MTH350

Discrete Mathematics

Module 7: Logic and Proofs, Part I

This module will build on your understanding of mathematical statements and give you tools to analyze them more in depth. In particular, you will learn how to use truth tables to analyze and compare statements. You will also learn about arguments and their validity.

Learning Outcomes

1. Construct a truth table for a given logical statement.
2. Analyze a mathematical statement using truth tables.
3. Prove logical equivalence without using truth tables.
4. Compare mathematical statements.
5. Assess the validity of an argument.

For Your Success & Readings

In Module 7, you will rely on your understanding of mathematical statements from Module 1. Make sure to review those concepts before beginning this module. You will also be presented with several rules and tables that will be useful to analyze and compare statements and arguments, so first make sure to have a solid grasp of previous definitions.

There will be no Critical Thinking Assignment this week, but the discussion board will give you the opportunity to engage with your peers about arguments and their validity. Make the most of this activity by asking your peers thought-provoking questions whenever possible.

Required

- Chapter 3, **Section 3.1** (http://discrete.openmathbooks.org/dmoi/sec_propositional.html) in *Discrete Mathematics: An Open Introduction*
- Blass, A. (2016). **Symbioses between mathematical logic and computer science** (https://www-sciencedirect-com.csuglobal.idm.oclc.org/science/article/pii/S0168007216300689). *Annals of Pure and Applied Logic*, 167(10), 868-878.
- Cruz Quiroga, L., & Moreno, W. (2016). **Chapter 15 – classic formal logic and nonclassical logics: Basis of research on neural networks** (https://www-sciencedirect-com.csuglobal.idm.oclc.org/science/article/pii/B9780128015599000156). In *Artificial Neural Network for Drug Design, Delivery and Disposition* (pp. 297-317). Boston, MA: Academic Press.

1. Truth Tables

In Module 1, we learned about mathematical statements and that they are either true or false. Now, we will take a deeper look into the truth values of statements by creating truth tables for them.

Let's start by recalling some of the important points about valid reasoning. Watch the following video to get started.

- **Valid Reasoning and Inference** (https://www.linkedin.com/checkpoint/enterprise/login/2245842?pathWildcard=2245842&application=learning&redirect=https%3A%2F%2Fwww%2Elinkedin%2Ecom%2Flearning%2Fprogramming-foundations-discrete-mathematics%2Fvalid-reasoning-and-inference%3Fu%3D2245842)

Also, there will be times when we need to determine the truth value of a complicated mathematical statement involving various connectives.

Before moving on, watch the following video that reviews the various molecular statements and their connectives. In this video, you will also be introduced to the notion of **truth tables**.

- **Conditional Propositions** (https://www.linkedin.com/checkpoint/enterprise/login/2245842?pathWildcard=2245842&application=learning&redirect=https%3A%2F%2Fwww%2Elinkedin%2Ecom%2Flearning%2Fprogramming-foundations-discrete-mathematics%2Fconditional-propositions%3Fu%3D2245842)

We have recalled that the truth value of a statement is completely determined by the truth values of its parts and how they are connected. Thus, all we really need to know is the truth tables for each of the logical connectives.

To see how this works, watch the following Linkedin.com video.

- **Truth Tables** (https://www.linkedin.com/checkpoint/enterprise/login/2245842?pathWildcard=2245842&application=learning&redirect=https%3A%2F%2Fwww%2Elinkedin%2Ecom%2Flearning%2Fprogramming-foundations-discrete-mathematics%2Ftruth-tables%3Fu%3D2245842)

Note that the video shows an example of how to create the truth table for a mathematical statement with three variables, but you can expect to work with mathematical statements involving only two variables for this course.

Here they are the truth tables for each logical connective for your reference:

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

P	Q	$P \rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

P	Q	$P \leftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T

P	$\neg P$
T	F
F	T

Let's see two examples of truth tables for statements $P \vee (\neg P \vee Q)$ and $P \wedge (\neg P \wedge Q)$.

P	Q	$\neg P$	$\neg P \lor Q$	$P \lor (\neg P \lor Q)$
T	T	F	T	T
T	F	F	F	T
F	T	T	T	T
F	F	T	T	T

P	Q	$\neg P$	$\neg P \land Q$	$P \land (\neg P \land Q)$
T	T	F	F	F
T	F	F	F	F
F	T	T	T	F
F	F	T	F	F

Notice that the truth table for $P \lor (\neg P \lor Q)$ is always true, which makes this statement a **tautology**. On the other hand, the truth table for $P \land (\neg P \land Q)$ is always false, so it is a **self-contradiction.**

Now, check your understanding by answering the following question.

