Lecture 5 Perron-Frobenius theorem

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Review: Perron-Frobenius theorem (classical)

A: non-negative matrix, $a_{ij} \geq 0$ Special case: $\sum_j a_{ij} = 1 \ \forall i$, transition matrix $\pi_n P = \pi_{n+1}$

- A is irreducible if (a_{ij}) formulates a strongly connected graph.

$\Leftrightarrow$ For any coordinate subspace $C = \mathrm{span}\{e_{i_1}, \dots, e_{i_k}\}$

$A(C) \nsubseteq C$. (It is not invariant space).

Thm: If A is non-negative, irreducible,

$\exists r \in \mathbb{R}_+$ s.t. ① r is an eigenvalue of A, $\rho(A) = \max\{|\lambda(A)|\} = r$

② r is simple eigenvalue (dim of eigenspace $= 1$)

③ $\min_i \sum_j a_{ij} \leq r \leq \max_i \sum_j a_{ij}$ $\Rightarrow$ If P is transition matrix, $r = 1$.

Goal: Extend this to positive maps on M_d.

$T: M_d(\mathbb{C}) \longrightarrow M_d(\mathbb{C})$ a positive linear map, i.e. $T(|\psi\rangle\langle\psi|) \geq 0$ $\forall |\psi\rangle$.

Thm 6.2 (Irreducibility). The following properties are equivalent:

(1). If P is a Hermitian projector such that $T(PM_dP) \subseteq PM_dP$, then $P \in \{0, \mathbb{I}\}$ $\rightarrow$ no non-trivial invariant subspace $\longrightarrow$ $\Rightarrow \mathrm{tr}((\mathbb{I}-P)\,T(P)) = 0$, P "reduce" T.

(2). For $\forall A \geq 0$, $(id + T)^{d-1}(A) > 0$

(3). For $\forall A \geq 0$, $t > 0$, $\exp(tT)(A) > 0$

(4). For any orthogonal pair of non-zero, positive definite $A, B \in M_d(\mathbb{C})$ $\exists t \in \{1, \dots, d-1\}$ s.t. $\mathrm{tr}(BT^t(A)) > 0$. $\rightarrow$ connected.

Re: 1. According to (4), T is irreducible $\Leftrightarrow T^*$ is irreducible.

2. According to (1). Given any invertible C, $T'[\cdot] = C^{-1}T(C[\cdot]C^\dagger)C^{-\dagger}$ is irreducible if and only if T is irreducible. (Prop 6.6)

Q: projection onto $C^{-1}PC^{-\dagger}$

$PA'P$

$T'(QAQ) = C^{-1}T(CQAQC^\dagger)C^{-\dagger}$
$= C^{-1}PBPC^{-\dagger}$
$= (C^{-1}PC^{-\dagger})C^\dagger BC(C^{-1}PC^{-\dagger})$
$= Q(C^\dagger C)Q$

Pf: (1) $\Rightarrow$ (2). Because $T(A) \geq 0$,

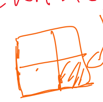
$\ker((id+T)(A)) = \ker(A + T(A)) \subseteq \ker(A)$ (Want to show $\ker(\sim) \subsetneq \ker(A)$) proper subset. or $A > 0$

① Suppose $\ker(A + T(A)) = \ker(A) \Rightarrow \mathrm{supp}\, T(A) \subseteq \mathrm{Supp}\, A$

Let P be the Hermitian projection on $\mathrm{supp}\, A$.

then $\boxed{T(PM_dP) \subseteq PM_dP}$ why?? $T(|\psi_i\rangle\langle\psi_i|) \subseteq PM_dP$, $\forall i$

$T(|\psi_i\rangle\langle\psi_j|) + T(|\psi_j\rangle\langle\psi_i|) \subseteq PM_dP$. $\rightarrow \mathrm{Re}[T(|\psi_i\rangle\langle\psi_j|)]$



① has support $|\psi_j\rangle \rightarrow -|\psi_j\rangle$

② $a|\psi_i\rangle + \sqrt{1-a^2}|\psi_j\rangle$

$2a\sqrt{1-a^2}\langle\psi_i|A|\psi_j\rangle + \langle\psi_j|A|\psi_j\rangle + a^2\langle\psi_i|B|\psi_i\rangle \geq 0$

$\Rightarrow A > 0$

② Otherwise $\ker((id+T)(A)) \subsetneq \ker(A)$, rank $\uparrow$ after applying $(id+T)$.

