MTH350

Discrete Mathematics

Module 1: Preliminaries

This module will introduce you to what discrete mathematics is and its various applications in real-world contexts. Additionally, you will learn about mathematical statements and sets as they are considered preliminaries for the study of the topics we will cover in this course.

Learning Outcomes

1. Classify mathematical statements.
2. Translate mathematical statements between English and symbols.
3. Compare mathematical statements.
4. Formulate distinct representations of sets.
5. Apply various set operations.

For Your Success & Readings

Module 1 will begin by providing an informative background on what discrete mathematics is and how it can be applied to real-world contexts. The recommended readings for this module will give you other perspectives on where this topic fits into the mathematics and computer sciences curricula. Keep this background in mind as you explore the various applications of discrete mathematics throughout the course.

Another goal of this module is to familiarize you with mathematical statements and set theory, which are topics you will need before moving on to the rest of the topics in this course. By learning about the structure of mathematical statements, you will essentially be learning the language needed to write mathematics.

In the introduction to set theory, you will learn about one of the most fundamental structures in mathematics, sets. It will be important that you become comfortable with the various representation of sets before learning how to perform operations with them. When it comes to learning about set operations, and just about every other mathematical concept, the key to success is practice, practice, and more practice!

The discussion board for this week will give you a chance to engage with your peers as you extend your understanding of sets. You can make the most of this opportunity by posting early and by asking follow-up questions that promote further conversations.

Required

- Chapter 0, Sections **0.1** (http://discretetext.oscarlevin.com/dmoi/sec_intro-intro.html), **0.2** (http://discretetext.oscarlevin.com/dmoi/sec_intro-statements.html), **0.3** (http://discretetext.oscarlevin.com/dmoi/sec_intro-sets.html), & **0.4** (http://discretetext.oscarlevin.com/dmoi/sec_intro-functions.html) in *Discrete Mathematics: An Open Introduction*
- Mathieu, J., & Théo, L. (2014). Teaching formal methods and discrete mathematics. *Electronic Proceedings in Theoretical Computer Science, 149*, 30-43. Retrieved from **https://arxiv.org/pdf/1404.6604v1.pdf** (https://arxiv.org/pdf/1404.6604v1.pdf)
- Vandrunen, T. (2017). **Functional programming as a discrete mathematics topic** (https://dl-acm-org.csuglobal.idm.oclc.org/citation.cfm?doid=3095781.3078325). *ACM Inroads,8*(2), 51-58.

1. Mathematical Statements

Before we can begin reading and writing about the mathematical concepts in this course, we need to establish some of the basic preliminaries. In this section, we are learning about the structure and properties of mathematical statements. Also, we will learn how to classify mathematical statements, translate them between English and symbols, and determine equivalency among them.

Let's start with some definitions first.

A **statement** is any declarative sentence which is either true or false. A statement is **atomic** if it cannot be divided into smaller statements; otherwise it is called **molecular**.

With the above definitions in mind, it is often helpful to understand what a statement is by way of examples. Consider the following list of sentences and decide whether or not each one is a statement before clicking on it. If it is a mathematical statement, think about whether it is atomic or molecular.

The first three letters of the alphabet are X, Y, Z.

This is an **atomic statement** since it is either true or false and there are no connectives.

The first three letters of the alphabet are X, Y, Z and the last three letters are A, B, C.

This is a **molecular statement** since it can be broken down into two atomic statements: "The first three letters of the alphabet are X, Y, Z." and "The last three letters are A, B, C."

The first three letters of the alphabet.

This is **not a statement** as it doesn't make sense to say it is true or false, so it is neither atomic nor molecular.

Now, given two or more statements (atomic or molecular), you can create a new molecular statement by using **logical connectives.** Below are the 5 connectives we will see in this course. (Note that the last one is technically a connective applied to a single statement, but it is still considered a molecular statement.)

Logical Connectives

- $P \wedge Q$ means P and Q, called a **conjunction**.
- $P \vee Q$ means P or Q, called a **disjunction**.
- $P \rightarrow Q$ means if P then Q, called an **implication** or **conditional**.
- $P \leftrightarrow Q$ means P if and only if Q, called a **biconditional**.
- $\neg P$ means not P, called a **negation**.

