Learn You Some Algebras for Glorious Good!

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January 4, 2015

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Chapter 1

Introduction

Before I bore you with a bunch of crap you don't care about, let's do some math, shall we?

There are basically three notions with which you need to be familiar in order to do anything interesting in math. Those three things are *sets*, *functions*, and *proofs*. Unfortunately, to be familiar with one, you have to be familiar with the other two.[1]

So, what are each of those things?

- A *set* is an unordered collection of things. There is also no repetition. For instance, $\{2, 5\}$ is the same as $\{5, 2\}$ (because order doesn't matter). $\{2, 5, 5\}$ would be the same set, because there's no notion of multiplicity.
- A *function* is a mathematical construct (well, obviously, else I wouldn't be talking about it). Basically, it takes some input, does something to it, and spits out some output. If you give the function the same input a bunch of times, you should get the same result each time. This concept is called "referential transparency." If the function is not referentially transparent, then it's not a function. It's something else.

[1] You'll learn as we go along, when math people use a common term like *set*, *function*, *proof*, *group*, *continuous* or *closed*, they usually mean something similar in concept to the colloquial term, but there are some strings attached. This is usually the case in the sciences too (e.g. *theory*, *hypothesis*, *experiment*).

- A *proof* is basically where you take a bunch of simple facts, called *axioms*, and chain them together to make *theorem*s. It's sort of like sticking puzzle pieces together to form a picture.

  The puzzle pieces (in this case, the axioms) aren't usually very interesting on their own. However, the picture they form (in this case, the theorem) can be really cool and enlightening. The proof would be analogous to an explicit set of instructions explaining how to put the pieces together.

Once you are familiar with each of those concepts, we can do all sorts of cool stuff. Throughout the book, we will prove all of the following:

- If you tap your finger against a bridge at exactly the right frequency, the bridge will collapse. (Resonance)
- The formula used to calculate the interest rate on your mortgage is actually just a fancy form of the ratios of angles in a triangle. (Euler's formula)
- Logic can't be used to prove everything we know to be true. (Gödel's incompleteness theorem)

1.1 Introduction (for real this time)

This is a math book. Well, duh. Why did I write it?

Most math (and science) books nowadays seem to value keeping an academic tone over ensuring that the reader understands the material, and — more importantly — enjoys reading the book.

I take the opposite approach. I want to create a book that is fun to read and easy to understand, while eschewing the practice of making myself look good.

The inspiration for this book is *Learn You a Haskell for Great Good!*, by Miran Lipovača. Haskell is a programming language, and LYAH is a great book for learning Haskell. If you are interested in a print copy of LYAH, see [2].

1.2 The community

Despite the fact that I used "I" in the first part of the book, LYSA is actually a community project, and many people participate in the writing of this book.

If you want to talk to us, or to other math people, come see us in `#lysa` on Freenode. If you don't know what IRC is, or you don't have a client set up, you can connect through Freenode's webchat (`http://webchat.freenode.net/?channels=lysa`).

If you have any questions about LYSA (or math), feel free to ask in the IRC channel (`#lysa` on FreeNode in case you forgot).

If you want to submit a correction, or have some issue, or want to add some content, really anything having to do with the content of the book, you can visit our GitLab page (`https://gitlab.com/lysa/lysa`). We also have a community on Reddit (`https://lysa.reddit.com/`).

1.3 Idris

In this book, we cover a lot of hard stuff.[2] Sometimes, it's useful to program your way through a problem. Every programmer will tell you that programming teaches a manner of thinking.

Many programmers will cite Steve Jobs[3] famous quote, regarding the use of programming in his job,

> *[sic] . . . much more importantly, it had nothing to do with using [the programs we wrote] for anything practical. It had to do with*

[2]This isn't actually true. Math isn't hard, stupid!

[3]For you youngsters, Steve Jobs is the former CEO of Apple. He's dead now.

> *using them to be a mirror of your thought process; to actually learn how to think. I think everybody in this country should learn how to program a computer — should learn a computer language — because it teaches you how to think.*

That first sentence or two is actually a pretty good description of mathematics (and programming). Both are incredibly useful, and have endless practical applications. That's not the point, though. The whole usefulness thing is a side gig. It's about learning how to think, and having a rigorous language through which to express your thoughts. Furthermore, the rigor of the language helps you build upon your current thoughts to find out even cooler things. That's what math is about.

Programming and math go hand-in-hand. Programmers and mathematicians will attest to this; I certainly can. For that reason, throughout this book, there will be coding exercises in the programming language Idris. Idris is an interesting programming language for many reasons. The chief of which is that it can be used to prove things mathematically. Most programming languages can't do this. Idris can, which is why it is special.[4]

1.3.1 Installing Idris

In our markup language, it's actually really difficult to put in the installation instructions. This is an open issue (`https://gitlab.com/lysa/lysa/issues/1`), with no solution in sight. Good luck getting Idris installed.

1.4 Licensing

This book is free, in the sense of freedom. You can copy this book and give it to your friend. You can even print it out and sell it to your friends.[5]

[4]There are other programming languages that can prove things, namely Coq and Agda. However, I'm most familiar with Idris, and Idris is probably the most useful, so I'm using Idris. Deal with it.

[5]There are some restrictions though, see § A.

If, for instance, you are a schoolteacher and want to use this for your class, you are free to edit it to your liking and give the modified copy to your students.

LYSA is licensed under the GNU Free Documentation License. § A contains the license. Please read the license; it's actually pretty comprehensible.

The source for this book can be downloaded at `https://gitlab.com/lysa/lysa/repository/archive.tar.gz`. If you are looking to contribute, it's probably best to clone the git repository. You can clone the git repository by running `git clone https://gitlab.com/lysa/lysa.git` in a terminal.

Chapter 2

Booleans, simple logic, and simple operators

Before we get into interesting content, you have to understand some stuff. This stuff is pretty easy. This will likely be the shortest and easiest chapter in the book.

You might think math is about dealing with numbers and pumping out formulas. Well, that's not what math is about. As said in § 1.3, it's about using math as a language to express your thoughts. Most people don't think about numbers all day; thus, we deal with things in math that aren't numbers.

