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18.01 Single Variable Calculus
Fall 2006

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18.01 Problem Set 1

Due Friday 9/15/06, 1:55 pm

18.01/18.01A Supplementary Notes, Exercises and Solutions are for sale. This is where to find the exercises labeled 1A, 1B, etc. You will need these for the first day's homework.

Part I consists of exercises given in the Notes and solved in section S of the Notes. It will be graded quickly, checking that all is there and the solutions not copied.
Part II consists of problems for which solutions are not given; it is worth more points. Some of these problems are longer multi-part exercises posed here because they do not fit conveniently into an exam or short-answer format. See the guidelines below for what collaboration is acceptable, and follow them.

To encourage you to keep up with the lectures, both Part I and Part II tell you for each problem on which day you will have the needed background for it.

Part I (30 points)

Notation: 2.1 = Section 2.1 of the Simmons book; Notes G = section G of the Notes;
1A-3 = Exercise 1A-3 in Section E (Exercises) of the Notes (solved in section S)
2.4/13; 81/4 = in Simmons, respectively, section 2.4 Problem 13; page 81 Problem 4

Recitation 0. Wed. Sept. 6 Graphing functions.
Read: Notes G, sections 1-4 Work: 1A-1a, 2a, 3abe, 6a, 7a

Lecture 1. Thurs., Sept. 7 Derivative; slope, velocity, rate of change.
Read: 2.1-2.4 Work: 1C-3abe, 4ab (use 3), 5, 6 (trace axes onto your answer sheet)
Work: 1B-2, 1C-1a (start from the definition of derivative)

Lecture 2. Fri. Sept. 8 Limits and continuity; some trigonometric limits
Read: 2.5 (bottom p.70-73; concentrate on examples, skip the $\epsilon - \delta$ def'n)
Read: 2.6 to p. 75; learn def'n (1) and proof "differentiable $\Longrightarrow$ continuous" at the end.
Read: Notes C Work: 1D-1bcefg, 4a; 1C-2, 1D-3ade, 6a, 8a (hint: "diff $\Longrightarrow$ cont.")

Lecture 3. Tues. Sept. 12 Differentiation formulas: products and quotients;
Derivatives of trigonometric functions.
In the exercises, an *antiderivative* of $f(x)$ is any $F(x)$ for which $F'(x) = f(x)$.
Read: 3.1, 3.2, 3.4 Work: 1E-1ac, 2b, 3, 4a, 5a; 1J-1e, 2

Lecture 4. Thurs. Sept. 14 Chain rule; higher derivatives.
Read: 3.3, 3.6 Work: 1F-1ab, 2, 6, 7bd; 1J- 1abm 1G-1b, 5ab

Lecture 5. Fri. Sept. 15 Implicit differentiation; inverse functions.
Read: 3.5, Notes G section 5 Work: given on Problem Set 2.

Part II (40 points)

Directions and Rules: Collaboration on problem sets is encouraged, **but**

a) Attempt each part of each problem yourself. Read each portion of the problem before asking for help. If you don't understand what is being asked, ask for help interpreting the problem and then make an honest attempt to solve it.

b) Write up each problem independently. On both Part I and II exercises you are expected to write the answer in your own words.

c) **Write on your problem set whom you consulted and the sources you used.** If you fail to do so, you may be charged with plagiarism and subject to serious penalties.

d) It is illegal to consult materials from previous semesters.

0. (not until due date; 3 points)

Write the names of all the people you consulted or with whom you collaborated and the resources you used, or say "none" or "no consultation". This includes visits outside recitation to your recitation instructor. If you don't know a name, you must nevertheless identify the person, as in, "tutor in Room 2-106," or "the student next to me in recitation." Optional: note which of these people or resources, if any, were particularly helpful to you.

This "Problem 0" will be assigned with every problem set. Its purpose is to make sure that you acknowledge (to yourself as well as others) what kind of help you require and to encourage you to pay attention to how you learn best (with a tutor, in a group, alone). It will help us by letting us know what resources you use.

1. (Wed, 3 pts) Express $(x-1)/(x+1)$ as the sum of an even and an odd function. (Simplify as much as possible.)

2. (Thurs, 6 pts: 3 + 3) Sensitivity of measurement: Suppose f is a function of x. If $x = x_0 + \Delta x$, then we define $\Delta f = f(x) - f(x_0)$ and $\Delta f/\Delta x$ measures how much changes in x affect the value of f.

The planet Quirk is flat. GPS satellites hover over Quirk at an altitude of 20,000 km (unlike Earth where the satellites circle twice a day). See how accurately you can estimate the distance L from the point directly below the satellite to a point on the planet surface knowing the distance h from the satellite to the point on the surface in two cases. (The letter h is for hypotenuse.)

a) Use a calculator to compute $\Delta L/\Delta h$ for $h = h_0 \pm \Delta h = 25,000 \pm \Delta h$, and $\Delta h = 1,\ 10^{-1},\ 10^{-2}$. Write an estimate for L in the form

$$|L - L_0| = |\Delta L| \leq C|\Delta h|$$

choosing the simplest round number C that works for all three cases.

b) Do the same for $h = 20,001 \pm \Delta h$, $\Delta h = 1,\ 10^{-1},\ 10^{-2}$. Is the value of L estimated more or less accurately than in part (a)? We will revisit this problem more systematically using calculus.

3. (Thurs, 4pts) On the planet Quirk, a cell phone tower is a 100-foot pole on top of a green mound 1000 feet tall whose outline is described by the parabolic equation $y = 1000 - x^2$. An ant climbs up the mound starting from ground level ($y = 0$). At what height y does the ant begin to see the tower?

4. (Thurs, 6 pts) 3.1/21 (parabolic mirrors)

5. (Thurs, 4pts: 2 + 2)

a) A water cooler is leaking so that its volume at time t in minutes is $(10 - t)^2/5$ liters. Find the average rate at which water drains during the first 5 minutes.

b) At what rate is the water flowing out 5 minutes after the tank begins to drain.

6. Friday (8 pts: 1 + 1 + 1 + 1 + 1 + 1 + 2) 2.5/19d (put $u = 1/x$), 19f, 19g, 20c, 20g (show work); 22a (needs a calculator), 22b (see the proof on page 73).

