Learn You Some Algebras for Glorious Good!

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December 30, 2014

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Contents

Chapter 1

Introduction

Before I bore you with a bunch of crap you don't care about, let's do some math, shall we?

There are basically three notions with which you need to be familiar in order to do anything interesting in math. Those three things are *sets*, *functions*, and *proofs*. Unfortunately, to be familiar with one, you have to be familiar with the other two.[1]

So, what are each of those things?

- A *set* is an unordered collection of things. There is also no repetition. For instance, $\{2, 5\}$ is the same as $\{5, 2\}$ (because order doesn't matter). $\{2, 5, 5\}$ would be the same set, because there's no notion of multiplicity.

- A *function* is a mathematical construct (well, obviously, else I wouldn't be talking about it). Basically, it takes some input, does something to it, and spits out some output. If you give the function the same input a bunch of times, you should get the same result each time. This concept is called "referential transparency." If the function is not referentially transparent, then it's not a function. It's something else.

[1] You'll learn as we go along, when math people use a common term like *set*, *function*, *proof*, *group*, *continuous* or *closed*, they usually mean something similar in concept to the colloquial term, but there are some strings attached. This is usually the case in the sciences too (e.g. *theory*, *hypothesis*, *experiment*).

- A *proof* is basically where you take a bunch of simple facts, called *axioms*, and chain them together to make *theorem*s. It's sort of like sticking puzzle pieces together to form a picture.

  The puzzle pieces (in this case, the axioms) aren't usually very interesting on their own. However, the picture they form (in this case, the theorem) can be really cool and enlightening. The proof would be analogous to an explicit set of instructions explaining how to put the pieces together.

Once you are familiar with each of those concepts, we can do all sorts of cool stuff. Throughout the book, we will prove all of the following:

- If you tap your finger against a bridge at exactly the right frequency, the bridge will collapse. (Resonance)
- The formula used to calculate the interest rate on your mortgage is actually just a fancy form of the ratios of angles in a triangle. (Euler's formula)
- Logic can't be used to prove everything we know to be true. (Gödel's incompleteness theorem)

1.1 Introduction (for real this time)

This is a math book. Well, duh. Why did I write it?

Most math (and science) books nowadays seem to value keeping an academic tone over ensuring that the reader understands the material, and — more importantly — enjoys reading the book.

I take the opposite approach. I want to create a book that is fun to read and easy to understand, while eschewing the practice of making myself look good.

The inspiration for this book is *Learn You a Haskell for Great Good!*, by Miran Lipovača. Haskell is a programming language, and LYAH is a great book for learning Haskell. If you are interested in a print copy of LYAH, see [2].

1.2 The community

Despite the fact that I used "I" in the first part of the book, LYSA is actually a community project, and many people participate in the writing of this book.

If you want to talk to us, or to other math people, come see us in `#lysa` on Freenode. If you don't know what IRC is, or you don't have a client set up, you can connect through Freenode's webchat (`http://webchat.freenode.net/?channels=lysa`).

If you have any questions about LYSA (or math), feel free to ask in the IRC channel (`#lysa` on FreeNode in case you forgot).

If you want to submit a correction, or have some issue, or want to add some content, really anything having to do with the content of the book, you can visit our Gitorious page at `https://gitorious.org/lysa/lysa`. We also have a community on Reddit (`https://lysa.reddit.com/`).

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This book is free, in the sense of freedom. You can copy this book and give it to your friend. You can even print it out and sell it to your friends.[2]

If, for instance, you are a schoolteacher and want to use this for your class, you are free to edit it to your liking and give the modified copy to your students.

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[2] There are some restrictions though, see § A.

1.4 Conventions used in this book

Chapter 2

Sets

Anyway, now that that boring introduction is out of the way, we can get to some math! The first thing we are going to cover are *set*s.

Sets were first studied by Georg Cantor, a German mathematician, in the second half of the nineteenth century. Back in his own day, the patterns Cantor found by studying set theory were considered so thoroughly bizarre that many of his colleagues simply refused to believe that Cantor could be right. In the end, Cantor turned out to be right all along. His ideas can be found in any introductory text on mathematics—including this one.

Sets are basically like lists, except there's no duplication, and there's no order. Examples:

1. $\{0, 1, 2, 3\}$ is a set.
2. $\{1, 2, 0, 3\}$ is the same set.
3. $\{1, 1, 2, 0, 3\}$ is not a set, because there's duplication.

