MTH350

Discrete Mathematics

Module 6: Graph Theory, Part II

This module will continue to introduce graph theory concepts and provide examples of real-world applications of them. In particular, this module will cover graph coloring, Euler paths and circuits, and Hamilton paths.

Learning Outcomes

1. Determine the chromatic number of a graph.
2. Use graph coloring principles to solve real-world applications.
3. Assess whether or not a given real-world scenario is possible using principles of graph theory.
4. Examine various graphs.
5. Create real-world problems that can be solved using graph theory principles.

For Your Success & Readings

Module 6 will build on your understanding of basic graph theory concepts discussed in the previous module. You will be given several examples of how real-world problems can be answered using the graph theory concepts covered in this module. The required readings for the module will also give you an idea of the graph theory applications that are currently being studied.

The Critical Thinking Assignment this week will allow you to work through a real-world application of graph theory on your own. You will be able to choose between an IT-related and a non-IT application. The discussion board this week will expand your understanding of how graph coloring can be applied to the real-world. Make sure to post early so you have plenty of time to engage with your peers about the scenarios everyone comes up with.

Required

- Chapter 4, Sections **4.3** (http://discrete.openmathbooks.org/dmoi/sec_coloring.html) & **4.4** (http://discrete.openmathbooks.org/dmoi/sec_paths.html) in *Discrete Mathematics: An Open Introduction*
- Bensouyad, M., Guidoum, N., & Saidouni, D. E. (2015). **Strict strong graph coloring** (https://dl-acm-org.csuglobal.idm.oclc.org/citation.cfm?id=2833020). *Proceedings of the International Conference on Engineering & MIS 2015 - ICEMIS.*
- Sungu, G., & Boz, B. (2015). **An evolutionary algorithm for weighted graph coloring problem** (https://dl-acm-org.csuglobal.idm.oclc.org/citation.cfm?id=2768488). *Proceedings of the Companion Publication of the 2015 on Genetic and Evolutionary Computation Conference - GECCO Companion 15.*

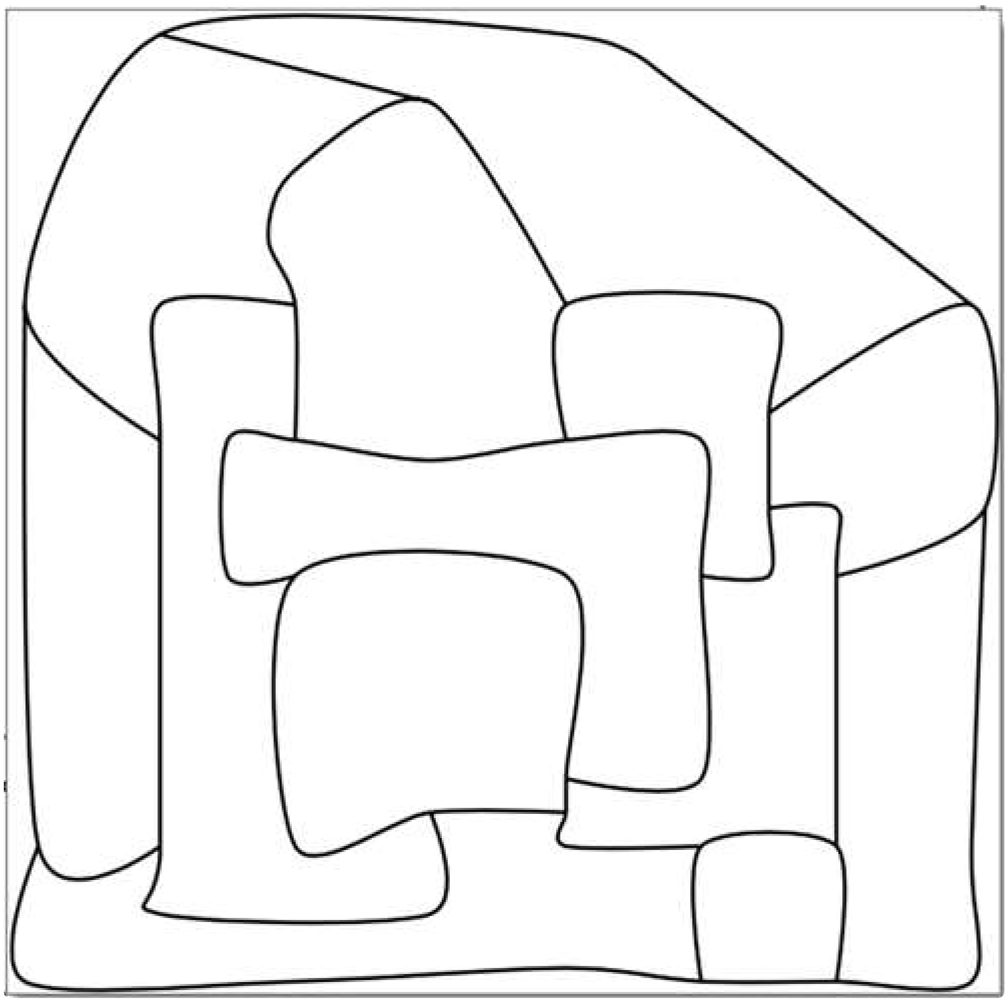
1. Coloring

One of the most useful tools in graph theory is the concept of graph coloring. We will see various examples of real-world applications of this topic, but first we need a few definitions.

Vertex coloring is an assignment of colors to each of the vertices of a graph. A vertex coloring is **proper** if adjacent vertices are always colored differently.

The **chromatic number** is the minimum number of colors required in a proper vertex coloring of the graph.

A good application to get started with is coloring a map so that neighboring regions are not colored the same. Consider the following fictional map of 13 regions:



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We could represent the map as a graph with 13 vertices as the regions and edges connecting bordering regions.

Notice that this map (and all maps, for that matter) will always have a planar graph representation since the edges will never cross, and, for planar graphs, we have the following important property:

The Four-Color Theorem

If G is a planar graph, then the chromatic number of G is less than or equal to 4.

Hence, the above theorem tells us that any map can be properly colored with four or fewer colors.

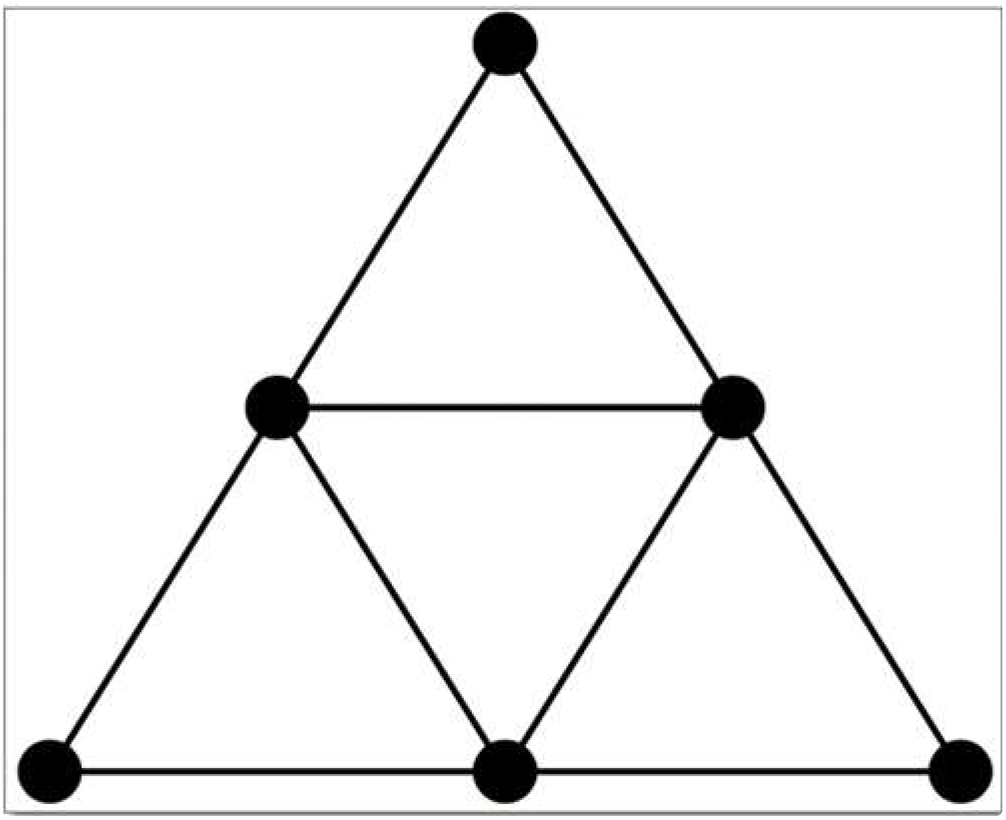
Another useful theorem in finding the chromatic number of a graph is as follows:

The 2-Colorable Graph Theorem

A graph is 2-colorable if and only if it has no circuits that contain an odd number of vertices.