In this next section, we're going to outline some basic rules for reasoning about things. You need to know these rules to do really cool stuff. Although, as you will (hopefully) see, these rules can be fun to toy around with on their own.

2.1 Basic Logic

2.1.1 True and False

True or False - in your education, you've been faced with questions where the answer is "True" or "False"? Well, hopefully the answer is "True". In math, we're going to deal with things where we have to decide whether something is true or not. Moreover, there are going to be a bunch of statements that may or may not be true, and we'll have to figure out how they relate to each other. With that in mind, this section has some rules for dealing with truth and falsehood.

We're going to create two "values", and they are called True and False, respectively. These values are called "Booleans". They are named after a mathematician named George Boole, who studied them rather extensively.

True and False are not very interesting on their own. However, when we have a bunch of things that are either True or False, we can combine them together in three basic ways:

- $A \wedge B$ should be read as "A logical-and B." Both have to be True. If one of them is False, than $A \wedge B$ is False. Likewise, if A and B are both True, then $A \wedge B$ is True.
- $A \vee B$ should be read as "A logical-or B." With $A \vee B$, if one of A and B is True, then $A \vee B$ is True. It's okay if both of them are True.
- Sometimes we are going to want to say 'A is not True'. Instead of writing that out each time, I'm instead going to use the symbol $\neg$. So, $\neg A$ should be read 'logical-not A'.

2.1.2 Logic

Mathematicians are too lazy to write things like "if A is True, then B is True, but if B is True, it doesn't necessarily imply that A is True." So, in math, they use these symbols. I'm also lazy, so I'm going to use these symbols.

- $A \implies B$ means that "if A is True, that means that B must be True." Note that this **does not** mean that "if B is True, then A is True." Always follow the arrow. $A \implies B$ should be read as "A implies B."
- $A \impliedby B$ is the same thing as writing $B \implies A$. It's often convenient to write $A \impliedby B$ though. $A \impliedby B$ should be read "A is implied by B."
- $A \iff B$ means that $A \implies B$ and $B \implies A$. You can think of $A \iff B$ as meaning "Saying A is the same thing as saying B." You should read $\iff$ as "if (and only if)." "If (and only if)" takes a while to write, and $\iff$ is often contextually inappropriate. So, I'll sometimes use iff (with two 'f's) in the place of "if (and only if)".
- $A \not\Longrightarrow B$ means "A does not imply B." **However**, this does not mean, A implies that B is false. It simply means that knowing something about A doesn't tell you anything about B. Got it? The analog is what you'd expect for $A \not\Longleftarrow B$.
- For any Boolean A, it is always true that $A \implies A$.
- For any A, it's always true that $A \not\Longrightarrow \neg A$

Now, I'm very lazy, so I'm going to quit writing "for every A,B, and C ..." Instead, I'm going to use the symbol $\forall$. It's an upside-down A, and it stands for 'all'. You should read it as 'for all'. So, the above statement is $\forall A, B, C \ldots$

- Alright, time for some notation:

$$\forall A, B, C;\ ((A \implies B) \wedge (B \implies C)) \implies (A \implies C)$$

For this reason, we can write things like $A \implies B \implies C$. [1]

You were probably really confused by that last glob of math. Let me read it out for you. "for all A, B, and C, if we know that $A \implies B$,

[1] A common critique of this practice has to do with associativity. That is, many people read $A \implies B \implies C$ as $(A \implies B) \implies C$. This translates to "if A implies B, then C is True", which isn't *quite* what we want. The solution is to not try to group the operators like that, or use parentheses when you do want to group them.

and $B \implies C$, then it's true that $A \implies C$." Sometimes, I'll write the $\forall \ldots$ part after the statement. So, I should have written the above as

$$((A \implies B) \wedge (B \implies C)) \implies (A \implies C);\ \forall A, B, C$$

- If you know that $A \implies B$, then $\neg A \implies \neg B$.

2.1.3 Exercises

Ex. 1 — Given A, is it always the case that $A \iff A$?

Ex. 2 — Given A, is it the case that $A \wedge A \iff A$?

Ex. 3 — Given A and B, is it always the case that $A \wedge B \iff B \wedge A$?

Ex. 4 — Given A, B, and C, is it always the case that $(A \wedge B) \wedge C \iff A \wedge (B \wedge C)$?

Ex. 5 — Given A, is it the case that $A \vee A \iff A$?

Ex. 6 — Given A and B, which are both True/False values, is it always the case that $A \vee B \iff B \vee A$?

Ex. 7 — Given A, B, and C, is it always the case that $(A \vee B) \vee C \iff A \vee (B \vee C)$?

Ex. 8 — Given A, B, and C, what is the result of $A \wedge (B \vee C)$?

Ex. 9 — What do you think the result of $\neg(A \wedge B)$ is?

Ex. 10 — What do you think the result of $\neg(A \vee B)$ is?

Answers

Answer (Ex. 1) — Yes. We know that $A \implies A$. Remember, $A \iff A$ is just saying that $(A \implies A) \wedge (A \impliedby A)$. Also recall that $A \implies B$ is the lazyman's way of writing $B \implies A$. Thus, if you know $A \implies A$, then it must be true that $A \implies A$, and therefore $A \iff A$.

Answer (Ex. 2) — Yes. There are two cases we need to deal with here:

$$\begin{cases} \text{True} \rightarrow & \text{True} \wedge \text{True} & \iff & \text{True} \\ \text{False} \rightarrow & \text{False} \wedge \text{False} & \iff & \text{False} \end{cases}$$

In both cases, $A \wedge A \iff A$. Q.E.D. This technique is called "proof by exhaustion". We named every possible case — in this case, there were only two — and proved the theorem for each of them.

Answer (Ex. 3) — Yes. In the previous problem, we showed that $A \wedge A \iff A$. Thus, if A and B are both True, or are both False, the answer is yes. Thus, the only case we need to consider is that in which A is Trueand B is False.[2]
If True $\wedge$ False $\iff$ False, and False $\wedge$ True $\iff$ False.

Answer (Ex. 4) — Yes.

Answer (Ex. 5) — Yes.

Answer (Ex. 6) — Yes.

Answer (Ex. 7) — Yes.

