Introduction to Linear Algebra and Convex Optimization

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Module objectives

- Become familiar with linear algebra concepts
- Understand how dimensions play a role in linear algebra
- Understand types of functions
- Appreciate optimization objectives
- Understand the basics of convex optimization

Outline

- Scalars, vectors, matrices, tensors
- Linear operations on vectors and matrices
- Vector and matrix properties
- Functions and derivatives
- Optimizing a continuous function

Scalars, vectors, matrices and tensors

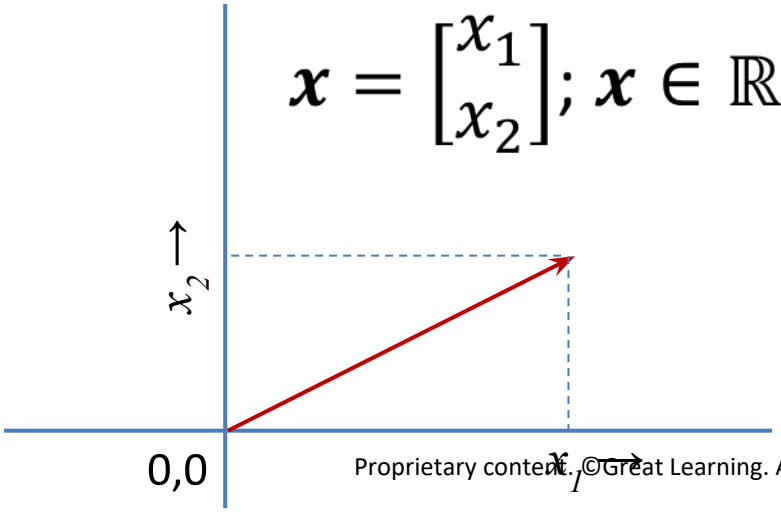
- $x; \quad x \in \mathbb{R}$

- $\boldsymbol{X} = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \end{bmatrix}$

$$\boldsymbol{X} \in \mathbb{R}^{M \times N}$$

Vectors, matrices, and tensors represent multi-dimensional ordered data.
Bold-face is used for vectors and matrices.

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}; \boldsymbol{x} \in \mathbb{R}^N$$

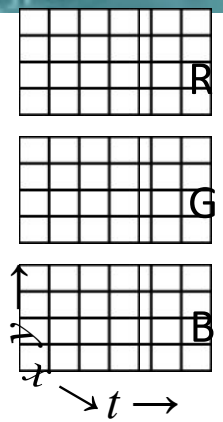


$$\begin{bmatrix} x_{111} & x_{121} & x_{131} \\ x_{211} & x_{221} & x_{231} \end{bmatrix}$$

$$\begin{bmatrix} x_{112} & x_{122} & x_{132} \\ x_{212} & x_{222} & x_{232} \end{bmatrix}$$

$$\boldsymbol{X} \in \mathbb{R}^{L \times M \times N \times \cdots}$$

An example from videos



- R value (red) of one pixel is a scalar
- RGB values of one pixel is a vector
- A row of R-values is a vector
- A frame of R-values is a matrix
- A frame of RGB values is a 3-d tensor
- A video of R-value frames is a 3-d tensor
- A video of RGB values and several frames is a 4-d tensor

N-D array

- 1-d, 2-d, and higher-d arrays are used to represent vectors, matrices, and tensors
- Sometimes n-d arrays are also used to represent scalars
- An array is a data structure, not a mathematical entity
- 1-d array is used interchangeably with a vector
- 2-d array is used interchangeably with a matrix

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Scalar multiplication

- ax $\quad aX = \begin{bmatrix} ax_{11} & ax_{12} & ax_{13} \\ ax_{21} & ax_{22} & ax_{23} \end{bmatrix}$

Scalar scales.

$a\boldsymbol{x} = \begin{bmatrix} ax_1 \\ ax_2 \end{bmatrix}$ $\quad \begin{bmatrix} ax_{111} & ax_{121} & ax_{131} \\ ax_{211} & ax_{221} & ax_{231} \end{bmatrix}$

$\begin{bmatrix} ax_{112} & ax_{122} & ax_{132} \\ ax_{212} & ax_{222} & ax_{232} \end{bmatrix}$

Adding a scalar to a vector or a matrix

- Not valid in mathematics, but used in programming and ML
- How it is denoted sometimes: $\boldsymbol{x} + a$
- Mathematically correct operation:

$$\boldsymbol{x} + a \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} x_1 + a \\ x_2 + a \end{bmatrix}$$

Point-wise operations require two tensors of the same size

- Addition and subtraction require two matrices (or vectors) of the same size: $\mathbb{R}^{M\times N\times\cdots} \times \mathbb{R}^{M\times N\times\cdots} \rightarrow \mathbb{R}^{M\times N\times\cdots}$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \end{bmatrix}; \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 - y_1 \\ x_2 - y_2 \end{bmatrix}$$

Point-wise multiplication is not valid in mathematics, but is important in ML, where it can be used for masking:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \odot \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 y_1 \\ x_2 y_2 \end{bmatrix}$$

Matrix transpose

- Transposition is exchange of rows and columns of a matrix (or a vector)

$$\boldsymbol{X} = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \end{bmatrix}; \quad \boldsymbol{X}^T = \begin{bmatrix} x_{11} & x_{21} \\ x_{12} & x_{22} \\ x_{13} & x_{23} \end{bmatrix}$$

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}; \quad \boldsymbol{x}^T = \begin{bmatrix} x_1 & x_2 \end{bmatrix}$$

Permuting dimensions for higher-dimensional tensors is defined in some python libraries

Matrix multiplication

- Matrix multiplication requires matching of the columns of one matrix with the rows of another: $\mathbb{R}^{M\times N} \times \mathbb{R}^{N\times P} \rightarrow \mathbb{R}^{M\times P}$

$$\boldsymbol{X} = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \end{bmatrix}; \quad \boldsymbol{Y} = \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \\ y_{31} & y_{32} \end{bmatrix}$$

$$\boldsymbol{XY} = \begin{bmatrix} x_{11}y_{11} + x_{12}y_{21} + x_{13}y_{31} & x_{11}y_{12} + x_{12}y_{22} + x_{13}y_{32} \\ x_{21}y_{11} + x_{22}y_{21} + x_{23}y_{31} & x_{21}y_{12} + x_{22}y_{22} + x_{23}y_{32} \end{bmatrix}$$

$$\boldsymbol{x}.\boldsymbol{y} = \boldsymbol{x}^T\boldsymbol{y} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = x_1y_1 + x_2y_2$$

$$\boldsymbol{x}.\boldsymbol{x} = \boldsymbol{x}^T\boldsymbol{x} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = {x_1}^2 + {x_2}^2$$

In general: $\boldsymbol{AB} \neq \boldsymbol{BA}$

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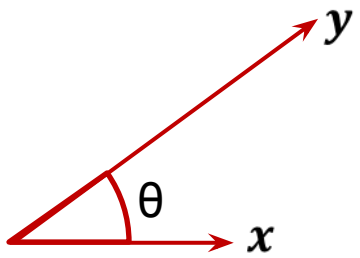
Vector norm

- L_p norm of a vector: $\|\boldsymbol{x}\|_p = \left(\sum_{i=1}^{d} |x_i|^p\right)^{\frac{1}{p}}$

- By default, norm means L_2 norm, which is the length of a vector (or Euclidean distance)
  - $\sqrt{{x_1}^2 + {x_2}^2 + \cdots + {x_d}^2}$
- L_1 norm is the sum of absolute values (or Manhattan distance)
  - $|x_1| + |x_2| + \cdots + |x_d|$
- L_∞ norm is max absolute value of all dimensions. Why? Because max raised to the power infinity will dominate all other values in the sum

