MTH350

Discrete Mathematics

Module 5: Graph Theory, Part I

This module introduces the basic definitions of graph theory. Students will also learn about various types of graphs and their uses in solving real-world applications.

Learning Outcomes

1. Create a graph given a list of properties.
2. Compare graphs.
3. Apply graph theory definitions to solve real-world problems.
4. Utilize Euler's Formula to determine whether or not a graph is planar.
5. Determine additional properties of planar graphs.

For Your Success & Readings

Module 5 introduces one of the most important topics of discrete mathematics for IT-related applications. In order to learn about those applications, you will first need to become familiar with the many definitions involved in graph theory. Make sure you are comfortable with these definitions before moving on to the next module where we will look at more complex application problems.

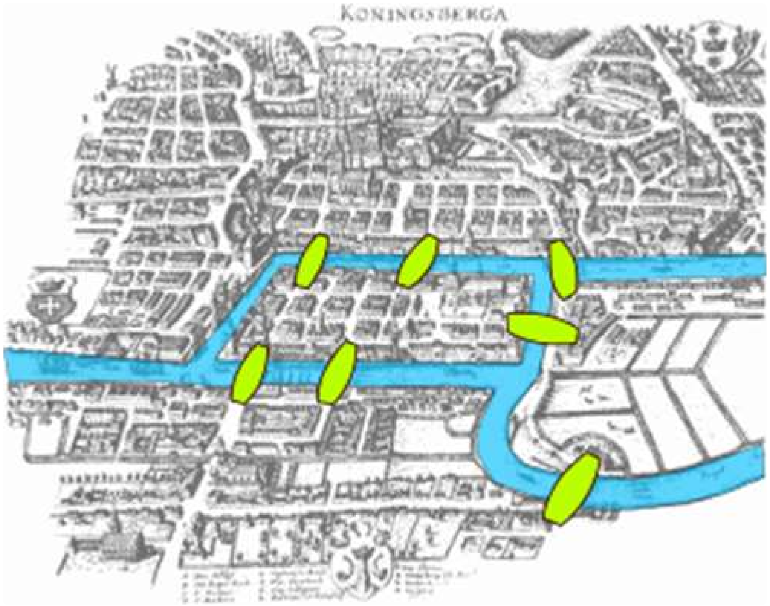
The Critical Thinking Assignment for this week will give you a chance to explore an example of an application of graph theory. The readings for this week will also give you an idea of the types of recent real-world problems that have been worked on using tools from graph theory.

You will also get the chance to write your own real-world problem that can be solved using graph theory concepts in the discussion board for this week. Make the most of this opportunity by engaging with your peers about the various uses of graph theory in the applications.

Required

- Chapter 4, Sections **4.1** (http://discrete.openmathbooks.org/dmoi/sec_gt-intro.html) & **4.2** (http://discrete.openmathbooks.org/dmoi3/sec_trees.html) in *Discrete Mathematics: An Open Introduction*
- Gutierrez, J. M., Jensen, M., & Riaz, T. (2016). **Applied graph theory to real smart city logistic problems** (https://www-sciencedirect-com.csuglobal.idm.oclc.org/science/article/pii/S1877050916324632). *Procedia Computer Science*, 95, 40-47.
- Song, C., Goswami, K., Park, Y., Chang, S., & Choo, E. (2017). **Graphic model analysis of frauds in online consumer reviews** (https://dl-acm-org.csuglobal.idm.oclc.org/citation.cfm?id=3018942). *Proceedings of the Second International Conference on Internet of things and Cloud Computing* - ICC 17.

1. What is a graph?



Bogdan Giuşcă, 2005, CC SA-BY 3.0

The study of graph theory dates back to 1735 when Leonhard Euler became inspired by a problem called *Seven Bridges of Königsberg*. Take a look at the following video on how this problem gave rise to the tools in graph theory.

How the Königsberg bridge problem changed mathematics - Dan Van der Vieren (https://ed.ted.com/lessons/how-the-konigsberg-bridge-problem-changed-mathematics-dan-van-der-vieren)

Euler took the scenario of the seven bridges that connected four islands, and reduced it to a picture of four dots connected by seven lines. This resulting picture is an example of a **graph**, and it is defined here.

Graphs are made up of a collection of dots, called **vertices**, and lines connecting those dots, called **edges**. When two vertices are connected by an edge, we say they are **adjacent**.

Let's look at how a graph can be used to represent another situation.

Consider the following scenario:

Al, Bob, Cam, Dan, and Euclid are all members of the social networking website Facebook. The site allows members to be “friends” with each other. It turns out that Al and Cam are friends, as are Bob and Dan. Euclid is friends with everyone.

What are the vertices?

Each person will be represented by a vertex.

How many vertices are there?

There are 5 people involved, so we need 5 vertices. Let's label them A, B, C, D,and E.

What are the edges?

Each friendship will be represented by an edge.

How many edges are there?