An important skill to master before moving on is translating mathematical statements between English and symbols. **Propositional variables** (often represented by capital letters P, Q, R, ... and can take on "T" or "F" values) and logical connectives will be used to represent the words of a mathematical statement written in English.

For example, consider statements:

P: It rains.

and

Q: I open my umbrella.

Then we would have:

- $P \wedge Q$: It rains and I open my umbrella.
- $P \vee Q$: It rains or I open my umbrella.
- $P \rightarrow Q$: If it rains, then I open my umbrella.
- $P \leftrightarrow Q$: It rains if and only if I open my umbrella.
- $\neg P$: It does not rain.
- $\neg Q$: I do not open my umbrella.

We said that a statement can either be true or false. This is called the **truth value** of a statement. In the case of a molecular statement that includes logical connectives, its truth value will depend on the truth values of each propositional variable and the logical connective used. Here are the truth conditions for the connectives:

Truth Conditions for Connectives

- $P \wedge Q$ is true when both P and Q are true
- $P \vee Q$ is true when P or Q or both are true.
- $P \rightarrow Q$ is true when P is false or Q is true or both.
- $P \leftrightarrow Q$ is true when P and Q are both true, or both false.
- $\neg P$ is true when P is false.

We say that two statements are logically equivalent if their truth tables are identical. We will discuss truth tables in Module 7, but for now, think of two logically equivalent statements as "meaning the same thing" and one being true (or false) when the other is true (or false).

One important type of mathematical statement is an **implication**, which is one of the molecular statements listed above of the form $P \rightarrow Q$.

An implication or conditional is a molecular statement of the form $P \rightarrow Q$ where P and Q are statements. We say that

- P is the **hypothesis** (or **antecedent**).
- Q is the **conclusion** (or **consequent**).

An implication is true provided P is false or Q is true (or both), and false otherwise. In particular, the only way for $P \rightarrow Q$ to be false is for P to be true and Q to be false.

An implication can also be discussed in terms of its **converse** ($Q \rightarrow P$) and **contrapositive** ($\neg Q \rightarrow \neg P$)**.** Note that the converse of an implication *is not* logically equivalent to the original implication, but its contrapositive *is* logically equivalent to it.

Now, think about how the converse and contrapositive of the implication $P \rightarrow Q$ shown above with P: It rains. and Q: I open my umbrella. Think about how each would be written in English before clicking through the activity below.

Converse, $Q \rightarrow P$

The converse of this statement in English is "If I open my umbrella, then it rains." Note that this *is not* logically equivalent to the original implication.

Contrapositive, $\neg Q \rightarrow \neg P$

The contrapositive of this statement in English is "If I don't open my umbrella, then it does not rain." Note that this *is* logically equivalent to the original implication.

One final aspect of mathematical statements that will be important for this course, is the use of **quantifiers**. There are two main ones that will be used here.

The **existential quantifier** is ∃ and is read “there exists” or “there is.”
The **universal quantifier** is∀ and is read “for all” or “every.”

For example, the statement “there exists a number that is greater than 2” can be written in symbols as “$\exists x(x > 2)$”.

As with implications, there are logical equivalences that may be useful for working with quantifiers. In this case, we know that:

$\neg\forall x P(x)$ is equivalent to $\exists x \neg P(x)$.
$\neg\exists x P(x)$ is equivalent to $\forall x \neg P(x)$.

All of the logical equivalences we have talked about so far are going to be especially useful when elaborating proofs. For example, there may be times when proving the contrapositive of a statement is easier than proving the original implication, and, since we know they are logically equivalent, proving one is enough to establish the other.