7. Tuesday (6 pts: 2 + 4)

a) If u, v and w are differentiable functions, find the formula for the derivative of their product, $D(uvw)$.

b) Generalize your work in part (a) by guessing the formula for $D(u_1 u_2 \cdots u_n)$—the derivative of the product of n differentiable functions.

Then prove your formula by mathematical induction (i.e., prove its truth for the product of $n+1$ functions, assuming its truth for the product of n functions).

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18.01 Problem Set 2A

Due Friday 9/29/06, 1:55 pm

2A is the first half of Problem Set 2, all of which is due a week after Exam 1 (the second half, 2B, will be issued at the exam, or the day before). Even though it won't be collected until later, you should do 2A before the exam, to prepare for it.

Part I (15 points)

Lecture 5. Fri. Sept. 15 Implicit differentiation; inverse functions and their derivatives.
Read: 3.5, Notes G section 5, 9.5 (bottom p.913 - 915)
Work: 1F-3,5,8c; 1A-5b; 5A-1a,b,c(just sin, cos, sec); 5A-3f,h

Lecture 6. Tues. Sept, 19 Exponentials and logs: def'n, algebra, applications, derivatives.
Read: Notes X (8.2 has some of this), 8.3 to middle p. 267; 8.4 to top p. 271
Work: 1H-1, 2, 3a, 5b; 1I-1c,d,e,f,m; 1I-4a

Lecture 7. Thurs. Sept. 21 Logarithmic differentiation. Hyperbolic functions (not on exam). Review.
Read: 9.7 to p. 326 Work: 5A-5abc

Lecture 8. Fri. Sept. 22 **Exam 1** covering 0-7.

Students not passing will get e-mail on Friday evening. Make-up exams are offered Monday-Thursday of the week following at times posted at the web site. (see "Exams" on Syllabus sheet).

Part II (30 points)

Directions: Attempt to solve *each part* of each problem yourself. If you collaborate, solutions must be written up independently. It is illegal to consult materials from previous semesters. With each problem is the day it can be done.

0. (not until due date; 2 points) Write the names of all the people you consulted or with whom you collaborated and the resources you used, or say "none" or "no consultation". (See full explanation on PS1).

1. (now; 4 pts) Graph the even and odd functions you found in Problem 1, Part II of PS1. Directly below, graph their derivatives. Do this qualitatively using your estimation of the slope. Do not use the formulas for the derivatives (except to check your work if you want). You can use a graphing calculator to check your answer, provided that you mention it in Problem 0. (Note, however, that you may not use books, notes or calculators during tests, so it is unwise to rely on a graphing calculator here.)

2. (before Fri; 5 pts = 2 + 3) Compute

a) $(d/dx)\tan^3(x^4)$

b) $(d/dy)(\sin^2 y \cos^2 y)$

(Do this two ways: first use the product rule, then write it as $f(2y)$. Show that the answers agree.)

3. (before Fri; 3pts = 1 + 2)

a) If $y = uv$, show that $y'' = u''v + 2u'v' + uv''$

b) Find y'''.

4. (Fri; 4pts 3 +1)

a) The function $\cos^{-1} x$ is the inverse of the $\cos\theta$ on $0 \le \theta \le \pi$. Use implicit differentiation to derive the formula for $(d/dx)\cos^{-1} x$. Pay particular attention to the sign of the square root. (See the book or lecture for the case of the inverse of sine.)

b) Without calculation, explain why $(d/dx)\cos^{-1} x + (d/dx)\sin^{-1} x = 0$

5. (Tues + Thurs; 10pts = 2+2+2+2+2) Do 8.2/8ac, 10, 11; 8.4/18,19a.

6. (Thurs; 2pts) Derive the formula for $D(u_1 u_2 \cdots u_n)$ from PS1, Part II, 7b, using logarithmic differentiation.

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18.01 Problem Set 2B

Due Friday 9/29/06, 1:55 pm

2B is the second half of Problem Set 2, all of which is due along with the first half 2A.

Part I (10 points)

Lecture 9. Tues. Sept. 26. Linear and quadratic approximations.
Read: Notes A Work: 2A-2, 3, 7, 11, 12ade

Lecture 10. Thurs. Sept. 28. Curve-sketching.
Read: 4.1, 4.2 Work: 2B-1,2: a,e,h; 2B-4, 6ab, 7ab

Lecture 11. Fri. Sept. 29. Maximum-minimum problems.
Read: 4.3, 4.4 Work: assigned on PS3

Part II (16 points + 3 extra)

Directions: Attempt to solve *each part* of each problem yourself. If you collaborate, solutions must be written up independently. It is illegal to consult materials from previous semesters. With each problem is the day it can be done.

0. (not until due date; 2 points) Write the names of all the people you consulted or with whom you collaborated and the resources you used, or say "none" or "no consultation". (See full explanation on PS1).

1. (10 points: 2 + 4 + 4; and 3 extra) **Golf balls**

The area of a section of a sphere of radius R between two parallel planes that are a distance h apart is [1]

$$\text{area of a spherical section } = 2\pi hR$$

Slice the sphere of radius R by a horizontal plane. The portion of the plane inside the sphere is a disk of radius $r \le R$. The portion of the spherical surface above the plane is called a *spherical cap*. For example, if the plane passes through the center, then the disk has radius $r = R$, its circumference is the equator, and the spherical cap is the Northern Hemisphere. More generally, a spherical cap is the portion of surface of the Earth north of a latitude line. The formula above applies to regions between two latitude lines, and, in particular, to spherical caps.

a) Consider a spherical cap which is the portion of the surface of the sphere above horizontal plane that slices the sphere at or above its center. Find the area of the cap as a function of R and r. Do this by finding first the formula for the height h of the spherical cap in terms of r and R. (This height is the vertical distance from the horizontal slicing plane to the North Pole.) Then use your formula for h and the formula above for the area of spherical sections.

