 2+n$.

Step 2: Determine the last term and the number of terms we want to add.

We are given the last term in the question prompt as 103.

To find the number of terms in the sequence 2, 3, ..., 103, we can take the last term, 103, and set it equal to its explicit formula $a_n = 2+n$. So we can set 103 = 2 + n, then solve for n to get $n = 101$.

So, we know that n is the index of the last term, (i.e., n goes from 0 to 101), so there are actually **102** terms in this sequence since we started at 0.

Step 3: Call the sum we are looking for S. Then reverse S and add it to itself.

$$
\begin{array}{lcccccccccc}
S & = & 2 & + & 3 & + & .... & + & 102 & + & 103 \\
S & = & 103 & + & 102 & + & .... & + & 3 & + & 2 \\
\hline
2S & = & 105 & + & 105 & + & ..... & + & 105 & + & 105
\end{array}
$$

Notice that by reversing and adding S to itself, we obtain an equation involving S that we can solve. From Step 2 we know that there are 102 of the 105's on the right hand side, so we can solve for S in Step 4.

Step 4: Solve the equation for S.

Now we have $2S = 105 \times 102$ and can solve for S to get, $S =$ **5,355** as the desired sum.

Module Outcome #4: **Apply the Multiply, Shift and Subtract technique to find geometric sequence sums.**

Find the sum: 5 + 10 + 20 + ... + 2,560.

Step 1: Determine the closed formula for the sequence.

This is a geometric sequence with initial term 5 and common ratio 2 since you multiply by 2 to get to each next term.

Step 2: Multiply the sum, S, by the common ratio.

The common ratio is 2 in this case, so we have:

$2S = 2\times5 + 2\times10 + \ldots + 2\times2{,}560 = 10 + 20 + \ldots + 5{,}120$.

Step 3: Subtract $2S - S$ to get the answer.

When subtracting $2S - S$, everything will cancel except for the first term and the last term multiplied by the common ratio.

Hence, $2S - S = -5 + 5{,}120 =$ **5,1115** is the desired sum.

Module Outcome #5: Analyze applications of the Fibonacci Sequence in real-world contexts.

What is ϕ?

The ratio between the numbers in the Fibonacci Sequence gets closer and closer to this irrational number as the sequence progresses.

In what contexts can the golden ratio be found?

There are numerous places where the golden ratio is present, such as in spirals of galaxies, hurricanes, and flower petals. It can also be found in our DNA structure and other proportions of our bodies.

Check Your Understanding

Embedded Media Content! Please use a browser to view this content.

References

Picknick. (2014, December). Sequência de Fibonacci pétalas [Image file]. Retrieved from https://commons.wikimedia.org/wiki/File:Sequ%C3%AAncia_de_Fibonacci_p%C3%A9talas.gif (CC-BY-SA-3.0)