Learn You Some Algebras for Glorious Good!

Peter Harpending <pharpend2@gmail.com>

January 11, 2015

Contents

Chapter 1

Introduction

Before I bore you with a bunch of crap you don't care about, let's do some math, shall we?

There are basically three notions with which you need to be familiar in order to do anything interesting in math. Those three things are *sets*, *functions*, and *proofs*. Unfortunately, to be familiar with one, you have to be familiar with the other two.[1]

So, what are each of those things?

- A *set* is an unordered collection of things. There is also no repetition. For instance, $\{ 2, 5 \}$ is the same as $\{ 5, 2 \}$ (because order doesn't matter). $\{ 2, 5, 5 \}$ would be the same set, because there's no notion of multiplicity.
- A *function* is a mathematical construct (well, obviously, else I wouldn't be talking about it). Basically, it takes some input, does something to it, and spits out some output. If you give the function the same input a bunch of times, you should get the same result each time. This concept is called "referential transparency." If the function is not referentially transparent, then it's not a function. It's something else.

[1] You'll learn as we go along, when math people use a common term like *set*, *function*, *proof*, *group*, *continuous* or *closed*, they usually mean something similar in concept to the colloquial term, but there are some strings attached. This is usually the case in the sciences too (e.g. *theory*, *hypothesis*, *experiment*).

- A *proof* is basically where you take a bunch of simple facts, called *axioms*, and chain them together to make *theorem*s. It's sort of like sticking puzzle pieces together to form a picture.

  The puzzle pieces (in this case, the axioms) aren't usually very interesting on their own. However, the picture they form (in this case, the theorem) can be really cool and enlightening. The proof would be analogous to an explicit set of instructions explaining how to put the pieces together.

Once you are familiar with each of those concepts, we can do all sorts of cool stuff. Throughout the book, we will prove all of the following:

- If you tap your finger against a bridge at exactly the right frequency, the bridge will collapse. (Resonance)
- The formula used to calculate the interest rate on your mortgage is actually just a fancy form of the ratios of angles in a triangle. (Euler's formula)
- Logic can't be used to prove everything we know to be true. (Gödel's incompleteness theorem)

1.1 Introduction (for real this time)

This is a math book. Well, duh. Why did I write it?

Most math (and science) books nowadays seem to value keeping an academic tone over ensuring that the reader understands the material, and — more importantly — enjoys reading the book.

I take the opposite approach. I want to create a book that is fun to read and easy to understand, while eschewing the practice of making myself look good.

The inspiration for this book is *Learn You a Haskell for Great Good!*, by Miran Lipovača. Haskell is a programming language, and LYAH is a great

book for learning Haskell. If you are interested in a print copy of LYAH, see [3].

1.2 The community

Despite the fact that I used "I" in the first part of the book, LYSA is actually a community project, and many people participate in the writing of this book.

If you want to talk to us, or to other math people, come see us in `#lysa` on Freenode. If you don't know what IRC is, or you don't have a client set up, you can connect through Freenode's webchat (`http://webchat.freenode.net/?channels=lysa`).

If you have any questions about LYSA (or math), feel free to ask in the IRC channel (`#lysa` on FreeNode in case you forgot).

If you want to submit a correction, or have some issue, or want to add some content, really anything having to do with the content of the book, you can visit our GitLab page (`https://gitlab.com/lysa/lysa`). We also have a community on Reddit (`https://lysa.reddit.com/`).

1.3 Idris

In this book, we cover a lot of hard stuff.[2] Sometimes, it's useful to program your way through a problem. Every programmer will tell you that programming teaches a manner of thinking.

Many programmers will cite Steve Jobs[3] famous quote, regarding the use of programming in his job,

> *[sic] . . . much more importantly, it had nothing to do with using [the programs we wrote] for anything practical. It had to do with*

[2] This isn't actually true. Math isn't hard, stupid!

[3] For you youngsters, Steve Jobs is the former CEO of Apple. He's dead now.

> *using them to be a mirror of your thought process; to actually learn how to think. I think everybody in this country should learn how to program a computer — should learn a computer language — because it teaches you how to think.*

That first sentence or two is actually a pretty good description of mathematics (and programming). Both are incredibly useful, and have endless practical applications. That's not the point, though. The whole usefulness thing is a side gig. It's about learning how to think, and having a rigorous language through which to express your thoughts. Furthermore, the rigor of the language helps you build upon your current thoughts to find out even cooler things. That's what math is about.

Programming and math go hand-in-hand. Programmers and mathematicians will attest to this; I certainly can. For that reason, throughout this book, there will be coding exercises in the programming language Idris. Idris is an interesting programming language for many reasons. The chief of which is that it can be used to prove things mathematically. Most programming languages can't do this. Idris can, which is why it is special.[4]

1.3.1 Installing Idris

This is something that is actually rather difficult to summarize, because it varies from operating system to operating system. I will put down the instructions for the operating systems I use. If you come upon this and don't see your operating system, please report this on the issue tracker (`https://gitlab.com/lysa/lysa/issues/new`). Better yet, you could add the instructions yourself, and ask me to merge your changes.

Linux

Gentoo You will need the Haskell platform, along with the GNU Multiple Precision (GMP) library. As of 5 January 2015, the Haskell platform is only

[4]There are other programming languages that can prove things, namely Coq and Agda. However, I'm most familiar with Idris, and Idris is probably the most useful, so I'm using Idris. Deal with it.

available on `~ARCH`, where `ARCH` is your processor architecture (e.g. `amd64`, `x86`). If you use `ARCH`, you can enable these by adding the following to `/etc/portage/package.keywords`:[5]

```
dev-lang/ghc
dev-haskell/cabal-install
```

Regardless of your `ACCEPT_KEYWORDS` variable, you should add the following to `/etc/portage/package.use`:

```
dev-lang/ghc binary
```

Otherwise, you have to compile GHC (the Haskell compiler) from scratch, and that takes forever.

Once you have that all out of the way, you'll want to run the following command as root:

```
# emerge -jav dev-lang/ghc dev-haskell/cabal-install
```

Warning: `-j` will make the installation a lot faster, but is more resource-intensive. If your power usage is precious, omit it (i.e. use `-av` instead).

Once GHC and cabal-install are installed, you'll want to run the following as a normal user:

```
% cabal update
% cabal install alex happy haddock hscolour
% cabal install idris
```

You can then get at the Idris shell by running `idris`.

[5] If you already use `~ARCH`, you can ignore this

1.4 Licensing

This book is free, in the sense of freedom. You can copy this book and give it to your friend. You can even print it out and sell it to your friends.[6]

If, for instance, you are a schoolteacher and want to use this for your class, you are free to edit it to your liking and give the modified copy to your students.

LYSA is licensed under the GNU Free Documentation License. § A contains the license. Please read the license; it's actually pretty comprehensible.

The source for this book can be downloaded at `https://gitlab.com/lysa/lysa/repository/archive.tar.gz`. If you are looking to contribute, it's probably best to clone the git repository. You can clone the git repository by running `git clone https://gitlab.com/lysa/lysa.git` in a terminal.

[6] There are some restrictions though, see § A.

Chapter 2

Booleans, simple logic, and simple operators

Before we get into interesting content, you have to understand some stuff. This stuff is pretty easy. This will likely be the shortest and easiest chapter in the book.

You might think math is about dealing with numbers and pumping out formulas. Well, that's not what math is about. As said in § 1.3, it's about using math as a language to express your thoughts. Most people don't think about numbers all day; thus, we deal with things in math that aren't numbers.

In this next section, we're going to outline some basic rules for reasoning about things. You need to know these rules to do really cool stuff. Although, as you will (hopefully) see, these rules can be fun to toy around with on their own.

2.1 Basic Logic

2.1.1 True and False

True or False - in your education, you've been faced with questions where the answer is "True" or "False"? Well, hopefully the answer is "True". In math, we're going to deal with things where we have to decide whether something is true or not. Moreover, there are going to be a bunch of statements that may or may not be true, and we'll have to figure out how they relate to each other. With that in mind, this section has some rules for dealing with truth and falsehood.

We're going to create two "values", and they are called True and False, respectively. These values are called "Booleans". They are named after a mathematician named George Boole, who studied them rather extensively.

True and False are not very interesting on their own. However, when we have a bunch of things that are either True or False, we can combine them together in three basic ways:

- $A \wedge B$ should be read as "A logical-and B." Both have to be True. If one of them is False, than $A \wedge B$ is False. Likewise, if A and B are both True, then $A \wedge B$ is True.
- $A \vee B$ should be read as "A logical-or B." With $A \vee B$, if one of A and B is True, then $A \vee B$ is True. It's okay if both of them are True.
- Sometimes we are going to want to say 'A is not True'. Instead of writing that out each time, I'm instead going to use the symbol $\neg$. So, $\neg A$ should be read 'logical-not A'.

2.1.2 Logic

Mathematicians are too lazy to write things like "if A is True, then B is True, but if B is True, it doesn't necessarily imply that A is True." So, in math, they use these symbols. I'm also lazy, so I'm going to use these symbols.