We are told that there are the following friendships:

- Al and Cam
- Bob and Dan
- Euclid and Al
- Euclid and Bob
- Euclid and Cam
- Euclid and Dan

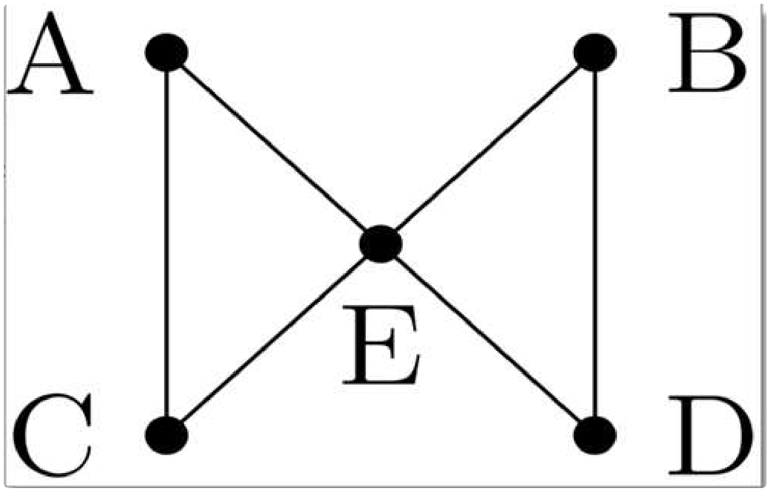
There is a total of 6 friendships, so we need 6 edges.

Which vertices are adjacent (connected by an edge)?

Based on our labeling of each person and the friendships we listed above, we need the following vertices to be adjacent (connected by an edge):

- A and C
- B and D
- E and A
- E and B
- E and C
- E and D

After considering the above characteristics of the scenario, we are ready to draw the graph representation of this situation as follows:



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Formally, we should think of a graph as an ordered pair of two sets. The first set is the collection of vertices, and the second set is the collection of edges.

Graph Definition

A **graph** is an ordered pair $G = (V, E)$ consisting of a nonempty set V (called the **vertices**) and a set E (called the **edges**) of two-element subsets of V.

In the example we saw above, we can call our graph, G, and represent G as a set in the following way:

$G = (\{A, B, C, D, E\}, \{\{A, C\}, \{B, D\}, \{E, A\}, \{E, B\}, \{E, C\}, \{E, D\}\})$

Notice that the first part of the ordered pair, G, is the set $V = \{A, B, C, D, E\}$ which is just the collection of all vertices of the graph.

The second set of the ordered pair is the set $E = \{\{A, C\}, \{B, D\}, \{E, A\}, \{E, B\}, \{E, C\}, \{E, D\}\}$ which is a set consisting of 2-element subsets of V. In other words, in the set E, we show which pairs of vertices are connected by including them in the same set.

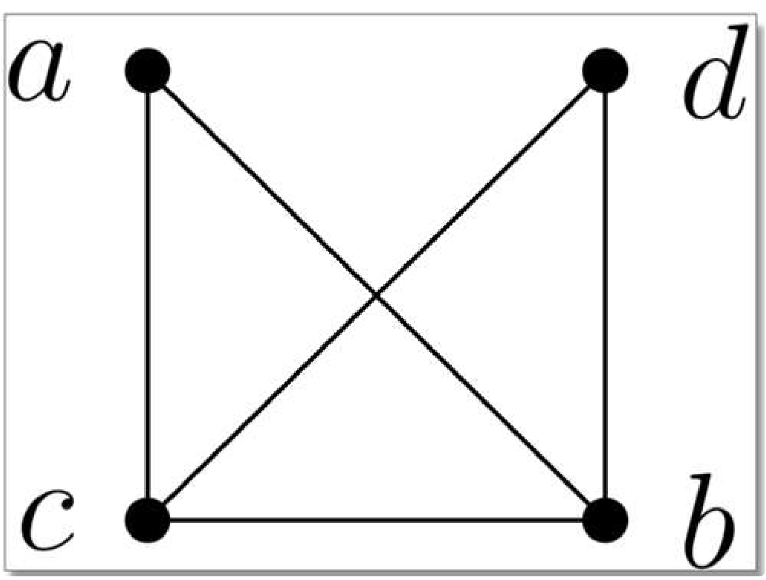
Now, recall that a set is an *unordered* collection of elements, so there is more than one way to represent the same set by rearranging its elements.

For example, the set $\{a, b, c\}$ is equivalent to the set $\{b, c, a\}$ since they both contain the exact same elements.

Since we are representing graphs in terms of sets, we can similarly represent the same graph in various ways.

For example, the graph $G_1 = (\{a, b, c, d\}, \{\{a, b\}, \{a, c\}, \{b, c\}, \{b, d\}, \{c, d\}\})$ is equivalent to the graph $G_2 = (\{a, d, c, b\}, \{\{a, b\}, \{a, c\}, \{b, c\}, \{b, d\}, \{c, d\}\})$ since their corresponding sets V_1, V_2 and $E_1, E_2$are equal.

Similarly, can see how two drawings can represent the same graph. Here are sample illustrations for $G_1$and G_2.



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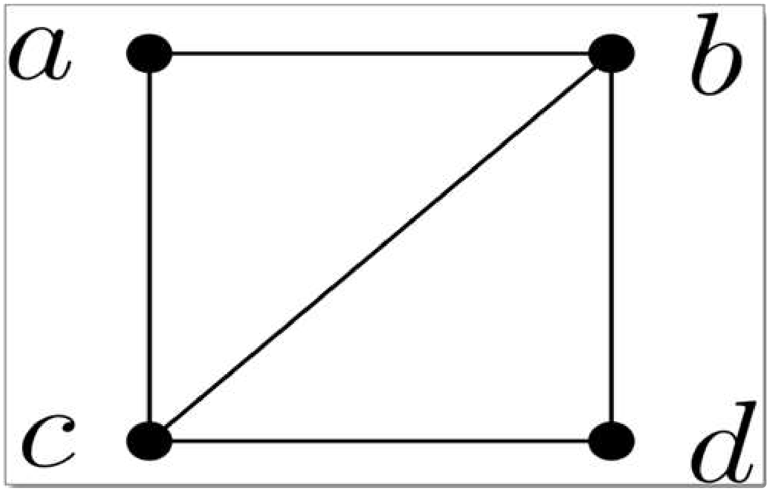

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Notice that the graph illustrations also show that they are equal, since all of the same vertices are connected to each other. The only difference between the two illustrations is the position of the vertices.

