1.1

MTH350

Discrete Mathematics

Module 2: Counting

This module will cover counting concepts, such as the additive and multiplicative principles, the principle of inclusion/exclusion, permutations, and combinations. You will learn how these methods can be used to answer real-world problems and in counting discrete structures.

Learning Outcomes

1. Apply additive and multiplicative principles to solve counting problems.
2. Utilize the principle of inclusion/exclusion to solve real-world applications.
3. Make use of counting techniques to count various discrete structures.
4. Determine the appropriate counting technique for solving a given real-world scenario.
5. Solve real-world problems with permutation and combination formulas.

For Your Success & Readings

Module 2 will cover the chapter on counting in the textbook. In this chapter, you will learn about various counting methods that can be applied to real-world contexts. One of the challenges of this topic is to navigate the various methods and decide which one is appropriate given a real-world scenario. For this, you might find it helpful to organize the information you learn in a way that helps you decide which method to select.

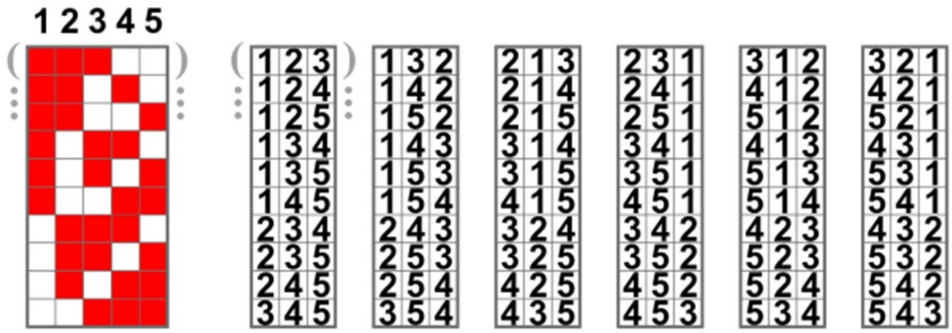
In this module, you will also learn about how to count various discrete structures. The lecture will focus on subsets and bit strings, but you can learn more about other structures from the textbook and required readings. Also, the critical thinking assignment this week will guide you in discovering many of the interesting connections between these discrete structures through an analysis of Pascal's Triangle.

In discussion board for this week you will be able to write your own examples of real-world scenarios that require one of the counting methods we learned. You will also get the chance to engage with your peers and contrast your understanding of when each method is appropriate. Make sure to interact with your peers as much as possible in the discussion board so you can broaden your understanding of these topics.

Required

- Chapter 1, Sections **1.** (http://discrete.openmathbooks.org/dmoi/sec_counting-addmult.html)**1** (http://discrete.openmathbooks.org/dmoi/sec_counting-addmult.html), **1.2** (http://discrete.openmathbooks.org/dmoi/sec_counting-binom.html), & **1.3** (http://discrete.openmathbooks.org/dmoi/sec_counting-combperm.html) in *Discrete Mathematics: An Open Introduction*
- Banderier, C., & Wallner, M. (2017). **Lattice paths with catastrophes** (https://www-sciencedirect-com.csuglobal.idm.oclc.org/science/article/pii/S1571065317300811?via%3Dihub). *Electronic Notes in Discrete Mathematics*, 59, 131-146.
- Yang, X. (2017). **Chapter 3 - binomial theorem and expansions** (https://www-sciencedirect-com.csuglobal.idm.oclc.org/science/article/pii/B9780128097304000045). In *Engineering Mathematics with Examples and Applications* (pp. 31-35). Boston, MA: Academic Press.

1. Additive and Multiplicative Principles



This module covers some of the concepts in counting, or **combinatorics**, which is the theory of ways things combine; in particular, how to count these ways.

Let's start with some basic counting principles that can help us count the number of ways events can occur. In particular, we will talk about sets of different ways the event can happen.

For example, if you want to count the number of ways to select an outfit consisting of a shirt and pants, you need to consider the set of different ways to select shirts (i.e., the set of all different shirts) and the set of different ways to select pants (i.e., the set of all different pants).

In that way, we will present counting principles through the use of sets.