- $A \implies B$ means that "if A is True, that means that B must be True." Note that this **does not** mean that "if B is True, then A is True." Always follow the arrow. $A \implies B$ should be read as "A implies B."
- $A \impliedby B$ is the same thing as writing $B \implies A$. It's often convenient to write $A \impliedby B$ though. $A \impliedby B$ should be read "A is implied by B."
- $A \iff B$ means that $A \implies B$ and $B \implies A$. You can think of $A \iff B$ as meaning "Saying A is the same thing as saying B." You should read $\iff$ as "if (and only if)." "If (and only if)" takes a while to write, and $\iff$ is often contextually inappropriate. So, I'll sometimes use iff (with two 'f's) in the place of "if (and only if)".
- $A \not\implies B$ means "A does not imply B." **However**, this does not mean, A implies that B is false. It simply means that knowing something about A doesn't tell you anything about B. Got it? The analog is what you'd expect for $A \not\impliedby B$.
- For any A, it is always true that $A \implies A$.
- For any A, it's always true that $A \not\implies \neg A$

Now, I'm very lazy, so I'm going to quit writing "for every A,B, and C ..." Instead, I'm going to use the symbol $\forall$. It's an upside-down A, and it stands for 'all'. You should read it as 'for all'. So, the above statement is $\forall A, B, C \ldots$

- Alright, time for some notation:

  $$\forall A, B, C;\ ((A \implies B) \wedge (B \implies C)) \implies (A \implies C)$$

  For this reason, we can write things like $A \implies B \implies C$. [1]

  You were probably really confused by that last glob of math. Let me read it out for you. "for all A, B, and C, if we know that $A \implies B$,

[1] A common critique of this practice has to do with associativity. That is, many people read $A \implies B \implies C$ as $(A \implies B) \implies C$. This translates to "if A implies B, then C is True", which isn't *quite* what we want. The solution is to not try to group the operators like that, or use parentheses when you do want to group them.

and $B \implies C$, then it's true that $A \implies C$." Sometimes, I'll write the $\forall \ldots$ part after the statement. So, I should have written the above as

$$((A \implies B) \wedge (B \implies C)) \implies (A \implies C)\,;\ \forall A, B, C$$

- If you know that $A \implies B$, then $\neg A \impliedby \neg B$.

Be careful. If two independent statements happen to be True, it doesn't mean that one implies the other. Thus, the values A, B, and C are not themselves the values True and False, but are instead statements, that, when evaluated, happen to be True or False.

2.1.3 Exercises

Ex. 1 — Given A, is it always the case that $A \iff A$?

Ex. 2 — Given A, is it the case that $A \wedge A \iff A$?

Ex. 3 — Given A and B, is it always the case that $A \wedge B \iff B \wedge A$?

Ex. 4 — Given A, B, and C, is it always the case that $(A \wedge B) \wedge C \iff A \wedge (B \wedge C)$?

Ex. 5 — Given A, is it the case that $A \vee A \iff A$?

Ex. 6 — Given A and B, which are both True/False values, is it always the case that $A \vee B \iff B \vee A$?

Ex. 7 — Given A, B, and C, is it always the case that $(A \vee B) \vee C \iff A \vee (B \vee C)$?

Ex. 8 — Given A, B, and C, what is the result of $A \wedge (B \vee C)$?

Ex. 9 — What do you think the result of $\neg (A \wedge B)$ is?

Ex. 10 — What do you think the result of $\neg (A \vee B)$ is?

Ex. 11 — We know, from one of the laws

$$(A \implies B) \implies (\neg A \impliedby \neg B)\,;\ \forall A, B$$

Is the following true?

$$(A \implies B) \iff (\neg A \impliedby \neg B)\,;\ \forall A, B$$

In the future, I won't put "is the following true…", mostly because I'm lazy, but it also means that you, the reader, might have a difficult time trying to figure out what the point of controversy is. So, in the future, I'll instead write:

$$(A \implies B) \overset{?}{\iff} (\neg A \impliedby \neg B)\,;\ \forall A, B$$

(Notice the ? over $\iff$). Now you know exactly what the question is, and I don't have to write as much.

Answers

Answer (Ex. 1) — Yes. We know that $A \implies A$. Remember, $A \iff A$ is just saying that $(A \implies A) \wedge (A \impliedby A)$. Also recall that $A \impliedby B$ is the lazyman's way of writing $B \implies A$. Thus, if you know $A \implies A$, then it must be true that $A \impliedby A$, and therefore $A \iff A$.

Answer (Ex. 2) — Yes. There are two cases we need to deal with here:

$$\begin{cases} A := \text{True} \rightarrow & \text{True} \wedge \text{True} & \iff & \text{True} \\ A := \text{False} \rightarrow & \text{False} \wedge \text{False} & \iff & \text{False} \end{cases}$$

In both cases, $A \wedge A \iff A$. Q.E.D. This technique is called "proof by exhaustion". We named every possible case — in this case, there were only two — and proved the theorem for each of them.
You might be wondering what $A := \text{True} \rightarrow \ldots$ is. It should be read "let A be True; then, ..."

Answer (Ex. 3) — Yes. In the previous problem, we showed that $A \wedge A \iff A$. Thus, if A and B are both True, or are both False, the answer is yes. Thus, the only case we need to consider is that in which A is Trueand B is False.[2]
If $\text{True} \wedge \text{False} \iff \text{False}$, and $\text{False} \wedge \text{True} \iff \text{False}$.

Answer (Ex. 4) — Yes.

Answer (Ex. 5) — Yes.

Answer (Ex. 6) — Yes.

[2]You may be wondering why I'm not considering the case when A is False, and B is True. However, as we showed earlier, it's the case that if $A \iff B$, then $B \iff A$. Thus if $A \wedge B \iff B \wedge A$, then $B \wedge A \iff A \wedge B$

Answer (Ex. 7) — Yes.

Answer (Ex. 8) — $A \wedge (B \vee C) \iff (A \wedge B) \vee (A \wedge C)$

Answer (Ex. 9) — I will explain in § 2.2, but the answer is

$$\neg (A \wedge B) \iff (\neg A) \vee (\neg B) ; \forall A, B$$

Answer (Ex. 10) — I will explain in § 2.2, but the answer is

$$\neg (A \vee B) \iff (\neg A) \wedge (\neg B) ; \forall A, B$$

Answer (Ex. 11) — Yes, by just applying the law again. Let's restate the law:

$$\begin{aligned}
(A \implies B) &\implies (\neg A \impliedby \neg B) ; \forall A, B \\
(\neg A \impliedby \neg B) &\impliedby (A \implies B) ; \forall A, B \\
(\neg B \implies \neg A) &\impliedby (B \impliedby A) ; \forall A, B \\
(\neg B \implies \neg A) &\impliedby (B \impliedby A) ; \forall A, B
\end{aligned}$$

Let $C := \neg B$, $D := \neg A$

$$(C \implies D) \impliedby (\neg C \impliedby \neg D) ; \forall C, D$$

We know, from the law,

$$(C \implies D) \implies (\neg C \impliedby \neg D) ; \forall C, D$$

Therefore,

$$\begin{aligned}
(C \implies D) &\iff (\neg C \impliedby \neg D) ; \forall C, D \\
(A \implies B) &\iff (\neg A \impliedby \neg B) ; \forall A, B
\end{aligned}$$

Q.E.D.

2.2 Combining not, and, and or

In the previous section, we took a cursory look into the operators $\implies$, $\impliedby$, $\iff$, $\wedge$, $\vee$, and $\neg$.

I'm only going to introduce 2 new rules in this section, and here they are

- $\neg(\neg A) \iff A$
- $\neg(A \wedge B) \iff (\neg A) \vee (\neg B)$

  Intuitively you can think of this law this way: both A and B have to be true for $A \wedge B$ to be true. So, if one of them isn't, the entire condition is false. The way to say "the entire condition isn't true" is $\neg(A \wedge B)$. The way to say "one of them isn't true" is $(\neg A) \vee (\neg B)$.

The last rule is

$$\neg(A \vee B) \iff (\neg A) \wedge (\neg B)\,;\ \forall A, B$$

Proof. This can be derived from the last law. Let $C = \neg A$, and $D = \neg B$.

$$\begin{array}{rcll}
\neg(A \wedge B) & \iff & (\neg A) \vee (\neg B) & ;\ \forall C, D \\
\neg((\neg C) \wedge (\neg D)) & \iff & C \vee D & ;\ \forall C, D \\
C \vee D & \iff & \neg((\neg C) \wedge (\neg D)) & ;\ \forall C, D \\
\neg(C \vee D) & \iff & \neg(\neg((\neg C) \wedge (\neg D))) & ;\ \forall C, D \\
\neg(C \vee D) & \iff & (\neg C) \wedge (\neg D) & ;\ \forall C, D
\end{array}$$

□

So what the hell is all that? Well, this is your first taste of a *mathematical proof*. Basically, we start with a set of rules which we know are true (the *laws* or *axioms*). We build them together to form *theorems*. That little white box at the bottom right is to say "okay, here's where I'm done proving stuff."

I will also use the acronym Q.E.D., which is short for "quod erat demonstrandum", which is fancy-talk for "I'm done proving stuff".

This really is your second taste of proofs, provided you did the exercises for § 2.1. You may have seen the terms "proof" and "Q.E.D" thrown around a lot. Well, now you know what they mean — hopefully.

2.2.1 Exercises

Ex. 12 — Show that $(A \not\Leftrightarrow B) \iff ((A \not\Rightarrow B) \vee (B \not\Rightarrow A))$

Answers

Answer (Ex. 12) — Let's restate the problem.

$$(A \not\Longleftrightarrow B) \iff ((A \not\Longrightarrow B) \vee (B \not\Longrightarrow A))$$

$$\neg(A \iff B) \iff (\neg(A \implies B) \vee \neg(B \implies A))$$

$$\neg(A \iff B) \iff \neg((A \implies B) \wedge (B \implies A))$$

$$\neg(A \iff B) \iff \neg(A \iff B)$$

Q.E.D.

2.3 Idris

This section provides a "working example" of the above in Idris. If you don't know what that is, you are a bad person. Go back and read § 1.3!