Now, check your understanding of this section by answering the following question:

1.1. Comparing Graphs

In the previous section we talked about a graph as pair of sets: one set of vertices and one set of edges. So, when comparing graphs, we are actually comparing their corresponding sets of vertices and edges.

From Module 1, recall that two sets are equal if they contain exactly the same elements. Hence, two graphs are **equal** if they contain the same edges and vertices.

Let's consider the following example:

G_1= $(\{a, b, c\}, \{\{a, b\}, \{a, c\}, \{b, c\}\})$
G_2= $(\{u, v, w\}, \{\{u, v\}, \{u, w\}, \{v, w\}\})$

Are these graphs the same?

The two graphs are NOT equal. It is enough to notice that a is a vertex of G_1 but is not a vertex of G_2.

How are they similar?

While they are not equal, we can rename the vertices of one graph and get the second graph as the result.

Now, check your understanding of how to compare graphs by answering the following question:

2. Graph Theory Definitions

In this section, we will look at various graph theory definitions that are useful in applying graph theory to the real world. Note that there are many different types of graphs and properties of graphs, but in this course we focus on just a few.

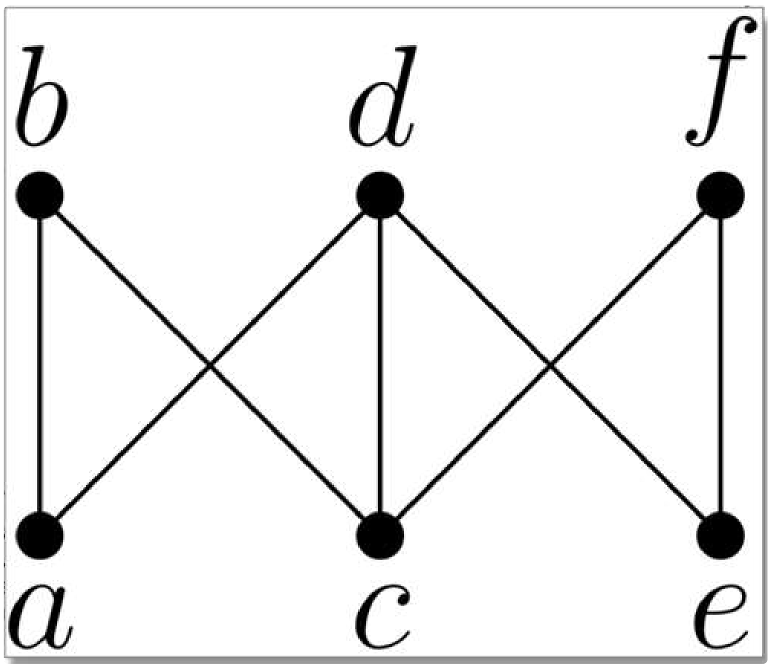
Let's begin with the following basic definitions of graph structures:

The number of edges incident to a vertex is the **degree of a vertex**. In general, if we know the degrees of all the vertices in a graph, we can find the number of edges. **The sum of the degrees of all vertices will always be** *twice* **the number of edges,** since each edge adds to the degree of two vertices. Notice this means that **the sum of the degrees of all vertices in any graph must be even!**

A **walk, (path)** is a sequence of vertices such that consecutive vertices (in the sequence) are adjacent (in the graph). A walk in which no vertex is repeated is called **simple walk**. A path that starts and stops at the same vertex, but contains no other repeated vertices is called a **cycle or circuit**.

A **simple graph** has no pair of vertices that are connected more than once and no vertex is connected to itself. A **multigraph** is just like a graph but can contain multiple edges between two vertices as well as single edge loops (that is, an edge from a vertex to itself).

To see these definitions in an example, consider the following graph:



Levin, 2017,

What is the degree of each vertex of the graph?

The graph has 6 vertices, each with the following degrees:

Vertex	Degree
a	2
b	2
c	3
d	3
e	2
f	2

Give an example of two different paths of this graph.

One path could start from a, then go to b, and end at c. A different path could start from a, then go to d, then go to e, and then stop at f.

Given an example of a circuit on this graph.

A possible circuit on this graph could start at a, then go to b, then go to c, then go to f, then go to e, then go to d, and finally return back to a.