[1] This formula will be derived in Unit 4. Two examples may convince you that it is reasonable. For $h = R$, it gives the area of the hemisphere, $2\pi R^2$. For $h = 2R$ it gives $4\pi R^2$, the area of the whole sphere.

b) Express the formula for the area of a spherical cap in terms of R^2 and r/R. (This is natural because the proportional scaling cr and cR changes the area by the factor c^2.) Then use the linear and quadratic approximations to $(1+x)^{1/2}$ near $x = 0$ to find a good and an even better approximation to the area of the spherical cap, appropriate when the ratio r/R is small. (Hint: What is x?) Simplify your answers as far as possible: the approximation corresponding to the linear approximation to $(1+x)^{1/2}$ should be very familiar.

c) The following problem appeared on a middle school math contest exam. The numbers have been changed to protect the innocent. Consider a golf ball that is 3 centimeters in diameter with 100 hemispherical dimples of diameter 3 millimeters. (Note that this is not a realistic golf ball because the dimples are too deep). Find the area of the golf ball rounded to the nearest 1/100 of a square centimeter using the approximation $\pi \approx 3.14$. (The students were given three minutes. We are spending more time on it.)

Under the rules of the contest, an incorrectly rounded answer was counted as wrong with no partial credit, so correct numerical approximation was crucial. Some students objected that they could not figure out the area of portion of the large sphere that is removed when a dimple is inserted. A careless examiner had assumed that the students would use the approximation that the area removed for each dimple was nearly the same as the area of a flat disk. We are going to figure out whether this approximation is adequate or gives the wrong answer according to the rules.

Write down formulas for the surface area of the golf ball in the three cases listed below. (Put in 100 dimples, but leave r, R, and π as letters.)

i) the approximation pretending that the removed surface is flat (what is the relationship between this and the approximations of part (b)?)

ii) the higher order approximation you derived in part (b)

iii) the exact formula

Finally, evaluate each of the answers for the given values $r = .15$ and $R = 1.5$ centimeters, and find the accuracy of the approximations.

d) (extra credit:[2] 3pts) Although nobody noticed at the time, the examiner who created this problem made a much bigger mistake. With the diameters actually given, it would have been impossible for the number of dimples given to be placed on the golf ball without overlap. Give a (reasonable) estimate for the largest number of dimples that can fit on our golf ball.

2. (4 points) Draw the graph of $f(x) = 1/(1 + x^2)$ and, directly underneath, it the graphs of $f'(x)$ and $f''(x)$. Label critical points and inflection points on the graph of f with their coordinates. Draw vertical lines joining these special points of the graph of f to the corresponding points on the graphs below.

[2] Extra credit points are tabulated separately and only added to your final score at the end of the semester so that they do not influence any grading cutoffs.

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18.01 Problem Set 3

Due Friday 10/06/06, 1:55 pm

Part I (10 points)

Lecture 11. Fri. Sept. 29. Maximum-minimum problems.
Read: 4.3, 4.4 Work: 2C-1, 2, 5, 11, 13.

Lecture 12. Tue. Oct. 3 Related rate problems.
Read: 4.5 Work: 2E-2, 3, 5, 7

Lecture 13. Thu. Oct. 5 Newton's method.
Read: 4.6, (4.7 is optional)

Lecture 14. Fri. Oct. 6 Mean-value theorem. Inequalities.
Read: 2.6 to middle p. 77, Notes MVT Work: assigned on PS4

Part II (31 points + 8 extra credit)

Directions: Attempt to solve *each part* of each problem yourself. If you collaborate, solutions must be written up independently. It is illegal to consult materials from previous semesters. With each problem is the day it can be done.

0. (not until due date; 3 pts) Write the names of all the people you consulted or with whom you collaborated and the resources you used, or say "none" or "no consultation". (See full explanation on PS1).

1. (Friday, 6pts: 3 + 3)

a) 4.3/28 (Use as variable the distance x from the foot of the ladder to the house. Check endpoints.)

b) 4.4/28

2. (Tuesday, 2pts) **Hypocycloid.** Show that every tangent line to the curve $x^{2/3} + y^{2/3} = 1$ in first quadrant has the property that portion of the line in the first quadrant has length 1. (Use implicit differentiation; this is the same as problem 45 page 114 of text.)

3. (Tuesday, 7pts: 3 + 3 + 1) **Sensitivity of measurement, revisited.**

a) Recall that in problem 2, PS1/Part II, $L^2 + 20,000^2 = h^2$. Use implicit differentiation to calculate dL/dh. Compare the linear approximation dL/dh to the error $\Delta L/\Delta h$ computed in examples on PS1. Explain why $\Delta L/\Delta h \leq dL/dh$ if the derivative is evaluated at the left endpoint of the interval of uncertainty (or, in other words, $\Delta h > 0$). In what range of values of h is it true that $|\Delta L| \leq 2|\Delta h|$?

b) Suppose that the Planet Quirk is a not only flat, but one-dimensional (a straight line). There are several satellites at height $20,000$ kilometers and you get readings saying that satellite 1 is directly above the point $x_1 \pm 10^{-10}$ and is at a distance $h_1 = 21,000 \pm 10^{-2}$ from you, satellite

2 is directly above $x_2 \pm 10^{-10}$ and at a distance $h_2 = 52,000 \pm 10^{-2}$. Where are you and to what accuracy? Hint: Consider separately the cases $x_1 < x_2$ and $x_2 > x_1$.

c) Express dL/dh in terms of the angle between the line of sight to the satellite and the horizontal from the person on the ground. (When expressed using the line-of-sight angle, the formula also works for a curved planet like Earth.)

4. (Tuesday, 5pts: 3 + 2 + 0) **More sensitivity of measurement.** Consider a parabolic mirror with equation $y = -1/4 + x^2$ and focus at the origin. (See Problem Set 1.) A ray of light traveling down vertically along the line $x = a$ hits the mirror at the point (a, b) where $b = -1/4 + a^2$ and goes to the origin along a ray at angle θ measured from the positive x-axis.

a) Find the formula for $\tan\theta$ in terms of a and b, and calculate $d\theta/da$ using implicit differentiation. (Express your answer in terms of a and θ.)

b) If the telescope records a star at $\theta = -\pi/6$ and the measurement is accurate to 10^{-3} radians, use part (a) to give an estimate as to the location of the star in the variable a.

c) (optional; no credit) Solve for a as a function of θ alone and doublecheck your answers to parts (a) and (b).

