Lecture 2: von Neumann algebras, Conditional Expectation

Ref: Carlen, Inequalities in Matrix Algebras, ch. 3

Last time...

- C^*-algebras $\left(\text{Think } \begin{array}{l} C(X) \text{ - continuous functions on compact } X \\ B(\mathcal{H}) \text{ - bounded operators on Hilbert } \mathcal{H} \end{array}\right)$

- Spectrum & Positivity.

  Key fact: positive elements $X \geq 0$ can always be written as

  $$B = A^* A$$

  - Positive linear maps preserve positivity: $T: \mathcal{A} \to \mathcal{B}$

  $$T(A^*A) \geq 0$$

  - $*$-Morphisms $\phi: \mathcal{A} \to \mathcal{B}$.
    - Linear: $\phi(zA+B) = z\phi(A) + \phi(B)$
    - $\phi(AB) = \phi(A)\phi(B)$
    - $\phi(A^*) = \phi(A)^*$
    - Automatically positive (actually, they are CP).

  - States are positive linear functionals $\omega: \mathcal{A} \to \mathbb{C}$.

Def: (Carlen Def 3.2)

A von Neumann algebra on a finite-dim $\mathcal{H}$ is a C^*-subalgebra $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ with $\mathbb{1} \in \mathcal{A}$.

- General def: require $\mathcal{A}$ to be closed in the weak operator topology* on $B(\mathcal{H})$, which ensures $\mathcal{A}$ contains the spectral projectors for any $A = A^* \in \mathcal{A}$.

  *weakest topology on $B(\mathcal{H})$ s.t. the "matrix element" functionals $T \mapsto \langle x, Ty \rangle$ are continuous for all $x, y \in \mathcal{H}$.

Revisiting Commutative $\mathcal{A}$:

Let $A = A^* \in B(\mathcal{H})$. By spectral Thm,

$$A = \sum_{\lambda \in \mathrm{spec}(A)} \lambda P_\lambda$$

with P_λ mutually commuting $\perp$ proj and $\sum_\lambda P_\lambda = \mathbb{1}$.

Then the algebra $\mathcal{A}$ generated by $\{P_\lambda : \lambda \in \mathrm{spec}(A)\}$ is a commutative vN algebra.

- This is the same as $C(\mathrm{spec}(A))$

Def: Commutant

Let $\mathcal{A} \subseteq B(\mathcal{H})$ be a C^*-subalgebra.

The <u>commutant</u> is

$$\mathcal{A}' = \{ C \in B(\mathcal{H}) : AC = CA \quad \forall A \in \mathcal{A} \}$$

The <u>center</u> is

$$Z(\mathcal{A}) = \mathcal{A} \cap \mathcal{A}'$$

- vN Double Commutant Thm: if $\mathcal{A}$ is a vN algebra,

$$\mathcal{A}'' = \mathcal{A}$$

<u>Ex:</u>

- $\mathcal{A} = M_n(\mathbb{C}) \subseteq M_n(\mathbb{C})$ has $\mathcal{A}' = Z(\mathcal{A}) = \mathbb{C}\mathbb{1}$

- (Amplification) Consider the $*$-rep $\pi : B(\mathcal{H}_1) \to B(\mathcal{H}_1) \otimes B(\mathcal{H}_2)$

$$A \longmapsto A \otimes \mathbb{1}_{\mathcal{H}_2} = \begin{bmatrix} A & & & \\ & A & & \\ & & \ddots & \\ & & & A \end{bmatrix}$$

Let $\mathcal{A} = \pi(B(\mathcal{H}_1))$. Then

$$\mathcal{A}' = \{ \mathbb{1}_{\mathcal{H}_1} \otimes B : B \in B(\mathcal{H}_2) \}$$

and $Z(\mathcal{A}) = \mathbb{C}\mathbb{1}$

We call a vN alg. $\mathcal{A}$ with a trivial $(Z(\mathcal{A}) = \mathbb{C}\mathbb{1})$ center a <u>factor</u>. For finite dim, factors are always of the form

$$\mathcal{A} = B(\mathcal{H}_1) \otimes B(\mathcal{H}_2)$$

- Ex: Let $A = (M_n(\mathbb{C}) \otimes \mathbb{1}_2) \oplus M_k(\mathbb{C}) \subseteq M_{2n+k}(\mathbb{C})$

These are block matrices

Corresponding projectors:

$$A = \left[\begin{array}{c|c|c} A_1 & & \\ \hline & A_1 & \\ \hline & & A_2 \end{array}\right]\begin{array}{l} \}\, n \\ \}\, n \\ \}\, k \end{array} \qquad \begin{array}{l} \left.\begin{array}{c} \\ \\ \end{array}\right\} P_{2n} \\ \left.\right\} P_k \end{array}$$

