MTH410

Quantitative Business Analysis

Module 4: Discrete and Continuous Probability

In this module we continue our exploration and understanding of probability calculations using random variables and probability distributions. Specifically, we will examine binomial, Poisson, and hypergeometric distributions, and how expected value, variance, and standard deviation are incorporated into the distribution calculations.

Learning Outcomes

1. Understand random variables and how they are used in decision making.
2. Analyze and apply probability scenarios.
3. Calculate expected value, variance, and standard deviation.
4. Recognize probability distributions and analyze how they are used in decision making.

For Your Success & Readings

This module addresses different types of probability distributions. So that you don't become too confused, it may help you to identify each type and the fundamentals of each so that you can become more comfortable with them and be able to identify which calculation should be applied for which type of circumstance.

In this module we will begin using the statistical tables found in the back of your textbook. Today there are software programs such as SPSS and even Excel that can automatically calculate your statistics for you so that you don't have to reference table information; however, to help you understand the process of statistical calculation, we will sometimes compute them manually in this course and use the tables. You are welcome to use Excel, too.

To navigate through this module successfully, keep the following in mind:

- This week, you will complete your second Critical Thinking Assignment. Review the assignment early in the week and contact your instructor if you have any questions or concerns.

Required

- Chapters 4, 5 and Sections 6.1 and 6.2 in *Introductory Business Statistics*

Recommended

- Habibi, M., & Asgharzadeh, A. (2018). **Exponential-uniform distribution** (https://search-proquest-com.csuglobal.idm.oclc.org/docview/2133379936?rfr_id=info%3Axri%2Fsid%3Aprimo). *Iranian Journal of Science and Technology, Transactions A: Science, 42*(3), 1439–1450.
- Taylor, C. (2018, June 27). When do you use a binomial distribution? *ThoughtCo*. Retrieved from **https://www.thoughtco.com/when-to-use-binomial-distribution-3126596** (https://www.thoughtco.com/when-to-use-binomial-distribution-3126596)

1. Combinations, Random Variables

In a subsequent topic (the binomial distribution), we will need a formula for counting the number of ways that a specified task can be done where the order of operations does not matter.

If order does not matter, then the number of possible arrangements is defined by a combination. The number of combinations of *n* objects taken *x* at a time is written as and is defined as

is often phrased as "*n choose x*" to stress that what is chosen matters, and not the order. The

above formula for a combination is useful when you study the binomial and hypergeometric distributions.

For example, suppose a box has 12 books and Joe is given a reading assignment to read 10 of them. How many ways can Joe select 10 out of 12 books to read? Note that the order in which he reads the books does not matter. It only matters which 10 out of 12 books were chosen. Thus, the answer is "12 choose 10:"

Here is a combinations calculator to make computations quicker: **Combinations Calculator (nCr)** (https://www.calculatorsoup.com/calculators/discretemathematics/combinations.php)

With Excel, you would type in =COMBIN(12,10)

You can also find the above function in Excel by pressing Formulas at the top and then go to math & trig.

Example: A room has 100 people. Ten of them will win a trip to Hawaii. In how many ways can this be done?

Solution: The order in which the winners are chosen does not matter. Thus, this is a combinations problem, 100 choose 10. Using the above calculator, we get:

A **random variable** provides a way to describe experimental outcomes using numerical values. These values can be used for a variety of decision-making purposes.

A random variable can be called a **discrete variable** or a **continuous variable**.

Click the tabs below to learn more about each.

Discrete VariableContinuous Variable

A discrete random variable can assume either a finite number of values or an infinite sequence of values that can be listed one by one.

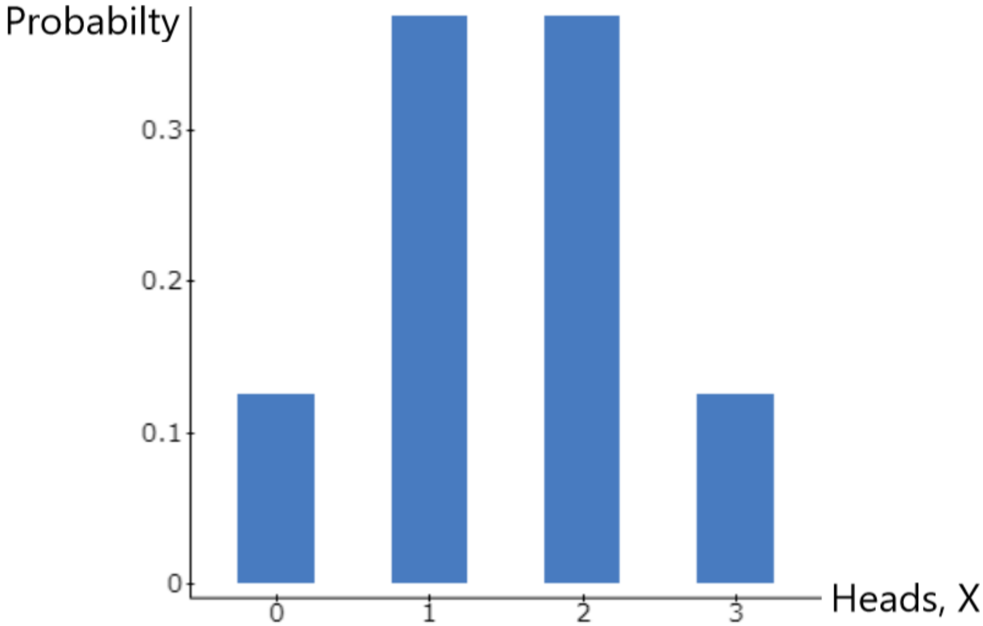
A continuous random variable may assume any numerical value in an interval or collection of intervals.

Similarly, **probability distributions** can be classified as discrete or continuous. If a random variable is a discrete variable, its probability distribution is called a **discrete probability distribution**. For example, if you flip three coins at the same time you can have eight possible outcomes: HHH, HHT, HTT, HTH, THH, TTH, THT, and TTT. If the random variable X represents the number of Heads that result from this experiment, then the random variable X can only take on the values 0, 1, 2, or 3. Hence, X is a discrete random variable. Each possible value of the discrete random variable can be associated with a probability. Therefore, a discrete probability distribution can be presented in tabular form.

Here is the probability distribution for the number of Heads when one tosses three coins at the same time:

Number of Heads, x	Probability, $P(x)$
0	1/8 = 0.125
1	3/8 = 0.375
2	3/8 = 0.375
3	1/8 = 0.125

$P(x)$ is also called a **probability density function**. The corresponding graph of the distribution is shown:



Properties of a discrete probability distribution or probability density function:

1. For any outcome value x in a discrete probability distribution we have $0 \le P(x) \le 1$.
2. The sum of all the probabilities in the table must equal 1.

For example, the following is not a probability distribution:

Value, x	Probability, $P(x)$
0	0.3
1	0.1
2	0.2

Each probability is between 0 and 1. However, the sum of the probabilities is 0.6 and not 1.
Here is a valid probability distribution:

Value, x	Probability, $P(x)$
0	0.3
1	0.1
2	0.6

Each probability is between 0 and 1. The sum of all the probabilities is 1.

If a random variable is a continuous variable, its probability distribution is called a **continuous probability distribution**. A continuous probability distribution differs from a discrete probability distribution because the probability that a continuous random variable will assume any single value is zero. Therefore, a continuous probability distribution cannot be expressed in tabular form, but rather as an equation or formula to describe a continuous probability distribution.

Mean for a Discrete Random Variable or of a Probability Density Function

The general formula for the mean of a discrete probability distribution is as follows:

The summation is for all possible values, *x*, of the probability distribution.

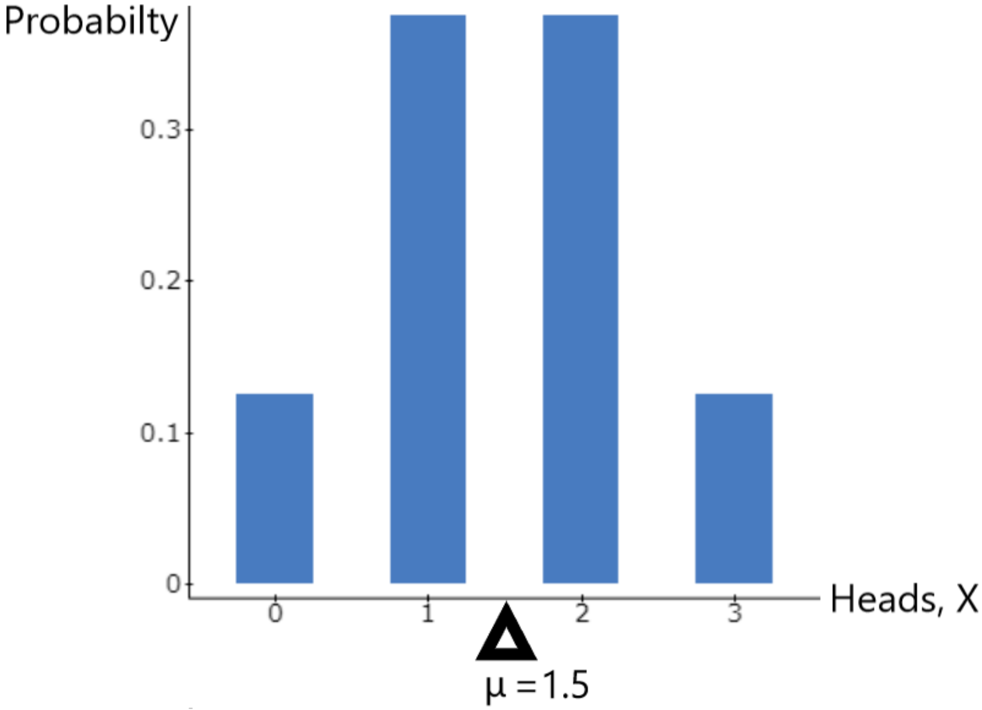
E(*X*) is called the **expected value** of the random variable *X* and *P*(*x*) is the probability of value *x*. The expected value of the random variable *X* is the same as the mean, μ. *E*(*X*) is the value of *X* that one expects, on average, when the experiment is performed many times.