Now, let's look at how we can use these theorems to find the chromatic number of a graph.

Consider the following graph:



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Does the graph contain any circuits with an odd number of vertices?

Yes. Each triangle of the graph is a circuit with three vertices. Hence, this graph is **not** 2-colorable, so its vertices cannot be colored with only two colors.

Is the graph planar?

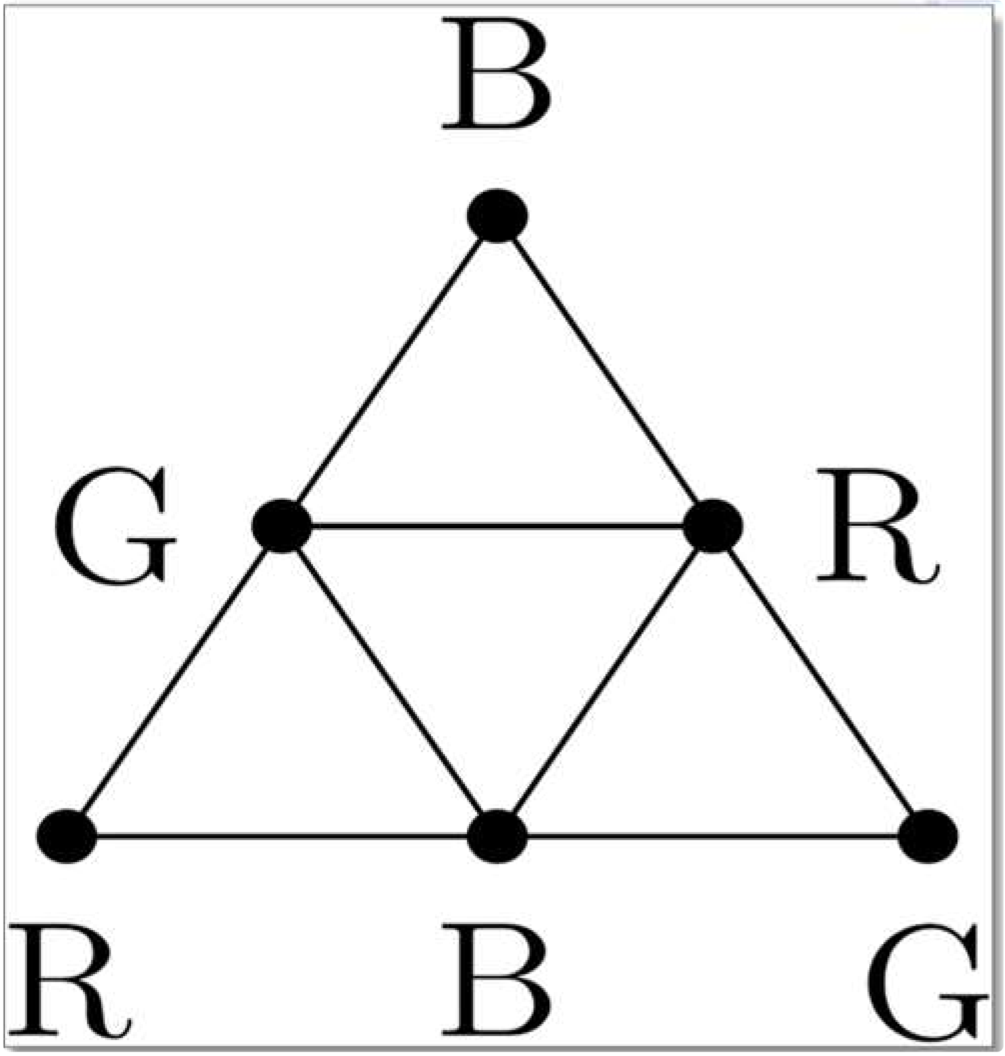
Yes. So we can apply The Four-Color Theorem to know that its chromatic number is 4 or less. In other words, its vertices can be colored with at most four colors.

Using trial and error, find the minimum number of colors needed to color this graph.

From the above two points, we know that the chromatic number of this graph is either 3 or 4. We can use trial and error to see if its vertices can be colored with just 3 colors. (Note that there is no simple theorem to determine if a graph is 3-colorable.)

The color verdict?

It turns out that you can color this graph with just 3 colors (such as red, green, and blue) as shown below:



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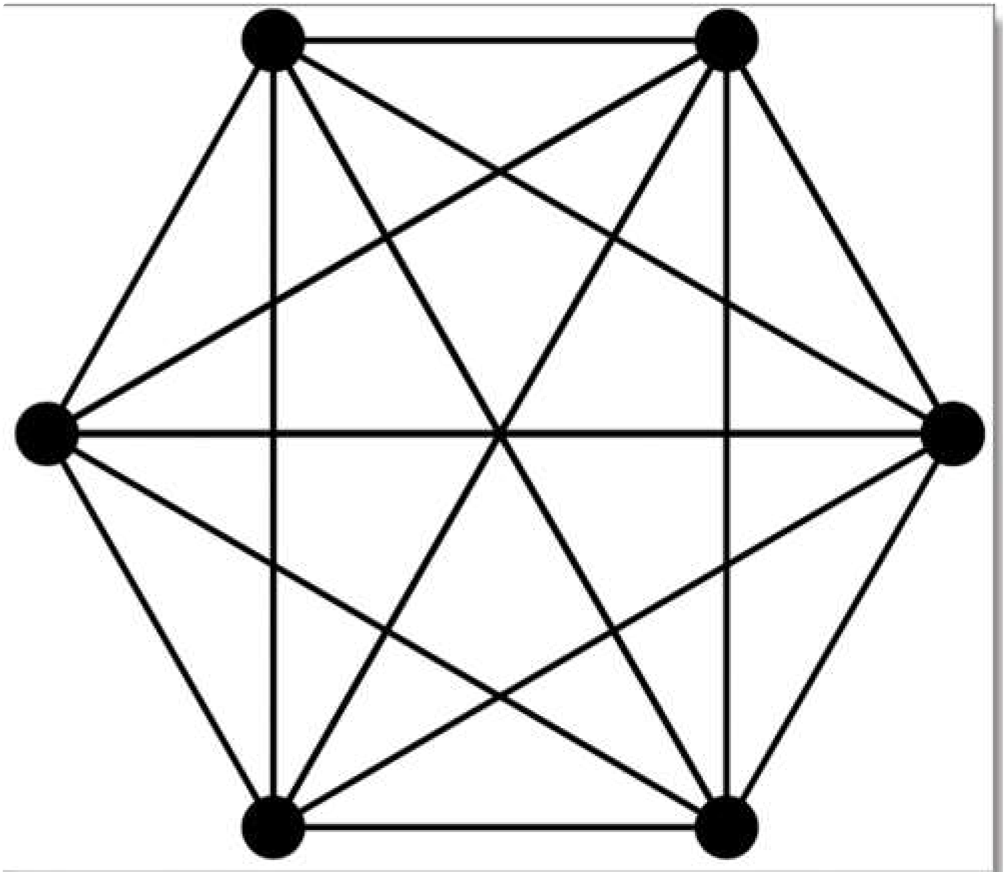
Thus, the chromatic number of this graph is 3.

Now, check your understanding of graph coloring by answering the following:

1.1. Chromatic Number of a Graph

Let’s look at another example of a graph:



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Does the graph contain any circuits with an odd number of vertices?

Yes. Each triangle of the graph is a circuit with three vertices. Hence, this graph is **not** 2-colorable, so its vertices cannot be colored with only two colors.

