MTH350

Discrete Mathematics

Module 4: Sequences, Part II

This module will cover proof by induction. You will learn the formal induction proof structure as well as the formal strong induction proof structure. Also, you will learn about how to validate induction proofs and what real-world scenarios can be shown using induction.

Learning Outcomes

1. Determine whether or not a mathematical statement can be proven by mathematical induction.
2. Apply the mathematical induction proof technique to solve applied problems.
3. Prove mathematical statements by mathematical induction.
4. Prove mathematical statements by strong induction.
5. Assess the validity of induction proofs.

For Your Success & Readings

Module 4 will cover the section on induction. This proof method involves considering a statement that is true for specific cases and generalizing the statement for all possible cases. This technique can take some time to get used to, so this module is full of worked examples for you to practice as you proceed.

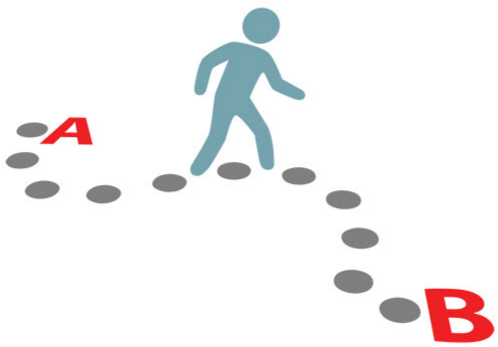
This week there is a midterm exam instead of a critical thinking assignment, so make sure to read the week's material and complete the mastery exercises early. This will allow you time to review for the midterm.

The discussion board this week will give you another chance to practice proving your own statement using induction. You will also get to reflect on the process and discuss your experience with your peers.

Required

- Chapter 2, **Section 2.5** (http://discrete.openmathbooks.org/dmoi/sec_seq-induction.html) in *Discrete Mathematics: An Open Introduction*
- García-Martínez, I., & Parraguez, M. (2017). **The basis step in the construction of the principle of mathematical induction based on APOS theory** (https://www-sciencedirect-com.csuglobal.idm.oclc.org/science/article/pii/S0732312316301080). *The Journal of Mathematical Behavior, 46*, 128-143.
- Stylianides, G. J., Sandefur, J., & Watson, A. (2016). **Conditions for proving by mathematical induction to be explanatory** (https://www-sciencedirect-com.csuglobal.idm.oclc.org/science/article/pii/S0732312316300098). *The Journal of Mathematical Behavior, 43*, 20-34.

1. Mathematical Induction



In the previous module, we learned how to define a sequence recursively by saying how the nth term relates to the previous term. This same recursive reasoning is what we will use in this module to prove mathematical statements by an argument style called **induction**.

07:04

Proof by induction can be useful when you have a situation involving many cases and when you know how each case relates to the previous one. A common example of such a situation is when trying to prove statements about natural numbers such as in the following example:

Prove for each natural number $n \geq 1$ that $1 + 2 + 3 + \ldots + n =$.

Before we look at the formal induction structure, let's go through the general idea of how induction can be used to prove this statement.

What is the big picture of this statement?

First, notice that the big picture of this statement is: For any natural number n greater or equal to 1, the sum of n numbers (on the left) is equal to a quantity involving n (on the right).

In this example, the cases that are possible are the different values that n can take: $n = 1$, $n = 2$, etc.... .

Consider the first case. (Base Case)

We begin by looking at the base case, which is the first possible case we want to prove. Here, that is the case when $n = 1$.

When $n = 1$, the left side will just be 1, and the right side will also be . So we know that this statement is true for the first case, when $n = 1$.

Build from one case to the next. (Inductive Step)

Next, the idea is to build from case to case and show that if the statement is true for $n = k$, then it is also true for $n = k + 1$.

To do this, we start by assuming that the statement is true for $n = k$. Assume that for any natural number $k \geq 1$, $1 + 2 + 3 + \ldots + k =$.

Now, based on the above statement being true, let's prove that it is also true for $k + 1$. (Note: this new statement is obtained by replacing k with $k+1$ in the assumed statement).

We want to prove that for any natural number $k \geq 1$, $1 + 2 + 3 + \ldots + k + (k+1)=$.

To do this, we will start with the left-hand side and work our way to obtain the right-hand side of the equation.

Notice that the left side of the equation includes the sum, $1 + 2 + \ldots + k$ which we already know is equal to . So, we can replace the sum $1 + 2 + \ldots + k$ with on the left hand side of the equation we want to prove to get:

$1 + 2 + 3 + \ldots + k + (k+1)=$ $+ (k+1)$

Now, combining and simplifying we have:

$+ (k+1) =$ which is the desired right hand side we wanted to obtain.

