MTH350

Discrete Mathematics

Module 8: Logic and Proofs, Part II

This module will cover three mathematical proof methods: direct proof, proof by contraposition, and proof by contradiction. Using these methods, you will learn how to select, plan, assess, and develop the proofs of mathematical statements.

Learning Outcomes

1. Select a proof method given a mathematical statement.
2. Modify a given mathematical statement into its contrapositive, converse, and negation.
3. Plan the proof of a mathematical statement using various proof methods.
4. Assess the validity of proofs of mathematical statements.
5. Develop proofs of mathematical statements.

For Your Success & Readings

Module 8 will introduce you to three proof methods. You will see various proofs worked out in the lecture as well as in the videos. It is important that you pay attention to the details of the proofs shown in this lecture so you can develop your own proofs of mathematical statements.

There is no Critical Thinking Assignment this week since you will be taking the final exam. Note that the final exam is cumulative and includes material from Module 8.

In the discussion board this week, you will get to re-create a proof from the textbook and discuss alternative proof methods that may also work. This will be an opportunity to expand your understanding of the proof method we will learn in this module.

Required

- Chapter 3, **Section 3.2** (http://discrete.openmathbooks.org/dmoi/sec_logic-proofs.html) in *Discrete Mathematics: An Open Introduction*
- Brown, S. A. (2018). **Are indirect proofs less convincing? A study of students' comparative assessments** (https://www.sciencedirect.com/science/article/pii/S0732312316302000?via%3Dihub). *The Journal of Mathematical Behavior, 49*, 1-23.
- McCartin-Lim, M., Woolf, B., & McGregor, A. (2018). **Connect the dots to prove it** (https://dl-acm-org.csuglobal.idm.oclc.org/citation.cfm?id=3159609). *Proceedings of the 49th ACM Technical Symposium on Computer Science Education - SIGCSE 18*.

1. Direct Proof

In this module, we will learn about ways to prove mathematical statements. Before reading the rest of the lecture, watch the following Linkedin.com video on the general idea behind proofs.

Write a General Outline for a Proof (https://www.linkedin.com/checkpoint/enterprise/login/2245842?pathWildcard=2245842&application=learning&redirect=https%3A%2F%2Fwww%2Elinkedin%2Ecom%2Flearning%2Fprogramming-foundations-discrete-mathematics%2Fwrite-a-general-outline-for-a-proof%3Fu%3D2245842)

In this section, we learn how to prove statements that can be phrased as implications. Recall from Module 1 that an implication is a molecular statement of the form $P \rightarrow Q$ where P and Q are statements and P is the hypothesis and Q is the conclusion.

The first proof style we will learn is called a **direct proof.**

The simplest (from a logic perspective) style of proof is a **direct proof**. Often all that is required to prove something is a systematic explanation of what everything means. The general format to prove $P \rightarrow Q$ is this:

Assume P. Explain, explain, ..., explain. Therefore Q.

Let's see how this works with an example. Suppose we want to prove the following statement:

For all integers n, if n is even, then n^2 is even.

Where do we start?

In a direct proof of an implication, we begin with the hypothesis:

Let n be an arbitrary integer. Suppose n is even.

Note that when the statement we want to prove involves the quantifier “for all,” the proof must always begin with an assumption that n is arbitrary. In this case, we want to prove the statement is true for all integers, so we begin by saying “Let n be an arbitrary integer.”

Where do we want to end?

In a direct proof of an implication, we want to end at the conclusion:

Therefore, n^2 is even.

How are the start and end statements related?

This is the heart of writing a proof. Often times, this connection won’t be immediately clear, and you won’t discover it until you begin a process of trial and error in getting from the start statement to the end statement.

In this case, we notice that the start statement deals with n, and the end statement deals with n^2. This leads us to want to find a connection through algebraic manipulation between the two statements.

What conclusions can we draw from the start statement?

With the above thought in mind, we can start to think about what conclusions can be drawn from the assumed start statement “n is even.”

Since we want to work algebraically with that statement to get to the end statement, we can think about how to represent n as an even integer.

This can be done by setting $n = 2k$ where k is an integer since this means n is a multiple of 2. So our proof so far looks like this:

Let n be an arbitrary integer. Suppose n is even.

Then $n = 2k$ for some integer k.

We are now ready to fill in the details and connect the assumption to the conclusion by working from the fact that $n = 2k$ for some integer k. On your own, fill in the details of the proof. Then click the tab below to check your work.

The Details of the Proof

To fill in the details of the proof, we notice that if we square both sides of the equation, we get $n^2 = (2k)^2 = 4k^2$. So, we have $n^2 = 4k^2$. Now, $4k^2$ can be written as $2(2k^2)$, and $2k^2$ is also an integer, so n^2 is also a multiple of 2, hence, n^2 is also even.

With this missing piece complete, we can fill in the details of the proof. The complete proof is as follows:

Proof:
Let n be an arbitrary integer. Suppose n is even.

