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# Stable Regimes of Dynamic Systems with Impulsive Influences

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**Abstract**—Let us consider a mathematical model of dynamic system, which is presented as a chain of three connected, singularly perturbed nonlinear differential equations. In the further text there were researched the questions of existence and stability of periodic solutions of this system due to a bifurcational analysis of special two-dimensional map. Also the special attention is paid to the number of coexisting stable regimes.

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## 1. PROBLEM DEFINITION

Let us consider a chain of three connected, singularly perturbed oscillators with a delay:

$$\dot{u}_j = d(a_1 u_{j-1} - a_2 u_j + u_{j+1}) + \lambda(-1 + \alpha f(u_j(t-1)) - \beta g(u_j))u_j, \quad j = \overline{1,3}, \tag{1}$$

where $u_j = u_j(t) > 0$, parameters $a_1, a_2 \in \{0, 1, 2\}$, $\lambda \gg 1$, $\beta > 0$, $\alpha > 1 + \beta$ and smooth functions $f(u), g(u) \in C^2(\mathbb{R}_+)$ have entry conditions: $0 < \beta g(u) < \alpha$, $f(0) = g(0) = 1$ and $f(u), g(u), uf'(u)$, $ug'(u) = O(1/u)$ as $u \to +\infty$. In this article there are researched three types of system (1) for different values of parameters $a_1, a_2$ and conditions on $u_0$, $u_4$: a) $a_1 = 1$, $a_2 = 2$, $u_0 = u_1$, $u_3 = u_4$; b) $a_1 = 1$, $a_2 = 2$, $u_0 = u_3$, $u_1 = u_4$; c) $a_1 = 0$, $a_2 = 1$, $u_1 = u_4$.

In [1–3] there was proved, when $\lambda$ is sufficiently great, by means of following substitutions

$$u_1 = \exp\left(\frac{x}{\varepsilon}\right), \quad u_j = \exp\left(\frac{x}{\varepsilon} + \sum_{k=1}^{j-1} y_k\right), \quad j = 2, 3, \quad \varepsilon = \frac{1}{\lambda} \ll 1,$$

where $x, y_1, ..., y_{m-1}$ are new variables, system (1) can be transformed to the two-dimensional system of differential equations without small parameters, but with impulsive influences

$$\dot{y}_1 = d(e^{y_2} + a_1 e^{-y_1} - e^{y_1} - a_1 e^{-y_0}), \quad \dot{y}_2 = d(e^{y_3} + a_1 e^{-y_2} - e^{y_2} - a_1 e^{-y_1}),$$
$$y_j(+0) = \frac{\alpha - 1}{\alpha - \beta - 1} y_j(-0), \quad y_j(1+0) = y_j(1-0) - \frac{\alpha}{\alpha - 1} y_j(+0),$$
$$y_j(\alpha + 0) = (1 + \beta) y_j(\alpha - 0), \quad y_j(\alpha + 1 + 0) = y_j(\alpha + 1 - 0) - \frac{\alpha}{1 + \beta} y_j(\alpha + 0), \tag{2}$$

where values of $y_0$, $y_3$ depend on entry conditions on $u_0$ and $u_4$: a) $u_0 = u_1$, $u_3 = u_4$: $y_0 = y_3 = 0$; b) $u_0 = u_3$, $u_1 = u_4$: $y_0 = y_3 = -(y_1 + y_2)$; c) $u_1 = u_4$: $y_3 = -(y_1 + y_2)$.

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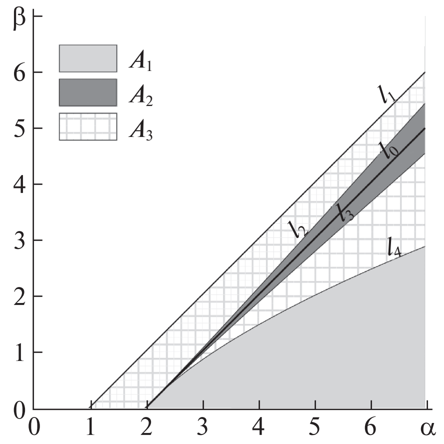



**Fig. 1.** Regions with the same bifurcations.

Let us consider solutions of system (2) $y_1(t, z_1, z_2)$, $y_2(t, z_1, z_2)$ with entry conditions $y_1(-0, z_1, z_2) = z_1$, $y_2(-0, z_1, z_2) = z_2$. For map

$$\Phi(z) : \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \to \begin{pmatrix} y_1(T_0, z_1, z_2) \\ y_2(T_0, z_1, z_2) \end{pmatrix} \tag{3}$$

in [1–3] there was proved, that exponentially stable points of map (3) are satisfied the orbitally, asymptotically stable cycles of systems (1) and (2). In other words it is enough to research stable points of map (3) instead of stable cycles of system (1). In map (3) $y_1(t)$ and $y_2(t)$ have entry conditions $y_1(-0) = z_1$, $y_2(-0) = z_2$. These functions are connected with initial variables by means of approximate equations $y_1 \approx \ln u_2 - \ln u_1$, $y_2 \approx \ln u_3 - \ln u_2$ and describe phase shifts of components in system (1). $T_0 = \alpha + 1 + (\beta + 1)/(\alpha - \beta - 1)$ is the first approximation of stable cycle for a single oscillator of system (1).

