Lecture 1: C^* algebras and basic notions.

Ref: Section 1.6 Wolf Quantum Channels (also, Murphy C^* algebras)

Def: C^* algebra $\mathcal{A}$ (over $\mathbb{C}$) ← algebra = vector space + multiplication.

- $(\mathcal{A}, \|\cdot\|)$ is a Banach space, and multiplication obeys

$$\|AB\| \leq \|A\|\,\|B\| \quad \forall A, B \in \mathcal{A}.$$ ← ensures continuity

- $*$-algebra: there is an involution $(\cdot)^*: \mathcal{A} \to \mathcal{A}$ s.t.

$$(AB)^* = B^* A^* \qquad \forall A, B \in \mathcal{A},$$
$$(cA)^* = \bar{c} A^* \qquad c \in \mathbb{C}.$$
$$(A+B)^* = A^* + B^*$$

- C^* property:

$$\|A^* A\| = \|A\|^2$$

A C^*-algebra satisfies all of these.
Often ask for unital, i.e. $\exists\, \mathbb{1} \in \mathcal{A}$ s.t. $A\mathbb{1} = \mathbb{1}A = A$.

Two central examples:

- $C(X)$, continuous $f: X \to \mathbb{C}$ with X compact

abelian →
  - Sup norm: $\|f\| := \sup_{x \in X} |f(x)|$
  - Complex conjugation: $f^* := \bar{f}$

- $\mathcal{A} \subseteq B(\mathcal{H})$, a norm-closed subalgebra of bounded operators on $\mathcal{H}$. (Hilbert)

nonabelian →
  - Operator norm: $\|A\| := \sup_{\substack{v \in \mathcal{H} \\ v \neq 0}} \frac{\|Av\|}{\|v\|}$
  - Adjoint: $A^* := A^\dagger$

Gelfand: every (unital) C^* algebra is of this form.

Def: • Let $\mathcal{A}$ be unital. $A \in \mathcal{A}$ is invertible if $\exists A^{-1}$ s.t. $AA^{-1} = \mathbb{1} = A^{-1}A$.

• The spectrum of an $A \in \mathcal{A}$ is

$$\mathrm{spec}(A) = \{\lambda \in \mathbb{C} : \lambda\mathbb{1} - A \text{ is not invertible}\}$$

and the resolvent set is $\mathrm{res}(A) = \mathbb{C} \setminus \mathrm{spec}(A)$.

easily adapted for nonunital $\mathcal{A}$. One may uniquely "adjoin" an identity $\mathbb{1}$ to $\mathcal{A}$.

Ex:

① Let $\mathcal{A} = C([0,1])$. Then $\forall f \in \mathcal{A}$,

$$\mathrm{spec}(f) = f([0,1]),$$

since if $\lambda \in f([0,1])$, $\exists x \in [0,1]$ s.t.

$$f(x) = \lambda, \text{ so } \lambda - f(x) = 0$$

$$\Rightarrow \lambda - f \text{ is not invertible.}$$

However, if $\mu \notin f([0,1])$, $\frac{1}{\mu - f} \in \mathcal{A}$, so $\mu \notin \mathrm{spec}(f)$.

② $\mathcal{A} = M_n(\mathbb{C})$, then $\mathrm{spec}(A)$ is the set of eigenvalues of A.

③ Let $\mathcal{A} = B(L^2([0,1]))$, and let $\hat{x} \in \mathcal{A}$ be

$$\hat{x} \cdot f = xf \quad \text{(position operator)}.$$

• $\hat{x}$ has no eigenvalues: if it did,

$$\hat{x} f = \lambda f \Rightarrow f = 0 \text{ a.e.}$$

• $\hat{x}$ has (continuous) spectrum $\mathrm{spec}(\hat{x}) = [0,1]$:

· If $\lambda \notin [0,1]$, then $(x-\lambda)^{-1} f \in L^2([0,1])$ for all $f \in L^2([0,1])$, since

$$\|(x-\lambda)^{-1} f\| \leq \mathrm{dist}(\lambda, [0,1])^{-1} \|f\|.$$

- But if $\lambda \in [0,1]$, $\hat{x} - \lambda \mathbb{1}$ is not onto, since e.g. constant functions c have that

$$(x-\lambda)^{-1} c \notin L^2([0,1]).$$

(λ in the cts spectrum, since $\operatorname{ran}(\hat{x} - \lambda \mathbb{1})$ is dense)

Thus $\lambda \in \operatorname{spec}(\hat{x})$.

$\Rightarrow \operatorname{spec}(\hat{x}) = [0,1]$.

Note: technically, $\operatorname{spec}(A)$ depends on the algebra $\mathcal{A}$.

- But if $\tilde{\mathcal{A}}$ is unital and $\mathcal{A} \subseteq \tilde{\mathcal{A}}$,

$$\operatorname{spec}_{\tilde{\mathcal{A}}}(A) \cup \{0\} = \operatorname{spec}_{\mathcal{A}}(A) \cup \{0\}.$$

(See Murphy Thm 2.1.11 + bottom of pg 44 & top of pg 45)

- So it usually is fine to just write $\operatorname{spec}(A)$.

- <u>Neumann Series:</u> given $A \in \mathcal{A}$ and $\lambda \in \mathbb{C}$ with $\|A\| < |\lambda|$,

$$\frac{1}{\lambda} \sum_{n=0}^{\infty} \left(\frac{A}{\lambda}\right)^n = (\lambda \mathbb{1} - A)^{-1}$$

(Just think geometric series $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$)

- Using this, not too hard to show
  - $\operatorname{spec}(A) \subseteq \{\lambda \in \mathbb{C} : |\lambda| \leq \|A\|\}$
  - $\operatorname{res}(A)$ is open
  - $\operatorname{spec}(A)$ is closed

- <u>Spectral Radius:</u>

$$\rho(A) := \sup(|\lambda| : \lambda \in \operatorname{spec}(A))$$

<u>Fact:</u>
- $\rho(A) \leq \|A\|$
- $\rho(A) = \lim_{n\to\infty} \|A^n\|^{1/n}$. $\Rightarrow$ norm making $\mathcal{A}$ a C^* algebra is unique
- If $AA^* = A^*A$, $\rho(A) = \|A\|$.

Usual defs and results:

- Hermitian/self-adjoint if $A = A^*$.

  $\mathrm{spec}(A) \subseteq [-\|A\|, \|A\|]$.

- Unitary if $A^*A = AA^* = \mathbb{1}$.

  $\mathrm{spec}(A) \subseteq \{\lambda \in \mathbb{C} : |\lambda| = 1\}$

Positivity:

Def:

We say $A \in \mathcal{A}$ is positive if $A = A^*$ and $\mathrm{spec}(A) \subseteq [0, \infty)$.

write $A \geq 0$, and call $\mathcal{A}_+$ the set of positive elements.

Thm: (Murphy 2.2.1, 2.2.4)

- If $A \geq 0$, there is a unique $B \geq 0$ with $B^2 = A$. ← write: $B = A^{1/2}$
  - This defines the square root $B = A^{1/2}$.
- $A^*A \geq 0$ for all $A \in \mathcal{A}$.
  - This allows us to define $|A| = (A^*A)^{1/2}$

    If A is invertible, we have the polar decomposition

$$A = U|A|$$

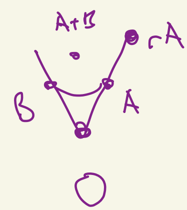
The relation $\geq$ defines a partial ordering on Hermitian operators,

where $A \geq B$ means $A - B \geq 0$.

- $\mathcal{A}_+$ forms a cone, since if $A \geq 0$ and $B \geq 0$, then

  $A + B \geq 0$, and

  $rA \geq 0$ for all $r \in [0, \infty)$.