Cosine between two vectors

- Cosine of the angle between $\boldsymbol{x}$ and $\boldsymbol{y}$ is $\frac{\boldsymbol{x}.\boldsymbol{y}}{\|\boldsymbol{x}\|\|\boldsymbol{y}\|}$



If cosine is zero then the vectors are orthogonal

That is, their dot product is zero

Cosine is a measure of similarity (direction) of vectors, when we want to ignore their magnitudes

Some special vectors and matrices

- Zero vector: $\begin{bmatrix}0\\0\\0\end{bmatrix}$ Ones: $\begin{bmatrix}1\\1\\1\end{bmatrix}$

One-hot bit: $\begin{bmatrix}1\\0\\0\end{bmatrix}$ $\begin{bmatrix}0\\1\\0\end{bmatrix}$ $\begin{bmatrix}0\\0\\1\end{bmatrix}$

$\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$; s.t. $x_1{}^2 + x_2{}^2 + x_3{}^2 = 1$

Diagonal matrix:

$$\boldsymbol{\Lambda} = \begin{bmatrix}\lambda_1 & 0 & 0\\0 & \lambda_2 & 0\\0 & 0 & \lambda_3\end{bmatrix}$$

Identity matrix:

$$\boldsymbol{I} = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$$

$$\boldsymbol{AI} = \boldsymbol{IA} = \boldsymbol{A}$$

Linear independence

- A set of vectors are linearly independent if none of them can be expressed as a linear combination of the others
- Example: $\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}$, and $\begin{bmatrix}0\\0\\1\end{bmatrix}$
- But not: $\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}$, and $\begin{bmatrix}2\\0\\1\end{bmatrix}$
- And nor: $\begin{bmatrix}1\\0\\0\end{bmatrix}$ and $\begin{bmatrix}2\\0\\0\end{bmatrix}$

Rank of a matrix

- For a square matrix: $\boldsymbol{A}$
- Rank is the minimum number of vectors needed to express its columns as their linear combination
- Example: $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & 2 \end{bmatrix}$ has rank 2, and $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 2 \\ 2 & 0 & 0 \end{bmatrix}$

  has rank 3 (a.k.a full rank)
- Eigenvalue analysis not only finds the rank, it also finds an orthonormal set of such vectors
  - Sort of like, modes of variation

Eigen decomposition of a square matrix

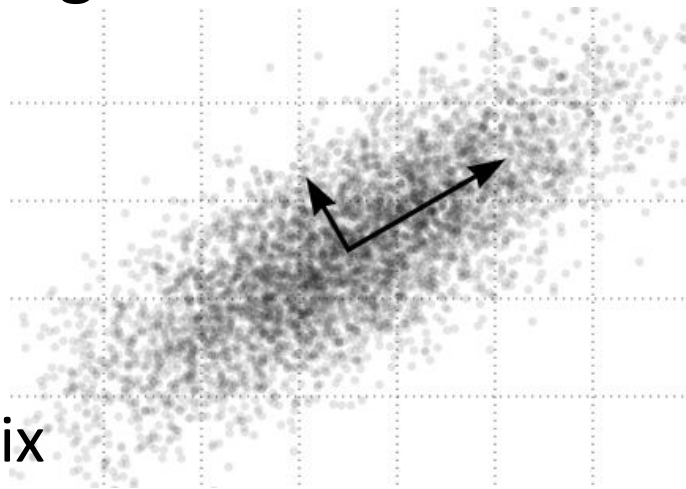
- Eigen decomposition expresses a matrix as:

$$\boldsymbol{A} = \boldsymbol{U}\boldsymbol{\Lambda}\boldsymbol{U}^{-1}$$

- Where, $\boldsymbol{\Lambda}$ is a diagonal matrix containing eigenvalues (usually sorted in descending order)
- And, $\boldsymbol{U}$ is a matrix of orthonormal eigenvectors $\boldsymbol{U} = [\boldsymbol{u_1} \ldots \boldsymbol{u_N}]$ such that $\boldsymbol{u}_i.\boldsymbol{u}_i = 1,\ \boldsymbol{u}_i.\boldsymbol{u}_j = 0,$ $\boldsymbol{A}\boldsymbol{u}_i = \lambda_i \boldsymbol{u}_i,\ \ \forall i \neq j$
- Number of non-zero eigenvalues is called the rank of the matrix (number of independent dimensions)

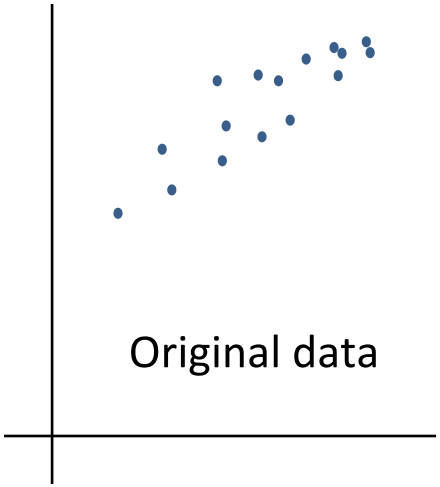
Principal component analysis

- Finding orthogonal directions of maximum variance of a set of data points
- Rotating the reference frame
- Finding which directions can be neglected

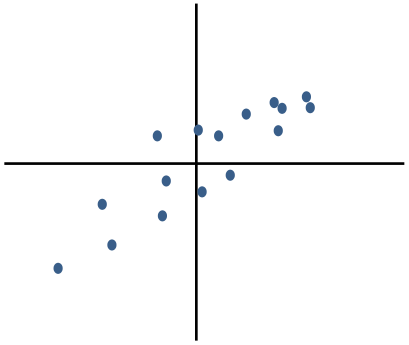


- Steps:
  - Center points (subtract mean)
  - Compute covariance matrix
  - Eigen decompose covariance matrix
  - Find projections (dot products) along eigenvectors corresponding to highest eigenvalues

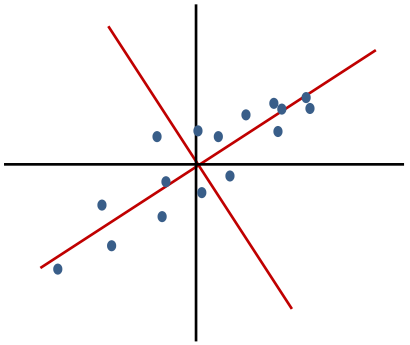
Dimension reduction using PCA



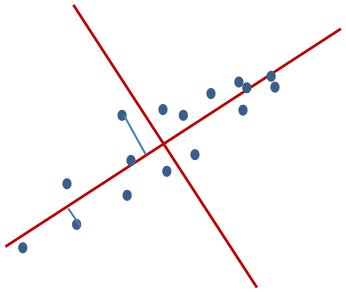
Original data



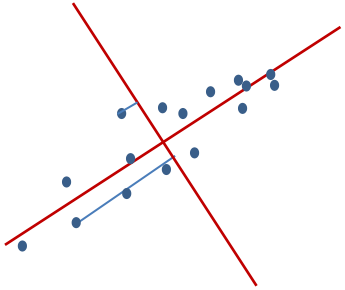
Mean subtraction



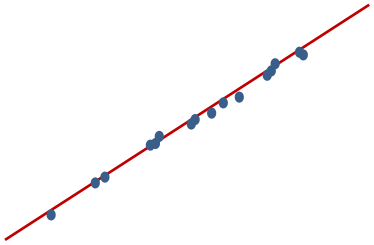
Principal components
of the covariance matrix



