*** INDUCTION

Base Case

Prove $S(n_0)$

Inductive Hypothesis

Assume $S(n)$ holds for an arbitrary $n \geq n_0$

Inductive Step

Given $S(n)$, prove $S(n+1)$

Axiom of Completeness

If S has an upper bound, then it has a *least* upper bound, namely the supremum of S.

Likewise if S has a lower bound, then it has a *greatest* lower bound, namely the infimum of S.

*** ARCHIMEDEAN PRINCIPLE

If c is any positive number, then there is an integer $n > c$, equivalently

$\forall x \in \mathbb{R}, \exists n \in \mathbb{N}$ such that $n > x$

Additionally, if c is any positive number, then there is an integer n with $0 < \frac{1}{n} < c$, equivalently

$\forall \varepsilon > 0, \exists n \in \mathbb{N}$ such that $\frac{1}{n} < \varepsilon$

Triangle Inequality

$$|a+b| \leq |a| + |b|$$

for all $a, b \in \mathbb{R}$. The following can also be proven from the Triangle Inequality.

$$\begin{aligned} |a| - |b| &\leq |a+b| \\ ||a| - |b|| &\leq |a-b| \\ |a| - |b| &\leq |a-b| \end{aligned}$$

Misc Binomial Formulas

$$\begin{aligned} a^n - b^n &= (a-b)\left(\sum_{i=1}^{n} a^{n-i}b^{i-1}\right) \\ &= (a-b)\left(a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1}\right) \end{aligned}$$

for all $a, b \in \mathbb{R}$ with $n \geq 2$.

Factoring $a^n + b^n$

This is only possible if n is **odd**.

Factoring $1 - r^n$

$$\begin{aligned} 1 - r^n &= (1-r)\left(\sum_{k=0}^{n-1} r^k\right) \\ &= (1-r)\left(1 + r + r^2 + \cdots + r^{n-1}\right) \end{aligned}$$

If $r \neq 1$, then this is a **geometric sum** and the following is true

$$\sum_{k=0}^{n-1} r^k = \frac{1-r^k}{1-r}$$

Binomial Formula

Recall

$${}_nC_k = \binom{n}{k} = \frac{n!}{k!(n-k)!}$$

for $0 \leq k \leq n$.

Now

$$\begin{aligned} &(a+b)^n \\ =&\sum_{k=0}^{n} \binom{n}{k} a^{n-k}b^k \\ =&\binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{n}b^n \end{aligned}$$

Consequence

Let $\frac{p}{q} \in \mathbb{Q}$ be a solution to

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0$$

with $a_i \in \mathbb{Z}$ for all $0 \leq i \leq n$. Then $p | a_0$ and $q | a_n$

Proof

$$a_n\left(\frac{p}{q}\right)^n + a_{n-1}\left(\frac{p}{q}\right)^{n-1} + \cdots + a_1\left(\frac{p}{q}\right) + a_0 = 0$$

Multiplying by q^n, we get

$$a_np^n + a_{n-1}p^{n-1}q + \cdots + a_1pq^{n-1} + a_0q^n = 0$$

With some manipulation, we get

$$\begin{aligned} a_np^n &= -\left(a_{n-1}p^{n-1}q + \cdots + a_0q^n\right) \\ &= -q\left(a_{n-1}p^{n-1} + \cdots + a_0q^{n-1}\right) \end{aligned}$$

so $q|a_np^n$ and thus $q|a_n$.

A similar manipulation from (1) can be done with a_0q^n to obtain

$$\begin{aligned} a_0q^n &= -\left(a_np^n + \cdots + a_1pq^{n-1}\right) \\ &= -p\left(a_{n-1}p^{n-1} + \cdots + a_0p^{n-1}q^{n-1}\right) \end{aligned}$$

so $p|a_0q^n$ and thus $p|a_0$.

Application

Does $2x^2 - 5x - 3 = 0$ have a rational solution $\frac{p}{q}$?