Now, A' also includes elements like $\left[\begin{array}{cc|c} c_1\mathbb{1} & c_2\mathbb{1} & \\ c_3\mathbb{1} & c_4\mathbb{1} & \\ \hline & & \mathbb{1} \end{array}\right]$

or $\mathbb{1}_n \otimes \begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix}$

$\Rightarrow \quad Z(A) = \text{span}\{P_{2n}, P_k\}$

For finite dimensions, this is all that can happen

Structure Thm: (Carlen Thm 3.23)

Let $\mathcal{A}$ be a von Neumann algebra on $\mathcal{H}$ with $\dim(\mathcal{H})<\infty$.
Then $\exists$ finite set of projectors $\{P_1,\dots,P_m\}$ with

$$Z(\mathcal{A}) = \mathrm{span}\{P_1,\dots,P_m\}$$

Letting $\mathcal{H}_j := \mathrm{ran}(P_j)$, each $\mathcal{H}_j$ has the form

$$\mathcal{H}_j = \mathcal{H}_j^{(l)} \otimes \mathcal{H}_j^{(r)}$$

and $\mathcal{A}$ consists of operators of $\mathcal{B}(\mathcal{H})$ of the form

$$A = \bigoplus_{j=1}^{m} A_j \otimes \mathbb{1}_{\mathcal{H}_j^{(r)}} \quad , \quad A_j \in \mathcal{B}(\mathcal{H}_j^{(l)})$$

and the commutant $\mathcal{A}'$ consists of operators of the form

$$B = \bigoplus_{j=1}^{m} \mathbb{1}_{\mathcal{H}_j^{(l)}} \otimes B_j \quad , \quad B_j \in \mathcal{B}(\mathcal{H}_j^{(r)})$$

Conditional Expectation

Let $\mathcal{B} \subseteq \mathcal{A}$ be v.N. alg acting on $\mathcal{H}$.
finite dim.

Equip $\mathcal{B}(\mathcal{H})$ with Hilbert-Schmidt inner product

$$\langle A, B \rangle = \mathrm{Tr}\, A^\dagger B.$$

Def: (concrete)

The conditional expectation $\mathbb{E}_{\mathcal{B}} : \mathcal{A} \to \mathcal{B}$ is the orthogonal projection onto $\mathcal{B}$, i.e. for all $A \in \mathcal{A}$, $\mathbb{E}_{\mathcal{B}}(A)$ is the unique element in $\mathcal{B}$ s.t.

$$\mathrm{Tr}\, B^\dagger \mathbb{E}_{\mathcal{B}}(A) = \mathrm{Tr}\, B^\dagger A \quad \forall B \in \mathcal{B}.$$

Def: (abstract)

$\mathbb{E}_{\mathcal{B}} : \mathcal{A} \to \mathcal{B}$ is the unique linear map s.t.

i) $\forall A \in \mathcal{A}, B, C \in \mathcal{B}$,

$$\mathbb{E}_{\mathcal{B}}(BAC) = B\, \mathbb{E}_{\mathcal{B}}(A)\, C$$

ii) $\mathrm{Tr}\, \mathbb{E}_{\mathcal{B}}(A) = \mathrm{Tr}\, A \quad \forall A \in \mathcal{A}.$

Thm: these are equivalent definitions.

Ex: Classical Prob:

Ω_1, Ω_2 are two finite sets.

Let's think about (unnormalized) probability measures on $\Omega = \Omega_1 \times \Omega_2$

i.e. $\mathcal{A} = C(\Omega) = C(\Omega_1) \otimes C(\Omega_2)$ (a basis is given by

Let $\mathcal{B} = C(\Omega_1) \cong C(\Omega_1) \otimes 1 \subseteq \mathcal{A}$. $e_{ij}(x,y) = \delta_i(x)\delta_j(y)$

$i = 1, \dots, |\Omega_1|$

$j = 1, \dots, |\Omega_2|$)

Claim:

$$\mathbb{E}_{\mathcal{B}}(f) = \frac{1}{|\Omega_2|} \sum_{y \in \Omega_2} f(x,y)$$

The marginal.

check:

i) $\mathbb{E}_{\mathcal{B}}^2(f) \overset{(*)}{=} \mathbb{E}_{\mathcal{B}}(f)$

$$\frac{1}{|\Omega_2|} \sum_{y \in \Omega_2} \underbrace{\sum_{ij} a_{ij} \delta_i(x) \delta_j(y)}_{f} = \frac{1}{|\Omega_2|} \sum_{ij} a_{ij} \delta_i(x) \sum_{y \in \Omega_2} \delta_j(y)$$

$$= \frac{1}{|\Omega_2|} \sum_{ij} a_{ij} \delta_i(x) \cdot 1$$

From here, not to hard to see

ii) $\langle g, \mathbb{E}_{\mathcal{B}}(f) \rangle = \langle \mathbb{E}_{\mathcal{B}}(g), f \rangle \quad \forall g, f \in C(\Omega)$