Open up an Idris prompt, and enter `:type True`. That is basically asking Idris "what type of thing is True?" Idris will tell you. Also do the same thing for False. Here's what happens when I do it:

```
Idris> :type True
Prelude.Bool.True : Bool
Idris> :type False
Prelude.Bool.False : Bool
```

If you ask Idris what the type of `True` or `False` is, it will tell you that they are `Bools`.[3] You're probably thinking "what the hell is a `Bool`?" `Moreover, why the hell is this guy printing all this stuff in monospace`? Well, explaining this sorta stuff, that's what I'm here for. `Bool` is short for Boolean, which is what this chapter is about.

The reason I'm printing stuff `in monospace` is to say "hey, this is code." More importantly, printing stuff in monospace eschews some formatting flukes caused by variable-width text. In normal paragraph text, these flukes are fine — they are even helpful — but they cause some ambiguities in code. For that reason, the standard thing to do is to write code in monospace.

In Idris, and in most programming languages, the $\wedge$ operator is replaced with `&&`. We know that, in Idris True and False are both Boolean values. What about True $\wedge$ False?

```
Idris> :type (True && False)
True && Delay False : Bool
```

[3]For the most part, you can ignore all the weird stuff on the left-hand-side, for the time being. For instance, when I ran `:type True`, Idris switched `True` to `Prelude.Bool.True`. These are odd caveats of Idris's syntax, which I don't have time to explain right now. We'll get to them later, I promise.

So, wait, True *and* False *are both Boolean values, but* True ∧ False *is also a Boolean value?*

Yes, Timmy, now try to keep up.

Now, from the explanation of ∧ in § 2.1, True ∧ False should only be true if both True and False are True. Well, that's obviously not the case, so True ∧ False should turn out to be False, right? Let's see!

```
Idris> True && False
False : Bool
```

What about ∨? In Idris — and most programming languages — ∨ is replaced with `||`.

```
Idris> True || False
True : Bool
```

Chapter 3

Sets, Proofs, and Functions

Alright, we're done with Booleans! Sort of. The next thing we are going to look at are *sets*.

Sets were first studied by Georg Cantor, a German mathematician, in the second half of the nineteenth century. Back in his own day, the results Cantor found by studying sets were considered so thoroughly bizarre that many of his colleagues simply refused to believe that Cantor could be right. In the end, Cantor turned out to be right all along. His ideas can be found in any introductory text on mathematics—including this one.

Sets are basically like lists— think "your grocery list" or "your to-do-list" — except there's no multiplicity, and there's no intrinsic order. A "list" is exactly what you think it is. It's a bunch of things. The standard notation is to use { Braces } for sets, and (Parentheses) for lists. Lists can have duplication, and order does matter.

(4), (4, 4), and (4, 4, 4) are all different **lists**; { 4 }, { 4, 4 }, and { 4, 4, 4 } are the same **set**. In a **list**, each 4 is considered a separate item. In a **set**, 4 can appear a billion times, but it's only counted once.

(1, 2, 3), (3, 1, 2), and (2, 3, 1) are all different **lists**. { 1, 2, 3 }, { 3, 1, 2 }, and { 2, 3, 1 } are all the same **set**. In a **set**, the order in which items appear doesn't matter; all that matters is that the items are there. In a **list**, however, the order is important.

In a list, order and multiplicity matter. In a set, order and multiplicity are ignored. If you can't remember whether to use braces {the curly things}, or parentheses (the round things), remember: *a **brace** is used to **set** a broken bone.* I don't have a horrible pun having to do with parentheses and lists, and for that, I apologize.

3.1 Elements, Subsets, and Supersets

Lists become important later on. Sets are, in general, much more important, so we'll focus on them for the time being. I'm going to throw a bunch of examples at you, then give you formal definitions. I think the examples are less confusing, but the definitions are more useful, hence the order.

3.1.1 Examples

1. $0, 1, 2, 3$ **is not** a set, it is instead the numbers 0 through 3, which are to be considered separately.
2. $\{\, 0, 1, 2, 3 \,\}$ **is** indeed a set (notice the braces).
3. If some object is in a set, the notation is 'OBJECT $\in$ SET', which should be read "OBJECT is an element of SET".
4. If the object is not in the set, the notation is 'OBJECT $\notin$ SET', which should be read "OBJECT is not an element of SET".
5. $0 \in \{\, 0, 1, 2, 3 \,\}$ should be read "0 is an element of $\{\, 0, 1, 2, 3 \,\}$", and indeed it is.
6. $0, 1 \in \{\, 0, 1, 2, 3 \,\}$ should be read "0 and 1 **are both** elements of $\{\, 0, 1, 2, 3 \,\}$". They are.
7. $\{\, 0, 1 \,\} \notin \{\, 0, 1, 2, 3 \,\}$ should be read "$\{\, 0, 1 \,\}$ **is not** an element of $\{\, 0, 1, 2, 3 \,\}$". $\{\, 0, 1 \,\}$ is indeed not an element of $\{\, 0, 1, 2, 3 \,\}$.

Now, here, we're faced with an interesting problem. $\{\, 0, 1, 2, 3 \,\}$ encapsulates $\{\, 0, 1 \,\}$, but $\{\, 0, 1 \,\}$ is not an element of $\{\, 0, 1, 2, 3 \,\}$. So, how do we

express this notion of 'encapsulation'? The answer is by using "subset" notation.

1. $\{0,1\} \subseteq \{0,1,2,3\}$is **true**. $\{0,1\} \subseteq \{0,1,2,3\}$should be read as "$\{0,1\}$ is an **improper** subset of $\{0,1,2,3\}$."

2. $\{0,1\} \subset \{0,1,2,3\}$is **true**. $\{0,1\} \subset \{0,1,2,3\}$should be read as "$\{0,1\}$ is a **proper** subset of $\{0,1,2,3\}$."

3. $\{0,1,2,3\} \subseteq \{0,1,2,3\}$ is **true**. Intuitively, you should think "$\{0,1,2,3\}$ is entirely encapsulated inside of $\{0,1,2,3\}$."

4. $\{0,1,2,3\} \subset \{0,1,2,3\}$ is **false**. This highlights the difference between $\subset$and $\subseteq$.

   $\{0,1\} \subset \{0,1,2,3\}$implies that $\{0,1,2,3\}$ contains some stuff that is not contained in $\{0,1\}$. $\{0,1\} \subseteq \{0,1,2,3\}$does not have this implication. That is, $\{0,1\} \subseteq \{0,1,2,3\}$does not say for sure that $\{0,1,2,3\}$ has more stuff than $\{0,1\}$, it just acknowledges it as a possibility.

Alright, my apologies, that was a lot of stuff to throw at you at once, and you probably remember absolutely none of it. However, you'll get used to that particular *set* of stuff[1] as time goes along.

3.1.2 Formal Definitions

I'm all for formalism up to the point at which it hinders your understanding of the material. In most cases, formalism does hinder your understanding. However, in this case — when talking about definitions — precise and formal definitions are usually the most helpful.

Set A collection of items, called "elements". The elements have no intrinsic order, and no multiplicity.

[1] I, for one, would never condone such terrible puns.

Elements Given a set S and an element x, $x \in S$ is the formal notation for saying "S contains x."

Non-elements Given a set S and an element x, $x \notin S$ is the formal notation for saying "S does not contain x."[2]

Improper subsets If you want to say "all of the elements in S are in T, but not necessarily the other way around" the notation is $S \subseteq T$. This is to be read "S is an improper subset of T".

More formally, $S \subseteq T$ means that for all elements x, such that $x \in S$, it is true that $x \in T$. In purely mathematical notation, this is written

$$S \subseteq T \iff \forall \{ x \mid x \in S \}, x \in T$$

That should be read "S is an improper subset of T if and only if for all x, such that x is an element of S, x is an element of T."

Improper supersets If you want to say "S contains all of the elements in T", you write $S \supseteq T$. $S \supseteq T$ should be read "S is an improper superset of T." Note that $S \supseteq T$ is the same as writing $T \subseteq S$, but it's often convenient to have the superset notation.

Equality of Sets A set S is equal to another set T if and only if $S \subseteq T$ and $S \supseteq T$. The notation for this is $S = T$, which is to be read "S is equal to T." That is,

$$S = T \iff (\forall \{ x \mid x \in S \}, x \in T) \wedge (\forall \{ x \mid x \in T \}, x \in S)$$

Proper subsets S is a proper subset of T if and only if $S \subseteq T$ and $S \neq T$. This is to be written $S \subset T$

Proper supersets S is a proper superset of T if and only if $S \supseteq T$, and $S \neq T$ This is to be written $S \supset T$.

[2] In general, if you want to say "something is not true," you cross out the operator in question. For instance, if you want to say "$x = y$ is not true", the notation is $x \neq y$.

3.1.3 Properties of Equality

This section doesn't quite fit in anywhere, but it's needed right now, so here it goes for the time being. I'm going to state a few properties of equality, which you should already know and/or should be obvious.

- $a = a;\ \forall a$
- $(\, a = b \,) \iff (\, b = a \,);\ \forall a, b$
- $(\, (\, a = b \,) \wedge (\, b = c \,) \,) \implies (\, a = c \,);\ \forall a, b, c$
- $(\, a \neq b \,) \iff \neg (\, a = b \,);\ \forall a, b$
- $\neg (\, a \neq a \,);\ \forall a$

3.1.4 Exercises

Ex. 13 — Given a set S, and some object n. If it is known that $n \in S$, does it make sense to ask "how many times does S contain n?" Why or why not?

Ex. 14 — $S \subset T \overset{?}{\iff} \exists \{ x \mid (x \in T) \wedge (x \notin S) \}$

Ex. 15 — $S \subset T \overset{?}{\implies} S \supseteq T$

Ex. 16 — $S = T \overset{?}{\implies} S \subseteq T$

Ex. 17 — $S \subseteq T \overset{?}{\implies} S \subset T$

Ex. 18 — $((n \in S) \wedge (S \subseteq T)) \overset{?}{\implies} n \in T$

Ex. 19 — $((n \in T) \wedge (S \subseteq T)) \overset{?}{\implies} n \in S$

Ex. 20 — $S \subseteq T \overset{?}{\impliedby} \{ n \in S, n \in T \}$

Answers

Answer (Ex. 13) — No, because repetition doesn't matter.