[2]You may be wondering why I'm not considering the case when A is False, and B is True. However, as we showed earlier, it's the case that if $A \iff B$, then $B \iff A$. Thus if $A \wedge B \iff B \wedge A$, then $B \wedge A \iff A \wedge B$

Answer (Ex. 8) — $A \wedge (B \vee C) \iff (A \wedge B) \vee (A \wedge C)$

Answer (Ex. 9) — I will explain in § 2.2, but the answer is

$$\neg (A \wedge B) \iff (\neg A) \vee (\neg B);\ \forall A, B$$

Answer (Ex. 10) — I will explain in § 2.2, but the answer is

$$\neg (A \vee B) \iff (\neg A) \wedge (\neg B);\ \forall A, B$$

2.2 Combining not, and, and or

In the previous section, we took a cursory look into the operators $\implies$, $\impliedby$, $\iff$, $\wedge$, $\vee$, and $\neg$.

I'm only going to introduce 2 new rules in this section, and here they are

- $\neg(\neg A) \iff A$
- $\neg(A \wedge B) \iff (\neg A) \vee (\neg B)$

  Intuitively you can think of this law this way: both A and B have to be true for $A \wedge B$ to be true. So, if one of them isn't, the entire condition is false. The way to say "the entire condition isn't true" is $\neg(A \wedge B)$. The way to say "one of them isn't true" is $(\neg A) \vee (\neg B)$.

The last rule is

$$\neg(A \vee B) \iff (\neg A) \wedge (\neg B);\ \forall A, B$$

Proof. This can be derived from the last law. Let $C = \neg A$, and $D = \neg B$.

$$\begin{aligned}
\neg(A \wedge B) &\iff (\neg A) \vee (\neg B) && ;\ \forall C, D \\
\neg((\neg C) \wedge (\neg D)) &\iff C \vee D && ;\ \forall C, D \\
C \vee D &\iff \neg((\neg C) \wedge (\neg D)) && ;\ \forall C, D \\
\neg(C \vee D) &\iff \neg(\neg((\neg C) \wedge (\neg D))) && ;\ \forall C, D \\
\neg(C \vee D) &\iff (\neg C) \wedge (\neg D) && ;\ \forall C, D
\end{aligned}$$

$\square$

So what the hell is all that? Well, this is your first taste of a *mathematical proof*. Basically, we start with a set of rules which we know are true (the *laws* or *axioms*). We build them together to form *theorems*. That little white box at the bottom right is to say "okay, here's where I'm done proving stuff."

I will also use the acronym Q.E.D., which is short for "quod erat demonstrandum", which is fancy-talk for "I'm done proving stuff".

This really is your second taste of proofs, provided you did the exercises for § 2.1. You may have seen the terms "proof" and "Q.E.D" thrown around a lot. Well, now you know what they mean — hopefully.

2.2.1 Exercises

Ex. 11 — Show that $(A \not\Leftrightarrow B) \iff ((A \not\Rightarrow B) \vee (B \not\Rightarrow A))$

Answers

Answer (Ex. 11) — Let's restate the problem.

$$(A \not\Leftrightarrow B) \iff ((A \not\Rightarrow B) \vee (B \not\Rightarrow A))$$

$$\neg(A \iff B) \iff (\neg(A \implies B) \vee \neg(B \implies A))$$

$$\neg(A \iff B) \iff \neg((A \implies B) \wedge (B \implies A))$$

$$\neg(A \iff B) \iff \neg(A \iff B)$$

Q.E.D.

2.3 Idris

This section provides a "working example" of the above in Idris. If you don't know what that is, you are a bad person. Go back and read § 1.3!

Open up an Idris prompt, and enter `:type True`. That is basically asking Idris "what type of thing is True?" Idris will tell you. Also do the same thing for False. Here's what happens when I do it:

```
Idris> :type True
Prelude.Bool.True : Bool
Idris> :type False
Prelude.Bool.False : Bool
```

If you ask Idris what the type of `True` or `False` is, it will tell you that they are `Bool`s.[3] You're probably thinking "what the hell is a `Bool`?" `Moreover, why the hell is this guy printing all this stuff in monospace`? Well, explaining this sorta stuff, that's what I'm here for. `Bool` is short for Boolean, which is what this chapter is about.

The reason I'm printing stuff `in monospace` is to say "hey, this is code." More importantly, printing stuff in monospace eschews some formatting flukes caused by variable-width text. In normal paragraph text, these flukes are fine — they are even helpful — but they cause some ambiguities in code. For that reason, the standard thing to do is to write code in monospace.

In Idris, and in most programming languages, the $\wedge$ operator is replaced with `&&`. We know that, in Idris True and False are both Boolean values. What about True $\wedge$ False?

```
Idris> :type (True && False)
True && Delay False : Bool
```

[3] For the most part, you can ignore all the weird stuff on the left-hand-side, for the time being. For instance, when I ran `:type True`, Idris switched `True` to `Prelude.Bool.True`. These are odd caveats of Idris's syntax, which I don't have time to explain right now. We'll get to them later, I promise.

So, wait, True *and* False *are both Boolean values, but* True $\wedge$ False *is also a Boolean value?*

Yes, Timmy, now try to keep up.

Now, from the explanation of $\wedge$ in § 2.1, True $\wedge$ False should only be true if both True and False are True. Well, that's obviously not the case, so True $\wedge$ False should turn out to be False, right? Let's see!

```
Idris> True && False
False : Bool
```

Yay! You figured out some stuff! This dealingswith of True and False is called "Boolean algebra." Boolean algebra deals only with two values, Trueand False, so it's understandably one of the simpler algebras. Anyway, more Boolean algebra to follow:

We've discussed that True $\iff$ True, and False $\iff$ False. We've also shown how True $\wedge$ False $\iff$ False. What about True $\vee$ False?

Remember, True $\vee$ False is True if one of them is True. Both of them can be True; only one of them has to be. You *do* remember, right? Since one of them is True, True $\vee$ False should therefore be True.

In Idris — and most programming languages — $\vee$ is replaced with `||`.

```
Idris> True || False
True : Bool
```

Chapter 3

Sets, Proofs, and Functions

Anyway, now that that boring introduction is out of the way, we can get to some math! The first thing we are going to cover are *set*s.