If so, $p| - 3$ so $p = \pm 1$ or $p = \pm 3$ and $q|2$ so $q = \pm 1$ or $q = \pm 2$.

*** MONOTONE CONVERGENCE THEOREM

Let $\{a_n\}_{n=1}^{\infty}$ be monotone.

$\{a_n\}_{n=1}^{\infty}$ converges **if and only if** it is bounded.

** Nested Interval Theorem

Let $I_n = [a_n, b_n]$ with $I_n \subseteq I_{n+1}$ for $n \geq 1$.

If $\lim_{n\to\infty}(b_n - a_n) = 0$.

⟨ We need this condition to ensure the interval converges to a point, and not a sub-interval ⟩

Then there is a unique x^* in $\bigcap_{n=1}^{\infty} I_n = \{x : x \in I_n \text{ for all } n\}$

⟨ There is a unique point x^* that is inside all intervals I_n ⟩

and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = x^*$.

⟨ a_n and b_n converge to the same value: x^* ⟩

TODO Boundedness, Sum Rule, Product Rule, Quotient Rule, Convergence...

Subsequences

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence and let $\{n_k\}_{k=1}^{\infty}$ be a sequence of positive integers. Then, $\{a_{n_k}\}$ is a subsequence of $\{a_n\}_{n=1}^{\infty}$.

Note: A subsequence is infinite.

Peak Index

A peak index n_* for a sequence $\{a_n\}_{n=1}^{\infty}$ is such that $n_* \geq 1$ with $a_m \leq a_{n_*}$ for all $m \geq n_*$.

⟨ All values that come **after** value n_* are less than or equal to it. ⟩

Every sequence has a monotone subsequence

1. If $\{a_n\}$ has infinitely many peak indices,

   ⟨ for example $\{\frac{1}{n}\}$ ⟩

   then the subsequence $\{a_{n_k}\}$ with $\{n_k\}_{k=1}^{\infty}$ is a decreasing subsequence.

2. If $\{a_n\}$ has finitely many peak indices,

   ⟨ for example $\{n\}$ ⟩

   then there is a subsequence that is strictly increasing.

** Corollary 2.33

Every bounded sequence $\{a_n\}$ has a convergent monotone subsequence.

1. If $\{a_n\}$ has infinitely many peak indices, the subsequence is decreasing.
2. If $\{a_n\}$ has finitely many peak indices, the subsequence is also decreasing.

Then, you can use the Monotone Convergence Theorem so show that this subsequence converges.

Sequential Compactness

Let $S \subseteq \mathbb{R}$ be a set. S is sequentially compact if every sequence has a subsequence that converges.

Any set $[a, b] \subseteq \mathbb{R}$ is sequentially compact.

Density in $\mathbb{R}$

A set S is dense in $\mathbb{R}$ if and only if for each $x \in \mathbb{R}$, there is a sequence $\{a_n\}_{n=1}^{\infty} \subseteq S$ with $a_n \to x$

Sequential Continuity

Let $f : D \to \mathbb{R}$ be a function. f is continuous at $x_0 \in D$ if, for any sequence $\{x_n\}_{n=1}^{\infty} \subseteq D$ with $x_n \to x_0$, then $f(x_n) \to f(x_0)$.

⟨ f is continuous at x_0 if, for all sequences subset of the domain that converge to x_0, applying the f to the sequence converges to $f(x_0)$ ⟩

Rules of Continuity

Let f, g be continuous functions at $x_0 \in D$.

- $f + g$ is continuous at x_0
- $f \cdot g$ is continuous at x_0
- If $g(x_0) \neq 0$, then $\frac{f}{g}$ is continuous at x_0

Maximum Value

$f : D \to \mathbb{R}$ has a maximum value $f(x_0)$ if $x_0 \in D$ and $f(x) \leq f(x_0)$ for all $x \in D$. There is only ever **at most one** maximum value.