Answer (Ex. 15) — Yes. $S \subseteq T$ means that one of $\{ S \subset T, S = T \}$ is true. If $S \subset T$ is true, then the condition of $S \subseteq T$ is satisfied. Thus, $S \subset T \implies S \subseteq T$.

Answer (Ex. 16) — Yes, by the same logic above. If $S = T$, then the conditions for $S \subseteq T$ are satisfied.

Answer (Ex. 17) — No.

- $S \subset T$ can be stated as "all of $\{ S \subseteq T, S \neq T \}$ are true."
- By the previous problem, if $S \neq T$ is false (i.e. $S = T$ is true), then $S \subseteq T$ is true.
- In this case, the set above could be written as $\{ S \subseteq T, S \neq T \} = \{ True, False \}$. $S \subset T$ means that both of those have to be true. That is, $S \subset T$ is true if (and only if) $\{ S \subseteq T, S \neq T \} = \{ True, True \}$.
- In this case, we have $\{ S \subseteq T, S \neq T \} = \{ True, False \}$, which means $S \not\subset T$.
- Thus, we have just found a case in which $S \subseteq T$ is true, but $S \subset T$ is false. Therefore, it cannot be the case that $S \subseteq T \implies S \subset T$. Or, to put it another way, $S \subseteq T \not\implies S \subset T$.

Answer (Ex. 18) — Yes. $S \subseteq T$ means that all of the elements in S are also in T, but not necessarily the other way around.. Thus, if $n \in S$, it must be true that $n \in T$.

Answer (Ex. 19) — No. $S \subseteq T$ means that it is possible that $S \subset T$ is true. $S \subset T$ means that T has all of the elements in S, but there are also

more elements. It's entirely plausible that n is one of the additional elements. Thus,

$$((n \in T) \wedge (S \subseteq T)) \not\Rightarrow n \in S$$

Answer (Ex. 20) — No. $n \in S$ can be stated as $\{n\} \subseteq S$. Likewise, $n \in T \iff \{n\} \subseteq T$. This means there are 3 possible cases for the relation between S and T:

1. $S = T$
2. $S \subset T$
3. $S \supset T$

$S \subseteq T$ means that either 1 or 2 is true, but the third one is false. The third one can be true when $\{n \in S, n \in T\}$ is true. Thus, we have found a case in which $\{n \in S, n \in T\}$ is true, but $S \subseteq T$ is false. Thus, it must be that.

$$S \subseteq T \not\Leftarrow \{n \in S, n \in T\}$$

3.2 Operators and Functions

This section will go over the general idea of functions — and therefore operators. Remember back to § 2, I yammered on about the idea where you could take a Boolean value, call it A, and then you could do $\neg A$, which would still be a Boolean value. Well, most people would call $\neg$ an "operator." Basically, it takes a Boolean, and gives you another Boolean. I'm lazy, so instead of "Boolean value", I'm just going to write "Boolean," or sometimes "Bool."

It's convenient for us lazy math people to have a notation for "$\neg$ takes a Boolean, and gives you another Boolean." First, we need a symbol for "this is a Boolean value." $\mathbb{B}$. There we go. So, here's the notation for "$\neg$ takes in a Boolean and spits out a Boolean."

$$\neg : \mathbb{B} \to \mathbb{B}$$

This property is called *algebra*, in the sense that the operator $\neg$ is *algebraic*. An operator is *algebraic* if it takes an element from a given set, and then spits back out an element of that set.

So, what really is $\mathbb{B}$? Well, it's the set of Booleans.

Wait, we're using sets already for stuff?

Yes, silly, why do you think I was talking about them?

So, basically:

$$\mathbb{B} = \{ \text{True}, \text{False} \}$$

Here's something super cool and fun to do in math, that we'll do a lot. We're going to take concepts that seem impossible to define, and define them. I bet the average mathematician is capable of defining irony. It seems boring, but it's actually pretty fun.

So, try to write down a definition of $\neg$.

You probably can't do it, can you? It just seems so obviously true. Well,

the easiest thing to do is to just list some properties of $\neg$, and then usually a definition emerges.

We already know

What's the result of $\neg$True? Well, False, of course! So:

$$\neg\text{True} = \text{False}$$

We also know that $\neg\,(\,\neg A\,) = A;\ \forall A$, so we can write down:

$$\neg\,(\,\neg A\,) = A;\ \forall A$$

(you should actually be writing this down on paper, by the way)

So, we're lazy, and we don't like writing $;\ \forall A$ or $;\ \forall A, B, C$ or whatever every time we write something down. It just gets annoying. Instead, we're going to use the symbol $\equiv$. You can read $\equiv$ as "identically equal to." So, $A \equiv B$ is just the lazyman's way of writing $A = B;\ \forall A, B$. Thus, we can restate the previous definition as:

$$\neg\,(\,\neg A\,) \equiv A$$

Let's recollect all the information we have so far:

$$\begin{aligned}
&\mathbb{B} = \{\,\text{True}, \text{False}\,\}\\
&\neg : \mathbb{B} \to \mathbb{B}\\
&\neg\text{True} = \text{False}\\
&\neg\,(\,\neg A\,) \equiv A
\end{aligned}$$

This is almost enough. We need one more thing. You could state that $\neg$False = True, which is True, but not as useful as what we're going to say:

$$(\,A = B\,) \implies (\,\neg A = \neg B\,);\ \forall A, B$$

You might be asking "he used a $\forall$ and $=$, why didn't he use the cool $\equiv$ symbol?" Really, because it would have been wrong. $\equiv$ is really a lazyman's amalgamation of $=$ and $\forall$. So, in that last thing, I was talking about a situation involving three separate operators: the $=$ on the left, the $\implies$, and the $=$ on the right. It was important to note that the $\forall A, B$ applies to all of those operators collectively. $\equiv$ would mean that the $\forall$ only applies to the $=$ operator.

Anyway, here's what we know:

$$\begin{aligned}
&\mathbb{B} = \{\, \text{True}, \text{False} \,\} \\
&\neg : \mathbb{B} \to \mathbb{B} \\
&\neg \text{True} = \text{False} \\
&\neg\, (\, \neg A \,) \equiv A \\
&(\, A = B \,) \implies (\, \neg A = \neg B \,)\,;\ \forall A, B
\end{aligned}$$

This is enough to define $\mathbb{B}$. Here's one quick exercise, then we'll move on.

3.2.1 Exercises

Ex. 21 — Using only the definition above, (and the definition of $=$), show that $\neg False = True$.

Ex. 22 — Use the following definition of $\wedge$ to prove:

- $A \wedge A \equiv A$
- $(\, A \wedge B \,) \wedge C \equiv A \wedge (\, B \wedge C \,)$

$$\begin{array}{l} \wedge : \mathbb{B}^2 \to \mathbb{B} \\ A \wedge \text{True} \equiv A \\ A \wedge B \equiv B \wedge A \end{array}$$

By the way, the $\mathbb{B}^2$ thing just means it needs two elements of $\mathbb{B}$ to do its job.

Ex. 23 — Do the same thing for $\vee$

Ex. 24 — Use this definition (and those aforementioned):

$$\neg\, (\, A \wedge B \,) \equiv \neg A \vee \neg B$$

to prove

$$\neg\, (\, A \vee B \,) \equiv \neg A \wedge \neg B$$

Answers

Answer (Ex. 21) — *Proof.*From the definition of ¬:

$$\neg\text{True} = \text{False}$$

By the commutativity of equality:

$$\text{False} = \neg\text{True}$$

From the definition of ¬:

$$(\, A = B \,) \implies (\, \neg A = \neg B \,);\ \forall A, B$$

Therefore,

$$\neg\text{False} = \neg\,(\,\neg\text{True}\,)$$

From the definition of ¬:

$$\neg\,(\,\neg A\,) \equiv A$$

Therefore,

$$\neg\,(\,\neg\text{True}\,) = \text{True}$$

By the transition of equality,

$$\neg\text{False} = \text{True}$$

□

3.2.2 Moving on

Before we got distracted with all that proof stuff, we were talking about operators. $\neg$ is rather narrow in scope. It deals with literally two values. There are some operators that deal with more values.

You remember numbers, right? Well, let's look at an operator, which takes a number, and adds 5 to it. We're going to call that operator f. Here's how we're going to write f:

$$f = \lambda\,(\,x\,) \to x + 5$$

If we want to say f evaluated where $(\,x = 1\,)$, we'll write

$$f_{[\,x=1\,]}$$

Let's carry this out:

$$\begin{array}{rcl} f & = & \lambda\,(\,x\,) \to x + 5 \\ f_{[\,x=1\,]} & = & \lambda\,(\,x = 1\,) \to x + 5 \\ & = & 1 + 5 \\ & = & 6 \end{array}$$

Hopefully you get the idea. f evaluated where $x = \mathit{something}$ basically means "replace x with *something*, and carry out the calculation."

That thing where you have to wrap the parentheses around both of the things:

I'm going to write another operator, g, which takes two numbers. It adds 5 to the first number, and subtracts 2 from the second. Then it multiplies the intermediate results together.

$$g = \lambda\,(\,x, y\,) \to (\,x + 5\,) \times (\,y - 2\,)$$

So, let's see what happens when we carry out g when $x = 4$ and $y = 6$. First of all, how do you write that?

$$\begin{aligned} g_{[x=4,y=6]} &= \lambda\,(\,x = 4, y = 6\,) \to (\,x + 5\,) \times (\,y - 2\,) \\ &= (\,4 + 5\,) \times (\,6 - 2\,) \\ &= 9 \times 4 \\ &= 36 \end{aligned}$$

What are the signatures[3] for these functions?

