Def: A quantum channel $T : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}')$ is a linear CPTP map. A map T is CP (completely positive) if $T \otimes \mathrm{id}_n$ is positive for all n. A map T is TP (trace preserving) if

$$\mathrm{Tr}(T(A)) = \mathrm{Tr}(A) \qquad \forall\, A \in \mathcal{B}(\mathcal{H}).$$

Def: Every $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ has orthonormal bases $\{|e_j\rangle\} \subseteq \mathcal{H}_A$ and $\{|f_j\rangle\} \subseteq \mathcal{H}_B$ such that

$$|\psi\rangle = \sum_{j=1}^{d} \sqrt{\lambda_j}\, |e_j\rangle \otimes |f_j\rangle \quad \text{with } \lambda_j \geq 0, \ \sum_i \lambda_i = \|\psi\|^2 \quad (1)$$

and $d = \min\{\dim(\mathcal{H}_A), \dim(\mathcal{H}_B)\}$. The λ_j are called Schmidt coefficients.

Fact: Given a density matrix ρ_A with spectral decomposition

$$\rho_A = \sum_j \lambda_j |e_j\rangle\langle e_j|,$$

eqn (1) gives a purification of ρ_A. I.e. a pure state st $\rho_A = \mathrm{Tr}_B(|\psi\rangle\langle\psi|)$. The minimal dilation space has $\dim(\mathcal{H}_B^{min}) = \mathrm{rank}(\rho_A)$. Moreover, if $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B^{m}$ is a purification of ρ_A, then all other purifications are of the form $|\psi'\rangle = (\mathbb{I} \otimes V)|\psi\rangle$ with $V \in \mathcal{B}(\mathcal{H}_B^{min}, \mathcal{H}_B)$ an isometry.

Def: If all Schmidt coefficients of a pure state are $\lambda_j = 1/d$, then the state is said to be maximally entangled. Every maximally entangled state is of the form

$$|\phi\rangle = (\mathbb{I} \otimes U)|\Omega\rangle$$

where U is any unitary and

$$|\Omega\rangle = \frac{1}{\sqrt{d}} \sum_{j=1}^{d} |jj\rangle.$$

Fact: $|\Omega\rangle$ satisfies the following $\forall A, B \in \mathcal{B}(\mathbb{C}^d)$

i) $\langle\Omega|A\otimes B|\Omega\rangle = \frac{1}{d}\operatorname{Tr}(A^T B)$

ii) $(A\otimes \mathbb{I})|\Omega\rangle = (\mathbb{I}\otimes A^T)|\Omega\rangle$

iii) Every pure state $|\psi\rangle_{AB}$ with reduced density operator ρ_B can be expressed as

$$|\psi\rangle = (\mathbb{I}\otimes R)|\Omega\rangle$$

where $R = \sqrt{d\rho_B}\, V$ for an appropriate isometry V.

Def: The flip (or swap) operator $\mathbb{F}$ is defined via $\mathbb{F}|ij\rangle = |ji\rangle$. I.e $\mathbb{F} = \sum_{ij} |ji\rangle\langle ij|$. It is related to $|\Omega\rangle$ via

$$d|\Omega\rangle\langle\Omega|^{T_A} = \mathbb{F} \qquad \left(d|\Omega\rangle\langle\Omega|^{T_A} = \sum_{ij} |i\rangle\langle j|\otimes|i\rangle\langle j|^{T_A} = \sum_{ij} |j\rangle\langle i|\otimes|i\rangle\langle j|\right)$$

where $(\cdot)^{T_A}$ denotes partial transpose

$$\langle ij|C^{T_A}|kl\rangle = \langle kj|C|il\rangle.$$

The relations in the previous fact can equivalently be expressed as

$$\operatorname{Tr}((A\otimes B)\mathbb{F}) = \operatorname{Tr}(AB), \qquad (A\otimes \mathbb{I})\mathbb{F} = \mathbb{F}(\mathbb{I}\otimes A).$$

Proposition: A linear map $T:\mathcal{B}(\mathcal{H})\to\mathcal{B}(\mathcal{H}')$ is completely positive iff $(T\otimes \mathrm{id}_d)(|\Omega\rangle\langle\Omega|)\geq 0$ where $|\Omega\rangle$ is a maximally entangled state of dimension $d=\dim(\mathcal{H})$.