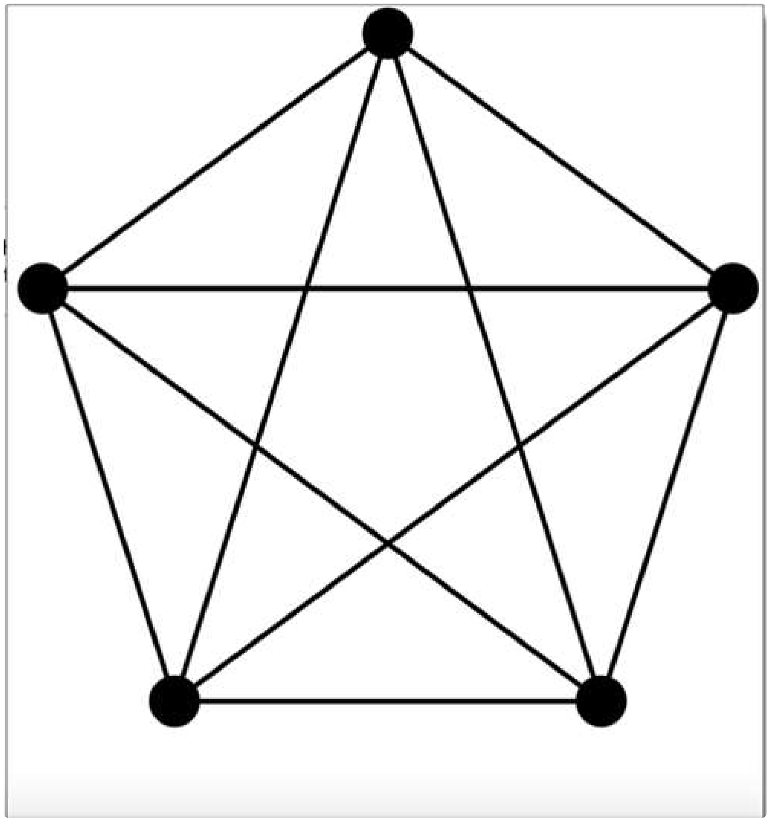
Is this a simple or a multigraph?

This is a simple graph since no pair of vertices is connected more than once, and no vertex is connected to itself.

Next, we will look at various types of graphs that have vast applications to real-world scenarios.

A **complete graph** is a graph in which every pair of vertices is adjacent.

This is an example of a complete graph, since all possible edges are drawn between its vertices.



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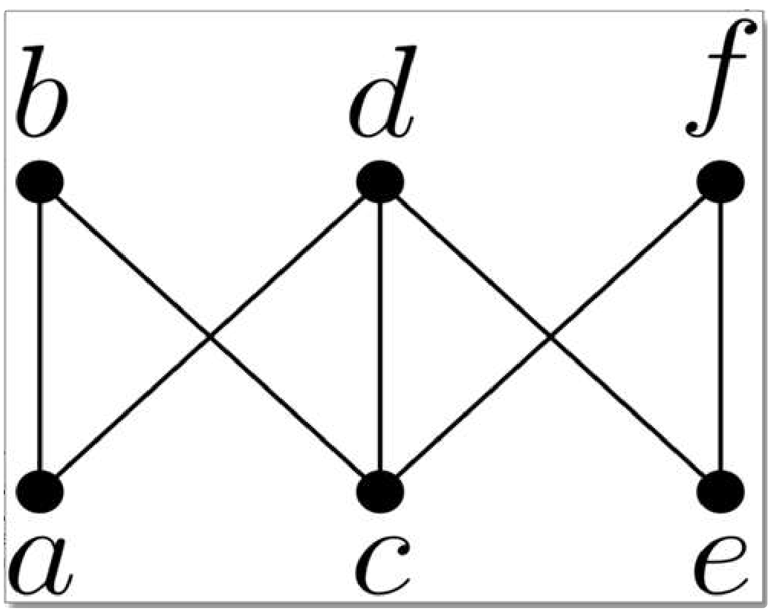
A graph is **connected** if there is a path from any vertex to any other vertex.

Here is an example of a connected graph. Notice that it is essentially in "one piece."



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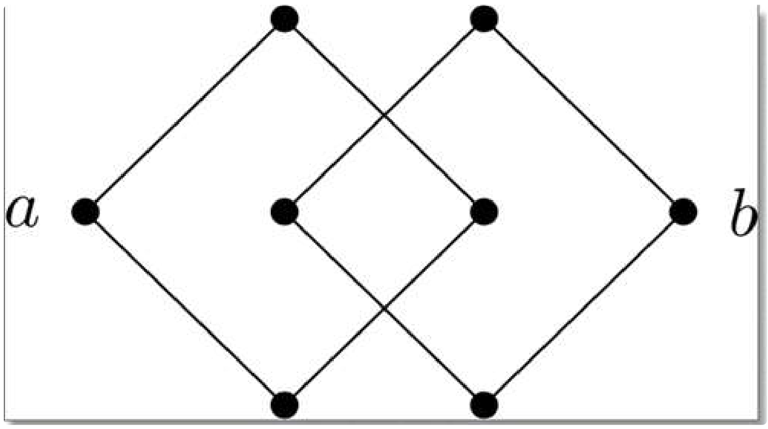
Now, here is an example of a graph that is not connected since there is no way to draw a path from vertex *a* to vertex *b*.