1.1. Constructing Truth Tables

Let’s look at another example of how to construct the truth table. Consider the following statement:

$\neg P \wedge (Q \rightarrow P)$

What is the main connective involved?

It is a good idea to look at the “big picture” of the statement we want to work with first. In this case, since $Q \to P$ is enclosed in parentheses, we can see that this statement is of the form:

(a molecular statement) ∧ (another molecular statement)

This helps us with the order in which each column of the truth table should be carried out. In particular, we know that we need the truth columns for the two molecular statements first that will be combined using the “∧” operation at the end.

Make a column for $\neg P$.

We can start with the column for $\neg P$ by using its truth table shown above.

P	Q	$\neg P$
T	T	F
T	F	F
F	T	T
F	F	T

Make a column for $Q \to P$.

We can now add the truth column for the other molecular statement that will be combined at the end with “∧”.

Before doing so, let’s recall the truth table for the connective “→”.

P	Q	$P \to Q$
T	T	T
T	F	F
F	T	T
F	F	T

Unlike the other truth tables, this truth table relies on the order in which the hypothesis (P) and conclusion (Q) are placed. So, when working with this connective, it is very important that you pay close attention to which propositional variable is the hypothesis and which is the conclusion.

In this case, we want to find $Q \to P$, so we need to apply the above truth table where Q is the hypothesis (P), and P is the conclusion (Q).

P	Q	$\neg P$	$Q \to P$
T	T	F	T
T	F	F	T
F	T	T	F
F	F	T	T

Make a column for the entire statement, $\neg P \wedge (Q \rightarrow P)$.

Now, we have both sides of the "∧" connective, so we are ready to join those columns to obtain the final statement.

P	Q	$\neg P$	$Q \rightarrow P$	$\neg P \wedge (Q \rightarrow P)$
T	T	F	T	F
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

Now, check your understanding by answering the following question.

1.2. Analyzing Mathematical Statements Using Truth Tables

Let's take a deeper look into truth tables and analyze them.

Recall the truth table we constructed above for the statement $\neg P \wedge (Q \rightarrow P)$:

P	Q	$\neg P$	$Q \rightarrow P$	$\neg P \wedge (Q \rightarrow P)$
T	T	F	T	F
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

Is the statement a tautology?

No, since the column for $\neg P \wedge (Q \rightarrow P)$ is not all true.

Is the statement a self-contradiction?

No, since the column for $\neg P \wedge (Q \rightarrow P)$ is not all false.

What is the truth value for the statement $\neg P \wedge (Q \rightarrow P)$ when P and Q are both true?

When both P and Q are true, the column for $\neg P \wedge (Q \rightarrow P)$ shows that it is false.

For which truth values of P and Q is the statement $\neg P \wedge (Q \rightarrow P)$ false?

The statement $\neg P \wedge (Q \rightarrow P)$ is false when:

- P is true and Q is true.
- P is true and Q is false.
- P is false and Q is true.

Now, check your understanding of using truth tables by answering the following.

2. Logical Equivalence

Truth tables are also useful for determining if two statements are **logically equivalent**.

Logical Equivalence

Two (molecular) statements P and Q are **logically equivalent** provided P is true precisely when Q is true. That is, P and Q have the same truth value under any assignment of truth values to their atomic parts.

To verify that two statements are logically equivalent, you can make a truth table for each and check whether the columns for the two statements are identical.

Let's see an example of this. Consider the following two statements:

"It will not rain or snow."
"It will not rain and it will not snow."

We can define propositional variables P and Q as follows:

P: It will rain.
Q: It will snow.

Then each statement can be written in symbols. Here is how we would write the first statement, "It will not rain or snow," in symbols: $\neg(P \vee Q)$.

How would you write the second statement statements in symbols? Click the tab below to check your answer.

"It will not rain and it will not snow."

"It will not rain and it will not snow." is written in symbols as $\neg P \wedge \neg Q$.

Now, try creating the truth table for each statement. Here is how the truth table for the first statement would be crafted:

P	Q	$\neg(P \vee Q)$
T	T	F
T	F	F
F	T	F
F	F	T

What would a truth table look like for the second statement? Try drawing one now, then click the tab below to check your work.

Truth Table for $\neg P \wedge \neg$Q

P	Q	$\neg P \wedge \neg Q$
T	T	F
T	F	F
F	T	F
F	F	T

Compare the truth tables to determine if they are logically equivalent. Are they logically equivalent?

YesNo

Correct! The columns for $\neg(P \vee Q)$ and $\neg P \wedge \neg Q$ are the same. Hence, the two statements are logically equivalent.

Actually, if you look again, notice that the columns for $\neg(P \vee Q)$ and $\neg P \wedge \neg Q$ are the same. Hence, the two statements are logically equivalent.