For example, recall the probability distribution of the number of Heads, *X*, when three coins are tossed at the same time:

Number of Heads, *x*	Probability, *P*(*x*)
0	1/8 = 0.125
1	3/8 = 0.375
2	3/8 = 0.375
3	1/8 = 0.125

The mean of the above probability density function is

The meaning of the above number is that if we tossed three coins many times and recorded the number of heads, then the average of all samples will be approximately 1.5. The geometric meaning of the mean is the balance point of the graph of the probability distribution:



In a certain lottery game, 100 tickets are sold at $2 each. Eighty-five of the tickets are losing tickets. Fourteen tickets will win $3. One ticket will win $50. Determine the expected winnings by a player.

Click “Solution” to check your thinking.

Solution

Let X denote the net payoff, the net amount of what the player receives or loses. If a player gets a losing ticket, then he/she loses \$2 (x=−2). The probability of losing is 85/100=17/20. If a player wins \$3, the net payoff to each such winner is \$1 (x=1) since he/she paid \$2 to play (\$3 -\$2=\$1). The probability of this happening is 14/100=7/50. Finally, if a player wins \$50, the net payoff is \$48 (x=48). The probability of winning the \$50 amount is 1/100.

We then obtain the following discrete probability density function:

x	−2	1	48
$P(X)$	17/20	7/50	1/100

The mean is the expected winning:

.

Thus, a player can expect to lose \$1.08, on average. This means that if a player played many times, then the average net payoff will be approximately a loss of \$1.08.

1.1. More on the Mean of a Discrete Random Variable from the Probability Density Function

Recall that the general formula for the mean of a discrete probability density function is as follows:

The summation is for all possible values, x, of the probability distribution.

The expected value is useful in insurance, too. An insurance company can use it to assess the profit of certain policies. Here is a simple example.

EXAMPLE

Joe bought a $50 yearly policy to insure car stereo system for theft. The insurance company will write joe a check of $1,000 if the system gets stolen. From police records, the probability of a such a theft is 0.01. Find the expected amount made by the insurance company by insuring a stereo system for one year to one client.

Click "Solution" to check your thinking.

Solution

Let X denote the amount made by the company. If the stereo system gets stolen, the insurance company loses $950 ($X$=-950) since the company already made $50 from the policy (1000-50=950). The probability of the system getting stolen is 0.01. If the system does not get stolen, the company makes $50 ($X$=50). The probability of the system not getting stolen is 1.0 -.01 =0.99. We then obtain the following discrete probability density function:

x	-950	50
$P(X)$	0.01	0.99

The mean is the expected profit:
The insurance company expects to make $40 per client.

2. Discrete Probability Distributions

Binomial Probability Distribution

Suppose you are tossing a die 7 times. If you roll a 3 you get $1. Otherwise, you get nothing. This is an example of a binomial experiment. Each of the 7 trials is either a "success" or "failure." The probability of success is 1/6 for each trial. This sequence of trials is an example of a **binomial experiment**.

A **binomial experiment** is a statistical experiment that has four properties. *Click on each tab to learn more.*

The experiment consists of *n* repeated trials.

Each trial can result in just two possible outcomes. We call one of these outcomes a success and the other, a failure.

The probability of success, denoted by *p*, is the same on every trial.

The trials are independent; that is, the outcome on one trial does not affect the outcome of other trials.

For example, for a statistical experiment, you flip a coin two times and count the number of times the coin lands on heads. This is a **binomial experiment** because:

- The experiment consists of repeated trials. We flip a coin two times.
- Each trial can result in just two possible outcomes—heads or tails.
- The probability of success is constant (0.5) on every trial.
- The trials are independent; that is, getting heads on one trial does not affect whether we get heads on other trials.

A **non-binomial experiment** is picking five chips **without replacement** from a box containing six red chips and seven blue chips. On each trial, there are two possible outcomes (red or blue). However, the trials are **not independent**. For example, if red chip appears on trial 1, then that means there are now five reds on trial 2. The outcome on trial 1 affects the probability of getting a red chip on trial 2.

If the chips were chosen **with replacement**, then we would have a binomial experiment. For example, we could let success be an outcome of "blue." The trials are independent because the chips are placed back into the box after each trial.

A **binomial random variable** is the number of successes x in n repeated trials of a binomial experiment. The probability distribution of a binomial random variable is called a **binomial distribution**.

Suppose we flip a coin two times and count the number of heads. The binomial random variable is the number of heads, which can take on values of 0, 1, or 2. The binomial distribution is:

Number of Heads, x	Probability, $P(x)$
0	0.25
1	0.50
2	0.25

For example, in the above table, we see that the binomial probability of getting exactly one head in two coin flips is 0.50.

Given x, n, and P, we can compute the binomial probability based on the following formula:

n = number of trials, x = number of successes
p = probability of success on each trial, $q=1-p$ = probability of failure on each trial

Recall that is n choose x:

(number of combinations of n objects taken x at a time)

Recall, is the number of ways to choose x objects out of n without regard to order. is the number of ways of obtaining x successes out of n trials.

It turns out that the mean and standard deviation of the binomial distribution are:
Mean of the Binomial Distribution: $\mu = E(X) = np$
Standard Deviation of the Binomial Distribution:

For example, if a die is tossed five times, what is the probability of getting exactly two sixes? Suppose "success" is getting a six on the toss of a die. This is a binomial experiment in which the number of trials is equal to five (n=5), the number of successes is equal to two (x=2), and the probability of success on a single trial is 1/6 or about 0.167 (p=0.167). Therefore, the binomial probability is:

You can also use tables to determine binomial probabilities. The tables for these probabilities include data regarding the number of values (n), the number of successes (9), and the probability of success (p). For quicker results, one may also use an **online binomial distribution probability calculator**. For example, here is one such online calculator: **Stat Trek Binomial Probability Calculator** (http://stattrek.com/online-calculator/binomial.aspx)

EXAMPLE

Suppose you are trying to determine the probability that the number 5 would appear exactly 3 (x=3) times in ten rolls of a die (n=10). Suppose "success" is getting a 5 on the toss of a die. The probability of success on each trial is 1/6 = 0.167 (p=0.167). You could use the above online calculator to obtain the probability of exactly 3 successes in 10 trials: $P(3) = 0.1555$

In the above example, find the probability that the number 5 will appear at most 6 times. The answer is of the form:
$P(X \leq 6) = P(0) + P(1) + P(2) + P(3) + P(4) + P(5) + P(6)$

However, it is tedious to compute and add those seven probabilities. Fortunately, we can use an online calculator like the one above to quickly get the answer:

$P(X \leq 6) = 0.9997$

As an alternative, we may use Excel to find $P(X \leq 6)$: = BINOM.DIST(6, 10, 0.167, TRUE)

EXAMPLE

In the town of Hickoryville, an adult citizen has a 1% chance of being selected for jury duty next year. If jury members are selected at random, and the Anderson family includes three adults, determine the probability that

1. None of the Andersons will be chosen.
2. Exactly one of the Andersons will be chosen.
3. Exactly two of the Andersons will be chosen.
4. All three of the Andersons will be chosen.

This is a binomial experiment with n=3 and p=0.01. Let x denote the number of Andersons chosen. We need the binomial probability formula to determine all of the above probabilities:

1. 1. $f(0)$ is the probability that none of the Andersons will be chosen:

2. 2. $f(1)$ is the probability that exactly one of the Andersons will be chosen.

We can continue the last two in the same manner by plugging in the appropriate values into $P(x)$.

3.

4.

Poisson Distribution:

If, however, we have an experiment in which the occurrence or nonoccurrence in any interval is independent of the occurrence or nonoccurrence in any other interval, and the probability of an occurrence of the event is the same for any two intervals of equal length, we can use the **Poisson Probability Distribution**. In other words, a **Poisson experiment** is a statistical experiment having the following properties.

Click on each tab to learn more.

The experiment results in outcomes that can be classified as successes or failures.

The average number of successes (μ) that occurs in a specified region is known.

The probability that a success will occur is proportional to the size of the region.

The probability that a success will occur in an extremely small region is virtually zero.