Minimum Value

$f : D \to \mathbb{R}$ has a minimum value $f(x_0)$ if $x_0 \in D$ and $f(x) \geq f(x_0)$ for all $x \in D$. There is only ever **at most one** minimum value.

Extreme Values

Extreme values are the maximum and minimum values of f

Maximizer

x_0 is a maximizer of f if $f(x_0)$ is the maximum value of f. There can be **infinitely many** maximizers.

Example

Let $f(x) = sin(x)$. Then $\frac{\pi}{2} + 2\pi n$ are all the maximizers of f, and the maximum value is 1.

Minimizer

x_0 is a minimizer of f if $f(x_0)$ is the minimum value of f. There can be **infinitely many** minimizers.

Example

Let $f(x) = sin(x)$. Then $-\frac{\pi}{2} + 2\pi n$ are all the minimizers of f, and the minimum value is -1.

Example

Let $f(x) = x$ with $0 < x < 1$. Then f has no maximum nor minimum values.

** Lemma 3.10

If $f : [a, b] \to \mathbb{R}$ is continuous, then it is bounded.

*** EXTREME VALUE THEOREM

Let $f : [a, b] \to \mathbb{R}$ be continuous. Then, f has a maximum value, and a minimum value.

*** INTERMEDIATE VALUE THEOREM

Let $f : [a, b] \to \mathbb{R}$ be continuous. Let $c \in [f(a), f(b)]$ be arbitrary, with $f(a) < c < f(b)$.

Then there is an $x_0 \in [a, b]$ such that $f(x_0) = c$.

Note: If f is a polynomial of odd degree, then the range of f is $(-\infty, \infty)$.

Bisection Method

This is used to approximate the zeroes of a function f that is continuous on $[a, b]$, with $f(a)f(b) < 0$.

Let $c_1 = \frac{a+b}{2}$ be the midpoint of $[a, b]$. If $f(c_1) = 0$, then we are done.

If $f(c_1)f(a) > 0$, let $c_2 = \frac{a+c_1}{2}$, and repeat.

If $f(c_1)f(b) > 0$, let $c_2 = \frac{c_1+b}{2}$, and repeat.

If $b - a = 1$, then c_n is within $\frac{b-a}{2^n}$ of a zero of f.

*** UNIFORM CONTINUITY

The function $f : D \to \mathbb{R}$ is *uniformly continuous* on D if, whenever $\{u_n\}_{n=1}^{\infty}$ and $\{v_n\}_{n=1}^{\infty}$ are sequences in D such that $|u_n - v_n| \to 0$ as $n \to \infty$, then $|f(u_n) - f(v_n)| \to 0$ as $n \to \infty$.

Note: If f is uniformly continuous on D, then it is continuous on D.

Note: Neither $\{u_n\}_{n=1}^{\infty}$ nor $\{v_n\}_{n=1}^{\infty}$ need to be convergent, or even bounded.

** Theorem 3.17

Let $f : [a, b] \to \mathbb{R}$ be continuous. Then f is uniformly continuous.

$\varepsilon - \delta$ Continuity

For all $\varepsilon > 0$, there exists $\delta > 0$ such that, if $0 < |x - a| < \delta$ then $|f(x) - f(a)| < \varepsilon$.

This definition is equivalent to the sequential definition of continuity.

Theorem 3.20

The sequence definition of continuity is equivalent to the $\varepsilon - \delta$ definition.

Monotone Functions

$f : D \to \mathbb{R}$ is **monotonically increasing** if for any $x, z \in D$, $x < z \Rightarrow f(x) \leq f(z)$.

$f : D \to \mathbb{R}$ is **strictly monotonically increasing** if for any $x, z \in D$, $x < z \Rightarrow f(x) < f(z)$.

$f : D \to \mathbb{R}$ is **monotonically decreasing** if for any $x, z \in D$, $x < z \Rightarrow f(x) \geq f(z)$.

$f : D \to \mathbb{R}$ is **strictly monotonically decreasing** if for any $x, z \in D$, $x < z \Rightarrow f(x) > f(z)$.