Now that we have the basic idea, let's look at the formal **Induction Proof Structure:**

Induction Proof Structure

Start by saying what the statement is that you want to prove: "Let $P(n)$ be the statement…" To prove that $P(n)$ is true for all, $n \geq 0$, you must prove two facts:

1. **Base case:** Prove that $P(0)$ is true. You do this directly. This is often easy.
2. **Inductive case:** Prove that $P(k) \rightarrow P(k+1)$ for all $k \geq 0$. That is, prove that for any $k \geq 0$ if $P(k)$ is true, then $P(k+1)$ is true as well. This is the proof of an if … then … statement, so you can assume $P(k)$ is true ($P(k)$ is called the *inductive hypothesis*). You must then explain why $P(k+1)$ is also true, given that assumption.

Assuming you are successful on both parts above, you can conclude, "Therefore, by the principle of mathematical induction, the statement $P(n)$ is true for all $n \geq 0$."

To see how this formal proof looks with an example, here is the formal induction proof of the example we saw above.

Prove for each natural number $n \geq 1$ that $1 + 2 + 3 + \ldots + n =$.

Proof:
Let $P(n)$ be the statement $1 + 2 + 3 + \ldots + n =$.
We will show that $P(n)$ is true for all natural numbers $n \geq 1$.

Base case: $P(1)$ is the statement $1 =$, which is clearly true.
Inductive case: Let $k \geq 1$ be a natural number. Assume (for induction) that $P(k)$ is true. That means $1 + 2 + 3 + \ldots + k =$. We will prove that $P(k+1)$ is true as well. That is, we must prove that $1 + 2 + 3 + \ldots + k + (k+1) =$. To prove this equation, start by adding $k+1$ to both sides of the inductive hypothesis:
$1 + 2 + 3 + \ldots + k + (k+1) = \quad + (k+1)$

Now, simplifying the right side we get:
$+ (k+1) =$
Thus $P(k+1)$ is true, so by the principle of mathematical induction, $P(n)$ is true for all natural numbers $n \geq 1$.

There are times when proving $P(k+1)$ requires knowing that all values less than $k+1$ are all also true. This is known as **Strong Induction** and is formally structured as follows:

Strong Induction Proof Structure

Again, start by saying what you want to prove: "Let $P(n)$ be the statement…" Then establish two facts:

1. **Base case:** Prove that $P(0)$ is true.
2. **Inductive case:** Assume $P(k)$ is true for all $k < n$. Prove that $P(n)$ is true.

Conclude, "therefore, by strong induction, $P(n)$ is true for all $n > 0$."

Now, check your understanding of how to write a formal induction proof by answering the following:

1.1. Using Mathematical Induction

Before looking at the formal structure of an induction proof, let's look at another example of how to work through an induction proof.

Prove for each natural number $n \geq 1$ that $3 + 6 + 9 + \ldots + 3n =$.

What is the big picture of this statement?

First, notice that the big picture of this statement is: For any natural number n greater or equal to 1, the sum of the positive multiples of 3 (on the left) is equal to a quantity involving n (on the right).

In this example, the cases that are possible are the different values that n can take: n = 1, n = 2, etc. … .

Consider the first case. (Base Case)

We begin by looking at the base case, which is the first possible case we want to prove. Here, that is the case when n = 1.

When n = 1, the left side will be 3×1 = 3, and the right side will also be . So we know that this statement is true for the first case, when n = 1.

Build from one case to the next. (Inductive Step)

Next, the idea is to build from case to case and show that if the statement is true for $n = k$, then it is also true for $n = k + 1$.

To do this, we start by assuming that the statement is true for $n = k$. Assume that for any natural number $k \geq 1$, $1 + 2 + 3 + \ldots + 3k =$.

Now, based on the above statement being true, let's prove that it is also true for $k + 1$. (Note: this new statement is obtained by replacing k with $k+1$ in the assumed statement).

We want to prove that for any natural number $k \geq 1$, $1 + 2 + 3 + \ldots + 3k + 3(k+1) =$.

To do this, we will start with the left-hand side and work our way to obtain the right-hand side of the equation.

Notice that the left side of the equation includes the sum, $1 + 2 + \ldots + 3k$ which we already know is equal to . So, we can replace the sum $1 + 2 + \ldots + 3k$ with on the left-hand side of the equation we want to prove to get:

$1 + 2 + 3 + \ldots + 3k + 3(k+1) = \quad + 3(k+1)$

Now, combining and simplifying we have:

$+ 3(k+1) =$ which is the desired right-hand side we wanted to obtain.