So, suppose you own 7 different t-shirts, 6 different pairs of short pants and 5 different pairs of long pants. In terms of sets, we can define the following:

T = {the set of t-shirts}

S = {the set of short pants}

L = {the set of long pants}

Now, recall from the previous module that the cardinality of each set is just the number of elements in the set. Before moving on, let's find the cardinality of each set.

$|T| = 7$

$|S| = 6$

$|L| = 5$

With this in mind, let's try to answer the following questions and discover the principles needed to answer them along the way.

How many different choices do you have for the pants you wear tomorrow?

Here, you want to select *either* one short pant **or** one long pant. Since you can't wear both short pants and long pants at the same time, we know that the set of short pants and the set of long pants are *disjoint* (i.e., there is no element in common to both sets, and their intersection is the empty set, $\emptyset$) .

Since we want to choose one element from one set **or** another, we are actually choosing an element from the **union** of the two sets. To do this, we can apply the following **additive principle**.

Additive Principle (with sets)

Given two sets A and B, if $A \cap B = \emptyset$ (that is if A and B are disjoint), then $|A \cup B| = |A| + |B|$.

In this case, our sets are S and L, with $|S| = 6$ and $|L| = 5$. Thus, the number of ways of selecting pants from either S *or* L, is $|S \cup L| = |S| + |L| = 6 + 5 =$ **11**.

How many different outfits can you make with one t-shirt and one long pants?

Here we want to select a t-shirt **and** one long pant. We are basically drawing one of each from their respective sets and creating a set of outfits which consists of *pairs* of the form (t-shirt, long pant).

This process is described as a formal set operation called **Cartesian product,** defined below:

Cartesian Product

Given sets A and B, we can form the *set* $A \times B = \{(x, y) : x \in A \wedge y \in B\}$ to be the set of all ordered pairs (x, y) where x is an element of A and y is an element of B. We call $A \times B$ the **Cartesian product** of A and B.

For example, let $A = \{a, b\}$ and $B = \{c, d, e\}$. Then $A \times B = \{(a, c), (a, d), (a, e), (b, c), (b, d), (b, e)\}$.

Now, the principle we can use to count the number of pairs in $A \times B$, is as follows:

Multiplicative Principle (with sets)

Given two sets A and B, we have $|A \times B| = |A| \cdot |B|$.

So, in our case, we have $|T \times L| = |T| \cdot |L| = 7 \cdot 5 =$ **35**.

Now, what about when you want to use the additive principle, but the sets are *not* disjoint? In other words, how do we count the number of elements in sets A or B ($A \cup B$) when they have elements in common?

For example, suppose a survey of students' course schedules showed that 20 students are currently enrolled in a math course, 30 in a science course, and 5 in both a math course and a science course. So, if we want to count the number of students who are taking either a math course **or** a science course, we need to make sure not to double count those taking both.

In other words, if we add 20 students taking a math course plus 30 students taking a science course, we will end up counting the 5 students taking both courses twice. Thus, we need to then subtract those 5 students once in order to make up for this.

Formally, the **Principle of Inclusion/Exclusion** gives us a way to find the cardinality of a union of two sets that are not necessarily disjoint as follows:

Cardinality of a union (2 sets)

For any finite sets A and B,
$|A \cup B| = |A| + |B| - |A \cap B|$.

So, here we can let M = the set of students taking a math course and S = the set of students taking a science course. Then the number of students taking a math course **or** a science course is:

$|M \cup S| = |M| + |S| - |M \cap S|$=30 + 20 - 5 = **45**.

Note that this can be done in a similar fashion with more than two sets, but for the purposes of this course, we will focus on just two sets.

Now, check your understanding of the concepts in this section by working through the following exercise:

1.1. Counting with Sets

When working with the additive and multiplicative principles, it is important to note when to use each one.

If you want to count the number of ways to select one element from one **or** another set, then you use the **additive principle** and **add** the number of items in one set with the number of items in the other set. However, this only applies when the sets are **disjoint**, meaning that they have no elements in common.

Now, when you want to select one item from one set **and** one item from another set, you use the **multiplicative principle** and **multiply** together the number of items in each set.

Let's look at an example.