Proof: ($\Rightarrow$) Trivial. ($\Leftarrow$) Let n be arbitrary and let $\rho\in\mathcal{B}(\mathcal{H}\otimes\mathbb{C}^n)$. Write ρ in its spectral decomposition as $\rho = \sum_{i=1}^{d''} \lambda_i |\psi_i\rangle\langle\psi_i|$. Then, $(T\otimes \mathrm{id}_n)(\rho)\geq 0$ if $(T\otimes\mathrm{id}_n)(|\psi_i\rangle\langle\psi_i|)\geq 0$ for each $i=1,\dots,d''$. By the previous Fact, we have $|\psi_i\rangle = (\mathbb{I}_d\otimes R_i)|\Omega\rangle$ for an appropriate isometry R_i. Thus,

$$\begin{aligned}(T\otimes \mathrm{id}_n)(|\psi_i\rangle\langle\psi_i|) &= (T\otimes\mathrm{id}_n)\big((\mathbb{I}_d\otimes R_i)|\Omega\rangle\langle\Omega|(\mathbb{I}_d\otimes R_i^\dagger)\big)\\ &= (\mathbb{I}_d\otimes R_i)(T\otimes \mathrm{id}_d)(|\Omega\rangle\langle\Omega|)(\mathbb{I}_d\otimes R_i^\dagger).\end{aligned}$$

Since we assume $(T\otimes\mathrm{id}_d)(|\Omega\rangle\langle\Omega|)\geq 0$ we see that this last term is positive. $\square$

The operator $\tau = (T \otimes id_d)(|\Omega\rangle\langle\Omega|)$ encodes every property of T, not just complete positivity. $d\tau$ is often called the Choi matrix. If T is CPTP, then τ is its Jamilkowski state.

<u>Proposition</u>: (Choi-Jamilkowski Isomorphism)

The following provides a one-to-one correspondence between linear maps $T \in \mathcal{B}(\mathcal{M}_d, \mathcal{M}_{d'})$ and operators $\tau \in \mathcal{B}(\mathbb{C}^d \otimes \mathbb{C}^{d'})$:

$$\tau = (T \otimes id_d)(|\Omega\rangle\langle\Omega|), \quad Tr(A T(B)) = d \cdot Tr(\tau\, A \otimes B^T) \tag{2}$$

for all $A \in \mathcal{M}_{d'}$, $B \in \mathcal{M}_d$, and $\Omega \in \mathbb{C}^d \otimes \mathbb{C}^d$ a maximally entangled state. The maps $T \mapsto \tau$ and $\tau \mapsto T$ defined by (2) are mutual inverses and lead to the following correspondences.

- Hermiticity: $\tau = \tau^\dagger$ iff $T(B^\dagger) = T(B)^\dagger \;\; \forall B \in \mathcal{M}_d$.
- Complete positivity: T is CP iff $\tau \geq 0$.
- Doubly Stochastic: $T(\mathbb{I}) \propto \mathbb{I}$ and $T^*(\mathbb{I}) \propto (\mathbb{I})$ iff $Tr_A(\tau) \propto \mathbb{I}$ and $Tr_B(\tau) \propto \mathbb{I}$.
- Unitality: $T(\mathbb{I}) = \mathbb{I}$ iff $Tr_B(\tau) = \mathbb{I}_{d'}/d$.
- Preservation of trace: $Tr_A(\tau) = T^*(\mathbb{I})^T/d$, ie. $T^*(\mathbb{I}) = \mathbb{I}$ iff $Tr_A(\tau) = \mathbb{I}_d/d$.
- Normalization: $Tr(\tau) = Tr(T^*(\mathbb{I}))/d$.