We will talk more about proof methods in Modules 4, 7, and 8. For now, it will be important to understand how to represent statements using symbols and how to find logically equivalent statements. Check your understanding of the some of the concepts from this section by working through the following question:

1.1. Atomic and Molecular Statements

Mathematical statements can be thought of as the type of sentences used to express mathematical ideas. These statements must be written in a way that is exact, and that can be deemed true or false. Now, statements can be more or less complex, but still be exact in nature. If a mathematical statement can be divided into smaller statements, then it is called a **molecular statement.** Otherwise, it is considered an **atomic statement.**

Now, suppose you have two atomic statements. There are several ways you can combine them to create a molecular statement by using **logical connectives.** These are often represented in symbols as follows:

Also, capital letters (typically P, Q, R, S, ...) are used as **propositional variables** that represent the statements. Note that each of those propositional variables can either take values "T" or "F".

Now, consider the following two atomic statements represented by P and Q:

P: The grass is green.
Q: The sky is blue.

Let's see how the logical connectives would be used to create new molecular statements out of them:

$P \wedge Q$

This reads "P and Q". In English this would be "The grass is green *and* the sky is blue.".

$P \vee Q$

This reads "P or Q". In English this would be "The grass is green *or* the sky is blue.".

$P \rightarrow Q$

This reads "if P then Q". In English this would be "If the grass is green, then the sky is blue.".

$P \leftrightarrow Q$

This reads "P if and only if Q". In English this would be "The grass is green if and only if the sky is blue."

$\neg P$

This reads "not P". In English this would be "The grass is *not* green."

Now, recall that the **truth value** of a statement is determined by the truth values of its parts, so we can determine the truth value of molecular statements using the following guide:

- $P \wedge Q$ is true when both P and Q are true.
- $P \vee Q$ is true when P or Q or both are true.
- $P \rightarrow Q$ is true when P is false or Q is true or both.
- $P \leftrightarrow Q$ is true when P and Q are both true, or both false.
- $\neg P$ is true when P is false.

We will discuss truth values more in detail in Module 7 where we will use truth tables to work with more complex examples. For now, make sure you are comfortable with the following exercises.

1.2. Implications

An implication is one of the most commonly used structures for mathematical statements. It can also be one of the trickiest ones when thinking about its truth values. We noted in the first page that an implication is true when P is false or Q is true (or both). Let's take a closer look at why this is the case through the following example:

Suppose we have the following statements:
P: It is April.
and
Q: It is raining.

Then all possibilities for the implication $P \rightarrow Q$: If it is April, then it is raining. are as follows:

Both P and Q are true: It is April, and it is raining. This is exactly the implication, so it is true.

P is true, and Q is false: It is April, and it is not raining. Here, the statement that P implies Q is false since we said if it was April, then it would rain.

P is false, and Q is true: It is not April, and it is raining. For this one, notice that our implication does not state that it cannot rain when it isn't April. In other words, our implication still holds true when the conclusion occurs without the hypothesis occurring.

P is false, and Q is false: It is not April, and it is not raining. Similar to the above case, it can rain or not rain regardless of whether or not it is April. So, the implication holds true in both cases when the hypothesis is false.
So, we see that $P \rightarrow Q$ is true when P is false or Q is true (or both).

Now, we know that, given an implication $P \rightarrow Q$, its converse $Q \rightarrow P$, is not logically equivalent to it, but its contrapositive, $\neg Q \rightarrow \neg P$ is logically equivalent to it.

Let's look at this a bit more using an example in English. Which of the following statements are equivalent to the implication "If you study math, then you are smart." and which are equivalent to the converse of the implication?

In the above example, we also noted additional ways to state implications. In particular, we can make use of the words "necessary" and "sufficient" in the following manner:

- "P is necessary for Q" means $Q \rightarrow P$.
- "P is sufficient for Q" means $P \rightarrow Q$.
- If P is necessary and sufficient for Q, then $P \leftrightarrow Q$.

Here is a final check to make sure you are comfortable with finding logically equivalent statements involving implications.

1.3. Quantifiers

The two main quantifiers we will work with in this course are the **existential quantifier** written in symbols as $\exists$ and is read "there exists" or "there is", and the **universal quantifier** written as $\forall$ and read as "for all" or "every."

As with implications, there are certain cases where we know that two statements involving quantifiers are logically equivalent. In particular, we know that:

$\neg \forall x P(x)$ is equivalent to $\exists x \neg P(x)$.
and
$\neg \exists x P(x)$ is equivalent to $\forall x \neg P(x)$.

