

Learn You Some Algebras for Glorious Good!

Peter Harpending <**peter@harpending.org**>

February 25, 2015

Copyright © 2014-2015 Peter Harpending <peter@harpending.org>

Permission is granted to copy, distribute and/or modify this document under the terms of the GNU Free Documentation License, Version 1.3 or any later version published by the Free Software Foundation; with no Invariant Sections, no Front-Cover Texts, and no Back-Cover Texts. A copy of the license is included in § A.

Contents

1	Introduction	9
1.1	How to read the book	10
1.2	Introduction (for real this time)	11
1.3	The community	11
1.4	Haskell	12
1.4.1	Installing Haskell	13
1.4.2	Install a text editor	15
1.5	Target audience	17
1.6	Licensing	18
1.7	Conventions used throughout	18
2	Booleans, simple logic, and simple operators	23
2.1	Implications	23
2.2	And and or	26
2.2.1	Exercises	29
3	Sets	33

3.1	Elements	34
3.2	Subsets and Supersets	35
3.2.1	Exercises	37
3.3	Combining sets together	39
3.3.1	Exercises	41
3.4	Special kinds of sets	45
4	Functions	49
4.0.1	Functions with multiple arguments	50
4.0.2	Eta-reductions	52
4.1	Currying	53
4.2	Vocabulary	55
4.2.1	Exercises	57
5	More stuff about sets	61
5.1	Set subtraction	62
5.1.1	Complement	63
5.1.2	Exercises	64
	Appendices	69
A	GNU Free Documentation License	73
1.	APPLICABILITY AND DEFINITIONS	73
2.	VERBATIM COPYING	75

<i>CONTENTS</i>	7
3. COPYING IN QUANTITY	75
4. MODIFICATIONS	76
5. COMBINING DOCUMENTS	77
6. COLLECTIONS OF DOCUMENTS	78
7. AGGREGATION WITH INDEPENDENT WORKS	78
8. TRANSLATION	78
9. TERMINATION	79
10. FUTURE REVISIONS OF THIS LICENSE	79
11. RELICENSING	80
ADDENDUM: How to use this License for your documents	80
B How to learn math	83
C Philosophy and/or FAQ	87

Chapter 1

Introduction

Before I bore you with a bunch of crap you don't care about, let's do some math, shall we?

There are basically three notions with which you need to be familiar in order to do anything interesting in math. Those three things are *sets*, *functions*, and *proofs*. Unfortunately, to be familiar with one, you have to be familiar with the other two.¹

So, what are each of those things?

- A *set* is an unordered collection of things. There is also no repetition. For instance, $\{2, 5\}$ is the same as $\{5, 2\}$ (because order doesn't matter). $\{2, 5, 5\}$ would be the same set, because there's no notion of multiplicity.

I'm going to use some notation for sets later in the book. The explanation for the notation doesn't really fit anywhere, so here it is:

$$A = \{ x \mid y \}$$

¹You'll learn as we go along, when math people use a common term like *set*, *function*, *proof*, *group*, *continuous* or *closed*, they usually mean something similar in concept to the colloquial term, but there are some strings attached. This is usually the case in the sciences too (e.g. *theory*, *hypothesis*, *experiment*).

This should be read as “ A is the set of all values x , such that y is True. This is called “set comprehension notation.” It’s not difficult, and you’ll pick it up as we go along.

- A *function* is a mathematical construct (well, obviously, else I wouldn’t be talking about it). Basically, it takes some input, does something to it, and spits out some output. If you give the function the same input a bunch of times, you should get the same result each time. This concept is called “referential transparency.” If the function is not referentially transparent, then it’s not a function. It’s something else.
- A *proof* is basically where you take a bunch of simple facts, called *axioms*, and chain them together to make *theorems*. It’s sort of like sticking puzzle pieces together to form a picture.

The puzzle pieces (in this case, the axioms) aren’t usually very interesting on their own. However, the picture they form (in this case, the theorem) can be really cool and enlightening. The proof would be analogous to an explicit set of instructions explaining how to put the pieces together.

Once you are familiar with each of those concepts, we can do all sorts of cool stuff. Throughout the book, we will prove all of the following:

- If you tap your finger against a bridge at exactly the right frequency, the bridge will collapse. (Resonance)
- The formula used to calculate the interest rate on your mortgage is actually just a fancy form of the ratios of angles in a triangle. (Euler’s formula)

1.1 How to read the book

The best way to read this book is to just read it. Don’t skip sections, or look ahead, or anything like that. Just read it straight through. It’s also pretty important that you read the rest of this chapter. I promise it’s not too boring.

Do all of the exercises. There aren't that many. However, they are pretty difficult. The exercises all have solutions, which are on the page after the exercises.

The exercises are designed to make you think, and widen your perspective on the topic at hand. They are not designed to be tedious. They are difficult, but the good kind of difficult.

It would be perfectly okay to just do the exercises (all of them), and then go back and read the text when you don't understand something.

1.2 Introduction (for real this time)

This is a math book. Well, duh. Why did I write it?

Most math (and science) books nowadays seem to value keeping an academic tone over ensuring that the reader understands the material, and — more importantly — enjoys reading the book.

I take the opposite approach. I want to create a book that is fun to read and easy to understand, while eschewing the practice of making myself look good.

The inspiration for this book is *Learn You a Haskell for Great Good!*, by Miran Lipovača. Haskell is a programming language, and LYAH is a great book for learning Haskell. If you are interested in a print copy of LYAH, see [5].

There is also an incomplete and unofficial Russian translation (<https://github.com/gazay/lysa>), courtesy of Alexey Gaziev.

1.3 The community

Despite the fact that I used “I” in the first part of the book, LYSA is actually a community project, and many people participate in the writing of this book.

If you want to talk to us, or to other math people, come see us in **#lysa** on Freenode. If you don't know what IRC is, or you don't have a client set up, you can connect through Freenode's webchat (<http://webchat.freenode.net/?channels=lysa>).

If you have any questions about LYSA (or math), feel free to ask in the IRC channel (**#lysa** on FreeNode in case you forgot).

If you want to submit a correction, or have some issue, or want to add some content, really anything having to do with the content of the book, you can visit our GitLab page (<https://gitlab.com/lysa/lysa>). We also have a woefully incomplete website (<http://learnyou.org>) and a community on Reddit (<https://lysa.reddit.com/>).

1.4 Haskell

In this book, I cover a lot of hard stuff.² Sometimes, it's useful to program your way through a problem. Every programmer will tell you that programming teaches a manner of thinking.

Many programmers will cite Steve Jobs³ famous quote, regarding the use of programming in his job,

[sic] . . . much more importantly, it had nothing to do with using [the programs we wrote] for anything practical. It had to do with using them to be a mirror of your thought process; to actually learn how to think. I think everybody in this country should learn how to program a computer — should learn a computer language — because it teaches you how to think.

That first sentence or two is actually a pretty good description of mathematics (and programming). Both are incredibly useful, and have endless practical applications. That's not the point, though. The whole usefulness

²This isn't actually true. Math isn't hard, stupid!

³For you youngsters, Steve Jobs is the former CEO of Apple. He's dead now.

thing is a side gig. It's about learning how to think, and having a rigorous language through which to express your thoughts. Furthermore, the rigor of the language helps you build upon your current thoughts to find out even cooler things. That's what math is about.

Programming and math go hand-in-hand. Programmers and mathematicians will attest to this; I certainly can. For that reason, throughout this book, there will be coding exercises in the programming language Haskell.⁴

1.4.1 Installing Haskell

This is something that is actually rather difficult to summarize, because it varies from operating system to operating system. I will put down the instructions for the operating systems I use. If you come upon this and don't see your operating system, please report this on the issue tracker (<https://gitlab.com/lysa/lysa/issues/new>). Better yet, you could add the instructions yourself, and ask me to merge your changes.

Linux

Arch

```
# pacman -S ghc cabal-install gmp
% \cabal update
% \cabal install -j cabal-install
```

At this point, you'll want to add `~/.cabal/bin` to your `$PATH` variable.

```
% cabal install -j alex happy haddock hscolour
```

`alex` and `happy` are dependencies for a number of Haskell packages. However, due to a long-standing bug in `cabal` (<https://github.com/haskell/cabal/issues/220>), they don't get pulled in when packages depend on them.

⁴I was originally going to use another language called Idris, but Idris is, at the time of this writing, so buggy that it is unusable.

Gentoo As of 5 January 2015, the Haskell platform is only available on `~ARCH`, where `ARCH` is your processor architecture (e.g. `amd64`, `x86`). If you use `ARCH`, you can enable these by adding the following to `/etc/portage/package.keywords`:⁵

```
dev-lang/ghc
dev-haskell/cabal-install
```

Regardless of your `ACCEPT_KEYWORDS` variable, you should add the following to `/etc/portage/package.use`:

```
dev-lang/ghc binary
```

Otherwise, you have to compile GHC (the Haskell compiler) from scratch, and that takes forever.

Once you have that all out of the way, you'll want to run the following command as root:

```
# emerge -jav dev-lang/ghc dev-haskell/cabal-install
```

Warning: `-j` will make the installation a lot faster, but is more resource-intensive. If your power usage is precious, omit it (i.e. use `-av` instead).

Once GHC and `cabal-install` are installed, you'll want to run the following as a normal user:

```
% cabal update
% cabal install alex happy haddock hscolour
```

⁵If you already use `~ARCH`, you can ignore this

1.4.2 Install a text editor

In order to edit Idris code, you need a plain-text editor (as opposed to a word processor).

Some popular plain-text editors are:

1. Gedit (<https://wiki.gnome.org/Apps/Gedit>) - very easy to use. I recommend either Gedit or Kate for beginners.
2. Kate (<http://kate-editor.org/get-it/>) - marginally harder than Gedit, but it has more features.

