CLASS

the Cosmological Linear Anisotropy Solving System[1]

Julien Lesgourgues
TTK, RWTH Aachen University

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[1]code developed by Julien Lesgourgues & Thomas Tram plus many others...

1. Essential steps in an Einstein-Boltzmann solver
2. Details on few useful aspects for each of them

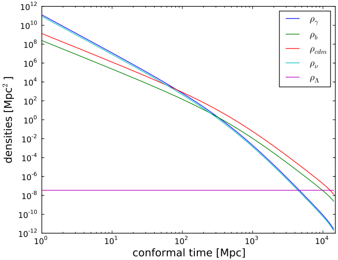
Essential steps in Einstein-Boltzmann solver

A. Background

Get all background quantities as function of a time variable (`class` $\rightarrow$ conformal time τ) after integrating differential equation like Friedmann:

$$\frac{da}{d\tau} = \left(\frac{8\pi G}{3} a^2 \sum_X \rho_X(a) - \frac{K}{a^2} \right)^{1/2}$$

Gives mapping between $\tau \leftrightarrow a \leftrightarrow z (\leftrightarrow t)$

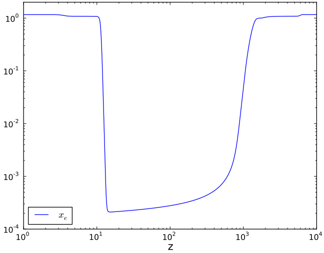


Essential steps in Einstein-Boltzmann solver

B. Thermodynamics

Get all thermodynamics quantities as a function of a time variable (`class` $\rightarrow$ redhsift z) after integrating differential equations like recombination equations:

$$\frac{dx_e}{dz} = \mathrm{excitation}, \mathrm{ionization}, \mathrm{heating}, ...$$



Then $x_e(z) \rightarrow \kappa'(z)$ (Thomson scattering rate)
$\rightarrow \kappa(z)$ (Optical depth)
$\rightarrow \exp(-\kappa(z))$ (factor for Integrated Sachs-Wolfe effect)
$\rightarrow g(z)$ (visibility function for Sachs-Wolfe effect)
$\rightarrow g'(z)$ (factor for Doppler effect)

Essential steps in Einstein-Boltzmann solver

C. Perturbations

- Find all perturbations ($\delta_X(\tau, k)$, $\phi(\tau, k)$, ...) by integrating ODEs for each independent wavenumber k, each mode (scalar/vector[1]/tensor), each initial condition (adiabatic/isocurvature):
  - Boltzmann (non-perfect fluids: photon temperature/polarization, massless/massive neutrino temperature)
  - Continuity + Euler (perfect fluid: baryons, hypothetical (DE/DM/DR) fluid) or approximatively pressureless species: (CDM)
  - linearized Einstein equations (one = differential equation, others = constraint equations)

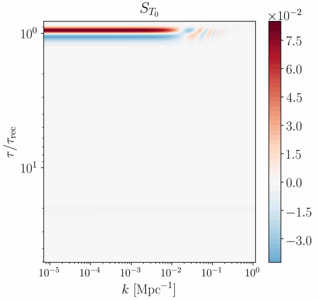
  Linear perturbations $\Rightarrow$ perturbations normalized to trivial initial condition (`class` $\rightarrow$ curvature $\mathcal{R} = 1$ for scalar with adiabatic I.C.)

[1] in `class` $\rightarrow$ vector perturbation equations present just in case, but never used: no implemented scenario where vectors are relevant, no vector I.C. and observables.

C. Perturbations

- Keep memory not of everything, but anything useful for final calculation of observables:
  - raw transfer function ($\delta_m(\tau, k) \rightarrow P_m(k, z)$)
  - non-trivial combinations (photon, baryon, metric, thermodynamical functions $\rightarrow$ CMB source functions $S_{T_i}(k, \tau)$)

All these are called *source functions* in `class`



D. Primordial spectra

Initial conditions for scalars (adiabatic, isocurvature) and tensors. Linear theory ⇔ Gaussian independent Fourier modes ⇔ only need primordial power spectra

- analytic mode: primordial power spectra as parametric functions (e.g. power-law)
- inflation mode: solve background+perturbation equation for single-field inflation and compute primordial scalar/tensor spectrum numerically

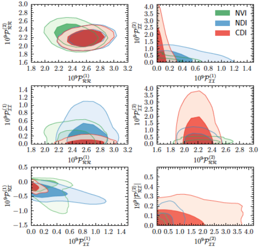


Fig. 22. Two dimensional distributions for power in isocurvature modes, using *Planck*+WP data.

