Learn You Some Algebras for Glorious Good!

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Chapter 1

Introduction

Before I bore you with a bunch of crap you don't care about, let's do some math, shall we?

There are basically three notions with which you need to be familiar in order to do anything interesting in math. Those three things are *sets*, *functions*, and *proofs*. Unfortunately, to be familiar with one, you have to be familiar with the other two.[1]

So, what are each of those things?

- A *set* is an unordered collection of things. There is also no repetition. For instance, $\{2, 5\}$ is the same as $\{5, 2\}$ (because order doesn't matter). $\{2, 5, 5\}$ would be the same set, because there's no notion of multiplicity.

  I'm going to use some notation for sets later in the book. The explanation for the notation doesn't really fit anywhere, so here it is:

$$A = \{ x \mid y \}$$

[1]You'll learn as we go along, when math people use a common term like *set*, *function*, *proof*, *group*, *continuous* or *closed*, they usually mean something similar in concept to the colloquial term, but there are some strings attached. This is usually the case in the sciences too (e.g. *theory*, *hypothesis*, *experiment*).

This should be read as "A is the set of all values x, such that y is True. This is called "set comprehension notation." It's not difficult, and you'll pick it up as we go along.

- A *function* is a mathematical construct (well, obviously, else I wouldn't be talking about it). Basically, it takes some input, does something to it, and spits out some output. If you give the function the same input a bunch of times, you should get the same result each time. This concept is called "referential transparency." If the function is not referentially transparent, then it's not a function. It's something else.

- A *proof* is basically where you take a bunch of simple facts, called *axioms*, and chain them together to make *theorem*s. It's sort of like sticking puzzle pieces together to form a picture.

  The puzzle pieces (in this case, the axioms) aren't usually very interesting on their own. However, the picture they form (in this case, the theorem) can be really cool and enlightening. The proof would be analogous to an explicit set of instructions explaining how to put the pieces together.

Once you are familiar with each of those concepts, we can do all sorts of cool stuff. Throughout the book, we will prove all of the following:

- If you tap your finger against a bridge at exactly the right frequency, the bridge will collapse. (Resonance)

- The formula used to calculate the interest rate on your mortgage is actually just a fancy form of the ratios of angles in a triangle. (Euler's formula)

1.1 How to read the book

The best way to read this book is to just read it. Don't skip sections, or look ahead, or anything like that. Just read it straight through. It's also pretty important that you read the rest of this chapter. I promise it's not too boring.

Do all of the exercises. There aren't that many. However, they are pretty difficult. The exercises all have solutions, which are on the page after the exercises.

The exercises are designed to make you think, and widen your perspective on the topic at hand. They are not designed to be tedious. They are difficult, but the good kind of difficult.

It would be perfectly okay to just do the exercises (all of them), and then go back and read the text when you don't understand something.

1.2 Introduction (for real this time)

This is a math book. Well, duh. Why did I write it?

Most math (and science) books nowadays seem to value keeping an academic tone over ensuring that the reader understands the material, and — more importantly — enjoys reading the book.

I take the opposite approach. I want to create a book that is fun to read and easy to understand, while eschewing the practice of making myself look good.

The inspiration for this book is *Learn You a Haskell for Great Good!*, by Miran Lipovača. Haskell is a programming language, and LYAH is a great book for learning Haskell. If you are interested in a print copy of LYAH, see [7].

There is also an incomplete and unofficial Russian translation (`https://github.com/gazay/lysa`), courtesy of Alexey Gaziev.

1.3 The community

Despite the fact that I used "I" in the first part of the book, LYSA is actually a community project, and many people participate in the writing of this book.

If you want to talk to us, or to other math people, come see us in `#lysa` on Freenode. If you don't know what IRC is, or you don't have a client set up, you can connect through Freenode's webchat (`http://webchat.freenode.net/?channels=lysa`).

If you have any questions about LYSA (or math), feel free to ask in the IRC channel (`#lysa` on FreeNode in case you forgot).

If you want to submit a correction, or have some issue, or want to add some content, really anything having to do with the content of the book, you can visit our GitLab page (`https://gitlab.com/lysa/lysa`). We also have a woefully incomplete website (`http://learnyou.org`) and a community on Reddit (`https://lysa.reddit.com/`).

1.4 Idris

In this book, I cover a lot of hard stuff.[2] Sometimes, it's useful to program your way through a problem. Every programmer will tell you that programming teaches a manner of thinking.

Many programmers will cite Steve Jobs[3] famous quote, regarding the use of programming in his job,

> *[sic] . . . much more importantly, it had nothing to do with using [the programs we wrote] for anything practical. It had to do with using them to be a mirror of your thought process; to actually learn how to think. I think everybody in this country should learn how to program a computer — should learn a computer language — because it teaches you how to think.*

That first sentence or two is actually a pretty good description of mathematics (and programming). Both are incredibly useful, and have endless practical applications. That's not the point, though. The whole usefulness

[2] This isn't actually true. Math isn't hard, stupid!

[3] For you youngsters, Steve Jobs is the former CEO of Apple. He's dead now.

thing is a side gig. It's about learning how to think, and having a rigorous language through which to express your thoughts. Furthermore, the rigor of the language helps you build upon your current thoughts to find out even cooler things. That's what math is about.

Programming and math go hand-in-hand. Programmers and mathematicians will attest to this; I certainly can. For that reason, throughout this book, there will be coding exercises in the programming language Idris. Idris is an interesting programming language for many reasons. The chief of which is that it can be used to prove things mathematically. Most programming languages can't do this. Idris can, which is why it is special.[4]

1.4.1 Installing Idris

This is something that is actually rather difficult to summarize, because it varies from operating system to operating system. I will put down the instructions for the operating systems I use. If you come upon this and don't see your operating system, please report this on the issue tracker (`https://gitlab.com/lysa/lysa/issues/new`). Better yet, you could add the instructions yourself, and ask me to merge your changes.

Linux

Arch You need the Haskell platform and the GNU Multiple-Precision (GMP) library.

```
# pacman -S ghc cabal-install gmp
% \cabal update
% \cabal install -j cabal-install
```

At this point, you'll want to add `~/.cabal/bin` to your `$PATH` variable.

[4]There are other programming languages that can prove things, namely Coq and Agda. However, I'm most familiar with Idris, and Idris is probably the most useful, so I'm using Idris. Deal with it.

```
% cabal install -j alex happy haddock hscolour idris
```

`alex` and `happy` are dependencies for a number of Haskell packages. However, due to a long-standing bug in `cabal` (`https://github.com/haskell/cabal/issues/220`), they don't get pulled in when packages depend on them.

If the installation doesn't work, please report the bug to the Idris people (`https://github.com/idris-lang/Idris-dev/issues/new`).

Gentoo You will need the Haskell platform, along with the GNU Multiple Precision (GMP) library. As of 5 January 2015, the Haskell platform is only available on `~ARCH`, where `ARCH` is your processor architecture (e.g. `amd64`, `x86`). If you use `ARCH`, you can enable these by adding the following to `/etc/portage/package.keywords`:[5]

```
dev-lang/ghc
dev-haskell/cabal-install
```

Regardless of your `ACCEPT_KEYWORDS` variable, you should add the following to `/etc/portage/package.use`:

```
dev-lang/ghc binary
```

Otherwise, you have to compile GHC (the Haskell compiler) from scratch, and that takes forever.