(2) $\Rightarrow$ (3). $\exists c > 0$, $\exp(tT)(A) \geq c\,(id+T)^{d-1}(A)$.

(3) $\Rightarrow$ (1) Suppose such nontrivial P exists, $\exists c$

$T(P) \leq cP$

$\Rightarrow \exp(tT)(P) \leq \exp(tc)P$ not positive strictly contradiction

(4) $\Rightarrow$ (1). - - - - - - - - - - - - -

$\mathrm{tr}((\mathbb{1}-P)T^t(P)) = 0$, $\forall t$

(2) $\Rightarrow$ (4). Expand: $\mathrm{tr}[B(id+T)^{d-1}A] > 0$, since $\mathrm{tr}[BA] = 0$

$\exists t \in [1, d-1]$, $\mathrm{tr}[BT^tA] > 0$ $\square$

Spectral properties of T: Given $X \geq 0$,

$r(X) = \sup\{\lambda \in \mathbb{R} \mid (T - \lambda id)(X) \geq 0\}$

$\tilde{r}(x) = \inf\{\lambda \in \mathbb{R} \mid (T-\lambda id)(x) \le 0\}$ $\tilde{r}(x) \geq r(x)$

we define: $r = \sup_{X \geq 0} r(x)$, $\tilde{r} = \overset{\text{inf ??}}{\sup_{X\geq 0}} \tilde{r}(x)$. $\Rightarrow \tilde{r} \geq r$??

can be achieved. $tr(x)=1$

Thm: G3 (Spectrum radius of T). $T: M_d(\mathbb{C}_d) \to M_d(\mathbb{C}_d)$ be irreducible.

Then.

① $r = \tilde{r}$

② r is non-degenerate eigenvalue of T, $\exists X > 0$. $TX = rX > 0$

③ If $\exists Y \geq 0$, $\lambda > 0$ s.t $TY = \lambda Y$, then $\lambda = r$.

④ $r = \rho(T)$

Pf: We first show r is achieved by some $X > 0$ and $TX = rX$.

We notice: $(T+id)^{d-1}(T-\lambda id)(x) = (T-\lambda id)(T+id)^{d-1}(x)$ (✱)

$\Rightarrow$ ① Given $X \geq 0$, if $\lambda = r(x)$, ① If (✱) $\geq 0, \neq 0$ $\underbrace{(T-\lambda id)(x) \geq 0, \neq 0}$ ② If (✱) $= 0$, $(T+id)^{d-1}(x)$ is eigvec with λ eigenvalue.

$r((T+id)^{d-1}(x)) > \lambda$. $\Rightarrow$ r can be achieved by some $X > 0$

② If $r(x) = r$, we have $(T - r id)(x) = 0$. otherwise

$r((T+id)^{d-1}(x)) > r$. contradiction.

we show the same result for $\tilde{r}$. ✱ correct proof.

① Given $X \geq 0$, $\lambda = \tilde{r}(x)$, (✱) < 0

$\Rightarrow \tilde{r}((T+id)^{d-1}(x)) < \lambda$ $\Rightarrow$ $\tilde{r}$ can be achieved by some $X > 0$

② If $\tilde{r}(x) = \tilde{r}$, we must have $(T - \tilde{r} id)(x) = 0$

In addition. if $X \geq 0$ is an eigenvector, $\tilde{r}(x) = r(x)$. $\Rightarrow \tilde{r} = r$.

Proof (2): Suppose X' is another eigenvector. that is not multiple of X.

Assume $X' = (X')^{+}$ w.l.o.g.

since $X > 0$, $\exists c \in \mathbb{R}$. $X + cX' \geq 0$ has a kernel

However. $(r+1)^{d-1}(X + cX') = (T+id)^{d-1}(X + cX') > 0$. contradiction.