Sets were first studied by Georg Cantor, a German mathematician, in the second half of the nineteenth century. Back in his own day, the results Cantor found by studying sets were considered so thoroughly bizarre that many of his colleagues simply refused to believe that Cantor could be right. In the end, Cantor turned out to be right all along. His ideas can be found in any introductory text on mathematics—including this one.

Sets are basically like lists— think "your grocery list" or "your to-do-list" — except there's no duplication, and there's no intrinsic order. Examples:

1. $\{0, 1, 2, 3\}$ is a set.

2. $\{1, 2, 0, 3\}$ is the same set, because the order is of no consequence.

3. $\{1, 1, 2, 0, 3\}$ is the same set, because the duplication can be ignored.

4. $\{8, 1, 2, 0, 3\}$ is a different set, because there's a new element.

3.1 Elements, Subsets, and Supersets

1. $0, 1, 2, 3$ **is not** a set, it is instead the numbers 0 through 3, which are to be considered separately.

2. $\{0, 1, 2, 3\}$ **is** indeed a set (notice the braces).

3. If some object is in a set, the notation is 'OBJECT $\in$ SET', which should be read "OBJECT is an element of SET".

4. If the object is not in the set, the notation is 'OBJECT $\notin$ SET', which should be read "OBJECT is not an element of SET".

5. $0 \in \{0, 1, 2, 3\}$ should be read "0 is an element of $\{0, 1, 2, 3\}$", and indeed it is.

6. $0, 1 \in \{0, 1, 2, 3\}$ should be read "0 and 1 **are both** elements of $\{0, 1, 2, 3\}$". They are.

7. $\{0, 1\} \notin \{0, 1, 2, 3\}$ should be read "$\{0, 1\}$ **is not** an element of $\{0, 1, 2, 3\}$". $\{0, 1\}$ is indeed not an element of $\{0, 1, 2, 3\}$.

Now, here, we're faced with an interesting problem. $\{0, 1, 2, 3\}$ encapsulates $\{0, 1\}$, but $\{0, 1\}$ is not an element of $\{0, 1, 2, 3\}$. So, how do we express this notion of 'encapsulation'? The answer is by using "subset" notation.

1. $\{0, 1\} \subseteq \{0, 1, 2, 3\}$is **true**. $\{0, 1\} \subseteq \{0, 1, 2, 3\}$should be read as "$\{0, 1\}$ is an **improper** subset of $\{0, 1, 2, 3\}$."

2. $\{0, 1\} \subset \{0, 1, 2, 3\}$is **true**. $\{0, 1\} \subset \{0, 1, 2, 3\}$should be read as "$\{0, 1\}$ is a **proper** subset of $\{0, 1, 2, 3\}$."

3. $\{0, 1, 2, 3\} \subseteq \{0, 1, 2, 3\}$ is **true**. Intuitively, you should think "$\{0, 1, 2, 3\}$ is entirely encapsulated inside of $\{0, 1, 2, 3\}$."

4. $\{0, 1, 2, 3\} \subset \{0, 1, 2, 3\}$ is **false**. This highlights the difference between $\subset$and $\subseteq$.

$\{0,1\} \subset \{0,1,2,3\}$implies that $\{0,1,2,3\}$ contains some stuff that is not contained in $\{0,1\}$. $\{0,1\} \subseteq \{0,1,2,3\}$does not have this implication. That is, $\{0,1\} \subseteq \{0,1,2,3\}$does not say for sure that $\{0,1,2,3\}$ has more stuff than $\{0,1\}$, it just acknowledges it as a possibility.

Alright, my apologies, that was a lot of stuff to throw at you at once, and you probably remember absolutely none of it. However, you'll get used to that particular *set* of stuff[1] as time goes along.

3.1.1 Formal Definitions

I'm all for formalism up to the point at which it hinders your understanding of the material. In most cases, formalism does hinder your understanding. However, in this case — when talking about definitions — precise and formal definitions are usually the most helpful.

Set A collection of items, called "elements". The elements have no intrinsic order, and no multiplicity.

Elements Given a set S and an element x, $x \in S$ is the formal notation for saying "S contains x."

Non-elements Given a set S and an element x, $x \notin S$ is the formal notation for saying "S does not contain x."[2]

Questions If you want to pose the question "does S contain x?", the notation is $x \overset{?}{\in} S$.[3]

Improper subsets If you want to say "all of the elements in S are in T, but not necessarily the other way around" the notation is $S \subseteq T$. This is to be read "S is an improper subset of T".

[1] I, for one, would never condone such terrible puns.

[2] In general, if you want to say "something is not true," you cross out the operator in question. For instance, if you want to say "$x = y$ is not true", the notation is $x \neq y$.

[3] In general, if you want to pose a question, you put a ? over the operator in question.

More formally, $S \subseteq T$ means that for all elements x, such that $x \in S$, it is true that $x \in T$. In purely mathematical notation, this is written

$$S \subseteq T \iff \forall \{ x \mid x \in S \}, x \in T$$

That should be read "S is an improper subset of T if and only if for all x, such that x is an element of S, x is an element of T."

Improper supersets If you want to say "S contains all of the elements in T", you write $S \supseteq T$. $S \supseteq T$ should be read "S is an improper superset of T." Note that $S \supseteq T$ is the same as writing $T \subseteq S$, but it's often convenient to have the superset notation.

Equality of Sets A set S is equal to another set T if and only if $S \subseteq T$ and $S \supseteq T$. The notation for this is $S = T$, which is to be read "S is equal to T." That is,

$$S = T \iff (\forall \{ x \mid x \in S \}, x \in T) \wedge (\forall \{ x \mid x \in T \}, x \in S)$$

Proper subsets S is a proper subset of T if and only if $S \subseteq T$ and $S \neq T$. This is to be written $S \subset T$

Proper supersets S is a proper superset of T if and only if $S \supseteq T$, and $S \neq T$ This is to be written $S \supset T$.

3.1.2 Exercises

The next page has some exercises. I would recommend that you do all of the exercises. Many books berate you with dozens of exercises all exercising the same problem. I'm too lazy to do that; instead, I'm going to give you a small number of exercises. The reason I do that is, I have to solve the exercises, too. The solutions are on the page after the exercises.