Linux/BSD users: If you are not a KDE user, then don't use Kate. It brings in a ton of KDE dependencies. Here's the result of trying to install it on my machine:

```
% sudo pacman -S kate
[sudo] password for pete:
resolving dependencies...
:: There are 2 providers available for phonon-qt5-backend:
:: Repository extra
   1) phonon-qt5-gstreamer  2) phonon-qt5-vlc

Enter a number (default=1): 2
looking for conflicting packages...

Packages (54) attica-qt5-5.6.0-1  gamin-0.1.10-8  karchive
-5.6.0-1  kauth-5.6.0-1  kbookmarks-5.6.0-1
           kcodecs-5.6.0-1  kcompletion-5.6.0-1  kconfig
-5.6.0-1  kconfigwidgets-5.6.0-1
           kcoreaddons-5.6.0-1  kcrash-5.6.0-2
kdbusaddons-5.6.0-1  kded-5.6.0-1  kglobalaccel-5.6.0-1
           kguiaddons-5.6.0-1  kiln-5.6.0-1  kiconthemes
-5.6.0-1  kinit-5.6.0-1  kio-5.6.0-1
           kitemmodels-5.6.0-1  kitemviews-5.6.0-1
kjobwidgets-5.6.0-1  knewstuff-5.6.0-1
           knotifications-5.6.0-1  kparts-5.6.0-1
kservice-5.6.0-1  ktexteditor-5.6.0-1
           ktextwidgets-5.6.0-1  kwallet-5.6.0-1
kwidgetsaddons-5.6.0-1  kwindowsystem-5.6.0-3
           kxmlgui-5.6.0-1  libdbusmenu-qt5
-0.9.3+14.10.20140619-1  libgit2-1:0.21.5-1
           libimobiledevice-1.1.7-1  libplist-1.11-1
libusbmuxd-1.0.9-1  libxkbcommon-x11-0.5.0-1
           media-player-info-19-1  phonon-qt5-4.8.3-1
phonon-qt5-vlc-0.8.2-1  polkit-qt5-0.112-2
           qt5-base-5.4.0-3  qt5-declarative-5.4.0-3  qt5
-script-5.4.0-3  qt5-svg-5.4.0-3
           qt5-x11extras-5.4.0-3  qt5-xmlpatterns-5.4.0-3
qtchooser-48-1  solid-5.6.0-1  sonnet-5.6.0-1
           threadweaver-5.6.0-1  upower-0.99.2-1  kate
-14.12.2-2

Total Download Size:    33.42 MiB
Total Installed Size:  178.32 MiB

:: Proceed with installation? [Y/n] n
```


3. Vim (<http://www.vim.org/>) - It has a sharp, but not steep learning curve.
4. GNU Emacs (<https://www.gnu.org/software/emacs/>) has an absolutely insane learning curve, but is a wonderful editor once you spend 3 years learning how to use it.

1.5 Target audience

The explanation of why programming is useful is a good segue into discussing the target audience.

When I was first writing the book, I wrote it in an effort to strengthen my own understanding. So, the target audience was me. The very first versions of this book were about a abstractish branch of math called commutative algebra. Later on, it seemed more fitting to abstractly go over the basics of math. That's what the current version of the book does.

That doesn't answer the question: who is the target audience? Well, people who want to learn basic and intermediate algebra, and to learn why it's so interesting.

Most books treat math as a tool you can use for calculations. I treat math as a language you can use to express your ideas. That's the core difference. This book will hopefully give you an interest in math itself, rather than just a cursory knowledge of it.

With that in mind, my book is going to approach the topics much differently than other books on the same topic. I rely very much on abstraction and intuition.

1.6 Licensing

This book is libre⁶. You can copy this book and give it to your friend. You can even print it out and sell it to people.⁷ If, for instance, you are a schoolteacher and want to use this for your class, you are free to edit it to your liking and give the modified copy to your students. The only string I attach is, you have to allow anyone to whom you give the book do the same thing (i.e. they have to be free to copy/modify/change your version). The details of this can be found in § A.

LYSA is licensed under the GNU Free Documentation License. § A contains the license. Please read the license; it's actually pretty comprehensible.

The source for this book can be downloaded at <https://gitlab.com/lysa/lysa/repository/archive.tar.gz>. If you are looking to contribute, it's probably best to clone the git repository. You can clone the git repository by running `git clone https://gitlab.com/lysa/lysa.git` in a terminal.

1.7 Conventions used throughout

You don't actually have to read this section, but it would be useful.

1. Things in monospace are either code snippets or commands to be run in a terminal. I have separate stylings for `terminal commands` and `inline code snippets`. That said, they are separate but equal, at least for the time being.
2. The § symbol refers to a section. So § 3.2 means “chapter 3, section 2”.
3. Even though most of the writers are American, I still use the British convention of putting periods after quotation marks: “like this”. The British convention is less ambiguous. If you see the American convention anywhere in this book, please report it.

⁶*Libre* is a French word, which, translated to English, means *free* in the sense of liberty, as opposed to price. Think *free speech*, not *free beer*.

⁷There are some restrictions though, see § A.

4. I will often recommend software. However, I will not recommend any non-libre software, or any software that costs money.
5. “I” refers to me. “We” refers to both me and you, the reader. “You” refers to, you guessed it, the reader. It’s the convention in academia to use the so-called “royal we”, such as “we subtract 2 from both sides of the equation to obtain the result ...”.

Sometimes, we will accidentally use the royal we, out of habit. Crap, I just did it there! See? It’s very difficult to avoid. Like any of the other conventions herein, if you see it broken, please report the error to the authors. You can use the bug tracker (<https://gitlab.com/lysa/lysa/issues/new>), or, if you don’t want to make a (free and libre) GitLab account, you can email me at peter@harpending.org.

6. Oh yeah, sometimes I’ll use `monospace` in things like URLs or emails for the sake of disambiguity.
7. Most of the authors use some version of Linux. Hence, when there are instructions for computer things (such as installing Idris), I’ll write instructions for Linux, because that’s what I know. There are two solutions here:
 - (a) You could try out Linux (it’s gratis, and it’s easy).
 - (b) If you know how to do the thing on your OS, and there aren’t instructions for your OS, you could write up instructions and add them to the book. If you don’t know how to do that, you could bring it up in the bug tracker (<https://gitlab.com/lysa/lysa/issues/new>) or email me at peter@harpending.org.
8. If you see some number as a superscript in the middle of text: like this⁸, then the number refers to a footnote. If the superscript number is in the middle of math, it’s probably just math.
9. If there’s some number in brackets, like this: [5], then it’s a citation. If you’re reading this as a PDF, you can actually click on the number, and your PDF reader will take you to the relevant bibliography entry. Go

⁸Hey, you found me!

ahead, check it out! I'll wait. You can do the same thing for footnotes and URLs.⁹

⁹Well, clicking the URL will open up your web browser, but you get the point

Chapter 2

Booleans, simple logic, and simple operators

Before we get into interesting content, you have to understand some stuff. This stuff is pretty easy. This will likely be the shortest and easiest chapter in the book.

You might think math is about dealing with numbers and pumping out formulas. Well, that's not what math is about. As said in § 1.4, it's about using math as a language to express your thoughts. Most people don't think about numbers all day; thus, we deal with things in math that aren't numbers.

In this next section, we're going to outline some basic rules for reasoning about things. You need to know these rules to do really cool stuff. Although, as you will (hopefully) see, these rules can be fun to toy around with on their own.

2.1 Implications

The first thing you need to understand is the notion that “if x is true, then y is also true. But, if y is true, it's not necessarily true that x is false.” As always, mathematicians are too lazy to write this stuff out by hand, so they

have notation for it.

1. $a \implies b$ means that “ a implies b ”. It doesn’t necessarily mean that b implies a . It means that if a is true, then b is also true.

If someone is decapitated, then they will die. So,

$$\text{Decapitated} \implies \text{Dead}$$

However, if someone is dead, it doesn’t necessarily mean that they were decapitated. They could have been shot, or stabbed, or had a heart attack. There are endless possibilities.

2. $a \impliedby b$ is the same as writing $b \implies a$. It’s sometimes convenient to use $a \impliedby b$ instead. $a \impliedby b$ should be read “ a is implied by b ”.
3. When I strike through some mathematical operator, like this: $\not\implies$, it means that you can semantically put “not” in front of whatever the operator says. So, $\not\implies$ means “not implies”. That doesn’t make much grammatical sense in English, so “does not imply” might be better. Nonetheless, you get the point.

Moving on from the example above:

$$\text{Decapitated} \implies \text{Dead}$$

If someone is decapitated, then they’re also dead (at least within a few seconds). However, if someone is dead, it’s not necessarily true that they were decapitated.

$$\text{Decapitated} \not\impliedby \text{Dead}$$

4. If something is not true, then I’ll put a \neg in front of it. So, if I want to say that a is false, then I’ll write $\neg a$.
5. If I want to pose a question, I could just ask the question. For instance, “Is $\neg \text{Decapitated} \impliedby \neg \text{Dead}$ true?”.

However, that quickly becomes difficult, usually when there are multiple assertions in a mathematical expression, and you don’t know which

one I'm asking about. Moreover, since I use the same font for text and math, if I have both, it might be hard to tell which is math and which is text. So, to help with ambiguity, I'll put a ? over the operator I'm asking about:

$$\neg\text{Decapitated} \stackrel{?}{\Leftarrow} \neg\text{Dead}$$

See, that's much easier.

6. Now, on to that question - Is “not decapitated” implied by “not dead”. Well, let's think about it. If someone is not dead, then they couldn't have been decapitated, because if they were decapitated, then they would be dead. Therefore, if someone is not dead, then they weren't decapitated.

That word jumble was probably confusing. Mathematicians don't like to be confused. I'll make it symbolic for you.

$$\begin{array}{c} \text{Decapitated} \implies \text{Dead} \\ \Downarrow \\ \neg\text{Decapitated} \Leftarrow \neg\text{Dead} \end{array}$$

Okay, I just used a vertical arrow. I'm sure you can figure out what it means.

7. So, hopefully you agree that

$$\begin{array}{c} \text{Decapitated} \implies \text{Dead} \\ \Downarrow \\ \neg\text{Decapitated} \Leftarrow \neg\text{Dead} \end{array}$$

However,

$$\begin{array}{c} \text{Decapitated} \implies \text{Dead} \\ \stackrel{?}{\Uparrow} \\ \neg\text{Decapitated} \Leftarrow \neg\text{Dead} \end{array}$$

Hm, the question mark doesn't work so well there. Oh well! Anyway, the answer is actually yes. We can figure this out by learning a rule about \neg . Namely, that

$$\neg\neg a = a$$

In this case, we know

$$\neg\text{Decapitated} \iff \neg\text{Dead}$$

So, if we just “not” both sides, and flip \implies to \iff ,

$$\begin{aligned}\neg\neg\text{Decapitated} &\implies \neg\neg\text{Dead} \\ \text{Decapitated} &\implies \text{Dead}\end{aligned}$$

This is basically just the rule mentioned in #3. Yay, we learned something!

