Solving Lindblad dynamics

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February 16, 2024

Contents

We discuss solving Lindblad dynamics (in some sense), by closely following the derivations in [Barthel & Zhang 2022 (arXiv:2112.08344)]. This note does not provide any more useful physics information than this wonderful paper. The only purpose of this note is to write out some calculation details that the authors kindly leave to us as exercises, while omitting some other details that can be found in this paper.

For simplicity, we only discuss fermionic systems.

1 Majorana Operators

Let us start with the definition of Majorana operators. Let $\hat{a}_i, \hat{a}_j^\dagger$ $(i, j = 1, \cdots, n)$ be the annihilation and creation operator in the second quantization. They satisfy the canonical anti-commutation rule:

$$\{\hat{a}_i, \hat{a}_j^\dagger\} = \delta_{ij}, \quad \{\hat{a}_i, \hat{a}_j\} = 0, \quad \{\hat{a}_i^\dagger, \hat{a}_j^\dagger\} = 0$$

Note that $\hat{a}_j^\dagger$ is the Hermitian conjugate of $\hat{a}_j$. The Majorana operators are definied as follows:

$$\hat{w}_{j+} = \frac{1}{\sqrt{2}}\left(\hat{a}_j + \hat{a}_j^\dagger\right), \quad \hat{w}_{j-} = \frac{1}{\sqrt{2}}\left(\mathrm{i}\hat{a}_j - \mathrm{i}\hat{a}_j^\dagger\right) \tag{1}$$

In matrix form, this is

$$\begin{pmatrix} \hat{w}_{j+} \\ \hat{w}_{j-} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ \mathrm{i} & -\mathrm{i} \end{pmatrix} \begin{pmatrix} \hat{a}_j \\ \hat{a}_j^\dagger \end{pmatrix}, \quad \begin{pmatrix} \hat{a}_j \\ \hat{a}_j^\dagger \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -\mathrm{i} \\ 1 & \mathrm{i} \end{pmatrix} \begin{pmatrix} \hat{w}_{j+} \\ \hat{w}_{j-} \end{pmatrix} \tag{2}$$

In fermionic systems, we often let

$$\hat{w}_k := \hat{w}_{k+}, \quad \hat{w}_{k+n} := \hat{w}_{k-}.$$

The Majorana operators have the following nice propertities:

- Self-adjointness:
$$(\hat{w}_k)^\dagger = \hat{w}_k, \quad k = 1, \cdots, 2n.$$
- Canonical anti-commutation:
$$\{\hat{w}_i, \hat{w}_j\} = \delta_{ij}$$

To get us familiar with the properties of Majorana operators, let us do the following useful exercises:

Prob 1: Prove that

$$[\hat{w}_i\hat{w}_j, \hat{w}_m] = \hat{w}_i\delta_{jm} - \hat{w}_j\delta_{im}$$

Proof.

$$[\hat{w}_i\hat{w}_j, \hat{w}_m] = \hat{w}_i\{\hat{w}_j, \hat{w}_m\} - \{\hat{w}_i, \hat{w}_m\}\hat{w}_j = \hat{w}_i\delta_{jm} - \hat{w}_j\delta_{im}.$$

□

Prob 2: For quadratic Hermitian Hamiltonian

$$\hat{H} = \sum_{ij=1}^{n} h_{ij}\hat{a}_i^\dagger\hat{a}_j, \quad h_{ij} = h_{ji}^*$$

$\hat{H}$ could be rewritten in the following form:

$$\hat{H} = \sum_{i,j=1}^{2n} H_{ij}\hat{w}_i\hat{w}_j, \quad \text{s.t. } H = H^\dagger = -H^T \tag{3}$$

In other words, the matrix H is anti-symmetric and Hermitian, thus purely imaginary.

Proof. Left as an exercise. □

PROB 3: Prove that with the above quadratic Hamiltonian, let $\hat{\gamma}_{mn} = \hat{w}_m \hat{w}_n$, we have

$$[\hat{H}, \hat{\gamma}_{mn}] = -2(H\hat{\gamma} + \hat{\gamma}H^T)_{mn}. \tag{4}$$

Proof.

$$\begin{aligned}
[\hat{H}, \hat{w}_m \hat{w}_n] &= [\hat{H}, \hat{w}_m]\hat{w}_n + \hat{w}_m[\hat{H}, \hat{w}_n] \\
&= \sum_{ij} H_{ij}(\hat{w}_i \delta_{jm} - \hat{w}_j \delta_{im})\hat{w}_n + \sum_{ij} \hat{w}_m H_{ij}(\hat{w}_i \delta_{jn} - \hat{w}_j \delta_{in}) \\
&= \sum_i H_{im}\hat{w}_i\hat{w}_n - \sum_j H_{mj}\hat{w}_j\hat{w}_n + \sum_i H_{in}\hat{w}_m\hat{w}_i - \sum_j H_{nj}\hat{w}_m\hat{w}_j \\
\text{(using } H = -H^T\text{)} \quad &= -2\sum_j H_{mj}\hat{w}_j\hat{w}_n + 2\sum_i \hat{w}_m\hat{w}_i H_{in} \\
&= -2(H\hat{\gamma} + \hat{\gamma}H^T)_{mn}.
\end{aligned}$$

□

Now let us re-introduce the Lindblad equation with the above notation:

$$\partial_t \hat{\rho} = \mathscr{L}(\hat{\rho}) = -\mathrm{i}[\hat{H}, \hat{\rho}] + \sum_r \left(\hat{L}_r \hat{\rho} \hat{L}_r^\dagger - \frac{1}{2}\left\{ \hat{L}_r^\dagger \hat{L}_r, \hat{\rho} \right\} \right) + \sum_s \left(\hat{M}_s \hat{\rho} \hat{M}_s^\dagger - \frac{1}{2}\left\{ \hat{M}_s^\dagger \hat{M}_s, \hat{\rho} \right\} \right) \tag{5}$$

In this note, we only consider quadratic Hamiltonian $\hat{H}$ (as in eq. (3)), linear Lindbladian $\hat{L}_r$, and quadratic Lindbladian $\hat{M}_s$:

$$\hat{L}_r = \sum_j L_{r,j}\hat{w}_j, \quad \hat{M}_s = \sum_{ij} \hat{w}_i (M_s)_{ij} \hat{w}_j,$$

Here like the quadratic Hamiltonian, M_s also satisfies $M_s = M_s^\dagger = -M_s^T$.