Now, check your understanding by selecting the statement from the list that is equivalent to the negation of the statement, "$\forall x$(x is even)".

2. Sets

Many of the concepts we will learn about in this course involve sets, as they are one of the most fundamental structures in mathematics. Let's begin with a formal definition.

A **set** is an unordered collection of objects.

Think about the following example of a set: S = {California, Colorado, Connecticut}

Notice that this set has 3 elements, so we say that the **cardinality** of S is 3.

Also, note that the set S was defined in **list** by using curly braces "{ }" to enclose the elements. Click through the following accordion to view two other common ways to describe this and all sets:

{x: x is the name of a U.S. state that begins with the letter C}

This notation reads "x is such that x is the name of a U.S. state that begins with the letter C". This notation is called **set builder** notation. By setting a logical statement after the ":", you are describing a condition for what objects should be included in the set. In this example, the condition is that the elements must be the name of a U.S. state that begins with the letter C. Since California, Colorado, and Connecticut are the only three that satisfy this condition, they are the only three elements of the set S.

The set of the names of all U.S. states that begin with the letter C.

You can always describe a set in **word** form as shown here. However, it sometimes leads to less exactness in how a set is defined, so it is more common to use list form or set builder notation.

Here are some special sets that are important to be familiar with as we move forward.

The empty set is the set which contains no elements. This is represented by the symbol .
The universe set is the set of all elements.
The power set of any set A is the set of all subsets of A. We typically write this as $P(A)$.
is the set of natural numbers = {0, 1, 2, ...}.
is the set of integers. That is, = {..., −2, −1, 0, 1, 2, 3, ...}.
is the set of rational numbers.
is the set of real numbers.

Below is a list of important set theory notation that you should become familiar with by the end of this module:

Set Theory Notation

{,}	We use these *braces* to enclose the elements of a set. So, {1, 2, 3} is the set containing 1, 2, and 3.
:	$\{x: x > 2\}$ is the set of all *x such that x* is greater than 2.
$\in$	$2 \in \{1, 2, 3\}$ asserts that 2 is *an element of* the set {1, 2, 3}.
$\notin$	$4 \notin \{1, 2, 3\}$ because 4 *is not an element of* the set {1, 2, 3}.
$\subseteq$	$A \subseteq B$ asserts that *A is a subset of B:* every element of *A* is also an element of *B*.
$\subset$	$A \subset B$ asserts that *A is a proper subset of B:* every element of *A* is also an element of *B,* but $A \neq B$.
$\cap$	$A \cap B$ is the *intersection of A and B:* the set containing all elements which are elements of both *A* and *B*.
$\cup$	$A \cup B$ is the *union of A and B:* is the set containing all elements which are elements of *A* or *B* or both.
x	*A* x *B* is the *Cartesian product of A and B:* the set of all ordered pairs (a, b) with $a \in$ A and $b \in B$.
\	$A \backslash B$ is *A set-minus B:* the set containing all elements of *A* which are not elements of *B*.
$\bar{A}$	The *complement of A* is the set of everything which is not an element of *A*.
$\|A\|$	The *cardinality (or size) of A is the number of elements in A*.

Now, let's take two sets $A = \{a, b, c\}$ and $B = \{a, b, c, d, e\}$.

Are A and B equal?

No, since they do not include the exact same elements.

Is $A \subset B$?

Yes, since every element in A is also an element in B.

Is $B \subset A$?

No, since there exists an element in B that is not in A. For example, d $\in$ B, but d $\notin$ A.

Is $\emptyset \subset A$?

Yes, the empty set is a subset of every set.

What is the cardinality of B?

5, since there are 5 elements in B.

What is $A \cup B$?

The elements that are in A, B, or both are {a, b, c, d, e}, which is just the set B.

What is $A \cap B$?

The elements that are in both A and B are $\{a, b, c\}$, which is just the set A.

What is $B \backslash A$?

The elements that are in B, but not in A are $\{d, e\}$.

Give the following problem a try before deciding whether or not you'd like to explore some of these concepts in depth.