Projection



Projection



Dimension Reduction

PCA FAQs

- Why perform mean-centering?
  - Co-variance is defined on mean-centered data
- What is special about principal component directions?
  - The directions are orthogonal, just like original axes
  - The data is decorrelated – in some sense independent – along these directions
  - Co-variance in the new directions is $\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$
  - Notice the zero terms in off-diagonal entries
- Which can we ignore some dimensions?
  - The ones along which the variance is the smallest
  - This will lead to the smallest mean square error between original and approximated data

Inverse and pseudo inverse

- For square matrices that are full rank (all rows and columns are linearly independent of each other) an inverse matrix exists such that:
- $\boldsymbol{AA^{-1} = A^{-1}A = I}$
- Similar to:

  $$xx^{-1} = x^{-1}x = 1$$
- And just like inverse of 0 does not exist, the inverse of rank deficient matrices does not exist

- For non-square matrices of full rank, there is something called a pseudo inverse (assuming $N_{rows} > N_{columns}$ and rank is $N_{columns}$):
- $\boldsymbol{A^{+} = (A^{T}A)^{-1}A^{T}}$
- Note that $\boldsymbol{A^{+}A = I}$

Singular Value Decomposition

- SVD is a generalization of eigendecomposition
- It can be applied to any sized matrix
- It expresses a matrix as: $\boldsymbol{M = U \Sigma V^*}$
  - $\boldsymbol{U}$ and $\boldsymbol{V}$ are square matrices
  - $\boldsymbol{\Sigma}$ is a diagonal matrix, but need not be square
  - $\boldsymbol{V^*}$ is the conjugate transpose of $\boldsymbol{V}$
  - $\boldsymbol{UU^*}$ and $\boldsymbol{VV^*}$ are identity matrices of different sizes
- Non-zero singular values of $\boldsymbol{M}$ are square root of eigenvalues of $\boldsymbol{MM^*}$ and $\boldsymbol{M^*M}$
- It is used to find finding pseudo inverse and least square fitting

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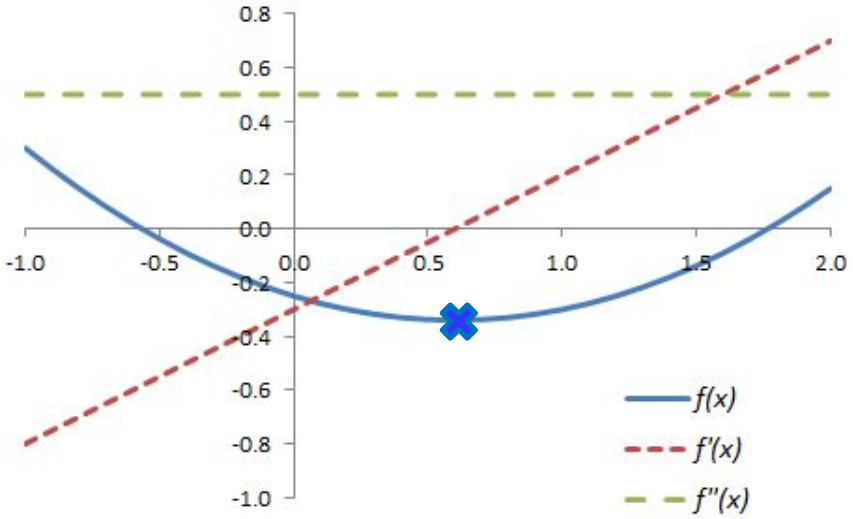
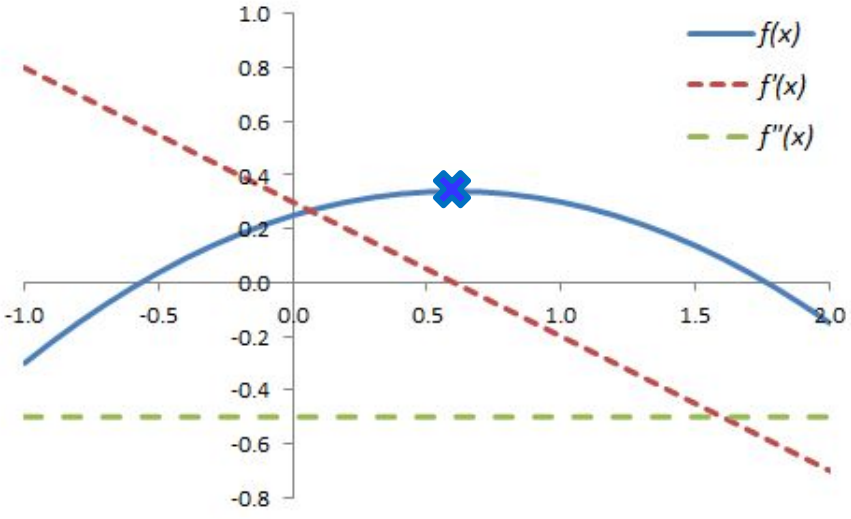
Function usually is a mapping from a vector to a scalar

- Usual definition of a function:
  - Input $\boldsymbol{x}$
  - Output $y = f(\boldsymbol{x})$
- Examples:
  - $f(x) = w\,x + b$ or $w\,x + b\,1$
  - $f(x) = w_2 x^2 + w_1 x^1 + w_0 x^0$
  - $f(x) = \boldsymbol{w}^T\boldsymbol{x}+b$
  - $f(x) = g(\boldsymbol{w}^T\boldsymbol{x}+b)$, where g is a nonlinear function

Derivative of a function of a scalar





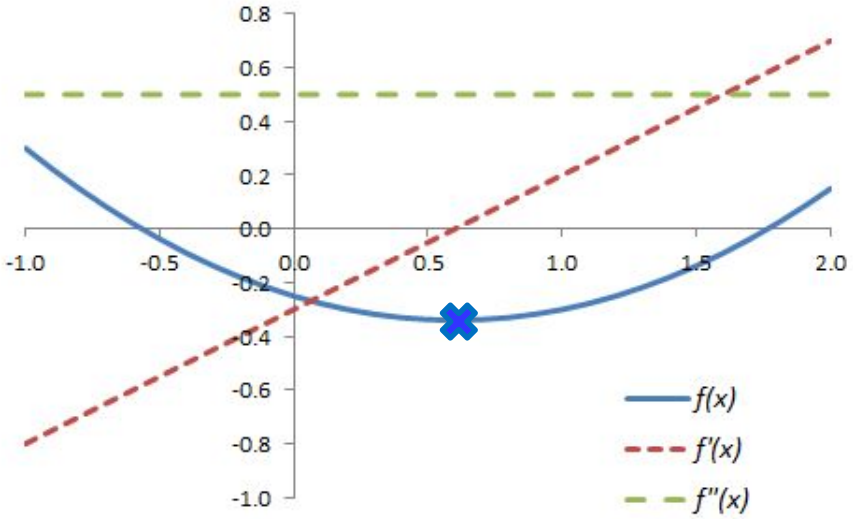
E.g. $f(x) = ax^2 + bx + c, \quad f'(x) = 2ax + b, \quad f''(x) = 2a$

