

1 Function Symbol Elimination in EPR

It is well-known that validity of $\forall\exists$ -formulas over a relational vocabulary (only relation and constant symbols) with equality is decidable. We try to recognize some general cases in which the addition of unary function symbols preserves the decidability. Moreover, as we prefer avoid devising new decision procedures, we are interested in performing reductions to EPR, and apply an off-the-shelf EPR-solver.

Assume a first-order signature $\Sigma_{\mathcal{F}}$ consisting of relation symbols, constant symbols and a set \mathcal{F} of unary function symbols. Suppose we are given a $\forall\exists$ -formula $\varphi_{\mathcal{F}}$ and a (finite) set $\Gamma_{\mathcal{F}}$ of universal formulas in this signature. We are interested to decide whether $\Gamma_{\mathcal{F}} \vdash \varphi_{\mathcal{F}}$ (i.e. $\varphi_{\mathcal{F}}$ follows from $\Gamma_{\mathcal{F}}$ in FOL). Initially, unary function symbols can be replaced by binary relations (that correspond to the graphs of the functions). We first produce an equivalent unnested formula φ and theory Γ (in which all atomic formulas have the form $R(x_1, \dots, x_k)$, $f(x) \approx y$, or $c \approx y$). By choosing the quantifiers correctly we can ensure that φ and Γ remain a $\forall\exists$ -formula and a set of universal formulas respectively (e.g. in φ we only add universal quantifiers in negative occurrences and existential quantifiers in positive occurrences). Next, for every function symbol in \mathcal{F} , we replace in Γ and φ the atomic formulas of the form $f(x) \approx y$ by $R_f(x, y)$, where R_f is a new binary relation symbol. We denote the new signature (with binary relations instead of unary functions) by Σ . Each new relation R_f should have two properties:

- $PF^f = \forall x, y, z. R_f(x, y) \wedge R_f(x, z) \rightarrow y \approx z$
- $TOT^f = \forall x. \exists y. R_f(x, y)$

We now have that $\Gamma_{\mathcal{F}} \vdash \varphi_{\mathcal{F}}$ iff $PF, TOT, \Gamma \vdash \varphi$ where $PF = \bigwedge_f PF^f$ and $TOT = \bigwedge_f TOT^f$. PF and Γ are universal and cause no problem. On the other hand, TOT is $\forall\exists$, and thus it cannot work as an assumption (entailment of $\forall\exists$ formulas is decidable when the assumptions are $\exists\forall$ formulas).

We will assume certain (universal) properties on our functions (that are naturally translated to assumptions on the new binary relations), that will allow us to get rid of TOT .

Notation 1.1. Given a formula ψ and a formula $P(x)$ (with one free variable x), $\psi_{P(x)}$ denotes the relativization (=restriction) of ψ to $P(x)$.

Proposition 1.2. Suppose that there is an existential formula $P(x)$ (over Σ) such that the following hold:

1. $PF, TOT, \Gamma \vdash \forall x. P(x)$.
2. $PF, \Gamma \vdash TOT_{P(x)}^f$ (that is $PF, \Gamma \vdash \forall x. (P(x) \rightarrow \exists y. (P(y) \wedge R_f(x, y)))$) for every function symbol $f \in \mathcal{F}$.

3. $PF, \Gamma \vdash P(c)$ for every constant symbol c .
4. $PF, \Gamma \vdash \exists x.P(x)$ (this follows from the previous item, provided that there is at least one constant symbol).

Then $PF, TOT, \Gamma \vdash \varphi$ iff $PF, \Gamma \vdash \varphi_{P(x)}$.

Note that if φ is $\forall\exists$ and $P(x)$ is existential, then $\varphi_{P(x)}$ is $\forall\exists$. Since PF and Γ are universal, it is decidable whether $PF, \Gamma \vdash \varphi_{P(x)}$ or not. This way we can decide on $PF, TOT, \Gamma \vdash \varphi$.

Proof. We denote by $M_{P(x)}$ the substructure of a first-order structure M obtained restricting the domain of M to all elements that satisfy $P(x)$ (assuming a relational vocabulary and $M \models \exists x.P(x)$). Note that $M_{P(x)} \models \psi$ iff $M \models \psi_{P(x)}$ for every formula ψ and structure M such that $M \models P(c)$ for every constant symbol c and $M \models \exists x.P(x)$.

Suppose that $PF, TOT, \Gamma \vdash \varphi$. Let M be a model of PF and Γ . Property 2 ensures that $M \models TOT_{P(x)}$. By properties 3 and 4, we also have $M \models P(c)$ for every constant symbol c , and $M \models \exists x.P(x)$. Hence $M_{P(x)} \models TOT$. Since PF and Γ are universal, we also have $M_{P(x)} \models PF$ and $M_{P(x)} \models \Gamma$. It follows that $M_{P(x)} \models \varphi$, and so $M \models \varphi_{P(x)}$.

For the converse, suppose that $PF, \Gamma \vdash \varphi_{P(x)}$. Let M be a model of PF , TOT and Γ . Hence, $M \models \varphi_{P(x)}$. In addition, properties 3 – 4 ensure that $M \models P(c)$ for every constant symbol c , and $M \models \exists x.P(x)$. This implies that $M_{P(x)} \models \varphi$. Property 1 ensures that $M \models \forall x.P(x)$, and hence $M_{P(x)} = M$. It follows that $M \models \varphi$. \square

Example 1.3. Suppose that we have exactly one function symbol f and one constant symbol $null$, and Γ implies $R_f(null, null) \wedge \forall x, y. R_f(x, y) \rightarrow R_f(y, y)$ (i.e., $f(null) = null$ and f is idempotent). We can choose $P(x) = \exists z. R_f(x, z)$. Then, $\forall x. P(x) = TOT$, and thus property 1 obviously holds. Property 2 also holds: $\forall x, y. R_f(x, y) \rightarrow R_f(y, y) \vdash \forall x. ((\exists z. R_f(x, z)) \rightarrow \exists y. ((\exists z. R_f(y, z)) \wedge R_f(x, y)))$. Properties 3 – 4 clearly hold as well.