Then $n = 2k$ for some integer k.

Square both sides of the equation, to get $n^2 = (2k)^2 = 4k^2 = 2(2k^2)$. $2k^2$ is also an integer, so n^2 is a multiple of 2.

Therefore, n^2 is even.

QED (which stands for "quod erat demonstrandum" and is Latin for "that which was to be demonstrated." This is often used at the end of a proof to indicate it is completed.)

Now, check your understanding of direct proofs by answering the following.

1.1. Direct Proof Explained

For another look at how to write a direct proof. Take a look at the following video. In this first video in our mathematical proof series, (the slightly injured) Ben discusses direct proofs and shows two examples.

- **Direct Proofs** (https://www.youtube.com/watch?v=G2fBky3jgCk)

Now, check your understanding of direct proofs by answering the following.

1.2. Direct Proof Examples

For more examples of how to prove implications using direct proofs, watch the following video: **Evaluate Conditional Proofs** (https://www.linkedin.com/checkpoint/enterprise/login/2245842?pathWildcard=2245842&application=learning&redirect=https%3A%2F%2Fwww%2Elinkedin%2Ecom%2Flearning%2Fprogramming-foundations-discrete-mathematics%2Fevaluate-conditional-proofs%3Fu%3D2245842)

Now, check your understanding of how to write direct proofs by answering the following:

2. Proof by Contrapositive

In Module 1, we learned that implications are equivalent to their contrapositive. This fact can often be useful in proving implications where its contrapositive statement is easier to prove by direct proof.

A **proof by contrapositive** gives a direct proof of the contrapositive of the implication. This is enough because the contrapositive is logically equivalent to the original implication.

The skeleton of the proof of $P \rightarrow Q$ by contrapositive will always look roughly like this:

Assume $\neg Q$. Explain, explain, … explain. Therefore $\neg P$.

Let's try to prove the following statement by contraposition:

For all integers n, if n^2 is even, then n is even.

Is the contrapositive of this statement easier to prove?

Planning a proof should be the first step in helping you decide the proof method and develop the proof.

Here, we notice that a direct proof of the statement would involve fixing an arbitrary n and assuming n^2 is even. However, this does not seem to tell us anything about n being even.

On the other hand, we can consider the contrapositive of the statement:

For all integers n, if n is odd, then n^2 is odd.

This should give us a better start in writing the proof.

Write a direct proof of the contrapositive statement.

Now we are ready to write a direct proof of the contrapositive statement, "*For all integers n, if n is odd, then n^2 is odd.*"

We can proceed as shown in the previous section on direct proofs to obtain the following proof:

Proof:
We will prove the contrapositive. Let n be an arbitrary integer. Suppose that n is not even, and thus odd. Then $n = 2k+1$ for some integer k. Now $n^2 = (2k+1)^2 = 4k^2+4k+1 = 2(2k^2+2k)+1$. Since $2k^2+2k$ is an integer, we see that n^2 is odd and therefore not even.

QED

Now, check your understanding of proof by contraposition by answering the following.

2.1. Proof by Contrapositive Explained

For another look at how to write a proof by contrapositive. Take a look at the following video. Ben discusses proof by contraposition and works through a few examples.

- **Contraposition** (https://www.youtube.com/watch?v=YUk16io1CLE)

Now, check your understanding of proofs by contraposition by answering the following.

2.2. Proof by Contrapositive Examples

Now take a look at the following two videos for more examples on how to construct a proof by contraposition.

- This video describes proof by contraposition, a method of proving a conditional statement by constructing a direct proof of its contrapositive. **Proof by contraposition (Screencast 3.2.1)** (https://www.youtube.com/watch?v=hAFpc9abNFc)
- This video gives more examples of proofs by contraposition. **Proof by contraposition 2 (Screencast 3.2.2)** (https://www.youtube.com/watch?v=3ORYou8dc0s)

Check your understanding by answering the following question.

3. Proof by Contradiction

The last type of proof we will learn in this module is **proof by contradiction**. In this proof method, we want to prove a statement P in a way that $\neg P$ leads to a contradiction so that the only conclusion is that $\neg P$ is false, and thus P is true. That's what we wanted to prove. In other words, if it is impossible for P to be false, then P must be true.

Let's look at the step by step of the following proof by contradiction of the statement, "*There are infinitely many primes.*"

Proof

This proof is an example of a *proof by contradiction*, one of the standard styles of mathematical proof. First and foremost, the proof is an argument. It contains sequence of statements, the last being the *conclusion* which follows from the previous statements. The argument is valid, so the conclusion must be true if the premises are true. Let's go through the proof line by line.

Suppose there are only finitely many primes.

This is a premise. Note the use of "suppose."

There must be a largest prime, call it *p*.

This follows from line 1, by the definition of "finitely many."

Let $N = p! + 1$.