An asymptotic analysis shows, that map (3) has at least four stable points, when parameter $d$ is sufficiently small. Moreover zero balance state is stable for any values of $d$. It is satisfied a homogeneous synchronous cycle of system (1). The task of research is to find values of parameters $\alpha$ and $\beta$, when map (3) has maximal amount of stable points. Also there are researched bifurcations in a phase space of map (3). The research was implemented by means of special developed software. The calculation of coordinates of stable points was carried out on a large number of independent streams on CPU. Given numerical results are shown as a phase portrait of map (3).

## 2. RESULTS OF NUMERICAL RESEARCH

In the case $a_1 = 1$, $a_2 = 2$, $u_0 = u_1$, $u_3 = u_4$ on coordinate plane of parameters $(\alpha, \beta)$ there are regions $A_1$, $A_2$, $A_3$ and curves $l_0, ..., l_4$. They are shown on Fig. 1.

The borders of regions depend on maximal amount of stable points, which are detected there for map (3). For values of parameters $\alpha$ and $\beta$ from region $A_1$ it is possible to exist five stable points (Fig. 2a). In regions $A_2$ and $A_3$ it is possible to exist seven (Fig. 2b) and six stable points (Fig. 2c), respectively.

The most important element for building of regions is line $l_0 = \{(\alpha, \beta) : \beta = \alpha - 2\}$. Curves $l_2$ and $l_3$ are symmetric relative to line $l_0$ and touch each other in point $(2, 0)$. These curves are borders of region $A_2 = \{(\alpha, \beta) : \beta < l_2, \beta > l_3\}$. Also in point $(2, 0)$ curve $l_4$ traces to line $l_0$. It permits to determine region $A_1 = \{(\alpha, \beta) : \beta < l_4, \beta > l_0\}$. Doubly connected region $A_3 = \{(\alpha, \beta) : \beta > l_2, \beta < l_1, \beta > l_4, \beta < l_3\}$, where line $l_1 = \{(\alpha, \beta) : \beta = \alpha - 1\}$ describes one of conditions for $\alpha$ and $\beta$ in system (1).

In [6] there are examples of different bifurcations for certain values of initial parameters. In the case $a_1 = 1$, $a_2 = 2$, $u_0 = u_3$, $u_1 = u_4$ on coordinate plane of parameters $(\alpha, \beta)$ there are regions $A_1$, $A_2$, $A_3$ and curves $l_0, ..., l_3$. They are shown on Fig. 3.