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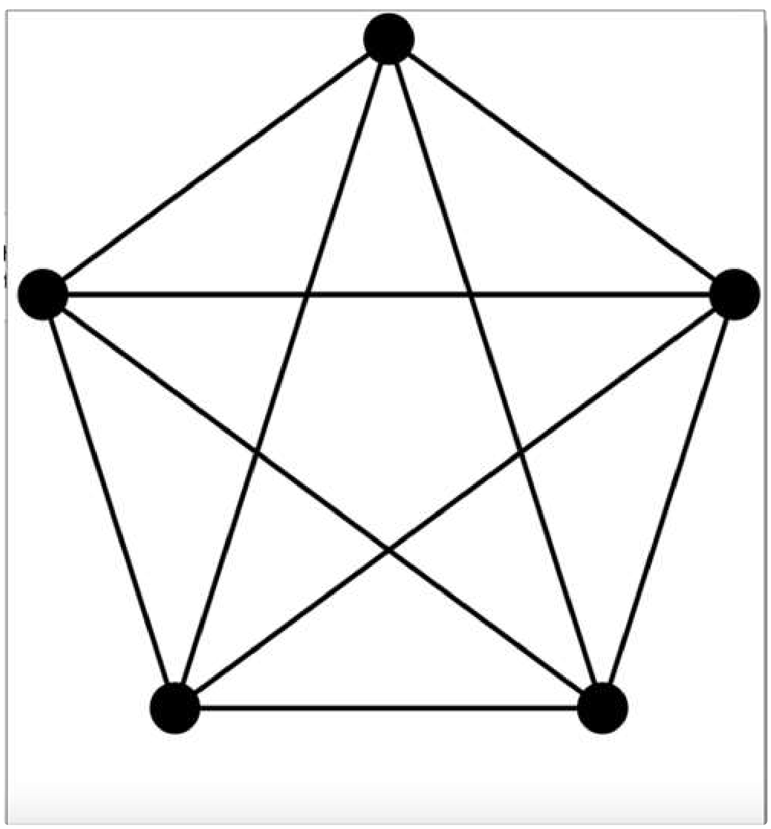
A path which uses each edge exactly once is called an **Euler path**. An Euler path which starts and stops at the same vertex is called an **Euler circuit**.

Euler Paths and Circuits

- A graph has an Euler circuit if and only if the degree of every vertex is even.
- A graph has an Euler path if and only if there are at most two vertices with odd degree.

Consider the following graph:



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Does the graph have an Euler circuit?

Yes, since all vertices are of even degree, the graph contains an Euler circuit.

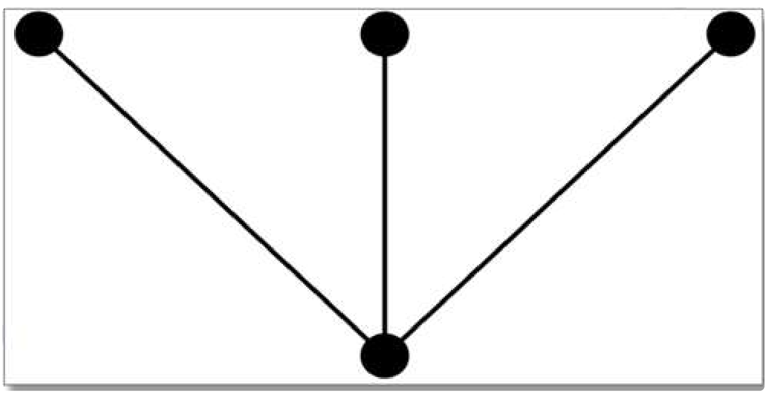
Does the graph have an Euler path?

Yes, since at most two vertices have odd degree, this graph contains an Euler path.

Tree Graphs

A **tree** is a (connected) graph with no cycles. (A non-connected graph with no cycles is called a **forest**.) The vertices in a tree with degree 1 are called **leaves**.

Here is an example of a tree with four vertices and three edges:



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K_n - The complete graph on n vertices.

C_n - The cycle on n vertices, just one big loop.

P_n -The path on n vertices, just one long path.

Now, check your understanding of graph definitions by answering the following:

2.1. Connected Graphs

Let's take a look at an example of a real-world scenario where connected graphs can be used.

Among a group of 5 people, is it possible for everyone to be friends with exactly 2 of the people in the group? What about 3 of the people in the group?

Is it possible for everyone to be friends with exactly 2 of the people in the group?

It is possible for everyone to be friends with exactly 2 people. You could arrange the 5 people in a circle and say that everyone is friends with the two people on either side of them (so you get the graph C_5).

What about 3 of the people in the group?

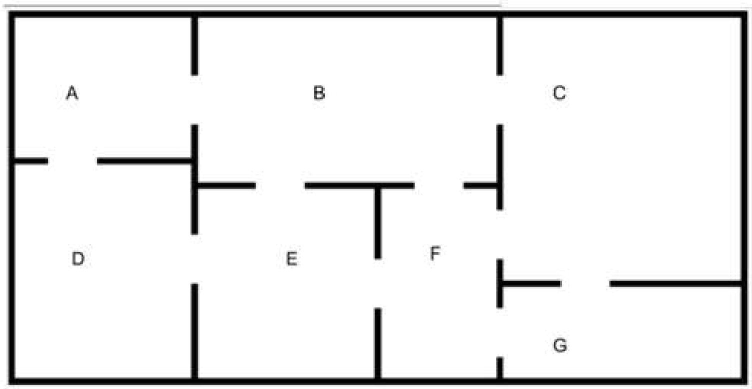
It is not possible for everyone to be friends with 3 people. That would lead to a graph with an odd number of odd degree vertices, which is impossible since the sum of the degrees must be even.