2.2 And and or

So, sometimes we need to combine two pieces of logic together. There are two ways we can do this - logical-or and logical-and.

I put logical- in front of them, because the mathematical meaning is slightly different than the colloquial meaning.

Mathematicians are lazy, so we don't like to write “logical-and” whenever we want to say it, so instead we use the symbol \wedge .

$A \wedge B$ is true if (and only if) both A and B are true. If one of them is false, then the entire thing is false.

On the other side, we have logical-or. The symbol for logical-or is \vee . $A \vee B$ is true if either A or B is true, or if both of them are true. You could think of logical-or as being equivalent to the colloquial “and/or”.

This monstrosity is called a “truth table”.

A	B	$A \wedge B$	$A \vee B$
True	True	True	True
True	False	False	True
False	True	False	True
False	False	False	False

This is pretty simple. If you're having trouble remembering which symbol is logical-and and which one is logical-or, remember that the logical-and symbol — \wedge — looks vaguely like an A.

I'm going to introduce some new notation: the \iff symbol.

$$(A \iff B) = (A \implies B) \wedge (A \impliedby B)$$

\iff should be read as “if (and only if)”. Sometimes I'll write the word iff - that's the same as “if (and only if)”.

In order to do the exercises (yes, there are exercises), you need to know these properties of \implies .

1. For all a , $a \implies a$.
2. For all a and b , and c , $((a \implies b) \wedge (b \implies c)) \implies (a \implies c)$.

I'm too lazy to type “For all” each time, so I'm going to use the \forall symbol; it's a common symbol in math that means “for all”. \forall looks like an upside-down A, so it should be easy to remember.

There are two more rules about \wedge and \vee :

1. $A \wedge A = A$; $\forall A$
2. $A \vee A = A$; $\forall A$
3. $A \wedge (B \wedge C) = (A \wedge B) \wedge C$; $\forall A, B, C$

Okay, even this is too difficult. Instead of saying $P = Q$; $\forall P, Q$, I'm going to write $P \equiv Q$. Yes, mathematicians are that lazy.

$$4. A \wedge (B \wedge C) \equiv (A \wedge B) \wedge C$$

$$5. A \vee (B \vee C) \equiv (A \vee B) \vee C$$

$$6. A \wedge (B \vee C) \equiv (A \wedge B) \vee (A \wedge C)$$

$$7. \neg(A \wedge B) \equiv \neg A \vee \neg B$$

Alright, cool, this was a pretty easy chapter. Only 200 or so lines of L^AT_EX! These exercises are somewhat tedious, as they require you to prove pretty obvious (i.e. tedious) things. However, you need to learn these techniques in order to be able to go on and prove less obvious things.

2.2.1 Exercises

Ex. 1 — I'm going to give you this one only in mathematical notation:

$$A \vee (B \wedge C) \stackrel{?}{\equiv} (A \vee B) \wedge (A \vee C)$$

Ex. 2 — $\neg(A \vee B) \stackrel{?}{\equiv} \neg A \wedge \neg B$

Answers

Chapter 3

Sets

In math, it's often useful to consider *collections* of objects. There are basically two types of collections: *sets* and *vectors*. Sets are unordered, and multiplicity doesn't matter. Vectors are ordered, and multiplicity does matter.

For instance, $\{3, 4\}$ and $\{4, 3\}$ are the same *set*, but $(3, 4)$ and $(4, 3)$ are different *vectors*.

Likewise, $\{20, 38\}$ and $\{38, 20, 38, 20, 20, 20, 20, 20\}$ are the same *set*. You guessed it, $(20, 38)$ and $(38, 20, 38, 20, 20, 20, 20, 20)$ are different *vectors*. Sets are, for basic stuff, much more important. Also, they are much easier to understand.

Sets were first studied to extent by Georg Cantor, a German mathematician, in the second half of the nineteenth century. Back in his own day, the results Cantor found by studying sets were considered so thoroughly bizarre that many of his colleagues simply refused to believe that Cantor could be right. In the end, Cantor turned out to be right all along. His ideas can be found in any introductory text on mathematics—including this one.

You probably figured it out from above: the notation is { Braces } for sets, and (Parentheses) for vectors.

If you can't remember whether to use braces {the curly things}, or parentheses (the round things), remember: *a **brace** is used to **set** a broken bone.*

I don't have a horrible pun having to do with parentheses and vectors, and for that, I apologize.

3.1 Elements

Let's invent a set.

$$Q = \{ 7, 7, 9, 5, 10, 1, 6, 6, 2, 10 \}$$

There we go. Remember, order and multiplicity don't matter. But, for the sake of clarity, let's put the elements in order, and deduplicate them.

$$Q = \{ 1, 2, 5, 6, 7, 9, 10 \}$$

Yay! You may have noticed that I slipped in the word *element* into the previous sentence. Objects in the set are called *elements*. Yay, we figured out what that word means!

It's too strenuous on our weak mathematical hands to write "10 is an element of Q ", so instead we have the symbol \in . \in is a very terrible attempt at drawing an E. If you can't remember what \in is, think "element of".¹

So, I'm going to ask you a question:

$$11 \stackrel{?}{\in} Q$$

(See, I used that thing from earlier with the question mark. I told you it would help.) Well, the answer is no, 11 is not an element of Q . As always, mathematicians are too weak to write "11 is not an element of Q " every time they want to say it, so instead they write

¹You better think this, because it took me 30 minutes to get the alignment right. So, you know, remember \in this way, or else...

$$11 \notin Q$$

By contrast,

$$6 \in Q$$

What if we want to say “both 6 and 2 are elements of Q ”? Well, again, we could write it out like:

$$(6 \in Q) \wedge (2 \in Q)$$

But that’s too cumbersome, so instead we’ll write

$$2, 6 \in Q$$

But won’t that get confusing?

Only if we put parentheses or braces around 2 or 6.

$$\{2, 6\} \in Q$$

That’s confusing, don’t do that (yet).

There’s one more thing I need to go over, which is the null set - it’s the simplest set, as it contains no elements. “Null set” takes too long to write, so we use \emptyset instead.

3.2 Subsets and Supersets

Remember when I said $\{2, 6\} \in Q$, and we were really confused? In case you don’t remember, $Q = \{1, 2, 5, 6, 7, 9, 10\}$. $\{2, 6\}$ is obviously not an element of Q . However, $\{2, 6\}$ is *in* Q but it’s not an element. It’s weird. How do we express this notion?

The answer is with *subsets*. “Sub” means “smaller”, so a “subset” would be a “smaller set”. A is a subset of B if all of the elements in A are also in B . The notation is $A \subseteq B$. Some people will read that as “ A is contained in B ”.

Referring to the previous example,

$$\{2, 6\} \subseteq Q$$

Wait, what is with the little line under the round half circley thing?

So, actually, there are two types of subsets - *proper* and *improper*. $A \subseteq B$ is for improper subsets. $A \subset B$ is for proper subsets. What’s the difference, then?

$A \subseteq B$ allows for the possibility that $A = B$. $A \subset B$ means that B is *strictly larger* than A ; there are some elements in B that are not in A .

Before I go on, I’m going to define a couple of other symbols. We’re constantly saying “this thing exists” and “for all”, so I’m going to add a couple of symbols. \exists is for “there exists” — notice the backwards E, as in “exists”. \forall is for “for all” — notice the upside-down A, as in “all”.

So, now that you’ve got all of this down (hopefully), I’ll give you some exercises.

3.2.1 Exercises

Ex. 3 — $\overset{?}{\exists} A; A \subset \emptyset$

Ex. 4 — $\forall A; \emptyset \overset{?}{\subseteq} A$

Answers

Answer (Ex. 3) — \emptyset , by definition, has no elements. For there to exist a proper subset of \emptyset , there would need to be a set with fewer elements. There cannot be fewer than zero elements, hence there is no proper subset of \emptyset .

Answer (Ex. 4) — Yes. It's obvious that $\emptyset \subset A$ if $A \neq \emptyset$. The only thing left is, $\emptyset \overset{?}{\subseteq} \emptyset$. $\emptyset \subseteq \emptyset$ allows for $\emptyset = \emptyset$, which is obviously true.

3.3 Combining sets together

Okay, hopefully those exercises weren't too bad (email me at peter@harpending.org if they were too hard). Next, we're going to talk about ways to combine two sets together. There are any number of ways to combine two sets. The two most common ways are through *unions* and *intersections*.

The union symbol is pretty easy to memorize — it looks like a giant U: \cup . Think “Union”.²

If A and B are sets, then $A \cup B$ is the set of elements that are in either A or in B . That is:

$$A \cup B = \{x \in \mathcal{A}; (x \in A) \vee (x \in B)\}$$

You might also remember this by the fact that the union symbol \cup looks vaguely like the or symbol \vee .

What the hell is that notation ?

Let's go over it.

\mathcal{A} is the “ambient set” – it's a set such that $A, B \subseteq \mathcal{A}$. I can't quite explain it right now, but if we don't have an ambient set, then things go horribly wrong. \mathcal{A} is supposed to be a cursive 'A', but the font is terrible.

Back to business – the intersection

The intersection is, you guessed it - what happens when we use \wedge in the above definition instead of \vee . Can you guess what the symbol for “intersection” is? If you guessed \cap , then you are right!

$$A \cap B = \{x \in \mathcal{A}; (x \in A) \wedge (x \in B)\}$$

The standard example is to use “Venn diagrams”. I drew some myself in

²You don't have to remember it this way, because \cup is already aligned reasonably well with the letters. Thus I didn't have to spend 30 minutes getting the alignment correct. (See footnote 1.)