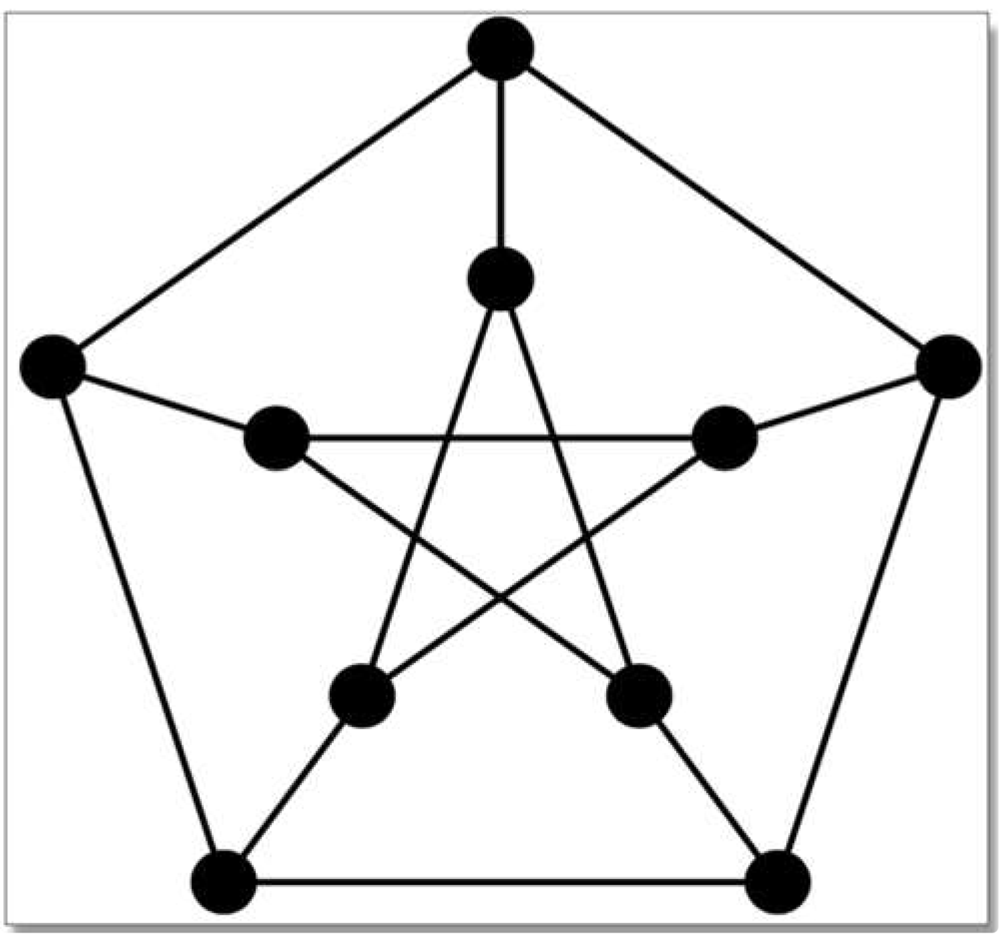
Is the graph planar?

No. So we can’t apply The Four-Color Theorem to gain information about its chromatic number.

Using trial and error, find the minimum number of colors needed to color this graph.

We know that the chromatic number is 3 or more. Using trial and error, we notice that the only possibility is to color each vertex a different color since all vertices are adjacent to each other. Hence, since this graph has six vertices, we would need six different colors. Thus, the chromatic number of this graph is **6.**

Now, check your understanding of how to find the chromatic number of a graph by answering the following question about the graph shown below.



1.2. Map Coloring

For another introduction to how map coloring works and to see an additional example, take a look at this video:

- **Coloring: Video (7:58 to end)** (https://youtu.be/Penh4mv5gAg?t=7m58s)

Now, check your understanding of map coloring by answering the following question:

1.3. Other Applications of Coloring

In general, any scenario that involves a partition of the objects in question so that related objects are not in the same set is an application of coloring concepts. For example, avoiding scheduling conflicts is one of those applications.

Let's look at the following video to see how airline schedules can be managed using vertex coloring.

- **Coloring: Video (0:00 to 7:57)** (https://youtu.be/Penh4mv5gAg)

Now, answer the following question to check your understanding of how to use graph coloring in scheduling applications.

2. Euler Paths and Circuits

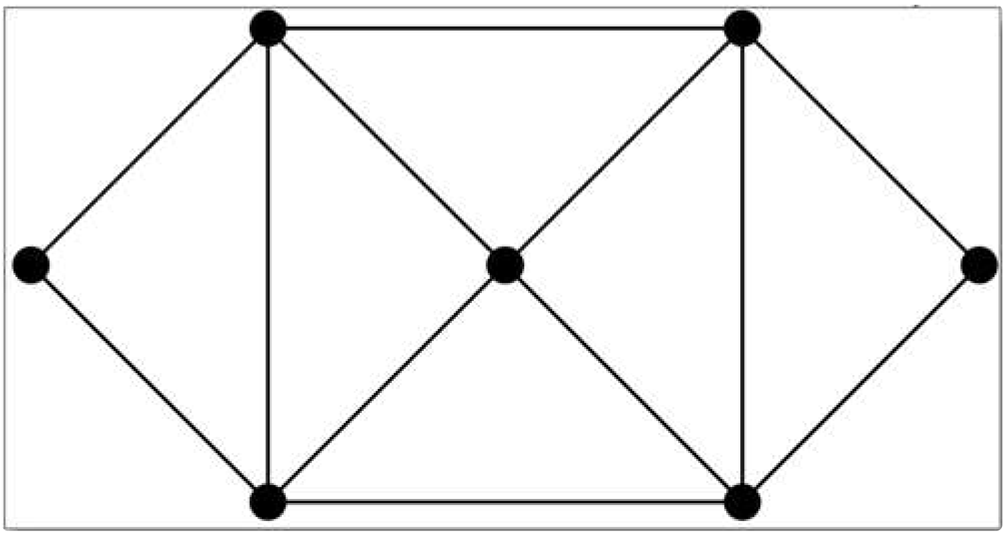
Recall from Module 5, the following definitions and facts about Euler Paths and Circuits:

An **Euler path**, in a graph or multigraph, is a walk through the graph which uses every edge exactly once. An **Euler circuit** is an Euler path which starts and stops at the same vertex.

Euler Paths and Circuits

- A graph has an Euler circuit if and only if the degree of every vertex is even.
- A graph has an Euler path if and only if there are at most two vertices with odd degree.

Let's practice determining if a graph has an Euler circuit and an Euler path by considering the following example:



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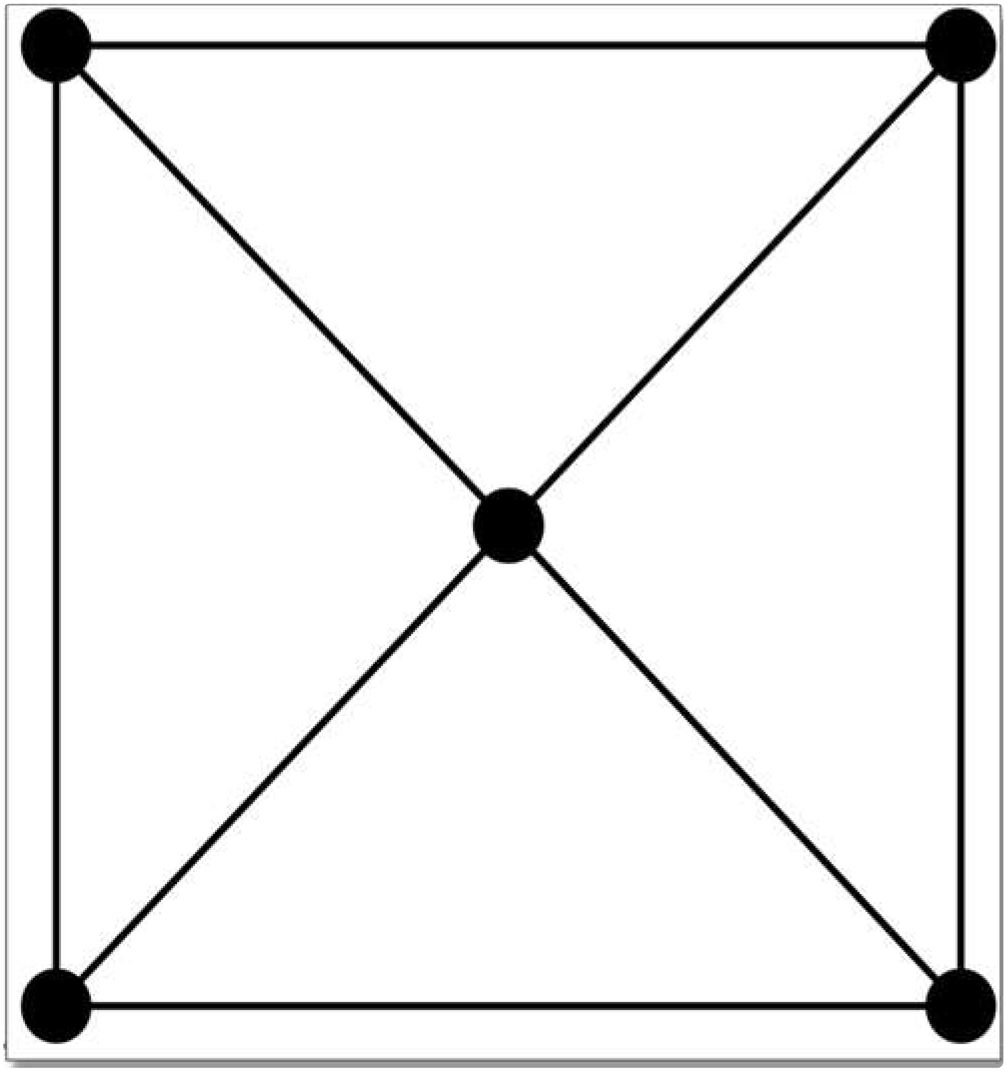
What are the degrees of all vertices of the graph?

The vertices of the graph we are given have degrees 2, 4, 4, 4, 2, 4, and 4.

Are there any vertices with odd degree?

In this case, no. All vertices have even degree. Hence, the graph has an Euler circuit, and thus, an Euler path.

Let's look at one more example using the graph below:



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What are the degrees of all vertices of the graph?

The vertices of the graph we are given have degrees 3, 3, 3, 3, and 4.

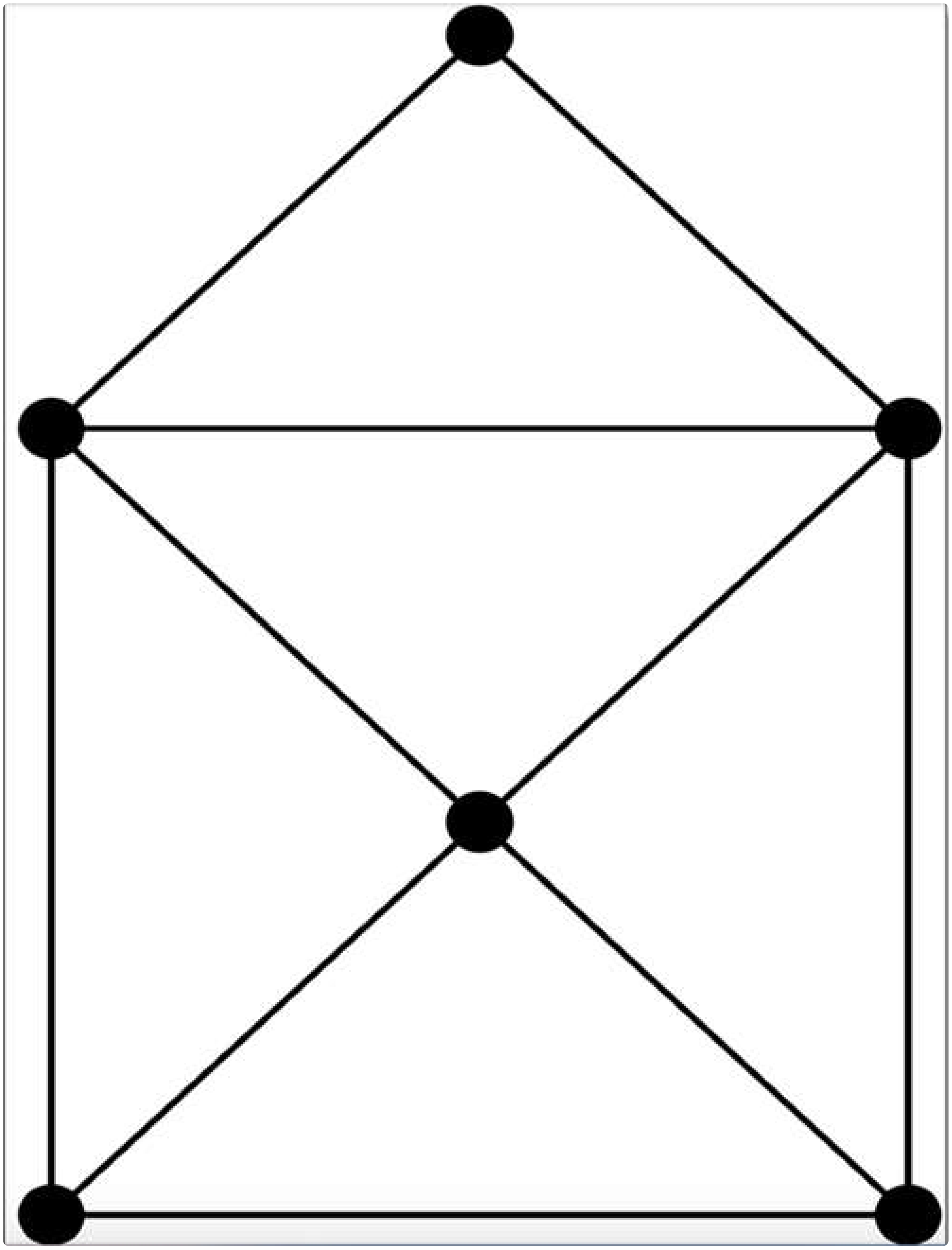
Are there any vertices with odd degree?

Yes, so we know that this graph does **not** contain an Euler circuit.

Are there at most 2 vertices with odd degree?

No, so we know that this graph also does **not** contain an Euler path.

Now, check your understanding of Euler paths and Euler circuits by answering the following question about the graph shown below:



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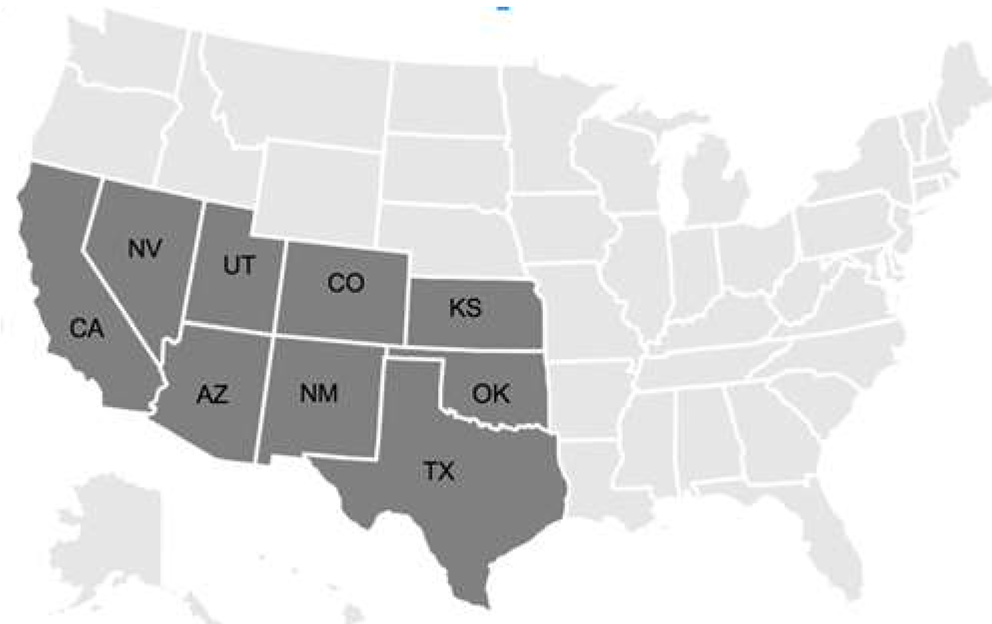
2.1. Applications of Euler Paths and Circuits

In this section, let’s look at an example of a real-world application of Euler paths.

