Last Time...

Examples:

	Classical	Quantum
• Ambient "phase space"	Manifold $M=\mathbb{R}$, position.	Lattice $\Gamma = \mathbb{Z}$
• Dense subset of observables	$C_c(\mathbb{R})$	$\mathcal{A}_{loc} = \bigcup_{\ell\in\mathbb{N}} \mathcal{A}_{[-\ell,\ell]}$, $\mathcal{A}_x \in B(\mathbb{C}^n)$
• C^*-Algebra of observables $\mathcal{A}$ (norm $\lVert\cdot\rVert$, adjoint $\dagger$, multiplication)	$C_0(\mathbb{R}) = \overline{C_c(\mathbb{R})}^{\lVert\cdot\rVert}$ (abelian)	$\mathcal{A} = \overline{\mathcal{A}_{loc}}^{\lVert\cdot\rVert}$ (nonabelian)
• States $\omega : \mathcal{A}\to\mathbb{R}$ (or $\mathbb{C}$) $\omega(\mathbb{1})=1$, and $A\geq 0 \Rightarrow \omega(A)\geq 0$ w/ weak-* topology: $\omega_n \to \omega$ if $\omega_n(A)\to\omega(A)\ \forall A\in\mathcal{A}$	Probability measures $\omega(A)=\int_{\mathbb{R}} A(x)\,d\mu(x)$ • Special case: pdfs $\rho(x)dx = d\mu$	Quantum states ω • Special case: $\omega(A)=\mathrm{Tr}\,\rho A$

Today: Dynamics and Spectrum!

Refs:

- Nachtergaele and Sims (lecture notes) "Introduction to quantum spin systems"
- Naaijkens "Quantum spin systems on Infinite lattices"
- Attal, Joye, Pillet "Open Quantum Systems I". ← Ch 2 and 3 for ideal quantum gasses!

We want dynamics and spectrum for our quantum system.

Working example: Heisenberg spin-1/2 chain:

- Lattice $\Gamma = \mathbb{Z}$
- On site $\mathcal{H}_x = \mathbb{C}^2$
- Finite chain $[-l,l]$

$$\mathcal{H}_{[-l,l]} = \bigotimes_{x=-l}^{l} \mathbb{C}^2$$

Hamiltonian:

$$H_l = \sum_{x=-l}^{l-1} h_{x,x+1}$$

with $h_{x,x+1} = \mathbb{1} - \text{SWAP} \geq 0$.

Finite chain ground states are easy.

SWAPS generate permutation group S_{2l} on $[-l,l]$,

$\Rightarrow \{\text{Ground states } |\psi\rangle\} = \{\text{vectors which are symmetric under all permutations of } [-l,l]\}$

H_l commutes with Number operator N_l which counts number of "minus signs"

(Take basis $|+\rangle, |-\rangle \in \mathbb{C}^2$)

$$N_l = \sum_{x=-l}^{l-1} \frac{\mathbb{1} - \sigma_x^z}{2}, \quad \text{spec } N_l = \{0,1,\dots,2l\}$$

Pauli $\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$

each eigenspace has 1 symmetric state

$\Rightarrow \dim(\ker H_l) = 2l+1$

For $n = 0, \ldots, 2l$, let

$\mathcal{H}^{(n)}_{[-l,l]}$ = eigenspace of eigenvalue n of N_l.

$\mathcal{H}^{(n)}_{[-l,l]}$ – space of n down spins.

Basis for $\mathcal{H}^{(n)}_{[-l,l]}$ = possible sets of n sites (down spin positions) in $[-l,l]$.

$$\Rightarrow \quad \dim \mathcal{H}^{(n)}_{[-l,l]} = \binom{2l+1}{n}$$

- $\mathcal{H}^{(0)}_{\Lambda}$ is 1-dimensional and spanned by $|+\rangle \otimes |+\rangle \otimes \cdots \otimes |+\rangle$.

$$\Updownarrow$$

$$\omega_+$$

ω_+ is uniquely determined by

$$\omega_+(\sigma^z_x) = 1 \quad \forall x \in \mathbb{Z}$$

$\omega_+ : \mathcal{A} \to \mathbb{C}$ is a state in thermo limit.