Now, check your understanding of this concept by answering the following:

2.2. Euler Paths and Circuits

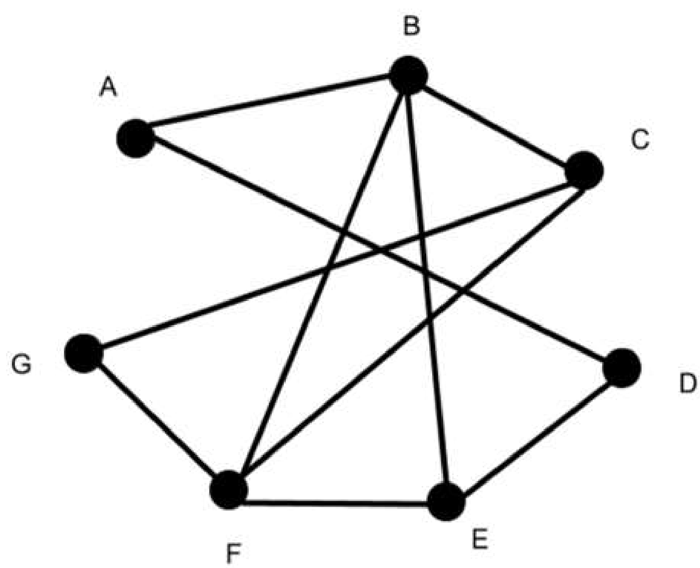
Now let's take a look at an example of a real-world problem involving Euler Paths.

Consider the following floor plan of a one-story home. Note: the spaces between rooms represent doorways.



Is it possible to tour the entire home by using each doorway only once?

We can use vertices to represent the rooms, and edges to represent the connecting doorways. Then, we could represent this floorplan with a graph with 7 vertices and 10 edges as follows:



First, notice that we want to find a path which uses each doorway (edge) exactly once, and this is called an Euler path. Also, we know that a graph has an Euler path if and only if there are, at most, two vertices with odd degree.

By looking at the graph we created to model this situation, we notice that there are exactly two vertices of odd degree, so this does, indeed, include an Euler path. Hence, there is a way to tour this home by using each doorway only once.

Now, check your understanding of Euler paths and circuits by answering the following:

2.3. Trees

To learn a bit more about trees, take a look at the following videos that discuss examples of them and how they can be used to solve network optimization problems in real-world contexts.

- **Visualize data with graph theory** (https://www.linkedin.com/checkpoint/enterprise/login/2245842?pathWildcard=2245842&application=learning&redirect=https%3A%2F%2Fwww%2Elinkedin%2Ecom%2Flearning%2Fprogramming-foundations-discrete-mathematics%2Fvisualize-data-with-graph-theory%3Fu%3D2245842)
- **Network optimization with trees** (https://www.linkedin.com/checkpoint/enterprise/login/2245842?pathWildcard=2245842&application=learning&redirect=https%3A%2F%2Fwww%2Elinkedin%2Ecom%2Flearning%2Fprogramming-foundations-discrete-mathematics%2Fnetwork-optimization-with-trees%3Fu%3D2245842)

Now, check your understanding of trees by answering the following question:

2.4. Named Graphs

Let’s look at some examples of named graphs to make sure you understand how the labelling works. Look at the image on the front of each card and describe the image in your mind. Flip the card to check your descriptions.

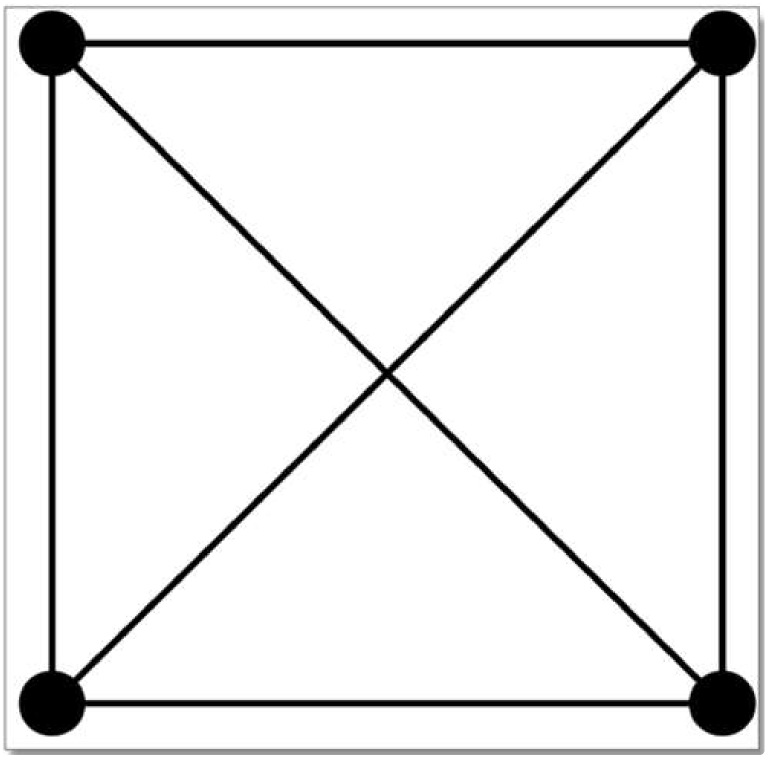
Now, check your understanding of named graphs by answering the following:

3. Planar Graphs

In this section, we talk about a type of graph that can be drawn on a single plane. In other words, it is a graph that can be drawn without any edges crossing. Here are a couple of definitions:

When a connected graph can be drawn without any edges crossing, it is called **planar**. When a planar graph is drawn in this way, it divides the plane into regions called **faces**.

