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STABLE REGIMES OF DYNAMIC SYSTEMS WITH IMPULSIVE INFLUENCES

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ABSTRACT. Let us consider a mathematical model of dynamic system, which is presented as a chain of three connected, singularly perturbed nonlinear differential equations. In the further text there were researched the questions of existence and stability of periodic solutions of this system due to a bifurcational analysis of special two-dimensional map. Also the special attention is paid to the number of coexisting stable regimes.

1. PROBLEM DEFINITION

Let us consider a chain of three connected, singularly perturbed oscillators with a delay:

$$(1) \quad \dot{u}_j = d(a_1 u_{j-1} - a_2 u_j + u_{j+1}) + \lambda(-1 + \alpha f(u_j(t-1)) - \beta g(u_j)) u_j, \quad j = \overline{1, 3},$$

where $u_j = u_j(t) > 0$, parameters $a_1, a_2 \in \{0, 1, 2\}$, $\lambda \gg 1$, $\beta > 0$, $\alpha > 1 + \beta$ and smooth functions $f(u), g(u) \in C^2(\mathbb{R}_+)$ have entry conditions: $0 < \beta g(u) < \alpha$, $f(0) = g(0) = 1$ and $f(u), g(u), u f'(u), u g'(u) = O(1/u)$ as $u \rightarrow +\infty$. In this article there are researched three types of system (1) for different values of parameters a_1, a_2 and conditions on u_0, u_4 : a) $a_1 = 1, a_2 = 2, u_0 = u_1, u_3 = u_4$; b) $a_1 = 1, a_2 = 2, u_0 = u_3, u_1 = u_4$; c) $a_1 = 0, a_2 = 1, u_1 = u_4$.

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In articles [1–3] there was proved, when λ is sufficiently great, by means of following substitutions

$$u_1 = \exp\left(\frac{x}{\varepsilon}\right), \quad u_j = \exp\left(\frac{x}{\varepsilon} + \sum_{k=1}^{j-1} y_k\right), \quad j = 2, 3, \quad \varepsilon = \frac{1}{\lambda} \ll 1,$$

where x, y_1, \dots, y_{m-1} are new variables, system (1) can be transformed to the two-dimensional system of differential equations without small parameters, but with impulsive influences

$$(2) \quad \begin{aligned} \dot{y}_1 &= d(e^{y_2} + a_1 e^{-y_1} - e^{y_1} - a_1 e^{-y_0}), \\ \dot{y}_2 &= d(e^{y_3} + a_1 e^{-y_2} - e^{y_2} - a_1 e^{-y_1}), \end{aligned}$$

$$y_j(+0) = \frac{\alpha - 1}{\alpha - \beta - 1} y_j(-0), \quad y_j(1+0) = y_j(1-0) - \frac{\alpha}{\alpha - 1} y_j(+0),$$

$$y_j(\alpha+0) = (1+\beta)y_j(\alpha-0), \quad y_j(\alpha+1+0) = y_j(\alpha+1-0) - \frac{\alpha}{1+\beta} y_j(\alpha+0),$$

where values of y_0, y_3 depend on entry conditions on u_0 and u_4 : a) $u_0 = u_1, u_3 = u_4$: $y_0 = y_3 = 0$; b) $u_0 = u_3, u_1 = u_4$: $y_0 = y_3 = -(y_1 + y_2)$; c) $u_1 = u_4$: $y_3 = -(y_1 + y_2)$.