2 Equation of motion

In the Heisenberg picture, the time evolution for operator $\hat{O}$ is

$$\begin{aligned}
\partial_t \hat{O} = \mathscr{L}^\dagger(\hat{O}) &= \mathrm{i}[\hat{H}, \hat{O}] + \sum_r \left(\hat{L}_r^\dagger \hat{O} \hat{L}_r - \frac{1}{2}\left\{ \hat{L}_r^\dagger \hat{L}_r, \hat{O} \right\} \right) + \sum_s \left(\hat{M}_s^\dagger \hat{O} \hat{M}_s - \frac{1}{2}\left\{ \hat{M}_s^\dagger \hat{M}_s, \hat{O} \right\} \right) \\
&=: \mathrm{i}[\hat{H}, \hat{O}] + \mathscr{D}_L^\dagger(\hat{O}) + \mathscr{D}_M^\dagger(\hat{O}).
\end{aligned} \tag{6}$$

In the exercises above, we have discussed operator $\hat{\gamma}_{ij} = \hat{w}_i\hat{w}_j$. In practice, however, we define the single-particle correlation operator as

$$\hat{\Gamma}_{ij} = \frac{\mathrm{i}}{2}(\hat{w}_i\hat{w}_j - \hat{w}_j\hat{w}_i) \tag{7}$$

Since $\{\hat{w}_i, \hat{w}_j\} = \delta_{ij}$, therefore

$$\hat{\Gamma}_{ij} = \frac{\mathrm{i}}{2}(\hat{w}_i\hat{w}_j - \hat{w}_j\hat{w}_i) = \mathrm{i}\hat{w}_i\hat{w}_j - \frac{\mathrm{i}}{2}\delta_{ij}$$

and

$$\hat{\Gamma}_{ij} = -\hat{\Gamma}_{ji}, \quad (\hat{\Gamma}_{ij})^\dagger = \hat{\Gamma}_{ij}.$$

Now we are ready to derive the equation of motion for $\hat{\Gamma}_{mn}$:

$$\partial_t \hat{\Gamma}_{mn} = \mathrm{i}[\hat{H}, \hat{\Gamma}_{mn}] + \mathscr{D}_L^\dagger(\hat{\Gamma}_{mn}) + \mathscr{D}_M^\dagger(\hat{\Gamma}_{mn}),$$

- First term $\mathrm{i}[\hat{H}, \hat{\Gamma}_{mn}]$: According to prob 3, we have

$$\mathrm{i}[\hat{H}, \hat{\Gamma}_{mn}] = -2\mathrm{i}(H\hat{\Gamma} + \hat{\Gamma}H^T)_{mn}. \tag{8}$$

- Second term $\mathscr{D}_L^\dagger(\hat{\Gamma}_{mn})$: with $\hat{L}_r = \sum_j L_{r,j}\hat{w}_j$, we have

$$\begin{aligned}\mathscr{D}_L^\dagger(\hat{\Gamma}_{mn}) &= \sum_r \left(\hat{L}_r^\dagger \hat{\Gamma}_{mn}\hat{L}_r - \frac{1}{2}\left\{\hat{L}_r^\dagger \hat{L}_r, \hat{\Gamma}_{mn}\right\}\right) \\ &= \frac{1}{2}\sum_r \left(\hat{L}_r^\dagger[\hat{\Gamma}_{mn}, \hat{L}_r] + [\hat{L}_r^\dagger, \hat{\Gamma}_{mn}]\hat{L}_r\right) = \frac{\mathrm{i}}{2}\sum_r \sum_{ij} L_{r,i}^* L_{r,j}\left(\hat{w}_i[\hat{w}_m\hat{w}_n, \hat{w}_j] + [\hat{w}_i, \hat{w}_m\hat{w}_n]\hat{w}_j\right)\end{aligned}$$

Let

$$B_{ji} = \sum_r L_{r,j}L_{r,i}^*, \quad \text{i.e. } B = \sum_r \boldsymbol{L}_r \boldsymbol{L}_r^\dagger$$

and using prob 3 again, we have

$$\begin{aligned}\mathscr{D}_L(\hat{\Gamma}_{mn}) &= \frac{\mathrm{i}}{2}\sum_{ij} B_{ji}\left(\hat{w}_i(\hat{w}_m\delta_{nj} - \hat{w}_n\delta_{mj}) - (\hat{w}_m\delta_{ni} - \hat{w}_n\delta_{mi})\hat{w}_j\right) \\ &= \frac{\mathrm{i}}{2}\left(\sum_i B_{ni}\hat{w}_i\hat{w}_m - \sum_i B_{mi}\hat{w}_i\hat{w}_n - \sum_j B_{jn}\hat{w}_m\hat{w}_j + \sum_j B_{jm}\hat{w}_n\hat{w}_j\right) \\ &= \frac{\mathrm{i}}{2}\left(\hat{\gamma}^T B^T - B\hat{\gamma} - \hat{\gamma}B + B^T\hat{\gamma}^T\right)_{mn} \\ &= \frac{1}{2}\left(\hat{\Gamma}^T B^T - B\hat{\Gamma} - \hat{\Gamma}B + B^T\hat{\Gamma}^T - \mathrm{i}(B - B^T)\right)_{mn} \quad \text{(Here we use } \Gamma = \mathrm{i}\gamma - \frac{\mathrm{i}}{2}I.\text{)} \\ &= \left(-\frac{\hat{\Gamma}(B + B^*)}{2} - \frac{(B^* + B)\hat{\Gamma}}{2} + \frac{B - B^*}{2\mathrm{i}}\right)_{mn} \quad \text{(Here we use } B = B^\dagger \text{ and } \hat{\Gamma}^T = -\hat{\Gamma}.\text{)}\end{aligned}$$

- Third term $\mathscr{D}_M^\dagger(\hat{\Gamma}_{mn})$: since We have

$$\mathscr{D}_M^\dagger(\hat{O}) = \sum_s \left(\hat{M}_s^\dagger \hat{O}\hat{M}_s - \frac{1}{2}\left\{\hat{M}_s^\dagger \hat{M}_s, \hat{O}\right\}\right) = -\frac{1}{2}\sum_s [\hat{M}_s, [\hat{M}_s, \hat{O}]]$$

Therefore

$$\mathscr{D}_M^\dagger(\hat{\Gamma}_{mn}) = \sum_s [\hat{M}_s, (M_s\hat{\Gamma} + \hat{\Gamma}M_s^T)_{mn}] = -2(M_s^2\hat{\Gamma} + 2M_s\hat{\Gamma}M_s^T + \hat{\Gamma}M_s^2)_{mn}.$$

Here we have used eq. (8) twice.

