Banach Algebra: $\mathcal{A}$ over $\mathbb{C}$

- $(\mathcal{A}, \|\cdot\|)$ is a Banach space with multiplication, s.t.

$$\|AB\| \leq \|A\| \|B\|$$

- Banach $*$-algebra has an "adjoint" $*: \mathcal{A} \to \mathcal{A}$, antilinear
  - $(A^*)^* = A$
  - $(AB)^* = B^* A^*$
  - $(\lambda A + B)^* = \bar{\lambda} A^* + B^*$

- C^*-algebra satisfies C^* property:

$$\|A^* A\| = \|A\|^2$$

we will consider unital C^* algebras, i.e. there is $1 \in \mathcal{A}$.

Key examples:

- Continuous complex-valued functions $C(X)$ with ptwise multiplication & sup norm & complex conjugate:
  - $(fg)(x) = f(x) g(x)$
  - $\|\bar{f} f\| = \|f\|^2$

- Norm-closed subalgebras of $B(\mathcal{H})$, bounded linear operators on Hilbert space $\mathcal{H}$ with adjoint A^+ and op. norm.

C^*-algebras for Stat. Mech.

First: a classical story.

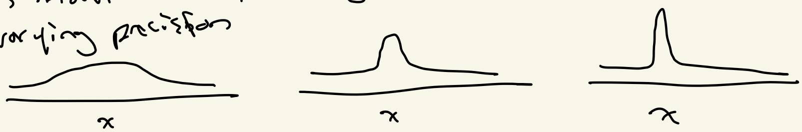
"States are determined by expectation values, and we want to take limits of states, so find a Banach space"

- We want to statistically describe ensembles of particles on phase space M. Think "particles moving in a potential with various initial conditions". Let's just take $M = \mathbb{R}$, position.
- We have measurement tools: observables.

These are certain functions on M, e.g. the

"Is there a particle at position x" function,

  - I don't have perfect measurement tools. Let's model this by saying I have continuous functions (even smooth) of varying precision



ultimately, the choice of observables and topology is dictated by application.

- My lab is finite, so I can only measure compactly supported observables.

I want to measure moments and correlations of ensembles, I can "multiply" these observables (pointwise)

- Finally, I should equip this space of observables with a norm $\|\cdot\|$ to measure distances, and $\lim_{n\to\infty} A_n = A \in \mathcal{A}$

$\Rightarrow$ So, our observables form a Banach $\underbrace{\text{algebra}}$:

can multiply elements

Banach space: vector space + norm $\|\cdot\|$, complete space.

with sup norm $\|\cdot\|$

$$\mathcal{A} = (C_0(\mathbb{R}), \cdot), \text{ where } (f\cdot g)(x) = f(x)g(x).$$

What can we obtain from experiments?

Expectations of observables, which are real (or complex) numbers

- Demand: these should depend continuously on our observables.

$\Rightarrow$ States (ensembles) live inside the (continuous dual) space

$$A^* = \{ \omega : A \to \mathbb{R} : \omega \text{ is continuous} \}$$

What are these?

Riesz-Markov-Kakutani

Let $\omega \in A^*$. Then there is a unique (regular, Borel) measure μ on $\mathbb{R}$ such that for all $A \in A$,

$$\omega(A) = \int_{x \in \mathbb{R}} A(x)\, d\mu(x).$$

- We don't have "negative" particles. Enforce positivity:
  - if $A \geq 0$, then $\omega(A) \geq 0$.
- We want to "normalize", so enforce
  - $\omega(1) = \int d\mu = 1$

$\Rightarrow$ These are probability measures μ!

Some ensembles...

- $\omega(\cdot) = \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} (\cdot)\, e^{-x^2}\, dx$ Gaussian
- $\omega(A) = \int_{\mathbb{R}} A(x) \underbrace{\delta_p(x)}\, dx = A(p)$ Dirac at point p

Note - not a continuous function!

We want to take limits of states!