Thm 3.23

Let f be monotone and $f(D)$ is an **interval** $\langle$ no unions, must be a contiguous interval $\rangle$. Then f is continuous.

Injective Functions

$f : D \to \mathbb{R}$ is **injective** $\langle$ one to one $\rangle$ if for any $x, z \in D$, $x \neq z \Rightarrow f(x) \neq f(z)$.

Surjective Functions

$f : D \to \mathbb{R}$ is **surjective** $\langle$ onto $\rangle$ if for all $y \in \mathbb{R}$, there exists $x \in D$ such that $f(x) = y$. $\langle$ Note that x may not be unique. $\rangle$

Inverse Functions

Let $f : D \to \mathbb{R}$ be one to one. Then f has an inverse f^{-1}, defined by

$$f^{-1}(y) = x \Leftrightarrow f(x) = y$$

Limit Points

If $D \subseteq \mathbb{R}$, then x_0 is a **limit point** of D if there exists a sequence $\{x_n\}_{n=1}^{\infty} \subseteq D$ with $\{x_n\} \to x_0$ and $x_n \neq x_0$ for any n. $\langle$ The sequence approaches x_0, but is never x_0. $\rangle$

Limits

Let $f : D \to \mathbb{R}$ and x_0 be a limit point of D. Then $\lim_{x \to x_0} f(x) = L$ if for each $\{x_n\} \subseteq D$ with $x_n \neq x_0$ for all n, then $x_n \to x_0$ *implies that* $f(x_n) \to L$. $\langle$ The limit of all sequences $\{x_n\}$ must converge $\rangle$

Composition of Limits

Let $f : D \to \mathbb{R}$ and $g : U \to \mathbb{R}$ with $f(D) \subseteq U$. Suppose that x_0 is a limit point of D, $\lim_{x \to x_0} f(x) = y_0$, and $\lim_{y \to y_0} g(y) = L$. Then $\lim_{x \to x_0} (g \circ f)(x) = L$.

TODO Quotient Rule, Product Rule, Sum Rule of limits.

Chapter 4

Neighborhoods

If $x_0 \in I = (a,b)$, then I is a **neighborhood** of x_0.

The Derivative

Let $f : D \to \mathbb{R}$, $x_0 \in D$ and D contains a neighborhood of x_0. If $\lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0}$ exists, then it is the **derivative** of f at x_0; it is denoted as $f'(x_0)$.

We say that f is differentiable at x_0. $\langle$ Likewise, if this is true of all points on an interval I, we say that f is differentiable on I. $\rangle$

** Proposition 4.5

If $f'(a)$ exists, then f is continuous at a.

** Sum Rule

$$(f+g)'(x_0) = f'(x_0) + g'(x_0)$$

** Product Rule

$$(fg)'(x_0) = f(x_0)g'(x_0) + f'(x_0)g(x_0)$$

** Quotient Rule

If $g(x_0) \neq 0$, then

$$\left(\frac{f}{g}\right)'(x_0) = \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{g^2(x_0)}$$

** Chain Rule

Let $f : I \to \mathbb{R}$ and $g : J \to \mathbb{R}$ with $f(I) \subseteq J$. Additionally, assume that $f'(x_0), g'(f(x_0))$ exist. Then,

$$(f \circ g)'(x_0) = [g'(f(x_0))]f'(x_0)$$

Rolle's Theorem

Let $f : A \to B$ be continuous on $[A,B]$ and differentiable on (A,B). Further assume that $f(a) = f(b)$. Then, there exists an $x_0 \in (a,b)$ with $f'(x_0) = 0$.