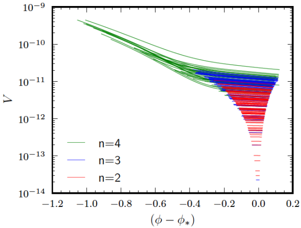
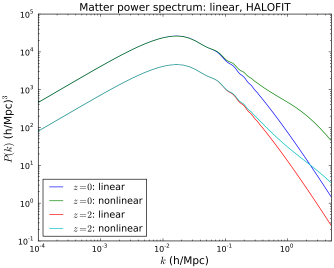


Fig. 14. Observable range of the best-fitting inflaton potentials, when $V(\phi)$ is Taylor expanded at the nth order around the pivot value ϕ_*, in natural units (where $\sqrt{8\pi}M_{\rm pl} = 1$), assuming a flat prior on ϵ_V, η_V, ξ_V^2, and ϖ_V^3, and using *Planck*+WP data.

Essential steps in Einstein-Boltzmann solver

E. Power spectra in Fourier space

- Linear matter power spectrum $P_m(k,z) \rightarrow$ integrated quantities $\sigma(R,z)$, $\sigma_8(z)$
- Linear baryon+CDM power spectrum $P_{cb}(k,z) \rightarrow$ integrated quantities $\sigma_{cb,8}(z)$
- Approximation for non-linear spectrum $P_m^{NL}(k,z)$ based on prescriptions like HALOFIT, HMCODE...
- Keep in memory non-linear correction factors like $R^{NL}(k,z) = \left(P_m^{NL}(k,z)/P_m(k,z)\right)^{1/2}$ for e.g. CMB lensing, cosmic shear, number count C_ℓ's



Essential steps in Einstein-Boltzmann solver

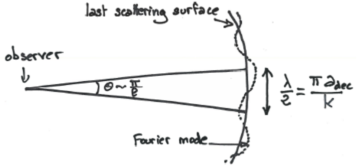
F. Transfer functions in harmonic space

CMB spectrum depends on $\Delta_\ell^X(k) = \ell$-th multipole of anisotropy of photon temperature and polarisation ($X \in \{T, E, B\}$) for each mode (scalar/tensor) and initial condition (adiabatic/isocurvature) today ($\tau = \tau_0$).

- In `COSMICS`: integrate equations for each k, ℓ, X, mode, I.C. until today.
- Since `CMBFAST` (Seljak & Zaldarriaga 1996): use "line-of-sight integral", more precisely and exact implicit solution of Boltzmann equation (here in flat space):

$$\Delta_\ell^X(k) = \int_\epsilon^{\tau_0} d\tau \; S^X(\tau, k) \; j_l(k(\tau_0 - \tau))$$

$S(\tau, k)$ only depends on thermodynamical functions, first few multipoles, baryons flux divergence and metric perturbations. Role of Bessel: projection from Fourier to harmonic space ($\theta \, d_a(z_{\rm rec}) = \frac{\lambda}{2}$ gives precisely $l = k(\tau_0 - \tau_{\rm rec})$):



Curved space: spherical bessel functions $\rightarrow$ modified Bessel functions (hypergeometric)

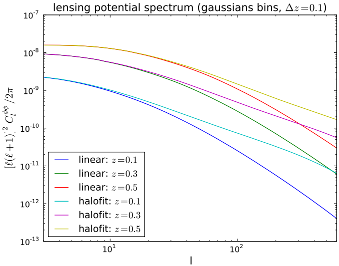
Essential steps in Einstein-Boltzmann solver

F. Transfer functions in harmonic space

$$\Delta_\ell^X(k) = \int_\epsilon^{\tau_0} d\tau \; S^X(\tau, k) \; j_l(k(\tau_0 - \tau))$$

Applies not just to CMB $X \in \{T, E, B\}$ but also all LSS C_ℓ's (one X per type of observable and redshift bin).

- CMB lensing + cosmic shear: similar formulation, $S(\tau, k)$ depends on metric fluctuation and window function (intrinsic to lensing + source selection function)
- number count (galaxy clustering): $S(\tau, k)$ depends on baryon+CDM density fluctuation and selection function in each bin plus corrections from matter flux divergence and metric perturbations (RSD, Doppler, lensing, other GR effects)
- may include non-linear correction factors $R^{NL}(k, z)$

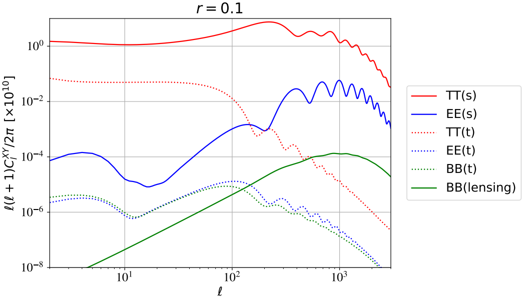


G. Harmonic power spectra (C_ℓ's)

Trivial:

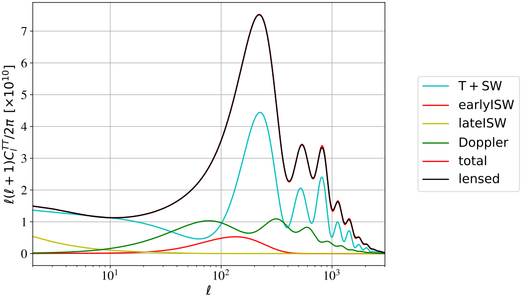
$$C_\ell^{XY} = \int \frac{dk}{k} \sum_{ij} \Delta^X_{\ell\, i}(k) \Delta^Y_{\ell\, j}(k) \mathcal{P}_{ij}(k)$$

with sum running over modes (scalar/tensor) and I.C. (adiabatic/isocurvature).



H. Lensed CMB C_ℓ's

- metric fluctuations (ϕ, ψ) $\rightarrow$ lensing potential source function $\rightarrow$ CMB lensing potential spectrum C_ℓ^{PP}
- several fluctuations $\rightarrow$ CMB source functions $\rightarrow$ unlensed CMB spectra $C_\ell^{TT,TE,EE,BB}$
- several quadratic sums over $C_{\ell_1}^{XY} C_{\ell_2}^{PP}$ $\rightarrow$ lensed CMB spectra $C_\ell^{TT,TE,EE,BB}$. Full-sky approach of Challinor & Lewis 2005.



Essential steps in Einstein-Boltzmann solver

A. Background: formalizing problem and classifying quantities

In general, three types of parameters:

- $\{A\}$ which can be expressed directly as a function of some variables $\{B\}$.
- $\{B\}$, which need to be integrated over τ through first-order differential equation
- $\{C\}$, which also need to be integrated but are not used to compute $\{A\}$.

ΛCDM and many simple extensions:

- $\{A\} = \{\rho_i(a), p_i(a), H(a), ...,\}$ with e.g. $H(a) = \left(\sum_X \rho_x(a) - \frac{K}{a^2}\right)^{1/2}$
- $\{B\} = \{a\}$ since $\frac{da}{d\tau} = a^2 H(a)$
- $\{C\} = \{t, r_s, \text{linear growth factor}\}$ with e.g. $\frac{dt}{d\tau} = a$, $\frac{dr_s}{d\tau} = c_s^2(a)$

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Exemple of DE/DM/DR fluid:

- $\{A\} = \{\rho_i(a), p_i(a), H(a), ..., w_{\rm fld}(a)\}$
- $\{B\} = \{a, \rho_{\rm fld}\}$ with $\frac{d\rho_{\rm fld}}{d\tau} = -3aH(a)\,(1 + w_{\rm fld}(a))\rho_{\rm fld}$

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Exemple of extended cosmology with quintessence ϕ:

- $\{A\} = \{\rho_i, p_i, H, ..., V(\phi), \rho_\phi(\phi, \phi')\}$ with e.g. $\rho_\phi(\phi, \phi') = \frac{1}{2}(\phi')^2 + V(\phi)$
- $\{B\} = \{a, \phi, \phi'\}$ with $\frac{d\phi}{d\tau} = \phi'$, $\frac{d\phi'}{d\tau} = -2aH(a)\,\phi' - a^2 V(\phi)$

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Also Cold Dark Matter decaying into Dark Radiation...

- $\{A\} = \{\rho_i, p_i, H, ...\}$
- $\{B\} = \{a, \rho_{\rm dcdm}, \rho_{\rm dr}\}$ with $\frac{d\rho_{\rm dcdm}}{d\tau} = -3aH(a)\,\rho_{\rm dcdm} - a\Gamma(a)\,\rho_{\rm dcdm}$

Essential steps in Einstein-Boltzmann solver

B. Thermodynamics: three approaches to recombination

User can choose to model approximate recombination and get $x_e(z)$, $T_b(z)$ from:

- RECFAST (Wong, Moss & Scott 2008): in-built
- HyRec (Y. Ali-Haïmoud) : download with CLASS, more accurate
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Recombination needs one more cosmological parameter: the primordial Helium fraction $Y_{\rm He}$.