When we use the Poisson distribution we assume:

- e: A constant equal to approximately 2.71828. (Actually, e is the base of the natural logarithm system.)
- μ: The mean number of successes that occur in a specified region.
- The notation for the mean number of successes, μ, is also denoted with the Greek letter lambda: $\mu = \lambda$
- x: The actual number of successes that occur in a specified region.
- $f(x)$: The **Poisson probability** that exactly x successes occur in a Poisson experiment, when the mean number of successes is μ.

EXAMPLE

Suppose the number of clients your sales representative will see each five-day week (Monday through Friday) is distributed according to a Poisson distribution with an average of 10 every five days.

a. What is the average number of clients the sales representative will see per day?
b. What is the probability that your sales representative will see three clients on any given day?
c. What is the probability that your sales representative will see more than three clients on any given day?

Click on "Solution" to check your thinking.

Solution

- This is a Poisson experiment. Let *X* denote the number of customers the sales representative will see on any given day. In this part of the problem, we need to find the parameter, , for the average number of customers that the sales representative sees per day. Since the sales representative sees an average of 10 customers every 5 days, then he/she will see an average of 10/5 = 2 customers per day.

  Thus, the parameter for the mean number of customers per day is customers.

- To answer part b, this is the information that we have:

  μ = 2; since your sales representative will see two customers per day on average.
  x = 3; since we want to find the likelihood that three clients will be seen on any given day.
  e = 2.71828; since e is a constant equal to approximately 2.71828.

  We plug these values into the Poisson formula as follows:

  Thus, the probability of your sale representative seeing 3 clients on any given day is 0.180.

- Use the rule of complements:

  Thus,

  As an alternative, we may use Excel to find *P(X≤3):* = POISSON(3, 2, TRUE)

Hypergeometric Probability Distribution

Recall the following example:

A non-binomial experiment is picking five chips without replacement from a box containing six red chips and seven blue chips. On each trial, there are two possible outcomes (red or blue). However, the trials are not independent. For example, if a red chip appears on trial 1, then that means there are now five reds on trial 2. The outcome on trial 1 affects the probability of getting a red chip on trial 2. Thus, one may not use the binomial probability function.

The **hypergeometric probability function** is used to compute the probability of x successes out of n trials. Thus, $n–x$ is the number of failures. Let r denote the number of successes in the population, and let N be the number of elements in the population. Then $N–r$ is the total number of failures in the population. The following is the **hypergeometric probability function:**

where

x = the number of successes, n = the number of trials

$f(x)$ = the probability of x successes in n trials

N = the total number of elements in the population, r = the total number of successes in the population

is the number of ways to get x successes out of a total of r in the population.

is the number of ways get n-x failures out of a total of N-r (failures) in the population.

is the number of ways to choose n objects out of a total of N in the population.

EXAMPLE

Consider the above example. Suppose five chips are chosen without replacement from a box containing six red chips and seven blue chips. Find the probability of choosing three blue chips. To solve this problem, one would need the Hypergeometric Distribution.

Click on “Solution” to check your thinking.

Solution

Let *x* denote the number of blue chips. There is a total of 6 + 7 = 13 chips in the box (*N*=13).

Since one wants to find the probability of getting three blue chips and there are five trials, then the other two chips will be red. We have the following:

x = 3 successes (blue chips), n = 5 trials

N = 13 total number of chips in the population, r = 7 total number of successes (blue chips) in the population

$N - r = 13 – 7 = 6$ total number of failures (red chips) in the population

$n - x = 5 - 3 = 2$ failures (red chips) out of 5.

The key here is to think of the experiment as one where all five chips are selected from the box at the same time:

$f(3) = \quad = \quad = \quad = 0.408$

is the number of ways to choose (without regard to order) five chips out of 13.

is the number of ways to choose (without regard to order) three blue chips out of seven.

is the number of ways to choose (without regard to order) two red chips out of six.

The Hypergeometric distribution is like the Binomial Distribution, except that the trials are **not independent**, and the **probability of success varies for each trial**.

Discrete probability distributions, such as the Binomial Probability Distribution, Poisson Probability Distribution, and Hypergeometric Distribution can provide information as to how often an event occurs. Your ability to calculate these distributions can provide important insight needed for decisions in planning and those that concern intervals of time and space.

2.1. Mean of the Poisson Distribution

Sometimes it is not clear what the mean of the Poisson Distribution is. The following is an example of such situation.

Suppose a center receives calls according to the Poisson Distribution of 15 calls in a three-hour period. Find the probability that there are 5 calls in a two-hour period.

Click "Solution" to check your thinking.

Solution

Let X denote the number of calls during a two-hour period. Finding the mean, μ, number of hours during a two-hour period is not clear because we are given that there is an average of 15 calls during a three-hour period, not a one-hour period. To find μ, we solve the following ratio:

Solving for μ in the above equation, we obtain μ = 10 calls in a two-hour period. We then use the Poisson distribution with x=5 and μ=10:

The above method can be used to find the mean for the Poisson distribution.

For example, if we want to find the probability that there are 5 calls in a four-hour period, we would solve for μ using this ratio:

The mean number of calls in a four-hour period would then be μ =20. The probability that there are 5 calls in a four-hour period would then be

2.2. More Examples of the Binomial Distribution

Recall that given x, n, and P, we can compute the **binomial probability** based on the following formula:

n = number of trials, x = number of successes
p = probability of success on each trial, $q=1-p$ = probability of failure on each trial
Mean of the Binomial Distribution: $\mu = E(K) = np$

Standard Deviation of the Binomial Distribution:

At a tube factory, approximately 3% of the tubes produced are defective. If 300 tubes are produced, find the probability that more than 7 tubes are defective. Also, find the mean number of defectives and the standard deviation, too.

Click "Solution" to check your thinking.

Solution

These are binomial trials since tube defects are independent and the probability of a tube being defective is approximately constant. Let X be the number of defective tubes. In this case, n=300 trials and the probability of success is p=0.03. We want to find $P(X>7)$.

It is a large amount of work to compute

.

One may use the complement:

However, the computation might still be tedious. One may use a binomial calculator like the following: **Stat Trek Binomial Probability Calculator** (http://stattrek.com/online-calculator/binomial.aspx)

The answer will be

As an alternative, we may use Excel to find : = BINOM.DIST(7, 300, 0.03, TRUE)

The mean number of defectives is $\mu = np = 300 \cdot 0.03 = 9$ defective tubes.

The standard deviation of the number of defective tubes is

EXAMPLE

At a restaurant, approximately 5% of the orders are returned because they are too cold. If 30 orders are placed, what is the probability that at least one order is returned? How many returned orders are expected?

Click "Solution" to check your thinking.

Solution

These are binomial trials since returned orders are independent and the probability of an order being returned is approximately constant. The probability of "success" is $p = 0.05$ and the number of trials is 30 (n=30). Let X denote the number of returned orders. We want to find $P(X \geq 1)$. By the rule of complements we can rewrite the problem in this way that is easier to compute:

The expected number of returned orders is orders.

EXAMPLE

(Quality Control): The quality control technician will reject a batch of 20 aluminum bottles if two or more filled bottles are found to have imperfections. Approximately 1% of bottles have imperfections. Find the probability that a batch will be rejected.

Click "Solution" to check your thinking.

Solution

These are binomial trials since bottle imperfections are independent and the probability of a bottle having an imperfection is approximately constant. By the rule of complements, we can first find the probability that a batch will be accepted:

$P(rejected)=1-P(accepted)$

To find the probability that a batch will be accepted, the probability of failure is p=0.01 and n=20 trials. Let X denote the number of rejected bottles. A batch will be accepted if there are fewer than 2 imperfections:

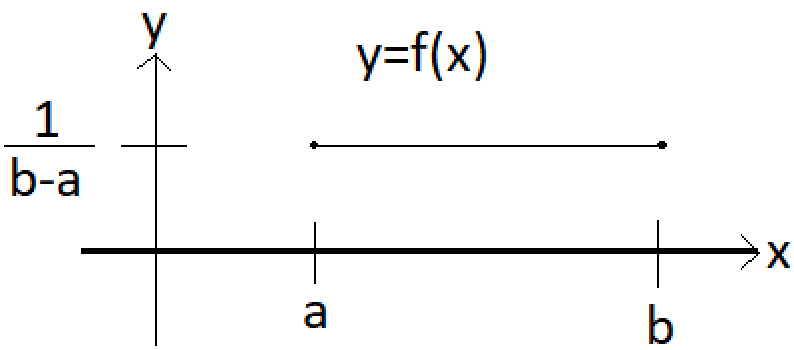
Thus,
$P(rejected)=1-P(accepted)=1-0.983=0.017$

3. Continuous Probability Distributions

A **continuous random variable** is one whose values can be any value over a number line (or union of number lines). For example, the weight of a randomly chosen person is a continuous random variable. This is because weight can be any number over an interval.

The **Uniform Probability Distribution Function** is defined as the following piecewise-defined function:

The following is the graph of the Uniform Probability Density Function. Notice that there is a horizontal line for x values between a and b. For x-values outside of the interval from a to b, the values of the function are zero.



It is a continuous distribution since x can be any number in the interval $[a, b]$. The area under the probability density function represents a probability. The mean of the uniform distribution is the midpoint of the interval $[a, b]$:

The standard deviation of the uniform distribution is:

The area under the above curve represents a probability.