The above two statements also happen to be examples of logically equivalent statements due to the following important laws.

De Morgan's Laws

$\neg(P \wedge Q)$ is logically equivalent to $\neg P \vee \neg Q$.

$\neg(P \vee Q)$ is logically equivalent to $\neg P \wedge \neg Q$.

You can think of the above statements as rules for distributing the negation connective to a parentheses involving $\wedge$ and $\vee$. In particular, when we distribute the negation, the connective inside the parentheses changes.

Now, to put everything together, take a look at the following Linkedin.com video on how logical equivalence can be determined using truth tables as well as other important logic rules. Keep in mind that the rules shown in the video can serve as shortcuts as you compare various mathematical statements, but that you can always resort to comparing their truth tables.

Prove Logical Equivalence (https://www.linkedin.com/checkpoint/enterprise/login/2245842?pathWildcard=2245842&application=learning&redirect=https%3A%2F%2Fwww%2Elinkedin%2Ecom%2Flearning%2Fprogramming-foundations-discrete-mathematics%2Fprove-logical-equivalence%3Fu%3D2245842)

Now, check your understanding of logical equivalence by answering the following:

2.1. Logical Equivalence Using Truth Tables

Take a look at the following video that shows several examples of how to use truth tables to determine logical equivalence of two statements.

How to create a truth table for a proposition involving three variables. **Logical equivalence with truth tables** (https://www.youtube.com/watch?v=D72f9azH2UI)

Now, check your understanding by answering the following question.

2.2. De Morgan's Laws

portrait of Augustus De Morgan, British mathematician who developed De Moran's laws and coined the term "mathematical induction."

Now, let's look at the following video that goes over how to use De Morgan's Laws and other logic rules to show logical equivalence.

This video explores how to use existing logical equivalences to prove new ones, without the use of truth tables. **Logical equivalence without truth tables (Screencast 2.2.4)** (https://www.youtube.com/watch?v=iPbLzl2kMHA)

Now, check your understanding of De Morgan's Laws by answering the following.

3. Arguments

Now, let’s look at a more complete logical structure, called an **argument**.

An **argument** is a set of statements, one of which is called the **conclusion** and the rest of which are called **premises**. An argument is said to be **valid** if the conclusion must be true whenever the premises are all true. An argument is **invalid** if it is not valid; it is possible for all the premises to be true and the conclusion to be false.

For an introduction to arguments and how to determine whether an argument is valid, take a look at the following Linkedin.com video.

- **Valid Arguments** (https://www.linkedin.com/checkpoint/enterprise/login/2245842?pathWildcard=2245842&application=learning&redirect=https%3A%2F%2Fwww%2Elinkedin%2Ecom%2Flearning%2Fprogramming-foundations-discrete-mathematics%2Fvalid-arguments%3Fu%3D2245842)

Now, let’s look at an example. Consider the following argument:

If Edith eats her vegetables, then she can have a cookie.

Edith eats her vegetables.

$\therefore$ Edith gets a cookie.

What are the premises of the argument?

This argument has two premises:

If Edith eats her vegetables, then she can have a cookie.

and

Edith eats her vegetables.

What is the conclusion of the argument?

The conclusion is the last statement, “Edith gets a cookie”.
(Note: The symbol “∴” means “therefore”.)

How can this argument be written in symbols?

Let P denote “Edith eats her vegetables” and Q denote “Edith can have a cookie.” The logical form of the argument is then:

$$P \rightarrow Q$$

$$P$$

$$\therefore \quad Q$$

Make a truth table that has a column for each line of the argument. Click the tab to compare your table with the one shown.

Truth Table – Show Me

P	Q	$P \to Q$
T	T	T
T	F	F
F	T	T
F	F	T

Is this a valid argument?

Remember that an argument is valid when the conclusion is true given that the premises are true. The premises in this case are $P \to Q$ and P.

Now we look at which **rows** of the truth table correspond to both of these being true.

P is true in the first two rows, and of those, only the first row has $P \to Q$ true as well.

Then, we look at the truth value of Q in that case when both $P \to Q$ and P are true. It is also true. So, if $P \to Q$ and P are both true, we see that Q must be true as well.

Hence, this is a valid argument.

The above argument is an example of a **deduction rule**, an argument form that is always valid. There are several deduction rules that can help us determine the validity of arguments. To learn more about these rules, watch the following video:

Rules of Inference (https://www.linkedin.com/checkpoint/enterprise/login/2245842?pathWildcard=2245842&application=learning&redirect=https%3A%2F%2Fwww%2Elinkedin%2Ecom%2Flearning%2Fprogramming-foundations-discrete-mathematics%2Frules-of-inference%3Fu%3D2245842)

Now, check your understanding of arguments by answering the following.