Once you have that all out of the way, you'll want to run the following command as root:

```
# emerge -jav dev-lang/ghc dev-haskell/cabal-install
```

[5] If you already use `~ARCH`, you can ignore this

Warning: `-j` will make the installation a lot faster, but is more resource-intensive. If your power usage is precious, omit it (i.e. use `-av` instead).

Once GHC and cabal-install are installed, you'll want to run the following as a normal user:

```
% cabal update
% cabal install alex happy haddock hscolour
% cabal install idris
```

You can then get at the Idris shell by running `idris`.

1.4.2 Install a text editor

In order to edit Idris code, you need a plain-text editor (as opposed to a word processor).

Some popular plain-text editors are:

1. Gedit (`https://wiki.gnome.org/Apps/Gedit`) - very easy to use. I recommend either Gedit or Kate for beginners.

2. Kate (`http://kate-editor.org/get-it/`) - marginally harder than Gedit, but it has more features.

   Linux/BSD users: If you are not a KDE user, then don't use Kate. It brings in a ton of KDE dependencies. Here's the result of trying to install it on my machine:

```
% sudo pacman -S kate
[sudo] password for pete:
resolving dependencies...
:: There are 2 providers available for phonon-qt5-backend:
:: Repository extra
   1) phonon-qt5-gstreamer  2) phonon-qt5-vlc

Enter a number (default=1): 2
looking for conflicting packages...

Packages (54) attica-qt5-5.6.0-1  gamin-0.1.10-8  karchive
    -5.6.0-1  kauth-5.6.0-1  kbookmarks-5.6.0-1
              kcodecs-5.6.0-1  kcompletion-5.6.0-1  kconfig
    -5.6.0-1  kconfigwidgets-5.6.0-1
              kcoreaddons-5.6.0-1  kcrash-5.6.0-2
    kdbusaddons-5.6.0-1  kded-5.6.0-1  kglobalaccel-5.6.0-1
              kguiaddons-5.6.0-1  ki18n-5.6.0-1  kiconthemes
    -5.6.0-1  kinit-5.6.0-1  kio-5.6.0-1
              kitemmodels-5.6.0-1  kitemviews-5.6.0-1
    kjobwidgets-5.6.0-1  knewstuff-5.6.0-1
              knotifications-5.6.0-1  kparts-5.6.0-1
    kservice-5.6.0-1  ktexteditor-5.6.0-1
              ktextwidgets-5.6.0-1  kwallet-5.6.0-1
    kwidgetsaddons-5.6.0-1  kwindowsystem-5.6.0-3
              kxmlgui-5.6.0-1  libdbusmenu-qt5
    -0.9.3+14.10.20140619-1  libgit2-1:0.21.5-1
              libimobiledevice-1.1.7-1  libplist-1.11-1
    libusbmuxd-1.0.9-1  libxkbcommon-x11-0.5.0-1
              media-player-info-19-1  phonon-qt5-4.8.3-1
    phonon-qt5-vlc-0.8.2-1  polkit-qt5-0.112-2
              qt5-base-5.4.0-3  qt5-declarative-5.4.0-3  qt5
    -script-5.4.0-3  qt5-svg-5.4.0-3
              qt5-x11extras-5.4.0-3  qt5-xmlpatterns-5.4.0-3
      qtchooser-48-1  solid-5.6.0-1  sonnet-5.6.0-1
              threadweaver-5.6.0-1  upower-0.99.2-1  kate
    -14.12.2-2

Total Download Size:    33.42 MiB
Total Installed Size:  178.32 MiB

:: Proceed with installation? [Y/n] n
```

3. Vim (`http://www.vim.org/`) - It has a sharp, but not steep learning curve.

4. GNU Emacs (`https://www.gnu.org/software/emacs/`) has an absolutely insane learning curve, but is a wonderful editor once you spend 3 years learning how to use it.

1.4.3 Hello World in Idris

Pretty much every programmer in the world is familiar with the "Hello, World!" program. It's a program that exists in every language that just prints out `hello, world`.[6]

Access the Idris shell by running `idris` in a terminal. Here's what mine looks like:

```
% idris
     ____    __     _
    /  _/___/ /____(_)____
    / // __  / ___/ / ___/     Version 0.9.16-git:fca8309
  _/ // /_/ / /  / (__  )      http://www.idris-lang.org/
 /___/\__,_/_/  /_/____/       Type :? for help

Idris is free software with ABSOLUTELY NO WARRANTY.
For details type :warranty.
Idris>
```

Note that the `%` is there to indicate that the rest of the line should typed in a terminal. You shouldn't type the `%`.

There will be a flashing cursor after `Idris>`. That's where you type stuff.[7] Here's hello, world!

[6] Prints it in the terminal, not on a printer. Seriously, who uses paper these days?

[7] Note that, during the writing of this book, I've encountered a number of bugs in Idris. Trying to fix these bugs requires that I use the same version of Idris as the Idris developers, hence my using version `0.9.16-git:fca8309`. Your version will probably be different.

```
Idris> "hello, world"
"hello, world" : String
```

That was pretty simple. Note that you only have to type the thing following `Idris>` , then hit `Return`. You shouldn't type `Idris>` The stuff below it is what `idris` prints. By the way, it's incredibly important that you type this stuff out yourself, rather than just reading it (or copying & pasting).

If things are preceded by `Idris>`, that means that it's run in the interactive environment.[8] If you see a bunch of Idris code, without `Idris>` in front of each line, that means you should put the code in a file.

Speaking of files, let's make an actual program that you can run on the command line. Write this in an editor, and save it to a file called `helloworld.idr`:

```
module Main

main : IO ()
main = putStrLn "hello, world"
```

helloworld.idr

The computer can't understand the code; the code exists for the benefit of humans. With that in mind, we need to turn the code into binary language, which the computer can understand. Doing this by hand would be a nightmare. Luckily, we have a program to do it for us. To compile the program we wrote, run `idris helloworld.idr -o helloworld`. To run it, run `./helloworld`.

```
% idris helloworld.idr -o helloworld
% ./helloworld
hello, world
```

[8] In case you forgot, the way you access the interactive environment is to run `idris` in a terminal.

1.5 Target audience

The explanation of why programming is useful is a good segue into discussing the target audience.

When I was first writing the book, I wrote it in an effort to strengthen my own understanding. So, the target audience was me. The very first versions of this book were about a abstractish branch of math called commutative algebra. Later on, it seemed more fitting to abstractly go over the basics of math. That's what the current version of the book does.

That doesn't answer the question: who is the target audience? Well, people who want to learn basic and intermediate algebra, and to learn why it's so interesting.

Most books treat math as a tool you can use for calculations. I treat math as a language you can use to express your ideas. That's the core difference. This book will hopefully give you an interest in math itself, rather than just a cursory knowledge of it.