<u>Proof</u>: Complete positivity was proven in the previous proposition and the other correspondences are easy enough to prove. It only remains to prove that the relations in (2) are mutual inverses.

$$\begin{aligned}
d\, Tr(\tau\, A \otimes B^T) &= Tr\big((T \otimes id_d)(d|\Omega\rangle\langle\Omega|)\, A \otimes B^T\big) \\
&= Tr\big(\mathbb{F}^{T_B} (T^* \otimes id_d)(A \otimes B^T)\big) \\
&= Tr\big(\mathbb{F}^{T_B}\, T^*(A) \otimes B^T\big) \\
&= Tr\big(\mathbb{F}\, T^*(A) \otimes B\big)
\end{aligned}$$

def of adjoint + $d|\Omega\rangle\langle\Omega| = \mathbb{F}^{T_B}$

$$
\begin{aligned}
\mathrm{Tr}(\mathbb{F}\, T^*(A)\otimes B) &= \mathrm{Tr}\Big(\sum_{ij} |ji\rangle\langle ij|\; T^*(A)\otimes B\Big)\\
&= \sum_{ij} \mathrm{Tr}\big(|j\rangle\langle i| T^*(A) \otimes |i\rangle\langle j| B\big)\\
&= \sum_{ij} \mathrm{Tr}\big(|j\rangle\langle i| T^*(A)\big)\cdot \mathrm{Tr}\big(|i\rangle\langle j| B\big)\\
&= \sum_{ij} \langle i| T^*(A) |j\rangle\langle j| B |i\rangle\\
&= \sum_{i} \langle i| T^*(A) B |i\rangle\\
&= \mathrm{Tr}(T^*(A) B)\\
&= \mathrm{Tr}(A\, T(B)).
\end{aligned}
$$

So, if the map $T \to \tau$ is surjective, then $T\to\tau$ and $\tau \to T$ are inverses. This follows from the fact that the space of linear maps $T: \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}')$ has dimension $d^2 d'^2$ since each map can be described by a $d'^2 \times d^2$ matrix acting on the vectorized matrices. The space of Choi matrices $\tau \in \mathcal{B}(\mathbb{C}^d \otimes \mathbb{C}^{d'})$ also has dimension $d'^2 d^2$. So, injectivity implies surjectivity and proving injectivity is simple: if $(T\otimes \mathrm{id}_d)(|\Omega\rangle\langle\Omega|) = 0$

$$
\begin{aligned}
&\Rightarrow (T\otimes \mathrm{id}_d) = \frac{1}{d}\sum_{ij} T(|i\rangle\langle j|)\otimes |i\rangle\langle j| = 0\\
&\Rightarrow T(|i\rangle\langle j|) = 0.
\end{aligned}
$$

This can also be proven as follows:

write
$$
\begin{aligned}
\tau &= \sum_\alpha c_\alpha |\psi_\alpha\rangle\langle \psi_\alpha'|\\
&= \sum_\alpha c_\alpha (X_\alpha \otimes \mathbb{I})|\Omega\rangle\langle\Omega|(X_\alpha'^{*}\otimes \mathbb{I})\\
&= (T\otimes \mathrm{id}_d)(|\Omega\rangle\langle\Omega|)
\end{aligned}
$$
where $T(\rho) = \sum_\alpha c_\alpha X_\alpha \rho X_\alpha'$. $\square$

This corresponce between maps T and states τ allows us to show that every linear map admits a decomposition into at most four CP maps.

Proposition: (Decomposition into CP Maps)

Every linear map $T \in \mathcal{B}(M_d, M_{d'})$ can be written as a complex linear combination of four CP maps. If T is Hermitian (i.e. $T(B^\dagger) = T(B)^\dagger \; \forall B$), then T can written as a real linear combination of two CP maps.

Proof: Since $T \leftrightarrow \tau$ is one-to-one and linear, it suffices to decompose τ. We do this by decomposing τ into Hermitian and anti-Hermitian parts, then decomposing these further into positive and negative parts. Explicitly,

$$\tau = \frac{1}{2}(\tau + \tau^\dagger) + \frac{i}{2}(i\tau^\dagger - i\tau)$$

gives the Hermitian and anti-Hermitian parts. The positive and negative parts can be attained via the spectral decomposition.