To give you some more notation:

1. $0, 1, 2, 3$ **is not** a set, it is instead the numbers 0 through 3, which are to be considered separately.
2. $\{0, 1, 2, 3\}$ **is** indeed a set (notice the braces).

3. If some object is in a set, the notation is 'OBJECT $\in$ SET', which should be read "OBJECT is an element of SET".

4. If the object is not in the set, the notation is 'OBJECT $\notin$ SET', which should be read "OBJECT is not an element of SET".

5. $0 \in \{0, 1, 2, 3\}$ should be read "0 is an element of $\{0, 1, 2, 3\}$", and indeed it is.

6. $0, 1 \in \{0, 1, 2, 3\}$ should be read "0 and 1 **are both** elements of $\{0, 1, 2, 3\}$". They are.

7. $\{0, 1\} \notin \{0, 1, 2, 3\}$ should be read "$\{0, 1\}$ **is not** an element of $\{0, 1, 2, 3\}$". $\{0, 1\}$ is indeed not an element of $\{0, 1, 2, 3\}$.

Now, here, we're faced with an interesting problem. $\{0, 1, 2, 3\}$ encapsulates $\{0, 1\}$, but $\{0, 1\}$ is not an element of $\{0, 1, 2, 3\}$. So, how do we express this notion of 'encapsulation'? The answer is by using "subset" notation.

1. $\{0, 1\} \subseteq \{0, 1, 2, 3\}$is **true**. $\{0, 1\} \subseteq \{0, 1, 2, 3\}$should be read as "$\{0, 1\}$ is an **improper** subset of $\{0, 1, 2, 3\}$."

2. $\{0, 1\} \subset \{0, 1, 2, 3\}$is **true**. $\{0, 1\} \subset \{0, 1, 2, 3\}$should be read as "$\{0, 1\}$ is a **proper** subset of $\{0, 1, 2, 3\}$."

3. $\{0, 1, 2, 3\} \subseteq \{0, 1, 2, 3\}$ is **true**. Intuitively, you should think "$\{0, 1, 2, 3\}$ is entirely encapsulated inside of $\{0, 1, 2, 3\}$."

4. $\{0, 1, 2, 3\} \subset \{0, 1, 2, 3\}$ is **false**. This highlights the difference between $\subset$and $\subseteq$.

   $\{0, 1\} \subset \{0, 1, 2, 3\}$implies that $\{0, 1, 2, 3\}$ contains some stuff that is no$\underline{\text{t}}$ contained in $\{0, 1\}$. $\{0, 1\} \subseteq \{0, 1, 2, 3\}$does not have this implication. That is, $\{0, 1\} \subseteq \{0, 1, 2, 3\}$does not say for sure that $\{0, 1, 2, 3\}$ has more stuff than $\{0, 1\}$, it just acknowledges it as a possibility.

Alright, my apologies, that was a lot of crap to throw at you at once, and you probably remember absolutely none of it. However, you'll get used to that particular *set* of crap[1] as time goes along.

[1] I, for one, would never condone such terrible puns.

Now, I'm going to throw more crap at you. I'm sorry for all the notation, but it is important that you get used to it. There are a bunch of common sets you'll encounter in the real world.

1. First of all, there are the integers: these are just the whole numbers, like $-3, -2, -1, 0, 1, 2$. Traditionally, this set is denoted $\mathbb{Z}$.
2. The natural numbers, denoted $\mathbb{N}$, are all non-negative integers[2].
3. The set of "rational" numbers, denoted $\mathbb{Q}$ is the set of all numbers that can be written as a quotient (or fraction) $\frac{p}{q}$, where p and q are integers.

   In mathematical notation, you would write this as $\left\{\frac{p}{q} \mid p,\ q \in \mathbb{Z}\right\}$. You should read that as "the set of all numbers p over q, such that p and q are elements of $\mathbb{Z}$."
4. Finally, the real numbers, denoted $\mathbb{R}$, contain everything on the number line. Equivalently, a real number is just any number you can write down by writing down an integer, a decimal point, and then writing any sequence of digits after the decimal point (even an infinite sequence).

   Let us show you an example. Let's write down an integer. 2. There we go. Now, let's just write down a bunch of digits all in a row. 215455211. By the definition in the previous paragraph, 2.215455211 is a real number. You should get in the habit of writing "2.215455211 is a real number" as $2.215455211 \in \mathbb{R}$.

Sets certainly look simple enough to understand intuitively, but are they also interesting enough to be worth studying? The answer to that is a resounding 'yes.' Exciting, right?