The above function is often used when a number is randomly chosen from a bounded interval. Here is an example:

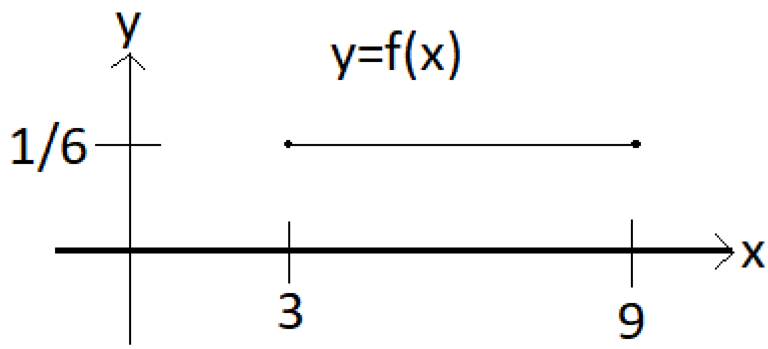
Suppose that a number is randomly chosen from the interval [3, 9]. Find the probability that a number between 4 and 8 is chosen. Find the average number chosen and the standard deviation.

Click "Solution" to check your thinking.

Solution

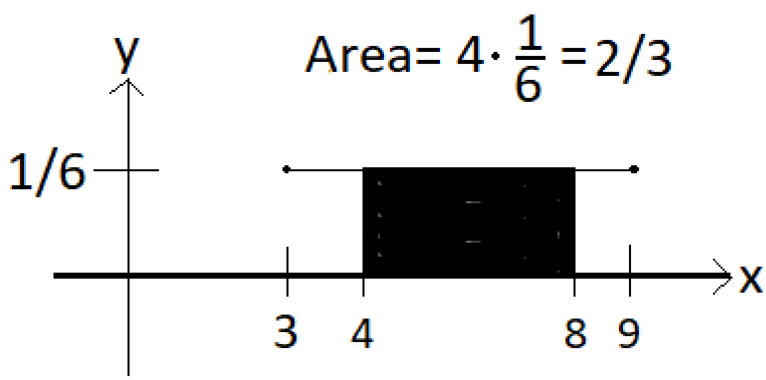
Let the random variable X denote the number that appears. Then X is a continuous random variable since it can attain any number in the interval [3, 9].

The graph of the probability density function is



To find the probability that a number between 4 and 8 appears, $P(4 \leq x \leq 8)$, you can find the area under the probability distribution function:

Use that the area of a rectangle is the product of its base and height:



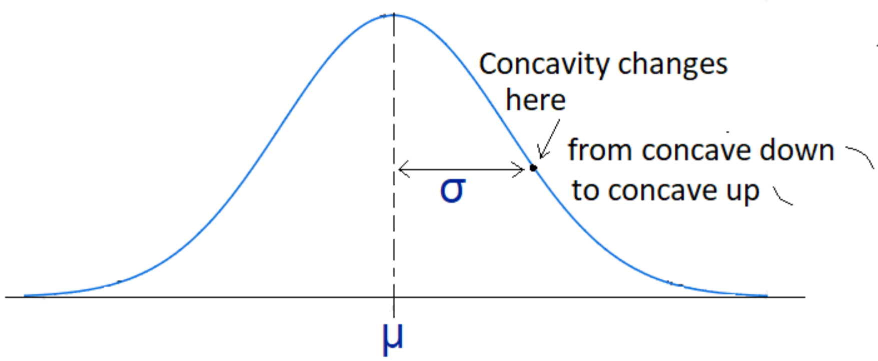
Since the length of [4,8] is 4 and that of [3,9] is 6, then the probability that a number between 4 and 8 is chosen is:

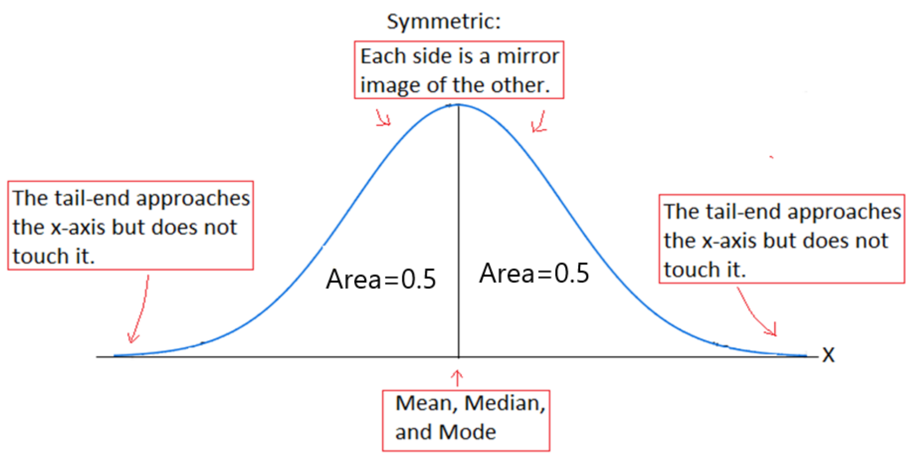
A way to understand the above method is to think of throwing darts. A dart that must land in the interval [3,9] will be thrown. To find the probability that the dart will land somewhere in the subinterval [4,8], you determine what proportion of the entire interval [3,9] is occupied by [4,8]: (8-4)/(9-3)= 4/6= 2/3

The mean and standard deviation are:

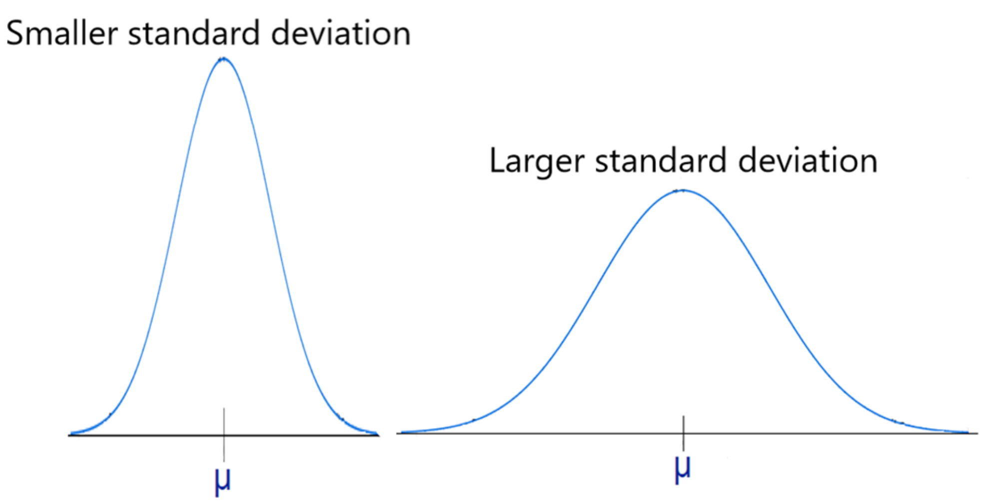
Another continuous distribution is the **Normal Probability Distribution**. It is an important distribution used to describe many real-life applications. It has important applications in probability and inferential statistics.

The normal distribution is symmetric about the vertical line (axis of symmetry). **Its mean and standard deviation are μ and σ, respectively.** It is bell-shaped. The standard deviation can be measured as the distance from the axis of symmetry to the point where concavity changes:

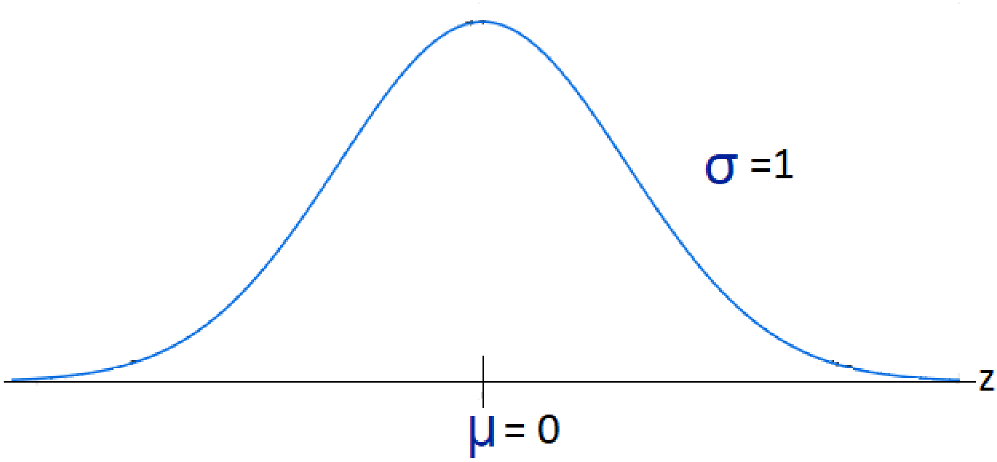




A larger standard deviation corresponds to a wider bell-shaped curve. A smaller standard deviation produces a narrower yet taller bell-shaped curve. Both curves have a total area of 1 under the curve:



A normal distribution is completely determined by its mean and standard deviation. Among all of the normal distribution functions, the most widely used is the **Standard Normal Distribution**. It has a mean of 0 and standard deviation of 1:



The random variable associated with the Standard Normal Distribution will be denoted by the letter Z. One reason why the Standard Normal Distribution is important is because one can use it to find areas under any normal distribution. If one applies the z-score formula to the random variable X (with mean μ and standard deviation σ), $Z = (X - \mu)/\sigma$ then the new random variable Z has a mean of 0 and standard deviation of 1. In other words, Z is the random variable associated with the Standard Normal Distribution. The above is called the z-score formula: $Z = (X - \mu)/\sigma$

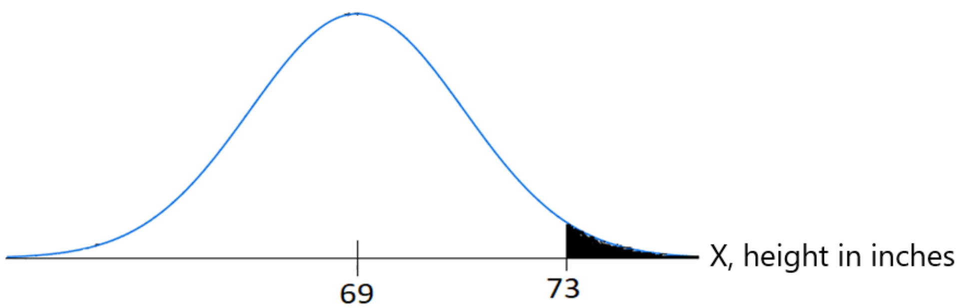
Suppose the height of an adult male is normally distributed with a mean μ = 69 in. and standard deviation of σ = 2.5 in. Find the probability that a randomly chosen person has height greater than 73 in. (6' 1").

Click "Solution" to check your thinking

Solution

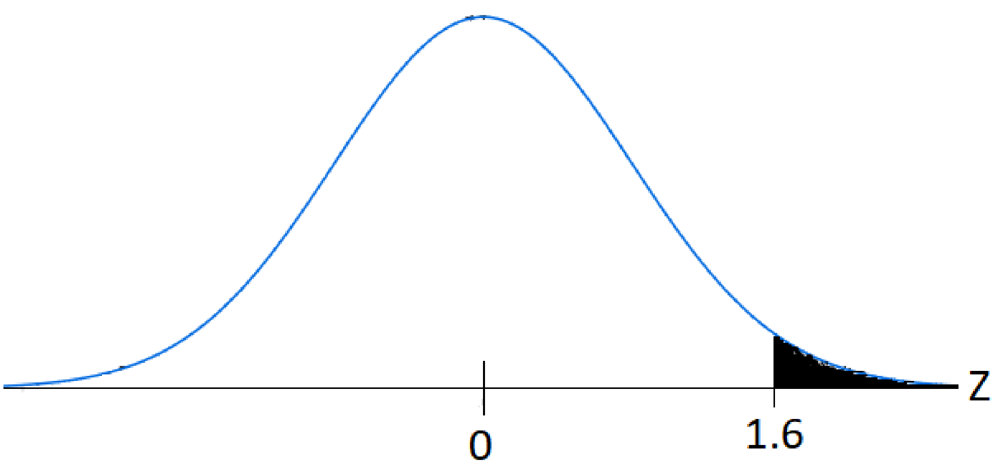
Let X denote the weight of a randomly chosen male.

Find $P(X > 73)$.



To find the probability, we cannot use a table of areas under the normal distribution. This is because such tables only work for the Standard Normal Distribution. To fix that problem, we apply the z-score formula to the value x=73:

Thus, finding $P(X > 73)$ is equivalent to computing $P(Z > 1.6)$:



Use Table A11 on page 607 of Appendix A of the textbook (Standard Normal Probability Distribution: Z Table). The table only has positive z-values, to the right of z=0 under the standard normal distribution (see Figure A2 on page 606).

Thus, to find the shaded area above, we need to subtract the area on the table from 0.5:

Therefore, the probability that height is greater than 73 inches is as follows:

Note: The z-score of 1.6 in the above example means that the height of 73 inches is approximately **1.6 standard deviations above the mean**:

EXAMPLE

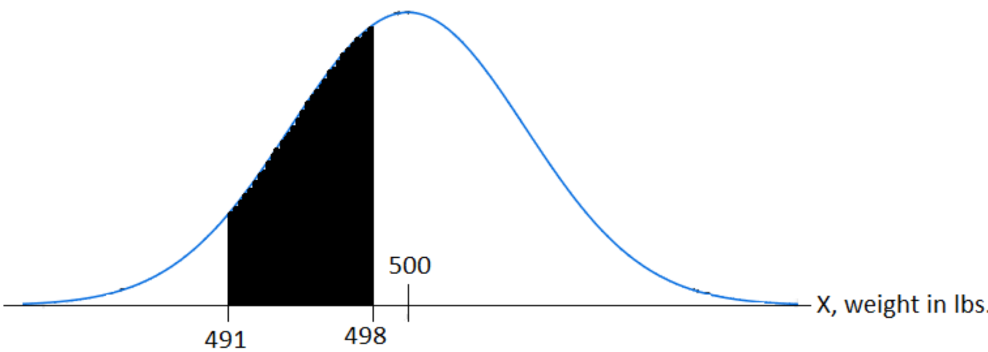
A certain business sells packaged food in large quantities. The weight of food packaged in its containers is normally distributed with a mean μ = 500 lbs. and standard deviation σ = 5 lbs. Find the probability that a randomly chosen container contains between 491 and 498 lbs.

Click "Solution" to check your thinking.

Solution

Let *X* denote the weight of a randomly chosen container.

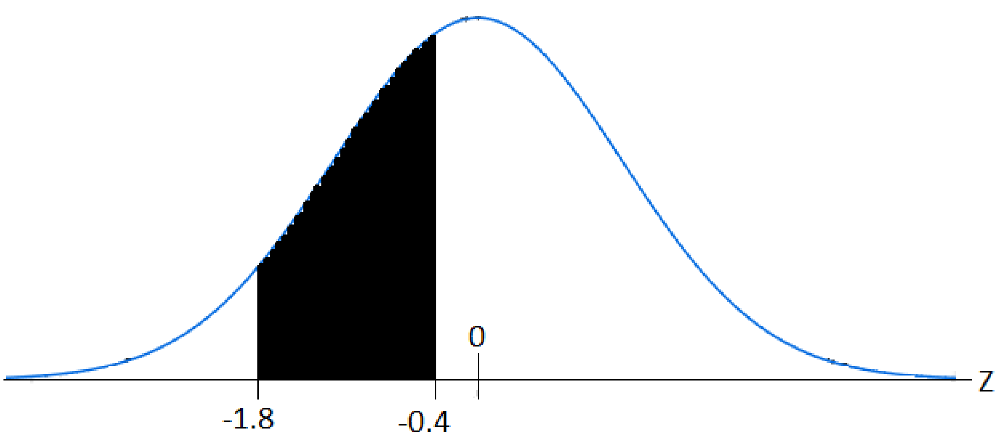
Find P(491 < X < 498).



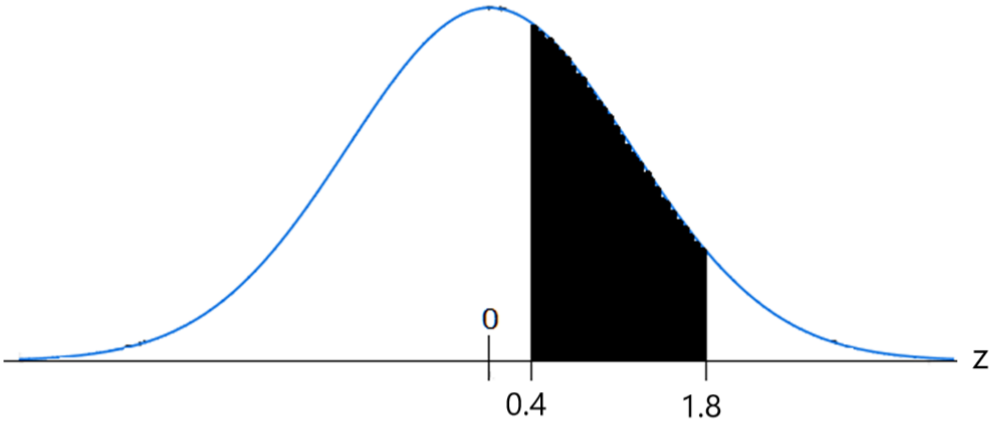
The two *z*-scores are

$z1 = (x1 - \mu)/\sigma = (491-500)/5 = -1.8$
$z2 = (x1 - \mu)/\sigma = (498-500)/5 = -0.4$

Computing $P(491<X<498)$ is equivalent to finding $P(-1.8 < Z < -0.4)$ using the standard normal distribution:



Use Table A11 on page 607 of Appendix A of the textbook (Standard Normal Probability Distribution: *Z* Table). The table only has positive *z*-values. Thus, use symmetry of the normal distribution. The area under the curve between -1.8 and -0.4 is equal to the area under the curve between 0.4 and 1.8: $P(-1.8 < Z < -0.4) = P(0.4 < Z < 1.8)$



Then subtract the area between 0 and 1.8 from the area between 0 and 0.4. This will produce the "between" area shaded above.