- Want to talk about thermodynamic limit!
  - (Same reasons from last week)
  - Meaningful notion of spectrum of H.
  - Spontaneous Symmetry breaking (Gibbs states need not be unique in thermo limit, even for classical Ising $\uparrow\uparrow\uparrow$ and $\downarrow\downarrow\downarrow$.)

A few more subtleties...

1. No a priori "physical" Hilbert space
2. Hamiltonians are generally unbounded operators (think $p = -i\frac{\partial}{\partial x}$). So they aren't in $\mathcal{A}$... How are we supposed to define time evolution?

Finite chains first. $\mathcal{H}_{[-l,l]} = \bigotimes_{x=-l}^{l} \mathbb{C}^2$, $\mathcal{A} = \mathcal{B}(\mathcal{H}_{[-l,l]})$, all states are given by $\omega^{(l)}(A) = \mathrm{Tr}\,\rho^{(l)} A$.

Let's work in Heisenberg picture:

- States ω are fixed.
- Observables $A \in \mathcal{A}_{[-l,l]}$ evolve over time, $A(t)$.

Here, $(\omega(A))(t) = \mathrm{Tr}\,\rho\, U^\dagger(t) A U(t) = \omega(A(t))$

We will define Hamiltonian through an interaction:

$$\underbrace{\Phi : P_0([-l,l])}_{\{\text{Finite Subsets of } [-l,l]\}} \longrightarrow \mathcal{A}_{[-l,l]}$$

Heisenberg Chain:

$$\Phi(X) = h_{x,x+1} = \mathbb{1} - \mathrm{SWAP}, \quad X = \{x, x+1\}$$
$$\Phi(X) = 0 \quad \text{otherwise}$$

$$H_l = \sum_{X \in P_0([-l,l])} \Phi(X) = \sum_{x=-l}^{l-1} h_{x,x+1}$$

Finite-chain Dynamics:

$\tau_t^{(\ell)} : \mathcal{A}_{[-\ell,\ell]} \to \mathcal{A}_{[-\ell,\ell]}$ is a (strongly continuous) 1-parameter group of $*$-automorphisms

$$\tau_t^{(\ell)}(A) = U_\ell^\dagger(t)\, A\, U_\ell(t)$$

where $U_\ell(t) = e^{-itH_\ell} A e^{itH_\ell} \in \mathcal{A}_{[-\ell,\ell]}$.

These solve the Heisenberg eqn:

$$\frac{d}{dt}\tau_t^{(\ell)}(A) = i\left[H_\ell, \tau_t^{(\ell)}(A)\right]$$

- 1-parameter gp: if $t, s \in \mathbb{R}$,

$$\tau_t \circ \tau_s(A) = \tau_{t+s}(A)$$

- strong continuity: $\forall A \in \mathcal{A}$,

  $t \mapsto \tau_t(A)$ is continuous.

- $*$-automorphism: $\tau_t : \mathcal{A} \to \mathcal{A}$
  - τ_t is linear, bounded (continuous)
  - $\tau_t(AB) = \tau_t(A)\tau_t(B)$
  - $\tau_t(A^\dagger) = \tau_t(A)^\dagger$

- This is fine for finite chains.

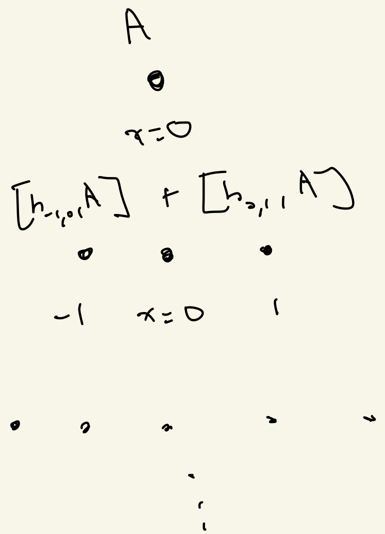
Here, $\tau_t^\ell : \mathcal{A}_\ell \to \mathcal{A}_\ell$.