Now, check your understanding of how induction works by answering the following:

1.2. Induction Proof Structure

Let's look at one more example of an induction proof.

Statement:

Prove for each natural number $n \geq 4$ that $n! > 2^n$.

Proof:

Let $P(n)$ be the statement $n! > 2^n$

We will show that $P(n)$ is true for all natural numbers $n \geq 4$.

Base case: $P(4)$ is the statement $4! > 2^4$

Since $4! = 24$, and $2^4 = 16$, this is true.

Inductive case: Let $k \geq 4$ be a natural number. Assume (for induction) that $P(k)$ is true. That means $k! > 2^k$

We will prove that P(k+1) is true as well. That is, we must prove that $(k+1)! > 2^{k+1}$.

To prove this, start with our assumption that $k! > 2^k$ and multiply both sides of the inequality by $k + 1$ to get:

$(k + 1)(k!) > (k + 1)(2k)$

Also note that $2^{k+1} = (2)(2^k)$.

Now, since $k \geq 4$, $k + 1 \geq 2$, so $(k + 1)(2^k) > (2)(2^k) = 2^{k+1}$

Hence, $(k+1)! = (k + 1)(k!) > (k + 1)(2^k) > (2)(2^k) = 2^{k+1}$

Thus P(k+1) is true, so by the principle of mathematical induction, $P(n)$ is true for all natural numbers $n \geq 4$.

Now, check your understanding of the induction proof structure by answering the following:

2. Examples

The key to success in understanding how proof by induction works is to practice. Hence, this section will include worked examples of statements that are proven using mathematical induction.

03:03

So far, we have seen several examples of mathematical statements that can be proven using induction. It is important to think about what those statements have in common and why they can be proven using induction.

The main idea is that there must be a sequence of n cases to consider, and each of those cases must be related to the previous one in some way. Also, there must be an established starting point for n.

Now, consider the following examples of statements, and click through the interactive to determine if it can be proven using induction.

For all $n \in \mathbb{N}$, $6^n - 1$ is a multiple of 5.

This statement can be proven using induction since every time we increase n, we multiply by one more six and still get a number with last digit 6, so subtracting 1 gives us another multiple of 5.

For all integers n, if n is even, then n^2 is even.

This statement **cannot** be proven using mathematical induction since there is no way to establish a base case. If this were dealing with natural numbers, then we could prove this with a base case of $n = 0$.

For all $n \in \mathbb{N}$ with $n \geq 1$, $1^3+2^3+3^3+\cdots+n^3 =$ [Math Processing Error]

This statement can be proven using induction since each time n increases by 1, the desired sum increases by $(n + 1)^3$. In other words, we have a way to relate the statement $P(k+1)$ with the assumed statement $P(k)$.

Now check your understanding of when mathematical induction can be useful by answering the following:

2.1. Induction Example 1

Here is an example of a proof by induction:

Prove $1^3+2^3+3^3+\cdots+n^3=$ *holds for all* $n \geq 1$*, by mathematical induction.*

Proof:
Let $P(n)$ be the statement $1^3+2^3+3^3+\cdots+n^3=$.

We will show that $P(n)$ is true for all natural numbers $n \geq 1$.

Base case: $P(1)$ is the statement $1^3 =$ *,* which is clearly true.

Inductive case: Let $k \geq 1$ be a natural number. Assume (for induction) that $P(k)$ is true. That means $1^3 + 2^3 + 3^3 + \ldots + k^3 =$. We will prove that $P(k+1)$ is true as well. That is, we must prove that $1^3 + 2^3 + 3^3 + \ldots + k^3 + (k+1)^3 =$.

To prove this equation, start by adding $(k+1)^3$ to both sides of the inductive hypothesis:
$1^3 + 2^3 + 3^3 + \ldots + k^3 + (k+1)^3 =$ $+ (k+1)^3$.

Now, simplifying the right side we get:
$+ (k+1)^3 =$

Thus $P(k+1)$ is true so, by the principle of mathematical induction, $P(n)$ is true for all natural numbers $n \geq 1$.

Now, check your understanding by completing the following:

2.2. Induction Example 2

Here is another example of a proof by induction.

Prove that for all $n \in \mathbb{N}$, $6^n - 1$ is a multiple of 5.

Proof:
Let $P(n)$ be the statement, "$6^n - 1$ is a multiple of 5." We will prove that $P(n)$ is true for all $n \in \mathbb{N}$.

Base case: $P(0)$ is true since $6^0 - 1 = 0$ which is a multiple of 5.

Inductive case: Let $k \geq 1$ be a natural number. Assume, for induction, that $P(k)$ is true. That is, assume $6^k - 1$ is a multiple of 5. Then $6^k - 1 = 5j$ for some integer j. This means that $6k = 5j + 1$.