P(0.4 < Z < 1.8) = P(0<Z<1.8) – P(0<z<0.4) = 0.4641- 0.1554 = 0.3087

Thus,

P(491 < X < 498) = 0.3087

A quicker way to compute normal probabilities is to use a **normal distribution calculator**, such as this one: **Math Portal Normal Distribution Calculator** (https://www.mathportal.org/calculators/statistics-calculator/normal-distribution-calculator.php)

In the above example, we can input the mean and standard deviation into the calculator, μ = 500 and σ = 5. Then, select the radio button corresponding to the "between" probability and fill-in the two blanks for the endpoints, 491 and 498. Then, press "Compute" to obtain the answer: $P(491 < X < 498) = 0.3087$

In Excel, you can use this command to find the cumulative probability: =NORMDIST(*x*, mean, standard deviation, TRUE)

For example, to find

P(-1.8 < Z < -0.4) = P(Z < -0.4) - P(Z < -1.8)

The probability, P(Z < -0.4)= 0.344578

would be =NORMDIST(-0.4, 0, 1, TRUE)

EXAMPLE

In the above example, how many standard deviations above the mean is a can weighing 510 lbs.?

Click "Solution" to check your thinking.

Solution

Recall that the mean is μ = 500 lbs. and the standard deviation is σ = 5 lbs.

Thus, the can weighing 510 lbs. is **2 standard deviations above the mean**.

The **Exponential Probability Distribution** is another example of a continuous distribution.

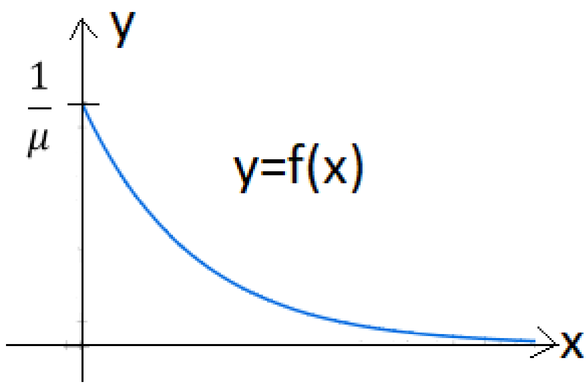
μ is the expected value or the mean of the distribution, and e is approximately 2.71828.

It is derived from the Poisson Distribution. However, for the Exponential Distribution, we are interested in the **waiting time** until the first success.

X is the waiting time until the first success.
μ is the average waiting time until the first success; μ also happens to be the **standard deviation of the distribution**:

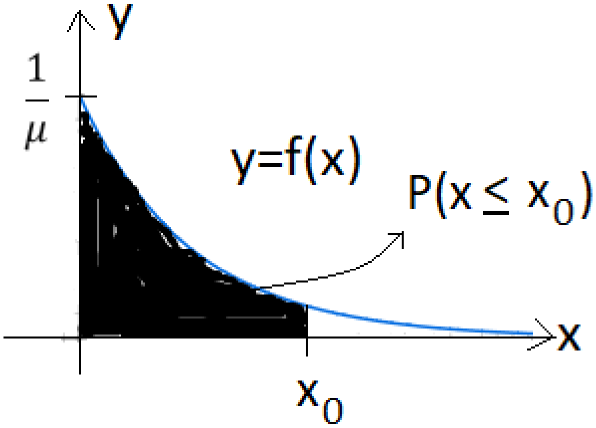
is called the **rate parameter**. λ is the expected number of successes per unit of time.

The graph of such a distribution looks like this. Notice that the y-intercept is and the graph approaches the x-axis (but never intersects the x-axis) as the values of x get larger.



Using calculus, one can obtain the following formula for the **cumulative density function** (cdf) of the Exponential Distribution:

The *cumulative* **probability represents the area** under the exponential distribution to the left of



Suppose the number of accidents at a busy street intersection is distributed according to a Poisson Process with a mean of two accidents per year.

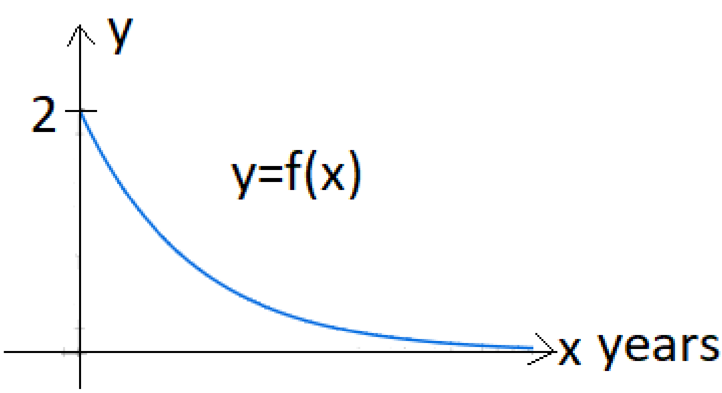
Let X denote the waiting time (in years) until the first accident. Since there are two accidents every year, then one would expect one accident every six months. Thus, the average waiting time until the first accident is $\mu = 1/2$ of a year.

The **rate parameter is** the expected (or average) number of accidents per year:

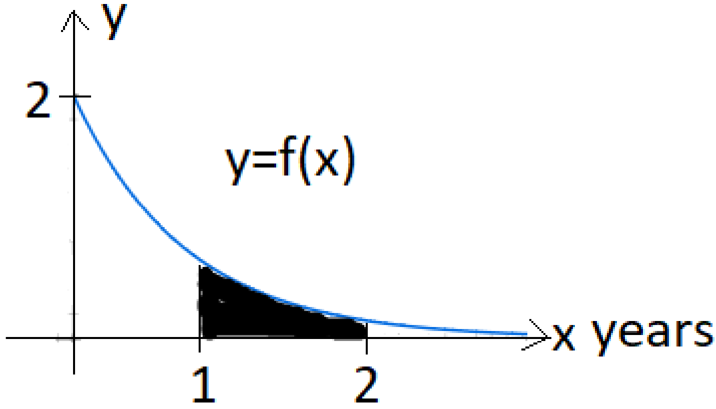
The **standard deviation is**

The corresponding Exponential Distribution function is:

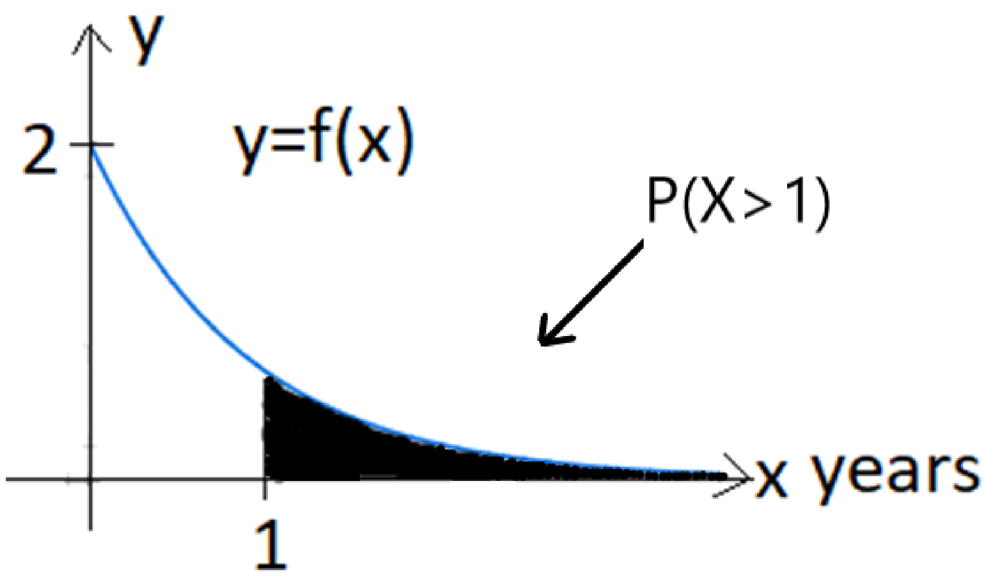
Its graph is



To find the probably that one must wait between one and two years for the first accident, the area under the curve from 1 to 2 must be found. The answer is found by **subtracting the two cumulative probabilities to obtain the "in-between" probability**:



In the same example, to find the probability that one must wait more than one week until the first accident, use the rule of complements:



3.1. More Examples of the Uniform Distribution

The uniform distribution is useful for events that are equally likely over an interval.

Suppose a machine makes metal containers that weigh an average of 2 lbs. and that the weight is uniformly distributed over the interval 1.9 lbs. to 2.1 lbs.