TODO check this definition

*** MEAN VALUE THEOREM

Let $f : [a,b] \to \mathbb{R}$ be continuous, and be such that f' exists on (a,b). Then, there is an $x_0 \in (a,b)$ with

$$f'(x_0) = \frac{f(b)-f(a)}{b-a}$$

$\langle$ There is a point in (a,b) such that the tangent line at that point is parallel to a line from $(a, f(a)), (b, f(b))$ $\rangle$

$\langle$ This is a generalization of Rolle's Theorem $\rangle$

Lemma 4.19

TODO Makes no sense in my notes

** Identity Criterion

Let I be an open interval, and $f : I \to \mathbb{R}$, $g : I \to \mathbb{R}$. Then, $f' = g'$ on I **if and only if** there is a constant c with $f(x) = g(x) + c$, for all $x \in I$.

2^{nd} Derivative Test

Let I be an open interval, let $x_0 \in I$, and let $f : I \to \mathbb{R}$ be such that f' and f'' exist on I, with the additional property that $f'(x_0) = 0$. Then

- $f''(x_0) < 0 \Rightarrow x_0$ is a local maximizer
- $f''(x_0) > 0 \Rightarrow x_0$ is a local minimizer

Cauchy Mean Value Theorem

Let $f : [a,b] \to \mathbb{R}$ and $g : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b), with $g'(x) \neq 0$ for all x and $g(b) \neq g(x)$. Then there is $x_0 \in (a,b)$ within

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(x_0)}{g'(x_0)}$$

Theorem 4.24

Let I be open, $n \neq 1$, $f : I \to \mathbb{R}$ with n^{nt} derivative of f existing on I. If $f^{(k)}(x_0) = 0$ for some $x_0 \in I$, $0 \leq k < n$, then for all $x \in I$ there is z_x between x and x_0 with

$$f(x) = \frac{f^{(n)}(z_n)}{n!}(x-x_0)^n$$

L'Hôpital's Rule

Let f, g be continuous on (a,b), f', g' exist on (a,b) assume $g(x) \neq 0$, also $g'(x) = 0$ for $x \in (a,b)$. Then,

$$\lim_{x \to a^+} \frac{f(x)}{g(x)} = \lim_{x \to a^+} \frac{f'(x)}{g'(x)}$$

If $f(a) = g(a) = 0$

Darboux Sums

Let $f : [a,b] \to \mathbb{R}$ be bounded and $P = \{a_1 = x_1, ..., x_n = b\}$ be a partition on $[a,b]$

Lower Sum

$$L(f,P) = \sum_{i=1}^{n} m_i(x_i - x_{i-1})$$

Upper Sum

$$U(f,P) = \sum_{i=1}^{n} M_i(x_i - x_{i-1})$$

Where $M_i = \sup(x), m_i = \inf(x)$ for $x \in (x_{i-1}, x_i)$.

Partition Refinment

A partition P^* is a refinment of P on $[a,b]$ if

$$P \subseteq P^* \subset [a,b]$$

Note

If P^* is a refinment of P on $[a,b]$, then for f bounded on $[a,b]$, $L(f,P) \leq L(f,P^*)$ and $U(f,P) \geq U(f,P^*)$.

Note

If $f : [a,b] \to \mathbb{R}$ is continuous and P, Q are partitions on $[a,b]$, then

$$L(f,P) \leq U(f,Q)$$

Lemma 6.3

Let $f : [a,b] \to \mathbb{R}$ be bounded. Let P, Q be partitions on $[a,b]$. Then

$$L(f,P) \leq U(f,Q)$$

Upper and Lower Integrals

Let $f : [a,b] \in \mathbb{R}$ be bounded, then

$$\int_{-a}^{b} f = \sup(L(f,P)) \qquad \int_{a}^{-b} f = \sup(L(f,P))$$

For a partition P

f is integrable on $[a,b]$ if

$$\int_{-a}^{b} f = \int_{a}^{-b} f$$

*** ARCHIMEDES RIEMANN THEOREM

Let $f : [a,b] \to \mathbb{R}$ be bounded. f is integrable **if and only if** there exists a sequence of partitions $\{P_n\}_{n=1}^{\infty}$ such that

$$\lim_{n \to \infty} (U(f,P_n) - L(f,P_n)) = 0$$

$\langle$ The difference of the upper and lower Darboux sums converges. $\rangle$

Partition Gaps

The gap of a partition on $[a,b]$ is the largest subinterval $[x_{i-1}, x_i]$ of P.