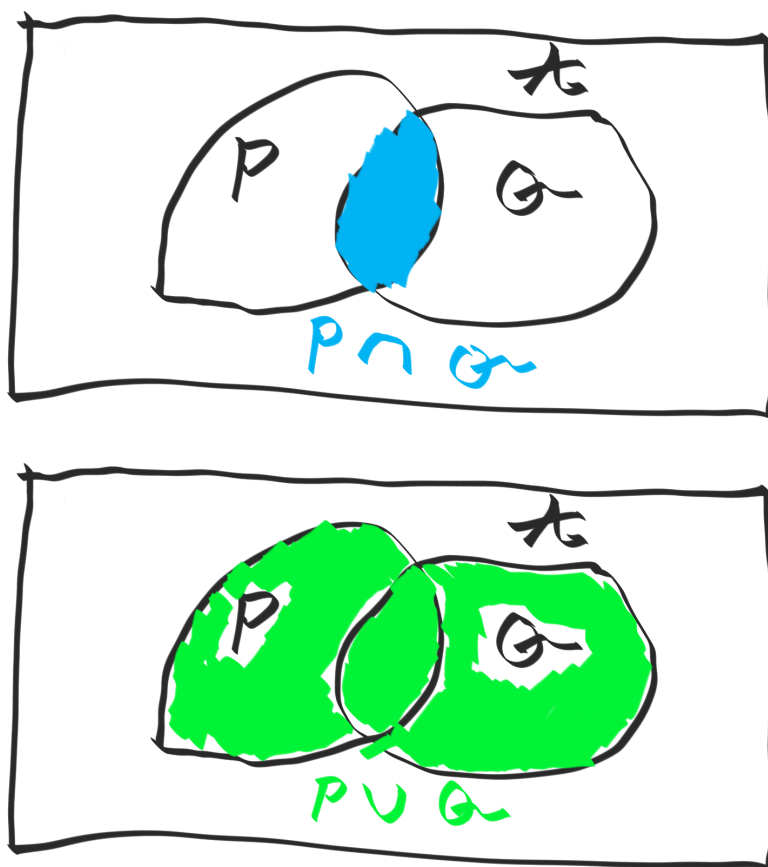
fig. 3.1.³

Figure 3.1: The Venn diagrams. That symbol in the left circle is supposed to be a Q . My handwriting sucks, deal with it.

³It's a tradition from *Learn You a Haskell* that each book include poorly-drawn explanatory graphics by the author.

3.3.1 Exercises**Ex. 5** — Given that

$$A \wedge (B \vee C) = (A \wedge B) \vee (A \wedge C)$$

Show that

$$A \vee (B \wedge C) = (A \vee B) \wedge (A \vee C)$$

Use these results to figure out

$$A \cap (B \cup C)$$

$$A \cup (B \cap C)$$

Ex. 6 — Is the following true:

$$A \cup \emptyset = A; \forall A \in \mathbf{Set}$$

Ex. 7 — What about the following:

$$A \cap \emptyset = \emptyset; \forall A \in \mathbf{Set}$$

Answers

Answer (Ex. 5) — Let's do this.

$$A \wedge (B \vee C) = (A \wedge B) \vee (A \wedge C)$$

$$\text{Let } P = B \vee C$$

$$Q = A \wedge B$$

$$R = A \wedge C$$

$$A \wedge P = Q \vee R$$

Use the definition of \neg

$$\neg(A \wedge P) = \neg(Q \vee R)$$

$$\neg A \vee \neg P = \neg Q \wedge \neg R$$

Expand out the definitions of P , Q , and R .

$$\neg A \vee \neg(B \vee C) = \neg(A \wedge B) \wedge \neg(A \wedge C)$$

$$\neg A \vee (\neg B \wedge \neg C) = (\neg A \vee \neg B) \wedge (\neg A \vee \neg C)$$

$$\text{Let } X = \neg A$$

$$Y = \neg B$$

$$Z = \neg C$$

$$X \vee (Y \wedge Z) = (X \vee Y) \wedge (X \vee Z)$$

Okay, we proved the first thing (yes, this is a proof).

Now, on to the other thing:

$$A \cap (B \cup C)$$

Let's write this out in comprehension notation:

$$A \cap (B \cup C) = \{x \in \mathcal{A}; (x \in A) \vee (x \in \{y \in \mathcal{A}; (y \in B) \vee (y \in C)\})\}$$

That probably looks horrifying to you. Trust me, it's just as bad typed out:

```

\begin{displaymath}
A \cap \text{\parens}{B \cup C} =
\text{\mset}{
  x \text{\inamb} \text{\semic}
  \text{\parens}{
    x \text{\in} A
  } \text{\land}
  \text{\parens}{
    x \text{\in} \text{\mset} {
      y \text{\inamb} \text{\semic}
      \text{\parens}{
        y \text{\in} B
      } \text{\lor}
      \text{\parens}{
        y \text{\in} C
      }
    }
  }
}
\end{displaymath}

```

We can simplify it a bit

$$A \cap (B \cup C) = \{x \in \mathcal{A}; (x \in A) \wedge ((x \in B) \vee (x \in C))\}$$

That's marginally less horrible

So we have some \wedge s and \vee s hanging around. Let's use that rule from above!

$$A \cap (B \cup C) = \{x \in \mathcal{A}; ((x \in A) \wedge (x \in B)) \vee ((x \in A) \wedge (x \in C))\}$$

But notice what we've done:

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

That result is totally worth 2 pages of math.

Answer (Ex. 6) — Yes - the union is the set of elements in either set. Since \emptyset has no elements, $A \cup \emptyset$ would just be A , no matter what A is.

Answer (Ex. 7) — Yes - the intersection of two sets is the set of objects in *both* sets. Because \emptyset has no elements, there wouldn't be any objects in both sets — the intersection would have no elements, which would make the intersection equal to \emptyset .

3.4 Special kinds of sets

Hopefully those last two exercises sparked a little lightbulb in your brain. If not, I'll try to lay it out for you:

$$\begin{aligned} A \cup \emptyset &= A; \forall A \in \mathbf{Set} \\ A \cap \emptyset &= \emptyset; \forall A \in \mathbf{Set} \end{aligned}$$

Let's look at the set \mathbb{Z} the set of integers:

$$\mathbb{Z} = \{ \dots, '6, '5, '4, '3, '2, '1, 0, 1, 2, 3, 4, 5, 6, \dots \}$$

Most people use $-$ instead of $'$ to indicate that something is negative. However, that becomes confusing, because $-$ is also used for subtraction (which requires two numbers).

$$\begin{aligned} a + 0 &= a; \forall a \in \mathbb{Z} \\ a \times 0 &= 0; \forall a \in \mathbb{Z} \\ A \cup \emptyset &= A; \forall A \in \mathbf{Set} \\ A \cap \emptyset &= \emptyset; \forall A \in \mathbf{Set} \end{aligned}$$

Don't those look really similar? It's aggravating, isn't it - there's some property they both share, but you can't articulate it! Luckily, you're a mathematician, so you can abstract things.

Fortunately for you, someone else recognized this pattern a long time ago, so they abstracted it away for you. This is called a *semirng* - specifically, a *commutative semirng*. Semirng is a joke term originated from "semiring" - it's "semiring" without the 'i' - for "identity".

Namely there's no fixed element $q : \mathbf{Set}$ for which

$$A \cap q = A; \forall A : \mathbf{Set}$$

(otherwise it would be a semiring)

Okay, what the hell are you talking about?

Mathematicians have invented all sorts of structures for these things which share properties. All of the structures are slightly different. I'll start with **Set** and \cup , and compare it to \mathbb{Z} and $+$:

1. First, we have to define \cup . It takes 2 sets, and spits out another set

$$\begin{aligned}\cup &: (\mathbf{Set}, \mathbf{Set}) \rightarrow \mathbf{Set} \\ + &: (\mathbb{Z}, \mathbb{Z}) \rightarrow \mathbb{Z}\end{aligned}$$

This properties is called “closure”, as in, “it’s closed”. Sets can’t use \cup to escape **Set**! The same is true with \mathbb{Z} .

2. We know that \cup is “associative”, which means that you can group operations weirdly.

$$\begin{aligned}A \cup (B \cup C) &= (A \cup B) \cup C \quad ; \forall A, B, C : \mathbf{Set} \\ a + (b + c) &= (a + b) + c \quad ; \forall a, b, c \in \mathbb{Z}\end{aligned}$$

For this reason, we can write $a + b + c + d + e$ without ambiguity.

3. We also know that \cup is “commutative”, which means order doesn’t matter:

$$\begin{aligned}A \cup B &= B \cup A \quad ; \forall A, B : \mathbf{Set} \\ a + b &= b + a \quad ; \forall a, b \in \mathbb{Z}\end{aligned}$$

Think of “commutative” as meaning “it commutes”, where “commutes” means “moves”.

4. We also know that \cup has an “identity” - meaning that there is an element in **Set** where \cup doesn’t have any effect:

$$\begin{aligned}A \cup \emptyset &= A \quad ; \forall A \in \mathbf{Set} \\ a + 0 &= a \quad ; \forall a \in \mathbb{Z}\end{aligned}$$

These qualities make $(\mathbb{Z}, +)$ and (\mathbf{Set}, \cup) *commutative monoids*. It's one of those fancy structures mathematicians invented. Nobody can remember the names for each one. Just Google it if you don't know. I personally use this chart from Wikipedia: fig. 3.2.

Group-like structures. The entries say whether the property is <i>required</i> .					
	Totality*	Associativity	Identity	Divisibility	Commutativity
Semicategory	No	Yes	No	No	No
Category	No	Yes	Yes	No	No
Groupoid	No	Yes	Yes	Yes	No
Magma	Yes	No	No	No	No
Quasigroup	Yes	No	No	Yes	No
Loop	Yes	No	Yes	Yes	No
Semigroup	Yes	Yes	No	No	No
Monoid	Yes	Yes	Yes	No	No
Group	Yes	Yes	Yes	Yes	No
Abelian Group	Yes	Yes	Yes	Yes	Yes

*Closure, which is used in many sources, is an equivalent axiom to totality, though defined differently.

Figure 3.2: I have to use this table every time. Source: [1] .

That chart doesn't even cover all of them. Anyway, we have our commutative monoids. I can't really go over much more without also talking about functions. So, the next chapter focuses on functions. We'll revisit this in Chapter 5.

Chapter 4

Functions

As promised, this chapter discusses functions.

So, what is a function?

So far, we've been dealing with *values* - like 2, { 3, 2, 5 }, and 90. They are static. Static things are fine, but they aren't very interesting. It's much more interesting to examine *changing things* — more specifically, things that change *predictably* and *transparently*.

Enter the *function*. It's a mathematical construct. A function takes some input, and maps it to an output. Functions are sometimes referred to as *mappings* or *morphisms*.