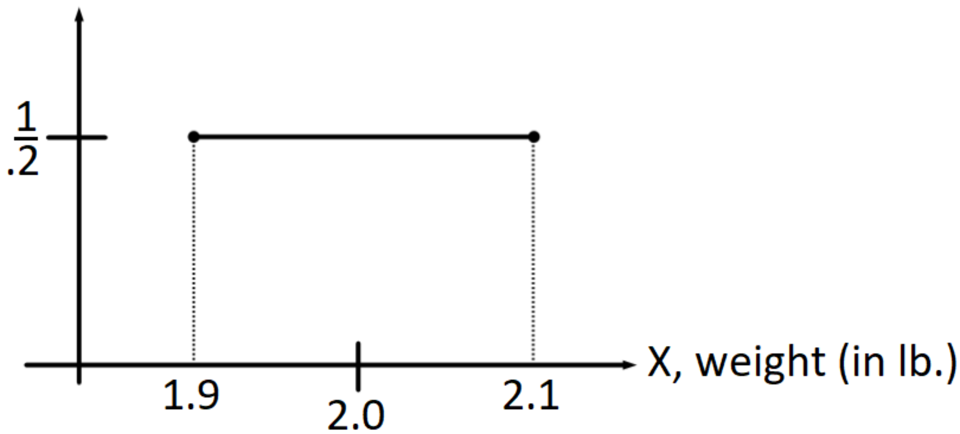
a. Find the mean and standard deviation of the weight.
b. Find the probability that a randomly chosen container will weigh between 1.95lbs. and 2.02 lbs.
c. Assume that the weights of containers are independent. Suppose three containers are randomly chosen. Find the probability that all three randomly chosen containers weigh between 1.95 lbs. and 2.02 lbs.

Click "Solution" to check your thinking.

Solution

- Let X denote the weight of a randomly chosen container. The probability density function is

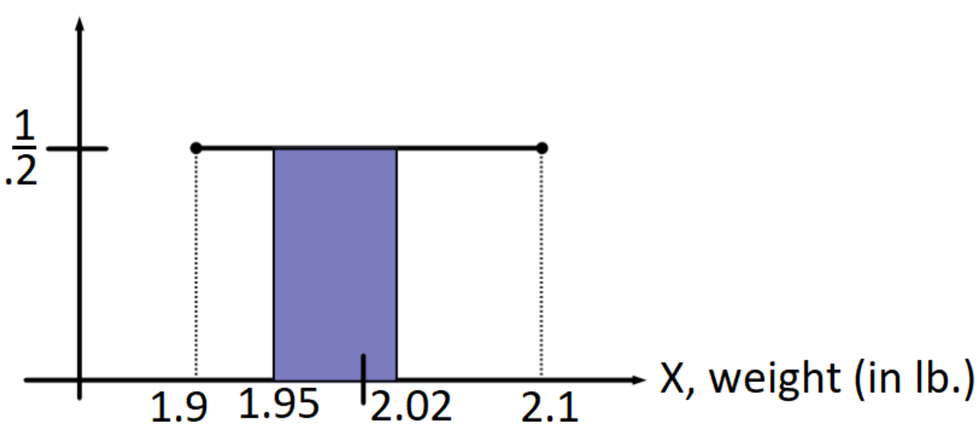
  The graph of the distribution is as shown:



  The midpoint of the interval [1.9, 2.1] is the mean:

  The standard deviation is:

- The probability *P(1.95<X<2.2)* is the rectangular area under the uniform distribution between 1.95 and 2.2:



- Let X_1, X_2, and X_3 denote the weight of the first, second, and third container, respectively. Since the weights of containers are independent of each other, then we may use the multiplication rule for independent events:

Then, from part (b) of this problem, use the fact that the probability of randomly chosen container weighing between 1.95lbs. and 2.02 lbs. is 0.35:

3.2. More Examples of the Normal Distribution

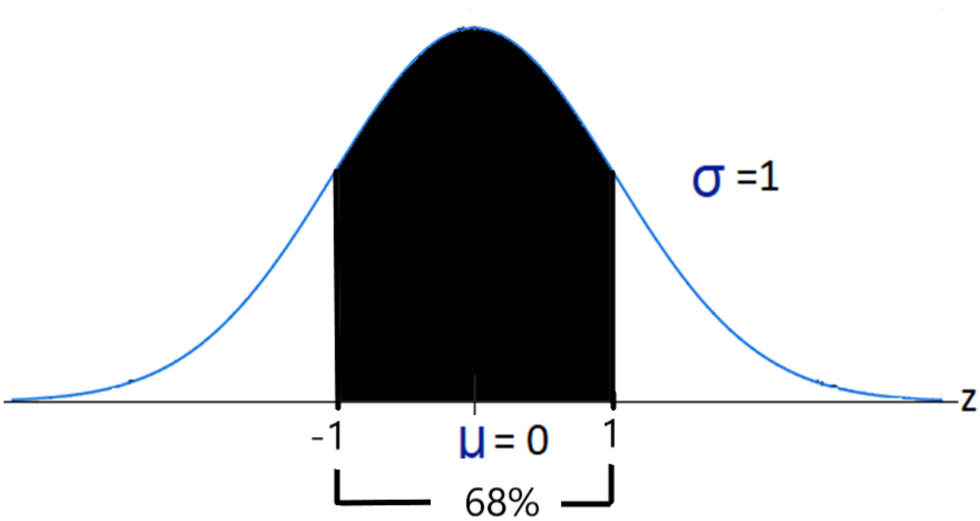
The normal distribution is important in statistics. In subsequent modules, we will use the normal distribution. We will cover more examples in this section.

Verify that this part of the Empirical Rule (for normally distributed data) is true: “Approximately 68% of data are within 1 standard deviation of the mean.”

Click “Solution” to check your thinking.

Solution

This means that approximately 68% of the *z*-scores are within 1 standard deviation of the mean:



Use Table A11 on page 607 of Appendix A of the textbook (Standard Normal Probability Distribution: *Z* Table). The table only has positive *z*-values.

Form Table A11, we get $P(0 < Z < 1) = 0.3413$

Using symmetry of the normal distribution and the above result, we get $P(-1 < Z < 0) = 0.3413$

Thus,

or approximately 68%

EXAMPLE

Gasoline prices in City A are approximately normally distributed with mean \$2.50 per gallon and standard deviation \$0.22. Gasoline prices in City B are approximately normally distributed with mean \$2.20 per gallon and standard deviation \$0.30. Quick Gas operates a gasoline store in both cities. In cities A and B, Quick Gas sells gasoline at \$2.28 and 1.98 per gallon, respectively. How do the two Quick Gas prices compare relative to their two distributions?

Click "Solution" to check your thinking.

Solution

The z-score of \$2.28 for city A is:

$z = (x - \mu)/\sigma = (2.28 - 2.50)/0.22 = -1.00$

The z-score of \$1.40 for city B is:

$z = (x - \mu)/\sigma = (1.98 - 2.20)/0.30 = -0.73$

Thus, in City A, Quick Gas gasoline prices are 1 standard deviation below the mean.

In city B, Quick Gas gasoline prices are 0.73 standard deviations below the mean.

Notice that both Quick Gas prices are \$0.22 below the mean, yet the City B price of \$2.28 is lower relative to the standard deviation.

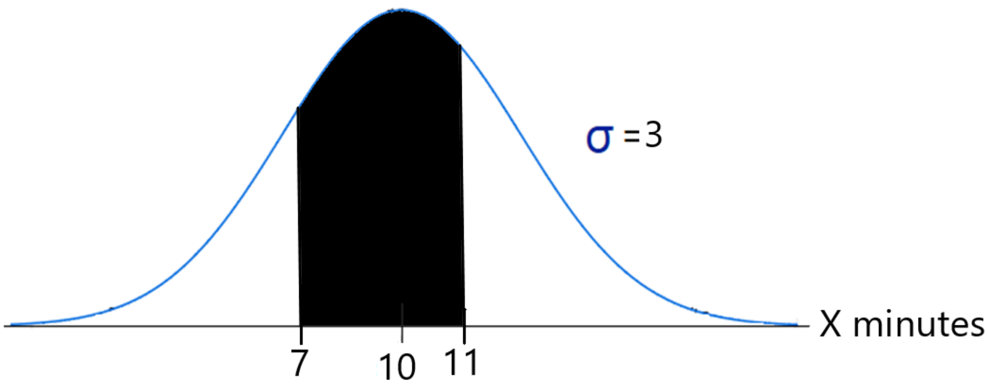
EXAMPLE

At a factory, it takes a worker an average of 10 minutes to assemble a certain component. Suppose the standard deviation of the assembly time is 3 minutes. Assuming assembly time is normally distributed, what is the probability that a worker assembles the component in 7 to 11 minutes?

Click "Solution" to check your thinking.