To give you a taste of the results, let's take a look at some of them.

Intuitively, sets can be of different sizes: it's clear that the empty set is the smallest possible set, and that $\{2, 4\}$ is smaller than $\{1, 2, 3\}$. When we get to infinite sets, however, the intuition fails.

[2]Conventions differ as to what the natural numbers are. Everyone agrees that positive whole numbers (1, 2, 3...) are natural. Some people say 0 is also a natural numbers, while others disagree. In this book, 0 *is* a natural number, so the natural numbers are 0, 1, 2, and so on. This might seem like a stupid fight, why can't we just choose a side? However, it actually turns out to be really important.

Cantor formalised this intuition, and the formal version also worked for infinite sets. The results he got were very surprising: for example, he found that there are as many natural numbers as rational numbers, but that there are strictly more real numbers. Even worse, he proved that given any set, there is always a set larger than it. If you've ever heard of people talking about "different kinds of infinity", this is probably what they meant.

Exercises

1. Suppose x is some object and S is a set. Does it make sense to ask "How many times does S contain x?"

2. Earlier, we constructed the set $\{3, 5\}$. We can also construct the set $\{5, 3\}$. Are these the same set?

3. Are the following true or false?

    - $\varnothing \in \varnothing$
    - $\varnothing \in \{\varnothing\}$
    - $\varnothing = \{\varnothing\}$
    - $\{\varnothing\} \in \{\varnothing\}$

2.1 Functions

A function from a set X to a set Y specifies for every element of X a corresponding element of Y. In other words, if f is a function from X to Y (denoted $f : X \to Y$) and $x \in X$, then $f(x) \in Y$. A lot of high school mathematics is about functions from $\mathbb{R}$ to $\mathbb{R}$, even if they are never presented that way. For example, solving linear equations is actually a question about functions: "given a function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = 5x - 10$$

find all $r \in \mathbb{R}$ such that $f(r) = 0$."

It turns out that by taking sets X and Y and asking what kind of functions exist between them, we can find out a lot about X and Y themselves. We'll continue this line of thought after some exercises.

Exercises

1. We can construct the set $\{0, 1\}$. What functions can you find from $\{0, 1\}$ to $\{0, 1\}$? How many such functions are there?

2. Can you find a function from $\varnothing$ to $\{0, 1\}$? What about in the other direction?

3. Suppose X is a set. Prove that there exists at least one function from X to X. How can this function be defined? Why does this work if X is the empty set?

4. Suppose X, Y, and Z are sets, and you are given functions $f : X \to Y$ and $g : Y \to Z$. Can you make a function from X to Z? How?

5. Suppose X and Y are sets, $f : X \to Y$, and $x, y \in X$. Do you think $x = y$ implies $f(x) = f(y)$? Why, or why not?

Solutions

2.2 Injective Functions

In the exercises, you saw that if $f : X \to Y$ and you had $x, y \in X$ such that $x = y$, then necessarily $f(x) = f(y)$. A function f is said to be injective if the converse is also true: if $x, y \in X$ and $x \neq y$, then $f(x) \neq f(y)$.

Let's consider some examples. Let $f : \mathbb{N} \to \mathbb{Z}$ be defined by $f(x) = x$. Is this definition correct? Well, we need to check that for any $n \in \mathbb{N}$, we have $f(n) \in \mathbb{Z}$. But by definition, $f(n) = n$, so we need to check that if $n \in \mathbb{N}$, then also $n \in \mathbb{Z}$. Every non-negative integer is certainly an integer, so yes, the definition is correct.

Now, is this function injective? For that, let $n, k \in \mathbb{N}$ so that $f(n) = f(k)$. We need to prove that $n = k$. Again, this is easy: $f(n) = n$ and $f(k) = k$, so by putting the three equalities together, we have $n = k$.

We can construct the set $\{f(0), f(1), f(2) \ldots\}$. If we want to be precise, we can use the *set comprehension* notation:

$$\{f(x) \mid x \in \mathbb{N}\}$$

You should read this as "the set that contains $f(x)$ for each $x \in \mathbb{N}$." That's rather a mouthful, so we'll abbreviate this by $f(\mathbb{N})$. Just imagine that instead of applying f to one value in $\mathbb{N}$, you're applying it to all the values and looking at the results as a set.