Let us consider solutions of system (2) $y_1(t, z_1, z_2), y_2(t, z_1, z_2)$ with entry conditions $y_1(-0, z_1, z_2) = z_1, y_2(-0, z_1, z_2) = z_2$. For map

$$(3) \quad \Phi(z) : \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \rightarrow \begin{pmatrix} y_1(T_0, z_1, z_2) \\ y_2(T_0, z_1, z_2) \end{pmatrix}$$

in articles [1–3] there was proved, that exponentially stable points of map (3) are satisfied the orbitally, asymptotically stable cycles of systems (1) and (2). In other words it is enough to research stable points of map (3) instead of stable cycles of system (1). In map (3) $y_1(t)$ and $y_2(t)$ have entry conditions $y_1(-0) = z_1, y_2(-0) = z_2$. These functions are connected with initial variables by means of approximate equations $y_1 \approx \ln u_2 - \ln u_1, y_2 \approx \ln u_3 - \ln u_2$ and describe phase shifts of components in system (1). $T_0 = \alpha + 1 + (\beta + 1)/(\alpha - \beta - 1)$ is the first approximation of stable cycle for a single oscillator of system (1).

An asymptotic analysis shows, that map (3) has at least four stable points, when parameter d is sufficiently small. Moreover zero balance state is stable for any values of d . It is satisfied a homogeneous synchronous cycle of system (1). The task of research is to find values of parameters α and β , when map (3) has maximal amount of stable points. Also there are researched bifurcations in a phase space of map (3). The research was implemented by means of special developed software. The

calculation of coordinates of stable points was carried out on a large number of independent streams on CPU. Given numerical results are shown as a phase portrait of map (3).

2. RESULTS OF NUMERICAL RESEARCH

In the case $a_1 = 1, a_2 = 2, u_0 = u_1, u_3 = u_4$ on coordinate plane of parameters (α, β) there are regions A_1, A_2, A_3 and curves l_0, \dots, l_4 . They are shown on fig.1.

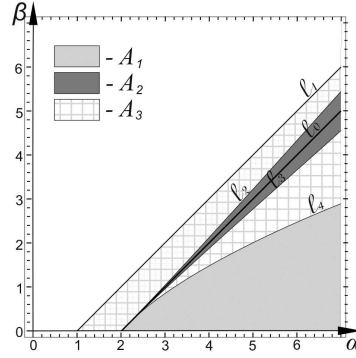


Fig. 1: Regions with the same bifurcations

The borders of regions depend on maximal amount of stable points, which are detected there for map (3). For values of parameters α and β from region A_1 it is possible to exist five stable points (fig.2a). In regions A_2 and A_3 it is possible to exist seven (fig.2b) and six stable points (fig.2c), respectively.

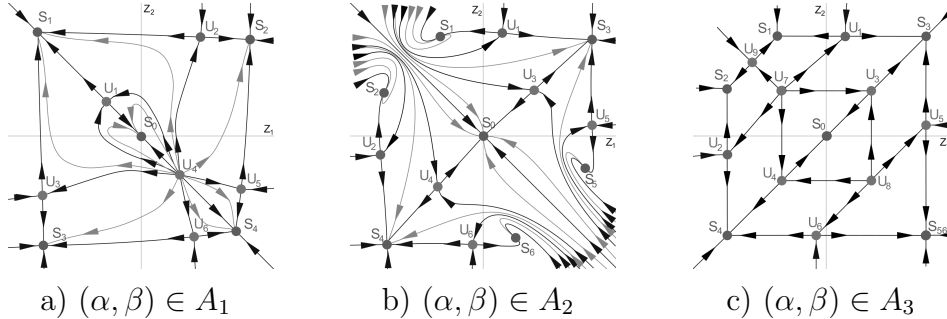


Fig. 2: Phase portraits of map

The most important element for building of regions is line $l_0 = \{(\alpha, \beta) : \beta = \alpha - 2\}$. Curves l_2 and l_3 are symmetric relative to line l_0 and touch each other in point $(2, 0)$. These curves are borders of region $A_2 = \{(\alpha, \beta) : \beta < l_2, \beta > l_3\}$. Also in point $(2, 0)$ curve l_4 traces to line l_0 . It permits to determine region $A_1 = \{(\alpha, \beta) : \beta < l_4, \beta > l_0\}$. Doubly connected region $A_3 = \{(\alpha, \beta) : \beta > l_2, \beta < l_1, \beta > l_4, \beta < l_3\}$, where line $l_1 = \{(\alpha, \beta) : \beta = \alpha - 1\}$ describes one of conditions for α and β in system (1).