With that in mind, my book is going to approach the topics much differently than other books on the same topic. I rely very much on abstraction and intuition.

1.6 Licensing

This book is libre[9]. You can copy this book and give it to your friend. You can even print it out and sell it to people.[10] If, for instance, you are a schoolteacher and want to use this for your class, you are free to edit it to your liking and give the modified copy to your students. The only string I attach is, you have to allow anyone to whom you give the book do the same thing (i.e. they have to be free to copy/modify/change your version). The details of this can be found in § A.

[9] *Libre* is a French word, which, translated to English, means *free* in the sense of liberty, as opposed to price. Think *free speech*, not *free beer*.

[10] There are some restrictions though, see § A.

LYSA is licensed under the GNU Free Documentation License. § A contains the license. Please read the license; it's actually pretty comprehensible.

The source for this book can be downloaded at `https://gitlab.com/lysa/lysa/repository/archive.tar.gz`. If you are looking to contribute, it's probably best to clone the git repository. You can clone the git repository by running `git clone https://gitlab.com/lysa/lysa.git` in a terminal.

1.7 Conventions used throughout

You don't actually have to read this section, but it would be useful.

1. `Things in monospace` are either code snippets or commands to be run in a terminal. I have separate stylings for `terminal commands` and `inline code snippets`. That said, they are separate but equal, at least for the time being.

2. The § symbol refers to a section. So § 3.2 means "chapter 3, section 2".

3. Even though most of the writers are American, I still use the British convention of putting periods after quotation marks: "like this". The British convention is less ambiguous. If you see the American convention anywhere in this book, please report it.

4. I will often recommend software. However, I will not recommend any non-libre software, or any software that costs money.

5. "I" refers to me. "We" refers to both me and you, the reader. "You" refers to, you guessed it, the reader. It's the convention in academia to use the so-called "royal we", such as "we subtract 2 from both sides of the equation to obtain the result ...".

   Sometimes, we will accidentally use the royal we, out of habit. Crap, I just did it there! See? It's very difficult to avoid. Like any of the other conventions herein, if you see it broken, please report the error to the authors. You can use the bug tracker (`https://gitlab.com/lysa/lysa/issues/new`), or, if you don't want to make a (free and libre) GitLab account, you can email me at `peter@harpending.org`.

6. Oh yeah, sometimes I'll use `monospace` in things like URLs or emails for the sake of disambiguity.

7. Most of the authors use some version of Linux. Hence, when there are instructions for computer things (such as installing Idris), I'll write instructions for Linux, because that's what I know. There are two solutions here:

   (a) You could try out Linux (it's gratis, and it's easy).

   (b) If you know how to do the thing on your OS, and there aren't instructions for your OS, you could write up instructions and add them to the book. If you don't know how to do that, you could bring it up in the bug tracker (`https://gitlab.com/lysa/lysa/issues/new`) or email me at `peter@harpending.org`.

8. If you see some number as a superscript in the middle of text: like this[11], then the number refers to a footnote. If the superscript number is in the middle of math, it's probably just math.

9. If there's some number in brackets, like this: [7], then it's a citation. If you're reading this as a PDF, you can actually click on the number, and your PDF reader will take you to the relevant bibliography entry. Go ahead, check it out! I'll wait. You can do the same thing for footnotes and URLs.[12]

[11] Hey, you found me!

[12] Well, clicking the URL will open up your web browser, but you get the point

Chapter 2

Booleans, simple logic, and simple operators

Before we get into interesting content, you have to understand some stuff. This stuff is pretty easy. This will likely be the shortest and easiest chapter in the book.

You might think math is about dealing with numbers and pumping out formulas. Well, that's not what math is about. As said in § 1.4, it's about using math as a language to express your thoughts. Most people don't think about numbers all day; thus, we deal with things in math that aren't numbers.

In this next section, we're going to outline some basic rules for reasoning about things. You need to know these rules to do really cool stuff. Although, as you will (hopefully) see, these rules can be fun to toy around with on their own.

2.1 Implications

The first thing you need to understand is the notion that "if x is true, then y is also true. But, if y is true, it's not necessarily true that x is false." As always, mathematicians are too lazy to write this stuff out by hand, so they have notation for it.

1. $a \implies b$ means that "a implies b". It doesn't necessarily mean that b implies a. It means that if a is true, then b is also true.

   If someone is decapitated, then they will die. So,

   $$\text{Decapitated} \implies \text{Dead}$$

   However, if someone is dead, it doesn't necessarily mean that they were decapitated. They could have been shot, or stabbed, or had a heart attack. There are endless possibilities.

2. $a \impliedby b$ is the same as writing $b \implies a$. It's sometimes convenient to use $a \impliedby b$ instead. $a \impliedby b$ should be read "a is implied by b".

3. When I strike through some mathematical operator, like this: $\centernot\implies$, it means that you can semantically but "not" in front of whatever the operator says. So, $\centernot\implies$ means "not implies". That doesn't make much grammatical sense in English, so "does not imply" might be better. Nonetheless, you get the point.

   Moving on from the example above:

   $$\text{Decapitated} \implies \text{Dead}$$

   If someone is decapitated, then they're also dead (at least within a few seconds). However, if someone is dead, it's not necessarily true that they were decapitated.

   $$\text{Decapitated} \centernot\impliedby \text{Dead}$$

4. If something is not true, then I'll put a $\neg$ in front of it. So, if I want to say that a is false, then I'll write $\neg a$.

5. If I want to pose a question, I could just ask the question. For instance, "Is $\neg$Decapitated $\impliedby$ $\neg$Dead true?".

   However, that quickly becomes difficult, usually when there are multiple assertions in a mathematical expression, and you don't know which one I'm asking about. Moreover, since I use the same font for text and math, if I have both, it might be hard to tell which is math and which is text. So, to help with ambiguity, I'll put a ? over the operator I'm asking about:

$$\neg\text{Decapitated} \overset{?}{\Longleftarrow} \neg\text{Dead}$$

See, that's much easier.

6. Now, on to that question - Is "not decapitated" implied by "not dead". Well, let's think about it. If someone is not dead, then they couldn't have been decapitated, because if they were decapitated, then they would be dead. Therefore, if someone is not dead, then they weren't decapitated.

   That word jumble was probably confusing. Mathematicians don't like to be confused. I'll make it symbolic for you.

$$\begin{array}{rcl} \text{Decapitated} & \Longrightarrow & \text{Dead} \\ \Downarrow & & \\ \neg\text{Decapitated} & \Longleftarrow & \neg\text{Dead} \end{array}$$

   Okay, I just used a vertical arrow. I'm sure you can figure out what it means.