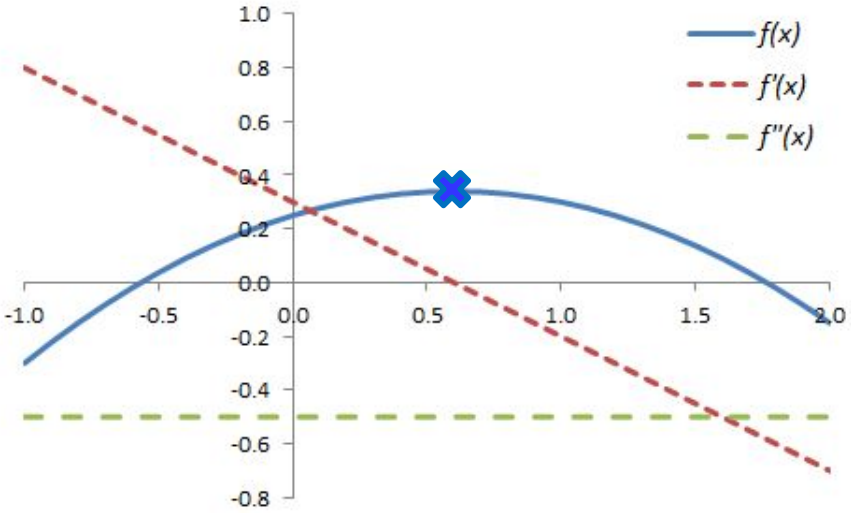
- Derivative $f'(x) = \frac{d\,f(x)}{d\,x}$ is the rate of change of $f(x)$ with x
- It is zero when then function is flat (horizontal), such as at the minimum or maximum of $f(x)$
- It is positive when $f(x)$ is sloping up, and negative when $f(x)$ is sloping down
- To move towards the maxima, taking a small step in a direction of the derivative

Chain rule of differentiation

- Very handy for complicated functions
  - Especially functions of functions
  - E.g. NN outputs are functions of previous layers
- For example: Let $f(x) = g\big(h(x)\big)$
  - Let $y = h(x), z = g(y) = g\big(h(x)\big)$
- Then $f'(x) = \frac{d\,z}{d\,x} = \frac{d\,z}{d\,y}\frac{d\,y}{d\,x} = g'(y)h'(x)$
- For example: $\frac{d\,\sin(x^2)}{d\,x} = 2x\cos(x^2)$

Double derivative

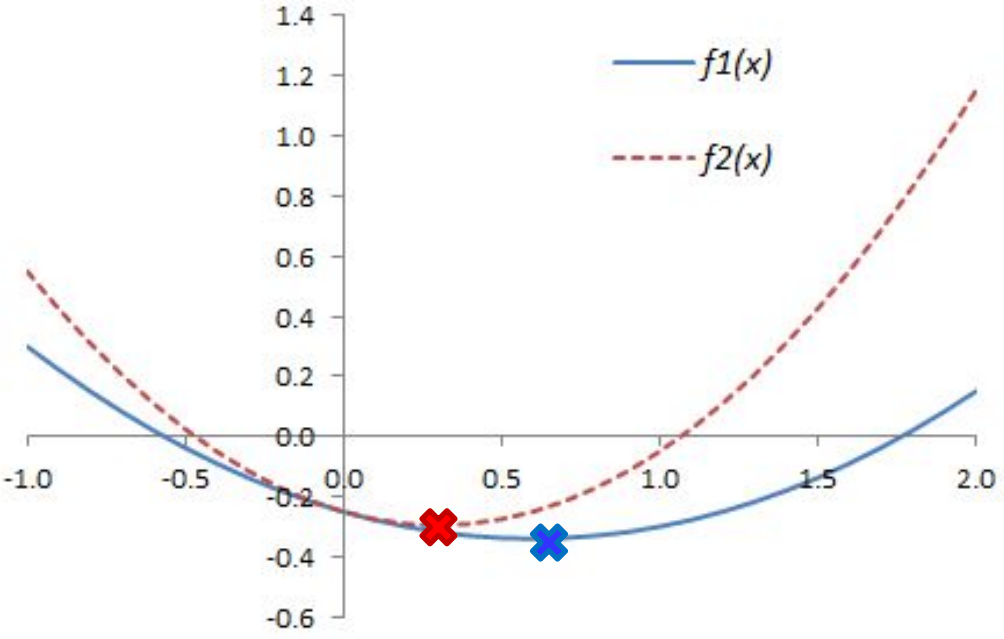




E.g. $f(x) = ax^2 + bx + c, \quad f'(x) = 2ax + b, \quad f''(x) = 2a$

- Double derivative $f''(x) = \frac{d^2 f(x)}{d x^2}$ is the derivative of derivative of $f(x)$
- Double derivative is positive for convex functions (have a single minima), and negative for concave functions (have a single maxima)

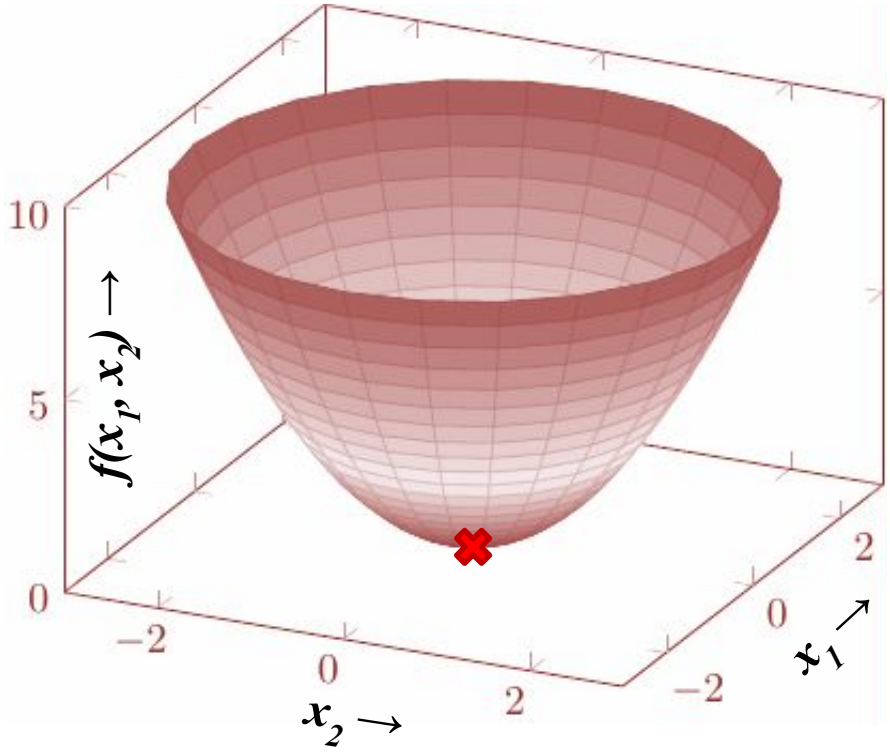
Double derivative



$$f(x) = ax^2 + bx + c,$$
$$f'(x) = 2ax + b,$$
$$f''(x) = 2a$$

- Double derivative tells how far the minima might be from a given point.
- From $x = 0$ the minima is closer for the red dashed curve than for the blue solid curve, because the former has a larger second derivative (its slope reverses faster)