Well, f takes a number, and gives back a number. Our fancy-schmancy symbol for "numbers" is $\mathbb{N}$.

$$f : \mathbb{N} \to \mathbb{N}$$

Thus, because g requires 2 numbers to do its job, its signature is:

$$g : \mathbb{N}^2 \to \mathbb{N}$$

3.2.3 Currying

This is a nice point to break and talk about Currying.

Currying is basically another instance of mathematicians being clever and lazy at the same time.

Remember g, right?

$$g = \lambda\,(\,x, y\,) \to (\,x + 5\,) \times (\,y - 2\,)$$

Well, what happens when we let $x = 4$, but leave y alone?

[3](the thing with the colon and the arrows)

$$\begin{aligned} g_{[x=4]} &= \lambda\,(\,x = 4, y\,) \to (\,x + 5\,) \times (\,y - 2\,) \\ &= \lambda\,(\,y\,) \to (\,4 + 5\,) \times (\,y - 2\,) \\ &= \lambda\,(\,y\,) \to 9 \times (\,y - 2\,) \end{aligned}$$

So, in this case, g took in a number, and gave us back a function with the signature $\mathbb{N} \to \mathbb{N}$. So, in this case, g's signature is actually:

$$g : \mathbb{N} \to (\,\mathbb{N} \to \mathbb{N}\,)$$

Thus it might be more semantic to write g as:

$$g = \lambda\,(\,x\,) \to (\,\lambda\,(\,y\,) \to (\,x + 5\,) \times (\,y - 2\,)\,)$$

But wait, isn't that notation rather constrictive? What if we want to do $g_{[y=3]}$*?*

Since it's pretty clear that $\to$ is right-associative — i.e. it doesn't make sense to write $(\,\lambda\,(\,x\,) \to \lambda\,(\,y\,)\,) \to (\,x + 5\,) \times (\,y - 2\,)$ — then we can be lazy and write:

$$\begin{aligned} &g : \mathbb{N} \to \mathbb{N} \to \mathbb{N} \\ &g = \lambda\,(\,x\,) \to \lambda\,(\,y\,) \to (\,x + 5\,) \times (\,y - 2\,) \\ &g = \lambda\,(\,x, y\,) \to (\,x + 5\,) \times (\,y - 2\,) \end{aligned}$$

And thus, it no longer seems wrong to write $g_{[y=3]}$.

What's that λ symbol?

It's the Greek letter lambda, pronounced "lamb-duh."

3.2.4 Moving on Again

So, we've — in very little depth — covered the idea of what a function looks like. What about operators?

Well, operators are actually just really sneaky functions. Remember, the definition for $\neg$ looked an awful lot like our function.

$$\begin{aligned}
&\mathbb{B} = \{\, \text{True}, \text{False} \,\} \\
&\neg : \mathbb{B} \to \mathbb{B} \\
&\neg \text{True} = \text{False} \\
&\neg\,(\,\neg A\,) \equiv A \\
&(\,A = B\,) \implies (\,\neg A = \neg B\,)\,;\ \forall A, B
\end{aligned}$$

However, it is really awkward to write unary operators (things that take only one argument) in λ-notation. So, we just don't. Especially when the definitions are recursive ($\neg$'s definition uses $\neg$ in the definition). Your English teacher probably told you that it's not okay to use a word in its own definition. Well, in math, it is okay.

3.2.5 Function Composition

Okay, just two more little subsections, then you get to do all the wonderful exercises! Yay!

Function composition is another example of mathematicians being lazy. Let's look at that definition of f from above:

$$f = \lambda\,(\,x\,) \to x + 5$$

$$\begin{aligned}
&f : \mathbb{N} \to \mathbb{N} \\
&f = \lambda\,(\,x\,) \to x + 5
\end{aligned}$$

In that case, because the function only does one thing (adds 5), the whole thing about explaining that there's an x there is sort of obvious

$$f = \lambda \to (\,+5\,)$$

And even the $\lambda \to$ is getting in the way. It's pretty obvious that $(+5)$ isn't a number, especially when we annotate f with its signature. So, we can be lazy, and write:

$$\begin{aligned} f &: \mathbb{N} \to \mathbb{N} \\ f &= (+5) \end{aligned}$$

Yay for laziness!

Now, what if we want to add 5 to the number and then multiply the intermediate result by 10? Well, the verboseman's way of writing this is:

$$\begin{aligned} f &: \mathbb{N} \to \mathbb{N} \\ f &= \lambda\,(x) \to (x + 5) \times 10 \end{aligned}$$

Well, let's see. There's probably an easier way to write that. First, we know that $\times$ is commutative (we'll prove this later, don't worry), so we can write

$$f = \lambda\,(x) \to 10 \times (x + 5)$$

Let's give the intermediate result a value:

$$f = \lambda\,(x) \to 10 \times (\lambda\,(z = x) \to z + 5)$$

Basically f says "do q first, then multiply the result of q by 10." Well, this seems a little verbose. We're sort of smushing the functions $(\times 10)$ and $(+5)$ together. Well, luckily there's a way to write this:

$$f = (\times 10) \circ (+5)$$

Note that order is important. When evaluating $f_{[\ldots]}$, you do the thing on the right-hand-side of $\circ$ first, then the thing on the left afterwards.

You can read $\circ$ as "compose" or "of", depending on your taste.

3.2.6 Formal definitions

Again, now that I've given you the intuition, let's go on with the formal definitions:

Cartesian products

The superduper formal definition of function relies on the definition of Cartesian products. The Cartesian product operates on two sets. The Cartesian product of two sets is the set of all ordered pairs between the two sets. So,

$$A \times B = \{ (x, y) \mid (x \in A) \wedge (y \in B) \}$$

A function $f : A \to B$ is the set

$$\left\{ \left(x, f_{[x]}\right) \mid (x \in A) \wedge \left(f_{[x]} \in B\right) \right\}$$

There is also the condition that

$$(x = y) \implies \left(f_{[x]} = f_{[y]}\right)$$

This is harmless formalism that, like I said, becomes useful when we graph stuff.

Functions If a function f is a mapping from a set X to another set Y, the notation is $f : X \to Y$. f, when given an argument x, gives the result $f_{[x]}$. That is,

$$f \equiv \lambda (x) \to f_{[x]}$$

That is, f is a function, when given a value x, yields the value $f_{[x]}$. It's also the case that if $f : X \to Y$, and $x \in X$, then $f_{[x]} \in Y$. That is

$$(f : X \to Y) \wedge (x \in A) \iff f_{[x]} \in Y$$

Inverse Functions Many functions have an inverse. That is, if $f : X \to Y$, then the inverse ${}^{\text{arc}}[\, f \,]$ has the signature ${}^{\text{arc}}[\, f \,] : Y \to X$

This brings up a couple of points about functions. Mostly, they have to be determinant. That is, if you have some value x which is in the domain, there can only be one value $f_{[x]}$. (There's a graphical interpretation of this which is much easier to understand). The phrase "referentially transparent" means the same thing as "determinant."

Anyway, if a function isn't determinant, then it's not a function, it something functionyish. Determinance can be stated as such:

$$(x = y) \implies \left(f_{[x]} = f_{[y]} \right)$$

The inverse of f, noted ${}^{\text{arc}}[\, f \,]$ has the following properties:

$$\begin{aligned} {}^{\text{arc}}[\, f \,]_{\left[f_{[x]} \right]} &\equiv x \\ {}^{\text{arc}}\left[{}^{\text{arc}}[\, f \,] \right] &\equiv f \end{aligned}$$

If there are two values x and y such that $x \neq y$ but $f_{[x]} = f_{[y]}$, then there would be two possible x's to fit the definition of ${}^{\text{arc}}[\, f \,]$, and therefore ${}^{\text{arc}}[\, f \,]$ wouldn't be a function. That is,

Let $p = f_{[x]}$, $q = f_{[y]}$, $x \neq y$ and $p = q$. Let ${}^{\text{arc}}[\, f \,] = h$

$$\begin{aligned} h_{[p]} &= x \\ h_{[q]} &= y \\ p &= q \end{aligned}$$

If h is a function, then it must be true that $x = y$, which we know is not true. This means that there exists an f such that $\nexists\, {}^{\text{arc}}[\, f \,]$. There are many functions that are not invertible. As a rule, an invertible function has the following property

$$(\, x = y \,) \impliedby \left(f_{[\, x \,]} = f_{[\, y \,]} \right)$$

Domain & Codomain If $f : X \to Y$, then X is the *domain* of f, and Y is the *codomain* of f.

Image & Preimage If $f : X \to Y$, then the set of elements in Y which can be expressed as $f_{[\, x \,]}$ is called the *image* of f. The image of f is written ${}^{\text{im}}[\, f \,]$ The set of elements in X which can be used as an argument to f is called the *preimage*, denoted ${}^{\text{preim}}[\, f \,]$. That is,

$$\begin{aligned} {}^{\text{im}}[\, f \,] &= \left\{ f_{[\, x \,]} \mid (\, x \in X \,) \wedge \left(f_{[\, x \,]} \in Y \right) \right\};\ \forall f : X \to Y \\ {}^{\text{preim}}[\, f \,] &= \left\{ x \mid (\, x \in X \,) \wedge \left(f_{[\, x \,]} \in {}^{\text{im}}[\, f \,] \right) \right\};\ \forall f : X \to Y \end{aligned}$$

Injection & Surjection If a function's preimage is equal to its domain, then the function is *injective*. If a function's image is equal to it's codomain, then the function is *surjective*. If the function is both injective and surjective, then the function is *bijective*.