Proposition: (No Information Without Disturbance)

Consider an instrument represented by a set of CP maps $\{T_\alpha : M_d \to M_d\}$. If there is no disturbance on average, i.e. $T = \sum_\alpha T_\alpha$ satisfies $T = \mathrm{id}$, then $T_\alpha \propto \mathrm{id}$ for each α and $\Pr[\text{outcome } \alpha] = \mathrm{Tr}(T_\alpha(\rho))$ is independent of ρ (hence, no information is gained).

Proof: $\tau = (T \otimes \mathrm{id})(|\Omega\rangle\langle\Omega|) = (\sum_\alpha T_\alpha \otimes \mathrm{id})(|\Omega\rangle\langle\Omega|) = \sum_\alpha \tau_\alpha$ where $\tau_\alpha \geq 0$ since T_α is CP. Also, $\tau = |\Omega\rangle\langle\Omega|$. So, $\sum_\alpha \tau_\alpha = |\Omega\rangle\langle\Omega|$ is a convex decomposition of a pure state. So, the decomposition is trivial, $\tau_\alpha = c_\alpha |\Omega\rangle\langle\Omega|$ for $c_\alpha \geq 0$. Thus, $T_\alpha = c_\alpha \cdot \mathrm{id}$ so $\mathrm{Tr}(T_\alpha(\rho)) = c_\alpha$ is independent of ρ. $\square$

Implementation by Teleportation

If T is a quantum channel, τ can be prepared by applying T to half of a mixed state. What about the converse? Given τ, can we implement T on a chosen input ρ?

Suppose Alice and Bob share $\tau = (T \otimes id)(|\Omega\rangle\langle\Omega|_{AB})$. Suppose Bob also holds ρ and performs a POVM on his composite subsystem ~~Then, Alice holds $T(\rho)$~~ which has $\omega = |\Omega\rangle\langle\Omega|$ as a POVM element. Note that

$$
\begin{aligned}
p(\omega) = \mathrm{Tr}\left(\omega \cdot \tfrac{\mathbb{I}}{d} \otimes \rho\right) &= \mathrm{Tr}\left(\left(\tfrac{1}{d}\sum_{ij} |i\rangle\langle j| \otimes |i\rangle\langle j|\right)\left(\tfrac{\mathbb{I}}{d} \otimes \rho\right)\right) \\
&= \frac{1}{d^2}\sum_{ij} \mathrm{Tr}\left(|i\rangle\langle j| \otimes |i\rangle\langle j|\rho\right) \\
&= \frac{1}{d^2}\sum_{ij} \mathrm{Tr}(|i\rangle\langle j|)\,\mathrm{Tr}(|i\rangle\langle j|\rho) \\
&= \frac{1}{d^2}\mathrm{Tr}(\rho) = \frac{1}{d^2}.
\end{aligned}
$$

Also, we can write

$$\tau \otimes \rho = \frac{1}{d}\sum_{ijkl} a_{kl} T(|i\rangle\langle j|) \otimes |i\rangle\langle j| \otimes |k\rangle\langle l|,$$

so upon outcome corresponding to ω, the state collapses to

$$\sum_{ij} a_{ij} T(|i\rangle\langle j|) \otimes \omega = T(\rho) \otimes \omega.$$

So, $T(\rho)$ is implemented with prob $= 1/d^2$.

However, if there is a set of local unitaries $\{V_i \otimes U_i\}_{i=1}^{N}$ such that $\tau = (V_i \otimes U_i)\tau(V_i \otimes U_i)^\dagger$ and $\mathrm{Tr}(U_i U_j^\dagger) = d\delta_{ij}$. Bob can construct a POVM with elements $(\mathbb{I} \otimes U_i)^\dagger \omega (\mathbb{I} \otimes U_i)$. Upon outcome i, Bob communicates this to Alice who then applies V_i. Then, $\rho_A = T(\rho)$ with prob N/d^2.