Again, this is dictated by expectation values:

want $\quad \omega_n \to \omega \quad$ to mean

$$\omega_n(A) \longrightarrow \omega(A) \quad \text{for all } A \in \mathfrak{A}$$

The weak-* topology

Let $\omega_n : \mathfrak{A} \to \mathbb{C}$ be a sequence of linear functionals on $\mathfrak{A}$.

We say $\omega_n \longrightarrow \omega$ in the weak-* topology if

$$\omega_n(A) \to \omega(A) \quad \text{for all } A \in \mathfrak{A}$$

Big Payoff!

easy to show weak-* closed.

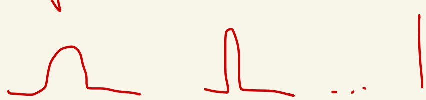
By Banach-Alaoglu, the set of states $S \subseteq \mathfrak{A}^*$ is weak-* compact!

- We can meaningfully take limits of states and get states:

$$\omega_n \longrightarrow \omega$$

- If we instead started with "a state is a probability density function $\rho(x)$", this is not true! Easy to build sequence

$$\omega_n(\cdot) = \int (\cdot)\, \rho_n(x)\, dx$$



$\rho_1 \quad \rho_2 \quad \ldots \quad \rho_\infty$ ← this is not a pdf.

... But it does limit to a state!

- Every sequence of states has a convergent subsequence (automatically guarantees existence of thermodynamic limiting states!)

- States form a convex set in A^*, and the extremal states are called pure.
  - In this example, pure states are dirac measures δ_x.
- The state space is automatically determined by the algebra!
  - In practice, this often arises a representation of a familiar algebra.
    - CAR, CCR!

Quantum Stat. Mech.

- The entire story here is the same, with one key difference: while the observable algebra for a classical system is commutative, quantum systems have noncommuting observables, like position and momentum

Ex: Observables and states for the AKLT chain.

Consider a spin-1 chain:

$$\begin{array}{cccccc} & & x & & & \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \mathbb{C}^3 & \mathbb{C}^3 & \mathbb{C}^3 & \dots & & \end{array}$$

Let's build our algebra of observables. Analogously to classical example, want to consider only local observables (analogous to compactly supported).

- On-site algebra: if $x \in \mathbb{Z}$,

$$\mathcal{A}_x = M_3(\mathbb{C}), \text{ with operator norm } \|\cdot\|.$$

- A local observable is $A \in \bigotimes_{x=-\ell}^{\ell} \mathcal{A}_x$.

Given $[-\ell,\ell] \subseteq \mathbb{Z}$, we may naturally embed

$$\mathcal{A}_{[-\ell,\ell]} = \mathbb{1} \otimes \mathcal{A}_{[-\ell,\ell]} \otimes \mathbb{1} \subseteq \mathcal{A}_{[-\ell,\ell]}$$

- Algebra of local observables: (inductive limit)

$$\mathcal{A}_{loc} = \bigcup_{\ell \in \mathbb{N}} \mathcal{A}_{[-\ell,\ell]}.$$

- Take norm closure to get algebra of quasi-local observables

$$\mathcal{A} := \overline{\mathcal{A}_{loc}}^{\|\cdot\|}$$

A state $\omega : \mathcal{A} \to \mathbb{C}$ is a linear functional satisfying

- if $A \geq 0$, then $\omega(A) \geq 0$

  Note: $A^\dagger A \geq 0$, and we have a notion of square root, so this is often written

  $$\omega(A^\dagger A) \geq 0$$

- $\omega(\mathbb{1}) = 1$.

· Notice that since $\mathcal{A}_{loc}$ is dense in $\mathcal{A}$, ω is determined by expectation values $\omega(A)$, $A \in \mathcal{A}_{loc}$

$\Rightarrow$ so ω can be described as a thermodynamic limit of finite chain states! [unclear how to do this for $|\psi^{(\ell)}\rangle$, but clear for $\omega^{(\ell)}$)

[A comment: if we had instead naively written

$$\mathcal{H} = \bigotimes_{x \in \mathbb{Z}} \mathbb{C}^3,$$

the algebra $\mathcal{B}(\mathcal{H})$ is far too big.