3.2.7 Exercises

Ex. 25 — What is the result of

$$\{\, 0, 1 \,\} \times \{\, 2, 3 \,\}$$

Ex. 26 — Write a bijective function f for the Cartesian product above. That is, write a function f such that

$$\begin{array}{l} \left\{ \left(x, f_{[x]} \right) \mid \left(x \in \{\, 0, 1 \,\} \right) \wedge \left(f_{[x]} \in \{\, 2, 3 \,\} \right) \right\} \\ \left(x = y \right) \iff \left(f_{[x]} = f_{[y]} \right) \end{array}$$

Ex. 27 — Given the f above, what is ${}^{\text{arc}}[\, f \,]$?

Ex. 28 — Show that a function is invertible iff it is bijective.

Ex. 29 — What are the signatures for each of the following?

1. $\subset$
2. $\subseteq$
3. $\supset$
4. $\supseteq$
5. $\not\subset$
6. $\not\subseteq$
7. $\not\supset$
8. $\not\supseteq$

(hint: they all have the same signature)

Ex. 30 — What is the signature for $\in$?

Answers

Answer (Ex. 25) — The result is the set of ordered pairs between the two sets, so:

$$\begin{aligned}\{\,0,1\,\}\times\{\,2,3\,\} \;=\; &\{\,(\,0,2\,)\\ &,\,(\,0,3\,)\\ &,\,(\,1,2\,)\\ &,\,(\,1,3\,)\\ &\}\end{aligned}$$

Answer (Ex. 26) — $f = (\,+2\,)$

Answer (Ex. 27) — ${}^{\mathrm{arc}}[\,f\,] = (\,-2\,)$

Answer (Ex. 28) — A function is **in**jective if its preimage equals its domain. It is **sur**jective if its image equals its codomain. This means that if f is bijective, then ${}^{\mathrm{arc}}[\,f\,]$ is also bijective.
This means that if a function is bijective, then it obviously has an inverse. (We've just shown that the inverse is bijective in this case, and that is predicated on the inverse existing.)
Let's assume that a function is not surjective. Then there is no inverse, because B is larger than ${}^{\mathrm{dom}}[\,{}^{\mathrm{arc}}[\,f\,]\,]$.
Let's assume that a function is not injective. Then there is no inverse, because there would then be two elements of B such that

$$\begin{aligned}&p, q \in B\\ &p = q\\ &{}^{\mathrm{arc}}[\,f\,]_{[\,p\,]} \neq {}^{\mathrm{arc}}[\,f\,]_{[\,q\,]}\end{aligned}$$

Which would mean that the inverse is not a function, and therefore doesn't exist.
Thus if a function has an inverse, then it must be bijective.
Q.E.D.

Answer (Ex. 29) — $\mathbf{Set} \to \mathbf{Set} \to \mathbb{B}$

Answer (Ex. 30) — Let a be any type of thing, like a number, or a boolean, or a set. Let $^{\mathbf{Set}}[\,a\,]$ be the set of sets where the elements are of type a.

$$\in \; : a \to {}^{\mathbf{Set}}[\,a\,] \to \mathbb{B}$$

(remember, $\mathbb{B}$ is the set of booleans.)

3.3 Unions and Intersections

So, we've talked briefly about sets. We're going to talk more about them a little bit, then move on to a very special set.

First, some notation. For now, we are going to denote "the set of sets" as **Set**. That is, for any set A, $A \in \mathbf{Set}$ is True. It's also true that $\mathbf{Set} \in \mathbf{Set}$, which is pretty weird, but we'll get to that later.

First, there's a trivial set, called $\varnothing$, which is just a set with no elements. It's obviously true that

$$\varnothing \subseteq A;\ \forall A \in \mathbf{Set}$$

We're going to define two operations here. The first is the "union:"

$$\begin{aligned} &\cup : \mathbf{Set} \to \mathbf{Set} \to \mathbf{Set} \\ &\cup = \lambda\,(\,A, B\,) \to \{\,x \mid (\,x \in A\,) \vee (\,x \in B\,)\,\} \end{aligned}$$

The $\cup$ looks vaguely like a U, so that's a mnemonic way you could think of it. You can also think of it as a "cup", and everything in either of the sets falls into the cup. Typically, we'll write $\cup_{[\,A,B\,]}$ as $A \cup B$, which should be read "A union B".

The other thing is for the intersect.

$$\begin{aligned} &\cap : \mathbf{Set} \to \mathbf{Set} \to \mathbf{Set} \\ &\cap = \lambda\,(\,A, B\,) \to \{\,x \mid (\,x \in A\,) \wedge (\,x \in B\,)\,\} \end{aligned}$$

It has the items that are in **both** of the sets. That is, if an item is in one set, but not the other, it's not in $A \cap B$, but it would be in $A \cup B$. $A \cap B$ should be read "A intersect B."

3.3.1 Exercises

Ex. 31 — $\varnothing \stackrel{?}{\in} \varnothing$

Ex. 32 — $\varnothing \stackrel{?}{\subset} A;\ \forall A \in \mathbf{Set}$

Ex. 33 — $(\, A \cap B \,) \stackrel{?}{\subseteq} (\, A \cup B \,);\ \forall A, B \in \mathbf{Set}$

Answers

Answer (Ex. 31) — No, because $\varnothing$ does not have any elements. To be even more general, there is no x such that $x \in \varnothing$. Therefore, $\varnothing \notin \varnothing$.

Answer (Ex. 32) — No, because $\varnothing \not\subset \varnothing$, and $\varnothing \in \mathbf{Set}$.

Answer (Ex. 33) — Yes. It's possible that there are elements in $A \cup B$ not in $A \cap B$, but it's not possible for something to be in $A \cap B$ but not in $A \cup B$.

Chapter 4

Proofs

4.1 Peano Axioms

This section covers the Peano axioms. As I said in **??**, these are a way for mathematicians to understand arithmetic.

Arithmetic (hopefully) seems simple enough, and easy to understand. Maybe an expression like

$$(2048282 \times 33221) + (3254 \times 11)$$

seems difficult to calculate, but you hopefully understand what each of the operators mean in concept. If you don't, well. In theory, reading this chapter alone will teach you arithmetic. However, I wrote this chapter assuming you already know arithmetic.

So, why is it important that you read this chapter?

Arithmetic is pretty simple and easy to understand. However, later on in this book, we're going to approach concepts that aren't so simple and easy to understand. Mathematicians have a systemic approach to these problems. This approach is called "mathematical proof." We prove things mathematically. Instead of approaching new concepts with proofs, I'm instead going to use proofs to illustrate some (hopefully) familiar concepts.

Alright, with all that out of the way, let's get started.

The basic idea of proofs is, you take a small set of obvious facts, called *axioms*, chain them together to make *theorems*. The following obvious facts, or axioms, are called the "Peano Axioms." They describe what we call "natural numbers." Natural numbers are the numbers $\{0, 1, 2, 3, 4, 5, 6, \ldots\}$.

Axiom 1 0 is a natural number. Again, obvious.

I'm going to use letters in the place of numbers right here. So, if I say "x is a natural number," that means that x is a placeholder for one of the numbers in $\{0, 1, 2, 3, 4, 5, 6, \ldots\}$. I could use any letter, such as a, b, q, r, θ, Γ, or $\aleph$. If I use a letter instead of a number, it usually means either

1. it doesn't matter which number I choose, or
2. it does matter which number I choose, but I don't know which number it is yet.

Axiom 2 If x is a natural number, it is true that $x \equiv x$.

You can read that $\equiv$ sign as $=$, for the time being. There are some subtle differences between the two, which I will get to in **??**. You are supposed to read $a \equiv b$ as one of these:

1. "a is equivalent to b,"
2. "a is identically equivalent to b,"
3. "a is congruent to b."

$=$ should be read as "a is equal to b," or "a equals b." Again the difference between $\equiv$ and $=$ isn't really important until **??**.

If you don't know what either of those signs are, $a \equiv b$ or $a = b$ means "a is the same thing as b." The difference can be summarized as $==\equiv\not=$.

So, in essence, this axiom says that each number is the same thing as itself. This is hopefully very obvious.

A math person would state this axiom as "congruence is reflexive."

Axiom 3 If x and y are both natural numbers, and $x \equiv y$, then it's true that $y \equiv x$. You can phrase this axiom as "if two numbers are the same number, then they are the same number." A math person would state this axiom as "congruence is symmetric."

Axiom 4 If x, y, and z are all natural numbers, and $x \equiv y$, and $y \equiv z$ then it's true that $x \equiv z$. You can phrase this axiom as "if three numbers are all the same number, then they are the same number." A math person would state this axiom as "congruence is transitive."

These last three axioms mean that we can be lazy, and write things like $a \equiv b \equiv c \equiv a$.

Axiom 5 If x is a natural number, and we know $x \equiv y$, then it's also true that y is a natural number. A math person would say "congruence forms a closure."

Axiom 6 If x is a natural number, then there is another number, $\text{suc}(x)$, which is also a natural number. suc is short for "successor." You should read $\text{suc}(x)$ as "the successor of x." You can think of the successor as "the next number." So, $\text{suc}(0) \equiv 1$, $\text{suc}(1) \equiv 2$, and so on.

Axiom 7 There isn't a number whose successor is 0. Basically this means "0 is the lowest natural number."

Axiom 8 If x and y are both natural numbers, and we know $\text{suc}(x) \equiv \text{suc}(y)$, then it's true that $x \equiv y$. This is what we would call the "converse" of Axiom 6. That is, Axiom 6 tells us that we can always go "up" a number. This axiom (almost) tells us that we can go "down" a number. Axiom 7 defines the limit of this, meaning that 0 is the only number where you can't go down any further.

Now, the previous 8 axioms have basically said "these numbers are all natural numbers." This next, and final axiom states "these numbers are all *of* the natural numbers."