Proof (3): If $TY = \lambda Y$, for $\lambda > 0$, $Y \geq 0$

chose $\hat{X} > 0$ be $T^{*}\hat{X} = r\hat{X} > 0$ must exist. (If not, apply above ~~the~~ proof again to find contradiction)

$r\,tr(\hat{X}Y) = tr(T^{*}(\hat{X})Y) = \lambda \underbrace{tr(\hat{X}Y)}_{>0}$ $\Rightarrow r = \lambda$.

Proof (4). $T'(\cdot) = X^{-1/2}\, T(X^{\frac{1}{2}}(\cdot)X^{\frac{1}{2}})\, X^{-\frac{1}{2}}/r$ with $X > 0$, $TX = rX$

$\Rightarrow \rho(T') = \rho(T)/r$

$T'(\mathbb{1}) = 1$ $\Rightarrow \rho(T') = 1$ $\Rightarrow \rho(T) = r$

Last Lecture, unital positive map has radius 1.

Theorem G4: T is a positive map with $r = \rho(T)$

T is irreducible $\Leftrightarrow$

r is a non-degenerate eigenvalue and the corresponding right and left eigenvectors are positive definite ($TX = rX > 0$, $T^*Y = rY > 0$)

Proof: "$\Rightarrow$" previous theorem

"$\Leftarrow$": $T' = \frac{1}{r} Y^{\frac{1}{2}} T(Y^{-\frac{1}{2}}[\cdot]Y^{-\frac{1}{2}})Y^{\frac{1}{2}}$

Trace preserving: $tr[T'(A)] = \frac{1}{r} tr(T^*(Y) Y^{-\frac{1}{2}} X Y^{-\frac{1}{2}}) = tr(A)$.

$\Rightarrow \rho(T') = 1$, → Property of PTP map

Also $T'(\sqrt{Y} X \sqrt{Y}) = \sqrt{Y} X \sqrt{Y} > 0$ (non-degenerate eigenvalue).

If T' is reducible, $\exists$ nontrivial P. $T' : PM_dP \to PM_dP$.

$X \notin$ because $X > 0$

Then we can find a fixed point density operator $\rho \geq 0$ $T'(\rho) = \rho$, contradicted to non-degenerate. prove in section 6.4.

Corollary 6.3. T is a PTP map

T is irreducible $\Leftrightarrow$ $\exists$ unique $\sigma > 0$ s.t.
$$\lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} T^t(\rho) = \sigma, \quad \forall \text{ density } \rho.$$

Proof: "$\Rightarrow$". trivial.

"$\Leftarrow$": Assume T is reducible, we can find the contradiction same as before.

claim: Irreducible map is dense: For positive T.

$T'(x) = T(x) + \varepsilon \mathbb{1} tr(X)$ is always irreducible. Since $T'(x) > 0, \forall x \geq 0$

denseness $\Rightarrow$ Thm 6.5. T is a positive map with $r = \rho(T)$. Then r is an eigenvalue, $\exists X \geq 0$, $T(X) = rX$.

Positive + unital + irreducible. + Schwartz inequality (*) $T^*(A^\dagger)T^*(A) \leq T^*(A^\dagger A)$

every eigenvalue $|\lambda| = \rho(T) = 1$ is non-degenerate.

Thm 6.6: If (*) is true. Define $S = spec(T) \cap \{|\lambda| = 1\}$ peripheral spectrum.

1. $\exists m \in \{1, \ldots, d^2\}$, $S = \{\exp(2\pi i k/m)\}$, $k = 0, 1, \ldots, m-1$
2. All eigenvalues on S are non-degenerate
3. $\exists$ unitary U, $T(U^k) = \gamma^k U^k$ $\Rightarrow$ eigenvector, $\gamma = \exp(2\pi i/m)$
4. U has spectral decomposition
$$U = \sum_{k \in \mathbb{Z}_m} \gamma^k P_k, \quad \text{where } T(P_{k+1}) = P_k.$$

Full characterization of eigenspace on S.

$T^m(P_k) = P_k \Rightarrow$ if $m > 1$, T^m is reducible.