For instance, if $Z = \begin{pmatrix} 1 & & \\ & 0 & \\ & & -1 \end{pmatrix}$ with $|1\rangle, |0\rangle, |-1\rangle$,

element $\bigotimes_{x \in \mathbb{Z}} Z_x$ is in $\mathcal{B}(\mathcal{H})$, but we cannot

approximate it by locals, since

$$\left\| \bigotimes_{x \in \mathbb{Z}} Z_x - \bigotimes_{x=-\ell}^{\ell} Z_x \right\| \geq 2, \text{ by picking } |\psi\rangle = |-1\rangle^{\otimes \ell+1}.$$

So we don't have good control on thermodynamic limiting states.]

Now, lets define our Hamiltonian!

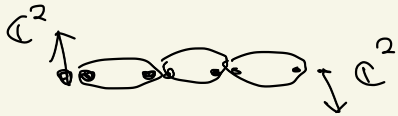
$$H_{[-\ell,\ell]} = \sum_{x=-\ell}^{\ell-1} h_{x,x+1}$$

$$= \sum_{x=-\ell}^{\ell-1} \frac{1}{3}\mathbb{1} + \frac{1}{2}\vec{S}_x \cdot \vec{S}_{x+1} + \frac{1}{6}\left(\vec{S}_x \cdot \vec{S}_{x+1}\right)^2$$

In particular, $H_{[-\ell,\ell]} \geq 0$.

we can describe $\ker H_{[-\ell,\ell]}$ in several ways, including as a valence bond state (or MPS)

VBS:



$$\dim \ker H_{[-\ell,\ell]} = 2 \cdot 2 = 4$$

MPS: Let $|\alpha\rangle, |\beta\rangle \in \mathbb{C}^2$.

$$\psi^{(\ell)}_{\alpha\beta} = \sum_{i_{-\ell},\dots,i_\ell} \operatorname{Tr} |\alpha\rangle\langle\beta| \, t_{i_\ell} \cdots t_{i_{-\ell}} \, |i_{-\ell} \cdots i_\ell\rangle$$

$$H_{[-\ell,\ell]} \psi^{(\ell)}_{\alpha\beta} = 0.$$

So it looks like there are 4 ground states:

$$\omega^{(\ell)}_{\alpha\beta}(A) = \underset{\substack{\uparrow \\ \text{normalizing}}}{c_{\alpha\beta}} \langle \psi^{(\ell)}_{\alpha\beta}, A \psi^{(\ell)}_{\alpha,\beta} \rangle \qquad A \in \mathcal{A}_{[-\ell,\ell]}$$

... but its only because of these edge modes.

<u>Thm</u>: forall $A \in \mathcal{A}$,

$$\lim_{\ell\to\infty} \omega^{(\ell)}_{\alpha\beta}(A) = \omega(A),$$ i.e. they all have the same limiting point!

In fact, one can show that ω is uniquely specified by

$$\omega(h_{x,x+1}) = 0 \quad \forall x \in \mathbb{Z}$$

- Thanks C^* algebra! Now these limiting states make sense and we can study them directly.
- Remark on ambiguity: while we thought H had degenerate ground states, it doesn't.

We can do much more with this language. We can meaningfully talk about the spectrum of a Hamiltonian in the thermo limit $\Rightarrow$ gaps and gapless! (Next time?)

- Another nice one: ideal quantum gas. Thermo limiting states don't have well defined particle number, and so they are not representable by density matrices on Fock space!

- Fact: (Banach-Alaoglu)
  The unit ball in A^* (so, the set of $\omega: A \to \mathbb{C}$ s.t. $\|\omega\| = \sup_{A \in A} |\omega(A)| \leq$
  is weak-* compact

  Cor: (Sorta like Heine-Borel)
  A subset of the dual space A^* is weak-* compact if it is norm bounded and weak-* closed.

  $\Rightarrow$ The set of states S is weak-* compact