Recall that sometimes a graph can be drawn in more than one way. That is, although a graph might be drawn in a way that doesn't seem planar, there may be a way to draw it so that it is planar. Consider the following example:



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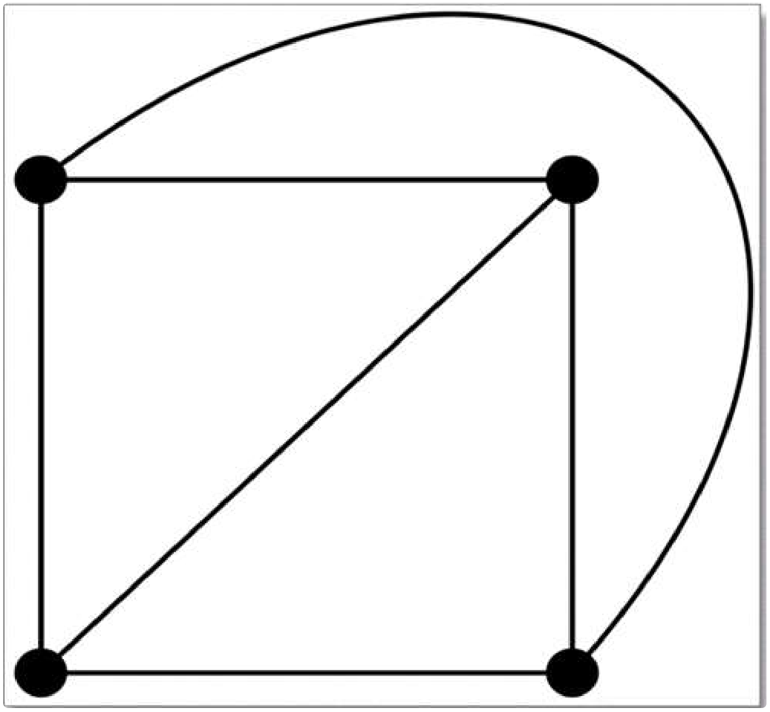
As it is drawn, this graph does not seem that it is planar since some of its edges cross. However, this same graph can be drawn in the following way so that the edges do not cross, and it proves to be, indeed, a planar graph:



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Note: In the above example, the number of faces is 4 since we include the outside region as a face.

Now, here is an important relationship between the number of vertices, the number of edges, and the number of faces of a planar graph:

Euler's Formula for Planar Graphs

For any (connected) planar graph with v vertices, e edges and f faces, we have:

$v - e + f = 2$

Click through the following interactive to see how this formula can be used:

Is it possible for a planar graph to have 6 vertices, 10 edges, and 5 faces?

No. A (connected) planar graph must satisfy Euler's formula: $v-e+f = 2$. Here $v-e+f = 6-10+5 = 1 \neq 2$.

Is it possible for a planar graph to have 4 vertices, 8 edges, and 6 faces?

Yes. This is a planar graph since it satisfies Euler's formula: $v-e+f = 4-8+6 = 2$.

Is the complete graph with 5 vertices (K_5) planar?

No. K_5 is not planar. Since it has 5 vertices and 10 edges, according to Euler's formula, it would need to have 7 faces. However, that is not possible. (For details on this proof, please see the proof of Theorem 4.2.1 in the textbook.)

We have seen some examples of graphs that are not planar. You may have noticed that one issue that can come up is when there are too many edges and not enough vertices. In that case, there is no choice but for some edges to cross.

Now, check your understanding of planar graphs by answering the following:

3.1. Euler's Formula for Planar Graphs

Recall Euler's formula for planar graphs: $v - e + f = 2$.

Let's use this formula to explore the following:

The graph G has 6 vertices with degrees 2, 2, 3, 4, 4, 5. How many edges does G have? Could G be planar? If so, how many faces would it have. If not, explain.

How many edges does G have?

Recall that the sum of the degrees of a graph is equal to twice its number of edges. Thus, in this case, G has 10 edges, since 10 =

.

Could G be planar? If so, how many faces would it have?

Yes, it could be planar, and using Euler's formula with $v = 6$ and $e = 10$ we have: $6–10+f = 2$. Thus, we have $f = 6$, so this graph would have 6 faces.

Now, check your understanding of Euler's formula for planar graphs by answering the following:

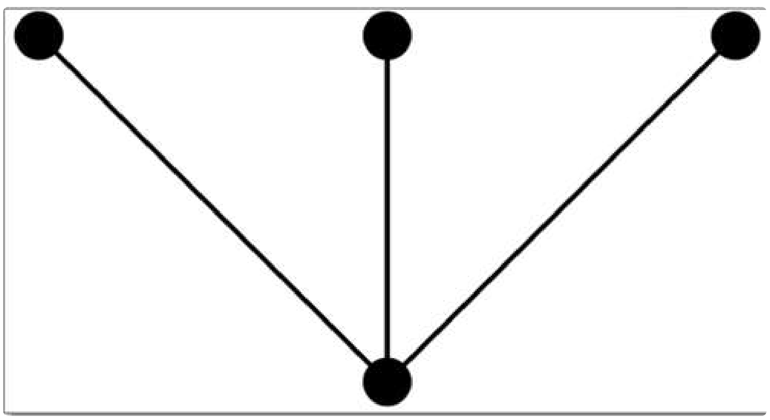
4. Summary

Click through the following interactives to view worked examples that cover key points for each Module Outcome.