- User can fix it to given value (e.g. `Y_He = 0.25`) or to `Y_He = BBN`. Then the value is infered from an interpolation table pre-pcomputed with a BBN code (Parthenope), for each given value of $N_{\rm eff}$, ω_b (assumes $\mu_{\nu_e} = 0$, easy to generalise).
- BBN interpolation table located in separate directory, in `bbn/bbn.dat`

Essential steps in Einstein-Boltzmann solver

C. Perturbations: the polarization hierarchy

Two approaches to polarization in Boltzmann hierarchy:

- Ma & Bertschinger 1994:
  $(F_\ell, G_\ell) \rightarrow (S_T, S_P) \rightarrow (\Delta_\ell^T, \Delta_\ell^E, \Delta_\ell^B)$: $2\ell_{\rm max}$ equations!
- Hu & White 1997:
  $(\Theta_\ell, E_\ell, B_\ell) \rightarrow (S_T, S_E, S_B) \rightarrow (\Delta_\ell^T, \Delta_\ell^E, \Delta_\ell^B)$: $3\ell_{\rm max}$ equations!

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CMBFAST: first in flat space, second in curved space

Essential steps in Einstein-Boltzmann solver

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CMBFAST: first in flat space, second in curved space

CAMB: always second case

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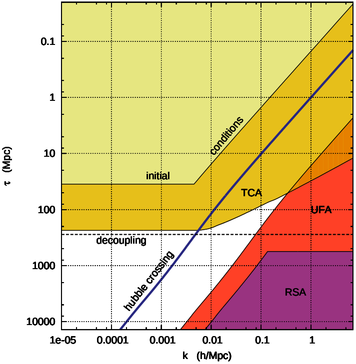
CMBFAST: first in flat space, second in curved space

CAMB: always second case

CLASS: always first case, thanks to new analytic results in curved space
(T. Tram & JL, JCAP 2013 [arXiv:1305.3261])

Essential steps in Einstein-Boltzmann solver

C. Perturbations: the approximation scheme (CLASS II & CLASS IV 2011)



- Tight Coupling Approximation for baryons and γ at 2nd order
- Ultrarelativistic Fluid Approximation (for massless ν, also one for massive ones): truncated Boltzmann, 3 equations
- Radiation Streaming Approximation (for photons and massless ν): test particles, 0 equations

Essential steps in Einstein-Boltzmann solver

C. Perturbations: an ODE solver customized for Einstein-Boltzmann solver:

- Stiff system require implicit method like backward Euler or more advanced:
  $\rightarrow$ find y_{n+1} as a solution of $y_{n+1} = y_n + y'(y_{n+1})\delta t$
- Should still be fast: Newton method with Jacobian recycling
- Robustness requires δt to be determined automatically (adaptive time step)
- Source function required at predefined t_i: integrator must interpolate on-the-fly at these valkues
- System is sparse: some algebra gives big speed up (sparse LU decomposition)

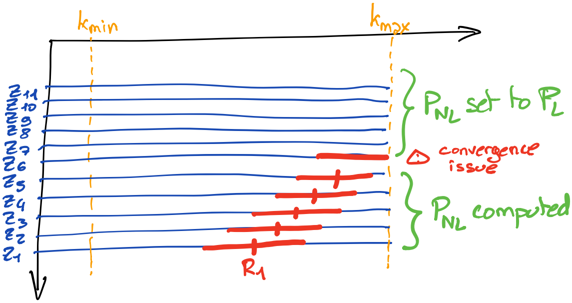
Everything gathered in `ndf15` by T. Tram (CLASS II 2011).
TCA could even be removed!

D. Primordial spectra: the different modes

P_k_ini type =	modes =	ic =
analytic_Pk	one or several of s,t	one or several of ad,bi,cdi,nid,niv
two_scales	one or several of s,t	at most two of ad,bi,cdi,nid,niv
inflation_V	s,t	ad
inflation_H	s,t	ad
inflation_V_end	s,t	ad
external_Pk	one or several of s,t	ad

E. Power spectra in Fourier space: the linear-to-non-linear transition

Halofit or HMcode require non-linearity scale $R(z)$ such that $\sigma(R(z), z) = 1$.



To get $P^{\mathrm{NL}}(k, z)$ ar higher z one should increase $k_{\max}$.