Want: Infinite Volume Dynamics

Problem: Observables can spread:

Ex: Heisenberg

$$\tau_t^{(\ell)}(A) = e^{it[H_\ell,\cdot]}A = A + it[H_\ell,\cdot] + \frac{(it)^2}{2}[H_\ell,[H_\ell,\cdot]] + \dots$$



- When interaction Φ is nearest-neighbor, get "light cone" dynamics which make it plausible that we could find infinite volume dynamics

$$\tau_t^{(\ell)} \xrightarrow{\ell\to\infty} \tau_t : \mathcal{A}\to\mathcal{A}.$$

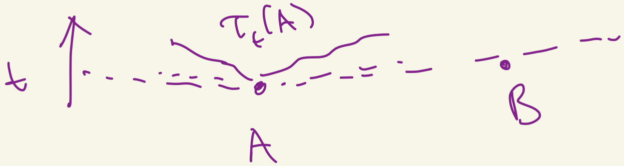
(Remember, $\mathcal{A}$ are quasi-local).

- Need control of spatial locality of dynamics. Otherwise, no hope of convergence to a $\tau_t : \mathcal{A}\to\mathcal{A}$.
  - Key definition: F-norm of interaction Φ.

Thm (Lieb-Robinson bound): (informal)

If Φ sufficiently local (e.g. decays like $\frac{1}{(1+r)^{\nu+\varepsilon}}$ on $\mathbb{Z}^{\nu}$), and X, Y are disjoint sets, $A \in \mathcal{A}_X$, $B \in \mathcal{A}_Y$,

$$\| [\tau_t(A), B] \| \lesssim \|A\| \, \|B\| \, D(A,B) \left(e^{C(\Phi)|t|} - 1 \right)$$



Cor:

Along any increasing, exhaustive sequence $[-l, l] \nearrow \mathbb{Z}$, the norm limit

$$\tau_t(A) = \lim_{l \to \infty} \tau_t^{(l)}(A)$$

exists for all $t \in \mathbb{R}$ and $A \in \mathcal{A}_{loc}$.

Def: C^* dynamical system $= \{ \mathcal{A}, \{\tau_t\} \}$

Note: At this point, it is not clear (and in fact untrue) that τ_t is implemented by a unitary $U(t)$!

Where's the Hamiltonian??

Semigp thy $\Rightarrow$ $\tau_t = e^{it\delta}$, $\quad \delta(A) = \lim_{l \to \infty} [H_l, A]$

$\forall A \in \mathcal{A}_{loc}$

(densely defined, closed operator, usually unbounded.)

Def: Ground state

ω is a ground state of $\{\mathcal{A}, \{\tau_t = e^{it\delta}\}\}$ if for all $A \in \mathcal{A}_{loc}$,

$$\omega(A^\dagger \underbrace{\delta(A)}) \geq 0$$

$$\omega(A^\dagger [H, A]) \geq 0$$ "Exciting by H can only raise energy"

The GNS construction:

The GNS construction allows us to return to somewhere more familiar.

Def:

$\mathcal{A}$ - unital C^*algebra.

A representation on a Hilbert space $\mathcal{H}$ is a linear map

$\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ s.t.

i) $\pi(\mathbb{1}) = \mathbb{1}$

ii) $\pi(A^\dagger) = \pi(A)^\dagger$

iii) $\pi(AB) = \pi(A)\pi(B)$

- A vector $\Omega \in \mathcal{H}$ is called cyclic for a rep π if

$$D_\Omega = \{\pi(A)\Omega : A \in \mathcal{A}\} \subseteq \mathcal{H}$$

is dense in $\mathcal{H}$.