Let B_r and B_i be the real and imaginary part of matrix B. Combing these, we have

$$\partial_t\hat{\Gamma} = -2\mathrm{i}(H\hat{\Gamma} + \hat{\Gamma}H^T) + \left(-\hat{\Gamma}B_r - B_r\hat{\Gamma} + B_i\right) - 2\sum_s (M_s^2\hat{\Gamma} + 2M_s\hat{\Gamma}M_s^T + \hat{\Gamma}M_s^2)$$

Let us define

$$X = -2\mathrm{i}H - B_r - 2\sum_s M_s^2, \quad Y = B_i, \quad Z_s = 2\mathrm{i}M_s,$$

Then we have the equation of motion for $\hat{\Gamma}$ in the following form:

$$\partial_t\hat{\Gamma} = X\hat{\Gamma} + \hat{\Gamma}X^T + Y + \sum_s Z_s\hat{\Gamma}Z_s^T.$$

3 Correlation function

The (single-particle) correlation function Γ_{ij} is defined as the expectation value of the correlation operator $\hat{\Gamma}_{ij}$:

$$\Gamma_{ij} = \langle\hat{\Gamma}_{ij}\rangle.$$

Recall that $(\hat{\Gamma}_{ij})^\dagger = \hat{\Gamma}_{ij}$ and $\hat{\Gamma}_{ij} = -\hat{\Gamma}_{ji}$, therefore the correlation function Γ as a matrix is real-valued and antisymmetric.

Linear algebra tells us that for a real anti-symmetric $2n \times 2n$ matrix Γ, there exists a real orthogonal matrix $U \in O(2n)$ such that

$$U\Gamma U^T = \begin{pmatrix} 0 & \nu \\ -\nu & 0 \end{pmatrix}, \quad \nu = \mathrm{diag}(\nu_1, \cdots, \nu_n),$$

and Γ's eigenvalues are $\pm \mathrm{i}\nu_k$. If we define a set of rotated Majorana operators $\hat{w}'_1, \cdots, \hat{w}'_n$ using the matrix U:

$$\hat{w}'_i = \sum_j U_{ij}\hat{w}_j$$

Then we see that ν_k is the correlation between the new Majorana operators:

$$\nu_k = \mathrm{i}\left\langle \hat{w}'_{k+}\hat{w}'_{k-}\right\rangle$$

Recall that $\mathrm{i}\hat{w}'_{k+}\hat{w}'_{k-} = \frac{1}{2}(\hat{a}_j^\dagger + \hat{a}_j)(\hat{a}_j^\dagger - \hat{a}_j) = \frac{1}{2} - \hat{a}_j^\dagger\hat{a}_j$, and $\hat{a}_j^\dagger\hat{a}_j$ has only two eigenvalues 0 and 1, therefore $\langle \hat{a}_j^\dagger\hat{a}_j\rangle \in [0,1]$, and therefore

$$\nu_k \in [-\frac{1}{2}, \frac{1}{2}].$$

The correlation function satisfies the same equation as the correlation operaetor:

$$\partial_t\Gamma = X\Gamma + \Gamma X^T + Y + \sum_s Z_s\Gamma Z_s^T. \tag{9}$$

The steady state correlation function, Γ_∞ satisfies

$$X\Gamma_\infty + \Gamma_\infty X^T + \sum_s Z_s\Gamma_\infty Z_s^T = -Y$$

If $Z_s = 0$, i.e. there is only linear Lindblad terms, we arrive at the following Sylvester equation:

$$X\Gamma_\infty + \Gamma_\infty X^T = -Y.$$

The equation of motion eq. (9) could be rewritten via *vectorizing*:

$$\partial_t\boldsymbol{\Gamma} = K\boldsymbol{\Gamma} + \boldsymbol{Y}, \quad K := X \otimes \mathbf{1} + \mathbf{1} \otimes X + \sum_s Z_s \otimes Z_s \tag{10}$$

Here $\boldsymbol{\Gamma}$ and $\boldsymbol{Y}$ is the vectorization of the matrix Γ, Y.

Note that for fermions, $\Gamma^T = -\Gamma$. Therefore only the subspace of antisymmetric matrices matter. Let $P_\pm$ be the projector onto the subspace of symmetric/anti-symmetric matrices. Note that the generator K maps (anti-)symmetric Γ to (anti-)symmetric subspaces, therefore K has the following block-diagonal form:

$$K = K_+ \oplus K_-, \quad K_\pm = P_\pm K P_\pm.$$

Assuming that K_- in invertible, then the solution of eq. (10) is

$$\boldsymbol{\Gamma}(t) = e^{K_- t}\left(\boldsymbol{\Gamma}(0) + K_-^{-1}\boldsymbol{Y}\right) - K_-^{-1}\boldsymbol{Y}. \tag{11}$$

This tells us that:

- The steady state Γ_∞ is

  $$\Gamma_\infty = -K_-^{-1}\boldsymbol{Y}$$

  and the steady state is unique if K_-^{-1} is invertible.

- The dynamics is stable if all eigenvalues of K_- has non-positive real parts.