7. So, hopefully you agree that

$$\begin{array}{rcl} \text{Decapitated} & \Longrightarrow & \text{Dead} \\ \Downarrow & & \\ \neg\text{Decapitated} & \Longleftarrow & \neg\text{Dead} \end{array}$$

   However,

$$\begin{array}{rcl} \text{Decapitated} & \Longrightarrow & \text{Dead} \\ \overset{?}{\Uparrow} & & \\ \neg\text{Decapitated} & \Longleftarrow & \neg\text{Dead} \end{array}$$

Hm, the question mark doesn't work so well there. Oh well! Anyway, the answer is actually yes. We can figure this out by learning a rule about $\neg$. Namely, that

$$\neg\neg a = a$$

In this case, we know

$$\neg\text{Decapitated} \impliedby \neg\text{Dead}$$

So, if we just "not" both sides, and flip $\implies$ to $\impliedby$,

$$\begin{aligned}\neg\neg\text{Decapitated} &\implies \neg\neg\text{Dead}\\ \text{Decapitated} &\implies \text{Dead}\end{aligned}$$

This is basically just the rule mentioned in #3. Yay, we learned something!

2.2 And and or

So, sometimes we need to combine two pieces of logic together. There are two ways we can do this - logical-or and logical-and.

I put logical- in front of them, because the mathematical meaning is slightly different than the colloquial meaning.

Mathematicians are lazy, so we don't like to write "logical-and" whenever we want to say it, so instead we use the symbol $\wedge$.

$A \wedge B$ is true if (and only if) both A and B are true. If one of them is false, then the entire thing is false.

On the other side, we have logical-or. The symbol for logical-or is $\vee$. $A \vee B$ is true if either A or B is true, or if both of them are true. You could think of logical-or as being equivalent to the colloquial "and/or".

This monstrosity is called a "truth table".

A	**B**	**A ∧ B**	**A ∨ B**
True	True	True	True
True	False	False	True
False	True	False	True
False	False	False	False

This is pretty simple. If you're having trouble remembering which symbol is logical-and and which one is logical-or, remember that the logical-and symbol — $\wedge$ — looks vaguely like an A.

I'm going to introduce some new notation: the $\iff$ symbol.

$$(A \iff B) = (A \implies B) \wedge (A \impliedby B)$$

In practice, $\iff$ is logically equivalent to $=$. That is, all of the properties of $=$ hold.[6]

1. For all a, $a = a$.
2. For all a and b, $(a = b) \implies (b = a)$.
3. For all a and b, and c, $((a = b) \wedge (b = c)) \implies (a = c)$.

You should go over those properties, and verify that they are true for $=$, and then verify that they are true for $\iff$.

Alright, cool, this was a pretty easy chapter. Only 240 lines of LaTeX! More importantly, I'm confident that this stuff is so easy, I won't even give you exercises![1]

[1]The real reason is, I can't come up with any exercises that aren't tedious.

Chapter 3

Sets

In math, it's often useful to consider *collections* of objects. There are basically two types of collections: *sets* and *vectors*. Sets are unordered, and multiplicity doesn't matter. Vectors are ordered, and multiplicity does matter.

For instance, $\{3, 4\}$ and $\{4, 3\}$ are the same *set*, but $(3, 4)$ and $(4, 3)$ are different *vectors*.

Likewise, $\{20, 38\}$ and $\{38, 20, 38, 20, 20, 20, 20, 20\}$ are the same *set*. You guessed it, $(20, 38)$ and $(38, 20, 38, 20, 20, 20, 20, 20)$ are different *vectors*. Sets are, for basic stuff, much more important. Also, they are much easier to understand.

Sets were first studied to extent by Georg Cantor, a German mathematician, in the second half of the nineteenth century. Back in his own day, the results Cantor found by studying sets were considered so thoroughly bizarre that many of his colleagues simply refused to believe that Cantor could be right. In the end, Cantor turned out to be right all along. His ideas can be found in any introductory text on mathematics—including this one.

You probably figured it out from above: the notation is $\{\text{Braces}\}$ for sets, and (Parentheses) for vectors.

If you can't remember whether to use braces {the curly things}, or parentheses (the round things), remember: *a **brace** is used to **set** a broken bone.*

I don't have a horrible pun having to do with parentheses and vectors, and for that, I apologize.

3.1 Elements

Let's invent a set.

$$Q = \{\, 7, 7, 9, 5, 10, 1, 6, 6, 2, 10 \,\}$$

There we go. Remember, order and multiplicity don't matter. But, for the sake of clarity, let's put the elements in order, and deduplicate them.

$$Q = \{\, 1, 2, 5, 6, 7, 9, 10 \,\}$$

Yay! You may have noticed that I slipped in the word *element* into the previous sentence. Objects in the set are called *elements*. Yay, we figured out what that word means!

It's too strenuous on our weak mathematical hands to write "10 is an element of Q", so instead we have the symbol $\in$. $\in$ is a very terrible attempt at drawing an E. If you can't remember what $\in$ is, think "$\in$lement of".[1]

So, I'm going to ask you a question:

$$11 \stackrel{?}{\in} Q$$

(See, I used that thing from earlier with the question mark. I told you it would help.) Well, the answer is no, 11 is not an element of Q. As always, mathematicians are too weak to write "11 is not an element of Q" every time they want to say it, so instead they write

[1] You better think this, because it took me 30 minutes to get the alignment right. So, you know, remember $\in$ this way, or else…

$$11 \notin Q$$

By contrast,

$$6 \in Q$$

What if we want to say "both 6 and 2 are elements of Q"? Well, again, we could write it out like:

$$(6 \in Q) \wedge (2 \in Q)$$

But that's too cumbersome, so instead we'll write

$$2, 6 \in Q$$

But won't that get confusing?

Only if we put parentheses or braces around 2 or 6.

$$\{2, 6\} \in Q$$

That's confusing, don't do that (yet).

There's one more thing I need to go over, which is the null set - it's the simplest set, as it contains no elements. "Null set" takes too long to write, so we use $\varnothing$ instead.

3.2 Subsets and Supersets

Remember when I said $\{2, 6\} \in Q$, and we were really confused? In case you don't remember, $Q = \{1, 2, 5, 6, 7, 9, 10\}$. $\{2, 6\}$ is obviously not an element of Q. However, $\{2, 6\}$ is *in* Q but it's not an element. It's weird. How do we express this notion?

The answer is with *subsets*. "Sub" means "smaller", so a "subset" would be a "smaller set". A is a subset of B is all of the elements in A are also in B. The notation is $A \subseteq B$. Some people will read that as "A is contained in B".

Referring to the previous example,

$$\{2, 6\} \subseteq Q$$

Wait, what is with the little line under the round half circley thing?

So, actually, there are two types of subsets - *proper* and *improper*. $A \subseteq B$ is for improper subsets. $A \subset B$ is for proper subsets. What's the difference, then?

$A \subseteq B$ allows for the possibility that $A = B$. $A \subset B$ means that B is *strictly larger* than A; there are some elements in B that are not in A.

Before I go on, I'm going to define a couple of other symbols. We're constantly saying "this thing exists" and "for all", so I'm going to add a couple of symbols. $\exists$ is for "there exists" — notice the backwards E, as in "exists". $\forall$ is for "for all" — notice the upside-down A, as in "all".

So, now that you've got all of this down (hopefully), I'll give you some exercises.