3.1.3 Exercises

Ex. 12 — Does it make sense to ask "how many times does S contain n?" Why or why not?

Ex. 13 —

1. If S is a proper superset of T, is it also true that $S \nsubseteq T$? In math notation,

$$S \supset T \overset{?}{\Longrightarrow} S \nsubseteq T$$

2. What about the opposite? That is, if $S \nsubseteq T$, is it also true that $S \supset T$? That is,

$$S \supset T \overset{?}{\Longleftarrow} S \nsubseteq T$$

Explain why or why not.

Ex. 14 —

1. If $S \subset T$, is it the case that there are elements of T which are not in S? Why or why not? To put this in mathematical notation:

$$S \subset T \overset{?}{\Longrightarrow} \exists \{ x \mid x \in T \wedge x \notin S \}$$

The expression above should be read "S is subset of T, question implies there exists a set of x, such that x is an element of T, and x is not an element of S"

2. What about the opposite, that is

$$S \subset T \overset{?}{\Longleftarrow} \exists \{ x \mid x \in T \wedge x \notin S \}$$

Ex. 15 — If you know $S \subset T$, is it also true that $S \subseteq T$? In math notation:

$$S \subset T \overset{?}{\Longrightarrow} S \supseteq T$$

Why or why not?

Ex. 16 — What about, if you know $S = T$, is it also true that $S \subseteq T$? Why or why not? In math notation,

$$S = T \overset{?}{\Longrightarrow} S \subseteq T$$

Ex. 17 — If it is known that $S \subseteq T$, is it also true that $S \subset T$? Why or why not?

$$S \subseteq T \overset{?}{\Longrightarrow} S \subset T$$

Ex. 18 — Let's look back at S, T, and n. Let's say that you know $n \in S$ (that is, n is in S). You also know that $S \subseteq T$. Does this also mean $n \in T$? Why or why not? To put this in mathematical notation:

$$\{\, n \in S,\ S \subseteq T \,\} \overset{?}{\Longrightarrow} n \in T$$

Ex. 19 — Since I'm lazy, and the math notation is probably much easier for you to understand,

$$\{\, n \in T,\ S \subseteq T \,\} \overset{?}{\Longrightarrow} n \in S$$

Ex. 20 — What about the opposite? That is,

$$S \subseteq T \overset{?}{\Longleftarrow} \{\, n \in S, n \in T \,\}$$

Answers

Answer (Ex. 12) — No, because repetition doesn't matter.

Answer (Ex. 15) — Yes. $S \subseteq T$ means that one of $\{ S \subset T, \; S = T \}$ is true. If $S \subset T$ is true, then the condition of $S \subseteq T$ is satisfied. Thus, $S \subset T \implies S \subseteq T$.

Answer (Ex. 16) — Yes, by the same logic above. If $S = T$, then the conditions for $S \subseteq T$ are satisfied.

Answer (Ex. 17) — No.

- $S \subset T$ can be stated as "all of $\{ S \subseteq T, S \neq T \}$ are true."
- By the previous problem, if $S \neq T$ is false (i.e. $S = T$ is true), then $S \subseteq T$ is true.
- In this case, the set above could be written as $\{ S \subseteq T, S \neq T \} = \{ True, False \}$. $S \subset T$ means that both of those have to be true. That is, $S \subset T$ is true if (and only if) $\{ S \subseteq T, S \neq T \} = \{ True, True \}$.
- In this case, we have $\{ S \subseteq T, S \neq T \} = \{ True, False \}$, which means $S \not\subset T$.
- Thus, we have just found a case in which $S \subseteq T$ is true, but $S \subset T$ is false. Therefore, it cannot be the case that $S \subseteq T \implies S \subset T$. Or, to put it another way, $S \subseteq T \;\not\Longrightarrow\; S \subset T$.

Answer (Ex. 18) — Yes. $S \subseteq T$ means that all of the elements in S are also in T, but not necessarily the other way around.. Thus, if $n \in S$, it must be true that $n \in T$.

Answer (Ex. 19) — No. $S \subseteq T$ means that it is possible that $S \subset T$ is true. $S \subset T$ means that T has all of the elements in S, but there are also more elements. It's entirely plausible that n is one of the additional elements. Thus,

$$\{\, n \in T,\, S \subseteq T \,\} \not\Rightarrow n \in S$$

Answer (Ex. 20) — No. $n \in S$ can be stated as $\{\, n \,\} \subseteq S$. Likewise, $n \in T \iff \{\, n \,\} \subseteq T$. This means there are 3 possible cases for the relation between S and T:

1. $S = T$
2. $S \subset T$
3. $S \supset T$

$S \subseteq T$ means that either 1 or 2 is true, but the third one is false. The third one can be true when $\{\, n \in S, n \in T \,\}$ is true. Thus, we have found a case in which $\{\, n \in S, n \in T \,\}$ is true, but $S \subseteq T$ is false. Thus, it must be that.

$$S \subseteq T \not\Leftarrow \{\, n \in S, n \in T \,\}$$

3.2 Numeric sets

Now, I'm going to throw more stuff at you. I'm sorry for all the notation, but it is important that you get used to it. There are a bunch of common sets you'll encounter in the real world.

1. First of all, there are the integers: these are just the whole numbers, like $-3, -2, -1, 0, 1, 2$. Traditionally, this set is denoted $\mathbb{Z}$.
2. The natural numbers, denoted $\mathbb{N}$, are all non-negative integers[4].
3. The set of "rational" numbers, denoted $\mathbb{Q}$ is the set of all numbers that can be written as a quotient (or fraction) $\frac{p}{q}$, where p and q are integers.

   In mathematical notation, you would write this as $\left\{\frac{p}{q} \mid p,\, q \in \mathbb{Z}\right\}$. You should read that as "the set of all numbers p over q, such that p and q are elements of $\mathbb{Z}$."
4. Finally, the real numbers, denoted $\mathbb{R}$, contain everything on the number line. Equivalently, a real number is just any number you can write down by writing down an integer, a decimal point, and then writing any sequence of digits after the decimal point (even an infinite sequence).