Regular Partitions

A partition P on $[a,b]$ is regular if all subuintervals have the same length

$$h = \frac{b-a}{n} = x_i - x_{i-1}$$

with h as the gap size.

Note

$$\int_{b}^{a} f := -\int_{a}^{b} f$$

** Integrability Theorem

Let $f : [a,b] \to \mathbb{R}$ be continuous. Let $G(x) \int_a^x f(t)dt$ for $x \in [a,b]$. Then

$$G'(x) = f(x)$$

for $x \in (a,b)$

Def

G is the **antiderivative** of g if $G' = g$

*** FUNDAMENTAL THM OF CALCULUS I

Let $F : [a, b] \to \mathbb{R}$ be continuous and let $f = F'$ on (a, b) with f continuous and bounded on the open interval. Then

$$\int_a^b f(t)dt = \int_a^b F'(t)dt = F(b) - F(a)$$

*** FUNDAMENTAL THM OF CALCULUS II

Let $f : [a, b] \to \mathbb{R}$ be contiuous. Then

$$\frac{d}{dx} \int_a^x f(t)dt = f(x)$$

for $x \in (a, b)$

Corollary

Let I, J be open intervals, let $f : I \to \mathbb{R}$, $\phi : J \to \mathbb{R}$ with $\phi(J) \subseteq I$. Assume that ϕ is differentiable and f is continuous. Then

$$\frac{d}{dx} \int_a^{\phi(x)} f(t)dt = [f(\phi(x))]\phi'(x)$$

Mean Value Thm for Integrals

Let $f : [a, b] \to \mathbb{R}$ be continuous. Then, there is an $x_0 \in [a, b]$ with

$$\frac{1}{b-a} \int_a^b f(t)dt = f(x_0)$$

or equivalently,

$$\int_a^b f(t)dt = f(x_0)(b-a)$$

$\langle$ You can turn the area into a rectangle. $\rangle$

Order of Contact

Let I be an open interval. Let $x_0 \in I$, and let $f : I \to \mathbb{R}$, $g : I \to \mathbb{R}$. Then f, g have **contact of order** n at x_0 if

$$f^{(k)}(x_0) = g^{(k)}(x_0) \text{ for } k = 0, ..., n$$

Taylor Polynomials

Let $x_0 \in I$ with I open, and $f : I \to \mathbb{R}$ with derivatives at x_0. Then there is a unique polynomial P_n with degree $P_n \leq n$ such that contact order with f at x_0 is n. P_n is called **the n^{th} Taylor Polynomial of f at** x_0, and

$$P_n(x) = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

*** LAGRANGE REMAINDER FORMULA

Let $f^{(n+1)}$ exist on an open neighborhood I for x_0. Then

$$\begin{aligned} f(x) =& P_n(x) + R_n(x) \\ =& \left[\sum_{k=0}^{n} \frac{f^{(n)(x_0)}}{k!}(x - x_0)^k\right] + \frac{f^{(n+1)}(c_x)}{(n+1)!}(x - x_0)^{n+1} \end{aligned}$$

Where R_n is the remainder, and $c_x \in (x, x_0)$

Thm 8.4

Let $f : I \to \mathbb{R}$ with I open, let $x_0 \in I$, and assume that $f^{(n)}(x_0)$ exists for all n. If there are $r > 0$ and $M < \infty$ so that $|x - x_0| < r$, then $|f^{(n)}(x)| \leq M^n$ for all $n \geq 0$, and $x \in I$. Then $\lim_{n\to\infty} R_n(x) = 0$ for such x some

$$f(x) = \sum_{k=1}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k$$

Cauchy Sequence

A sequence is Cauchy if, for any $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that, for any $n, m \in \mathbb{N}$, if $n, m > N$, then $|a_m - a_n| < \varepsilon$.

