Semigroup Structure

"Quantum Channels can be semigroups".

Continuous time evolution $\Rightarrow$ continuous 1-param semigroup.

Def. (Dynamical semigroups)

For a set Σ of 'observables' or 'states', a family of maps $T_t : \Sigma \to \Sigma$, $t \in \mathbb{R}_+$ is called <u>dynamical semigrp</u> if for all $t, s \in \mathbb{R}_+$

$$T_t \circ T_s = T_{t+s} \quad \text{and} \quad T_0 = id$$

{ explain solution to SE, or the evolution of its propagator.

think of time evolution.

$\uparrow$ semigroup property

$\Downarrow$ Associativity of semigroup multiplicatⁿ.

$\mapsto$ Markovian (doesn't depend on history)

$\mapsto$ Homogenous (doesn't depend on actual time)

} again think of PDE analogy.

→ So for purely algebraic definition.

{ e.g. flows of vector fields.

1-param group of diffeomorphisms.

e.g. $\frac{d}{dt}$

→ For now think of Σ as a finite dim Banach space.

Continuity & Differentiability

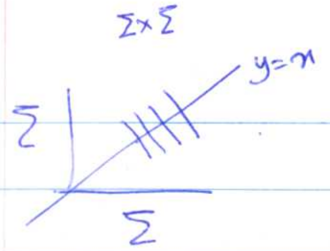
- T_t depends continuously on t. How to impose this? (diff. choices usually motivated by nature of problem we are studying)

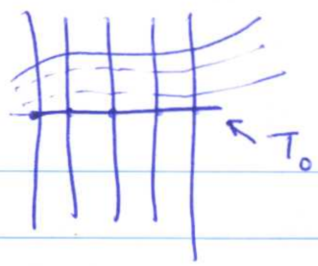
→ <u>Strong convergence</u>:

$T_t \to T_{t_0}$ converges strongly when $t \to t_0$ if $\| T_t(x) - T_{t_0}(x) \| \to 0 \quad \forall x \in \Sigma$

(think of pointwise convergence)

Fibre Bundle Picture : $\begin{array}{c}\Sigma \\ \downarrow \\ \Sigma\end{array}$





→ Uniform Convergence or Norm Convergence :

$$\|T_t - T_{t_0}\| \to 0, \qquad \|T\| := \sup_{x \in \Sigma} \frac{\|T(x)\|}{\|x\|}$$

Uniform $\Rightarrow$ Strong $\Rightarrow$ Weak

equivalent for dynamical semigroups

[Think of convergence in operator norm]

All of them coincid on linear spaces + linear maps.

[* The distinction is very important when talking about infinite dim spaces.] → Also define "strong continuity".

From now think of: $\Sigma = M_d(\mathbb{C}) \simeq \mathbb{C}^{d^2} \simeq \underbrace{\mathbb{C}^d \otimes \cdots \otimes \mathbb{C}^d}_{d \text{ times}}$

$\qquad\qquad\qquad\qquad\quad \overset{\shortparallel}{\rho^d}$

Infinitesimal generator.

e.g. $T_t = e^{tL}$, $L \in M_d(\mathbb{C})$, satisfies $\frac{d}{dt} T_t = L\, T_t$ → (*)

L is called "generator" or "infinitesimal generator" of the semigroup.

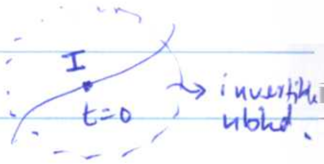
ODE Uniqueness ↑

Conversely, if (*) holds for a differentiable map $t \mapsto T_t \in M_d(\mathbb{C})$, $T_0 = id$, then $T_t = e^{tL}$ with $L = \left.\frac{d}{dt} T_t\right|_{t=0}$

(**) For finite dimensional dynamical semigroups, continuity $\Leftrightarrow$ differentiability.

Thm: $\{T_t \in M_d(\mathbb{C})\}$ some dynamical semigroup continuous in $t \in \mathbb{R}_+$. Then T_t is differentiable for $t \in \mathbb{R}_+$ and of the form $T_t = e^{tL}$ for some $L \in M_d(\mathbb{C})$.

Pf: $T_0 = \mathbb{I}$, T_t cont. in t, & so,



$$M_\epsilon := \int_0^\epsilon T_s \, ds \in M_d(\mathbb{C})$$ is invertible for $\epsilon > 0$, but sufficiently small.

For t fixed.

$$T_t = M_\varepsilon^{-1} M_\varepsilon T_t = M_\varepsilon^{-1} \int_0^\varepsilon T_{s+t} \, ds$$ [No issues in pulling T_t inside integral]

$$= M_\varepsilon^{-1} \int_t^{t+\varepsilon} T_s \, ds = M_\epsilon^{-1} (M_{t+\varepsilon} - M_t)$$

$\Rightarrow T_t$ is differentiable at $t \in \mathbb{R}_+$. $\Rightarrow$ rest by ODE uniqueness.