3.2.1 Exercises

Ex. 1 — $\overset{?}{\exists} A;\ A \subset \varnothing$

Ex. 2 — $\forall A;\ \varnothing \overset{?}{\subseteq} A$

Answers

Answer (Ex. 1) — $\varnothing$, by definition, has no elements. For there to exist a proper subset of $\varnothing$, there would need to be a set with fewer elements. There cannot be fewer than zero elements, hence there is no proper subset of $\varnothing$.

Answer (Ex. 2) — Yes. It's obvious that $\varnothing \subset A$ if $A \neq \varnothing$. The only thing left is, $\varnothing \overset{?}{\subseteq} \varnothing$. $\varnothing \subseteq \varnothing$ allows for $\varnothing = \varnothing$, which is obviously true.

3.3 Combining sets together

Okay, hopefully those exercises weren't too bad (email me at `peter@harpending.org` if they were too hard). Next, we're going to talk about ways to combine two sets together. There are any number of ways to combine two sets. The two most common ways are through *unions* and *intersections*.

The union symbol is pretty easy to memorize — it looks like a giant U: $\cup$. Think "$\cup$nion".[2].

If A and B are sets, then $A \cup B$ is the set of elements that are in either A or in B. That is:

$$A \cup B = \{\, x;\ (\, x \in A \,) \vee (\, x \in B \,) \,\}$$

This informal notation is called a "set comprehension". It's technically incorrect, but it works for intuitive purposes. That above is the set of all x such that $x \in A$ logical-or $x \in B$

You might also remember this by the fact that the union symbol $\cup$ looks vaguely like the or symbol $\vee$.

The intersection is, you guessed it - what happens when we use $\wedge$ in the above definition instead of $\vee$. Can you guess what the symbol for "intersection" is? If you guessed $\cap$, then you are right!

$$A \cap B = \{\, x;\ (\, x \in A \,) \wedge (\, x \in B \,) \,\}$$

The standard example is to use "Venn diagrams".

FIXME: Add venn diagrams explaining $\cup$ and $\cap$.

I should remind you of the fundamental rule of algebra - what you do to one side of $=$, you need to do to the other.

[2]You don't have to remember it this way, because $\cup$ is already aligned reasonably well with the letters. Thus I didn't have to spend 30 minutes getting the alignment correct. (See footnote 1.)

3.3.1 Exercises

Ex. 3 — Given that

$$A \wedge (B \vee C) = (A \wedge B) \vee (A \wedge C)$$

Show that

$$A \vee (B \wedge C) = (A \vee B) \wedge (A \vee C)$$

Use these results to figure out

$$A \cap (B \cup C)$$

$$A \cup (B \cap C)$$

Ex. 4 — Is the following true:

$$A \cup \varnothing = A; \ \forall A \in \mathbf{Set}$$

Ex. 5 — What about the following:

$$A \cap \varnothing = \varnothing; \ \forall A \in \mathbf{Set}$$

Answers

Answer (Ex. 3) — Let's do this.

$$
\begin{aligned}
A \wedge (B \vee C) &= (A \wedge B) \vee (A \wedge C) \\
\text{Let } P &= B \vee C \\
Q &= A \wedge B \\
R &= A \wedge C \\
A \wedge P &= Q \vee R \\
\text{Use the definition of } \neg & \\
\neg (A \wedge P) &= \neg (Q \vee R) \\
\neg A \vee \neg P &= \neg Q \wedge \neg R \\
\text{Expand out the definitions of } P, Q, \text{ and } R. & \\
\neg A \vee \neg (B \vee C) &= \neg (A \wedge B) \wedge \neg (A \wedge C) \\
\neg A \vee (\neg B \wedge \neg C) &= (\neg A \vee \neg B) \wedge (\neg A \vee \neg C) \\
\text{Let } X &= \neg A \\
Y &= \neg B \\
Z &= \neg C \\
X \vee (Y \wedge Z) &= (X \vee Y) \wedge (X \vee Z)
\end{aligned}
$$

Okay, we proved the first thing (yes, this is a proof).
Now, on to the other thing:

$$A \cap (B \cup C)$$

Let's write this out in comprehension notation:

$$A \cap (B \cup C) = \{x;\ (x \in A) \vee (x \in \{y;\ (y \in B) \vee (y \in C)\})\}$$

That probably looks horrifying to you. Trust me, it's just as bad typed out:

```
\begin{displaymath}
  A \cap \parens{B \cup C} =
  \mset{
    x \semic
    \parens{
      x \in A
    } \land
    \parens{
      x \in \mset {
        y \semic
        \parens{
          y \in B
        } \lor
        \parens{
          y \in C
        }
      }
    }
  }
\end{displaymath}
```

We can simplify it a bit

$$A \cap (B \cup C) = \{ x;\ (x \in A) \land ((x \in B) \lor (x \in C)) \}$$

That's marginally less horrible
So we have some $\land$s and $\lor$s hanging around. Let's use that rule from above!

$$A \cap (B \cup C) = \{ x;\ ((x \in A) \land (x \in B)) \lor ((x \in A) \land (x \in C)) \}$$

But notice what we've done:

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

That result is totally worth 2 pages of math.

Answer (Ex. 4) — Yes - the union is the set of elements in either set. Since $\varnothing$ has no elements, $A \cup \varnothing$ would just be A, no matter what A is.

Answer (Ex. 5) — Yes - the intersection of two sets is the set of objects in *both* sets. Because $\varnothing$ has no elements, there wouldn't be any objects in both sets — the intersection would have no elements, which would make the intersection equal to $\varnothing$.

3.4 Special kinds of sets

Hopefully those last two exercises sparked a little lightbulb in your brain. If not, I'll try to lay it out for you:

$$A \cup \varnothing = A;\ \forall A \in \mathbf{Set}$$
$$A \cap \varnothing = \varnothing;\ \forall A \in \mathbf{Set}$$

Let's look at the set $\mathbb{Z}$ the set of integers:

$$\mathbb{Z} = \{ \ldots \text{‘}6, \text{‘}5, \text{‘}4, \text{‘}3, \text{‘}2, \text{‘}1, 0, 1, 2, 3, 4, 5, 6 \ldots \}$$

Most people use $-$ instead of ‘ to indicate that something is negative. However, that becomes confusing, because $-$ is also used for subtraction (which requires two numbers).

$$a + 0 = a;\ \forall a \in \mathbb{Z}$$
$$a \times 0 = 0;\ \forall a \in \mathbb{Z}$$
$$A \cup \varnothing = A;\ \forall A \in \mathbf{Set}$$
$$A \cap \varnothing = \varnothing;\ \forall A \in \mathbf{Set}$$

Don't those look really similar? It's aggravating, isn't it - there's some property they both share, but you can't articulate it! Luckily, you're a mathematician, so you can abstract things.

Fortunately for you, someone else recognized this pattern a long time ago, so they abstracted it away for you. This is called a *semirng* - specifically, a *commutative semirng*. Semirng is a joke term originated from "semiring" - it's "semiring" without the 'i' - for "identity".

Namely there's no fixed element $q : \mathbf{Set}$ for which

$$A \cap q = A;\ \forall A : Set$$

(otherwise it would be a semiring)

Okay, what the hell are you talking about?