  In fact, for arbitrary vector $\boldsymbol{v}$,

  $$\begin{aligned}\mathrm{Re}(\boldsymbol{v}^\dagger K\boldsymbol{v}) &= -\boldsymbol{v}^\dagger\left(B_\mathrm{r} \otimes \mathbf{1} + \mathbf{1} \otimes B_\mathrm{r}\right)\boldsymbol{v} - 2\sum_s \boldsymbol{v}^\dagger\left(M_s^2 \otimes \mathbf{1} + \mathbf{1} \otimes M_s^2 + 2M_s \otimes M_s\right)\boldsymbol{v} \\ &= -\boldsymbol{v}^\dagger\left(B_\mathrm{r} \otimes \mathbf{1} + \mathbf{1} \otimes B_\mathrm{r}\right)\boldsymbol{v} - 2\sum_s \boldsymbol{v}^\dagger\left(M_s \otimes \mathbf{1} + \mathbf{1} \otimes M_s\right)^2\boldsymbol{v}\end{aligned}$$

  And this is non-negative since B_r is positive semi-definite.

- The dynamics is relaxing, i.e. it will converge to the steady state if all eigenvalues of K_- has negative real parts.

In the case of finite fermionic system, i.e. $n < \infty$, since $\mathrm{e}^{t\mathscr{L}}$ is a continuous map and the space of density operators of finite-dimensional systems is a compact convex set, therefore one can use Brouwer's fixed-point theorem to prove that there exists a steady state.

4 Third Quantization

Third quantization is not a quantization. It is introducing ladder super-operators to express the action of the Liouvillian on an operator $\hat{O}$.

Let us define the following super-operators $\mathbb{C}_j$:

$$\mathbb{C}_j(\hat{O}) = \frac{1}{\sqrt{2}}(\hat{w}_j\hat{O} - \hat{\Pi}\hat{O}\hat{\Pi}\hat{w}_j)$$

Here we use the particle-number parity operator $\hat{\Pi}$, which is defined as

$$\hat{\Pi} = (-1)^{\hat{N}}, \quad \hat{N} = \sum_{j=1}^{n} \hat{a}_j^\dagger \hat{a}_j.$$

$\hat{\Pi}$ satisfies that

$$\{\hat{w}_j, \hat{\Pi}\} = 0.$$

The conjugate of $\mathbb{C}_j$, denoted by $\mathbb{C}_j^\dagger$, is defined using the Hilbert-Schmidt inner product:

$$\langle\langle \hat{A} \mid \hat{B} \rangle\rangle := \operatorname{Tr}\left(\hat{A}^\dagger \hat{B}\right)$$

Therefore

$$\left\langle\left\langle \hat{A} \mid \mathbb{C}_j(\hat{B}) \right\rangle\right\rangle = \frac{1}{\sqrt{2}} \operatorname{Tr}\left(\hat{A}^\dagger \left(\hat{w}_j\hat{B} - \hat{\Pi}\hat{B}\hat{\Pi}\hat{w}_j\right)\right) = \frac{1}{\sqrt{2}} \operatorname{Tr}\left(\left(\hat{A}^\dagger\hat{w}_j + \hat{w}_j\hat{\Pi}^\dagger\hat{A}^\dagger\hat{\Pi}\right)\hat{B}\right) = \left\langle\left\langle \mathbb{C}_j^\dagger(\hat{A}) \mid \hat{B} \right\rangle\right\rangle$$

From which we see the definition of $\mathbb{C}_j^\dagger$:

$$\mathbb{C}_j^\dagger(\hat{O}) = \frac{1}{\sqrt{2}}\left(\hat{w}_j\hat{O} + \hat{\Pi}\hat{O}\hat{\Pi}\hat{w}_j\right).$$

This weird-looking definition is for the sake of obtaining canonical anti-commutation relations for these third quatization operators. We have (left as an exercise to verify):

$$\left\{\mathbb{C}_i, \mathbb{C}_j^\dagger\right\} = \delta_{i,j}, \quad \{\mathbb{C}_i, \mathbb{C}_j\} = 0, \quad \text{and} \quad \left\{\mathbb{C}_i^\dagger, \mathbb{C}_j^\dagger\right\} = 0 \quad \text{for} \quad i, j = 1, \ldots, 2n$$

Just like the introduction of Majorana operators, let us introduce the following self-adjoint super-operators:

$$\mathbb{W}_j(\hat{O}) = \frac{1}{\sqrt{2}}(\mathbb{C}_j^\dagger + \mathbb{C}_j)(\hat{O}) = \hat{w}_j\hat{O}$$

$$\widetilde{\mathbb{W}}_j(\hat{O}) = \frac{1}{\sqrt{2}}(\mathbb{C}_j^\dagger - \mathbb{C}_j)(\Pi\hat{O}\Pi) = \hat{O}\hat{w}_j$$

Let us define another superoperator $\mathbb{P}$ as $\mathbb{P}(\hat{O}) = \hat{\Pi}\hat{O}\hat{\Pi}$, then $\mathbb{P}^2(\hat{O}) = \hat{O}$ and

$$\widetilde{\mathbb{W}}_j(R) = \frac{1}{\sqrt{2}}(\mathbb{C}_j^\dagger - \mathbb{C}_j)\mathbb{P}(\hat{O}) = \frac{1}{\sqrt{2}}\mathbb{P}(\mathbb{C}_j - \mathbb{C}_j^\dagger)(\hat{O})$$

.

Now we are ready to rewrite the Lindblad equations using third quantization (superfermion representation).

- From the definition of $\mathbb{W}_i\widetilde{\mathbb{W}}_i$, we have

$$\left[\hat{w}_i\hat{w}_j, \hat{O}\right] = \left(\mathbb{W}_i\mathbb{W}_j - \widetilde{\mathbb{W}}_j\widetilde{\mathbb{W}}_i\right)(\hat{O}) \quad \text{and}$$

$$\hat{w}_i\hat{O}\hat{w}_j - \frac{1}{2}\left\{\hat{w}_j\hat{w}_i, \hat{O}\right\} = \left(\mathbb{W}_i\widetilde{\mathbb{W}}_j - \frac{1}{2}\left(\mathbb{W}_j\mathbb{W}_i + \widetilde{\mathbb{W}}_i\widetilde{\mathbb{W}}_j\right)\right)(\hat{O})$$

- Let us calculate $[H,\hat{O}]$.