Let's look at a simple function, which takes a number and adds 2 to it

$$\begin{aligned} f : \mathbb{Z} &\rightarrow \mathbb{Z} \\ f = \lambda (x) &\rightarrow x + 2 \end{aligned}$$

Pretty simple, right? Okay, so what happens when we send 28 to f ?

$$\begin{aligned} f(x = 28) &= \lambda(x = 28) \rightarrow 28 + 2 \\ &= 30 \end{aligned}$$

Alternatively, since it's obvious we're working with x :

$$\begin{aligned} f(28) &= 28 + 2 \\ &= 30 \end{aligned}$$

Using the lambda — λ — is common when I am using a function without giving it a name. However, usually I will use this notation:

$$\begin{aligned} f : \mathbb{Z} &\rightarrow \mathbb{Z} \\ f(x) &= x + 2 \end{aligned}$$

The whole $f : \mathbb{Z} \rightarrow \mathbb{Z}$ thing should be pretty obvious. If not, it means that f is a function that takes a member of \mathbb{Z} (the whole numbers, both negative and positive), and takes it to another member of \mathbb{Z} . Other people might use the notation

$$\mathbb{Z} \xrightarrow{f} \mathbb{Z}$$

That notation is undoubtedly easier to understand. However, as we'll see, that notation quickly becomes unfeasible.

4.0.1 Functions with multiple arguments

Remember my explanation of vectors earlier? If not, vectors are like sets, but order and repetition matter.

Here's a function that takes two arguments, and adds them to each other.

$$\begin{aligned} f : (\mathbb{Z}, \mathbb{Z}) &\rightarrow \mathbb{Z} \\ f(x, y) &= x + y \end{aligned}$$

Pretty easy to understand, right?

If you haven't figured it out from the context, the inputs to the function are called the *arguments*.

Here's a similar function that takes three arguments and adds them to each other

$$\begin{aligned} f : (\mathbb{Z}, \mathbb{Z}, \mathbb{Z}) &\rightarrow \mathbb{Z} \\ f(x, y, z) &= x + y + z \end{aligned}$$

You can name your function anything you want, same with the arguments (it doesn't have to be f). It's just a common convention, which you don't have to follow.

What if I want to add a bunch of things together?

Good idea!

$$\begin{aligned} f : (\mathbb{Z}, \mathbb{Z}, \dots, \mathbb{Z}) &\rightarrow \mathbb{Z} \\ f(x_1, x_2, x_3, \dots, x_n) &= x_1 + x_2 + x_3 + \dots + x_n \end{aligned}$$

That however isn't ideal, because we have no guarantee that the arguments in the \dots are actually integers. How about we have a *set* of integers, and we just take the sum? This has the added benefit of less typing

$$\begin{aligned} f : \mathbf{Set}(\mathbb{Z}) &\rightarrow \mathbb{Z} \\ f(s) &= \sum s \end{aligned}$$

So,

$$\begin{aligned} f(\{1, 2, 3, 4, 5\}) &= \sum \{1, 2, 3, 4, 5\} \\ &= 1 + 2 + 3 + 4 + 5 \\ &= 15 \end{aligned}$$

4.0.2 Eta-reductions

Mathematicians like to make themselves look smart. One such way is to invent fancy terms for simple things. One such term is the η -reduction.

Let's look at that function we just had

$$\begin{aligned} f &: \mathbf{Set}(\mathbb{Z}) \rightarrow \mathbb{Z} \\ f(s) &= \sum s \end{aligned}$$

Notice that we are repeating s on both sides of the equation. It would seem much simpler, and just as clear, to write:

$$\begin{aligned} f &: \mathbf{Set}(\mathbb{Z}) \rightarrow \mathbb{Z} \\ f &= \sum \end{aligned}$$

That's all an η -reduction is: if you see an extraneous argument, you remove it to make things simpler. As long as we have the signature — the $f : \mathbf{Set}(\mathbb{Z}) \rightarrow \mathbb{Z}$ thing — it's pretty clear what f does. This is a prime example of mathematicians being both lazy and pretentious at the same time: a practice designed to allow us to be lazier, to which mathematicians have assigned a ridiculous name to make it sound hard.

What the hell is η ?

η is the Greek letter eta; it's pronounced "eight-uh".

The ancient Greeks were too dumb to comprehend the concept of "eight". Every time someone brought it up, they said "uh" immediately thereafter. The sound "eight-uh" became so common that they decided to make it a letter.

The Greeks' poor comprehension of simple mathematics remains to this day, and is largely the reason for their current financial crisis.[4]

If you ever take a physics course, you will undoubtedly notice that Greek letters are used frequently in physics. This is the physicists way of subtly

hinting that they actually have no idea what they are talking about, and pleading for help from the mathematicians.

4.1 Currying

We sort of got side-tracked by toying around with sets and making fun of physicists. Hopefully that introduction introduced you to the basic concept of a function, and let you know that they can take multiple arguments

Let's look at that function again:

$$\begin{aligned} f : (\mathbb{Z}, \mathbb{Z}, \mathbb{Z}) &\rightarrow \mathbb{Z} \\ f(x, y, z) &= x + y + z \end{aligned}$$

What if you wanted to bind $x = 3$, but leave the rest “free”?

$$\begin{aligned} f : (\mathbb{Z}, \mathbb{Z}, \mathbb{Z}) &\rightarrow \mathbb{Z} \\ f(x = 3, y, z) &= 3 + y + z \end{aligned}$$

Okay, cool. We now have another function:

$$\begin{aligned} f(3) : (\mathbb{Z}, \mathbb{Z}) &\rightarrow \mathbb{Z} \\ f(3, y, z) &= 3 + y + z \end{aligned}$$

So, actually, instead of needing 3 integers to do its job, f only needed one. However, instead of spitting out another integer, it spit out a function. So, we could write f 's signature as:

$$\begin{aligned} f : \mathbb{Z} &\rightarrow ((\mathbb{Z}, \mathbb{Z}) \rightarrow \mathbb{Z}) \\ f(x, y, z) &= x + y + z \end{aligned}$$

Okay, that's sort of weird and unintuitive. Let's try writing f differently:

$$\begin{aligned} f &: \mathbb{Z} \rightarrow ((\mathbb{Z}, \mathbb{Z}) \rightarrow \mathbb{Z}) \\ f &= \lambda(x) \rightarrow (\lambda(y, z) \rightarrow x + y + z) \end{aligned}$$

Let's look at the second half of that:

$$\lambda(y, z) \rightarrow x + y + z : (\mathbb{Z}, \mathbb{Z}) \rightarrow \mathbb{Z}$$

(This assumes that we know what x is)

Let's try splitting this up again:

$$\lambda(y) \rightarrow (\lambda(z) \rightarrow x + y + z) : \mathbb{Z} \rightarrow (\mathbb{Z} \rightarrow \mathbb{Z})$$

You give this function a value for y , and instead of giving you a value, it gives you another function, hence the signature $\mathbb{Z} \rightarrow (\mathbb{Z} \rightarrow \mathbb{Z})$.

Let's plug this back into f :

$$\begin{aligned} f &: \mathbb{Z} \rightarrow (\mathbb{Z} \rightarrow (\mathbb{Z} \rightarrow \mathbb{Z})) \\ f &= \lambda(x) \rightarrow (\lambda(y) \rightarrow (\lambda(z) \rightarrow x + y + z)) \end{aligned}$$

So, instead of f taking three integers, it now only takes one, but spits out a function, which in turn spits out a function, which spits out an integer.

This idea of making a function into a chain of functions is called “Currying”.^[2] It's named after a dead mathematician named Haskell Curry (ca. 1900-1982), who developed the technique. The programming language Haskell is also named after Mr. Curry.

Getting back to that function, those parentheses are somewhat burdensome, let's get rid of them

$$\begin{aligned}
f &: \mathbb{Z} \rightarrow \mathbb{Z} \rightarrow \mathbb{Z} \rightarrow \mathbb{Z} \\
f &= \lambda(x) \rightarrow \lambda(y) \rightarrow \lambda(z) \rightarrow x + y + z \\
f(x, y, z) &= x + y + z
\end{aligned}$$

That's much easier to read. It should be understood that the parentheses are right-associative: the parentheses “associate” rightward — i.e. it's $a \rightarrow (b \rightarrow (c \rightarrow d))$, not $((a \rightarrow b) \rightarrow c) \rightarrow d$. [6]

That's Currying for you.

4.2 Vocabulary

This is important vocabulary that I'm going to use later:

1. All functions are *transparent* — $a = b \implies f(a) = f(b)$
2. If $f : A \rightarrow B$, then A is the *domain* of f and B is the *codomain* of f .
3. If $f : A \rightarrow B$, and there are no two distinct elements of A that map to the same thing in B , then f is *injective*.

$$\begin{aligned}
&f : A \rightarrow B \\
&\nexists (a, b); a, b \in A \wedge a \neq b \wedge f(a) = f(b) \iff f \text{ is injective}
\end{aligned}$$

4. If $f : A \rightarrow B$, then the elements in B that can be expressed as $f(x)$; $x \in A$ form the *image*.

$$\begin{aligned}
&f : A \rightarrow B \\
&\text{im}(f) = \{ f(x) \in B; x \in A \}
\end{aligned}$$

5. If the image of a function is equal to its codomain, then the function is *surjective*.

$$f : A \rightarrow B$$

$$B = \{ f(x) \in B; x \in A \} \iff f \text{ is surjective}$$

6. If a function is both injective and surjective, then it is *bijective*.
7. Some functions have *inverses*. That is, if

$$f : A \rightarrow B$$

$$\text{arc}(f) : B \rightarrow A$$

$$\text{arc}(f, x) = x; \forall x \in A$$

Remember that, because of currying, $\text{arc}(f, x) = \text{arc}(f)(x)$. That is:

$$f : A \rightarrow B$$

$$\text{arc} : (A \rightarrow B) \rightarrow B \rightarrow A$$

$$\text{arc}(f) : B \rightarrow A$$

If a function has an inverse, it is said to be *invertible*.

8. If a function is invertible, then the image of the inverse is called the *preimage*.

4.2.1 Exercises

Ex. 8 — I knew you were going to just gloss over those, so I made a really hard (i.e. fun) problem: prove that a function is invertible if (and only if) it is bijective. This is a very difficult proof, but you really need to understand it.