We can now ask ourselves: how big is $f(\mathbb{N})$? In this case, we can see that $\mathbb{N} = f(\mathbb{N})$, so of course they're exactly the same size. However, what if instead of the f we chose, we had chosen $f(x) = -x$, or $g : \mathbb{N} \to \mathbb{Q}$ defined by $g(x) = \frac{1}{1+x}$? Prove that both of these are injective functions, and imagine $f(\mathbb{N})$ and $g(\mathbb{N})$. How big are these sets?

It should be clear that given any sets X and Y and any function $f : X \to Y$, $f(X)$ cannot possibly be bigger than X. For every element $y \in f(X)$, there is some element $x \in X$ so that $f(x) = y$; that's how we defined $f(X)$ in the first place.

On the other hand, if f is injective, $f(X)$ also shouldn't be any smaller than X. Why? Well, we can just define a function the other way. Let

$g : f(X) \to X$, and define $g(f(x)) = x$. Now we know that $g(f(X))$ is no larger than $f(X)$. But $g(f(X)) = \{g(f(x)) \,|\, x \in X\} = \{x \,|\, x \in X\} = X$! That means, by the argument above, that $f(X)$ is no larger than X, just as we wanted.

The careful reader might notice that the definition of g is fishy. Does a function defined this way really work? The answer is that it does, but only because we were careful about f. We know that if we have an element $y \in f(X)$, then there is an element $x \in X$ so that $y = f(x)$. This means that g really is defined for all $y \in f(X)$. That's not all, though. Additionally, suppose there was also a $z \in X$ so that $y = f(z)$. Then $g(y) = g(f(x)) = x$, but also $g(y) = g(f(z)) = z$. For g to be a function, we need to prove that $x = z$. Here is where we need the injectivity of f: we know that since $f(x) = y = f(z)$, that $f(x) = f(z)$, and thus $x = z$. Therefore, g really is a function, and so our argument holds.

Our intuition thus tells us that however we define size, we should make sure that for any set X and any injective function $f : X \to Y$, we should have X and $f(X)$ be of equal size. As $f(X)$ is just a part of Y, this means that X should also be no bigger than Y. By themselves, all these loose bits of intuition might not seem very useful, but in the next section we'll use them to show that a certain approach to define size *doesn't* work. First, however, some exercises.

Exercises

1. In an older exercise, you saw there were four functions from $\{0, 1\}$ to itself. How many injective functions are there? What about between $\{0, 1, 2\}$ and itself? What if you have even more elements?

2.3 Subsets

When people first hear that there are as many integers as natural numbers, their reaction is often "Surely that can't be right? Every natural number is an integer, and there are some others, too!"

To make this argument formal we should introduce the notion of a subset. We say that X is a subset of Y if every element of X is also an element of Y. We denote this by $X \subseteq Y$. We have, then $\varnothing \subseteq \mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$. We will use $X \subset Y$ when X is a subset of Y, but X and Y are not equal.

Now let's denote the size of X by $|X|$. Can we define this size in such a way that for finite sets, $|X|$ is the number of elements in X, that for any sets X and Y such that $X \subset Y$ we have $|X| < |Y|$, and that if there exists an injective function $f : X - > Y$ then $|X| = |f(X)|$?

We've already given away that this isn't possible. The question becomes, how do we prove it? It isn't enough to just try a few definitions of 'size' and see that they don't work, as there'll always be a few that we haven't yet attempted. We need a way of showing that whatever definition we come up with, we'll be able to show it doesn't work.

To do that, we'll assume that we were given some definition of 'size' that satisfies the above requirements, but with no other assumptions. You can imagine it as being given the definition by a friend, and now you have to show why it's wrong. If the definition allows us to derive something that isn't true then there's certainly something wrong with it, so that's what we'll try to do.

Assume that for every set X, we defined $|X|$. Define $f : \mathbb{Z} \to \mathbb{Z}$ by $f(x) = 2x$. We required that $|\mathbb{Z}| = |f(\mathbb{Z})|$. But because $1 \notin f(\mathbb{Z})$, $f(\mathbb{Z}) \subset \mathbb{Z}$. This implies $|f(\mathbb{Z})| < |\mathbb{Z}|$, and so we conclude $|\mathbb{Z}| = |f(\mathbb{Z})| < |\mathbb{Z}|$. Surely, though, a set can't be smaller than itself, so we've just derived a falsehood, and the definition of 'size' that we started with can't work.

If this proof doesn't feel quite right to you, don't worry: we'll look at arguments like this in-depth in the next chapter. For now, the main result is that we are too strict in how we want to define size. We need to relax one of the requirements, and in the next section we'll see that if we remove the requirement that $X \subset Y$ implies $|X| < |Y|$, we can find a good definition.