5. (Thursday, 8 pts: 3 + 3 + 2) **Newton's method.**

a) Compute the cube root of 9 to 6 significant figures using Newton's method. Give the general formula, and list numerical values, starting with $x_0 = 2$. At what iteration k does the method surpass the accuracy of your calculator or computer? (Display your answers to the accuracy of your calculator or computer.)

b) For each step x_k, $k = 0, 1, \ldots$, say whether the value is i) larger or smaller than $9^{1/3}$; ii) larger or smaller than the preceding value x_{k-1}. Illustrate on the graph of $x^3 - 9$ why this is so.

c) Find a quadratic approximation to $9^{1/3}$, and estimate the difference between the quadratic approximation and the exact answer. (Hint: To get a reasonable quadratic approximation, use $9 = 8(1 + 1/8)$.)

6. (extra credit 8 pts: 3 + 2 + 2 + 1) **Hypocycloid, again.** Here we derive the equation for the hypocycloid of Problem 2 from the sweeping out property directly. This takes quite a bit longer. We will look at the hypocycloid from yet another (easier) point of view later on.

Think of the first quadrant of the xy-plane as representing the region to the right of a wall with the ground as the positive x-axis and the wall as the positive y-axis. A unit length ladder is placed vertically against the wall. The bottom of the ladder is at $x = 0$ and slides to the right along the x-axis until the ladder is horizontal. At the same time, the top of the ladder is dragged down the y-axis ending at the origin $(0, 0)$. We are going to describe the region swept out by this motion, in other words, the blurry region formed in a photograph of the motion if the eye of the camera is open the whole time.

a) Suppose that L_1 is the line segment from $(0, y_1)$ to $(x_1, 0)$ and L_2 is the line segment from $(0, y_2)$ to $(x_2, 0)$. Find the formula for the point of intersection (x_3, y_3) of the two line segments. Don't expect the formula to be simple: It must involve all four parameters x_1, x_2, y_1, and y_2. But simplify as much as possible!

It's important to make sure you have the right formulas before proceeding further. You can doublecheck your formulas in several ways. (This is optional.)

i) If $y_2 = 0$, then $x_3 = x_1$.

ii) When the x's and y's are interchanged the formulas should be the same. What transformation of the plane does the exchange of x and y represent?

iii) It is impossible to find x_3 and y_3 if the lines are parallel, so the denominator in the formula must be zero when L_1 and L_2 have the same slope.

iv) Rescaling all variables by a factor c leaves the formula unchanged, so the numerator of the formula for x_3 and y_3 should have degree (in all variables) one greater than the denominator.

b) Write the equation involving x_2 and y_2 that expresses the property that ladder L_2 has length one. We will suppose that L_1 represents the ladder at a fixed position, and L_2 tends to L_1. Thus

$$x_2 = x_1 + \Delta x; \quad y_2 = y_1 + \Delta y$$

Use implicit differentiation (related rates) to find

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

(Express the limit as a function of the fixed values x_1 and y_1.)

c) Substitute $x_2 = x_1 + \Delta x$ and $y_2 = y_1 + \Delta y$ into the formula in part (a) for x_3 and use part (b) to compute

$$X = \lim_{x_2 \to x_1} x_3 = \lim_{\Delta x \to 0} x_3$$

Simplify as much as possible. Deduce, by symmetry alone, the formula for

$$Y = \lim_{x_2 \to x_1} y_3$$

d) Show that $X^{2/3} + Y^{2/3} = 1$. (The limit point (X, Y) that you found in part (c) is expressed as a function of x_1 and y_1. This is the unique point of the ladder L_1 that is also part of the boundary curve of the region swept out by the family of ladders.)

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18.01 Problem Set 4

Due Friday 10/20/06, 1:55 pm

This is all of Problem Set 4 (not split into 4A and 4B). Although it is not due until after Exam 2, you should do all the Part I exercises through Lecture 16 and all the Part II problems through Problem 4 before the exam, in order to prepare for it. Practice exam problems and an actual past exam will be posted on line as usual.

Part I (20 points)

Lecture 14. Fri. Oct. 6 Mean-value theorem. Inequalities.
Read: 2.6 to middle p. 77, Notes MVT Work: 2G-1b, 2b, 5, 6

(Columbus Day Holiday. No classes Mon and Tues, Oct 9 and 10)

Lecture 15. Thurs. Oct. 12 Differentials and antiderivatives.
Read: 5.2, 5.3 Work: 3A-1de, 2acegik, -3aceg

Lecture 16. Fri. Oct. 13 Differential equations; separating variables.
Read: 5.4, 8.5 Work: 3F-1cd, 2ae, 4bcd, 8b

Lecture 17. Tues. Oct 17 **Exam 2** Covers Lectures 8–16.

Lecture 18. Thurs. Oct. 19 Definite integral; summation notation.
Read: 6.3 though formula (4); skip proofs; 6.4, 6.5
Work: 3B-2ab, 3b, 4a, 5 4J-1 (set up integral; do not evaluate)

Lecture 19. Fri. Oct. 20 First fundamental theorem. Properties of integrals.
Read: 6.6, 6.7 to top p. 215 (Skip the proof pp. 207-8, which will be discussed in Lec 20.)
Work: assigned on PS 5

Part II (36 points + 10 extra credit)

Directions: Attempt to solve *each part* of each problem yourself. If you collaborate, solutions must be written up independently. It is illegal to consult materials from previous semesters. With each problem is the day it can be done.

0. (not until due date; 3 pts) Write the names of all the people you consulted or with whom you collaborated and the resources you used, or say "none" or "no consultation". (See full explanation on PS1).