Axiom 9 Let's say K is a set of numbers (a bunch of numbers). If we know that

1. if 0 is in K, and
2. if some number x is in K, then $\text{suc}(x)$ is in K,
3. then K contains every single natural number.

Continue with...

1. $\mathbb{N}$ is a Monoid
2. Quasigroups
3. Loops
4. Groups
5. Abelian Groups
   (a) $\mathbb{Z}$
      i. $\mathbb{Z}$ is an Abelian Group
      ii. $\mathbb{Z}$ is a ring
6. Fields
   (a) $\mathbb{R}$
      i. $\mathbb{R}$ is a field
7. Categories
8. Groupoids

Appendices

Appendix A

GNU Free Documentation License

Preamble

The purpose of this License is to make a manual, textbook, or other functional and useful document "free" in the sense of freedom: to assure everyone the effective freedom to copy and redistribute it, with or without modifying it, either commercially or noncommercially. Secondarily, this License preserves for the author and publisher a way to get credit for their work, while not being considered responsible for modifications made by others.
This License is a kind of "copyleft", which means that derivative works of the document must themselves be free in the same sense. It complements the GNU General Public License, which is a copyleft license designed for free software.
We have designed this License in order to use it for manuals for free software, because free software needs free documentation: a free program should come with manuals providing the same freedoms that the software does. But this License is not limited to software manuals; it can be used for any textual work, regardless of subject matter or whether it is published as a printed book. We recommend this License principally for works whose purpose is instruction or reference.

1. APPLICABILITY AND DEFINITIONS

This License applies to any manual or other work, in any medium, that contains a notice placed by the copyright holder saying it can be distributed under the terms of this License. Such a notice grants a world-wide, royalty-free license, unlimited in duration, to use that work under the conditions stated herein. The "**Document**", below, refers to any such manual or work. Any member of the public is a licensee, and is addressed as "**you**". You accept the license if you copy, modify or distribute the work in a way requiring permission under copyright law.
A "**Modified Version**" of the Document means any work containing the Document or a portion of it, either copied verbatim, or with modifications and/or translated into another language.
A "**Secondary Section**" is a named appendix or a front-matter section of the Document that deals exclusively with the relationship of the publishers or authors of the Document to the Document's overall subject (or to related matters) and contains nothing that could fall directly within that overall subject. (Thus, if the Document is in part a textbook of mathematics, a Secondary Section may not explain any mathematics.) The relationship could be a matter of historical connection with the subject or with related matters, or of legal, commercial, philosophical, ethical or political position regarding them.
The "**Invariant Sections**" are certain Secondary Sections whose titles are designated, as being those of Invariant Sections, in the notice that says that the Document is released under this License. If a section does not fit the above definition of Secondary then it is not allowed to be designated as Invariant. The Document may contain zero Invariant Sections. If the Document does not identify any Invariant Sections then there are none.
The "**Cover Texts**" are certain short passages of text that are listed, as Front-Cover Texts or Back-Cover Texts, in the notice that says that the Document is released under this License. A Front-Cover Text may be at most 5 words, and a Back-Cover Text may be at most 25 words.
A "**Transparent**" copy of the Document means a machine-readable copy, represented in a format whose specification is available to the general public, that is suitable for revising the document straightforwardly with generic text editors or (for images composed of pixels) generic paint programs or (for drawings) some widely available drawing editor, and that is suitable for input to text formatters or for automatic translation to a variety of formats suitable for input to text formatters. A copy made in an otherwise Transparent file format whose markup, or absence of markup, has been arranged to thwart or discourage subsequent modification by readers is not Transparent. An image format is not Transparent if used for any substantial amount of text. A copy that is not "Transparent" is called "**Opaque**".

Examples of suitable formats for Transparent copies include plain ASCII without markup, Texinfo input format, LaTeX input format, SGML or XML using a publicly available DTD, and standard-conforming simple HTML, PostScript or PDF designed for human modification. Examples of transparent image formats include PNG, XCF and JPG. Opaque formats include proprietary formats that can be read and edited only by proprietary word processors, SGML or XML for which the DTD and/or processing tools are not generally available, and the machine-generated HTML, PostScript or PDF produced by some word processors for output purposes only.
The "**Title Page**" means, for a printed book, the title page itself, plus such following pages as are needed to hold, legibly, the material this License requires to appear in the title page. For works in formats which do not have any title page as such, "Title Page" means the text near the most prominent appearance of the work's title, preceding the beginning of the body of the text.
The "**publisher**" means any person or entity that distributes copies of the Document to the public.
A section "**Entitled XYZ**" means a named subunit of the Document whose title either is precisely XYZ or contains XYZ in parentheses following text that translates XYZ in another language. (Here XYZ stands for a specific section name mentioned below, such as "**Acknowledgements**", "**Dedications**", "**Endorsements**", or "**History**".) To "**Preserve the Title**" of such a section when you modify the Document means that it remains a section "Entitled XYZ" according to this definition.
The Document may include Warranty Disclaimers next to the notice which states that this License applies to the Document. These Warranty Disclaimers are considered to be included by reference in this License, but only as regards disclaiming warranties: any other implication that these Warranty Disclaimers may have is void and has no effect on the meaning of this License.

2. VERBATIM COPYING

You may copy and distribute the Document in any medium, either commercially or noncommercially, provided that this License, the copyright notices, and the license notice saying this License applies to the Document are reproduced in all copies, and that you add no other conditions whatsoever to those of this License. You may not use technical measures to obstruct or control the reading or further copying of the copies you make or distribute. However, you may accept compensation in exchange for copies. If you distribute a large enough number of copies you must also follow the conditions in section 3.
You may also lend copies, under the same conditions stated above, and you may publicly display copies.

3. COPYING IN QUANTITY

If you publish printed copies (or copies in media that commonly have printed covers) of the Document, numbering more than 100, and the Document's license notice requires Cover Texts, you must enclose the copies in covers that carry, clearly and legibly, all these Cover Texts: Front-Cover Texts on the front cover, and Back-Cover Texts on the back cover. Both covers must also clearly and legibly identify you as the publisher of these copies. The front cover must present the full title with all words of the title equally prominent and visible. You may add other material on the covers in addition. Copying with changes limited to the covers, as long as they preserve the title of the Document and satisfy these conditions, can be treated as verbatim copying in other respects.
If the required texts for either cover are too voluminous to fit legibly, you should put the first ones listed (as many as fit reasonably) on the actual cover, and continue the rest onto adjacent pages.
If you publish or distribute Opaque copies of the Document numbering more than 100, you must either include a machine-readable Transparent copy along with each Opaque copy, or state in or with each Opaque copy a computer-network location from which the general network-using public has access to download using public-standard network protocols a complete Transparent copy of the Document, free of added material. If you use the latter option, you must take reasonably prudent steps, when you begin distribution of Opaque copies in quantity, to ensure that this Transparent copy will remain thus accessible at the stated location until at least one year after the last time you distribute an Opaque copy (directly or through your agents or retailers) of that edition to the public.
It is requested, but not required, that you contact the authors of the Document well before redistributing any large number of copies, to give them a chance to provide you with an updated version of the Document.

4. MODIFICATIONS

You may copy and distribute a Modified Version of the Document under the conditions of sections 2 and 3 above, provided that you release the Modified Version under precisely this License, with the Modified Version filling the role of the Document, thus licensing distribution and modification of the Modified Version to whoever possesses a copy of it. In addition, you must do these things in the Modified Version:

A. Use in the Title Page (and on the covers, if any) a title distinct from that of the Document, and from those of previous versions (which should, if there were any, be listed in the History section of the Document). You may use the same title as a previous version if the original publisher of that version gives permission.

B. List on the Title Page, as authors, one or more persons or entities responsible for authorship of the modifications in the Modified Version, together with at least five of the principal authors of the Document (all of its principal authors, if it has fewer than five), unless they release you from this requirement.

C. State on the Title page the name of the publisher of the Modified Version, as the publisher.

D. Preserve all the copyright notices of the Document.

E. Add an appropriate copyright notice for your modifications adjacent to the other copyright notices.

F. Include, immediately after the copyright notices, a license notice giving the public permission to use the Modified Version under the terms of this License, in the form shown in the Addendum below.

G. Preserve in that license notice the full lists of Invariant Sections and required Cover Texts given in the Document's license notice.

H. Include an unaltered copy of this License.

I. Preserve the section Entitled "History", Preserve its Title, and add to it an item stating at least the title, year, new authors, and publisher of the Modified Version as given on the Title Page. If there is no section Entitled "History" in the Document, create one stating the title, year, authors, and publisher of the Document as given on its Title Page, then add an item describing the Modified Version as stated in the previous sentence.

J. Preserve the network location, if any, given in the Document for public access to a Transparent copy of the Document, and likewise the network locations given in the Document for previous versions it was based on. These may be placed in the "History" section. You may omit a network location for a work that was published at least four years before the Document itself, or if the original publisher of the version it refers to gives permission.

K. For any section Entitled "Acknowledgements" or "Dedications", Preserve the Title of the section, and preserve in the section all the substance and tone of each of the contributor acknowledgements and/or dedications given therein.

L. Preserve all the Invariant Sections of the Document, unaltered in their text and in their titles. Section numbers or the equivalent are not considered part of the section titles.

M. Delete any section Entitled "Endorsements". Such a section may not be included in the Modified Version.

N. Do not retitle any existing section to be Entitled "Endorsements" or to conflict in title with any Invariant Section.

O. Preserve any Warranty Disclaimers.

If the Modified Version includes new front-matter sections or appendices that qualify as Secondary Sections and contain no material copied from the Document, you may at your option designate some or all of these sections as invariant. To do this, add their titles to the list of Invariant Sections in the Modified Version's license notice. These titles must be distinct from any other section titles.