   Let us show you an example. Let's write down an integer. 2. There we go. Now, let's just write down a bunch of digits all in a row. 215455211. By the definition in the previous paragraph, 2.215455211 is a real number. You should get in the habit of writing "2.215455211 is a real number" as $2.215455211 \in \mathbb{R}$.

Sets certainly look simple enough to understand intuitively, but are they also interesting enough to be worth studying? The answer to that is a resounding 'yes.' Exciting, right?

To give you a taste of the results, let's take a look at some of them.

[4]Conventions differ as to what the natural numbers are. Everyone agrees that positive whole numbers (1, 2, 3...) are natural. Some people say 0 is also a natural numbers, while others disagree. In this book, 0 *is* a natural number, so the natural numbers are 0, 1, 2, and so on. This might seem like a stupid fight, why can't we just choose a side? However, it actually turns out to be really important.

Intuitively, sets can be of different sizes: it's clear that the empty set is the smallest possible set, and that $\{2, 4\}$ is smaller than $\{1, 2, 3\}$. When we get to infinite sets, however, the intuition fails.

Cantor formalised this intuition, and the formal version also worked for infinite sets. The results he got were very surprising: for example, he found that there are as many natural numbers as rational numbers, but that there are strictly more real numbers. Even worse, he proved that given any set, there is always a set larger than it. If you've ever heard of people talking about "different kinds of infinity", this is probably what they meant.

Exercises

1. Suppose x is some object and S is a set. Does it make sense to ask "How many times does S contain x?"

2. Earlier, we constructed the set $\{3, 5\}$. We can also construct the set $\{5, 3\}$. Are these the same set?

3. Are the following true or false?

    - $\varnothing \in \varnothing$
    - $\varnothing \in \{\varnothing\}$
    - $\varnothing = \{\varnothing\}$
    - $\{\varnothing\} \in \{\varnothing\}$

3.3 Functions

A function from a set X to a set Y specifies for every element of X a corresponding element of Y. In other words, if f is a function from X to Y (denoted $f : X \to Y$) and $x \in X$, then $f(x) \in Y$. A lot of high school mathematics is about functions from $\mathbb{R}$ to $\mathbb{R}$, even if they are never presented that way. For example, solving linear equations is actually a question about functions: "given a function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = 5x - 10$$

find all $r \in \mathbb{R}$ such that $f(r) = 0$."

It turns out that by taking sets X and Y and asking what kind of functions exist between them, we can find out a lot about X and Y themselves. We'll continue this line of thought after some exercises.

Exercises

1. We can construct the set $\{0, 1\}$. What functions can you find from $\{0, 1\}$ to $\{0, 1\}$? How many such functions are there?

2. Can you find a function from $\varnothing$ to $\{0, 1\}$? What about in the other direction?

3. Suppose X is a set. Prove that there exists at least one function from X to X. How can this function be defined? Why does this work if X is the empty set?

4. Suppose X, Y, and Z are sets, and you are given functions $f : X \to Y$ and $g : Y \to Z$. Can you make a function from X to Z? How?

5. Suppose X and Y are sets, $f : X \to Y$, and $x, y \in X$. Do you think $x = y$ implies $f(x) = f(y)$? Why, or why not?

Solutions

3.4 Injective Functions

In the exercises, you saw that if $f : X \to Y$ and you had $x, y \in X$ such that $x = y$, then necessarily $f(x) = f(y)$. A function f is said to be injective if the converse is also true: if $x, y \in X$ and $x \neq y$, then $f(x) \neq f(y)$.

Let's consider some examples. Let $f : \mathbb{N} \to \mathbb{Z}$ be defined by $f(x) = x$. Is this definition correct? Well, we need to check that for any $n \in \mathbb{N}$, we have $f(n) \in \mathbb{Z}$. But by definition, $f(n) = n$, so we need to check that if $n \in \mathbb{N}$, then also $n \in \mathbb{Z}$. Every non-negative integer is certainly an integer, so yes, the definition is correct.

Now, is this function injective? For that, let $n, k \in \mathbb{N}$ so that $f(n) = f(k)$. We need to prove that $n = k$. Again, this is easy: $f(n) = n$ and $f(k) = k$, so by putting the three equalities together, we have $n = k$.

We can construct the set $\{f(0), f(1), f(2) \ldots\}$. If we want to be precise, we can use the *set comprehension* notation:

$$\{f(x) \mid x \in \mathbb{N}\}$$

You should read this as "the set that contains $f(x)$ for each $x \in \mathbb{N}$." That's rather a mouthful, so we'll abbreviate this by $f(\mathbb{N})$. Just imagine that instead of applying f to one value in $\mathbb{N}$, you're applying it to all the values and looking at the results as a set.

We can now ask ourselves: how big is $f(\mathbb{N})$? In this case, we can see that $\mathbb{N} = f(\mathbb{N})$, so of course they're exactly the same size. However, what if instead of the f we chose, we had chosen $f(x) = -x$, or $g : \mathbb{N} \to \mathbb{Q}$ defined by $g(x) = \frac{1}{1+x}$? Prove that both of these are injective functions, and imagine $f(\mathbb{N})$ and $g(\mathbb{N})$. How big are these sets?

It should be clear that given any sets X and Y and any function $f : X \to Y$, $f(X)$ cannot possibly be bigger than X. For every element $y \in f(X)$, there is some element $x \in X$ so that $f(x) = y$; that's how we defined $f(X)$ in the first place.

On the other hand, if f is injective, $f(X)$ also shouldn't be any smaller than X. Why? Well, we can just define a function the other way. Let $g : f(X) \to X$, and define $g(f(x)) = x$. Now we know that $g(f(X))$ is no larger than $f(X)$. But $g(f(X)) = \{g(f(x)) \mid x \in X\} = \{x \mid x \in X\} = X$! That means, by the argument above, that $f(X)$ is no larger than X, just as we wanted.