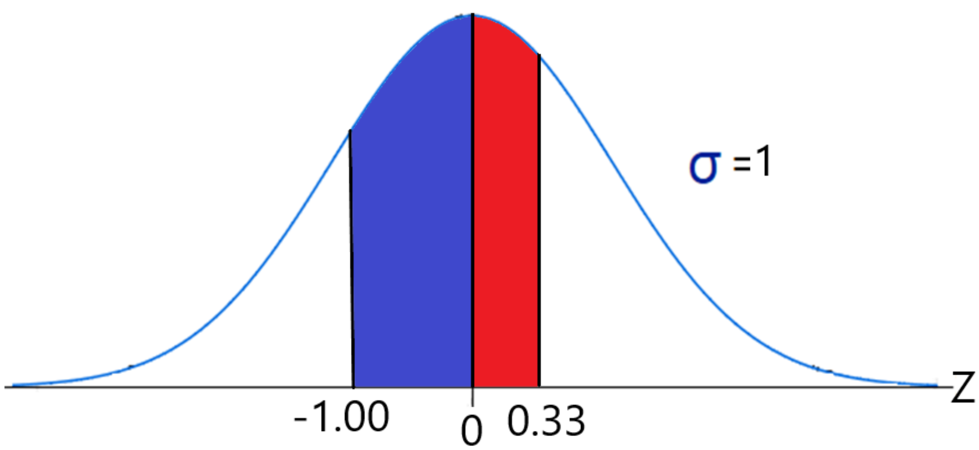
Solution

Let X denote the number of minutes that it takes a randomly chosen employee to assemble a component. To determine $P(7<X<11)$, we need to find the area under the normal curve:



The z-scores for 7 and 11 minutes, respectively, are as follows:

The probability $P(7< X < 11)= P(-1.00 < Z < 0.33)$ is the sum of the blue and red areas under the standard normal curve:



Use Table A11 on page 607 of Appendix A of the textbook (Standard Normal Probability Distribution: Z Table). The table only has positive z-values. Use symmetry of the normal distribution. The (blue) area under the curve between -1.00 and 0 is equal to the area between 0 and 1.00. Hence, look up 1.00 in Table A11:

$P(-1.00 < Z < 0) = P(0 < Z < 1.00) = 0.3413$

From Table A11, we can find the red area directly:

$P(0 < Z < 0.33) = 0.1293$

Thus, the probability that a worker assembles the component in 7 to 11 minutes is as follows:

$P(7< X < 11)= P(-1.00 < Z < 0.33) = 0.3413 + 0.1293 = 0.4706$

We can instead use a **normal distribution calculator** (https://www.mathportal.org/calculators/statistics-calculator/normal-distribution-calculator.php).

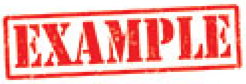
Input the mean and standard deviation into the calculator, $\mu = 10$ and $\sigma = 3$. Then, select the radio button corresponding to the "between" probability and fill in the two blanks for the endpoints, 7 and 11. Then press "Compute." The answer will be:
$P(7 < X < 11) = 0.4706$

As previously mentioned, you can use this Excel command to find the cumulative probability:

=NORMDIST(*x*, mean, standard deviation, TRUE)
For example, $P(X < 11)$ is this in Excel:
=NORMDIST(11, 10, 3, TRUE)

3.3. More on the Exponential Distribution

The exponential distribution is often useful when studying the time taken for a part or component to fail.



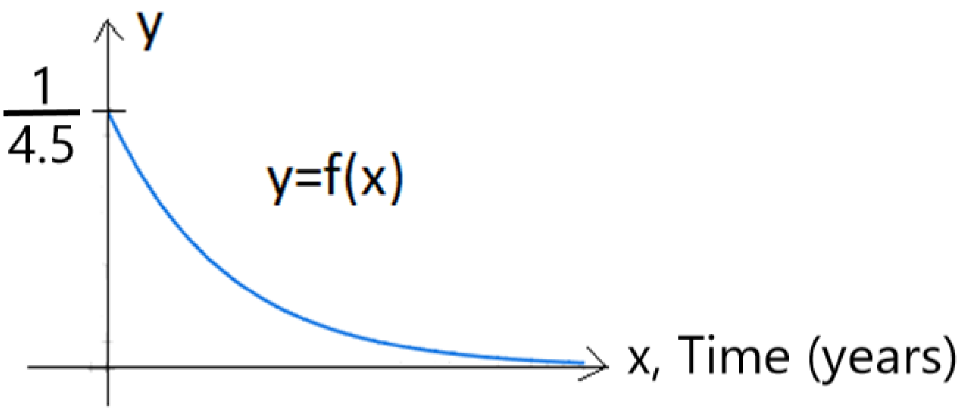
Suppose the time it takes a car battery to fail is distributed according to the exponential distribution with a mean of 4.5 years.

a. Find the probability that a car battery lasts more than 5 years.
b. What is the probability that the battery lasts longer than average?
c. Find the probability that the battery lasts longer than 3.12 years.

Click "Solution" to check your thinking.

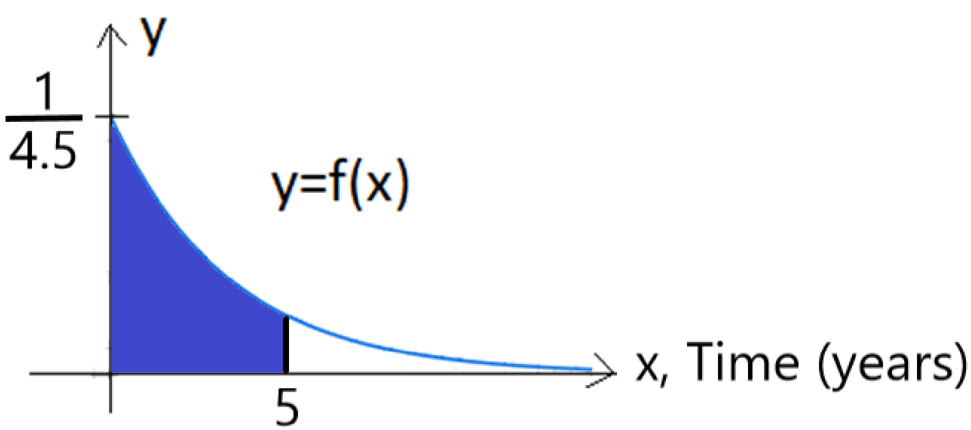
Solution

- Let X denote the amount of time it takes a car battery to fail. The mean is μ=4.5 years. We want to find the probability that a car battery lasts more than 5 years: P(X >5) The exponential probability distribution function is:



By the rule of complements:

The probability represents the area between x=0 and x=5 under the exponential curve:



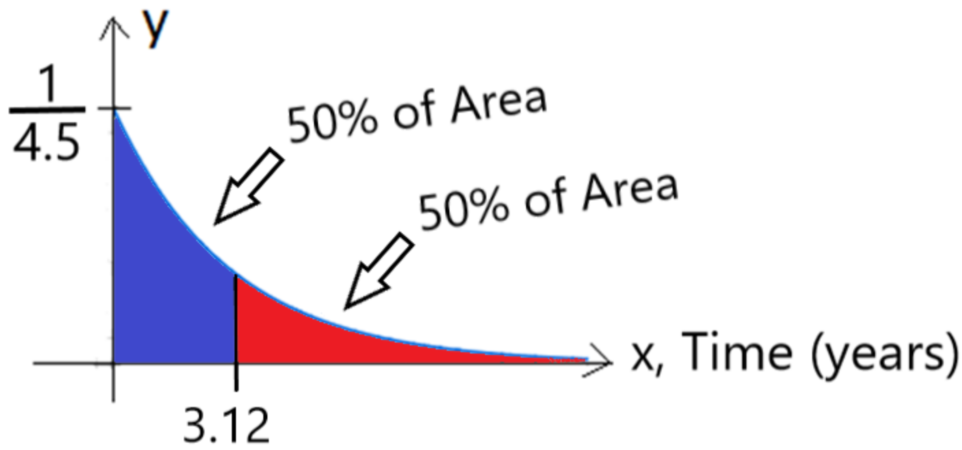
To find $P(X \leq 5)$, we may use the formula for the cumulative probability of the exponential distribution:

Thus,

- Since the mean lifetime is μ=4.5 years, we want to find the probability that a car battery lasts more than 4.5 years. Thus, this problem is solved using a method like the one in part a using the rule of complements and the formula for the cumulative probability of the exponential distribution:

- This problem is like part a:

Thus, 3.12 is the median of the distribution. This means that approximately 50% of the batteries last longer than 3.12 years and approximately 50% last less than 3.12 years. Fifty percent of the area under the probability density curve is to the left of 3.12 and 50% is to the right:


y
1/4.5
50% of Area
50% of Area
x, Time (years)
3.12

4. Summary

In Module 4, we covered what a discrete and probability distribution is and how to find its mean. In addition, we covered well-known discrete and continuous probability distributions:

Discrete:

a. Hypergeometric—Section 4.1
b. Binomial—Section 4.2
c. Poisson—Section 4.4

Continuous:

d. Uniform—Section 5.2
e. Exponential—Section 5.3
f. Normal—Sections 6.2 and 6.3

We will see the normal distribution in upcoming modules.

Here is the list of the objectives that we have covered and are part of the Mastery Exercises in Knewton Alta:

- Understand the properties of a discrete probability density function
- Find the mean of a discrete random variable from its probability density function
- Understand the parameters of the binomial distribution
- Use the binomial distribution to compute probability
- Understand the parameters of the Poisson distribution
- Use the Poisson distribution to compute probability
- Use the uniform distribution to compute probability
- Find the mean and standard deviation of the uniform distribution
- Understand the parameters of the exponential distribution
- Use the exponential distribution to compute probability
- Understand the notation and interpret the parameters of a normal distribution in business examples
- Standardize a normally distributed random variable in business contexts
- Calculate the mean and standard deviation of a standard normal distribution in business examples
- Use the normal distribution to compute probability in business examples