You may add a section Entitled "Endorsements", provided it contains nothing but endorsements of your Modified Version by various parties—for example, statements of peer review or that the text has been approved by an organization as the authoritative definition of a standard.

You may add a passage of up to five words as a Front-Cover Text, and a passage of up to 25 words as a Back-Cover Text, to the end of the list of Cover Texts in the Modified Version. Only one passage of Front-Cover Text and one of Back-Cover Text may be added by (or through arrangements made by) any one entity. If the Document already includes a cover text for the same cover, previously added by you or by arrangement made by the same entity you are acting on behalf of, you may not add another; but you may replace the old one, on explicit permission from the previous publisher that added the old one.

The author(s) and publisher(s) of the Document do not by this License give permission to use their names for publicity for or to assert or imply endorsement of any Modified Version.

5. COMBINING DOCUMENTS

You may combine the Document with other documents released under this License, under the terms defined in section 4 above for modified versions, provided that you include in the combination all of the Invariant Sections of all of the original documents, unmodified, and list them all as Invariant Sections of your combined work in its license notice, and that you preserve all their Warranty Disclaimers.

The combined work need only contain one copy of this License, and multiple identical Invariant Sections may be replaced with a single copy. If there are multiple Invariant Sections with the same name but different contents, make the title of each such section unique by adding at the end of it, in parentheses, the name of the original author or publisher of that section if known, or else a unique number. Make the same adjustment to the section titles in the list of Invariant Sections in the license notice of the combined work.

In the combination, you must combine any sections Entitled "History" in the various original documents, forming one section Entitled "History"; likewise combine any sections Entitled "Acknowledgements", and any sections Entitled "Dedications". You must delete all sections Entitled "Endorsements".

6. COLLECTIONS OF DOCUMENTS

You may make a collection consisting of the Document and other documents released under this License, and replace the individual copies of this License in the various documents with a single copy that is included in the collection, provided that you follow the rules of this License for verbatim copying of each of the documents in all other respects.

You may extract a single document from such a collection, and distribute it individually under this License, provided you insert a copy of this License into the extracted document, and follow this License in all other respects regarding verbatim copying of that document.

7. AGGREGATION WITH INDEPENDENT WORKS

A compilation of the Document or its derivatives with other separate and independent documents or works, in or on a volume of a storage or distribution medium, is called an "aggregate" if the copyright resulting from the compilation is not used to limit the legal rights of the compilation's users beyond what the individual works permit. When the Document is included in an aggregate, this License does not apply to the other works in the aggregate which are not themselves derivative works of the Document.
If the Cover Text requirement of section 3 is applicable to these copies of the Document, then if the Document is less than one half of the entire aggregate, the Document's Cover Texts may be placed on covers that bracket the Document within the aggregate, or the electronic equivalent of covers if the Document is in electronic form. Otherwise they must appear on printed covers that bracket the whole aggregate.

8. TRANSLATION

Translation is considered a kind of modification, so you may distribute translations of the Document under the terms of section 4. Replacing Invariant Sections with translations requires special permission from their copyright holders, but you may include translations of some or all Invariant Sections in addition to the original versions of these Invariant Sections. You may include a translation of this License, and all the license notices in the Document, and any Warranty Disclaimers, provided that you also include the original English version of this License and the original versions of those notices and disclaimers. In case of a disagreement between the translation and the original version of this License or a notice or disclaimer, the original version will prevail.
If a section in the Document is Entitled "Acknowledgements", "Dedications", or "History", the requirement (section 4) to Preserve its Title (section 1) will typically require changing the actual title.

9. TERMINATION

You may not copy, modify, sublicense, or distribute the Document except as expressly provided under this License. Any attempt otherwise to copy, modify, sublicense, or distribute it is void, and will automatically terminate your rights under this License.
However, if you cease all violation of this License, then your license from a particular copyright holder is reinstated (a) provisionally, unless and until the copyright holder explicitly and finally terminates your license, and (b) permanently, if the copyright holder fails to notify you of the violation by some reasonable means prior to 60 days after the cessation.
Moreover, your license from a particular copyright holder is reinstated permanently if the copyright holder notifies you of the violation by some reasonable means, this is the first time you have received notice of violation of this License (for any work) from that copyright holder, and you cure the violation prior to 30 days after your receipt of the notice.
Termination of your rights under this section does not terminate the licenses of parties who have received copies or rights from you under this License. If your rights have been terminated and not permanently reinstated, receipt of a copy of some or all of the same material does not give you any rights to use it.

10. FUTURE REVISIONS OF THIS LICENSE

The Free Software Foundation may publish new, revised versions of the GNU Free Documentation License from time to time. Such new versions will be similar in spirit to the present version, but may differ in detail to address new problems or concerns. See `http://www.gnu.org/copyleft/`.
Each version of the License is given a distinguishing version number. If the Document specifies that a particular numbered version of this License "or any later version" applies to it, you have the option of following the terms and conditions either of that specified version or of any later version that has been published (not as a draft) by the Free Software Foundation. If the Document does not specify a version number of this License, you may choose any version ever published (not as a draft) by the Free Software Foundation. If the Document specifies that a proxy can decide which future versions of this License can be used, that proxy's public statement of acceptance of a version permanently authorizes you to choose that version for the Document.

11. RELICENSING

"Massive Multiauthor Collaboration Site" (or "MMC Site") means any World Wide Web server that publishes copyrightable works and also provides prominent facilities for anybody to edit those works. A public wiki that anybody can edit is an example of such a server. A "Massive Multiauthor Collaboration" (or "MMC") contained in the site means any set of copyrightable works thus published on the MMC site.

"CC-BY-SA" means the Creative Commons Attribution-Share Alike 3.0 license published by Creative Commons Corporation, a not-for-profit corporation with a principal place of business in San Francisco, California, as well as future copyleft versions of that license published by that same organization.
"Incorporate" means to publish or republish a Document, in whole or in part, as part of another Document.
An MMC is "eligible for relicensing" if it is licensed under this License, and if all works that were first published under this License somewhere other than this MMC, and subsequently incorporated in whole or in part into the MMC, (1) had no cover texts or invariant sections, and (2) were thus incorporated prior to November 1, 2008.
The operator of an MMC Site may republish an MMC contained in the site under CC-BY-SA on the same site at any time before August 1, 2009, provided the MMC is eligible for relicensing.

ADDENDUM: How to use this License for your documents

To use this License in a document you have written, include a copy of the License in the document and put the following copyright and license notices just after the title page:

> Copyright © YEAR YOUR NAME. Permission is granted to copy, distribute and/or modify this document under the terms of the GNU Free Documentation License, Version 1.3 or any later version published by the Free Software Foundation; with no Invariant Sections, no Front-Cover Texts, and no Back-Cover Texts. A copy of the license is included in the section entitled "GNU Free Documentation License".

If you have Invariant Sections, Front-Cover Texts and Back-Cover Texts, replace the "with ... Texts." line with this:

> with the Invariant Sections being LIST THEIR TITLES, with the Front-Cover Texts being LIST, and with the Back-Cover Texts being LIST.

If you have Invariant Sections without Cover Texts, or some other combination of the three, merge those two alternatives to suit the situation.
If your document contains nontrivial examples of program code, we recommend releasing these examples in parallel under your choice of free software license, such as the GNU General Public License, to permit their use in free software.

Appendix B

How to learn math

Now that that's all out of the way, let's talk a little about math. When this chapter is over, we're going to dive right in to proving a bunch of things you already know to be true. We feel that without a little explanation, these proofs may leave you a little lost or confused. We'll save the explanation of the proofs for later, but right now, we're going to talk about how to actually learn math, and the proofs are a great example.

Most people's experience with math is through their primary and possibly secondary education, which is or was a dreary affair in general, and math probably even moreso, unless you're one of the lucky few. By lucky few, we don't mean those wizards with a sort of inherent ability to do math–the first thing you need to know about learning math is that math is for everyone with a brain–that's you, right? You see, your brain is a pattern recognition engine, and that's all math is: the study of patterns. Unlike reading or history, your body comes with a biological imperative to know math. There's some really great brain studies on the topic, but that's boring, and I said we're already done with the boring part, so let's move on.

In that last paragraph, we presented what we hold to be the proper answer to 'what is math': the study of patterns. This is completely different from most people's interaction with math: in primary school, we are taught how to apply four operations to solve math problems. You're given something about two trains leaving a station and going different speeds and different directions and yadda yadda yadda and before you know it your teacher turned

everything into a math problem and it all seemed so forced–a layer on top of what was intuitive, and made everything complicated. We agree–this is a counterintuitive approach to math, and it makes math very confusing and disconnected. Math is just the study of patterns. That is, math is not so much a way to solve a set of problems that exist in a sphere apart from what is natural, but a way to understand what's going on in the world around us. When you learn math, you should think of it as a science–another level of detail in the amazing world we live in.

That's how this book is written. It's written to reflect that math is a single unified study. While you're reading it, try to think of how what you're learning clarifies or refines early material. This is a big deal to us, because one thing we dislike most about the standard way of learning math is that at some point in everyone's math career, they learn they were taught something that wasn't actually true. We want to avoid that.

Bibliography

[1] Function (mathematics). URL: https://en.wikipedia.org/wiki/Function_%28mathematics%29 (visited on 01/10/2015).

[2] Edmund Landau. Foundations of Analysis. Providence, RI: AMS Chelsea Publishing, 1966.

[3] Miran Lipovača. Learn You a Haskell for Great Good! 245 8th Street, San Francisco, CA, 94103: No Starch Press, 2011.

[4] steve jobs on programming. URL: https://www.youtube.com/watch?v=5Z1gfgM7kzo (visited on 01/01/2015).