Answers

Answer (Ex. 8) — Let's look at $f : A \rightarrow B$. If f is surjective, then $B = \text{im}(f)$, so we can write

$$f : A \rightarrow \text{im}(f)$$

In other words

$$f : \mathbf{dom}(f) \rightarrow \text{im}(f)$$

It must be true that f is a surjection for f to be invertible. Else there would be elements in the codomain of f that were not in the domain of $\text{arc}(f)$. We've established

$$\text{arc}(f) : \text{im}(f) \rightarrow \mathbf{dom}(f)$$

Let's assume f is invertible. Then $\mathbf{dom}(f) = \text{preim}(f)$. Thus

$$\text{arc}(f) : \text{im}(f) \rightarrow \mathbf{dom}(f)$$

For $\text{arc}(f)$ to be a function — i.e. for f to be invertible, then it must be true that

$$\nexists a, b \in \text{im}(f); a = b; \text{arc}(f, a) \neq \text{arc}(f, b)$$

If we flip this around

$$\nexists a, b \in \text{preim}(f); a \neq b; f(a) = f(b)$$

That is, the definition of injectivity. Thus we have proven

$$f \text{ is invertible} \iff (f \text{ is an injection}) \wedge (f \text{ is a surjection}) \iff f \text{ is a bijection}$$

Chapter 5

More stuff about sets

By now, you hopefully have some idea into the basic intuition behind sets and functions. Moreover, you've proven some cool stuff about them – for instance, you proved that a function is invertible iff it is bijective.

That's kind of cool, right? It's much easier to verify that a function is bijective than it is to find its inverse. So, right off the bat, you can see if a function is invertible without trying to invert it. Despite what you may think, a common problem in math is to find inverse functions.

Speaking of functions, I'm going to define a really simple function:

$$\begin{aligned}\text{id} &: a \rightarrow a \\ \text{id}(x) &= x\end{aligned}$$

That function is about as simple as functions get. It's not a very interesting function, but it's handy when defining things.

This is a slightly less simple function, but nonetheless important

$$\begin{aligned}\text{flip} &: (a \rightarrow b \rightarrow c) \rightarrow b \rightarrow a \rightarrow c \\ \text{flip}(f, x, y) &= f(y, x)\end{aligned}$$

It takes a function, f , which takes an a and a b , and then returns another function with the arguments flipped. You won't usually see $\text{flip}(\lambda(x, y) \rightarrow x - y, 3, 5)$ floating around. Instead

$$\begin{aligned} f & : p \rightarrow q \rightarrow r \\ \text{flip}(f) & : q \rightarrow p \rightarrow r \end{aligned}$$

Here's an infix operator:

$$\begin{aligned} \circ & : (b \rightarrow c) \rightarrow (a \rightarrow b) \rightarrow a \rightarrow c \\ (f \circ g)(x) & = f(g(x)) \end{aligned}$$

We're going to look at some stuff with sets.

5.1 Set subtraction

First off, we have “set subtraction”. This is sometimes called the “relative complement”.

$$P \setminus Q = \{x \in P; x \notin Q\}$$

$P \setminus Q$ is all of the elements in P that are not in Q .

What would $P \setminus P$ be, then? Well, \emptyset , of course, right!

$$P \setminus P = \{x \in P; x \notin P\} = \emptyset$$

Well, that looks like a contradiction, doesn't it? Well, sort of. It's a contradiction if you think of it as “ x has to be both in P and not in P , so it can't be a set.” But, if you realize that the comprehension is the “set of objects satisfying the condition”, then you don't encounter a contradiction. $P \setminus P$ is the set of all objects in P that are not in P : there are no elements satisfying this condition, thus $P \setminus P \equiv \emptyset$.

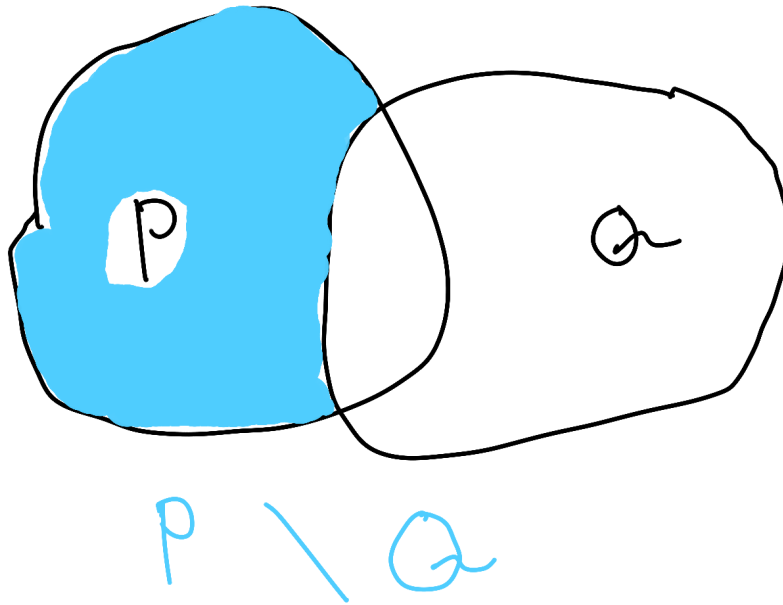


Figure 5.1: Set subtraction

Figure 5.1 explains this idea graphically.

5.1.1 Complement

If $P \subset \mathcal{A}$, then $\mathcal{A} \setminus P$ is called the “*complement* of P with respect to \mathcal{A} ”. Yes, that’s complement with an ‘e’, not compliment, with an ‘i’. Given a set P , and another set \mathcal{A} , such that $P \subseteq \mathcal{A}$,

P^c is the complement of P with respect to an unknown ambient set.

I drew another diagram to illustrate the complement in fig. 5.2.

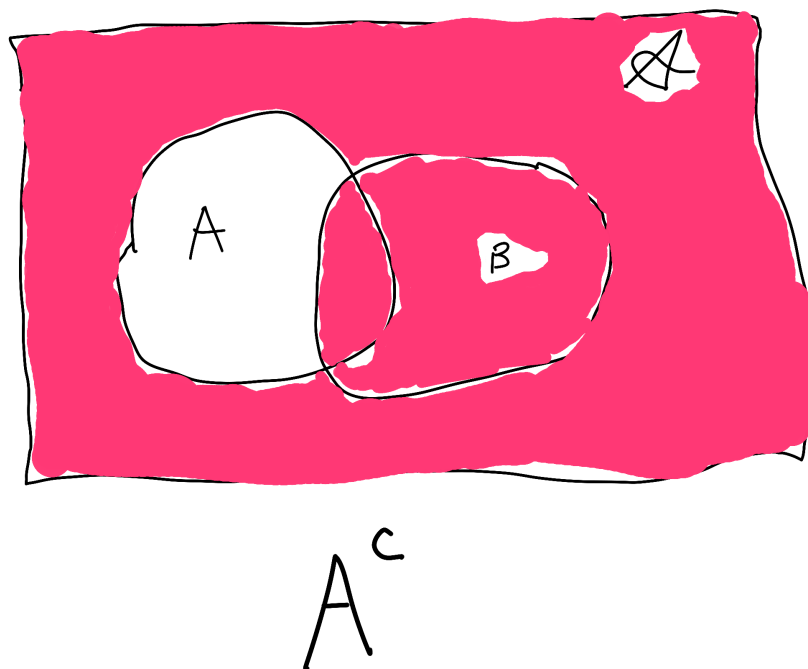


Figure 5.2: The complement, illustrated graphically. I tried to draw an \mathcal{A} in the upper right corner. It turned out terribly.

5.1.2 Exercises

Ex. 9 — $P^{cc} \stackrel{?}{=} P$

Ex. 10 — $C \setminus (A \cup B) \stackrel{?}{=} (C \setminus A) \cap (C \setminus B)$

What about

$$C \setminus (A \cap B) \stackrel{?}{=} (C \setminus A) \cup (C \setminus B)$$

Ex. 11 — $A \setminus (B \setminus C) \stackrel{?}{=} (A \setminus B) \setminus C$

Ex. 12 — $A \cap (B \setminus C) \stackrel{?}{=} (A \cap B) \setminus C$

Ex. 13 — $A \setminus (B \cap C) \stackrel{?}{=} (A \setminus B) \cap C$

Answers

Answer (Ex. 9) — Yes.

$$\begin{aligned} P^c &\equiv \{x \in \mathcal{A}; x \notin P\} \\ &\equiv \{x \in \mathcal{A}; \neg(x \in P)\} \\ P^{cc} &\equiv \{x \in \mathcal{A}; \neg\neg(x \in P)\} \\ &\equiv \{x \in \mathcal{A}; x \in P\} \\ &\equiv P \end{aligned}$$

Appendices

Appendix A

GNU Free Documentation License

Version 1.3, 3 November 2008

Copyright © 2000, 2001, 2002, 2007, 2008 Free Software Foundation, Inc.

[<http://fsf.org/>](http://fsf.org/)

Everyone is permitted to copy and distribute verbatim copies of this license document, but changing it is not allowed.

Preamble

The purpose of this License is to make a manual, textbook, or other functional and useful document “free” in the sense of freedom: to assure everyone the effective freedom to copy and redistribute it, with or without modifying it, either commercially or noncommercially. Secondly, this License preserves for the author and publisher a way to get credit for their work, while not being considered responsible for modifications made by others.

This License is a kind of “copyleft”, which means that derivative works of the document must themselves be free in the same sense. It complements the GNU General Public License, which is a copyleft license designed for free software.

We have designed this License in order to use it for manuals for free software, because free software needs free documentation: a free program should come with manuals providing the same freedoms that the software does. But this License is not limited to software manuals; it can be used for any textual work, regardless of subject matter or whether it is published as a printed book. We recommend this License principally for works whose purpose is instruction or reference.

1. APPLICABILITY AND DEFINITIONS

This License applies to any manual or other work, in any medium, that contains a notice placed by the copyright holder saying it can be distributed under the terms of this License. Such a notice grants a world-wide, royalty-free license, unlimited in duration, to use that work under the conditions stated herein. The “**Document**”, below, refers to any such manual or work. Any member of the public is a licensee, and is addressed as

“**you**”. You accept the license if you copy, modify or distribute the work in a way requiring permission under copyright law.

A “**Modified Version**” of the Document means any work containing the Document or a portion of it, either copied verbatim, or with modifications and/or translated into another language.

A “**Secondary Section**” is a named appendix or a front-matter section of the Document that deals exclusively with the relationship of the publishers or authors of the Document to the Document’s overall subject (or to related matters) and contains nothing that could fall directly within that overall subject. (Thus, if the Document is in part a textbook of mathematics, a Secondary Section may not explain any mathematics.) The relationship could be a matter of historical connection with the subject or with related matters, or of legal, commercial, philosophical, ethical or political position regarding them.