Thm (GNS construction)

- Let ω be a state on $\mathcal{A}$.
- Then there exists a Hilbert space $\mathcal{H}_\omega$, a rep $\pi_\omega : \mathcal{A} \to B(\mathcal{H}_\omega)$, and a vector $\Omega_\omega \in \mathcal{H}_\omega$ which is cyclic for π_ω and such that

$$\omega(A) = \langle \Omega_\omega, A \Omega_\omega \rangle \quad \forall A \in \mathcal{A}.$$

- The triple $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$ is uniquely determined by ω up to unitary equivalence. I.e. if there are two such $(\mathcal{H}_1, \pi_1, \Omega_1)$, $(\mathcal{H}_2, \pi_2, \Omega_2)$, then $\exists!$ unitary $U : \mathcal{H}_1 \to \mathcal{H}_2$ s.t.

$$\Omega_2 = U \Omega_1 \quad \text{and} \quad \pi_2(A) = U \pi_1(A) U^\dagger \quad \forall A \in \mathcal{A}.$$

Cor:

Let τ an automorphism of $\mathcal{A}$ s.t.

$$\omega \circ \tau = \omega.$$

Then $\exists$ unique unitary $U \in B(\mathcal{H}_\omega)$ s.t.

$$\pi_\omega(\tau(A)) = U \pi_\omega(A) U^\dagger, \quad U\Omega = \Omega$$

Stone's Thm:

U_t strongly cts unitary gp $\Leftrightarrow$ $\exists$ densely defined self-adjoint H st.

$$U_t = e^{-itH}$$

GNS cor + Stone's $\Rightarrow$ H_ω is the GNS (bulk) Hamiltonian!!

Back to reality. Take ω_+ and take its GNS representation $(\mathcal{H}, \pi, \Omega)$. We will describe it, then show it is (up to unitary equiv) the GNS triple of ω_+.

- Take $\mathcal{H}_{\omega_+} = \ell^2(P_0(\mathbb{Z}))$.

  onb $= \{ \xi_X : X \in P_0(\mathbb{Z}) \}$ Delta functions, signifying locations of down spins.

- ω_+ has no down spins

$\Rightarrow \Omega = \xi_\phi$, no down spins.

- Define the rep by

$$\pi(\sigma_x^-)\xi_X = \begin{cases} \xi_{X \cup \{x\}} & x \notin X \\ 0 & x \in X \end{cases}$$

Eg: $\pi(\sigma_1^-)|{+}{+}{+}{-}{+}\rangle = |{-}{+}{+}{-}{+}\rangle$

$\pi(\sigma_1^-)|{-}{+}{+}{-}{+}\rangle = 0$.

- π is a morphism, so we know $\pi(\sigma_x^+) = \pi(\sigma_x^-)^*$ and it's easy to define other observables.

- Since π is cyclic and

$$\pi\left(\prod_{x \in X} \sigma_x^-\right)\Omega = \xi_X \rightarrow \pi(A_{loc})\Omega$$

is dense in $\mathcal{H}$.

$\Rightarrow$ $(\mathcal{H}, \pi, \Omega)$ is the GNS Triple of ω_+.

In the GNS rep,

$$\#\text{ down spins} = \text{eigenvalues } (N)$$

number operator, now densely defined on core $\pi(A_{loc})\Omega$.

Eigenvectors of N = basis vectors ξ_X with eigenvalue $n = |X|$. } $\mathcal{H}^{(n)}$

Claim: Eigenspaces $\mathcal{H}^{(n)}$ are invariant subspaces of GNS Hamiltonian H_{ω_+} } we know $[N, h_{x,x+1}] = 0$.

Check:

We need an explicit description of H_{ω_+} on the core $\pi(A_{loc})\Omega$.