2.1. Notation

To gain a broader understanding of what a set is, begin by watching the following video by Fisher (2016). Here she emphasizes that sets can include all types of elements, not just numbers: **Objects as Sets** (https://www.linkedin.com/checkpoint/enterprise/login/2245842?pathWildcard=2245842&application=learning&redirect=https%3A%2F%2Fwww%2Elinkedin%2Ecom%2Flearning%2Fprogramming-foundations-discrete-mathematics%2Fobjects-as-sets%3Fu%3D2245842)

To extend this, consider the set of all students currently enrolled at CSU-Global. Let's call that set U. Then, suppose set A is all students currently enrolled in this course at CSU-Global. Before clicking through the following activity, think about each question carefully.

What does $U \setminus A$ represent?

All CSU-Global students *not* currently enrolled in this course.

What is the cardinality of A?

The number of students in this course.

How could we define the set A in list form?

You could list the names of each student within curly braces. This is an example of a set that lends itself best to using the word description to define it rather than list form or set-builder notation.

Before moving on, take a moment to check your understanding by answering the following question:

2.2. Relationships Between Sets

For some more examples on how to determine relationships between sets, take a look at the following video by Fisher (2016) that goes over the main relationships: **Set Notation** (https://www.linkedin.com/checkpoint/enterprise/login/2245842?pathWildcard=2245842&application=learning&redirect=https%3A%2F%2Fwww%2Elinkedin%2Ecom%2Flearning%2Fprogramming-foundations-discrete-mathematics%2Fset-notation%3Fu%3D2245842)

Consider the sets A = {a, b, c, d, e, f}, B = {b, d, f}, and C = {a, b, c}. Determine which of the following are true, false, or meaningless.

Before moving on, take a moment to check your understanding by answering the following question:

2.3. Operations on Sets

To dive a bit deeper into set operations, take a look at the following video by Fisher (2016) that shows how to utilize them in some examples. Note that we won't be using the laws explicitly in this course, but feel free to take note of them in case they can be useful for you in working with set operations. Note that the difference of sets is explained in the video with the notation of "$A - B$" rather than "$A \setminus B$" as in this course: **Set Operations** (https://www.linkedin.com/checkpoint/enterprise/login/2245842?pathWildcard=2245842&application=learning&redirect=https%3A%2F%2Fwww%2Elinkedin%2Ecom%2Flearning%2Fprogramming-foundations-discrete-mathematics%2Fset-operations%3Fu%3D2245842)

Finally, this video also by Fisher (2016) goes over a set's power set and how to determine its cardinality. Also, the video applies this concept to the real-world context of family photos: **Power Sets** (https://www.linkedin.com/checkpoint/enterprise/login/2245842?pathWildcard=2245842&application=learning&redirect=https%3A%2F%2Fwww%2Elinkedin%2Ecom%2Flearning%2Fprogramming-foundations-discrete-mathematics%2Fpower-sets%3Fu%3D2245842)

Before moving on, take a moment to check your understanding by answering the following question:

3. Summary

Click through the tabs below to view worked examples that discuss key points for each module outcome. Use this as your review before taking the final Check Your Understanding activity and moving on to the mastery exercise for Module 1.

Module Outcome #1: Classify mathematical statements.

Early in Module 1 we learned about **atomic statements** and **molecular statements**. We also learned that sometimes statements are actually not statements at all. Molecular statements can also be broken down into types: **conjunction, disjunction, conditional, biconditional, negation.**

Reach each of the following statements and mentally classify them as either atomic, molecular, or *not* a statement. If the statement is molecular, see if you can determine what kind it is (conjunction, disjunction, conditional, biconditional, negation). Then, click on the statement to check your ideas.

The first three letters of your name.

This is *not* a statement since it does not make sense to say it is true or false.

There is a prime number in the set, but only if the number 2 is in the set.

This is a molecular statement that has two atomic statements connected by a conditional connective "if...then". Thus, this is a conditional statement.

All dogs like cats.

This is an atomic statement since there are no **connectives**.

$x+2=4$ or $x+2=5$

This is a molecular statement, and in particular, a disjunction since it is composed of two atomic statements connected by an "or."