The careful reader might notice that the definition of g is fishy. Does a function defined this way really work? The answer is that it does, but only because we were careful about f. We know that if we have an element $y \in f(X)$, then there is an element $x \in X$ so that $y = f(x)$. This means that g really is defined for all $y \in f(X)$. That's not all, though. Additionally, suppose there was also a $z \in X$ so that $y = f(z)$. Then $g(y) = g(f(x)) = x$, but also $g(y) = g(f(z)) = z$. For g to be a function, we need to prove that $x = z$. Here is where we need the injectivity of f: we know that since $f(x) = y = f(z)$, that $f(x) = f(z)$, and thus $x = z$. Therefore, g really is a function, and so our argument holds.

Our intuition thus tells us that however we define size, we should make sure that for any set X and any injective function $f : X \to Y$, we should have X and $f(X)$ be of equal size. As $f(X)$ is just a part of Y, this means that X should also be no bigger than Y. By themselves, all these loose bits of intuition might not seem very useful, but in the next section we'll use them to show that a certain approach to define size *doesn't* work. First, however, some exercises.

Exercises

1. In an older exercise, you saw there were four functions from $\{0, 1\}$ to itself. How many injective functions are there? What about between $\{0, 1, 2\}$ and itself? What if you have even more elements?

3.5 Subsets

When people first hear that there are as many integers as natural numbers, their reaction is often "Surely that can't be right? Every natural number is an integer, and there are some others, too!"

To make this argument formal we should introduce the notion of a subset. We say that X is a subset of Y if every element of X is also an element of Y. We denote this by $X \subseteq Y$. We have, then $\varnothing \subseteq \mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$. We will use $X \subset Y$ when X is a subset of Y, but X and Y are not equal.

Now let's denote the size of X by $|X|$. Can we define this size in such a way that for finite sets, $|X|$ is the number of elements in X, that for any sets X and Y such that $X \subset Y$ we have $|X| < |Y|$, and that if there exists an injective function $f : X- > Y$ then $|X| = |f(X)|$?

We've already given away that this isn't possible. The question becomes, how do we prove it? It isn't enough to just try a few definitions of 'size' and see that they don't work, as there'll always be a few that we haven't yet attempted. We need a way of showing that whatever definition we come up with, we'll be able to show it doesn't work.

To do that, we'll assume that we were given some definition of 'size' that satisfies the above requirements, but with no other assumptions. You can imagine it as being given the definition by a friend, and now you have to show why it's wrong. If the definition allows us to derive something that isn't true then there's certainly something wrong with it, so that's what we'll try to do.

Assume that for every set X, we defined $|X|$. Define $f : \mathbb{Z} \to \mathbb{Z}$ by $f(x) = 2x$. We required that $|\mathbb{Z}| = |f(\mathbb{Z})|$. But because $1 \notin f(\mathbb{Z})$, $f(\mathbb{Z}) \subset \mathbb{Z}$. This implies $|f(\mathbb{Z})| < |\mathbb{Z}|$, and so we conclude $|\mathbb{Z}| = |f(\mathbb{Z})| < |\mathbb{Z}|$. Surely, though, a set can't be smaller than itself, so we've just derived a falsehood, and the definition of 'size' that we started with can't work.

If this proof doesn't feel quite right to you, don't worry: we'll look at arguments like this in-depth in the next chapter. For now, the main result is that we are too strict in how we want to define size. We need to relax one of the requirements, and in the next section we'll see that if we remove the requirement that $X \subset Y$ implies $|X| < |Y|$, we can find a good definition.

Exercises

In these exercises we'll explore some more operations for constructing sets. The three most important ones are *union*, *intersection*, and *difference*.

The union of X and Y is written as $X \cup Y$ and contains every element of X and every element of Y. We can write this using set comprehension notation:

$$X \cup Y = \{x \mid x \in X \text{ or } x \in Y\}$$

1. Write out the following sets:
   (a) $\{0, 1\} \cup \{2, 3\}$
   (b) $\{0, 1\} \cup \{1, 2\}$
   (c) $\{0, 1\} \cup \{0, 1\}$
   (d) $\{0, 1\} \cup \varnothing$
2. We say that a set A is the identity of $\cup$ if for every set X we have $X \cup A = X = A \cup X$. Does $\cup$ have an identity?
3. Suppose X and Y are sets. Convince yourself that $X \cup Y = Y \cup X$. This property of $\cup$ is called *commutativity*.
4. Suppose X, Y, and Z are sets. Convince yourself that $(X \cup Y) \cup Z = X \cup (Y \cup Z)$. This property is called *associativity*.

The intersection of X and Y is written as $X \cap Y$ and contains all elements that are both in X and Y.

1. Do the first exercise on unions again, but replace $\cup$ with $\cap$.
2. Let X and Y be sets. Write $X \cap Y$ as a set comprehension.
3. Does $\cap$ have an identity?
4. Is $\cap$ commutative and associative, just like $\cup$ is?

The difference between X and Y is written as $X - Y$ (or, sometimes, $X\ Y$) and contains all elements that are in X but are not in Y.

1. Do the first exercise on unions a third time, but now taking the difference.
2. Does set difference have an identity?
3. Is set difference commutative? What about associative?

3.6 Cardinalities

3.7 The Power Set

3.8 The Category of Sets

3.9 Special kinds of sets

3.9.1 Magmas

3.9.2 Semigroups

3.9.3 Monoids

Chapter 4

Proofs

4.1 Peano Axioms

This section covers the Peano axioms. As I said in **??**, these are a way for mathematicians to understand arithmetic.

Arithmetic (hopefully) seems simple enough, and easy to understand. Maybe an expression like

$$(2048282 \times 33221) + (3254 \times 11)$$

seems difficult to calculate, but you hopefully understand what each of the operators mean in concept. If you don't, well. In theory, reading this chapter alone will teach you arithmetic. However, I wrote this chapter assuming you already know arithmetic.

So, why is it important that you read this chapter?

Arithmetic is pretty simple and easy to understand. However, later on in this book, we're going to approach concepts that aren't so simple and easy to understand. Mathematicians have a systemic approach to these problems. This approach is called "mathematical proof." We prove things mathematically. Instead of approaching new concepts with proofs, I'm instead going to use proofs to illustrate some (hopefully) familiar concepts.

Alright, with all that out of the way, let's get started.