1.2. Principle of Inclusion/Exclusion

To understand how the Principle of Inclusion/Exclusion works, suppose we have two sets, A and B, with cardinalities $|A| = 10$ and $|B| = 15$. Click through the following interactive to explore what possibilities exist for elements they have in common.

What is the largest possible value for $|A \cap B|$?

To maximize the number of elements in common between A and B, it must be the case that all elements in A are also in B. In other words, we would make $A \subset B$. This would mean that the elements in common between A and B are all of those in A. Thus, we would have $|A \cap B| = |A| =$ **10** as the largest possible value for $|A \cap B|$.

What is the smallest possible value for $|A \cap B|$?

To minimize the number of elements in common between A and B, we could take the case when A and B have no elements in common (i.e. disjoint). In other words, we would make $A \cap B = \emptyset$. This would mean that $|A \cap B| =$ **0** is the smallest possible value for $|A \cap B|$.

What are the possible values for $|A \cup B|$?

Recall that the Principle of Inclusion/Exclusion gives us the following formula $|A \cup B| = |A| + |B| - |A \cap B|$.
So, possible values for $|A \cup B|$ will depend on the range of values for $|A \cap B|$. We saw above that we can minimize $|A \cap B|$ to 0 and maximize $|A \cap B|$ to 10.
When $|A \cap B| = 0$, $|A \cup B| = |A| + |B| - |A \cap B| = 10 + 15 - 0 =$ **25**.
When $|A \cap B| = 10$, $|A \cup B| = |A| + |B| - |A \cap B| = 10 + 15 - 10 =$ **15**.
Hence, $\mathbf{15 \leq |A \cup B| \leq 25}$.

Try the following problem before moving on to the next topic.

2. Counting Discrete Structures

In this section, we are going to learn about how to count various discrete structures and see examples of each. In particular, we will be using the following **binomial coefficients** formula that counts the number of ways to select k objects from a total of n objects.

Binomial Coefficients

For each integer $n \geq 0$ and integer k with $0 \leq k \leq n$ there is a number

read "n choose k."

However, first we need to establish a few concepts that we will be working with.

Factorial, n!SubsetsBit Strings

In case you are not familiar with factorial, $n!$ is defined as the product of all integers between 1 and n. So, 5! = 5 * 4 * 3 * 2 * 1 = 120. Also, note that 0! =1 .

Recall from Module 1, that A is a subset of B if every element in the set A is also in the set B. In this section, we will consider subsets of a particular cardinality, and we will refer to a subset with k elements as a k-element subset.

"Bit" is short for "binary digit," so a **bit string** is a string of binary digits. The **binary digits** are simply the numbers 0 and 1. The number of bits (0's or 1's) in the string is the **length** of the string. The number of 1's in a bit string is the **weight** of the string.

For example, the bit string 01101 has length 5 and weight 3.

A bit string of length n is referred to as an **n-bit string**, and is the set of all n-bit strings of weight k. For example, the elements of the set are the bit strings 001, 010, and 100. Those are the only strings containing three bits, exactly one of which is a 1.

Now, click through the following activity to discover how to use the binomial coefficients formula to find the number of subsets of a particular size and the number of bit strings of a particular weight.