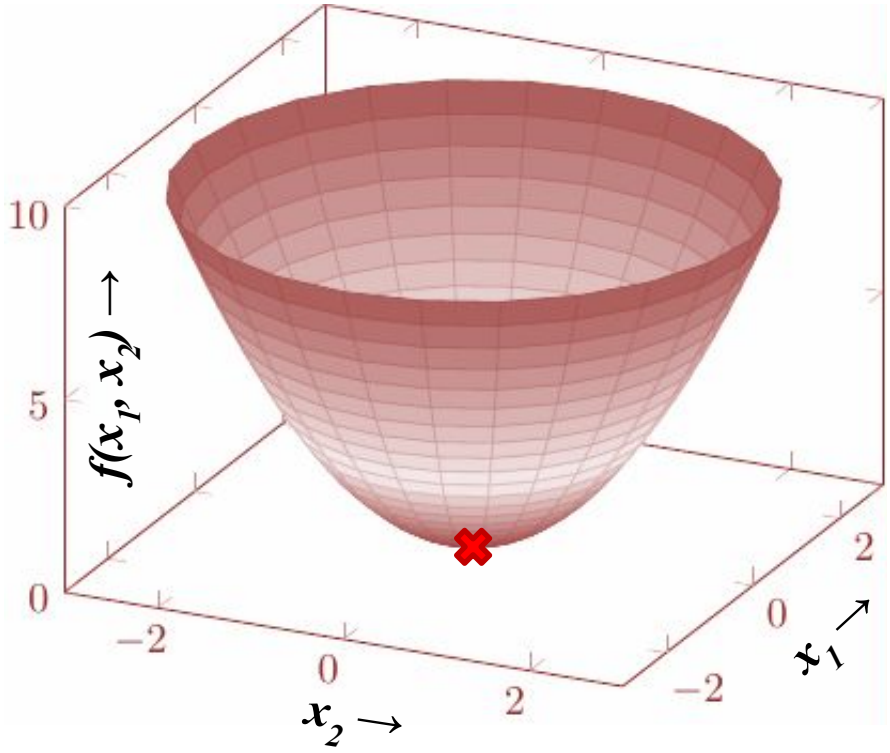
Gradient of a function of a vector



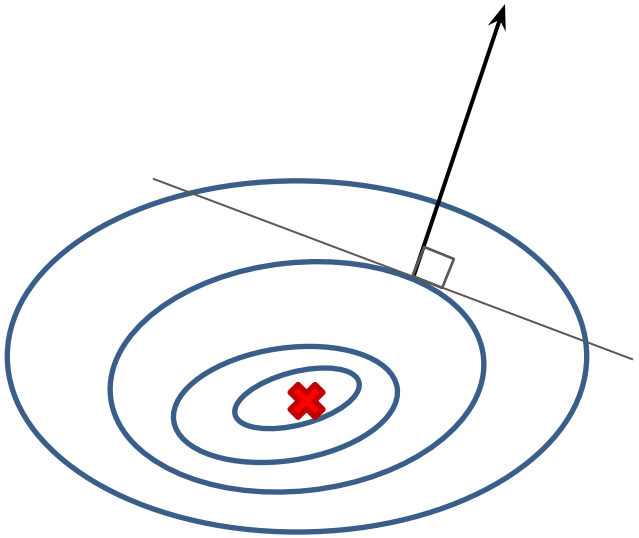
- Derivative with respect to each dimension, holding other dimensions constant
- $\nabla f(\boldsymbol{x}) = \nabla f(x_1, x_2) = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \end{bmatrix}$
- At a minima or a maxima the gradient is a zero vector

  The function is flat in every direction
- At a minima or a maxima the gradient is a zero vector

Gradient of a function of a vector



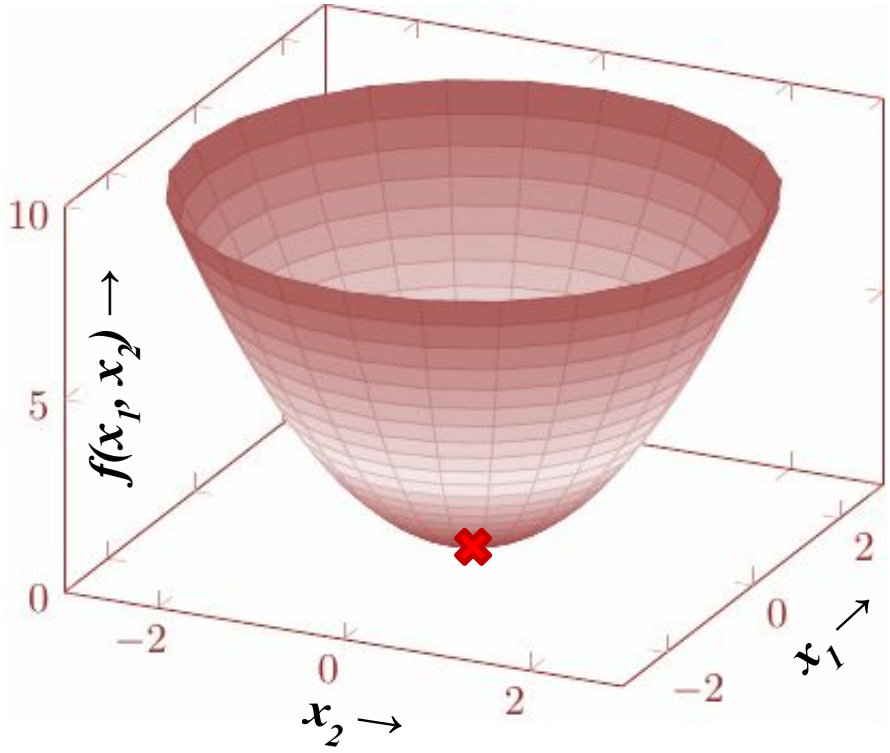
- Gradient gives a direction for moving towards the minima
- Take a small step towards negative of the gradient



Example of gradient

- Let $f(\boldsymbol{x}) = f(x_1, x_2) = 5{x_1}^2 + 3{x_2}^2$
- Then $\nabla f(\boldsymbol{x}) = \nabla f(x_1, x_2) = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 10x_1 \\ 6x_2 \end{bmatrix}$
- At a location $(2,1)$ a step in $\begin{bmatrix} 20 \\ 6 \end{bmatrix}$ or $\begin{bmatrix} 0.958 \\ 0.287 \end{bmatrix}$ direction will lead to maximal increase in the function

Hessian of a function of a vector



- Double derivative with respect to a pair of dimensions forms the Hessian matrix:

$$\mathbf{H} = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \, \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \, \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \, \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_2 \, \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \, \partial x_1} & \frac{\partial^2 f}{\partial x_n \, \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix}$$

- If all eigenvalues of a Hessian matrix are positive, then the function is convex

Example of Hessian

- Let $f(\boldsymbol{x}) = f(x_1, x_2) = 5{x_1}^2 + 3{x_2}^2 + 4x_1x_2$
- Then $\nabla f(\boldsymbol{x}) = \nabla f(x_1, x_2) = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \end{bmatrix} =$

$$\begin{bmatrix} 10x_1 + 4x_2 \\ 6x_2 + 4x_1 \end{bmatrix}$$

- And, $H(f(\boldsymbol{x})) = \begin{bmatrix} \frac{\partial^2 f}{\partial {x_1}^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial {x_2}^2} \end{bmatrix} = \begin{bmatrix} 10 & 4 \\ 4 & 6 \end{bmatrix}$

Vector valued functions and Jacobians

- We often deal with functions that give multiple outputs
- Let $\boldsymbol{f}(\boldsymbol{x}) = \begin{bmatrix} f_1(\boldsymbol{x}) \\ f_2(\boldsymbol{x}) \end{bmatrix} = \begin{bmatrix} f_1(x_1, x_2, x_3) \\ f_2(x_1, x_2, x_3) \end{bmatrix}$
- Thinking in terms of vector of functions can make the representation less cumbersome and computations more efficient
- Then Jacobian $\boldsymbol{J}(\boldsymbol{f}) = \begin{bmatrix} \frac{\partial \boldsymbol{f}}{\partial x_1} & \frac{\partial \boldsymbol{f}}{\partial x_2} & \frac{\partial \boldsymbol{f}}{\partial x_3} \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_3} \end{bmatrix}$