The basic idea of proofs is, you take a small set of obvious facts, called *axioms*, chain them together to make *theorems*. The following obvious facts, or axioms, are called the "Peano Axioms." They describe what we call "natural numbers." Natural numbers are the numbers $\{0, 1, 2, 3, 4, 5, 6, \ldots\}$.

Axiom 1 0 is a natural number. Again, obvious.

I'm going to use letters in the place of numbers right here. So, if I say "x is a natural number," that means that x is a placeholder for one of the numbers in $\{0, 1, 2, 3, 4, 5, 6, \ldots\}$. I could use any letter, such as a, b, q, r, θ, Γ, or $\aleph$. If I use a letter instead of a number, it usually means either

1. it doesn't matter which number I choose, or
2. it does matter which number I choose, but I don't know which number it is yet.

Axiom 2 If x is a natural number, it is true that $x \equiv x$.

You can read that $\equiv$ sign as $=$, for the time being. There are some subtle differences between the two, which I will get to in **??**. You are supposed to read $a \equiv b$ as one of these:

1. "a is equivalent to b,"
2. "a is identically equivalent to b,"
3. "a is congruent to b."

$=$ should be read as "a is equal to b," or "a equals b." Again the difference between $\equiv$ and $=$ isn't really important until **??**.

If you don't know what either of those signs are, $a \equiv b$ or $a = b$ means "a is the same thing as b." The difference can be summarized as $==\equiv\neq=$.

So, in essence, this axiom says that each number is the same thing as itself. This is hopefully very obvious.

A math person would state this axiom as "congruence is reflexive."

Axiom 3 If x and y are both natural numbers, and $x \equiv y$, then it's true that $y \equiv x$. You can phrase this axiom as "if two numbers are the same number, then they are the same number." A math person would state this axiom as "congruence is symmetric."

Axiom 4 If x, y, and z are all natural numbers, and $x \equiv y$, and $y \equiv z$ then it's true that $x \equiv z$. You can phrase this axiom as "if three numbers are all the same number, then they are the same number." A math person would state this axiom as "congruence is transitive."

These last three axioms mean that we can be lazy, and write things like $a \equiv b \equiv c \equiv a$.

Axiom 5 If x is a natural number, and we know $x \equiv y$, then it's also true that y is a natural number. A math person would say "congruence forms a closure."

Axiom 6 If x is a natural number, then there is another number, $\mathrm{suc}(x)$, which is also a natural number. suc is short for "successor." You should read $\mathrm{suc}(x)$ as "the successor of x." You can think of the successor as "the next number." So, $\mathrm{suc}(0) \equiv 1$, $\mathrm{suc}(1) \equiv 2$, and so on.

Axiom 7 There isn't a number whose successor is 0. Basically this means "0 is the lowest natural number."

Axiom 8 If x and y are both natural numbers, and we know $\mathrm{suc}(x) \equiv \mathrm{suc}(y)$, then it's true that $x \equiv y$. This is what we would call the "converse" of Axiom 6. That is, Axiom 6 tells us that we can always go "up" a number. This axiom (almost) tells us that we can go "down" a number. Axiom 7 defines the limit of this, meaning that 0 is the only number where you can't go down any further.

Now, the previous 8 axioms have basically said "these numbers are all natural numbers." This next, and final axiom states "these numbers are all *of* the natural numbers."

Axiom 9 Let's say K is a set of numbers (a bunch of numbers). If we know that

1. if 0 is in K, and
2. if some number x is in K, then $\mathrm{suc}(x)$ is in K,
3. then K contains every single natural number.

Continue with...

1. $\mathbb{N}$ is a Monoid
2. Quasigroups
3. Loops
4. Groups
5. Abelian Groups
   (a) $\mathbb{Z}$
      i. $\mathbb{Z}$ is an Abelian Group
      ii. $\mathbb{Z}$ is a ring
6. Fields
   (a) $\mathbb{R}$
      i. $\mathbb{R}$ is a field
7. Categories
8. Groupoids

Appendices

Appendix A

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Preamble

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Appendix B

How to learn math

Now that that's all out of the way, let's talk a little about math. When this chapter is over, we're going to dive right in to proving a bunch of things you already know to be true. We feel that without a little explanation, these proofs may leave you a little lost or confused. We'll save the explanation of the proofs for later, but right now, we're going to talk about how to actually learn math, and the proofs are a great example.

Most people's experience with math is through their primary and possibly secondary education, which is or was a dreary affair in general, and math probably even moreso, unless you're one of the lucky few. By lucky few, we don't mean those wizards with a sort of inherent ability to do math–the first thing you need to know about learning math is that math is for everyone with a brain–that's you, right? You see, your brain is a pattern recognition engine, and that's all math is: the study of patterns. Unlike reading or history, your body comes with a biological imperative to know math. There's some really great brain studies on the topic, but that's boring, and I said we're already done with the boring part, so let's move on.

In that last paragraph, we presented what we hold to be the proper answer to 'what is math': the study of patterns. This is completely different from most people's interaction with math: in primary school, we are taught how to apply four operations to solve math problems. You're given something about two trains leaving a station and going different speeds and different directions and yadda yadda yadda and before you know it your teacher turned everything into a math problem and it all seemed so forced–a layer on top of what was intuitive, and made everything complicated. We agree–this is a counterintuitive approach to math, and it makes math very confusing and disconnected. Math is just the study of patterns. That is, math is not so much a way to solve a set of problems that exist in a sphere apart from what is natural, but a way to understand what's going on in the world around us. When you learn math, you should think of it as a science–another level of detail in the amazing world we live in.

That's how this book is written. It's written to reflect that math is a single unified study. While you're reading it, try to think of how what you're learning clarifies or refines early material. This is a big deal to us, because one thing we dislike most about the standard way of learning math is that at some point in everyone's math career, they learn they were taught something that wasn't actually true. We want to avoid that.

Bibliography

[1] Edmund Landau. Foundations of Analysis. Providence, RI: AMS Chelsea Publishing, 1966.

[2] Miran Lipovača. Learn You a Haskell for Great Good! 245 8th Street, San Francisco, CA, 94103: No Starch Press, 2011.

[3] steve jobs on programming. URL: `https://www.youtube.com/watch?v=5Z1gfgM7kzo` (visited on 01/01/2015).