So, $\exists$ L s.t. $T_t = e^{tL}$

$$L = \frac{d}{dt} T_t \Big|_{t=0}$$

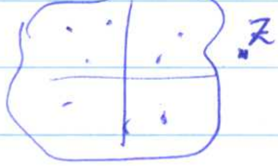
Cor: T_t can be embedded into a group by extending the range of t to $\mathbb{R}$ or $\mathbb{C}$.

Resolvents:

$L \in M_d(\mathbb{C})$, $\qquad R(z) = (zI - L)^{-1}$

$$\rho(L) = \mathbb{C} \setminus \sigma(L)$$

σ: spectrum.



Resolvent of generator of dynamical semigroup:

$$R(z) = \int_0^\infty e^{-zs} T_s \, ds \qquad z \in \rho(L)$$

if $\mathrm{Re}(z) > \sup\{\mathrm{Re}(\lambda) : \lambda \in \sigma(L)\}$

Growth bound $\|T(t)\| \le Me^{\omega t}$ [$\omega > \sup\{\mathrm{Re}(\lambda) : \lambda \in \sigma(L)\}$]

Pf. ① $\hat{T}(\lambda) := \int_0^\infty e^{-\lambda t} T_t \, dt$ → converges in op. norm due to exponential bd. on $\|T(t)\|$.

② Define $R(\lambda) := \hat{T}(\lambda)$.

$R(\lambda)(\lambda I - L)x = x$ [To show].

$(\lambda I - L) \int_0^\infty e^{-\lambda t} T(t) x \, dt$

$$= \int_0^\infty e^{-\lambda t} \left(\lambda T(t) x - T'(t) x \right) dt$$

by parts integrati

$$= \underbrace{\lambda \int_0^\infty e^{-\lambda t} T(t) x \, dt}_{} - \lambda \int_0^\infty e^{-\lambda t} T(t) x \, dt - \left[e^{-\lambda t} T(t) x \right]_{t=0}^{t=\infty}$$

$$\lambda \hat{T}(\lambda) x - \lambda \hat{T}(\lambda) x - 0 + T(0) x = x.$$

Conversely, if resolvent of L given, then we can obtain the dynamical semigroup via the expressions:

① $$T_t = \frac{1}{2\pi i} \int_{\partial \Delta} e^{zt} R(z) \, dz$$ [Cauchy integral formula].

$$= \frac{1}{2\pi i} \int_{\partial \Delta} e^{zt} (zI - L)^{-1} dz = e^{tL} = T_t$$

$\Delta \supset \sigma(L)$.

② $$T_t = \lim_{n \to \infty} \left(\frac{n}{t} \right)^n R\left(\frac{n}{t} \right)^n,$$ $$e^{tL} = \lim_{n \to \infty} \left(I - \frac{tL}{n} \right)^{-n}$$

[Euler's approximati]

Perturbations.

$$T_t = e^{tL}, \quad T_t' = e^{tL'}, \quad \Delta := L' - L$$

(Thm) $\{T_t \in M_d(\mathbb{C})\}$, $\{T_t' \in M_d(\mathbb{C})\}$ two continuous dynamical semigroups. Define $\Delta := \frac{d}{dt}(T_t' - T_t)\Big|_{t=0}$, i.e. diff of generators. Then.

$$T_t' = T_t + \int_0^t T_{t-s} \Delta T_s' \, ds. \quad (***)$$

Pf: Define $f(s) := T_{t-s} T_s'$, t fixed "think as a parameter"

$$\frac{d}{ds} f(s) =: f'(s) = T_{t-s}(L' - L) T_s', \quad \text{[simple diff of matrices]}$$

Then, $T_t' - T_t = f(t) - f(0) = \int_0^t f'(s)\, ds = \int_0^t T_{t-s} \Delta T_s' \, ds.$

(Corollary) (Perturbation of generators)

Setting same as above. Then for any norm and $t \in \mathbb{R}_+$

$$\| T_t' - T_t \| \leq t \|\Delta\| \sup_{s, s \in [0,t]} \|T_s\| \, \|T_s'\|.$$

$\downarrow$

further simplification when unitary evolution $\|T_s\| = 1$, $\|T_s'\| = 1$.

[e.g. cb-norm of quantum channels].

Another implication

"Dyson-Philips series"

$\searrow$ if both semigroups are unitary "Dyson series"

Insert $(***)$ into itself.

$$T_t' = \sum_{n=0}^{\infty} \tilde{T}_t^{(n)}, \quad \text{where, } \tilde{T}_t^{(n+1)} = \int_0^t T_{t-s} \Delta \tilde{T}_t^{(n)}, \quad \text{with } \tilde{T}_t^{(0)} = T_t.$$