Exercises

In these exercises we'll explore some more operations for constructing sets. The three most important ones are *union*, *intersection*, and *difference*.

The union of X and Y is written as $X \cup Y$ and contains every element of X and every element of Y. We can write this using set comprehension notation:

$$X \cup Y = \{x \mid x \in X \text{ or } x \in Y\}$$

1. Write out the following sets:
   (a) $\{0, 1\} \cup \{2, 3\}$
   (b) $\{0, 1\} \cup \{1, 2\}$
   (c) $\{0, 1\} \cup \{0, 1\}$
   (d) $\{0, 1\} \cup \varnothing$
2. We say that a set A is the identity of $\cup$ if for every set X we have $X \cup A = X = A \cup X$. Does $\cup$ have an identity?
3. Suppose X and Y are sets. Convince yourself that $X \cup Y = Y \cup X$. This property of $\cup$ is called *commutativity*.
4. Suppose X, Y, and Z are sets. Convince yourself that $(X \cup Y) = Y \cup X$. This property is called *associativity*.

The intersection of X and Y is written as $X \cap Y$ and contains all elements that are both in X and Y.

1. Do the first exercise on unions again, but replace $\cup$ with $\cap$.
2. Let X and Y be sets. Write $X \cap Y$ as a set comprehension.
3. Does $\cap$ have an identity?
4. Is $\cap$ commutative and associative, just like $\cup$ is?

The difference between X and Y is written as $X - Y$ (or, sometimes, $X\ Y$) and contains all elements that are in X but are not in Y.

1. Do the first exercise on unions a third time, but now taking the difference.
2. Does set difference have an identity?
3. Is set difference commutative? What about associative?

2.4 Cardinalities

2.5 The Power Set

2.6 The Category of Sets

2.7 Special kinds of sets

2.7.1 Magmas

2.7.2 Semigroups

2.7.3 Monoids

Chapter 3

Proofs

3.1 Peano Axioms

This section covers the Peano axioms. As I said in **??**, these are a way for mathematicians to understand arithmetic.

Arithmetic (hopefully) seems simple enough, and easy to understand. Maybe an expression like

$$(2048282 \times 33221) + (3254 \times 11)$$

seems difficult to calculate, but you hopefully understand what each of the operators mean in concept. If you don't, well. In theory, reading this chapter alone will teach you arithmetic. However, I wrote this chapter assuming you already know arithmetic.

So, why is it important that you read this chapter?

Arithmetic is pretty simple and easy to understand. However, later on in this book, we're going to approach concepts that aren't so simple and easy to understand. Mathematicians have a systemic approach to these problems. This approach is called "mathematical proof." We prove things mathematically. Instead of approaching new concepts with proofs, I'm instead going to use proofs to illustrate some (hopefully) familiar concepts.

Alright, with all that out of the way, let's get started.

The basic idea of proofs is, you take a small set of obvious facts, called *axioms*, chain them together to make *theorems*. The following obvious facts, or axioms, are called the "Peano Axioms." They describe what we call "natural numbers." Natural numbers are the numbers $\{0, 1, 2, 3, 4, 5, 6, \ldots\}$.

Axiom 1 0 is a natural number. Again, obvious.

I'm going to use letters in the place of numbers right here. So, if I say "x is a natural number," that means that x is a placeholder for one of the numbers in $\{0, 1, 2, 3, 4, 5, 6, \ldots\}$. I could use any letter, such as a, b, q, r, θ, Γ, or $\aleph$. If I use a letter instead of a number, it usually means either

1. it doesn't matter which number I choose, or
2. it does matter which number I choose, but I don't know which number it is yet.

Axiom 2 If x is a natural number, it is true that $x \equiv x$.

You can read that $\equiv$ sign as $=$, for the time being. There are some subtle differences between the two, which I will get to in **??**. You are supposed to read $a \equiv b$ as one of these:

1. "a is equivalent to b,"
2. "a is identically equivalent to b,"
3. "a is congruent to b."

$=$ should be read as "a is equal to b," or "a equals b." Again the difference between $\equiv$ and $=$ isn't really important until **??**.

If you don't know what either of those signs are, $a \equiv b$ or $a = b$ means "a is the same thing as b." The difference can be summarized as $= = \equiv \neq =$.

So, in essence, this axiom says that each number is the same thing as itself. This is hopefully very obvious.

A math person would state this axiom as "congruence is reflexive."