Mathematicians have invented all sorts of structures for these things which share properties. All of the structures are slightly different. I'll start with **Set** and $\cup$, and compare it to $\mathbb{Z}$ and $+$:

1. First, we have to define $\cup$. It takes 2 sets, and spits out another set

$$\begin{aligned} \cup &: (\,\mathbf{Set}, \mathbf{Set}\,) \to \mathbf{Set} \\ + &: (\,\mathbb{Z}, \mathbb{Z}\,) \to \mathbb{Z} \end{aligned}$$

   This properties is called "closure", as in, "it's closed". Sets can't use $\cup$ to escape **Set**! The same is true with $\mathbb{Z}$.

2. We know that $\cup$ is "associative", which means that you can group operations weirdly.

$$\begin{aligned} A \cup (\,B \cup C\,) = (\,A \cup B\,) \cup C \quad &; \ \forall A, B, C : \mathbf{Set} \\ a + (\,b + c\,) = (\,a + b\,) + c \quad &; \ \forall a, b, c \in \mathbb{Z} \end{aligned}$$

   For this reason, we can write $a + b + c + d + e$ without ambiguity.

3. We also know that $\cup$ is "commutative", which means order doesn't matter:

$$\begin{aligned} A \cup B = B \cup A \quad &; \ \forall A, B : \mathbf{Set} \\ a + b = b + a \quad &; \ \forall a, b \in \mathbb{Z} \end{aligned}$$

   Think of "commutative" as meaning "it commutes", where "commutes" means "moves".

4. We also know that $\cup$ has an "identity" - meaning that there is an element in **Set** where $\cup$ doesn't have any effect:

$$\begin{aligned} A \cup \varnothing = A \quad &; \ \forall A \in \mathbf{Set} \\ a + 0 = a \quad &; \ \forall a \in \mathbb{Z} \end{aligned}$$

These qualities make $(\mathbb{Z}, +)$ and $(\mathbf{Set}, \cup)$ *commutative monoids*. It's one of those fancy structures mathematicians invented. Nobody can remember the names for each one. Just Google it if you don't know. I personally use this chart from Wikipedia: fig. 3.1.

Group-like structures. The entries say whether the property is *required*.

	Totality*	Associativity	Identity	Divisibility	Commutativity
Semicategory	No	Yes	No	No	No
Category	No	Yes	Yes	No	No
Groupoid	No	Yes	Yes	Yes	No
Magma	Yes	No	No	No	No
Quasigroup	Yes	No	No	Yes	No
Loop	Yes	No	Yes	Yes	No
Semigroup	Yes	Yes	No	No	No
Monoid	Yes	Yes	Yes	No	No
Group	Yes	Yes	Yes	Yes	No
Abelian Group	Yes	Yes	Yes	Yes	Yes

*Closure, which is used in many sources, is an equivalent axiom to totality, though defined differently.

Figure 3.1: I have to use this table every time. Source: [1] .

That chart doesn't even cover all of them. Anyway, we have our commutative monoids. I can't really go over much more without also talking about functions. So, the next chapter focuses on functions. We'll revisit this in Chapter 5.

Chapter 4

Functions

As promised, this chapter discusses functions.

So, what is a function?

So far, we've been dealing with *values* - like 2, $\{3, 2, 5\}$, and 90. They are static. Static things are fine, but they aren't very interesting. It's much more interesting to examine *changing things* — more specifically, things that change *predictably* and *transparently*.

Enter the *function*. It's a mathematical construct. A function takes some input, and maps it to an output. Functions are sometimes referred to as *mappings* or *morphisms*.

Let's look at a simple function, which takes a number and adds 2 to it

$$
\begin{array}{l}
f : \mathbb{Z} \to \mathbb{Z} \\
f = \lambda\,(\,x\,) \to x + 2
\end{array}
$$

Pretty simple, right? Okay, so what happens when we send 28 to f?

$$
\begin{aligned}
f\,(\,x = 28\,) &= \lambda\,(\,x = 28\,) \to 28 + 2 \\
&= 30
\end{aligned}
$$

Alternatively, since it's obvious we're working with x:

$$\begin{aligned} f(28) &= 28+2 \\ &= 30 \end{aligned}$$

Using the lambda — λ — is common when I am using a function without giving it a name. However, usually I will use this notation:

$$\begin{aligned} &f : \mathbb{Z} \to \mathbb{Z} \\ &f(x) = x+2 \end{aligned}$$

The whole $f : \mathbb{Z} \to \mathbb{Z}$ thing should be pretty obvious. If not, it means that f is a function that takes a member of $\mathbb{Z}$ (the whole numbers, both negative and positive), and takes it to another member of $\mathbb{Z}$. Other people might use the notation

$$\mathbb{Z} \xrightarrow{f} \mathbb{Z}$$

That notation is undoubtedly easier to understand. However, as we'll see, that notation quickly becomes unfeasible.

4.0.1 Functions with multiple arguments

Remember my explanation of vectors earlier? If not, vectors are like sets, but order and repetition matter.

Here's a function that takes two arguments, and adds them to each other.

$$\begin{aligned} &f : (\mathbb{Z}, \mathbb{Z}) \to \mathbb{Z} \\ &f(x, y) = x+y \end{aligned}$$

Pretty easy to understand, right?

If you haven't figured it out from the context, the inputs to the function are called the *arguments*.

Here's a similar function that takes three arguments and adds them to each other

$$
\begin{aligned}
&f : (\,\mathbb{Z}, \mathbb{Z}, \mathbb{Z}\,) \to \mathbb{Z} \\
&f\,(\,x, y, z\,) = x + y + z
\end{aligned}
$$

You can name your function anything you want, same with the arguments (it doesn't have to be f). It's just a common convention, which you don't have to follow.

What if I want to add a bunch of things together?

Good idea!

$$
\begin{aligned}
&f : (\,\mathbb{Z}, \mathbb{Z}, \ldots, \mathbb{Z}\,) \to \mathbb{Z} \\
&f\,(\,x_1, x_2, x_3, \ldots, x_n\,) = x_1 + x_2 + x_3 + \cdots + x_n
\end{aligned}
$$

That however isn't ideal, because we have no guarantee that the arguments in the ... are actually integers. How about we have a *set* of integers, and we just take the sum? This has the added benefit of less typing

$$
\begin{aligned}
&f : \mathbf{Set}\,(\,\mathbb{Z}\,) \to \mathbb{Z} \\
&f\,(\,s\,) = \sum s
\end{aligned}
$$

So,

$$
\begin{aligned}
f\,(\,\{\,1, 2, 3, 4, 5\,\}\,) &= \sum \{\,1, 2, 3, 4, 5\,\} \\
&= 1 + 2 + 3 + 4 + 5 \\
&= 15
\end{aligned}
$$

4.0.2 Eta-reductions

Mathematicians like to make themselves look smart. One such way is to invent fancy terms for simple things. One such term is the η-reduction.