Basic Facts.

① $\mathcal{A}_+ = \{A^*A : A \in \mathcal{A}\}$

② If $A = A^*$, $B = B^*$ and $C \in \mathcal{A}$,

$$A \le B \Rightarrow C^*AC \le C^*BC$$

③ If $A \ge 0$ and $A \le 0$, $A = 0$.

④ If $A \ge 0$, then $A \le \|A\| \mathbb{1}$.

Def: Positive linear maps

We call a linear map $T : \mathcal{A} \to \mathcal{B}$ positive if

$$T(A^*A) \ge 0 \quad \forall A \in \mathcal{A}.$$

Def: $*$-morphisms

$\pi : \mathcal{A} \to \mathcal{B}$ is called a $*$-morphism if

① $\pi(\alpha A + \beta B) = \alpha \pi(A) + \beta \pi(B)$, $\quad \alpha, \beta \in \mathbb{C}$, $A, B \in \mathcal{A}$

② $\pi(AB) = \pi(A)\pi(B)$

③ $\pi(A)^* = \pi(A^*)$

- Ex: Unitary dynamics

  Let $U^*U = UU^* = \mathbb{1}$. Then

$$\pi(A) = UAU^*$$

  is a $*$-homomorphism.

- A representation is a $*$-hom $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$

  Hilbert space

Properties:

① If $A \geq 0$, then $\pi(A) \geq 0$. ($*$-homs are positive)

pf: write $A = B^2$, so $\pi(A) = \pi(B)^2$ with $\pi(B) = \pi(B^*) = \pi(B)^*$,

② Bounded:

$$\|\pi(A)\| \leq \|A\| \quad \forall A \in \mathfrak{A}$$

If $\pi : \mathfrak{A} \to \mathcal{B}(\mathcal{H})$ a rep and $\ker \pi = \{0\}$ (aka π is faithful)

$$\|\pi(A)\| = \|A\|$$

dual space of Banach space $\mathfrak{A}$

Def: State

$\mathfrak{A}$ a unital C^* algebra. A state $\omega \in \mathfrak{A}^*$ is a linear functional which is

- positive: $\omega(A^* A) \geq 0$
- normalized: $\|\omega\| = 1$

$$\|\omega\| := \sup_{\|A\|=1} |\omega(A)|$$

Thm (Russo-Dye)

Let $\mathfrak{A}, \mathfrak{B}$ be unital C^* algebras and $T : \mathfrak{A} \to \mathfrak{B}$ a positive linear map. Then

$$\|T\| = \|T(\mathbb{1})\|$$

- Cor: A positive linear functional is a state if $\omega(\mathbb{1}) = 1$.

Key ex:

- Vector states. Let $\mathcal{A} = \mathcal{B}(\mathcal{H})$, and pick $|\psi\rangle \in \mathcal{H}$. Then $\omega_\psi \in \mathcal{A}^*$ is a state:

$$\omega_\psi(A) = \langle \psi, A\psi \rangle$$

- If $\dim \mathcal{H} < \infty$, we can equip $\mathcal{A} = \mathcal{B}(\mathcal{H})$ with the Hilbert-Schmidt inner product $\mathrm{Tr}\, A^* B$ to make it a Hilbert space. Then Riesz rep thm guarantees

$$\begin{matrix} \text{states} & \longleftrightarrow & \text{density matrices} \\ \omega \in \mathcal{A}^* & & \rho \in \mathcal{A} \end{matrix}$$

Cauchy-Schwarz: $\omega \in \mathcal{A}^*$ a state. $\forall A, B \in \mathcal{A}$,

1. $\omega(A^* B) = \overline{\omega(B^* A)}$
2. $|\omega(A^* B)|^2 \leq \omega(A^* A)\, \omega(B^* B)$

Next time:
complete positivity, vN algebras, conditional expectation