3.1. Validity of Arguments

Before using truth tables to determine the validity of an argument, let's take a closer look at what an argument consists of and how we can check if it is valid.

Let's look at an example. Consider the following argument:

	If Aria studies for her math exam, then she will be happy.
	Aria is happy.
∴	Aria studies for her math exam.

Notice that this argument has two **premises**:

If Aria studies for her math exam, then she will be happy.

and

Aria is happy.

The **conclusion** of this argument is the last statement, "Aria studies for her math exam.".

In this case, even if both premises are true, we cannot conclude the conclusion. Thus, this is not a valid argument.

Now, check your understanding of this by answering the following.

3.2. Validity of Arguments Using Truth Tables

Let's work through another example to show that an argument is valid using truth tables.

Prove that the following argument is valid:

$P \to Q$

$\neg P \to Q$

$\therefore$ Q

Recall that to determine if an argument is valid, we need to make a truth table that contains a column for each line of the argument.

P	Q	$P \to Q$	$\neg P$	$\neg P \to Q$
T	T	T	F	T
T	F	F	F	T
F	T	T	T	T
F	F	T	T	F

Next, we look at which rows have both premises, $P \to Q$ and $\neg P \to Q$, being true.

This occurs in rows 1 and 3.

Finally, we look at the truth value of the conclusion, Q, in those rows and see that it is also true. Hence, this is a valid argument.

Now, check your understanding of valid arguments by answering the following:

4. Summary

Click through the following interactives to view worked examples that cover key points for each module outcome.

Module Outcome #1: Construct a truth table for a given logical statement.

Complete a truth table for the statement $\neg P \rightarrow (P \wedge Q)$. Do this on your own and then click "Show Me" to compare your table with the answer shown.

Show Me

We begin by setting up all possible combinations of truth values for P and Q. We then find the truth values for $\neg P$, and $P \wedge Q$. Finally, we can find truth values for $\neg P \rightarrow (P \wedge Q)$ by taking $\neg P$ as the hypothesis and $P \wedge Q$ as the conclusion when using the implication truth value table.

P	Q	$\neg P$	$P \wedge Q$	$\neg P \rightarrow (P \wedge Q)$
T	T	F	T	T
T	F	F	F	T
F	T	T	F	F
F	F	T	F	F

Module Outcome #2: Analyze a mathematical statement using truth tables.

In which cases is the statement $\neg P \rightarrow (P \wedge Q)$ false?

When P is false and Q is true or when P is false and Q is false.

Module Outcome #3: Prove logical equivalence without using truth tables.

Prove that the statements $\neg(\neg P \vee Q)$ and $P \wedge \neg Q$ are logically equivalent without using truth tables.

We want to start with one of the statements and transform it into the other through a sequence of logically equivalent statements. Start with $\neg(\neg P \vee Q)$.

Now apply De Morgan's law to get
$\neg\neg P \wedge \neg Q$.

Finally, use double negation to arrive at P ∧ ¬Q

Module Outcome #4: Compare mathematical statements.

Using truth tables, determine if the statements $\neg(P \rightarrow Q)$ and P ∧ $\neg Q$ logically equivalent.

We start by making the truth tables for both statements.

P	Q	$P \rightarrow Q$	$\neg(P \rightarrow Q)$
T	T	T	F
T	F	F	T
F	T	T	F
F	F	T	F

P	Q	$\neg Q$	P ∧ ¬Q
T	T	F	F
T	F	T	T
F	T	F	F
F	F	T	F

Now, we compare the last columns of the truth tables and see that they are equal. Hence, the statements were equivalent.

Module Outcome #5. Assess the validity of an argument.

Determine if the following argument is valid.

	$P \rightarrow Q$
	$\neg P$
∴	Q

First, we make a truth table that has a column for each line of the argument.

P	Q	$\neg P$	$P \rightarrow Q$
T	T	F	T
T	F	F	F
F	T	T	T
F	F	T	T

Next, remember that an argument is valid when the conclusion is true given that the premises are true. The premises in this case are $P \rightarrow Q$ and $\neg P$.

Now we look at which **rows** of the truth table correspond to both of these being true.

$\neg P$ is true in the last two rows, and of those, both have $P \rightarrow Q$ true as well.

Then, we look at the truth value of Q in that case when both $P \rightarrow Q$ and $\neg P$ are true. It is also true.

So, if $P \rightarrow Q$ and $\neg P$ are both true, we see that Q is **not** true in the last row.

Is this a valid argument?

Yes

Nope. Read through this example again. This is actually **not** a valid argument.

No

Correct! This is not a valid argument.

Check Your Understanding

Embedded Media Content! Please use a browser to view this content.

References

None