The “**Invariant Sections**” are certain Secondary Sections whose titles are designated, as being those of Invariant Sections, in the notice that says that the Document is released under this License. If a section does not fit the above definition of Secondary then it is not allowed to be designated as Invariant. The Document may contain zero Invariant Sections. If the Document does not identify any Invariant Sections then there are none.

The “**Cover Texts**” are certain short passages of text that are listed, as Front-Cover Texts or Back-Cover Texts, in the notice that says that the Document is released under this License. A Front-Cover Text may be at most 5 words, and a Back-Cover Text may be at most 25 words.

A “**Transparent**” copy of the Document means a machine-readable copy, represented in a format whose specification is available to the general public, that is suitable for revising the document straightforwardly with generic text editors or (for images composed of pixels) generic paint programs or (for drawings) some widely available drawing editor, and that is suitable for input to text formatters or for automatic translation to a variety of formats suitable for input to text formatters. A copy made in an otherwise Transparent file format whose markup, or absence of markup, has been arranged to thwart or discourage subsequent modification by readers is not Transparent. An image format is not Transparent if used for any substantial amount of text. A copy that is not “Transparent” is called “**Opaque**”.

Examples of suitable formats for Transparent copies include plain ASCII without markup, Texinfo input format, LaTeX input format, SGML or XML using a publicly available DTD, and standard-conforming simple HTML, PostScript or PDF designed for human modification. Examples of transparent image formats include PNG, XCF and JPG. Opaque formats include proprietary formats that can be read and edited only by proprietary word processors, SGML or XML for which the DTD and/or processing tools are not generally available, and the machine-generated HTML, PostScript or PDF produced by some word processors for output purposes only.

The “**Title Page**” means, for a printed book, the title page itself, plus such following pages as are needed to hold, legibly, the material this License requires to appear in the title page. For works in formats which do not have any title page as such, “Title Page” means the text near the most prominent appearance of the work’s title, preceding the beginning of the body of the text.

The “**publisher**” means any person or entity that distributes copies of the Document to the public.

A section “**Entitled XYZ**” means a named subunit of the Document whose title either is precisely XYZ or contains XYZ in parentheses following text that translates XYZ in another language. (Here XYZ stands for a specific section name mentioned below, such as “**Acknowledgements**”, “**Dedications**”, “**Endorsements**”, or “**History**”.) To “**Preserve the Title**” of such a section when you modify the Document means that it remains a section “Entitled XYZ” according to this definition.

The Document may include Warranty Disclaimers next to the notice which states that this License applies to the Document. These Warranty Disclaimers are considered to be included by reference in this License, but only as regards disclaiming warranties: any other implication that these Warranty Disclaimers may have is void and has no effect on the meaning of this License.

2. VERBATIM COPYING

You may copy and distribute the Document in any medium, either commercially or noncommercially, provided that this License, the copyright notices, and the license notice saying this License applies to the Document are reproduced in all copies, and that you add no other conditions whatsoever to those of this License. You may not use technical measures to obstruct or control the reading or further copying of the copies you make or distribute. However, you may accept compensation in exchange for copies. If you distribute a large enough number of copies you must also follow the conditions in section 3.

You may also lend copies, under the same conditions stated above, and you may publicly display copies.

3. COPYING IN QUANTITY

If you publish printed copies (or copies in media that commonly have printed covers) of the Document, numbering more than 100, and the Document’s license notice requires Cover Texts, you must enclose the copies in covers that carry, clearly and legibly, all these Cover Texts: Front-Cover Texts on the front cover, and Back-Cover Texts on the back cover. Both covers must also clearly and legibly identify you as the publisher of these copies. The front cover must present the full title with all words of the title equally prominent and visible. You may add other material on the covers in addition. Copying with changes limited to the covers, as long as they preserve the title of the Document and satisfy these conditions, can be treated as verbatim copying in other respects.

If the required texts for either cover are too voluminous to fit legibly, you should put the first ones listed (as many as fit reasonably) on the actual cover, and continue the rest onto adjacent pages.

If you publish or distribute Opaque copies of the Document numbering more than 100, you must either include a machine-readable Transparent copy along with each Opaque copy, or state in or with each Opaque copy a computer-network location from which the general network-using public has access to download using public-standard network protocols a complete Transparent copy of the Document, free of added material. If you use the latter option, you must take reasonably prudent steps, when you begin distribution of

Opaque copies in quantity, to ensure that this Transparent copy will remain thus accessible at the stated location until at least one year after the last time you distribute an Opaque copy (directly or through your agents or retailers) of that edition to the public.

It is requested, but not required, that you contact the authors of the Document well before redistributing any large number of copies, to give them a chance to provide you with an updated version of the Document.

4. MODIFICATIONS

You may copy and distribute a Modified Version of the Document under the conditions of sections 2 and 3 above, provided that you release the Modified Version under precisely this License, with the Modified Version filling the role of the Document, thus licensing distribution and modification of the Modified Version to whoever possesses a copy of it. In addition, you must do these things in the Modified Version:

- A. Use in the Title Page (and on the covers, if any) a title distinct from that of the Document, and from those of previous versions (which should, if there were any, be listed in the History section of the Document). You may use the same title as a previous version if the original publisher of that version gives permission.
- B. List on the Title Page, as authors, one or more persons or entities responsible for authorship of the modifications in the Modified Version, together with at least five of the principal authors of the Document (all of its principal authors, if it has fewer than five), unless they release you from this requirement.
- C. State on the Title page the name of the publisher of the Modified Version, as the publisher.
- D. Preserve all the copyright notices of the Document.
- E. Add an appropriate copyright notice for your modifications adjacent to the other copyright notices.
- F. Include, immediately after the copyright notices, a license notice giving the public permission to use the Modified Version under the terms of this License, in the form shown in the Addendum below.
- G. Preserve in that license notice the full lists of Invariant Sections and required Cover Texts given in the Document's license notice.
- H. Include an unaltered copy of this License.
- I. Preserve the section Entitled "History", Preserve its Title, and add to it an item stating at least the title, year, new authors, and publisher of the Modified Version as given on the Title Page. If there is no section Entitled "History" in the Document, create one stating the title, year, authors, and publisher of the Document as given on its Title Page, then add an item describing the Modified Version as stated in the previous sentence.

- J. Preserve the network location, if any, given in the Document for public access to a Transparent copy of the Document, and likewise the network locations given in the Document for previous versions it was based on. These may be placed in the “History” section. You may omit a network location for a work that was published at least four years before the Document itself, or if the original publisher of the version it refers to gives permission.
- K. For any section Entitled “Acknowledgements” or “Dedications”, Preserve the Title of the section, and preserve in the section all the substance and tone of each of the contributor acknowledgements and/or dedications given therein.
- L. Preserve all the Invariant Sections of the Document, unaltered in their text and in their titles. Section numbers or the equivalent are not considered part of the section titles.
- M. Delete any section Entitled “Endorsements”. Such a section may not be included in the Modified Version.
- N. Do not retitle any existing section to be Entitled “Endorsements” or to conflict in title with any Invariant Section.
- O. Preserve any Warranty Disclaimers.

If the Modified Version includes new front-matter sections or appendices that qualify as Secondary Sections and contain no material copied from the Document, you may at your option designate some or all of these sections as invariant. To do this, add their titles to the list of Invariant Sections in the Modified Version’s license notice. These titles must be distinct from any other section titles.

You may add a section Entitled “Endorsements”, provided it contains nothing but endorsements of your Modified Version by various parties—for example, statements of peer review or that the text has been approved by an organization as the authoritative definition of a standard.

You may add a passage of up to five words as a Front-Cover Text, and a passage of up to 25 words as a Back-Cover Text, to the end of the list of Cover Texts in the Modified Version. Only one passage of Front-Cover Text and one of Back-Cover Text may be added by (or through arrangements made by) any one entity. If the Document already includes a cover text for the same cover, previously added by you or by arrangement made by the same entity you are acting on behalf of, you may not add another; but you may replace the old one, on explicit permission from the previous publisher that added the old one.

The author(s) and publisher(s) of the Document do not by this License give permission to use their names for publicity for or to assert or imply endorsement of any Modified Version.

5. COMBINING DOCUMENTS

You may combine the Document with other documents released under this License, under the terms defined in section 4 above for modified versions, provided that you include in the combination all of the Invariant Sections of all of the original documents, unmodified,

and list them all as Invariant Sections of your combined work in its license notice, and that you preserve all their Warranty Disclaimers.

The combined work need only contain one copy of this License, and multiple identical Invariant Sections may be replaced with a single copy. If there are multiple Invariant Sections with the same name but different contents, make the title of each such section unique by adding at the end of it, in parentheses, the name of the original author or publisher of that section if known, or else a unique number. Make the same adjustment to the section titles in the list of Invariant Sections in the license notice of the combined work.

In the combination, you must combine any sections Entitled “History” in the various original documents, forming one section Entitled “History”; likewise combine any sections Entitled “Acknowledgements”, and any sections Entitled “Dedications”. You must delete all sections Entitled “Endorsements”.

6. COLLECTIONS OF DOCUMENTS

You may make a collection consisting of the Document and other documents released under this License, and replace the individual copies of this License in the various documents with a single copy that is included in the collection, provided that you follow the rules of this License for verbatim copying of each of the documents in all other respects.

You may extract a single document from such a collection, and distribute it individually under this License, provided you insert a copy of this License into the extracted document, and follow this License in all other respects regarding verbatim copying of that document.

7. AGGREGATION WITH INDEPENDENT WORKS

A compilation of the Document or its derivatives with other separate and independent documents or works, in or on a volume of a storage or distribution medium, is called an “aggregate” if the copyright resulting from the compilation is not used to limit the legal rights of the compilation’s users beyond what the individual works permit. When the Document is included in an aggregate, this License does not apply to the other works in the aggregate which are not themselves derivative works of the Document.

If the Cover Text requirement of section 3 is applicable to these copies of the Document, then if the Document is less than one half of the entire aggregate, the Document’s Cover Texts may be placed on covers that bracket the Document within the aggregate, or the electronic equivalent of covers if the Document is in electronic form. Otherwise they must appear on printed covers that bracket the whole aggregate.