$$[H,\hat{O}]=\sum_{ij}H_{ij}[\hat{w}_i\hat{w}_j,\hat{O}]=\left(\sum_{ij}H_{ij}\mathbb{W}_i\mathbb{W}_j-H_{ij}\widetilde{\mathbb{W}}_j\widetilde{\mathbb{W}}_i\right)(\hat{O})=\left(\sum_{ij}H_{ij}\mathbb{W}_i\mathbb{W}_j+H_{ji}\widetilde{\mathbb{W}}_j\widetilde{\mathbb{W}}_i\right)(\hat{O})$$

Here we use $H_{ij}=-H_{ji}$. Let $\boldsymbol{W}=(\mathbb{W}_1,\cdots,\mathbb{W}_{2n})^T$, $\widetilde{\boldsymbol{W}}=(\widetilde{\mathbb{W}}_1,\cdots,\widetilde{\mathbb{W}}_{2n})^T$, and we similarly define $\boldsymbol{C}$ and $\boldsymbol{C}^\dagger$, then we have

$$\boldsymbol{W}=\frac{1}{\sqrt{2}}(\boldsymbol{C}+\boldsymbol{C}^\dagger),\quad \boldsymbol{W}^T=\frac{1}{\sqrt{2}}(\boldsymbol{C}+\boldsymbol{C}^\dagger)^T,\quad \widetilde{\boldsymbol{W}}=\frac{1}{\sqrt{2}}P(\boldsymbol{C}-\boldsymbol{C}^\dagger),\quad \widetilde{\boldsymbol{W}}^T=\frac{1}{\sqrt{2}}(\boldsymbol{C}^\dagger-\boldsymbol{C})^TP,$$

$$\begin{aligned}[H,\hat{O}]=\left(\boldsymbol{W}\cdot H\boldsymbol{W}+\widetilde{\boldsymbol{W}}\cdot H\widetilde{\boldsymbol{W}}\right)(\hat{O})&=\frac{1}{2}\left((\boldsymbol{C}+\boldsymbol{C}^\dagger)\cdot H(\boldsymbol{C}+\boldsymbol{C}^\dagger)+((\boldsymbol{C}^\dagger)-\boldsymbol{C})\cdot P^2H(\boldsymbol{C}-\boldsymbol{C}^\dagger)\right)(\hat{O})\\&=\left(\boldsymbol{C}^\dagger\cdot H\boldsymbol{C}+\boldsymbol{C}\cdot H\boldsymbol{C}^\dagger\right)(\hat{O})=2\boldsymbol{C}^\dagger\cdot H\boldsymbol{C}(\hat{O})=2\boldsymbol{C}\cdot H\boldsymbol{C}^\dagger(\hat{O})\end{aligned}$$

- Using the above result twice, we have

$$D_M(\hat{O})=-\frac{1}{2}\sum_s\left[\hat{M}_s,\left[\hat{M}_s,\hat{O}\right]\right]=-2\sum_s\left(\boldsymbol{C}^\dagger\cdot M_s\boldsymbol{C}\right)^2(\hat{O})$$

- The linear Lindbladian becomes

$$D_L(\hat{O})=\sum_r\left(\hat{L}_r\hat{O}\hat{L}_r^\dagger-\frac{1}{2}\left\{\hat{L}_r^\dagger\hat{L}_r,\hat{O}\right\}\right)=\left(\boldsymbol{W}\cdot B\widetilde{\boldsymbol{W}}-\frac{1}{2}\left(\boldsymbol{W}\cdot B^T\boldsymbol{W}+\widetilde{\boldsymbol{W}}\cdot B\widetilde{\boldsymbol{W}}\right)\right)(\hat{O}).$$

Using $\boldsymbol{C},\boldsymbol{C}^\dagger$, we have

$$\begin{aligned}&\boldsymbol{W}\cdot B\widetilde{\boldsymbol{W}}-\frac{1}{2}\left(\boldsymbol{W}\cdot B^T\boldsymbol{W}+\widetilde{\boldsymbol{W}}\cdot B\widetilde{\boldsymbol{W}}\right)\\=&\frac{1}{2}(\boldsymbol{C}+\boldsymbol{C}^\dagger)\cdot B(\boldsymbol{C}^\dagger-\boldsymbol{C})P-\frac{1}{4}(\boldsymbol{C}+\boldsymbol{C}^\dagger)\cdot B^T(\boldsymbol{C}^\dagger+\boldsymbol{C})-\frac{1}{4}(\boldsymbol{C}^\dagger-\boldsymbol{C})\cdot P^2B(\boldsymbol{C}-\boldsymbol{C}^\dagger)\\=&\left(-\boldsymbol{C}^\dagger\cdot B_{\mathrm{r}}\boldsymbol{C}\mathbb{P}_++\boldsymbol{C}^\dagger\cdot \mathrm{i}B_{\mathrm{i}}\boldsymbol{C}^\dagger\mathbb{P}_+-\boldsymbol{C}\cdot B_{\mathrm{r}}\boldsymbol{C}^\dagger\mathbb{P}_-+\boldsymbol{C}\cdot\mathrm{i}B_{\mathrm{i}}\boldsymbol{C}\mathbb{P}_-\right)(\hat{O})\end{aligned}$$

Here we have defined:

$$\mathbb{P}_\pm(\hat{O}):=\frac{1\pm\mathbb{P}}{2}(\hat{O})=\frac{1}{2}(\hat{O}\pm\hat{\Pi}\hat{O}\hat{\Pi})$$

which is the projections onto the even and odd parity sectors, and we have $\mathbb{P}_\pm\boldsymbol{C}^{(\dagger)}=\boldsymbol{C}^{(\dagger)}\mathbb{P}_\mp$.

- One can see that all three terms above preserve the particle number parity, i.e. $\mathscr{L}P=P\mathscr{L}$.