In article [6] there are examples of different bifurcations for certain values of initial parameters.

In the case $a_1 = 1, a_2 = 2, u_0 = u_3, u_1 = u_4$ on coordinate plane of parameters (α, β) there are regions A_1, A_2, A_3 and curves l_0, \dots, l_3 . They are shown on fig.3.

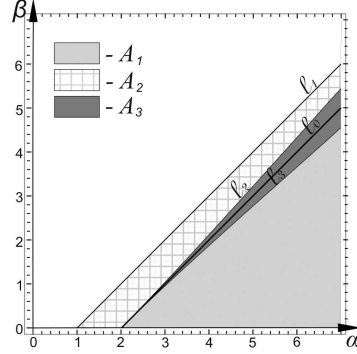


Fig. 3: Regions with the same bifurcations

In article [7] there are examples of different bifurcations for certain values of initial parameters.

For values of parameters α and β from regions $A_1 = \{(\alpha, \beta) : \beta > 0, \beta < l_3\}$ (fig.4a) and $A_2 = \{(\alpha, \beta) : \beta > l_2, \beta < l_1\}$ (fig.4b) it is possible to exist seven stable points for map (3). The difference between these cases is the types of unstable points U_1, \dots, U_6 . In region $A_3 = \{(\alpha, \beta) : \beta > l_3, \beta < l_2\}$ there is an unstable manifold around zero balance state S_0 instead of unstable points U_1, \dots, U_6 . Every point of this manifold is an unstable state.

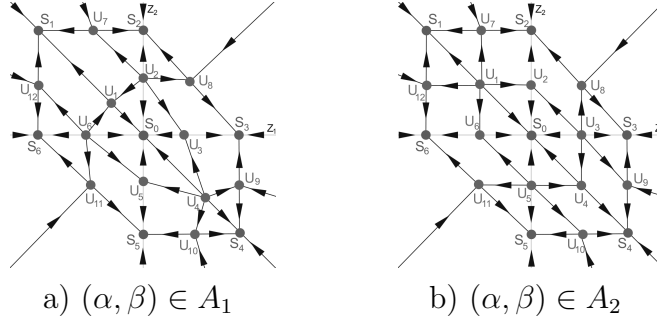


Fig. 4: Phase portraits of map

In the case $a_1 = 0, a_2 = 1, u_1 = u_4$ on coordinate plane of parameters (α, β) there are regions A_1, A_2, A_3 and curves l_0, \dots, l_5 . They are shown on fig.5.

Line l_5 comes nearer to curve l_2 , when parameter α increases. Curves l_0, \dots, l_5 permit to describe regions $A_1 = \{(\alpha, \beta) : \beta > 0, \beta < l_4, \beta > l_2, \beta < l_5\}$, $A_2 = \{(\alpha, \beta) : \beta > 0, \beta < l_4, \beta > l_2, \beta < l_5\}$ and $A_3 = \{(\alpha, \beta) : \beta > l_3, \beta < l_2\}$.

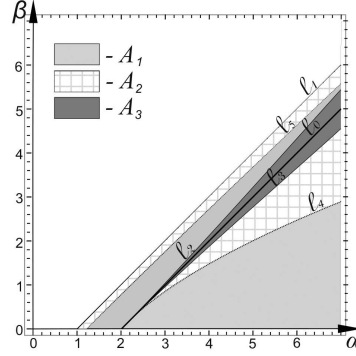


Fig. 5: Regions with the same bifurcations

For every region A_1 , A_2 and A_3 , as in articles [6, 7] there were given all possible bifurcations in a phase space of map (3). The difference is the bifurcational value of d . Let us consider one sequence of bifurcations for parameters $(\alpha, \beta) \in A_2$ in details.