This is basically just notation, although this is the inspired part of the proof; looking at p! + 1 is the key insight.

N is larger than *p*.

This is true by the definition of p! from step 3 above.

N is not divisible by any number less than or equal to *p*.

By definition, p! is divisible by each number less than or equal to p, so p! + 1 is not.

The prime factorization of *N* contains prime numbers greater than *p*.

This is true since N is divisible by each prime number in the prime factorization of N, and by line 5.

Therefore, *p* is not the largest prime.

This is true by line 6: N is divisible by a prime larger than p.

This is a contradiction.

From line 2 and line 7: the largest prime is p and there is a prime larger than p.

Therefore, there are infinitely many primes.

This follows from line 1 and line 8: our only premise lead to a contradiction, so the premise is false.

Let's look at the beginning of another proof by contradiction.

Prove: There are no integers *x* and *y* such that $x^2 = 4y+2$.

We proceed by contradiction. Suppose there *are* integers *x* and *y* such that $x^2 = 4y+2 = 2(2y+1)$. So, x^2 is even. We have seen that this implies that *x* is even. So, $x = 2k$ for some integer *k*. Then $x^2=4k^2$. This in turn gives $2k^2 = (2y+1)$.

Now, check your understanding of proof by contradiction by answering the following question.

3.1. Proof by Contradiction Explained

For a closer look at proofs by contradiction, take a look at the following video that walks through how this works. Ben discusses proof by contradiction and goes through some examples.

- **Proof by Contradiction** (https://www.youtube.com/watch?v=E7J_lhygeK4)

Now, check your understanding of this by answering the following.

3.2. Proof by Contradiction Examples

Now, let's look at the following video for an in-depth review of how to write a proof by contradiction. Additionally, this video discusses the proof by contradiction for the theorem which states, "For all positive integers m, n, we have $m^2 - n^2 \neq 1$.". Note that in this example, you will need to consider two cases:

1. Suppose $m + n = 1$ and $m - n = 1$.
2. Suppose $m + n = -1$ and $m - n = -1$.

- **Proof by contradiction (Screencast 3.3.1)** (https://www.youtube.com/watch?v=YUL6HMJmTM4)

Now, check your understanding by answering the following.

4. Summary

Click through the following interactives to view worked examples that cover key points for each module outcome.

Module Outcome #1: Select a proof method given a mathematical statement.

What would be the best proof method to prove the following statement? For all integers n, if $7n$ is odd, then n is odd.

This is best to prove by contraposition since the contrapositive of the statement is easier to prove.

Module Outcome #2: Modify a given mathematical statement into its contrapositive, converse, and negation.

Which of the following is the contrapositive of the mathematical statement, “For all integers n, if $7n$ is odd, then n is odd”?

The contrapositive of the statement is: “For all integers n, if n is even, then $7n$ is even.”

Module Outcome #3: Plan the proof of a mathematical statement using various proof methods.

Which of the following would be a correct way to set up a **proof by contraposition** for the following statement?

For all integers n, if $7n$ is odd, then n is odd.

The proof of this statement by contraposition would begin as follows: Let n be an integer. Assume n is even.

Module Outcome #4: Assess the validity of proofs of mathematical statements.

Determine whether the following proof of the statement, “For all integers n, if $7n$ is odd, then n is odd.” is valid.

Proof:
Let n be an integer. Assume n is odd. So $n=2k+1$ for some integer k. Then $7n = 7(2k+1) = 14k+7 = 2(7k+3)+1$.

Since $7k+3$ is an integer, we see that $7n$ is odd.

QED

Yes, this is valid.

Actually, look again. This is not a valid proof of the statement, “For all integers n, if $7n$ is odd, then n is odd,”. The original statement can be written as $P \rightarrow Q$ with P: $7n$ is odd. and Q: n is odd.

Now, this proof is of its inverse, $Q \rightarrow P$, which is not logically equivalent to $P \rightarrow Q$.

No, this is not valid.

Correct! This is not a valid proof of the statement, “For all integers n, if $7n$ is odd, then n is odd,”. The original statement can be written as $P \rightarrow Q$ with P: $7n$ is odd. and Q: n is odd.

Now, this proof is of its inverse, $Q \rightarrow P$, which is not logically equivalent to $P \rightarrow Q$.

Module Outcome #5: Develop proofs of mathematical statements.

Prove the statement: For all integers n, if $7n$ is even, then n is even. When you have your proof, click the tab below to check your answer.

Proof

Here is a proof by contraposition of this statement.

Proof:

Let n be an integer. Assume n is odd. So, $n = 2k+1$ for some integer k. Then $7n = 7(2k+1) = 14k+7 = 2(7k+3)+1$.

Since $7k+3$ is an integer, we see that $7n$ is odd. Hence, the contrapositive statement is also true.

QED

Check Your Understanding

Embedded Media Content! Please use a browser to view this content.

References

None