We want to show that $6^{k+1} - 1$ is also a multiple of 5, so we need an equation that shows $6^{k+1} - 1$ as a multiple of 5.

To get from the assumed equation, $6^k = 5j + 1$, we multiply both sides by 6:
$6^{k+1} = 6(5j + 1) = 30j + 6$.

Now, we want to know about $6^{k+1} - 1$, so subtract 1 from both sides:
$6^{k+1} - 1 = 30j + 5$

We notice that $30j + 5 = 5(6j + 1)$, so $6^{k+1} - 1$ is a multiple of 5.

Hence, $P(k + 1)$ is true. Thus, by the principle of mathematical induction, $P(n)$ is true for all $n \in \mathbb{N}$.

Now, check your understanding by completing the following:

2.3. Strong Induction Example

Here's an example of a proof by strong induction:

Prove that any natural number greater than 1 is either prime or can be written as the product of primes.

Proof:

Let $P(n)$ be the statement, "n is either prime or can be written as the product of primes." We will prove $P(n)$ is true for all $n \geq 2$.

Base case: $P(2)$ is true since 2 is a prime number.

Inductive case: Assume $P(k)$ is true for all $k < n$. We want to show that $P(n)$ is true (that n is either prime or is the product of primes).

If n is prime, we are done.

If not, then n has factors other than 1 and itself, so we can write $n = m_1 \cdot m_2$, with m_1 and m_2 less than n (and greater than 1). By the inductive hypothesis, m_1 and m_2 are each either prime or can be written as the product of primes. In either case, we have that n is written as the product of primes.

Thus, by the strong induction, $P(n)$ is true for all $n \geq 2$.

Now, check your understanding of strong induction by answering the following:

3. Validity of Induction Proofs

Beyond knowing how to write an induction proof, it is also important to be able to assess the validity of an induction proof.

Let's consider the following induction proof example and assess its validity.

Prove that for every $n \in \mathbb{N}$, $n < 100$.

Proof:
Let $P(n)$ be the statement, "for every $n \in \mathbb{N}$, $n < 100$".
First we establish the base case: when $n = 0$, $P(n)$ is true, because $0 < 100$.
Now for the inductive step, assume $P(k)$ is true. That is, $k<100$. Now if $k < 100$, then k is some number, like 80. Of course, $80 + 1 = 81$ which is still less than 100. So, $k + 1 < 100$ as well. But this is what $P(k+1)$ claims, so we have shown that $P(k) \rightarrow P(k+1)$.
Thus, by the principle of mathematical induction, $P(n)$ is true for all $n \in \mathbb{N}$.

Is there a valid base case?

Yes. $P(0)$ is true, so this step is valid.

Is the inductive step valid for all values of k?

No. While $P(k)$ implies $P(k+1)$ for some values of k, there is at least one value of k (namely $k = 99$) when that implication fails.

From the above example, we are reminded that in order for an induction proof to be valid, $P(k)$ must imply $P(k+1)$ for all values of k greater than or equal to the base case.

Now, check your understanding of checking the validity of an induction proof by answering the following:

3.1. Additional Example

Here is another example of an induction proof for which we will check the validity.

Prove that for all $n \in \mathbb{N}$*, the number* n^2+n *is odd.*

Proof:
Let $P(n)$ be the statement "for all $n \in \mathbb{N}$, the number n^2+n is odd." We will prove that $P(n)$ is true for all $n \in \mathbb{N}$. Suppose for induction that $P(k)$ is true, that is, that k^2+k is odd. Now consider the statement $P(k+1)$. Now $(k+1)^2+(k+1) = k^2+2k+1+k+1 = k^2+k+2k+2$.

By the inductive hypothesis, k^2+k is odd, and, of course, $2k+2$ is even. An odd plus an even is always odd, so therefore $(k+1)^2+(k+1)$ is odd.

Therefore, by the principle of mathematical induction, $P(n)$ is true for all $n \in \mathbb{N}$.

Is there a valid base case?

No. $P(0)$ is not considered in the proof. In fact, $P(0)$ is not true since $0^2+0 = 0$ is **not** odd. Hence, this step is invalid, and so is the proof.

Is the inductive step valid for all values of k?

Yes. The proof shows that $P(k)$ implies $P(k+1)$ for all values of k.

In this example, we saw that the base case is also necessary for induction to be valid.

Now, try the following example to check your understanding of how to validate induction proofs.

4. Summary

Click through the following interactives to view worked examples that cover key points for each module outcome.

Module Outcome #1: Determine whether or not a mathematical statement can be proven by mathematical induction.