Combining above, we can see that the Lindbladian $\mathscr{L}$ could be written as a direct sum $\mathscr{L}=\mathscr{L}_+\oplus\mathscr{L}_-$, where $\mathscr{L}_\pm$ is the Liouvillian for the even/odd parity sector:

$$\begin{aligned}\mathscr{L}_+&=\boldsymbol{C}^\dagger\cdot X_0\boldsymbol{C}+\boldsymbol{C}^\dagger\cdot\mathrm{i}B_{\mathrm{i}}\boldsymbol{C}^\dagger-2\sum_s\left(\boldsymbol{C}^\dagger\cdot M_s\boldsymbol{C}\right)^2\\\mathscr{L}_-&=\boldsymbol{C}\cdot X_0\boldsymbol{C}^\dagger+\boldsymbol{C}\cdot\mathrm{i}B_{\mathrm{i}}\boldsymbol{C}-2\sum_s\left(\boldsymbol{C}\cdot M_s\boldsymbol{C}^\dagger\right)^2\end{aligned}\tag{12}$$

and $X_0:=-2\mathrm{i}H-B_{\mathrm{r}}$.

5 Exact diagonalization for Quasi-free systems

Quasi-free systems are systems where there are quadratic Hamiltonian and linear Lindbladian. The Liouvillians become

$$\begin{aligned}\mathscr{L}_+&=\boldsymbol{C}^\dagger\cdot X_0\boldsymbol{C}+\boldsymbol{C}^\dagger\cdot\mathrm{i}Y\boldsymbol{C}^\dagger\\\mathscr{L}_-&=\boldsymbol{C}\cdot X_0\boldsymbol{C}^\dagger+\boldsymbol{C}\cdot\mathrm{i}Y\boldsymbol{C}\end{aligned}\tag{13}$$

Here $X=-2\mathrm{i}H-B_{\mathrm{r}}$, $Y=B_{\mathrm{i}}$.

Let us first write the Lindbladians in an anti-symmetric form. Note that $\boldsymbol{C}^\dagger\cdot X_0\boldsymbol{C}=-\boldsymbol{C}X_0\boldsymbol{C}^\dagger+\mathrm{Tr}(X_0)$, we have

$$\mathscr{L}_+=\frac{1}{2}\begin{pmatrix}\boldsymbol{C}\\\boldsymbol{C}^\dagger\end{pmatrix}\cdot\begin{pmatrix}&-X_0^T\\X_0&2\mathrm{i}Y\end{pmatrix}\begin{pmatrix}\boldsymbol{C}\\\boldsymbol{C}^\dagger\end{pmatrix}+\frac{\mathrm{Tr}\,X_0}{2},\quad \mathscr{L}_-=\frac{1}{2}\begin{pmatrix}\boldsymbol{C}\\\boldsymbol{C}^\dagger\end{pmatrix}\cdot\begin{pmatrix}2\mathrm{i}Y&X_0\\-X_0^T&\end{pmatrix}\begin{pmatrix}\boldsymbol{C}\\\boldsymbol{C}^\dagger\end{pmatrix}+\frac{\mathrm{Tr}\,X_0}{2},$$

5.1 Jordan normal form

Let us first discuss how to obtain the Jordan normal form of the above anti-symmetric matrices. In other words, we are looking for matrix V such that

$$(V^T)^{-1}\begin{pmatrix} & -X_0^T \\ X_0 & 2\mathrm{i}Y \end{pmatrix} V^{-1} = \begin{pmatrix} & -\xi^T \\ \xi & \end{pmatrix} \tag{14}$$

Here ξ, a $2n \times 2n$ matrix, is the Jordan normal (similarity) form of X_0, with N_J Jordan blocks with eigenvalues ξ_k and dimensions D_k. Note that these eigenvalues ξ_k has non-positive real parts (left as an exercise). With this matrix V, we can define new ladder super-operators $\mathbb{D}_j$, $\mathbb{D}'_j$:

$$\begin{pmatrix} \boldsymbol{D} \\ \boldsymbol{D}' \end{pmatrix} = V \begin{pmatrix} \boldsymbol{C} \\ \boldsymbol{C}^\dagger \end{pmatrix}$$

If the new superoperators still satisfy canonical anti-commutation relations:

$$\{\mathbb{D}_i, \mathbb{D}'_j\} = \delta_{ij}, \quad \{\mathbb{D}_i, \mathbb{D}_j\} = 0, \quad \{\mathbb{D}'_i, \mathbb{D}'_j\} = 0, \tag{15}$$

Then the Liouvillian becomes

$$\begin{aligned}
\mathscr{L}_+ &= \frac{1}{2}\begin{pmatrix} \boldsymbol{D} \\ \boldsymbol{D}' \end{pmatrix} \cdot \begin{pmatrix} & -\xi^T \\ \xi & \end{pmatrix}\begin{pmatrix} \boldsymbol{D} \\ \boldsymbol{D}' \end{pmatrix} + \frac{\operatorname{Tr} X_0}{2} \\
&= \frac{1}{2}(\boldsymbol{D}')^T \xi \boldsymbol{D} - \frac{1}{2}\boldsymbol{D}\xi^T \boldsymbol{D}' + \frac{\operatorname{Tr}\xi}{2} = \boldsymbol{D}'\xi\boldsymbol{D} = \sum_{k=1}^{N_J}\left(\xi_k \sum_{\ell=1}^{D_k} \mathbb{D}'_{k,\ell}\mathbb{D}_{k,\ell} + \sum_{\ell=1}^{D_k-1} \mathbb{D}'_{k,\ell}\mathbb{D}_{k,\ell+1}\right).
\end{aligned}$$

Let S be the similarity transformation that brings X_0 to Jordan normal form ξ:

$$S^{-1} X_0 S = \xi.$$

Now let us find this V that satisfies eq. (14) and eq. (15). This is a linear algebra exercise. Turns out if V is

$$V = \begin{pmatrix} S^{-1} & \\ & S^T \end{pmatrix}\begin{pmatrix} \mathbf{1}_{2n} & -2\mathrm{i}\Gamma \\ & \mathbf{1}_{2n} \end{pmatrix}$$

then

$$(V^T)^{-1}\begin{pmatrix} & -X_0^T \\ X_0 & 2\mathrm{i}Y \end{pmatrix} V^{-1} = \begin{pmatrix} & -\xi^T \\ \xi & Q \end{pmatrix}, \quad Q = 2\mathrm{i}S^{-1}\left(X_0\Gamma + \Gamma X_0^T + Y\right)\left(S^{-1}\right)^T.$$

Recall that if Γ is taken to be steady state Γ_∞, then $X_0\Gamma + \Gamma X_0^T + Y = 0$, therefore $Q = 0$. Also note that Γ_∞ exists for finite-dimensional system. One can also verify that this V satisfies eq. (15) (left as an exercise).