For any fixed values $(\alpha, \beta) \in A_2$ and change of parameter d in a phase space of map (3) there is the only sequence of bifurcations. Let us fix values $\alpha = 1.9$ and $\beta = 0.1$, parameter d will increase. As a result, there is the following sequence of bifurcations:

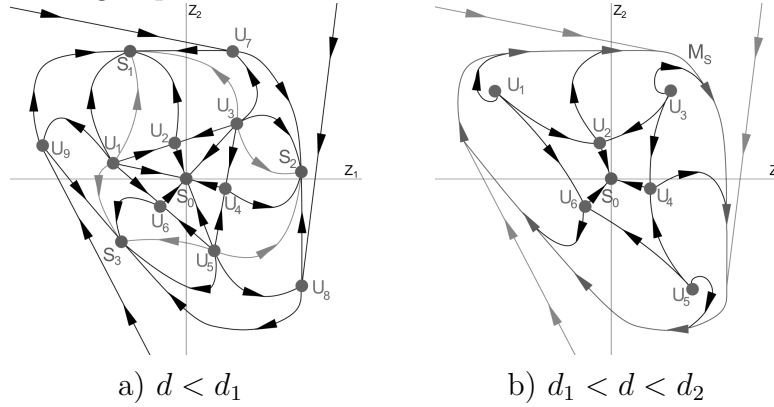


Fig. 6: Phase portraits of map

- (1) $d < d_1, d_1 \approx 0.316$: map (3) has 4 stable and 9 unstable points (fig.6a)
- (2) $d = d_1$: unstable saddles U_7, U_8 and U_9 come nearer to stable saddles S_1, S_2 and S_3 , merge with them and stable manifold M_S appears. The movement on manifold M_S is clockwise. Unstable points U_1, U_3 and U_5 become focuses.
- (3) $d_1 < d < d_2, d_2 \approx 0.317$: map (3) has 1 stable, 3 unstable points and stable manifold (fig.6b).
- (4) $d = d_2$: unstable manifolds separate from unstable focuses U_1, U_3, U_5 . Points U_1, U_3 and U_5 become stable focuses S'_1, S'_2 and S'_3 .

- (5) $d_2 < d < d_3$, $d_3 \approx 0.3174$: map (3) has 4 stable and 3 unstable points. Also it has 1 stable and 3 unstable manifolds (fig.7a).
- (6) $d = d_3$: 3 unstable manifolds around stable focuses S'_1, S'_2 and S'_3 merge with stable manifold M_S . All manifolds disappear.
- (7) $d_3 < d < d_4$, $d_4 \approx 0.3178$: map (3) has 4 stable and 3 unstable points (fig.7b).

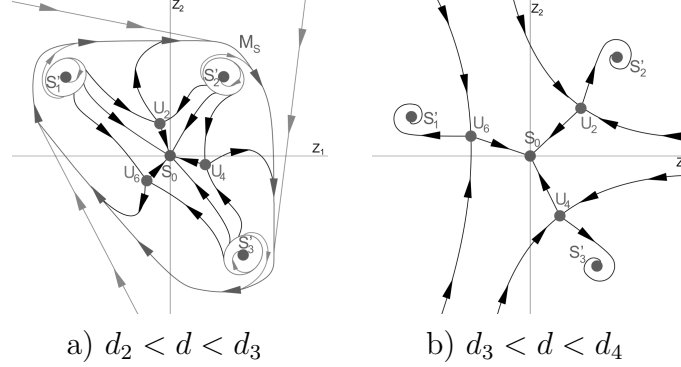


Fig. 7: Phase portraits of map

- (8) Last bifurcation takes places, when $d = d_4$. Unstable saddles U_2, U_4 and U_6 merge with stable focuses S'_2, S'_3 and S'_1 and disappear. When $d > d_4$ map (3) has only zero balance state S_0 .

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