Consider the following map of the united states with the southwest region shaded.



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Now, suppose you and your friends want to tour the southwest by car. You will visit the nine states below, with the following rather odd rule: you must cross each border between neighboring states exactly once (so, for example, you must cross the Colorado-Utah border exactly once). Can it be done? Why or why not?

Before we answer this question, let’s look at how it relates to graph theory.

How could this scenario be represented with a graph?

We could represent states as vertices and edges connecting the states that share a border.

What would an Euler path of the graph represent?

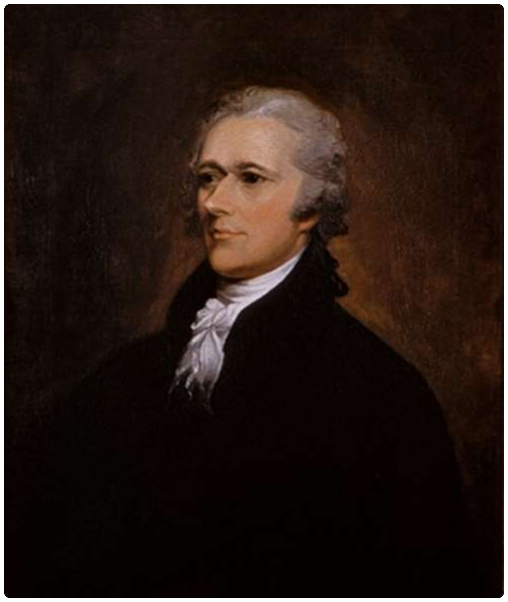
An Euler path is defined as a path which crosses every edge exactly once. In this scenario, this would be a tour that crosses every border exactly once.

How could we know if there is an Euler path for this graph?

If there are at most two vertices of odd degree, then we know that the graph contains an Euler path. In other words, if there are at most two states with an odd number of bordering states, then it is possible to tour the southwest region by crossing each border exactly once.

Now, check your understanding of Euler path applications by answering the following:

3. Hamilton Paths



No...not this Hamilton.

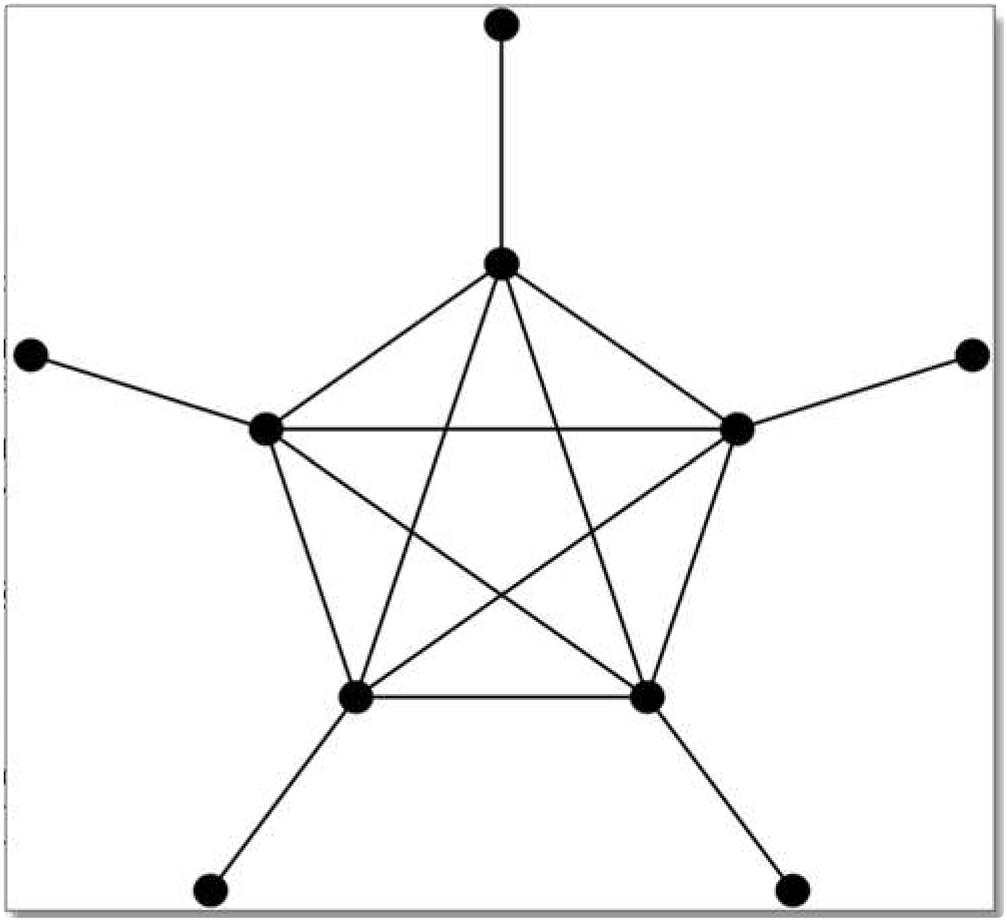
In this section, we are going to learn one final graph theory definition that also has many profound applications in the real-world.

A **Hamilton path** (or Hamiltonian path) is a path which visits every vertex exactly once. We could also consider **Hamilton cycles**, which are Hamilton paths which start and stop at the same vertex.

You may have noticed that the definitions of a Hamilton path and an Euler path are easily confused since they are so similar. So, it is important to note that Hamilton paths pass each **vertex** exactly once, while Euler paths pass each **edge** exactly once.

Also, unlike with Euler paths, there is no known simple test for checking if a graph has a Hamilton path. We need to use trial and error.

Let's consider the following graph as an example:



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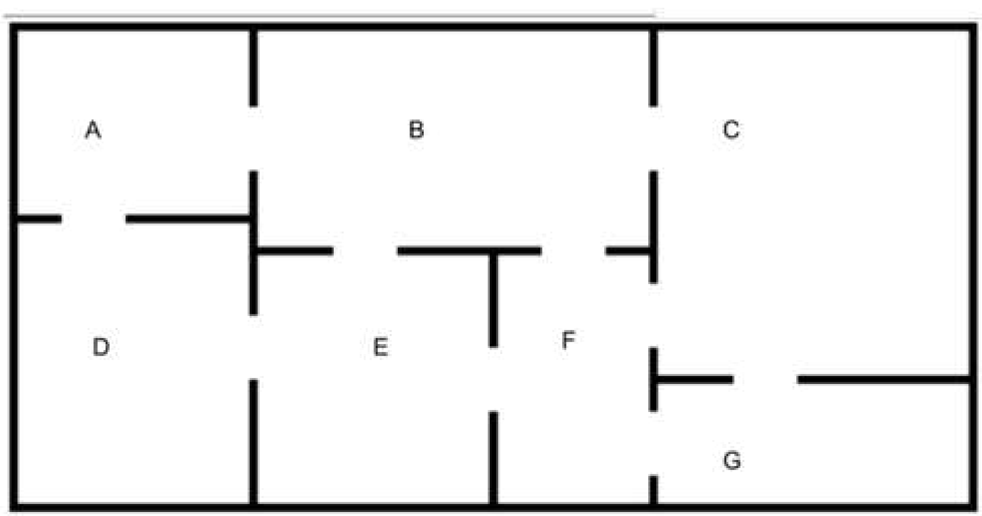
When using trial and error to find a Hamilton path for the graph above, you should have noticed that you get stuck when visiting an outside vertex since you wouldn't be able to revisit the vertex that connects to it.