Let's look at that function we just had

$$f : \mathbf{Set}(\mathbb{Z}) \to \mathbb{Z}$$
$$f(s) = \sum s$$

Notice that we are repeating s on both sides of the equation. It would seem much simpler, and just as clear, to write:

$$f : \mathbf{Set}(\mathbb{Z}) \to \mathbb{Z}$$
$$f = \sum$$

That's all an η-reduction is: if you see an extraneous argument, you remove it to make things simpler. As long as we have the signature — the $f : \mathbf{Set}(\mathbb{Z}) \to \mathbb{Z}$ thing – it's pretty clear what f does. This is a prime example of mathematicians being both lazy and pretentious at the same time: a practice designed to allow us to be lazier, to which mathematicians have assigned a ridiculous name to make it sound hard.

What the hell is η?

η is the Greek letter eta; it's pronounced "eight-uh".

The ancient Greeks were too dumb to comprehend the concept of "eight". Every time someone brought it up, they said "uh" immediately thereafter. The sound "eight-uh" became so common that they decided to make it a letter.

The Greeks' poor comprehension of simple mathematics remains to this day, and is largely the reason for their current financial crisis.[4]

If you ever take a physics course, you will undoubtedly notice that Greek letters are used frequently in physics. This is the physicists way of subtly hinting that they actually have no idea what they are talking about, and pleading for help from the mathematicians.

4.1 Currying

We sort of got side-tracked by toying around with sets and making fun of physicists. Hopefully that introduction introduced you to the basic concept of a function, and let you know that they can take multiple arguments

Let's look at that function again:

$$\begin{aligned} &f : (\,\mathbb{Z}, \mathbb{Z}, \mathbb{Z}\,) \to \mathbb{Z} \\ &f\,(\,x, y, z\,) = x + y + z \end{aligned}$$

What if you wanted to bind $x = 3$, but leave the rest "free"?

$$\begin{aligned} &f : (\,\mathbb{Z}, \mathbb{Z}, \mathbb{Z}\,) \to \mathbb{Z} \\ &f\,(\,x = 3, y, z\,) = 3 + y + z \end{aligned}$$

Okay, cool. We now have another function:

$$\begin{aligned} &f\,(\,3\,) : (\,\mathbb{Z}, \mathbb{Z}\,) \to \mathbb{Z} \\ &f\,(\,3, y, z\,) = 3 + y + z \end{aligned}$$

So, actually, instead of needing 3 integers to do its job, f only needed one. However, instead of spitting out another integer, it spit out a function. So, we could write f's signature as:

$$\begin{aligned} &f : \mathbb{Z} \to (\,(\,\mathbb{Z}, \mathbb{Z}\,) \to \mathbb{Z}\,) \\ &f\,(\,x, y, z\,) = x + y + z \end{aligned}$$

Okay, that's sort of weird and unintuitive. Let's try writing f differently:

$$
\begin{aligned}
&f : \mathbb{Z} \to ((\mathbb{Z}, \mathbb{Z}) \to \mathbb{Z}) \\
&f = \lambda (x) \to (\lambda (y, z) \to x + y + z)
\end{aligned}
$$

Let's look at the second half of that:

$$\lambda (y, z) \to x + y + z : (\mathbb{Z}, \mathbb{Z}) \to \mathbb{Z}$$

(This assumes that we know what x is)

Let's try splitting this up again:

$$\lambda (y) \to (\lambda (z) \to x + y + z) : \mathbb{Z} \to (\mathbb{Z} \to \mathbb{Z})$$

You give this function a value for y, and instead of giving you a value, it gives you another function, hence the signature $\mathbb{Z} \to (\mathbb{Z} \to \mathbb{Z})$.

Let's plug this back into f:

$$
\begin{aligned}
&f : \mathbb{Z} \to (\mathbb{Z} \to (\mathbb{Z} \to \mathbb{Z})) \\
&f = \lambda (x) \to (\lambda (y) \to (\lambda (z) \to x + y + z))
\end{aligned}
$$

So, instead of f taking three integers, it now only takes one, but spits out a function, which in turn spits out a function, which spits out an integer.

This idea of making a function into a chain of functions is called "Currying".[2] It's named after a dead mathematician named Haskell Curry (ca. 1900-1982), who developed the technique. The programming language Haskell is also named after Mr. Curry.

Getting back to that function, those parentheses are somewhat burdensome, let's get rid of them

$$\begin{aligned} f &: \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z} \\ f &= \lambda\,(\,x\,) \to \lambda\,(\,y\,) \to \lambda\,(\,z\,) \to x + y + z \\ f\,(\,x, y, z\,) &= x + y + z \end{aligned}$$

That's much easier to read. It should be understood that the parentheses are right-associative: the parentheses "associate" rightward — i.e. it's $a \to (\,b \to (\,c \to d\,)\,)$, not $(\,(\,a \to b\,) \to c\,) \to d$.[8]

That's Currying for you.

4.2 Vocabulary

This is important vocabulary that I'm going to use later:

1. All functions are *transparent* — $a = b \implies f\,(\,a\,) = f\,(\,b\,)$
2. If $f : A \to B$, then A is the *domain* of f and B is the *codomain* of f.
3. If $f : A \to B$, and there are no two distinct elements of A that map to the same thing in B, then f is *injective*.

$$\begin{array}{l} f : A \to B \\ \nexists\,(\,a, b\,);\ a, b \in A \wedge a \neq b \wedge f\,(\,a\,) = f\,(\,b\,) \iff f \text{ is injective} \end{array}$$

4. If $f : A \to B$, then the elements in B that can be expressed as $f\,(\,x\,);\ x \in A$ form the *image*.

$$\begin{array}{l} f : A \to B \\ \mathbf{im}\,(\,f\,) = \{\,f\,(\,x\,) \in B;\ x \in A\,\} \end{array}$$

5. If the image of a function is equal to its codomain, then the function is *surjective*.

$$
\begin{aligned}
&f : A \to B \\
&B = \{\, f\,(\,x\,) \in B;\; x \in A \,\} \iff f \text{ is surjective}
\end{aligned}
$$

6. If a function is both injective and surjective, then it is *bijective*.

7. Some functions have *inverses*. That is, if

$$f : A \to B$$

$$
\begin{aligned}
&\mathbf{arc}\,(\,f\,) : B \to A \\
&\mathbf{arc}\,(\,f, x\,) = x;\; \forall x \in A
\end{aligned}
$$

   Remember that, because of currying, $\mathbf{arc}\,(\,f, x\,) = \mathbf{arc}\,(\,f\,)\,(\,x\,)$. That is:

$$
\begin{array}{rcl}
f & : & A \to B \\
\mathbf{arc} & : & (\,A \to B\,) \to B \to A \\
\mathbf{arc}\,(\,f\,) & : & B \to A
\end{array}
$$

   If a function has an inverse, it is said to be *invertible*.

8. If a function is invertible, then the image of the inverse is called the *preimage*.

4.2.1 Exercises

Ex. 6 — I knew you were going to just gloss over those, so I made a really hard (i.e. fun) problem: prove that a function is invertible if (and only if) it is bijective. This is a very difficult proof, but you really need to understand it.