Here, we can think of $C(\Omega)$ as diagonal matrices acting on $\mathcal{H} = C(\Omega)$. Then

$$\langle g, f \rangle = \sum_{xy} g(x,y) \cdot f(x,y), \quad \text{usual } L^2\text{-inner product.}$$

So: $\langle g, \mathbb{E}_{\mathcal{B}}(f) \rangle = \sum_{x,y} \sum_{\substack{i,j,\\k,l}} g_{ij} \delta_i(x) \delta_j(y) \frac{1}{|\Omega_2|} f_{kl} \delta_k(x) \cdot 1 = \langle \mathbb{E}_{\mathcal{B}}(g), f \rangle$

Ex: Partial Trace

Let $\mathcal{H}_1, \mathcal{H}_2$ be finite dimensional.

$$\mathcal{A} = \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$$

$$\mathcal{B} = \mathcal{B}(\mathcal{H}_1) \otimes \mathbb{1}_{\mathcal{H}_2}$$

$$\begin{bmatrix} \boxed{B} & & & \\ & \boxed{B} & & \\ & & \boxed{B} & \\ & & & \boxed{B} \end{bmatrix}$$

Claim: $d_2 = \dim(\mathcal{H}_2)$

$$\mathbb{E}_{\mathcal{B}}(A) = \frac{1}{d_2} \operatorname{Tr}_{\mathcal{H}_2} A \otimes \mathbb{1}$$

Check:

i) $\mathbb{E}_{\mathcal{B}}^2(A) = \mathbb{E}_{\mathcal{B}}(A)$

check on basis of simple tensors $A = A_1 \otimes A_2$.

$$\Rightarrow \quad \mathbb{E}_{\mathcal{B}}(A_1 \otimes A_2) = \frac{1}{d_2} \operatorname{Tr}_{\mathcal{H}_2}(A_1 \otimes A_2) \otimes \mathbb{1}$$

$$= \frac{\operatorname{Tr} A_2}{d_2} A_1 \otimes \mathbb{1}$$

$$\mathbb{E}_{\mathcal{B}}^2(A_1 \otimes A_2) = \mathbb{E}_{\mathcal{B}}(A_1 \otimes A_2).$$

ii) $\langle A, \mathbb{E}_{\mathcal{B}}(B) \rangle = \langle \mathbb{E}_{\mathcal{B}}(A), B \rangle \quad \forall A, B \in \mathcal{A}.$

Again, check on basis.

Ex: Pinching & Measurement

Think about quantum measurement again, so $\mathcal{P} := \{P_1, \dots, P_m\}$ a set of mutually commuting $\perp$ proj. with $\sum_j P_j = \mathbb{1}$.

- Let $\mathcal{A} = \mathcal{B}(\mathcal{H})$, and define

$$M_{\mathcal{P}}(A) = \sum_{j=1}^{m} P_j A P_j \quad , \quad A \in \mathcal{B}(\mathcal{H}).$$

Claim:

$$M_{\mathcal{P}} = \mathbb{E}_{\mathcal{B}} , \text{ where } \mathcal{B} = \operatorname{ran}(M_{\mathcal{P}})$$

- First, need to show that $\mathcal{B}$ is a v.N. algebra. Observe

$$\mathcal{B} = \mathcal{P}'$$

since for all $k = 1, \dots, m$,

$$P_k \Big(\sum_j P_j A P_j \Big) = P_k A P_k = \Big(\sum_j P_j A P_j \Big) P_k$$

So $\mathcal{B} \subseteq \mathcal{P}'$, and if $A \in \mathcal{P}'$, then

$$A = \sum_j A P_j = \sum_j A P_j^2 \underset{\substack{\uparrow \\ A \in \mathcal{P}'}}{=} \sum_j P_j A P_j$$

so $\mathcal{B} \supseteq \mathcal{P}'$.

- Now check that for $A \in \mathcal{A}$ and $X, Y \in \mathcal{B}$

$$M_{\mathcal{P}}(XAY) = \sum_j P_j X A Y P_j \underset{\substack{\uparrow \\ \mathcal{B} = \mathcal{P}'}}{=} \sum_j X P_j A P_j Y = X M_{\mathcal{P}}(A) Y.$$

$$\operatorname{Tr} M_{\mathcal{P}}(A) = \sum_j \operatorname{Tr} P_j A P_j = \sum_j \operatorname{Tr} A P_j = \operatorname{Tr} A \cdot \mathbb{1} = \operatorname{Tr} A.$$

- Note: For every (fin. dim.) conditional expectation, there exists a finite group of unitaries G s.t.,

$$\mathbb{E}_{\mathcal{B}}(A) = \frac{1}{|G|} \sum_{g \in G} g^* A g$$

Thm 3.42 in Carlen

- This implies that $\mathbb{E}_{\mathcal{B}}$ is completely positive!

- More generally, Tomiyama's Thm guarantees that any such conditional expectation $\mathbb{E}_{\mathcal{B}}$ is completely positive. (Brown-Ozawa Thm 1.5.10).