A NOTE ON PRODUCTS OF POSITIVE OPERATORS

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Lemma 0.1. *Let $A, B \in \mathcal{H}$ where $A \succ 0$ and B is normal. Then $AB \succeq 0$ if and only if $B \succeq 0$ and $[A, B] = 0$.*

Remark 0.2. *The assumption B is normal is necessary, as a counter example consider*

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} \tag{1}$$

Since the trace and determinant of A are positive and A is Hermitian, $A \succ 0$. B is not a positive operator (even though its eigenvalues are positive) since

$$\begin{bmatrix} 1 & -i \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ i \end{bmatrix} = 3 - i \tag{2}$$

which is not real. Now notice that

$$AB = \begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix} \tag{3}$$

which is easily seen to be positive.

Proof. Since B is normal, we may diagonalize $B = UDU^\dagger$ where U is unitary and D is diagonal. Now

$$AB \succeq 0 \Leftrightarrow AUDU^\dagger \succeq 0 \Leftrightarrow U^\dagger AUD \succeq 0. \tag{4}$$

Now towards a contradiction, suppose D had a non-positive eigenvalue λ_i. We calculate

$$\langle e_i, U^\dagger AUDe_i \rangle = \lambda_i \langle U^\dagger e_i, AUe_i \rangle \tag{5}$$

which contradicts the assumption $AB \succeq 0$ since $A \succ 0$.

For the converse, since $[A, B] = 0$ and both are non-negative by spectral theorem $[A^{\frac{1}{2}}, B^{\frac{1}{2}}] = 0$. Therefore, since $A^\dagger = A$ and $B^\dagger = B$ we have

$$AB = A^{\frac{1}{2}} A^{\frac{1}{2}} B^{\frac{1}{2}} B^{\frac{1}{2}} = A^{\frac{1}{2}} B^{\frac{1}{2}} B^{\frac{1}{2}} A^{\frac{1}{2}} = \left(A^{\frac{1}{2}} B^{\frac{1}{2}}\right)\left(A^{\frac{1}{2}} B^{\frac{1}{2}}\right)^\dagger \tag{6}$$

which is clearly non-negative. ∎