Answers

Answer (Ex. 6) — Let's look at $f : A \to B$. If f is surjective, then $B = \mathbf{im}\,(\,f\,)$, so we can write

$$f : A \to \mathbf{im}\,(\,f\,)$$

In other words

$$f : \mathbf{dom}\,(\,f\,) \to \mathbf{im}\,(\,f\,)$$

It must be true that f is a surjection for f to be invertible. Else there would be elements in the codomain of f that were not in the domain of $\mathbf{arc}\,(\,f\,)$. We've established

$$\mathbf{arc}\,(\,f\,) : \mathbf{im}\,(\,f\,) \to \mathbf{dom}\,(\,f\,)$$

Let's assume f is invertible. Then $\mathbf{dom}\,(\,f\,) = \mathbf{preim}\,(\,f\,)$. Thus

$$\mathbf{arc}\,(\,f\,) : \mathbf{im}\,(\,f\,) \to \mathbf{dom}\,(\,f\,)$$

For $\mathbf{arc}\,(\,f\,)$ to be a function — i.e. for f to be invertible, then it must be true that

$$\nexists\, a, b \in \mathbf{im}\,(\,f\,)\,;\; a = b;\; \mathbf{arc}\,(\,f, a\,) \neq \mathbf{arc}\,(\,f, b\,)$$

If we flip this around

$$\nexists\, a, b \in \mathbf{preim}\,(\,f\,)\,;\; a \neq b;\; f\,(\,a\,) = f\,(\,b\,)$$

That is, the definition of injectivity. Thus we have proven

$$f \text{ is invertible } \iff (\,f \text{ is an injection}\,) \wedge (\,f \text{ is a surjection}\,) \iff f \text{ is a bijection}$$

Chapter 5

Categories

I'm briefly going to go over the history of set theory, bringing us up to date with modern research. We're then going to discuss the notion of a *category*, and explain how they can be used to model other parts of math.

5.1 Technically incorrect

??

Remember when I said that the set comprehension was "technically incorrect"? Well, the reason is Russell's paradox.[9, 5] There's a colloquial version of Russell's paradox called the "Barber paradox", and it is as follows:

> There's a barber in a small town - he cuts everybody's hair, but only if a given person person doesn't cut his own hair. Does the barber cut is own hair?

If the answer is "yes", we run into a problem – if the barber cuts his own hair, he doesn't cut his own hair, because he only cuts someone's hair if he don't cut his own hair.

If the answer is "no", then we run into the same problem – the barber cuts someone's hair if he doesn't cut it themselves. Since the barber doesn't cut his own hair, he therefore cuts his own hair.

Let's put this in mathematical notation

$$\text{People whose hair barber cuts} = \{\, \text{Person} \in \text{Town};\ \text{Person doesn't cut his own hair} \,\}$$

The question is

$$\text{Barber} \overset{?}{\in} \text{People whose hair barber cuts}$$

The paradox is

$$\text{Barber} \in \text{People whose hair barber cuts} \iff \text{Barber} \notin \text{People whose hair barber cuts}$$

The version with sets is:

$$\begin{array}{l} \text{let } \mathbf{Set} = \{\, x;\ x \text{ is a set} \,\} \\ \text{let } A = \{\, x \in \mathbf{Set};\ x \notin x \,\} \\ A \overset{?}{\in} A \end{array}$$

That is, A is the set of all sets x such that x is not an element of itself. Is A an element of itself?

If A is an element of itself, then A is not an element of itself, because A is the set of all sets in which the set is not an element of itself.

If A is not an element of itself, then A must be an element of itself, because A is the set of all sets for which that assertion holds true.

That is:

$$A \in A \iff A \notin A$$

This paradox was first proposed by a mathematician-stroke-philosopher named Bertrand Russell, in the early 20^{th} century.

Well, sets are still around in 2015, so someone must have solved this paradox, right? Indeed!

The solution is actually quite simple - comprehensions don't necessarily form *sets*, they instead form *classes*. Classes don't have things like $\in$ or $\subset$. Some classes are sets. Classes that are not sets are called *proper classes*.

In order for a class to be a set, the class must draw elements from an ambient set.

$$\{ x \in Z;\ x \geq 3 \}$$

I sort of spoiled the ending for you. A much more riveting explanation of this can be found in [9].

Axiomizing equality

Before we can go any further, we have to make some rules about the = sign. I mentioned these in **??**.

1. $a = a;\ \forall a$. This is called the *reflexive property*. "Reflective property" would be a better name, because if you put a in front of the = mirror, a is there.
2. $a = b \implies b = a;\ \forall a, b$. This is called the *commutative property*. "Commute" means "move", so the "commutative property" is the property of moving things around.
3. $(a = b) \wedge (b = c) \implies a = c;\ \forall a, b, c$. This is called the *transitive property*. "Transition" is what happens when things change. so "transitive" property is what happens when you change the variable name.

Good? You don't have to remember the names of the properties, but you do have to remember what the properties are.

Appendices

Appendix A

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Appendix B

How to learn math

Now that that's all out of the way, let's talk a little about math. When this chapter is over, we're going to dive right in to proving a bunch of things you already know to be true. We feel that without a little explanation, these proofs may leave you a little lost or confused. We'll save the explanation of the proofs for later, but right now, we're going to talk about how to actually learn math, and the proofs are a great example.

Most people's experience with math is through their primary and possibly secondary education, which is or was a dreary affair in general, and math probably even moreso, unless you're one of the lucky few. By lucky few, we don't mean those wizards with a sort of inherent ability to do math–the first thing you need to know about learning math is that math is for everyone with a brain–that's you, right? You see, your brain is a pattern recognition engine, and that's all math is: the study of patterns. Unlike reading or history, your body comes with a biological imperative to know math. There's some really great brain studies on the topic, but that's boring, and I said we're already done with the boring part, so let's move on.

In that last paragraph, we presented what we hold to be the proper answer to 'what is math': the study of patterns. This is completely different from most people's interaction with math: in primary school, we are taught how to apply four operations to solve math problems. You're given something about two trains leaving a station and going different speeds and different directions and yadda yadda yadda and before you know it your teacher turned everything into a math problem and it all seemed so forced–a layer on top of what was intuitive, and made everything complicated. We agree–this is a counterintuitive approach to math, and it makes math very confusing and disconnected. Math is just the study of patterns. That is, math is not so much a way to solve a set of problems that exist in a sphere apart from what is natural, but a way to understand what's going on in the world around us. When you learn math, you should think of it as a science–another level of detail in the amazing world we live in.

That's how this book is written. It's written to reflect that math is a single unified study. While you're reading it, try to think of how what you're learning clarifies or refines early material. This is a big deal to us, because one thing we dislike most about the standard way of learning math is that at some point in everyone's math career, they learn they were taught something that wasn't actually true. We want to avoid that.

Appendix C

Philosophy and/or FAQ

by Peter Harpending <peter@harpending.org>

This book is written with a certain philosophy in mind. Explaining my philosophy will answer a number of questions I am often asked.

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You would think that this freedom would be implicit in any academic work. Unfortunately, that's not the case.

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Anyway, the point of all this is, freedom is important, especially in academic works. Part of the reason I wrote this book is that there are very few free textbooks.

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