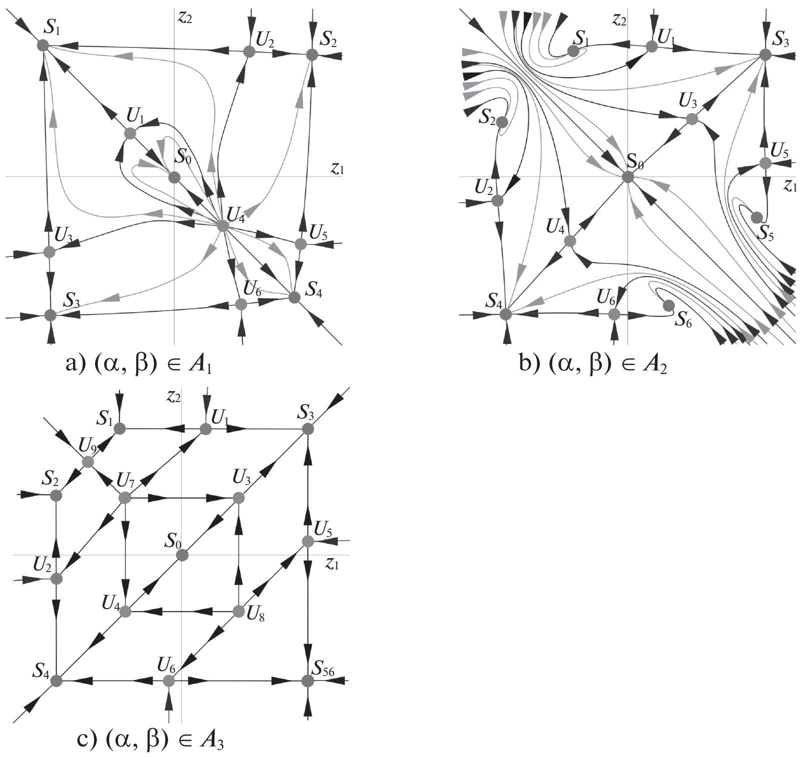



**Fig. 2.** Phase portraits of map.

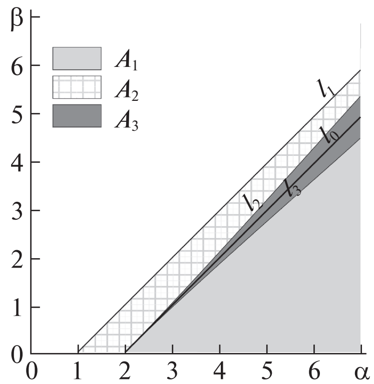
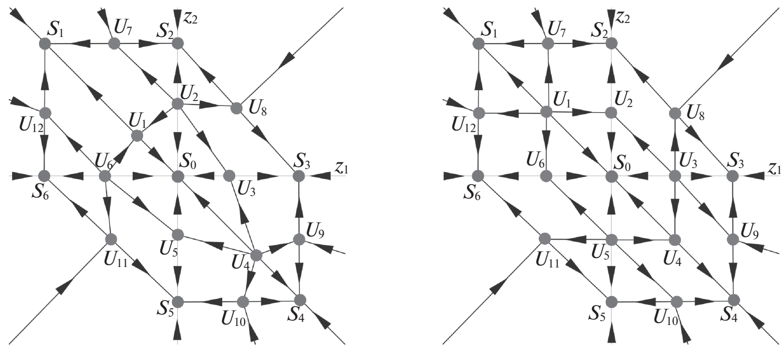



**Fig. 3.** Regions with the same bifurcations.

In [7] there are examples of different bifurcations for certain values of initial parameters. For values of parameters $\alpha$ and $\beta$ from regions $A_1 = \{(\alpha, \beta) : \beta > 0, \beta < l_3\}$ (Fig. 4a) and $A_2 = \{(\alpha, \beta) : \beta > l_2, \beta < l_1\}$ (Fig. 4b) it is possible to exist seven stable points for map (3). The difference between these cases is the types of unstable points $U_1, ..., U_6$. In region $A_3 = \{(\alpha, \beta) : \beta > l_3, \beta < l_2\}$ there is an unstable manifold around zero balance state $S_0$ instead of unstable points $U_1, ..., U_6$. Every point of this manifold is an unstable state.

In the case $a_1 = 0$, $a_2 = 1$, $u_1 = u_4$ on coordinate plane of parameters $(\alpha, \beta)$ there are regions $A_1, A_2, A_3$ and curves $l_0, ..., l_5$. They are shown on Fig. 5. Line $l_5$ comes nearer to curve $l_2$, when parameter $\alpha$ increases. Curves $l_0, ..., l_5$ permit to describe regions $A_1 = \{(\alpha, \beta) : \beta > 0, \beta < l_4, \beta > l_2, \beta < l_5\}$, $A_2 = \{(\alpha, \beta) : \beta > 0, \beta < l_4, \beta > l_2, \beta < l_5\}$ and $A_3 = \{(\alpha, \beta) : \beta > l_3, \beta < l_2\}$.

For every region $A_1$, $A_2$ and $A_3$, as in [6, 7] there were given all possible bifurcations in a phase space of map (3). The difference is the bifurcational value of $d$. Let us consider one sequence of bifurcations for parameters $(\alpha, \beta) \in A_2$ in details. For any fixed values $(\alpha, \beta) \in A_2$ and change of parameter $d$ in a