Relationship between Hessian, Jacobian, and gradient

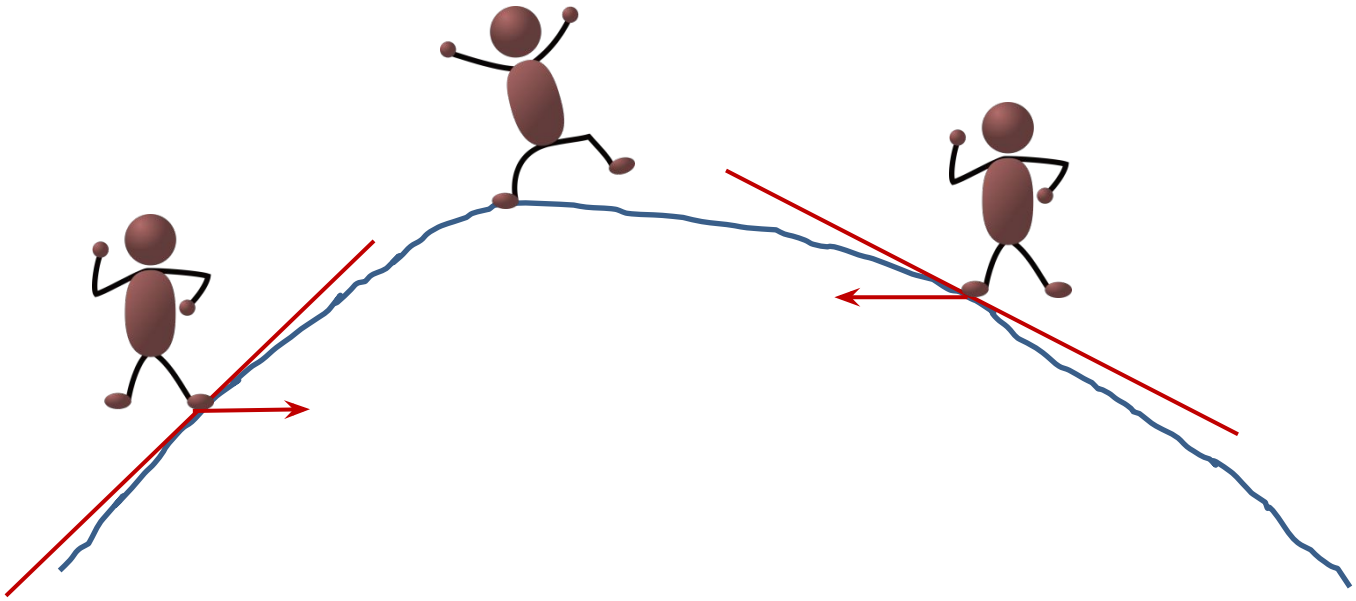
- Interestingly, Hessian is the transpose of Jacobian of the gradient
  - Gradient can be thought of as a multi-output function
  - So, its Jacobian makes sense
  - And derivative (Jacobian) of a derivative (gradient) is the second derivative (Hessian transposed)

Outline

- Scalars, vectors, matrices, tensors
- Linear operations on vectors and matrices
- Vector and matrix properties
- Functions and derivatives
- Optimizing a continuous function

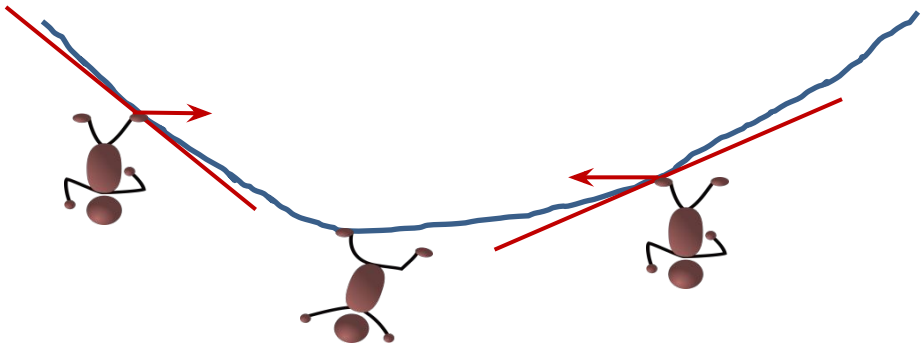
Gradient ascent

- If you didn't know the shape of a mountain
- But at every step you knew the slope
- Can you reach the top of the mountain?



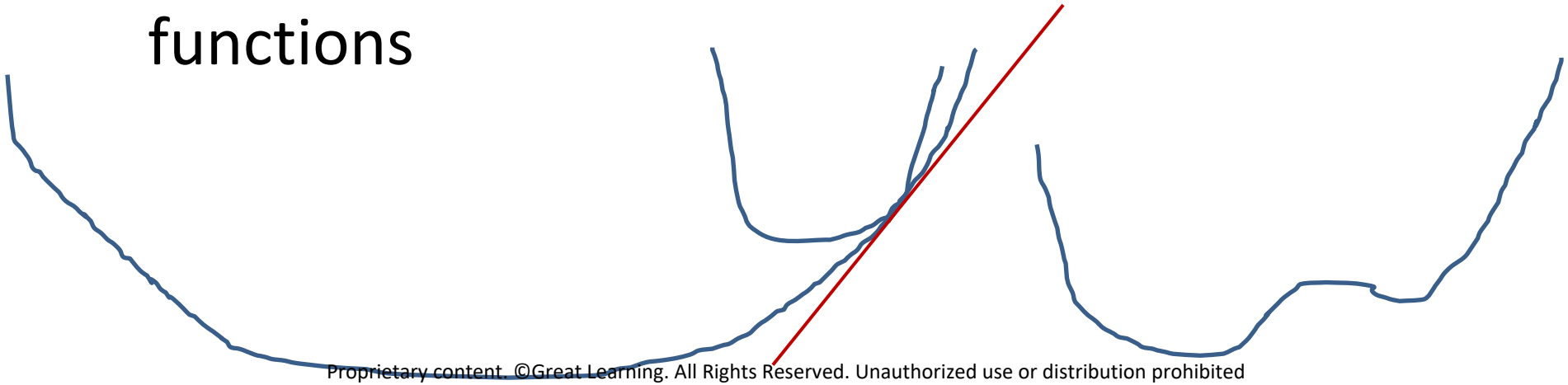
Gradient descent minimizes a function

- At every point, compute
  - $f(\boldsymbol{x})$
  - Gradient of loss with respect to weights (vector):
    $$\nabla_{\boldsymbol{x}} f(\boldsymbol{x})$$
  - Take a step towards negative gradient:
    $$\boldsymbol{x} \leftarrow \boldsymbol{x} - \eta \nabla_{\boldsymbol{x}} f(\boldsymbol{x})$$



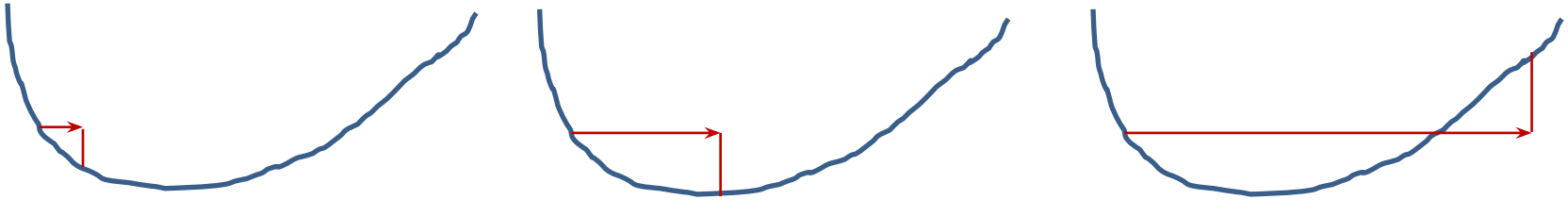
Role of step size and learning rate

- Tale of two loss functions
  - Same value, and
  - Same gradient (first derivative), but
  - Different Hessian (second derivative)
  - Different step sizes needed
- Success not guaranteed for non-convex functions