It is also easy to see that $\mathscr{L}_-$ could be handled similarly by switching $\mathbb{C}$ and $\mathbb{C}^\dagger$.

Let us summarize these results here: the Liouvillian for the even-parity sector has the following Jordan normal form

$$\mathscr{L}_+ = \sum_{k=1}^{N_J}\left(\xi_k \sum_{\ell=1}^{D_k} \mathbb{D}'_{k,\ell}\mathbb{D}_{k,\ell} + \sum_{\ell=1}^{D_k-1} \mathbb{D}'_{k,\ell}\mathbb{D}_{k,\ell+1}\right), \quad \begin{pmatrix} \boldsymbol{D} \\ \boldsymbol{D}' \end{pmatrix} = \begin{pmatrix} S^{-1} & \\ & S^T \end{pmatrix}\begin{pmatrix} \mathbf{1} & -2\mathrm{i}\Gamma_\infty \\ & \mathbf{1} \end{pmatrix}\begin{pmatrix} \boldsymbol{C} \\ \boldsymbol{C}^\dagger \end{pmatrix},$$

The new ladder super-operators, interpreted as the creation/annihilation super-operator for super-fermions, obey the anti-commutation rules:

$$\left\{\mathbb{D}_{k,\ell}, \mathbb{D}'_{k',\ell'}\right\} = \delta_{k,k'}\delta_{\ell,\ell'}, \quad \left\{\mathbb{D}_{k,\ell}, \mathbb{D}_{k',\ell'}\right\} = 0, \quad \text{and} \quad \left\{\mathbb{D}'_{k,\ell}, \mathbb{D}'_{k',\ell'}\right\} = 0 \quad \forall k, k', \ell, \ell'$$

while the Liouvillian for the odd-parity sector has the following Jordan normal form

$$\mathscr{L}_- = \sum_{k=1}^{N_J}\left(\xi_k \sum_{\ell=1}^{D_k} \overline{\mathbb{D}}'_{k,\ell}\overline{\mathbb{D}}_{k,\ell} + \sum_{\ell=1}^{D_k-1} \overline{\mathbb{D}}'_{k,\ell}\overline{\mathbb{D}}_{k,\ell+1}\right), \quad \begin{pmatrix} \overline{\boldsymbol{D}} \\ \overline{\boldsymbol{D}}' \end{pmatrix} = \begin{pmatrix} S^{-1} & \\ & S^T \end{pmatrix}\begin{pmatrix} \mathbf{1} & -2\mathrm{i}\Gamma_\infty \\ & \mathbf{1} \end{pmatrix}\begin{pmatrix} \boldsymbol{C}^\dagger \\ \boldsymbol{C} \end{pmatrix}.$$

and $\overline{\mathbb{D}}_{k,\ell}$ and $\overline{\mathbb{D}}'_{k,\ell}$ obey the same canonical anti-commutation rule.

5.2 Steady state

Let us first introduce the definition of a Gaussian state. A Gaussian state is fully determined by its correlation matrix Γ. Recall our previous notation:

$$U\Gamma U^T = \begin{pmatrix} & \nu \\ -\nu & \end{pmatrix}, \quad \nu = \mathrm{diag}\left(\nu_1, \dots, \nu_n\right), \quad \hat{w}_i' = \sum_j U_{i,j}\hat{w}_j.$$

Then the Gaussian state corresponding to the correlation matrix Γ is defined as:

$$\hat{\rho}(\Gamma) = \prod_k \left(1/2+\nu_k\right)\left(\frac{1/2-\nu_k}{1/2+\nu_k}\right)^{\mathrm{i}\hat{w}_{k-}'\hat{w}_{k+}'+1/2} = \prod_k \left(1/2+\nu_k\right)\left(\frac{1/2-\nu_k}{1/2+\nu_k}\right)^{1-\hat{a}_k'^{\dagger}\hat{a}_k'}$$

Note that $\mathrm{Tr}(\hat{\rho}) = 1$, and $\nu_k = \mathrm{i}\,\mathrm{Tr}\left(\hat{\rho}\hat{w}_{k+}'\hat{w}_{k-}'\right)$. Also, this $\hat{\rho}$ is in the even-parity sector.

We employ a Dirac notation with super-kets $|\hat{B}\rangle\rangle := \hat{B}$ and super-bras $\langle\langle\hat{A}|$, where $\hat{A}$ and $\hat{B}$ are operators on the Hilbert space, such that $\langle\langle\hat{A} \mid \hat{B}\rangle\rangle = \mathrm{Tr}\left(\hat{A}^{\dagger}\hat{B}\right)$.

We state the following fact:

- By definition, operator $\hat{O}$ in the odd-parity sector satisfies $\hat{\Pi}\hat{O}\hat{\Pi} = -\hat{O}$, therefore are traceless. Since $\mathrm{tr}(\hat{\rho}) = 1$, it must have support in the even-parity sector.
- The above Gaussian state defined by Γ_∞, i.e. $\hat{\rho}(\Gamma_\infty)$ is the steady state of the Lindblad dynamics. Let $|\mathbf{0}\rangle\rangle_{\mathbb{D}} := \hat{\rho}\left(\Gamma_\infty\right)$. In fact, we have
  - $\mathbb{D}_{k,l}$ are annihilation super-operators for this state $|\mathbf{0}\rangle\rangle_{\mathbb{D}}$:
  $$\mathbb{D}_{k,\ell}|\mathbf{0}\rangle\rangle_{\mathbb{D}} = 0, \quad \forall k, \ell$$
  We could also say that $|\mathbf{0}\rangle\rangle_{\mathbb{D}}$ is the vaccuum state of annihilation super-operators $\mathbb{D}_{k,l}$. This is proved with the help of Wick's theorem and direct calculations.
  - As a result, $|\mathbf{0}\rangle\rangle_{\mathbb{D}}$ is the steady state,
  $$\mathscr{L}|\mathbf{0}\rangle\rangle_{\mathbb{D}} = \mathscr{L}_+|\mathbf{0}\rangle\rangle_{\mathbb{D}} = 0.$$
  - If eigenvalue $\xi_k = 0$, then $(\mathbb{D}_{k,1}')^{2m}|\mathbf{0}\rangle\rangle_{\mathbb{D}}$ (and any linear combination with $|\mathbf{0}\rangle\rangle_{\mathbb{D}}$) is also a steady state of $\mathscr{L}$.
- Define $|\mathbf{0}\rangle\rangle_{\overline{\mathbb{D}}} := \hat{\Pi}\hat{\rho}\left(\Gamma_\infty\right)$. Then similarly we have
  - $\overline{\mathbb{D}}_{k,l}$ are annihilation super-operators for this state $|\mathbf{0}\rangle\rangle_{\overline{\mathbb{D}}}$:
  $$\overline{\mathbb{D}}_{k,\ell}|\mathbf{0}\rangle\rangle_{\overline{\mathbb{D}}} = 0, \quad \forall k, \ell$$
  We could also say that $|\mathbf{0}\rangle\rangle_{\mathbb{D}}$ is the vaccuum state of annihilation super-operators $\overline{\mathbb{D}}_{k,l}$.
  - If eigenvalue $\xi_k = 0$, then $(\overline{\mathbb{D}}_{k,1}')^{2m+1}|\mathbf{0}\rangle\rangle_{\overline{\mathbb{D}}}$ is in the odd-parity sector, is traceless and is a stationary state of $\mathscr{L}_-$. Linear combination of this state with $\hat{\rho}(\Gamma_\infty)$ is also steady state.
- $\langle\langle\mathbf{1}|$ and $\langle\langle\hat{\Pi}|$ are the left vacuum for the creation super-operators of the even and odd-parity sectors respectively:
$$\langle\langle\mathbf{1}|\mathbb{D}_{k,\ell}' = 0 \quad \text{and} \quad \langle\langle\hat{\Pi}|\overline{\mathbb{D}}_{k,\ell}' = 0 \quad \forall k, \ell.$$

5.3 Upper-Triangularization

In suitable basis, the Lindbladian could be triangularized. Let us define such basis.

Define right bases $|\boldsymbol{n}\rangle\rangle$ and left bases $\langle\langle\boldsymbol{n}|$ ($\boldsymbol{n} = (\cdots, n_{k,\ell}, \cdots)$) as:

$$|\boldsymbol{n}\rangle\rangle = \begin{cases} \overrightarrow{\prod}_{k,\ell}\left(\mathbb{D}_{k,\ell}'\right)^{n_{k,\ell}}|\mathbf{0}\rangle\rangle_{\mathbb{D}}, & \text{if } \sum_{k,\ell} n_{k,\ell} \text{ is even,} \\ \overrightarrow{\prod}_{k,\ell}\left(\overline{\mathbb{D}}_{k,\ell}'\right)^{n_{k,\ell}}|\mathbf{0}\rangle\rangle_{\overline{\mathbb{D}}}, & \text{if } \sum_{k,\ell} n_{k,\ell} \text{ is odd,} \end{cases}, \quad \langle\langle\boldsymbol{n}| = \begin{cases} \langle\langle\mathbf{1}|\overleftarrow{\prod}_{k,\ell}\left(\mathbb{D}_{k,\ell}\right)^{n_{k,\ell}}, & \text{if } \sum_{k,\ell} n_{k,\ell} \text{ is even,} \\ \langle\langle\Pi|\overleftarrow{\prod}_{k,\ell}\left(\overline{\mathbb{D}}_{k,\ell}\right)^{n_{k,\ell}}, & \text{if } \sum_{k,\ell} n_{k,\ell} \text{ is odd,} \end{cases}$$

Here $\boldsymbol{n} := \left(n_{1,1}, \dots, n_{1,D_1}, n_{2,1}, \dots, n_{N_J,D_{N_J}}\right)$ and $n_{k,\ell} \in \{0, 1\}$.

We have the following fact:

- This is a biorthogonal basis, i.e.

$$\langle\langle \boldsymbol{n} \mid \boldsymbol{n}' \rangle\rangle = \delta_{\boldsymbol{n},\boldsymbol{n}'}, \quad \forall \boldsymbol{n}, \boldsymbol{n}'$$

  This follows directly from the canonical anti-commutation relation (as in second quantization).

- If ordering this basis with increading $I_{\boldsymbol{n}} := \sum_{k=1}^{N_J} \sum_{\ell=1}^{D_k} \ell n_{k,\ell}$, then the matrix representation of the Lindbladian, i.e.

$$\mathscr{L}_{\boldsymbol{n},\boldsymbol{n}'} := \langle\langle \boldsymbol{n} | \mathscr{L} | \boldsymbol{n}' \rangle\rangle$$

  is a upper-triangular matrix. This is a result of direct calculation.

- The diagonal of this matrix, i.e. the generalized eigenvalues, are

$$\left\{ \lambda_{\boldsymbol{n}} = \sum_{k=1}^{N_J} \sum_{\ell=1}^{D_k} \xi_k n_{k,\ell} \mid n_{k,\ell} = 0, 1 \right\}.$$

- If $\operatorname{Re} \xi_k < 0$ for all k, then the system is relaxing with the unique steady state $|\boldsymbol{0}\rangle\rangle$, and the dissipative gap is

$$\Delta \equiv -\max_{\boldsymbol{n} \neq \boldsymbol{0}} \operatorname{Re} \lambda_{\boldsymbol{n}} = -\max_k \operatorname{Re} \xi_k$$

  which corresponds to a single super-fermion excitation in the odd-parity sector.

6 Overview of quadratic systems

If the Lindbladian also contain quadratic terms, we can no longer solve it analytically. However, there are a few things that can be said:

- The Lindbladian could still be block-triangularized.
- The hierarchy of multi-point correlations are closed, in the sense that $\Gamma^{(k)}$ only depends on $\Gamma^{(k)}, \Gamma^{(k-2)}, \Gamma^{(k-4)}$. In the even-parity sector, $\Gamma^{(k)}$ only depends on $\Gamma^{(k)}, \Gamma^{(k-2)}$.