a) $(\alpha, \beta) \in A_1$ b) $(\alpha, \beta) \in A_2$

**Fig. 4.** Phase portraits of map.

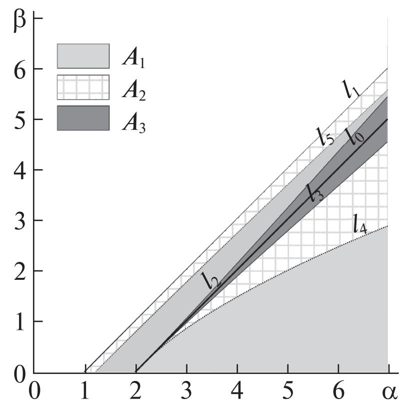
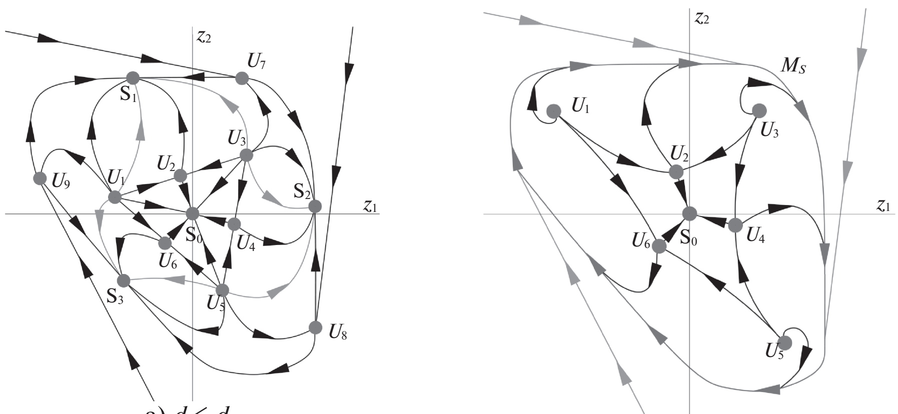



**Fig. 5.** Regions with the same bifurcations.




a) $d < d_1$ b) $d_1 < d < d_2$

**Fig. 6.** Phase portraits of map.

phase space of map (3) there is the only sequence of bifurcations. Let us fix values $\alpha = 1.9$ and $\beta = 0.1$, parameter $d$ will increase. As a result, there is the following sequence of bifurcations:

1. $d < d_1$, $d_1 \approx 0.316$: map (3) has 4 stable and 9 unstable points (Fig. 6a);

2. $d = d_1$: unstable saddles $U_7$, $U_8$ and $U_9$ come nearer to stable saddles $S_1$, $S_2$ and $S_3$, merge with them and stable manifold $M_S$ appears. The movement on manifold $M_S$ is clockwise. Unstable points $U_1$, $U_3$ and $U_5$ become focuses;

3. $d_1 < d < d_2$, $d_2 \approx 0.317$: map (3) has 1 stable, 3 unstable points and stable manifold (Fig. 6b);

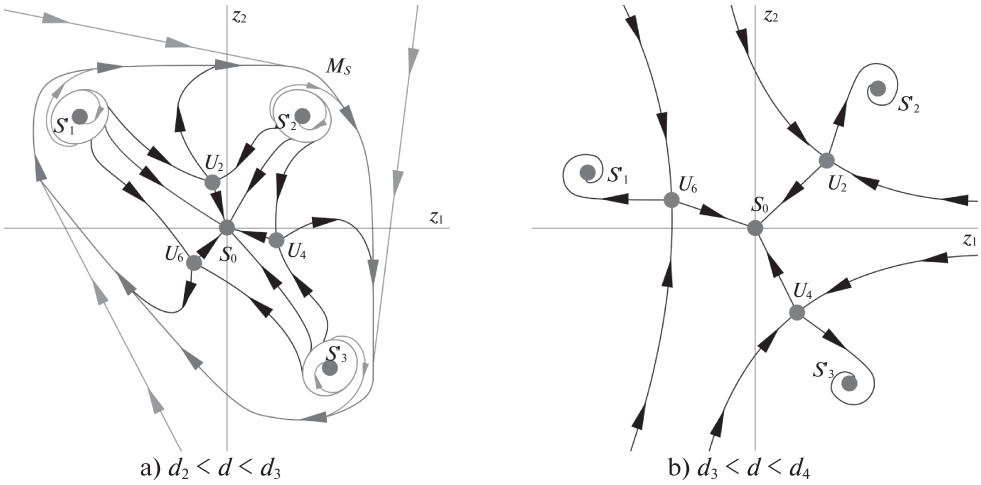



**Fig. 7.** Phase portraits of map.

4. $d = d_2$: unstable manifolds separate from unstable focuses $U_1$, $U_3$, $U_5$. Points $U_1$, $U_3$ and $U_5$ become stable focuses $S_1'$, $S_2'$ and $S_3'$;

5. $d_2 < d < d_3$, $d_3 \approx 0.3174$: map (3) has 4 stable and 3 unstable points. Also it has 1 stable and 3 unstable manifolds (Fig. 7a);

6. $d = d_3$: 3 unstable manifolds around stable focuses $S_1'$, $S_2'$ and $S_3'$ merge with stable manifold $M_S$. All manifolds disappear;

7. $d_3 < d < d_4$, $d_4 \approx 0.3178$: map (3) has 4 stable and 3 unstable points (Fig. 7b).

8. Last bifurcation takes places, when $d = d_4$. Unstable saddles $U_2$, $U_4$ and $U_6$ merge with stable focuses $S_2'$, $S_3'$ and $S_1'$ and disappear. When $d > d_4$ map (3) has only zero balance state $S_0$.

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