Now, recall the following floorplan from Module 5:



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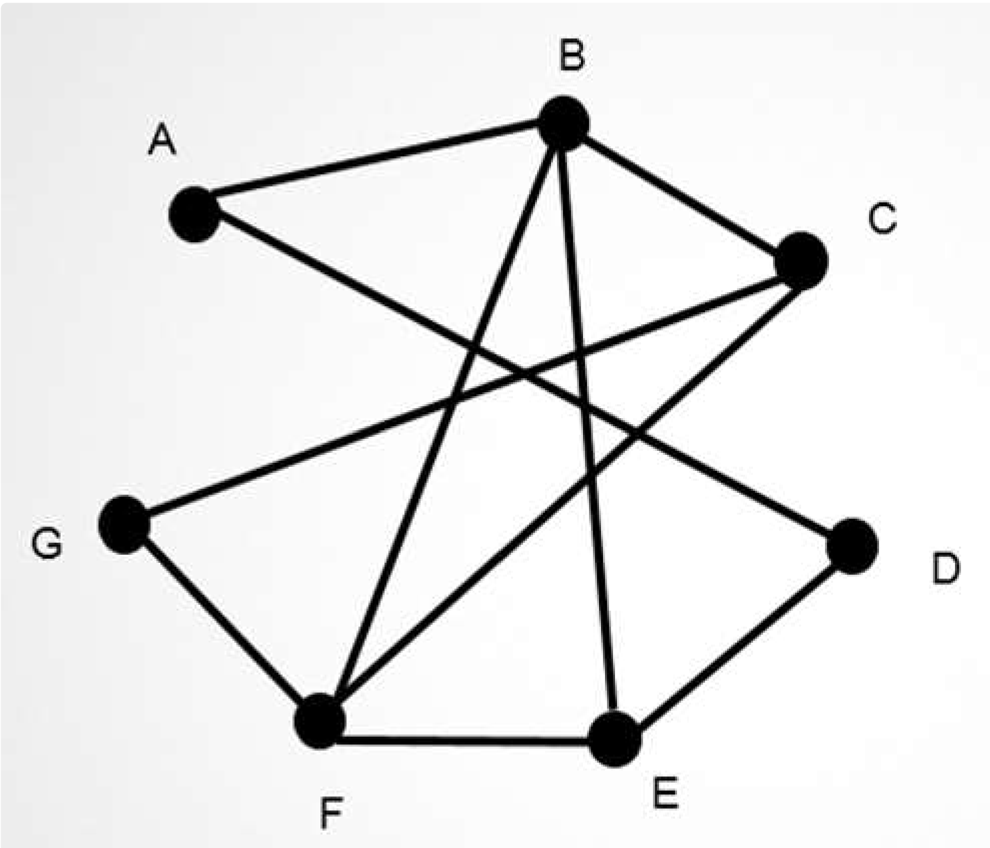
In the previous module, we determined that it was possible to tour the home by walking through every doorway exactly once. Since doorways are represented by edges, this entailed finding an Euler path for the graph that represents this floor plan.

This time, we will answer the following question:

Is it possible to tour the house visiting each room exactly once (not necessarily using every doorway)?

Since we are looking for a path that passes each **vertex** exactly once, we need to determine if a Hamilton path exists for the following graph.



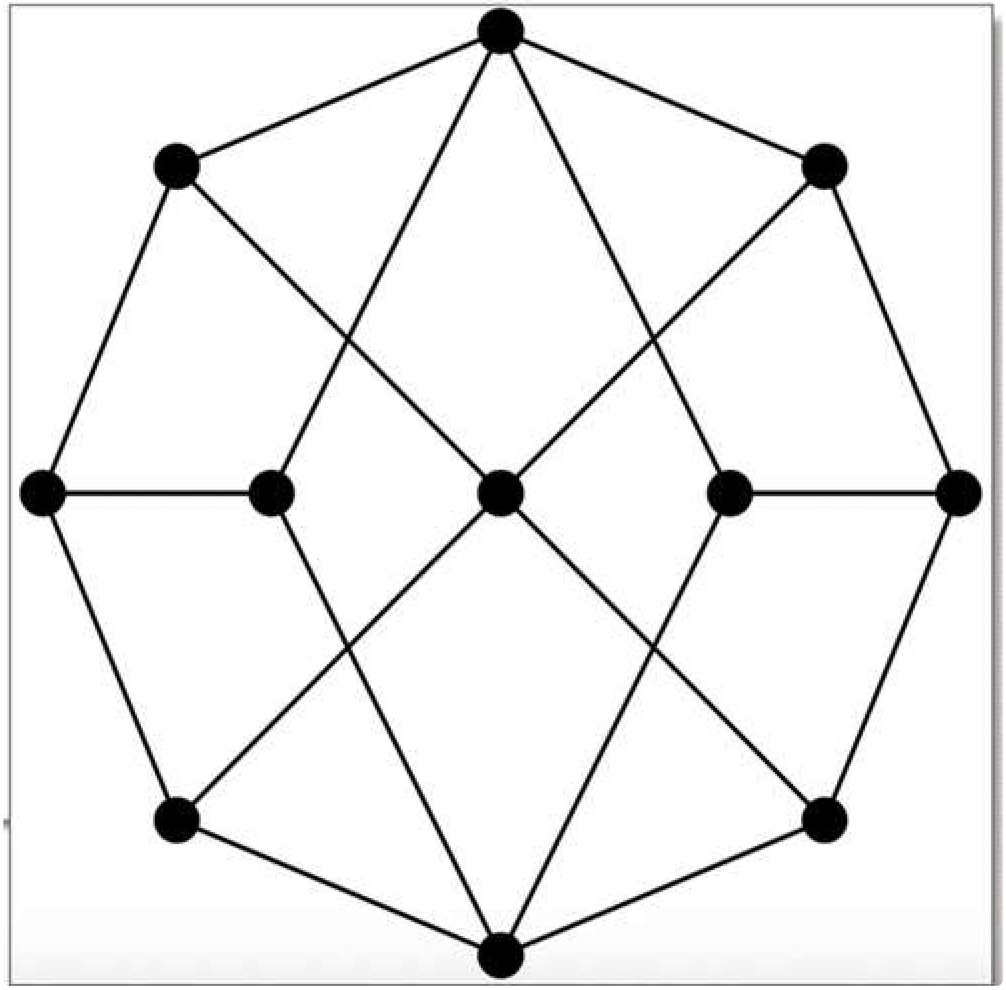
As mentioned above, there is no simple test to determine this, so all we can do is proceed through trial and error.

It turns out there is a Hamilton path that starts at *A*, then goes to *B*, then *C*, then *G*, then *F*, then *E*, and finally ends at *D*. In this way, every room will have been visited exactly once.

Now, check your understanding of Hamilton paths by answering the following:

3.1. Definition of Hamilton Paths

Here's another example of a graph to determine if it contains a Hamilton Path:



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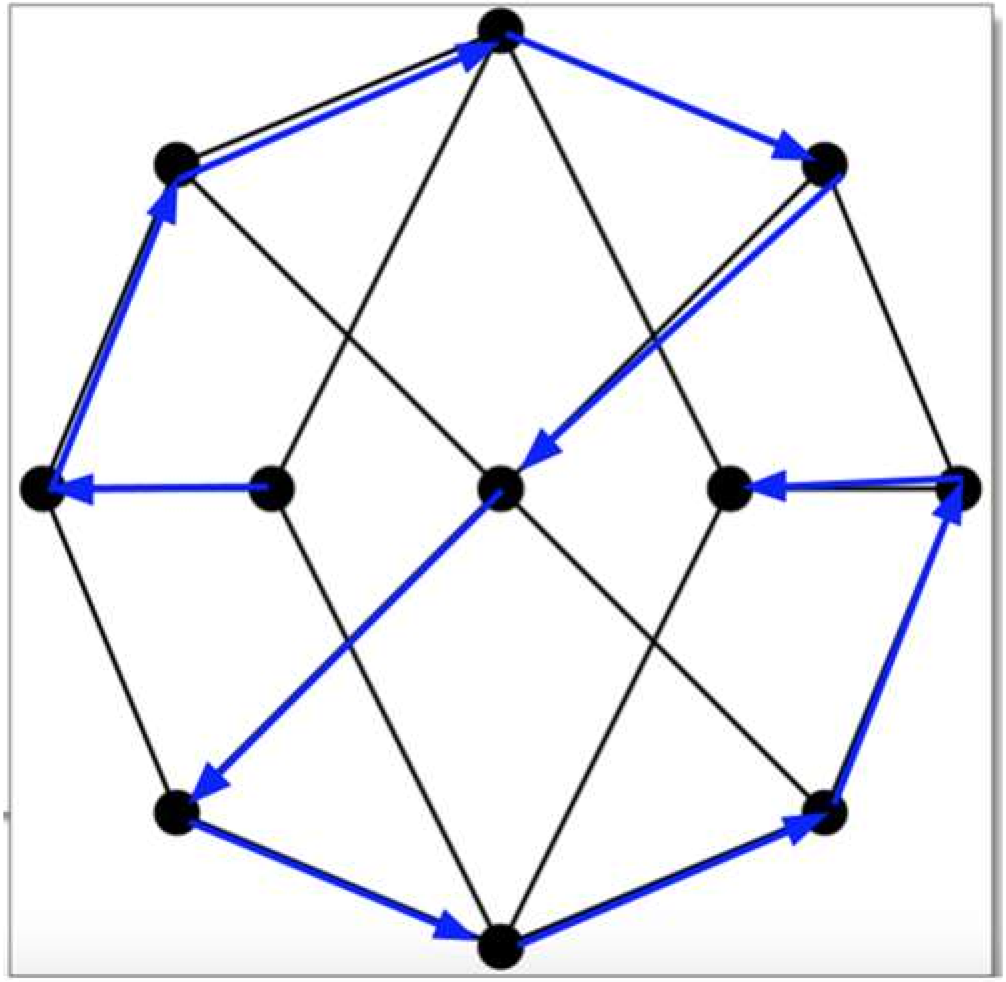
Through trial and error, we can see that it does contain a Hamilton path. One such path is shown below:



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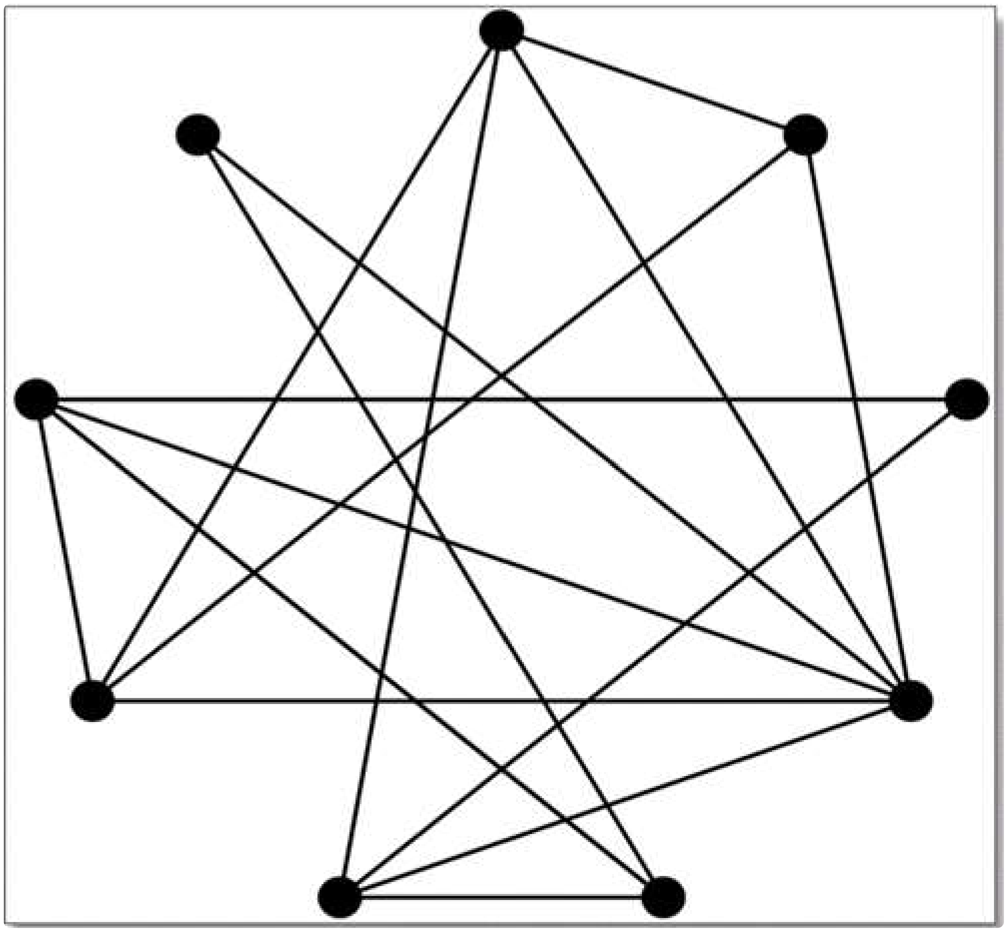
Now, check your understanding of Hamilton graphs by answering the following:

3.2. Applications of Hamilton Paths

Let's look at another application of Hamilton paths. Consider the following question:

Below is a graph representing friendships between a group of students (each vertex is a student and each edge is a friendship). Is it possible for the students to sit around a round table in such a way that every student sits between two friends?

Read the questions on the tabs below with regard to this scenario. When you have your answer, click the tab to check your ideas.



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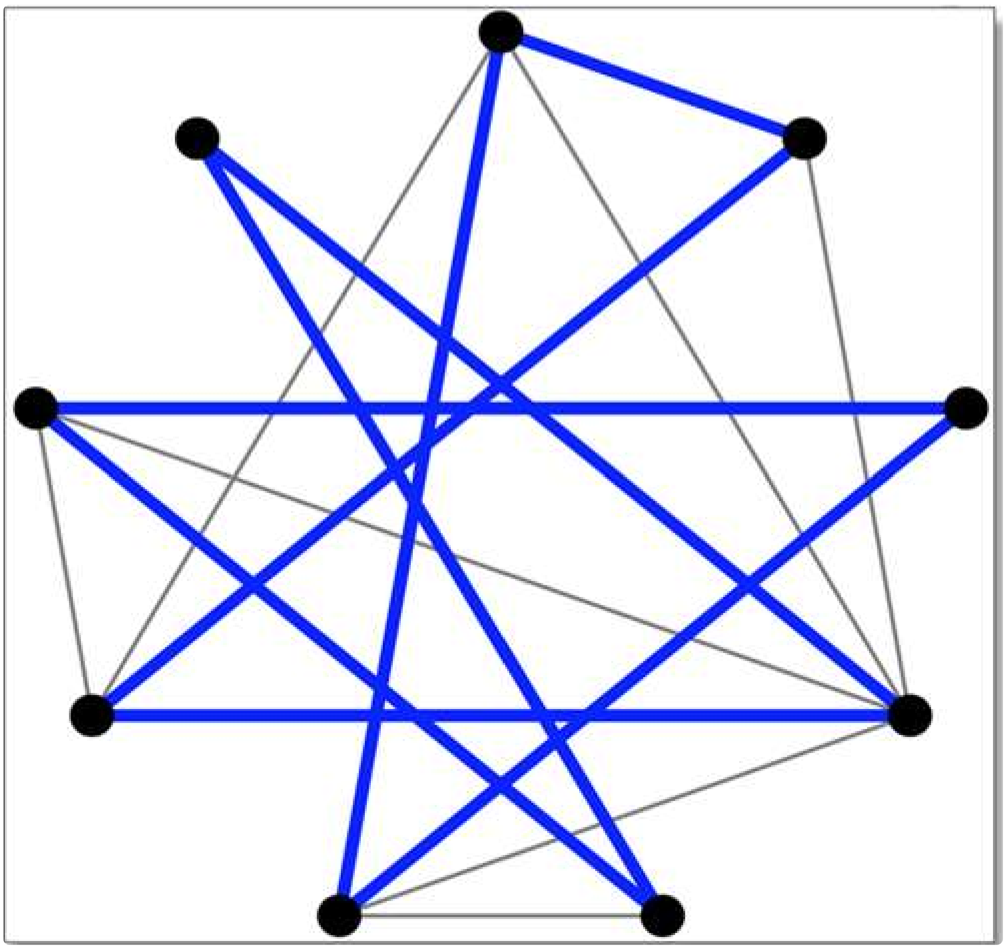
What type of path would represent the case when students can sit around a round table in such a way that every student sits between two friends?