Axiom 3 If x and y are both natural numbers, and $x \equiv y$, then it's true that $y \equiv x$. You can phrase this axiom as "if two numbers are the same number, then they are the same number." A math person would state this axiom as "congruence is symmetric."

Axiom 4 If x, y, and z are all natural numbers, and $x \equiv y$, and $y \equiv z$ then it's true that $x \equiv z$. You can phrase this axiom as "if three numbers are all the same number, then they are the same number." A math person would state this axiom as "congruence is transitive."

These last three axioms mean that we can be lazy, and write things like $a \equiv b \equiv c \equiv a$.

Axiom 5 If x is a natural number, and we know $x \equiv y$, then it's also true that y is a natural number. A math person would say "congruence forms a closure."

Axiom 6 If x is a natural number, then there is another number, $\mathrm{suc}(x)$, which is also a natural number. suc is short for "successor." You should read $\mathrm{suc}(x)$ as "the successor of x." You can think of the successor as "the next number." So, $\mathrm{suc}(0) \equiv 1$, $\mathrm{suc}(1) \equiv 2$, and so on.

Axiom 7 There isn't a number whose successor is 0. Basically this means "0 is the lowest natural number."

Axiom 8 If x and y are both natural numbers, and we know $\mathrm{suc}(x) \equiv \mathrm{suc}(y)$, then it's true that $x \equiv y$. This is what we would call the "converse" of Axiom 6. That is, Axiom 6 tells us that we can always go "up" a number. This axiom (almost) tells us that we can go "down" a number. Axiom 7 defines the limit of this, meaning that 0 is the only number where you can't go down any further.

Now, the previous 8 axioms have basically said "these numbers are all natural numbers." This next, and final axiom states "these numbers are all *of* the natural numbers."

Axiom 9 Let's say K is a set of numbers (a bunch of numbers). If we know that

1. if 0 is in K, and
2. if some number x is in K, then $\mathrm{suc}(x)$ is in K,
3. then K contains every single natural number.

Continue with...

1. $\mathbb{N}$ is a Monoid
2. Quasigroups
3. Loops
4. Groups
5. Abelian Groups
    (a) $\mathbb{Z}$
        i. $\mathbb{Z}$ is an Abelian Group
        ii. $\mathbb{Z}$ is a ring
6. Fields
    (a) $\mathbb{R}$
        i. $\mathbb{R}$ is a field
7. Categories
8. Groupoids

Appendices

Appendix A

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Preamble

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Appendix B

How to learn math

Now that that's all out of the way, let's talk a little about math. When this chapter is over, we're going to dive right in to proving a bunch of things you already know to be true. We feel that without a little explanation, these proofs may leave you a little lost or confused. We'll save the explanation of the proofs for later, but right now, we're going to talk about how to actually learn math, and the proofs are a great example.

Most people's experience with math is through their primary and possibly secondary education, which is or was a dreary affair in general, and math probably even moreso, unless you're one of the lucky few. By lucky few, we don't mean those wizards with a sort of inherent ability to do math–the first thing you need to know about learning math is that math is for everyone with a brain–that's you, right? You see, your brain is a pattern recognition engine, and that's all math is: the study of patterns. Unlike reading or history, your body comes with a biological imperative to know math. There's some really great brain studies on the topic, but that's boring, and I said we're already done with the boring part, so let's move on.

In that last paragraph, we presented what we hold to be the proper answer to 'what is math': the study of patterns. This is completely different from most people's interaction with math: in primary school, we are taught how to apply four operations to solve math problems. You're given something about two trains leaving a station and going different speeds and different directions and yadda yadda yadda and before you know it your teacher turned everything into a math problem and it all seemed so forced–a layer on top of what was intuitive, and made everything complicated. We agree–this is a counterintuitive approach to math, and it makes math very confusing and disconnected. Math is just the study of patterns. That is, math is not so much a way to solve a set of problems that exist in a sphere apart from what is natural, but a way to understand what's going on in the world around us. When you learn math, you should think of it as a science–another level of detail in the amazing world we live in.

That's how this book is written. It's written to reflect that math is a single unified study. While you're reading it, try to think of how what you're learning clarifies or refines early material. This is a big deal to us, because one thing we dislike most about the standard way of learning math is that at some point in everyone's math career, they learn they were taught something that wasn't actually true. We want to avoid that.

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[1] Edmund Landau. *Foundations of Analysis*. Providence, RI: AMS Chelsea Publishing, 1966.

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