SubsetsBit Strings

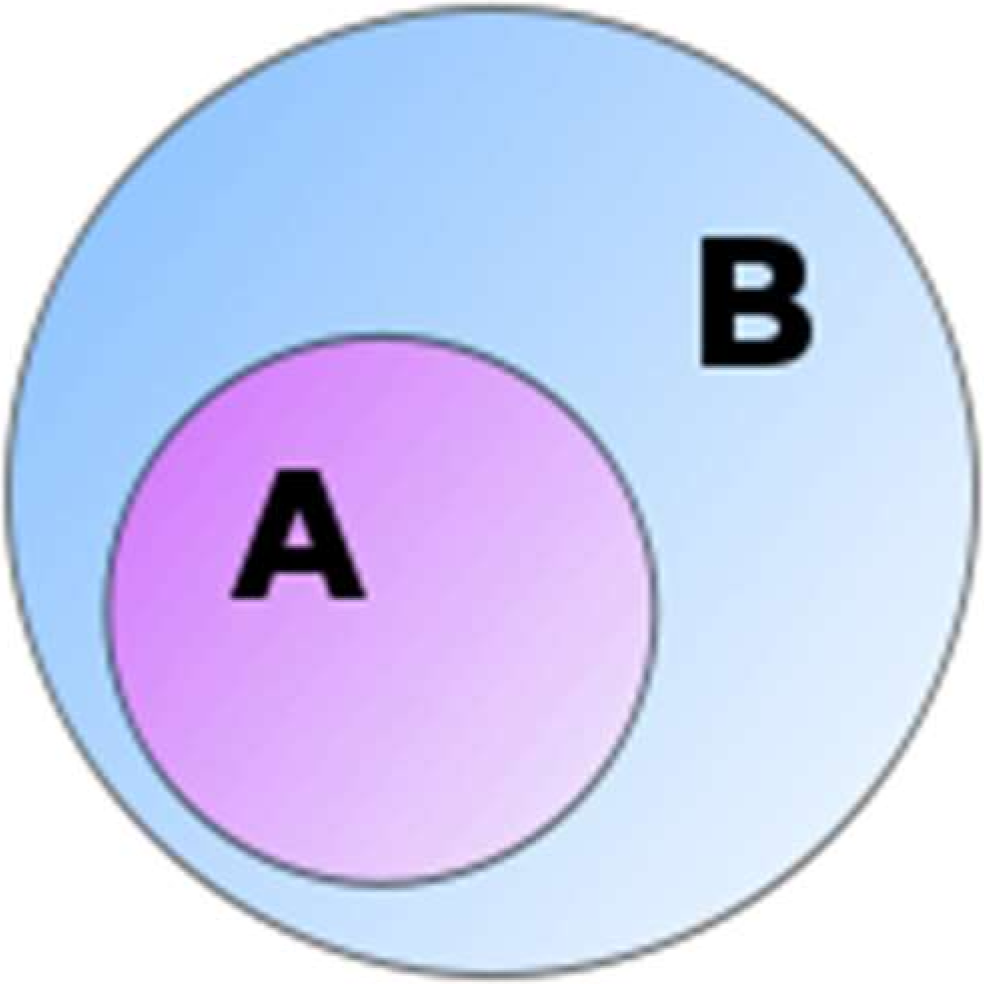
is the number of subsets of a set of size n each with cardinality k.

is the number of n-bit strings of weight k. You can also think of this as the cardinality of , or

.

As we will see a bit later, can also be used to count the number of combinations. For now, make sure you are comfortable with how to use to count subsets and bit strings before moving on.

2.1. Subsets



Let's look at the following example:

How many subsets of {0, 1, …, 9} have cardinality 6 or more?

To use the binomial coefficients formula, we need to determine the number of things we are choosing from, n, and the number of things we are choosing, k. In this case, the number of things to choose from is 10 since there are 10 elements in the set we are given. Hence, **n=10.**

Now, to find k, we need to take a closer look at the problem. We are asked for subsets that have a cardinality of 6 or more. First, we need to think about what options there are for the cardinality of the subsets we should be selecting. To do this, notice that the maximum cardinality of a subset is 10, since that is the size of the set we are choosing from. Thus, we need to calculate the number of 6-element, 7-element, 8-element, 9-element, and 10-element subsets separately, and then add them together to find our answer.

How many 6-element subsets of a set with 10 elements are there?

= 210 is the number of subsets of a set of size 10 each with cardinality 6.

How many 7-element subsets of a set with 10 elements are there?

= 120 is the number of subsets of a set of size 10 each with cardinality 7.

How many 8-element subsets of a set with 10 elements are there?

= 45 is the number of subsets of a set of size 10 each with cardinality 8.

How many 9-element subsets of a set with 10 elements are there?

= 10 is the number of subsets of a set of size 10 each with cardinality 9.

How many 10-element subsets of a set with 10 elements are there?

= 1 is the number of subsets of a set of size 10 each with cardinality 10. Note that this is the set itself.

Thus, the total number of subsets with cardinality of at least 6 is 210 + 120 + 45 + 10 + 1 = **386**.

Try this out yourself with the following question.