1. (Lec 14, 10pts: 2 + 2 + 2 + 2 + 2))

a) Use the mean value property to show that if $f(0) = 0$ and $f'(x) \geq 0$, then $f(x) \geq 0$ for all $x \geq 0$.

b) Deduce from part (a) that $\ln(1 + x) \leq x$ for $x \geq 0$. Hint: Use $f(x) = x - \ln(1 + x)$.

c) Use the same method as in (b) to show $\ln(1+x) \geq x - x^2/2$ and $\ln(1+x) \leq x - x^2/2 + x^3/3$ for $x \geq 0$.

d) Find the pattern in (b) and (c) and make a general conjecture.

e) Show that $\ln(1+x) \le x$ for $-1 < x \le 0$. (Use the change of variable $u = -x$.)

2. (Lec 15, 4 pts: 2 + 2)

a) Do 5.3/68

b) Show that both of the following integrals are correct, and explain.

$$\int \tan x \sec^2 x dx = (1/2)\tan^2 x; \qquad \int \tan x \sec^2 x dx = (1/2)\sec^2 x$$

3. (Lec 16, 6 pts: 3 + 3)

a) Do 8.6/5 (answer in back of text)

b) Do 8.6/6 (optional?)

4. (Lec 16, 7 pts: 2 + 3 + 2) Do 3F-5abc

STOP HERE. DO THE REST AFTER EXAM 2.

5. (Lec 18, 6 pts) Calculate $\int_0^1 e^x dx$ using lower Riemann sums. (You will need to sum a geometric series to get a usable formula for the Riemann sum. To take the limit of Riemann sums, you will need to evaluate $\lim_{n\to\infty} n(e^{1/n} - 1)$, which can be done using the standard linear approximation to the exponential function.)

6. (Lec 16; extra credit: 10 pts: 2 + 2 + 3 + 3) **More about the hypocycloid.** We use differential equations to find the curve with the property that the portion of its tangent line in the first quadrant has fixed length.

a) Suppose that a line through the point (x_0, y_0) has slope m_0 and that the point is in the first quadrant. Let L denote the length of the portion of the line in the first quadrant. Calculate L^2 in terms of x_0, y_0 and m_0. (Do not expand or simplify.)

b) Suppose that $y = f(x)$ is a graph on $0 \le x \le L$ satisfying $f(0) = L$ and $f(L) = 0$ and such that the portion of each tangent line to the graph in the first quadrant has the same length L. Find the differential equation that f satisfies. Express it in terms of L, x, y and $y' = dy/dx$. (Hints: This requires only thought, not computation. Note that $y = f(x)$, $y' = f'(x)$. Don't take square roots, the expression using L^2 is much easier to use. Don't expand or simplify; that would make things harder in the next step.)

c) Differentiate the equation in part (b) with respect to x. Simplify and write in the form

$$(\text{something})(xy' - y)y'' = 0$$

(This starts out looking horrendous, but simplifies considerably.)

d) Show that one solution to the equation in part (c) is $x^{2/3} + y^{2/3} = L^{2/3}$. What about two other possibilities, namely, those solving $y'' = 0$ and $xy' - y = 0$?

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18.01 Problem Set 5

Due Friday 10/27/06, 1:55 pm

Part I (20 points)

Lecture 19. Fri. Oct. 20 First fundamental theorem. Properties of integrals.
Read: 6.6, 6.7 (The second fundamental theorem, discussed in Lecture 20, is stated in the text as (13) at the bottom of page 215 and also as Step 1, page 207, of the proof of the first fundamental theorem.)
Work: 3C-1, 2a, 3a, 5a; 3E-6bc; 4J-2

Lecture 20. Tues. Oct. 24 Second Fundamental Theorem. Def'n of $\ln x$.
Read: Notes PI, p.2 [eqn.(7) and example]; Notes FT.
Work: 3E-1, 3a; 3D-1, 4bc, 5, 8a; 3E-2ac

Lecture 21. Thur. Oct. 26 Areas between curves. Volumes by slicing.
Read: 7.1, 7.2, 7.3 Work[1]: 4A-1b, 2, 4; 4B-1de, 6, 7

Lecture 22. Fri. Oct. 27 Volumes by disks and shells.
Read: 7.4 Work on PS 6

Part II (30 points + 5 extra credit)

Directions: Attempt to solve *each part* of each problem yourself. If you collaborate, solutions must be written up independently. It is illegal to consult materials from previous semesters. With each problem is the day it can be done.

0. (not until due date; 3 pts) Write the names of all the people you consulted or with whom you collaborated and the resources you used, or say "none" or "no consultation". (See full explanation on PS1).

1. (Lec 19, 6 pts: 3 + 3) (+ 5 extra for part (c))

a) Suppose that at the beginning of day 0, some time last summer, the temperature in Boston was $y(0) = 65^\circ$ Fahrenheit and that over a 50-day period, the temperature increased according to the rule $y'(t) = y(t)/100$, with time t measured in days. Find the formula for y, and draw a graph of temperature on days 3 and 4, $3 \le t \le 5$, and label with the correct day and shade in the regions whose areas represent the average temperature each of the two days.[2]

[1] A more colorful way of expressing 4B-6 is in terms of the volume of a tent, as in the textbook problem 7.3/7. Unfortunately, the problem is ill-posed and can't be done without pretending that the cross-sections are triangles. In real life, the canvas would have creases. In general, the shapes formed by stretching canvas or nylon over various arrays of tent poles are quite hard to compute.

[2] The continuous average of a function is

$$\frac{1}{b-a}\int_a^b f(x)dx$$

In this case $b - a = 1$, so the average is the same as integral. For more, see Notes, AV, and Lecture 23.

b) The number of *cooling degree days* is the sum for each day of the difference between the average temperature for that day and 65°. The number is used to estimate the demand for electricity for air conditioning. Draw a second graph of y for the whole 50 days and shade in the region whose area represents the total number of degree days. Write a formula for this total area as the difference between $65 \cdot 50$ and a definite integral. Evaluate the definite integral using the fundamental theorem of calculus. (Alternatively, write the whole quantity as an integral expressing the area between curves as in Lecture 21, and 7.2.)

c) (extra credit) Compute the definite integral in part (b) directly by evaluating a lower Riemann sum and taking a limit. Follow the procedure in Problem 5, PS4, but with different scale factors. This rather elaborate calculation shows how much time and effort we save every time we use the fundamental theorem and the change of variable formula in integrals.