Prove that 7^n - 1 is a multiple of 6 for all $n \in \mathbb{N}$.

This statement can be proven using mathematical induction since each time n increases, 7^n - 1 is still a multiple of 6. Also, since we have a starting point of n = 0, that allows for a valid base case.

Prove that $n < n^2$ for all integer values of n.

This statement **cannot** be proven using induction since there is no starting point for n. This is why the natural numbers (which begin at 0) are often involved in induction proofs.

Module Outcome #2: Apply the mathematical induction proof technique to solve applied problems.

Let's look at a real-world example that can be shown using mathematical induction.

Use induction to prove that if n people all shake hands with each other, that the total number of handshakes is $n(n-1)^2$.

Proof

Let $P(n)$ be the statement "when n people shake hands with each other, there is a total of handshakes." We will show that $P(n)$ is true for all natural numbers $n \geq 2$.

Base case

When $n = 2$, there will be one handshake, and = 1. Thus $P(2)$ is true.

Inductive case

Assume $P(k)$ is true for arbitrary $k \geq 2$ (that the number of handshakes among k people is . What happens if a $(k+1)^{st}$ person shows up? How many *new* handshakes take place? The new person must shake hands with everyone there, which is k new handshakes. So, the total is now , as needed.

Therefore, by the principle of mathematical induction, P(n) is true for all n ≥ 2.

Module Outcome #3: Prove mathematical statements by mathematical induction.

Prove $1^3+2^3+3^3+\cdots+n^3=$ *holds for all n ≥ 1, by mathematical induction.*

Proof

Let $P(n)$ be the statement $1^3+2^3+3^3+\cdots+n^3=$. We will show that $P(n)$ is true for all natural numbers $n \geq 1$.

Base case

$P(1)$ is the statement $1^3 =$, which is clearly true.

Inductive case

Let $k \geq 1$ be a natural number. Assume (for induction) that $P(k)$ is true. That means $1^3 + 2^3 + 3^3 + \ldots + k^3$ = . We will prove that $P(k+1)$ is true as well. That is, we must prove that $1^3 + 2^3 + 3^3 + \ldots + k^3 + (k+1)^3 =$.

Prove

To prove this equation, start by adding $(k+1)^3$ to both sides of the inductive hypothesis: $1^3 + 2^3 + 3^3 + \ldots + k^3 + (k + 1)^3 =$ $+ (k+1)^3$.

Simplify

Now, simplifying the right side we get: $+ (k+1)^3 =$

Thus, $P(k+1)$ is true so, by the principle of mathematical induction, $P(n)$ is true for all natural numbers $n \geq 1$.

Module Outcome #4. Prove mathematical statements by strong induction.

Prove that every positive integer is either a power of 2, or can be written as the sum of distinct powers of 2.

Proof

Let $P(n)$ be the statement, "n is either a power of 2 or can be written as the sum of distinct powers of 2." We will prove $P(n)$ is true for all $n \geq 1$.

Base case

$P(1)$ is true since 1 can be written as $1 = 2^0$.

Inductive case

Assume $P(k)$ is true for all $k < n$. We want to show that $P(n)$ is true (that n is either a power of 2 or can be written as the sum of distinct powers of 2).

If n is a power of 2, we are done.

If not, then suppose 2^x is the largest power of 2 that is strictly less than n. Then $n - 2^x$ is less than both n and 2^x. Thus $n - 2^x$ is either a power of 2 or can be written as the sum of distinct powers of 2, but none of them are going to be 2^x.

Hence, together with 2^x, we have written n as the sum of distinct powers of 2.

Thus by the strong induction, $P(n)$ is true for all $n \geq 2$.

Module Outcome #5. Assess the validity of induction proofs.

What is wrong with the following "proof" of the "fact" that $n + 3 = n + 7$ for all values of n (besides of course that the thing it is claiming to prove is false)?

Proof

Let P(n) be the statement that $n+3 = n+7$. We will prove that $P(n)$is true for all $n \in \mathbb{N}$. Assume, for induction, that $P(k)$ is true. That is, $k+3=k+7$. We must show that P(k+1) is true. Now since $k+3=k+7$, add 1 to both sides. This gives $k+3+1=k+7+1$. Regrouping $(k+1) + 3 = (k+1) + 7$. But this is simply $P(k+1)$. Thus by the principle of mathematical induction $P(n)$ is true for all $n \in \mathbb{N}$.

This induction proof is missing a base case. In fact, there is no case where a natural number, n, can satisfy the equation n+3 = n+7.

Check Your Understanding

Embedded Media Content! Please use a browser to view this content.

References

None