The perfect step size is impossible to guess

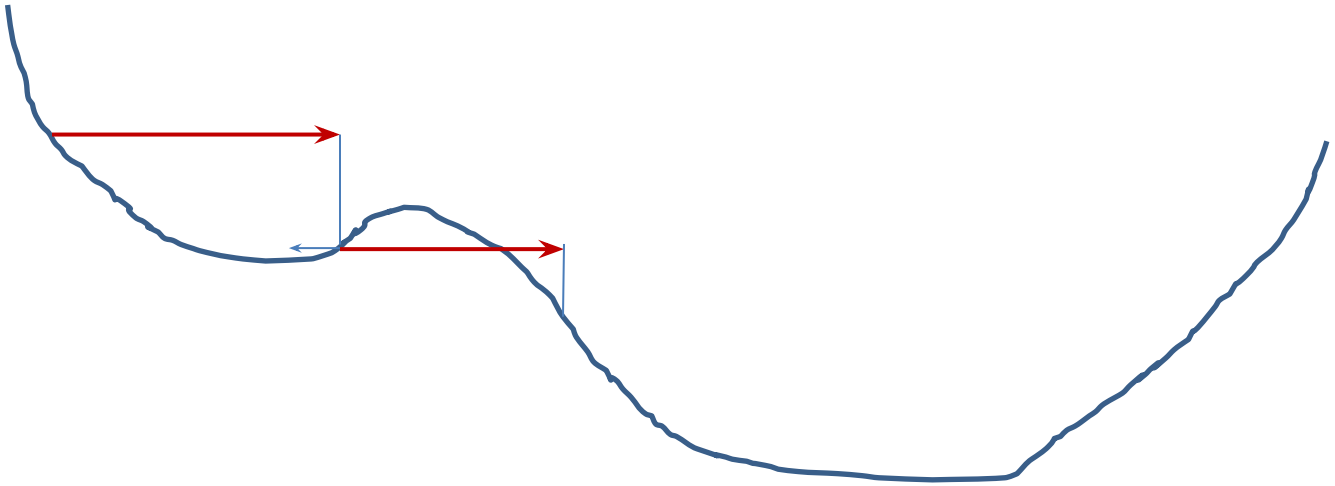
- Goldilocks finds the perfect balance only in a fairy tale



- The step size is decided by learning rate η and the gradient

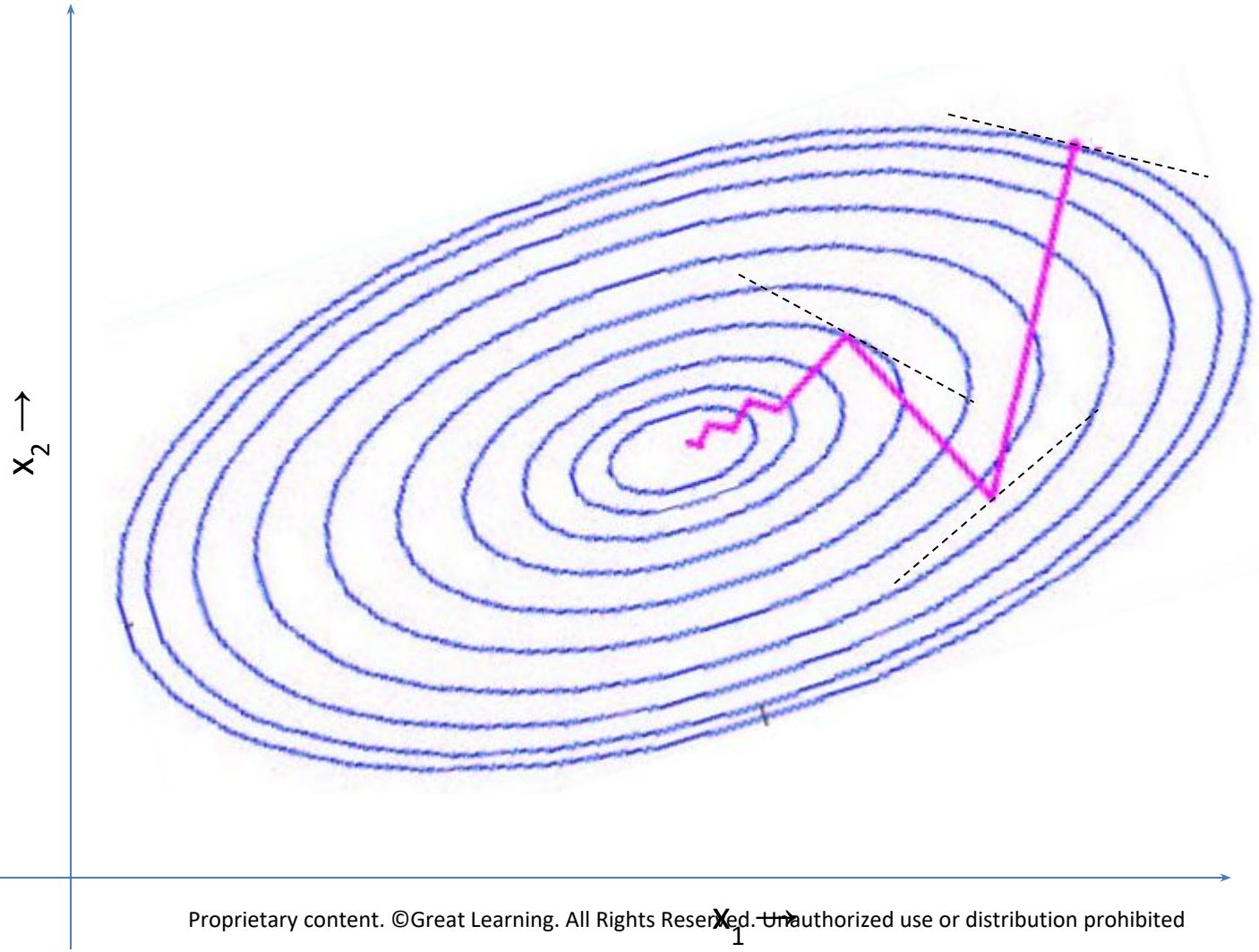
Momentum

- Momentum means using the memory of previous step to build up speed or to slow down with forgetting factor $\alpha;\ 0 \leq \alpha = 1$



$$\Delta \boldsymbol{x}^{(t)} = \alpha\ \Delta \boldsymbol{x}^{(t-1)} - \eta\ \nabla_{\boldsymbol{x}} f(\boldsymbol{x})$$