Module Outcome #2: Translate mathematical statements between English and symbols.

Early in Module 1 we learned about **atomic statements** and **molecular statements**. We also learned that sometimes statements are actually not statements at all. Molecular statements can also be broken down into types: **conjunction, disjunction, conditional, biconditional, negation.**

Reach each of the following statements and mentally classify them as either atomic, molecular, or *not* a statement. If the statement is molecular, see if you can determine what kind it is (conjunction, disjunction, conditional, biconditional, negation). Then, click on the statement to check your ideas.

$P \rightarrow \neg Q$

Since there is a "→" symbol, we know that everything to the left will be the **hypothesis** and everything to the right of it will be the **conclusion** so that it reads If ...(hypothesis)... then...(conclusion). Now, the hypothesis here is just P, which is "The sun is out.", but the conclusion is $\neg Q$, the **negation** of Q. So, the conclusion reads "It is not summer.". *P*utting it all together we have: "If the sun is out, then it is not summer."

It is not summer or the sun is not out.

Start by finding P and Q in the statement and deciding what connectives are added to P and Q. We see that both P and Q are negated with the word "not", so we know that in symbols, we will have $\neg P$ and $\neg Q$. Finally, we notice that the word "or" is what separates the two negated statements, so the translation becomes $\neg P \vee \neg Q$.

Module Outcome #3: Compare mathematical statements.

Let's now review how to compare mathematical statements.

Which of the following statements are equivalent to the implication, "If the word is bolded, then it is an important point"? (Hint: Before beginning this exercise, It is often helpful to convert the given statement into symbols first: Let P: The word is bolded. and Q: It is an important point. Then the given statement is $P \rightarrow Q$). You can check your ideas by clicking on the following tabs.

If it is an important point, then the word is bolded.

Converting this statement into symbols we have: $Q \rightarrow P$, which is the **converse** of the given statement $P \rightarrow Q$. Hence, we know this is *not* logically equivalent to the original implication.

If it is not an important point, then the word is not bolded.

Converting this statement into symbols we have: $\neg Q \rightarrow \neg P$, which is the **contrapositive** of the given statement $P \rightarrow Q$. Hence, we know this is logically equivalent to the original implication.

Module Outcome #4: Formulate distinct representations of sets.

We also learned how to formulate distinct representations of sets. Let's review. Describe the following sets in a different way than given (i.e.: By using **list form**, **set-builder notation**, or **word description**). Then click on each tab to check your ideas.

$\{4, 5, 6, 7, 8\}$

- Word description: The set of integers between 4 and 8, inclusive.
- Set-builder notation: $\{x\in \quad : 4\le x\le 8\}$

(Note that there could be alternate correct answers here since the set can be defined in more than one way.)

The set of all U.S. state names.

- Set-builder notation: {x: x is a U.S. state name.}
- List form: {Alabama, Alaska, Arizona, ... , Wyoming}

$\{x: x^2+2=11\}$

- Word description: The set of all values of x that satisfy the condition $x^2 + 2 = 11$.
- List form: $\{3, -3\}$

Module Outcome #5: Apply various set operations.

Finally, we learned how to apply various set options. See what you remember! Let $A = \{x\in \quad : 2\le x\le 6\}$, $B = \{x\in \quad : 4\le x\le 8\}$, and the universal set, $U = \{x\in \quad : 1\le x\le 10\}$. *P*erform the following operations:

1. $A\cup B$
   - This is the collection of all elements in either A, B, or both. Therefore, this would be the set $\{2, 3, 4, 5, 6, 7, 8\}$.
2. $A\cap B$
   - This is the collection of all elements in both A and B. Therefore, this would be the set $\{4, 5, 6\}$.
3. $A\setminus B$
   - This is the collection of all elements that are in A, but not in B. Therefore, this would be the set $\{2, 3\}$.
4. $U\setminus B$
   - This is the collection of all elements that are in U, but not in B. Therefore, this would be the set $\{1, 2, 3, 9, 10\}$.

Now, check your understanding of the concepts from Module 1 by answering the following questions.

Check Your Understanding

Embedded Media Content! Please use a browser to view this content.

References

None