8. TRANSLATION

Translation is considered a kind of modification, so you may distribute translations of the Document under the terms of section 4. Replacing Invariant Sections with translations requires special permission from their copyright holders, but you may include translations of some or all Invariant Sections in addition to the original versions of these Invariant Sections. You may include a translation of this License, and all the license notices in the Document, and any Warranty Disclaimers, provided that you also include the original English version of this License and the original versions of those notices and disclaimers. In case of a disagreement between the translation and the original version of this License or a notice or disclaimer, the original version will prevail.

If a section in the Document is Entitled “Acknowledgements”, “Dedications”, or “History”, the requirement (section 4) to Preserve its Title (section 1) will typically require changing the actual title.

9. TERMINATION

You may not copy, modify, sublicense, or distribute the Document except as expressly provided under this License. Any attempt otherwise to copy, modify, sublicense, or distribute it is void, and will automatically terminate your rights under this License.

However, if you cease all violation of this License, then your license from a particular copyright holder is reinstated (a) provisionally, unless and until the copyright holder explicitly and finally terminates your license, and (b) permanently, if the copyright holder fails to notify you of the violation by some reasonable means prior to 60 days after the cessation.

Moreover, your license from a particular copyright holder is reinstated permanently if the copyright holder notifies you of the violation by some reasonable means, this is the first time you have received notice of violation of this License (for any work) from that copyright holder, and you cure the violation prior to 30 days after your receipt of the notice.

Termination of your rights under this section does not terminate the licenses of parties who have received copies or rights from you under this License. If your rights have been terminated and not permanently reinstated, receipt of a copy of some or all of the same material does not give you any rights to use it.

10. FUTURE REVISIONS OF THIS LICENSE

The Free Software Foundation may publish new, revised versions of the GNU Free Documentation License from time to time. Such new versions will be similar in spirit to the present version, but may differ in detail to address new problems or concerns. See <http://www.gnu.org/copyleft/>.

Each version of the License is given a distinguishing version number. If the Document specifies that a particular numbered version of this License “or any later version” applies to it, you have the option of following the terms and conditions either of that specified version or of any later version that has been published (not as a draft) by the Free Software Foundation. If the Document does not specify a version number of this License, you may

choose any version ever published (not as a draft) by the Free Software Foundation. If the Document specifies that a proxy can decide which future versions of this License can be used, that proxy's public statement of acceptance of a version permanently authorizes you to choose that version for the Document.

11. RELICENSING

“Massive Multiauthor Collaboration Site” (or “MMC Site”) means any World Wide Web server that publishes copyrightable works and also provides prominent facilities for anybody to edit those works. A public wiki that anybody can edit is an example of such a server. A “Massive Multiauthor Collaboration” (or “MMC”) contained in the site means any set of copyrightable works thus published on the MMC site.

“CC-BY-SA” means the Creative Commons Attribution-Share Alike 3.0 license published by Creative Commons Corporation, a not-for-profit corporation with a principal place of business in San Francisco, California, as well as future copyleft versions of that license published by that same organization.

“Incorporate” means to publish or republish a Document, in whole or in part, as part of another Document.

An MMC is “eligible for relicensing” if it is licensed under this License, and if all works that were first published under this License somewhere other than this MMC, and subsequently incorporated in whole or in part into the MMC, (1) had no cover texts or invariant sections, and (2) were thus incorporated prior to November 1, 2008.

The operator of an MMC Site may republish an MMC contained in the site under CC-BY-SA on the same site at any time before August 1, 2009, provided the MMC is eligible for relicensing.

ADDENDUM: How to use this License for your documents

To use this License in a document you have written, include a copy of the License in the document and put the following copyright and license notices just after the title page:

Copyright © YEAR YOUR NAME. Permission is granted to copy, distribute and/or modify this document under the terms of the GNU Free Documentation License, Version 1.3 or any later version published by the Free Software Foundation; with no Invariant Sections, no Front-Cover Texts, and no Back-Cover Texts. A copy of the license is included in the section entitled “GNU Free Documentation License”.

If you have Invariant Sections, Front-Cover Texts and Back-Cover Texts, replace the “with ... Texts.” line with this:

with the Invariant Sections being LIST THEIR TITLES, with the Front-Cover Texts being LIST, and with the Back-Cover Texts being LIST.

If you have Invariant Sections without Cover Texts, or some other combination of the three, merge those two alternatives to suit the situation.

If your document contains nontrivial examples of program code, we recommend releasing these examples in parallel under your choice of free software license, such as the GNU General Public License, to permit their use in free software.

Appendix B

How to learn math

Now that that's all out of the way, let's talk a little about math. When this chapter is over, we're going to dive right in to proving a bunch of things you already know to be true. We feel that without a little explanation, these proofs may leave you a little lost or confused. We'll save the explanation of the proofs for later, but right now, we're going to talk about how to actually learn math, and the proofs are a great example.

Most people's experience with math is through their primary and possibly secondary education, which is or was a dreary affair in general, and math probably even moreso, unless you're one of the lucky few. By lucky few, we don't mean those wizards with a sort of inherent ability to do math—the first thing you need to know about learning math is that math is for everyone with a brain—that's you, right? You see, your brain is a pattern recognition engine, and that's all math is: the study of patterns. Unlike reading or history, your body comes with a biological imperative to know math. There's some really great brain studies on the topic, but that's boring, and I said we're already done with the boring part, so let's move on.

In that last paragraph, we presented what we hold to be the proper answer to 'what is math': the study of patterns. This is completely different from most people's interaction with math: in primary school, we are taught how to apply four operations to solve math problems. You're given something about two trains leaving a station and going different speeds and different directions and yadda yadda yadda and before you know it your teacher turned

everything into a math problem and it all seemed so forced—a layer on top of what was intuitive, and made everything complicated. We agree—this is a counterintuitive approach to math, and it makes math very confusing and disconnected. Math is just the study of patterns. That is, math is not so much a way to solve a set of problems that exist in a sphere apart from what is natural, but a way to understand what’s going on in the world around us. When you learn math, you should think of it as a science—another level of detail in the amazing world we live in.

That’s how this book is written. It’s written to reflect that math is a single unified study. While you’re reading it, try to think of how what you’re learning clarifies or refines early material. This is a big deal to us, because one thing we dislike most about the standard way of learning math is that at some point in everyone’s math career, they learn they were taught something that wasn’t actually true. We want to avoid that.

Appendix C

Philosophy and/or FAQ

by Peter Harpending <peter@harpending.org>

This book is written with a certain philosophy in mind. Explaining my philosophy will answer a number of questions I am often asked.

First, I'll start with the license. The license I chose for this book is the GNU Free Documentation License (FDL). Again, “free” refers to freedom, not price. The FDL is similar in spirit to another license, the Creative Commons Attribution-ShareAlike License (CC-SA). CC-SA is much more popular than the FDL, mostly because it is much more general (e.g. you could distribute a painting under CC-SA, but not the FDL). The CC-SA license and the FDL are both “copyleft” licenses, in that they require that derivative works be licensed under the same license.

So, why did you go with the FDL instead of CC-SA?

Simply put, the CC-SA license is too general to fit our purposes. The FDL is specifically designed for reference texts, so it has a clause requiring that the work be made available in source form. The CC-SA has no such requirement.

To go on about this, I need to define “freedom” in academic works. This is a modified version of the GNU Project’s definition of free software (<https://gnu.org/philosophy/free-sw.html>). In my view, an academic work is free

if you have:

- The freedom to use the digital document as you wish, for any purpose (freedom 0).
- The freedom to reproduce exact copies of the document in any medium. (e.g. print the book). (freedom 1)
- The freedom to distribute and/or sell exact copies of the document to whomever you choose, in any medium. (freedom 2).
- The freedom to modify your own copy of the book. Access to the source is a precondition for this. (freedom 3)
- The freedom to reproduce modified copies of the document in any medium. (e.g. print the book). (freedom 4)
- The freedom to distribute and/or sell exact copies of your modified version to others, in any medium (freedom 5).

To guarantee freedom for everyone, we unfortunately have to restrict freedom 3 a little bit. You can't modify the book in such a way that would restrict others' freedom. Such ways would include

- Implementing digital restrictions management, or DRM.
- Removing the license.
- Releasing your modified version under a different license.

I suppose someone who ran in a different clique would wonder why people put up so much fuss about something as silly as a license. The license explains exactly what the reader can and can't do with my work. When my objective is freedom, allowing everyone the maximum quantity of freedom requires some copyright trickery, hence the 10-page-long license.

You would think that this freedom would be implicit in any academic work. Unfortunately, that's not the case.

An extreme instance of this trope of freedom restriction was a wonderful company named Myriad Genetics. Myriad Genetics attempted to patent human genes. Fortunately, the United States Supreme Court struck down this class of patents in a unanimous decision (http://www.supremecourt.gov/opinions/12pdf/12-398_1b7d.pdf). For those of you who aren't American, unanimous US Supreme Court decisions are incredibly rare. Myriad Genetics is headquartered within walking distance from my house, so this was big news in my area.

Anyway, the point of all this is, freedom is important, especially in academic works. Part of the reason I wrote this book is that there are very few free textbooks.

To put it another way, it is of no benefit for the work to be nonfree. If the work is free, I, as a writer, benefit from people giving me feedback, and improving upon my work. You, as a reader, benefit from the freedom. The only people who don't benefit are distributors (e.g. a publisher). However, in the age of the internet, the need for a for-profit publisher isn't exactly clear.

To be clear, it is perfectly okay for a publisher to publish this book, and to attempt to profit off of it. However, the publisher wouldn't have the traditional nonfree monopoly over the book, which might discourage a publisher.

Bibliography

- [1] Category (mathematics). URL: https://en.wikipedia.org/wiki/Category_%28mathematics%29 (visited on 01/27/2015).
- [2] Currying. URL: <https://en.wikipedia.org/wiki/Currying> (visited on 02/23/2015).
- [3] Eta conversion. URL: https://wiki.haskell.org/Eta_conversion (visited on 02/23/2015).
- [4] Greek government-debt crisis. URL: https://en.wikipedia.org/wiki/Greek_financial_crisis (visited on 02/23/2015).
- [5] Miran Lipovača. Learn You a Haskell for Great Good! San Francisco, CA: No Starch Press, 2011.
- [6] Operator associativity. URL: <https://en.wikipedia.org/wiki/Operatorassociativity> (visited on 02/23/2015).
- [7] steve jobs on programming. URL: <https://www.youtube.com/watch?v=5Z1gfgM7kzo> (visited on 01/01/2015).