This story is unfolding in multiple dimensions



Perfect step size for a paraboloid

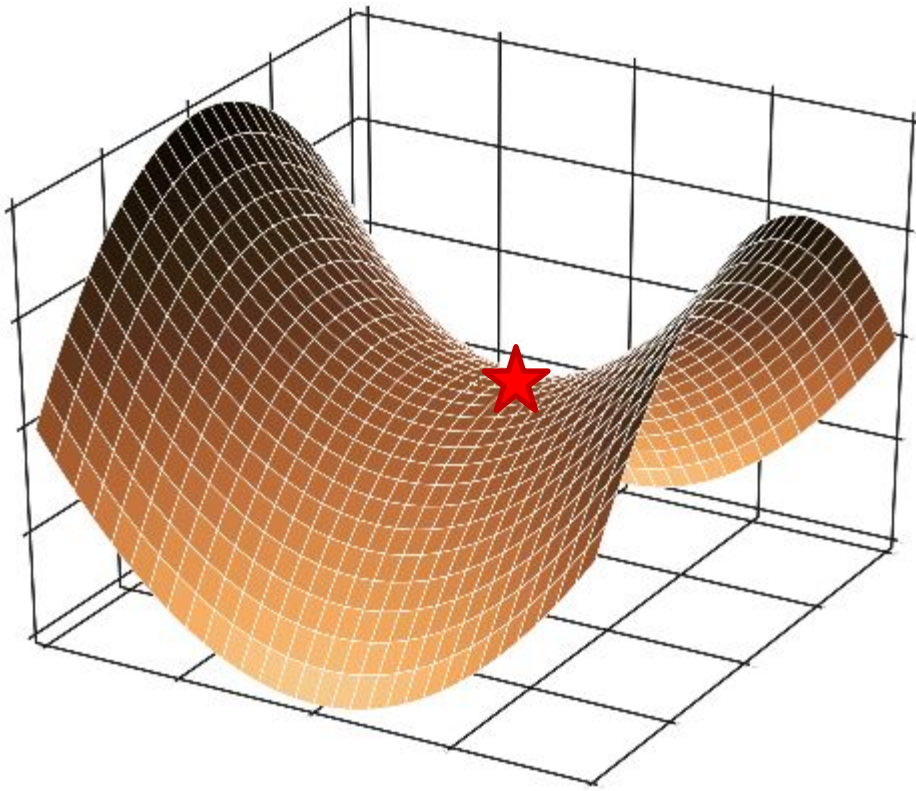
- Let $f(x) = ax^2 + bx + c$
- Assuming $a < 0$
- Minima is at: $x^* = -\frac{b}{2a}$
- For any x the perfect step would be:

$$-\frac{b}{2a} - x = -\frac{2ax+b}{2a} = -\frac{f'(x)}{f''(x)}$$

- So, the perfect learning rate is: $\eta^* = \frac{1}{f''(x)}$
- In multiple dimensions, $\boldsymbol{x} \leftarrow \boldsymbol{x} - H\big(f(\boldsymbol{x})\big)^{-1} \nabla(f(\boldsymbol{x}))$
- Practically, we do not want to compute the inverse of a Hessian matrix, so we approximate Hessian inverse

Saddle points, Hessian and long local furrows

- Some variables may have reached a local minima while others have not
- Some weights may have almost zero gradient
- At least some eigenvalues may not be negative



A realistic picture

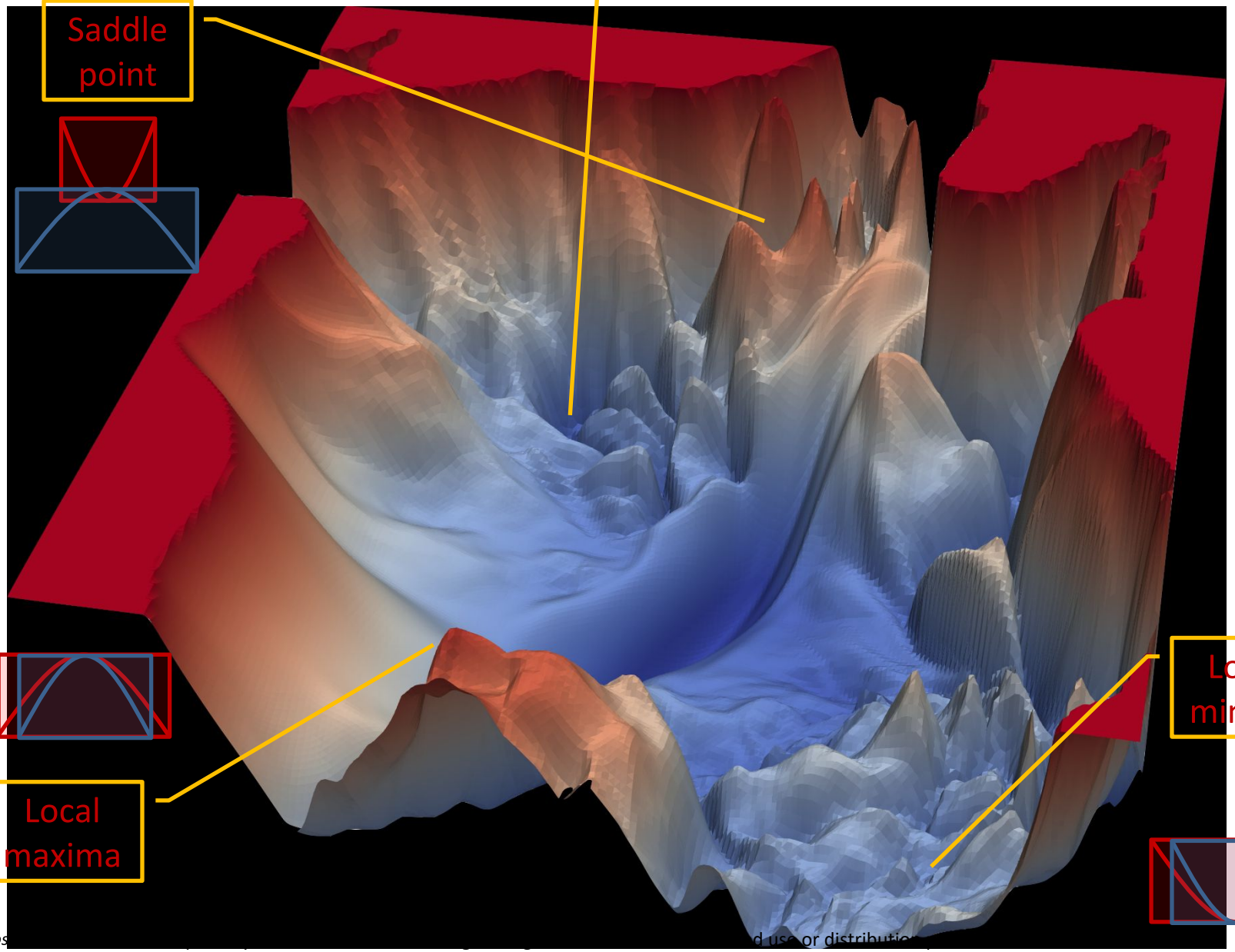


Image source: https

Adam optimizer

- For update step t
  - $\boldsymbol{g}_t \leftarrow \nabla_{\boldsymbol{x}} f(\boldsymbol{x}_{t-1})$
  - $\boldsymbol{m}_t \leftarrow \beta_1 \cdot \boldsymbol{m}_{t-1} + (1 - \beta_1)\, \boldsymbol{g}_t$
  - $\boldsymbol{v}_t \leftarrow \beta_2 \cdot \boldsymbol{v}_{t-1} + (1 - \beta_2) \cdot (\boldsymbol{g}_t \odot \boldsymbol{g}_t)$
  - $\hat{\boldsymbol{m}}_t \leftarrow \boldsymbol{m}_t / (1 - \beta_1^t)$
  - $\hat{\boldsymbol{v}}_t \leftarrow \boldsymbol{v}_t / (1 - \beta_2^t)$
  - $\boldsymbol{x}_t \leftarrow \boldsymbol{x}_{t-1} - \alpha\, \hat{\boldsymbol{m}}_t \,./ (\sqrt{\hat{\boldsymbol{v}}_t} + \varepsilon)$
- Good default values:
  - $\alpha = 0.001$, $\beta_1 = 0.9$, $\beta_2 = 0.999$ and $\varepsilon = 10^{-8}$

- Explanation
  - Gradient
  - Momentum
  - Moment (grad. sq.)
  - Bias correction
  - Bias correction
  - Update
- Why divide by moment?
  - It approximates Hessian inverse to give correct step size for each dimension