Kraus, Stinespring, and Neumark

Thm: (Kraus representation)

A linear map $T \in \mathcal{B}(M_d, M_{d'})$ is completely positive iff it admits a representation of the form

$$T(A) = \sum_{j=1}^{r} K_j A K_j^{\dagger}.$$

This decomposition satisfies:

1) Normalization: T is TP iff $\sum_j K_j^{\dagger} K_j = \mathbb{I}$ and unital iff $\sum_j K_j K_j^{\dagger} = \mathbb{I}$.
2) Kraus rank: The minimal number of Kraus operators is $r = \mathrm{rank}(\tau) \leq dd'$.
3) Orthogonality: There is always a representation with $r = \mathrm{rank}(\tau)$ orthogonal Kraus operators (i.e. $\mathrm{Tr}(K_i^{\dagger} K_j) \propto \delta_{ij}$).
4) Freedom: Two sets of Kraus operators $\{K_j\}$ and $\{\tilde{K}_\ell\}$ represent the same map T iff there is a unitary U so that $K_j = \sum_\ell U_{j\ell} \tilde{K}_\ell$.

Proof: ($\Rightarrow$) If T is CP, then $\tau \geq 0$, so

$$\tau = \sum_{j=1}^{r} |\psi_j\rangle\langle\psi_j| = \sum_{j=1}^{r} (K_j \otimes \mathbb{I})|\Omega\rangle\langle\Omega|(K_j \otimes \mathbb{I})^{\dagger}$$
$$= (T \otimes \mathrm{id})(|\Omega\rangle\langle\Omega|).$$

This shows that $T(A) = \sum_{j=1}^{r} K_j A K_j^{\dagger}$. It also shows that $r \geq \mathrm{rank}(\tau)$. If the $|\psi_j\rangle$ are orthogonal, then the K_j are as well.

($\Leftarrow$) If $T(A) = \sum_{j=1}^{r} K_j A K_j^{\dagger}$, then $\tau = \sum_{j=1}^{r} (K_j \otimes \mathbb{I})|\Omega\rangle\langle\Omega|(K_j \otimes \mathbb{I})^{\dagger} \geq 0$

The normalization conditions are easy to verify.

The unitary freedom of the Kraus operators follows from

$$\tau = \sum_{j=1}^{r} (K_j \otimes \mathbb{I})|\Omega\rangle\langle\Omega|(K_j \otimes \mathbb{I})^{\dagger} = \sum_{j=1}^{r} |\psi_j\rangle\langle\psi_j|$$

and the following proposition. $\square$

Proposition: (Equivalence of Ensembles)

Two ensembles of (not necessarily normalized) vectors $\{|\psi_j\rangle\}$ and $\{|\tilde{\psi}_\ell\rangle\}$ satisfy

$$\sum_j |\psi_j\rangle\langle\psi_j| = \sum_\ell |\tilde{\psi}_\ell\rangle\langle\tilde{\psi}_\ell| \qquad (3)$$

iff there exists a unitary U such that

$$|\psi_j\rangle = \sum_\ell U_{j\ell} |\tilde{\psi}_\ell\rangle.$$

Proof: Write $\rho = \sum_j |\psi_j\rangle\langle\psi_j| = \sum_\ell |\tilde{\psi}_\ell\rangle\langle\tilde{\psi}_\ell|$. Construct two purifications using the same refrence basis

$$|\Psi\rangle = \sum_j |\psi_j\rangle \otimes |j\rangle$$

$$|\tilde{\Psi}\rangle = \sum_\ell |\tilde{\psi}_\ell\rangle \otimes |\ell\rangle.$$

There exists a unitary U (or isometry) such that

$$|\Psi\rangle = (\mathbb{I} \otimes U)|\tilde{\Psi}\rangle.$$

If U is an isometry, we can ~~complete~~ embed it into a unitary by completing the orthonormal basis of columns. $\square$