So, for all $A \in A_\ell$,

$$H_{\omega_+}\pi(A)\Omega = \lim_{\ell\to\infty}[\pi(H_\ell), \pi(A)]\Omega$$

$\cdots A_\ell$

$\cdots\cdots A_{\ell+1}$

$H_\ell = \sum_{x=-\ell}^{\ell-1} h_{x,x+1}$

$$= [\pi(H_{\ell+1}), \pi(A)]\Omega$$

Abuse notation: $\pi(\mathrm{SWAP}) = \mathrm{SWAP}$

$$= \sum_{x=-(\ell+1)}^{\ell} (\mathbb{1} - \mathrm{SWAP})\,\pi(A)\Omega$$

since $(\mathbb{1} - \mathrm{SWAP})\Omega = 0$.

$\Rightarrow$ $\mathcal{H}^{(n)}$ are invariant under H_{ω_+} $\square$

We have:

$$\operatorname{spec}\left(H_{\omega_+}\big|_{\mathcal{H}^{(1)}}\right) \subseteq \operatorname{spec}\left(H_{\omega_+}\right)$$

<u>Now:</u>

$$\mathcal{H}^{(1)} = \overline{\operatorname{span}\{\xi_{\{x\}} : x \in \mathbb{Z}\}} \cong \ell^2(\mathbb{Z})$$

$\uparrow$

$|\ldots +-+++\ldots\rangle$

$|\ldots ++-++\ldots\rangle$

$|\ldots +++-+\ldots$

we can calculate matrix elements:

$$\langle \xi_{\{x\}}, H_{\omega_+}\xi_{\{y\}}\rangle = \begin{cases} -1 & \text{if } |x-y|=1 \\ 2 & \text{if } x=y \\ 0 & \text{else} \end{cases}$$

e.g. $\langle ++- | \mathbb{1}-SWAP_{23} | +-+\rangle = -\langle ++- | SWAP_{23} | +-+\rangle$

$= -1$

$$H_{\omega_+}\big|_{\mathcal{H}^{(1)}} = \begin{pmatrix} \ddots & & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & -1 & & \\ & & -1 & 2 & -1 & \\ & & & & & \ddots \end{pmatrix}$$

Discrete Laplacian on $\mathbb{Z}$! $\xrightarrow[\text{Fourier transform}]{} [0,4] \subseteq \operatorname{spec}(H_{\omega_+})$

absolutely cts.

<u>Result:</u> H_{ω_+} has no gap above the ground state ω_+

- Low-lying excitations are called "spin waves"

  (Imagine a negative charge propagating through sea of positive charges)

$$|-+++\ldots\rangle \longrightarrow |+-+++\ldots\rangle \longrightarrow |++-++\ldots\rangle$$

Generalized eigenfunctions = plane waves

(we are diagonalizing via Fourier series

$$-\Delta = \begin{pmatrix} -1 & 2 & -1 & & \\ & -1 & 2 & -1 & \\ & & -1 & 2 & -1 \end{pmatrix} = -(\underset{\substack{\uparrow \\ \text{left shift}}}{L} + \underset{\substack{\uparrow \\ \text{right shift}}}{L^*} - 2\mathbb{1})$$

$x = (x_k)_{k=-\infty}^{\infty}$, $x_k = e^{iky}$, $y \in [-\pi, \pi]$

$$-\Delta(e^{iky}) = -(e^{i(k-1)y} + e^{i(k+1)y} - 2e^{iky})$$
$$= -e^{ik}(e^{-iy} + e^{iy} - 2)$$
$$= -e^{iky}(2\cos y - 2)$$
$$= (2 - 2\cos y)e^{iky}$$

(Not ℓ^2 integrable, but $\operatorname{ran}(-\Delta - \lambda\mathbb{1})$ is dense, by truncation argument)

$$\Rightarrow \operatorname{spec} \Delta = [0, 4].$$

Comment: ω_+ and ω_- (and all the other SU(2) symmetry broken pure ground states) have <u>inequivalent</u> GNS reps.

i.e. they occupy different superselection sectors

(physically, flipping every spin is "too nonlocal" to be an inner automorphism $\alpha(\cdot) = U(\cdot)U^*$ on $\mathcal{A}$. It is an outer autom.)