2.2. Bit Strings

Another discrete structure we can count is **bit strings.** Now we are going to go through an equivalent example to the one on the previous page involving subsets. As you go through this example, think about the connections you might be able to make between these two discrete structures.

So, let's look at the following equivalent example involving bit strings.

How many 10-bit strings contain 6 or more 1's?

As mentioned before, to use the binomial coefficients formula, we need to determine the number of things we are choosing from, n, and the number of things we are choosing, k. In this case, the number of things to choose from is 10 since there are 10 digits in the 10-bit string. Hence, **n=10.**

Now, to find k, as was the case with subsets, we need to think about what options there are for the weights of 10-bit strings. To do this, notice that the maximum number of 1's that can be in each string is 10, so the maximum weight is 10. Thus, we need to calculate the number of 10-bit strings of weights, 6, 7, 8, 9, and 10 separately, and then add them together to find our answer.

How many 10-bit strings contain 6 or more 1's?

= 210 is the number of 10-bit strings of weight 6.

How many 10-bit strings contain 7 or more 1's?

= 120 is the number of 10-bit strings of weight 7.

How many 10-bit strings contain 8 or more 1's?

= 45 is the number of 10-bit strings of weight 8.

How many 10-bit strings contain 9 or more 1's?

= 10 is the number of 10-bit strings of weight 9.

How many 10-bit strings contain 10 or more 1's?

= 1 is the number of 10-bit strings of weight 10. Note that this is bit string 1111111111.

Thus, the total number of 10-bit strings with weight of at least 6 is....

The total number of 10-bit strings with weight of at least 6 is 210 + 120 + 45 + 10 + 1 = **386**.

Now, give this a try by completing the following question.

3. Combinations and Permutations

In this section, we will look at the number of ways to select and arrange objects.

Let's start with the case when we have n objects that we want to arrange in a certain order. Note that we are not making any choices here. Instead, we are re-arranging the n objects we already have.

A **permutation** is a (possible) rearrangement of objects.

Permutations of n elements
There are $n! = n \cdot (n-1) \cdot (n-2) \cdot \cdots \cdot 2 \cdot 1$ permutations of n (distinct) elements.

For example, in how many ways can you arrange the letters a, b, c, d, e?

Using the above definition, we know that there are 5! = **120** ways to arrange the five letters a, b, c, d, e.

Now, what if we want to know the number of *three letter "words"* we can make with the letters a, b, c, d, e? (Note that "words" are just any arrangement of letters and don't necessarily have to be accepted words in any dictionary.)

This time, we are first selecting 3 out of the 5 total letters, and then arranging them in all possible ways. For this, we need the following definition.

k-permutations of n elements

P(n,k) is the number of k-**permutations of** n **elements**, the number of ways to *arrange* k objects chosen from n distinct objects.

So, in this case we want to find the number of 3-permutations of 5 elements, (i.e., P(5,3)). Using the formula above we have:

So there are **60** ways to arrange a selection of 3-letter words chosen from 5 letters.

Continuing with the example above, let's say we want to find the number of ways to select 3 letters out of the 5 letters a, b, c, d, e. However, this time, suppose we don't care what order they are in. In other words, we would consider "abc" and "bca" as being the same. How many ways can we select 3 letters out of 5 total letters when order doesn't matter?

This question should sound familiar as it is one we answered in the previous section on counting discrete structures. Indeed, we would simply be using the binomial coefficients formula, to determine the number of **combinations** there are of 3 letters chosen from a total of 5 letters.

Combinations - Choosing k objects out of n total objects (Closed formula for)

So, we know that there are ways to select 3 letters out of 5 total letters with no regard for the order in which they are arranged.

To summarize:

1. P(n,k) counts *permutations*and should be used when **order matters**
2. counts *combinations* and should be used when **order does** <u>not</u> **matter**.

3.1. Permutations

For another look at permutations, let's work through one more example.

Scenario

A code consists of 4 digits between 0 and 9. How many different permutations are there if each digit may only be used once?

Does order matter?

Yes, in this case order matters since 0123 would be a different code than 2310.

What are n and r?

In this case, n = 10 since that is the total number of digits to choose from, and r = 4, since that is the number of digits we choose at a time for each code.

Apply the formula.