On the graph, a student being seated between friends can be represented by a vertex always being between adjacent vertices which occurs when going along a path. Also, since we want all nine students to be seated around a round table, we are looking for a path that includes all vertices exactly once. This type of path is a Hamilton path.

Does such a path exist?

Yes. Here is a Hamilton path of this graph:



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Is it possible for the students to sit around a round table in such a way that every student sits between two friends?

Yes. Since we were able to find a Hamilton path for the graph, we know that it is possible to seat the nine students around a round table in such a way that every student sits between two friends.

Now, check your understanding of Hamilton path applications by answering the following:

4. Summary

Click through the following interactives to view worked examples that cover key points for each module outcome.

Module Outcome #1: Determine the chromatic number of a graph.

What is the chromatic number of a complete graph K_n?

The chromatic number of complete graph with n vertices is n. This is because every vertex is adjacent to every other vertex, so we need a new color for each vertex.

Module Outcome #2: Use graph coloring principles to solve real-world applications.

Consider the following scenario:

Radio stations broadcast their signal at certain frequencies. However, there is a limited number of frequencies to choose from, so nationwide many stations use the same frequency. This works because the stations are far enough apart that their signals will not interfere; no one radio could pick them up at the same time.

Suppose 10 new radio stations are to be set up in a currently unpopulated (by radio stations) region. The radio stations that are close enough to each other to cause interference are recorded. What is the fewest number of frequencies the stations could use.

Which of the following is a way to model this situation with a graph?

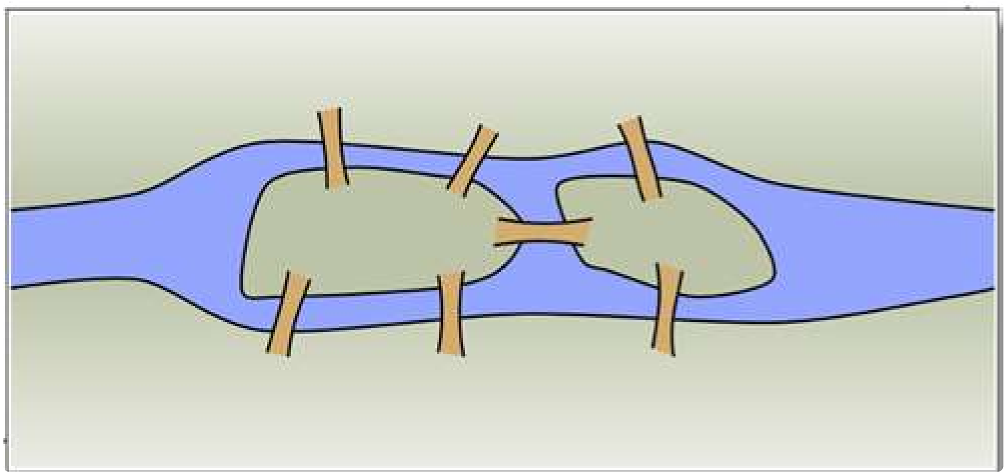
a. Vertices are stations, and edges connect stations that are far enough away to not cause interference.
b. Vertices are frequencies, and edges connect frequencies that share a station.
c. Vertices are stations, and edges connect stations that are close enough to cause interference.
d. Vertices are frequencies, and edges connect frequencies that don't share a station.

Answer

The correct answer is ‘c’. We can represent the problem as a graph with vertices as the stations and edges when two stations are close enough to cause interference. Then we can find the chromatic number of the graph to answer the question.

Module Outcome #3: Assess whether or not a given real-world scenario is possible using principles of graph theory.

Consider the following illustration of bridges connecting land masses.



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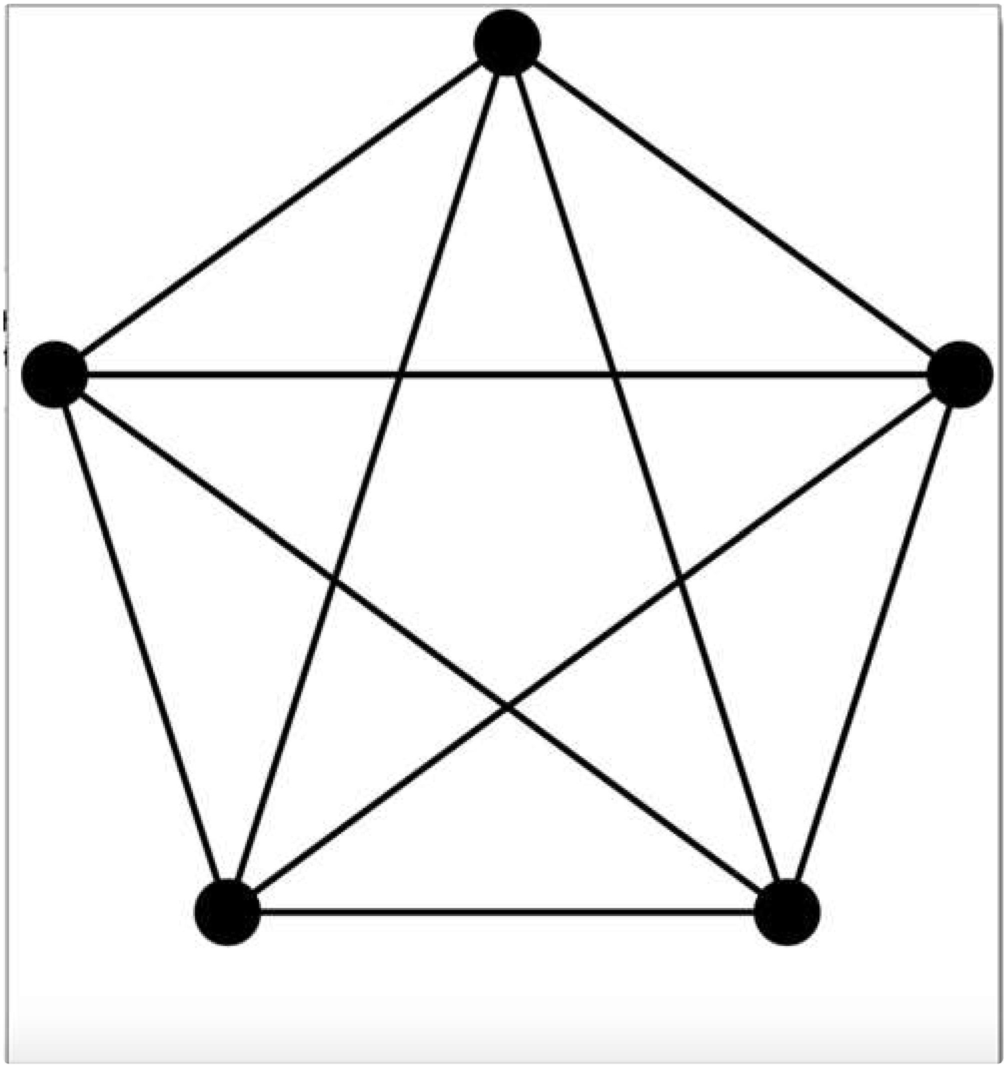
Is it possible to plan a walk so that you cross each bridge exactly once?

The problem above, known as the *Seven Bridges of Königsberg*, is the problem that originally inspired graph theory. When representing this as a graph with vertices as land masses and edges as bridges connecting them, we notice that there is no Euler path of the graph since more than two vertices have odd degree.

Module Outcome #4: Examine various graphs.

If possible, grab a piece of scratch paper and draw a graph which has an Euler circuit, but is *not* planar. When your drawing is finished, click “Show Me” below to check your drawing with the suggested drawing.

Show Me



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The complete graph with five vertices is an example of a graph that has an Euler circuit since all vertices are of even degree but is not planar since edges must cross.

Module Outcome #5: Create real-world problems that can be solved using graph theory principles.

Read the question on the tab below. When you have your response, click the tab to reveal the answer.

What are some examples of applications of graph theory?

There are many. Here are a few:

- Designing travel routes.
- Scheduling scenarios.
- Coloring maps.
- Efficient network connections.

Check Your Understanding

Embedded Media Content! Please use a browser to view this content.

References

Levin, O. (2017). *Discrete mathematics: An open introduction*. Retrieved from http://discrete.openmathbooks.org/dmoi/colophon-1.html (CC BY-SA 4.0)