2. (Lec 20, 16 pts: 2 + 2 + 4 + 2 + 6) Consider the function $f(x) = \int_0^x \cos(t^2)dt$. There is no expression for $f(x)$ in terms of standard elementary functions. It is known as a Fresnel integral, along with the corresponding sine integral.

a) Draw a rough sketch of $\cos(t^2)$, showing the first positive and negative zeros. What does the curve look like at $t = 0$? Is the function even or odd?

b) List the critical points of $f(x)$ in the entire range $-\infty < x < \infty$. Which critical points are local maxima and which ones are local minima?

c) Sketch the graph of f on the interval $-2 \le x \le 2$, with labels for the critical points **and inflection points**. (The drawing should be qualitatively correct, but just estimate the values of f at the labelled points.)

d) Estimate $f(0.1)$ to six decimal places.

e) Fresnel integrals are sometimes expressed using different scaling of the variables.

i) Let $g(x) = \int_0^x \cos((\pi/2)u^2)du$. Make a change of variables to show that $f(x) = c_1 g(c_2 x)$ for some constants c_1 and c_2. Why did we choose the factor $\pi/2$?

ii) Let $h(x) = \int_0^x \frac{\cos v}{\sqrt{v}} dv$. (This integral is called *improper* because $1/\sqrt{v}$ is infinite[3] at $v = 0$.) Make a different change of variable to show that $f(x) = ch(x^2)$ for some constant c (assume that $x > 0$).

iii) Let $k(x) = \sqrt{x} \int_0^1 \cos(xt^2)dt$, $x > 0$. Use the change of variable $z = xt^2$ and part (ii) to find the relationship between the functions k and f. Hint: Which quantities are variable and which are constant? (Not assigned: you can also work out this relationship using a change of variables like the one in part (i).)

3. (Lec 21, 5 pts: 2 + 3)

a) Do 7.3/22.

b) Find the volume of the region in 3-space with $x > 0$, $y > 0$ and $z > 0$ given by

$$z^2/2 < x + y < z$$

Hint: First find the area of the horizontal cross-sections.

[3]Although the integrand is infinite, the area under the curve is finite. The function h is continuous, $h(0) = 0$, but its graph has infinite slope at $x = 0$.

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18.01 Problem Set 6

Due THURSDAY 11/09/06, 12:55 pm

Warning: This problem set is due on a **THURSDAY** not Friday, because of the Veterans' Day holiday. It is due **before lecture**, which is at 1:05 on Thursday.

Even though this problem set is due two days after Exam 3, you will need to do most of it by Tuesday, in the process of preparing for Exam 3 — all except the Part I problems connected to Lecture 25.

Part I (22 points)

Lecture 22. Fri. Oct. 27 Volumes by disks and shells.
Read: 7.4 Work: 4B-2eg, 5; 4C-1a, 2, 3 4J-3

Lecture 23. Tues. Oct. 31 Work; average value; probability.
Read: 7.7, to middle p. 247 Notes AV.
Work: 249/5, 6, 15 (solutions posted at web site); 4D-2, 3, 5

Lecture 24. Thurs. Oct. 29 Numerical Integration.
Read 10.9 Work: 3G-1ad, 4

Lecture 25. Fri. Nov. 3 Trigonometric integrals. Direct substitution.
Read 10.2, 10.3 Work: 5B-9, 11, 13, 16; 5C-5, 7, 9, 11 (due after Exam 3)

Lecture 26. Tues. Nov. 7 **Exam 3 1:05-1:55** covering lectures 18–24.

Part II (30 points)

Directions: Attempt to solve *each part* of each problem yourself. If you collaborate, solutions must be written up independently. It is illegal to consult materials from previous semesters. With each problem is the day it can be done.

0. (not until due date; 3 pts) Write the names of all the people you consulted or with whom you collaborated and the resources you used, or say "none" or "no consultation". (See full explanation on PS1).

1. (Lec 22, 7pts: 3 + 4) Do 7.4/12 and 13.

2. (Lec 23, 4pts) The voltage V of house current is given by

$$V(t) = C\sin(120\pi t)$$

where t is time, in seconds and C is a constant amplitude. The square root of the average value of V^2 over one period of $V(t)$ (or cycle) is called the *root-mean-square* voltage, abbreviated RMS. This is what the voltage meter on a house records. For house current, find the RMS in terms of the constant C. (The peak voltage delivered to the house is $\pm C$. The units of V^2 are square volts; when we take the square root again after averaging, the units become volts again.)

3. (Lec 23, 6 pts: 1 + 2 + 1 + 2)

a) What is the probability that $x^2 < y$ if (x, y) is chosen from the unit square $0 \le x \le 1$, $0 \le y \le 1$ with probability equal to the area.

b) What is the probability that $x^2 < y$ if (x, y) is chosen from the square $0 \le x \le 2$, $0 \le y \le 2$ with probability **proportional** to the area. (Probability = Part/Whole).

c) Evaluate

$$W = \int_0^\infty e^{-at} dt = \lim_{N\to\infty} \int_0^N e^{-at} dt$$

This is known as an improper integral because it represents the area of an unbounded region. We are using the letter W to signify "whole."

The probability that a radioactive particle will decay some time in the interval $0 \le t \le T$ is

$$P([0, T]) = \frac{\text{PART}}{\text{WHOLE}} = \frac{1}{W} \int_0^T e^{-at} dt$$

Note that $P([0, \infty)) = 1 = 100\%$.

d) The half-life is the time T for which $P([0, T]) = 1/2$. Find the value of a and W for which the half-life is $T = 1$. Suppose that a radioactive particle has a half-life of 1 second. What is the probability that it survives to time $t = 1$, but decays some time during the interval $1 \le t \le 2$? (Give an integral formula, and use a calculator to get an approximate numerical answer.)

4. (Lec 24, 6pts) The basis for Simpson's rule is the following formula. Let x_1 be the midpoint of the interval $[x_0, x_2]$, and denote its length by $2h$. Consider any three points (x_0, y_0), (x_1, y_1), (x_2, y_2). There is a unique quadratic function (parabola)

$$y = Ax^2 + Bx + C$$

whose graph passes through the three points. Simpson's rule says that the area under the parabola above $[x_0, x_2]$ is

$$\frac{h}{3}(y_0 + 4y_1 + y_2)$$

This problem is devoted to proving this formula. It is significant because it illustrates how calculations can be simplified by using symmetry, and by looking ahead to see what you need.