The following corollary of ?? is particularly useful.

Notation 1.4. Given a composition $f \circ g$ of function symbols in $\Sigma_{\mathcal{F}}$ and variables x, y, z , $R_{f \circ g}(x, y, z)$ is a syntactic sugar for the formula (over Σ) $R_g(x, y) \wedge R_f(y, z)$. Similarly, given a composition sequence $f_m \circ \dots \circ f_1$ and variables x_0, \dots, x_m , $R_{f_m \circ \dots \circ f_1}(x_0, \dots, x_m)$ denotes the formula $\bigwedge_{1 \leq i \leq m} R_{f_i}(x_{i-1}, x_i)$.

Corollary 1.5. Let F_1, \dots, F_n be n composition sequences of function symbols from \mathcal{F} of lengths m_1, \dots, m_n (respectively). For every $1 \leq k \leq n$ and variable x , let $P_{F_k}(x)$ denote the formula $\exists y_1, \dots, y_{m_k}. R_{F_k}(x, y_1, \dots, y_{m_k})$. For every variable x , let $P_F(x)$ denote the conjunction $\bigwedge_{1 \leq k \leq n} P_{F_k}(x)$. Suppose that each of the following formulas follows from PF and Γ :

1. $\forall x. \exists x_1^1, \dots, x_1^{m_1}, \dots, x_n^1, \dots, x_n^{m_n}. \bigwedge_{1 \leq k \leq n} R_{F_k}(x, x_k^1, \dots, x_k^{m_k}) \rightarrow \bigwedge_{1 \leq i \leq n; 1 \leq j \leq m_i} P_F(x_i^j).$
2. For every function symbol $f \in \mathcal{F}$:
 $\forall x. \exists x_1^1, \dots, x_1^{m_1}, \dots, x_n^1, \dots, x_n^{m_n}. \bigwedge_{1 \leq k \leq n} R_{F_k}(x, x_k^1, \dots, x_k^{m_k}) \rightarrow \bigvee_{1 \leq i \leq n; 1 \leq j \leq m_i} R_f(x, x_i^j).$
3. For every constant symbol c : $P_F(c).$
4. $\exists x. P_F(x)$ (this follows from the previous item, provided that there is at least one constant symbol).

Then $PF, TOT, \Gamma \vdash \varphi$ iff $PF, \Gamma \vdash \varphi_{P_F(x)}$.

Proof. Use ?? with $P(x) = P_F(x)$. □

Example 1.6. Suppose that we have exactly one function symbol f and one constant symbol $null$, and Γ implies $R_f(null, null) \wedge \forall x, y, z. R_f(x, y) \wedge R_f(y, z) \rightarrow R_f(z, y)$ (i.e., f satisfies $f(f(f(x))) = f(x)$). Then, ?? can be used with $n = 1$ and $F_1 = f \circ f$.

Example 1.7. Suppose that we have two function symbols f and g and one constant symbol $null$, and Γ implies the following:

- $\forall x, y. R_f(x, y) \rightarrow R_f(y, y)$
- $\forall x, y. R_g(x, y) \rightarrow R_g(y, y)$
- $\forall x, y, z. R_f(x, y) \wedge R_g(x, z) \rightarrow R_g(y, z)$
- $\forall x, y, z. R_f(x, y) \wedge R_g(x, z) \rightarrow R_f(z, y)$
- $R_f(null, null) \wedge R_g(null, null)$

In this case, ?? can be used with $n = 2$, $F_1 = f$ and $F_2 = g$.

Example 1.8. Suppose that we have two function symbols f and g and one constant symbol $null$, and Γ implies the following:

- $\forall x, y. R_f(x, y) \rightarrow R_f(y, y)$
- $\forall x, y. R_g(x, y) \rightarrow R_g(y, y)$
- $\forall x, y, z. R_g(x, y) \wedge R_f(y, z) \rightarrow R_g(z, null) \vee R_g(z, z)$
- $\forall x, y, z. R_f(x, y) \wedge R_g(y, z) \rightarrow R_f(z, null) \vee R_f(z, z)$
- $R_f(null, null) \wedge R_g(null, null)$

In this case, ?? can be used with $n = 2$, $F_1 = f \circ g$ and $F_2 = g \circ f$.

Questions

1. Is it known? perhaps there is a simpler way to obtain a similar result?
2. Is it interesting? useful? so far, we had idempotent functions (Example ??) and the case of Example ?? for dealing with cycles.
3. Can it be extended? in particular, for binary function symbols?

E.g. if we know $f(x, y) = f(y, x)$ and $f(x, f(y, z)) = f(x, y) \vee f(x, f(y, z)) = f(x, z)$ like the case of “lowest common ancestor”.

A trivial example is when we know $f(x, y) \in \{x, y\}$ (like min or max functions), and in this case we can simply have $\forall x, y. R_f(x, y, x) \vee R_f(x, y, y)$ instead of $\forall x, y. \exists z. R_f(x, y, z).$