Thm 9.4: A sequence converges if and only if it is Cauchy.

Convergence Tests

k^{th} term test

If $\sum_{k=1}^{\infty} a_k$ converges, then $\lim_{n\to\infty} a_k = 0$. $\langle$ Note that **the converse is not always true**, i.e. Harmonic series. $\rangle$

Comparison Test

If $\sum_{k=1}^{\infty} b_k$ converges and $b_k \geq a_k$, then $\sum_{k=1}^{\infty} a_k$ converges.

Geometric Series

Let $c \neq 0$, if $0 \leq r < 1$, then

$\sum_{k=0}^{\infty} r^k = \frac{1}{1-r}$, and $\sum_{k=m}^{\infty} r^k = r^m \sum_{k=1}^{\infty} r^k = \frac{r^m}{1-r}$

$\sum_{k=1}^{\infty} cr^k = \frac{cr^m}{1-r}$ if $|r| < 1$.

Ratio Test

Let $a_k > 0$ for all $k \geq 1$. Assume that $\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = L$. If $0 \geq L < 1$, then $\sum_{k=1}^{\infty} a_k$ converges. If $L > 1$, $\sum_{k=1}^{\infty} a_k$ diverges. If $L = 1$ either could happen.

Integral Test

Let $\{a_k\}_{k=1}^{\infty}$ be a strictly decreasing sequence, and $\lim_{k\to\infty} a_k = 0$. Let $f : [1, \infty) \to \mathbb{R}$ be contiunous and decreasing, with $f(k) = a_k$ for all $k \geq 1$. Then

$\sum_{k=1}^{\infty} a_k$ converges if and only if $\int_1^{\infty} f(x)dx$ converges.

p **Test**

$\sum_{k=1}^{\infty} \frac{1}{k^p}$ converges if and only if $p > 1$ with $p \in \mathbb{R}$

Alternating Convergence Test

$\sum_{k=1}^{\infty} (-1)^k a_k$ is alternating if $a_k \geq 0$ for all k. If a_k is strictly decreasing, then the alternating series converges.

Root Test

Consider $\sum_{k=1}^{\infty} a_k$ with $a_k \geq 0$ for all k. Assume that $\lim_{k\to\infty} (a_k)^{1/k} = L$

- If $0 \geq L < 1$, then the series converges
- If $L > 1$, then the series diverges
- If $L = 1$, then the root test doesn't tell us anything

Pointwise Convergence

$$(\forall x \in D)(\forall \varepsilon > 0)(\exists N \in \mathbb{N})(\forall n \in \mathbb{N})[n > N \Rightarrow f_n(x) \in (f(x)-\varepsilon, f(x)+\varepsilon)]$$

Pick any x in your domain, then pick any margin $\varepsilon > 0$ around your domain, then you can go far enough out in your sequence of functions (specifically at least N far) so that every n larger than N guarantees that the f_n function at point x lives inside the ε margin around the true function f at point x.

Uniform Convergence

Uniform convergence is stronger. Everything that converges uniformly converges pointwise, but the converse is not necessarily true.

$$(\forall \varepsilon > 0)(\exists N \in \mathbb{N})(\forall x \in D)(\forall n \in \mathbb{N})[n > N \Rightarrow f_n(x) \in (f(x)-\varepsilon, f(x)+\varepsilon)]$$

Pick any ε margin, if you go far enough out in your sequence of functions (specifically N far) and then pick any x in your domain, then for all numbers n larger N it is guaranteed that the f_n function at point x lives inside the ε margin around the true function f at point x.

The distinction is that by picking a margin for the whole interval first, and then picking a point x in your domain, your entire function has to live in this ε margin in order to converge uniformly.

On the other hand, for pointwise convergence, the ability to pick a distinct N that satisfies the ε margin for each point x in our domain makes it "easier" for the function f to converge. Note that this ε margin is the same at every point, but the N value might differ at every point.