Since the area will be the same if the parabola is translated to the left or right, we may assume that $x_0 = -h$, $x_1 = 0$, and $x_2 = h$. Then in terms of the rest of the data (i.e., h and the y_i)

make a sketch and determine C;

show, by integrating, that to find the area we need only determine A (or better, $2Ah^2$);

determine $2Ah^2$ using the data;

put the results together to establish the formula for area.

5. (Lec 24, 4pts) Use a calculator to make a table of values of the integrand and find approximations to the Fresnel integral $\int_0^a \cos(t^2)dt$ for $a = \sqrt{\pi/2}$, using Simpson's rule with four and eight intervals. (The exact answer to five decimal places is 1.22505. Record your approximations to six decimal places to compare.)

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18.01 Problem Set 7

Due Friday 11/17/06, 1:55 pm

Part I (20 points)

Lecture 25. (We will begin here on Nov 9.) Trigonometric integrals. Direct substitution.
Read 10.2, 10.3 Work: 5B-9, 11, 13, 16; 5C-5, 7, 9, 11 (moved here from PS6)

Lecture 27. Thurs. Nov. 9 Inverse substitution. Completing the square.
Read 10.4 Work: 5D-1, 2, 7, 10

Lecture 28. Tues. Nov. 14 Integrating rational functions; partial fractions.
Read 10.6, Notes F Work: 5E-2, 3, 5, 6, 10h (complete the square)

Lecture 29. Thurs. Nov. 16 Integration by parts. Reduction formulas.
Read 10.7 Work: 5F-1a, 2d then 2b, 3

Lecture 30. Tues. Nov. 16 Parametric equations; arclength. Surface area
Read 17.1, 7.5 Work will be assigned on PS8

Part II (22 points)

Directions: Attempt to solve *each part* of each problem yourself. If you collaborate, solutions must be written up independently. It is illegal to consult materials from previous semesters. With each problem is the day it can be done.

0. (not until due date; 3 pts) Write the names of all the people you consulted or with whom you collaborated and the resources you used, or say “none” or “no consultation”. (See full explanation on PS1).

1. (Lec 27, 4pts) (from PS6) The voltage V of house current is given by

$$V(t) = C\sin(120\pi t)$$

where t is time, in seconds and C is a constant amplitude. The square root of the average value of V^2 over one period of $V(t)$ (or cycle) is called the *root-mean-square* voltage, abbreviated RMS. This is what the voltage meter on a house records. For house current, find the RMS in terms of the constant C. (The peak voltage delivered to the house is $\pm C$. The units of V^2 are square volts; when we take the square root again after averaging, the units become volts again.)

2. (Lec 27, 4pts) The solid torus is the figure obtained by rotating the disk $(x-b)^2 + y^2 \le a^2$ around the y-axis. Find its volume by the method of shells. (Hint: Substitute for $x - b$. As noted p. 229/11, the answer happens to be the area of the disk multiplied by the distance travelled by the center as it revolves.)

3. (Lec 27, 4pts: 2 + 2)

a) For any integer $n \ge 0$, use the substitution $\tan^2 x = \sec^2 x - 1$ to show that

$$\int \tan^{n+2} x\,dx = \frac{1}{n+1}\tan^{n+1} x - \int \tan^n x\,dx$$

b) Deduce a formula for $\int \tan^4 x\,dx$.

4. (Lec 28, 4pts: 3 + 1)

a) Derive a formula for $\int \sec x\,dx$ by writing $\sec x = \dfrac{\cos x}{1-\sin^2 x}$ (verify this), and then making a substitution for $\sin x$ and using partial fractions. (Your final answer must be expressed in terms of x.)

b) Convert the formula into the more familiar one by multiplying the fraction in the answer on both top and bottom by $1+\sin x$. (Note that $(1/2)\ln u = \ln\sqrt{u}$.)

5. (Lec 29, 3pts) Find the volume under the first hump of the function $y = \cos x$ rotated around the y-axis by the method of shells.

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18.01 Problem Set 8A

Due Friday 12/08/06, 1:55 pm

8A is the first half of Problem Set 8, all of which is due a week after Exam 4. (The second half, 8B, will be issued at the exam.) Even though it won't be collected until later, you should do 8A before the exam, to prepare for it.

Part I (15 points)

Lecture 30. Fri. Nov. 17 Parametric equations; arclength. Surface area.
Read 17.1, 7.5, 7.6 Work: 4E-2, 3, 8; 4F-1d, 4, 5, 8; 4G-2, 5.
If a curve is given by $x = x(t)$, $y = y(t)$, to find its arclength, use ds in the form

$$ds = \sqrt{(dx)^2 + (dy)^2} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt\ ,$$

and integrate from start to finish: from $t = t_0$ to $t = t_1$.

Lecture 31. Tues. Nov. 21 Polar coordinates; area in polar coordinates.
Read: 16.1, (16.2 lightly, for the pictures), 16.3 to top p.570, 16.5 to middle p.581
Work: 4H-1bfg; 4H-2a,3f; 4I-2,3

Lecture 32. Tues. Nov 28. Continuation and review.

Lecture 33. Thurs. Nov 30. **Exam 4, 1:05-1:55,** lectures 25–32.

Part II (30 points)

Directions: Attempt to solve *each part* of each problem yourself. If you collaborate, solutions must be written up independently. It is illegal to consult materials from previous semesters. With each problem is the day it can be done.

0. (not until due date; 3 pts) Write the names of all the people you consulted or with whom you collaborated and the resources you used, or say "none" or "no consultation". (See full explanation on PS1).

1. (Lec 30, 7 pts: 3 + 1 + 3)

a) Find the algebraic equation in x and y for the curve

$$x = a\cos^k t, \quad y = a\sin^k t.$$

Draw the portion of the curve $0 \le t \le \pi/2$ in the three cases $k = 1$, $k = 2$, $k = 3$.

b) Without calculation, find the arclength in the cases $k = 1$ and $k = 2$.

c) Find a definite integral formula for the length of the curve for general k. Then evaluate the integral in the three cases $k = 1$, $k = 2$, and $k = 3$. (Your answer in the first two cases should match what you found in part (b), but the calculation takes more time.)