Since we determined that order does matter, we calculate ${}_{10}P_4$ = **5,040**.

Now, check your understanding of permutations by attempting the following problem.

3.2. Combinations

For another look at combinations, let's work through one more example.

Scenario

There is a league of seven soccer teams that must all play each other during a single season. How many games take place if the teams all play each other once?

Does order matter?

Since there is no order involved in selecting two teams to play against each other, orde does not matter.

What are n and r?

In this case, n = 7 since that is the total number of teams to choose from, and r = 2, since two teams can play at once.

Apply the formula.

Since we determined that order doesn't matter, we calculate ${}_7C_2$ = **21**.

Now, check your understanding of combinations by attempting the following problem.

4. Summary

Click through the following interactive to view a worked example that discusses key points for each module outcome.

Module Outcome #1: Apply additive and multiplicative principles to solve counting problems.

Suppose your wardrobe consists of 2 shirts, 3 pairs of pants, and 7 bow ties.

How many different outfits consisting of a shirt, pair of pants, and bowtie can you make?

We want to select one shirt, one pair of pants, **and** one bow tie to make an outfit. We can use the multiplicative principle and multiply the number of shirts, pants, and bow ties together to get that there are **42** total possible outfits.

In how many ways can you select one article of clothing?

Here, you want to select *either* one shirt **or** pair of pants **or** bow-tie. Since all the sets you are choosing from are *disjoint* (i.e., there is no element in common to the sets), we can apply the following **additive principle** to find the answer. In this case, we can choose from 2 + 3 + 7 = **12** articles of clothing.

Module Outcome #2: Utilize the principle of inclusion/exclusion to solve real-world applications.

A group of college students was asked about course enrollment. Of those surveyed, 10 students were enrolled in a language course, 15 were enrolled in a biology course, and 7 were enrolled in both.

Why is the Principle of Inclusion/Exclusion needed here?

We want to select one shirt, one pair of pants, **and** one bow tie to make an outfit. We can use the multiplicative principle and multiply the number of shirts, pants, and bow ties together to get that there are **42** total possible outfits.

How many students surveyed were enrolled in at least one of those courses?

We can let L = the set of students enrolled in a language course, and B = the set of students enrolled in a biology course. Then $L \cup B$ represents the set of students enrolled in at least one of those courses. So we have: $|L \cup B| = |L| + |B| - |L \cap B|$ = 10 + 15 - 7 = **18**.

Module Outcome #3: Make use of counting techniques to count various discrete structures.

How many 4-elements subsets are there of the set {1, 2, ..., 9}?

- Here we want to count the number of ways to choose a subset of 4 elements out of a set of 9 elements. So we would calculate
- . Now, since we know that for every n and k, = , then we know that this answer is also equal to .

How many ways can you arrange exactly 4 ones in a string of 9 binary digits?

- don't be fooled by the “arrange” in there - you are picking 4 out of 9 spots to put the 1's.

Module Outcome #4: Determine the appropriate counting for solving a given real-world scenario.

For each of the following counting problems, say whether the answer is , P(9,5), or neither.

How many sets of 5 letters can be made from the first 9 letters of the alphabet?

- Recall that a set is an **unordered** collection of elements. Hence, the order in this case does not matter, so we use the combinations formula and calculate .

Suppose you have 9 shirts and you will wear a different one on each weekday. How many choices do you have?

- First, we notice that order is important in this case. (i.e., Wearing your red shirt on Monday is a different choice than wearing it on Wednesday.) Thus, we use the permutations formula **P(9,5)** to answer this question.

Module Outcome #5: Solve real-world problems with permutation and combination formulas.

Consider the following question:
How many different 2-topping pizzas can you make with 6 toppings to choose from?

Are we selecting some objects from a total?

Yes. We have 6 toppings to choose from, and will select 2 of them. So we know that we must either use P(n,k) or to answer this question.

Does order matter?

No. It doesn't matter what order the toppings are chosen in, so we use the **combinations** formula .

How many different 2-topping pizzas can you make with 6 toppings to choose from?

= **15**.

Now, check your understanding of the concepts from Module 2 by answering the following questions.

Check Your Understanding

Embedded Media Content! Please use a browser to view this content.

References

None