Module Outcome #1: Create a graph given a list of properties.

For each of the following, try to give two *different* unlabeled graphs with the given properties, or explain why doing so is impossible.

Two different trees with the same number of vertices and the same number of edges. A tree is a connected graph with no cycles.



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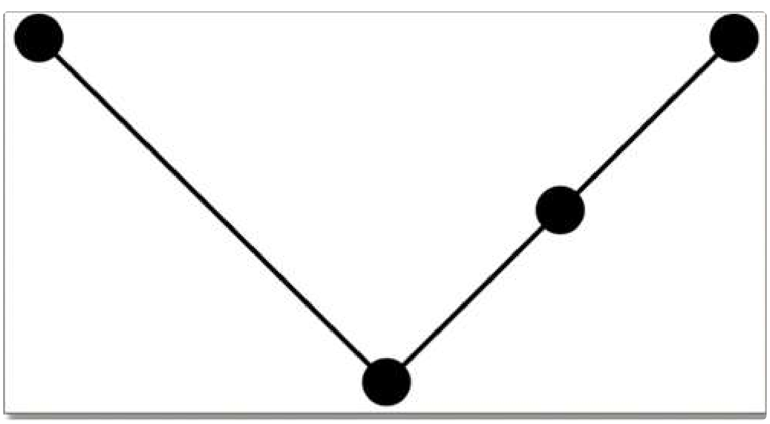

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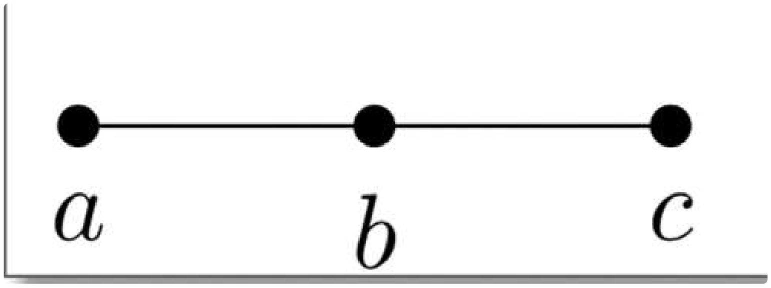
Two different graphs with 5 vertices all of degree 3.

This is not possible. In fact, there is not even one graph with this property (such a graph would have $5 \cdot 3/2 = 7.55 \cdot 3/2 = 7.5$ edges).

Module Outcome #2: Compare graphs.

Determine which of the following graphs are equal to the graph:



$G_1 = (\{a, b, c\}, \{\{a, b\}, \{b, c\}\})$

This is equal to the graph shown above since it has the same vertices and the same edges between them.

$G_2 = (\{a, b, c\}, \{\{c, b\}, \{b, a\}\})$

This is equal to the graph shown above since it has the same vertices and the same edges between them.

$G_3 = (\{a, b, c\}, \{\{b, c\}, \{a, c\}\})$

This graph is not equal to the graph shown above since; although it has the same vertices, the edges connect different vertices.

Module Outcome #3: Apply graph theory definitions to solve real-world problems.

If 10 people each shake hands with each other, how many handshakes took place?

How could this scenario be represented using a graph?

A graph could be made with vertices representing the people, and edges between people who shake hands.

What is the question asking for in terms of its graph?

The question is asking for the number of handshakes that took place. In the graph, this is represented by edges, so we would be counting the number of edges of a graph with 10 vertices. Each vertex will have degree 9 since each person will shake hands with 9 other people.

How can we find this amount?

Using the fact that the sum of the degrees of all vertices is equal to twice the number of edges, and knowing that each of the 10 vertices has degree 9, we have:

Twice the number of edges = sum of the degrees = 10 * 9 = 90

So, the number of edges is 45, and **45** handshakes took place.

Module Outcome #4: Utilize Euler's Formula to determine whether or not a graph is planar.

Read the following question related to Module Outcome #4 on the tab below and make a mental note of your answer. To check your ideas, click the tab to reveal the answer.

Is it possible for a planar graph to have 15 vertices, 25 edges and 12 faces? How do you now?

Yes. Since the number of vertices, edges, and faces satisfy Euler's formula: 15 – 25 + 12 = 2.

Module Outcome #5: Determine additional properties of planar graphs.

Read the following question related to Module Outcome #5 on the tab below and make a mental note of your answer. To check your ideas, click the tab to reveal the answer.

Consider the statement, "If a graph is planar, then it has an Euler path." Is this statement true or false? Explain your answer.

This statement is false. For example, the complete graph with 4 vertices, K_4 is planar but does not have an Euler path since all of its 4 vertices have degree 3. (Since this is more than 2 vertices having odd degree, we know it doesn't have an Euler path.)

Check Your Understanding

Embedded Media Content! Please use a browser to view this content.

References

Bogdan Giuşcă. (2005). Seven bridges of Königsberg [Image file]. Retrieved from https://en.wikipedia.org/wiki/Seven_Bridges_of_K%C3%B6nigsberg#/media/File:Konigsberg_bridges.png CC BY-SA-3.0

Levin, O. (2017). *Discrete mathematics: An open introduction*. Retrieved from http://discrete.openmathbooks.org/dmoi/colophon-1.html (CC BY-SA 4.0)