2. (Lec 30, 9 pts: 3 + 1 + 3 + 2) The hyperbolic sine and cosine are defined by

$$\cosh x = \frac{e^x + e^{-x}}{2}, \quad \sinh x = \frac{e^x - e^{-x}}{2}$$

a) Show that

i) $\dfrac{d}{dx}\sinh x = \cosh x$ and $\dfrac{d}{dx}\cosh x = \sinh x$.

ii) $\cosh^2 x = 1 + \sinh^2 x$ and

iii) $\cosh^2 x = \dfrac{1+\cosh 2x}{2}$

b) What curve is described parametrically by $x = \cosh t$, $y = \sinh t$? (Give the equation and its name.)

c) The curve $y = \cosh x$ is known as a *catenary.* It is the curve formed by a chain whose two ends are held at the same height.

i) Sketch the curve

ii) Find its arclength from the lowest point to the point $(x_1, \cosh x_1)$ for a fixed $x_1 > 0$.

d) Find the area of the surface of revolution formed by revolving the portion of the curve from part (c) around the x-axis. This surface is known as a catenoid. It is interesting because it is the surface of least area connecting the two circles that form its edges. If you dip two circles of wire in a soap solution, then (with some coaxing) a soap film will form in this shape. In general, the soap films try to span a frame of wires with a surface with the least area possible.

3. (Lec 31, 11 pts: 3 + 2 + 2 + 4) Area in polar coordinates.

a) Find the area of the right triangle with vertices at $(x,y) = (0,0)$, $(x,y) = (a,0)$ and at $(x,y) = (a,h)$, using polar coordinates. (One of the less convenient ways to find the area of a triangle.)

b) Find the equation in (x,y) coordinates for the curve $r = 1/(1+\sin\theta)$ and sketch it.

c) Find the area of the region $0 \le r \le 1/(1+\sin\theta)$, $0 \le \theta \le \pi$, using the *rectangular* coordinate formula you found in part (b).

d) Find the area of the region in part (c) using polar coordinates. One way to evaluate the integral is to change variables to $u = \theta - \pi/2$, and then use the half angle formula

$$1 + \cos u = 2\cos^2(u/2)$$

This area was already computed in part (c), but the polar coordinate formula is still valuable because it gives the area of any sector, not just the one that is bounded by a horizontal line. The area swept out from the viewpoint of the focus of an ellipse, parabola, or hyperbola is related by Kepler's law to the speed of planets and comets.

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18.01 Problem Set 8B

Due Friday 12/08/06, 1:55 pm

Part I (20 points)

Lecture 34. Fri. Dec 1 Indeterminate forms; L'Hospital's rule, growth rate of functions.
Read: 12.2, 12.3 (Examples 1-3, remark 1) Work: 6A-1befgj, 5, 6c

Lecture 35. Tue. Dec. 5 Improper integrals.
Read: 12.4, Notes INT Work: 6B-1,2,7afkm, 8c

Lecture 36. Thurs. Dec. 7 Infinite series; simple convergence tests
Geometric series; harmonic series. Read: pp. 439-442(top)
Comparison tests. pp. 451-3 (skip proof in Example 3)
Integral test. pp. 455-457(top)
Work: 7A-1abc; 7B-1abf 7B-2acde

Lecture 37. Fri. Dec. 8 Taylor series.
Read: 14.4 through p. 498 (bottom); skip everything involving the remainder term $R_n(x)$.
Differentiation and integration of series. Read: 14.3-p.490(top); Examples 1-5.
Work: see handout with remarks about the final exam

Lecture 38. Tues. Dec. 12 Final Review.

Part II (17 points + 5 extra)

Directions: Attempt to solve *each part* of each problem yourself. If you collaborate, solutions must be written up independently. It is illegal to consult materials from previous semesters. With each problem is the day it can be done.

0. (not until due date; 3 pts) Write the names of all the people you consulted or with whom you collaborated and the resources you used, or say "none" or "no consultation". (See full explanation on PS1).

1. (Lecs 34-36, 6 + 5 pts: 1 + 2 + 2 + 1 + (5 extra))

a) Use L'Hospital's rule to evaluate

$$\lim_{x\to\infty} x^m e^{-x}$$

b) Use part (a) and limit comparison to show that the improper integral $\displaystyle\int_0^\infty x^n e^{-x} dx$ converges for $n \geq 0$. (Do not integrate by parts.)

c) Denote $A(n) = \displaystyle\int_0^\infty x^n e^{-x}\, dx$. Use integration by parts to find the constant c_n for which

$$A(n+1) = c_n A(n)$$

(Explain what happens at the infinite limit using part (a).)

d) Find $A(0)$ and deduce from part (c) the formula for $A(n)$, $n = 0, 1, 2, \ldots$.

e) (optional; 5pts extra credit) Show that the improper integral representing $A(-1/2)$ converges and evaluate it using a change of variables.

2. (Review problem using solids and surfaces of revolution; 9pts: 4 + 2 + 3)

a) Show that the volume V of the solid of revolution enclosed on the top by a spherical cap of radius r of height h and underneath by a horizontal plane is

$$V = \pi \left(rh^2 - \frac{h^3}{3} \right)$$

Draw a picture. Doublecheck this answer in the three cases: the empty set ($h = 0$); the half ball ($h = r$); the whole ball ($h = 2r$). We will always assume that $0 \le h \le 2r$. Why?

b) Find the surface area of the spherical cap of radius r and height h. Doublecheck your answer by evaluating it in the three cases $h = 0$, $h = r$ and $h = 2r$.

c) We will now find the shape of a soap bubble sitting on a table. Suppose that the bubble is a portion of a sphere as above, that it encloses a fixed volume V, and that its surface area is as small as possible (area of the curved surface, not counting the flat bottom = table). Find the minimizing shape, and describe it geometrically. (Hint: find h/r.)