Essential steps in Einstein-Boltzmann solver

F. Transfer functions in harmonic space: compact source functions

Well known

$$\Delta_\ell(k) = \int_\epsilon^{\tau_0} d\tau \; S_T(\tau,k) \; j_\ell(k(\tau_0-\tau))$$

$$\text{with} \quad S_T(\tau,k) \equiv \underbrace{g\,(\Theta_0+\psi)}_{\text{SW}} + \underbrace{\left(g\,k^{-2}\theta_{\rm b}\right)'}_{\text{Doppler}} + \underbrace{e^{-\kappa}(\phi'+\psi')}_{\text{ISW}} + \text{polarisation}$$

comes from integration by part of:

$$\begin{aligned}\Delta_l(k) = \int_{\tau_{\rm ini}}^{\tau_0} d\tau \; \Big\{ & S_T^0(\tau,k)\; j_l(k(\tau_0-\tau)) \\ & + S_T^1(\tau,k)\; \frac{dj_l}{dx}(k(\tau_0-\tau)) \\ & + S_T^2(\tau,k)\; \frac{1}{2}\left[3\frac{d^2 j_l}{dx^2}(k(\tau_0-\tau)) + j_l(k(\tau_0-\tau))\right]\Big\}\end{aligned}$$

But $(S_T^1)'$, $(S_T^2)'$, $(S_T^2)''$ problematic! (Derivative of Einstein equation, massive neutrinos $\rightarrow$ finite differences...)

Essential steps in Einstein-Boltzmann solver

F. Transfer functions in harmonic space: compact source functions

Example of temperature source function in CAMB:

```
!Maple fortran output - see scal_eqs.map
        ISW = (4.D0/3.D0*k*EV%Kf(1)*sigma+(-2.D0/3.D0*sigma
            -2.D0/3.D0*etak/adotoa)*k &
              -diff_rhopi/k**2-1.D0/adotoa*dgrho/3.D0+(3.D0*
                  gpres+5.D0*grho)*sigma/k/3.D0 &
              -2.D0/k*adotoa/EV%Kf(1)*etak)*expmmu(j)
!The rest, note y(9)->octg, yprime(9)->octgprime (octopoles)
    sources(1)= ISW +  ((-9.D0/160.D0*pig-27.D0/80.D0*ypol
        (2))/k**2*opac(j)+(11.D0/10.D0*sigma- &
    3.D0/8.D0*EV%Kf(2)*ypol(3)+vb-9.D0/80.D0*EV%Kf(2)*octg
        +3.D0/40.D0*qg)/k-(- &
    180.D0*ypolprime(2)-30.D0*pigdot)/k**2/160.D0)*dvis(j)
        +(-(9.D0*pigdot+ &
    54.D0*ypolprime(2))/k**2*opac(j)/160.D0+pig/16.D0+clxg
        /4.D0+3.D0/8.D0*ypol(2)+(- &
    21.D0/5.D0*adotoa*sigma-3.D0/8.D0*EV%Kf(2)*ypolprime(3)+
        vbdot+3.D0/40.D0*qgdot- &
    9.D0/80.D0*EV%Kf(2)*octgprime)/k+(-9.D0/160.D0*dopac(j)*
        pig-21.D0/10.D0*dgpi-27.D0/ &
    80.D0*dopac(j)*ypol(2))/k**2)*vis(j)+(3.D0/16.D0*ddvis(j
        )*pig+9.D0/ &
    8.D0*ddvis(j)*ypol(2))/k**2+21.D0/10.D0/k/EV%Kf(1)*vis(j
```

Essential steps in Einstein-Boltzmann solver

F. Transfer functions in harmonic space: compact source functions

So we should rather stick to

$$\Delta_l(k) = \int_{\tau_{\rm ini}}^{\tau_0} d\tau \, \Big\{ S_T^0(\tau,k)\; j_l(k(\tau_0-\tau)) + S_T^1(\tau,k)\; \frac{dj_l}{dx}(k(\tau_0-\tau)) + S_T^2(\tau,k)\; \frac{1}{2}\left[3\frac{d^2 j_l}{dx^2}(k(\tau_0-\tau)) + j_l(k(\tau_0-\tau))\right]\Big\}$$

CLASS v2.0 stores separately $S_T^0(\tau,k)$, $S_T^1(\tau,k)$, $S_T^2(\tau,k)$, and the transfer module will convolve them individually with respective bessel functions.

$$S_T^0 = g\left(\frac{\delta_g}{4}+\psi\right) + e^{-\kappa}(\phi'+\psi') \qquad S_T^1 = g\frac{\theta_b}{k} \qquad S_T^2 = \frac{g}{8}\left(G_0+G_2+F_2\right)$$

or

$$S_T^0 = g\left(\frac{\delta_g}{4}+\phi\right) + e^{-\kappa}2\phi' + g'\theta_b + g\theta_b' \qquad S_T^1 = e^{-\kappa}k(\psi-\phi) \qquad S_T^2 = \frac{g}{8}\left(G_0+G_2+F_2\right)$$