Generalization of NLIN model for WDM systems considering fiber attenuation and Raman gain

Abstract

Starting from [2], in which a model for NLIN (nonlinear Interference Noise) is provided in the assumption of perfect amplification, we develop a generalized model for this phenomenon in the presence of wavelength and position-dependent attenuation and Raman gain. This analysis aims to open the possibility to address the optimization problem with respect to noise performance in multi-pump Raman amplifiers, which was a work already

I. Introduction

nonlinear Interference Noise (NLIN) is a phenomenon which is a crucial limit to the performances of modern WDM (Wavelength Division Multiplexing) links. In systems that implement Raman amplification, along with Amplified Stimulated Emission noise (ASE), NLIN is a major subject of interest in determining the noise performance of amplifiers. A time domain model for NLIN is proposed in [2]: the model proposes a coefficient of intrachannel interaction which can be used with general modulation schemes. However, in this work, the nonlinear interaction is modeled assuming a *perfect amplification*, which means that the coefficient of Raman amplification $g(z)$ is identically equal to α, and this relationship holds for *every* WDM channel. In order to drop this assumption, it is required to develop a slight modification of the model.

The aim of the article is twofold: in the first place is to generalize the model proposed in [2] to generic Raman gain and attenuation scenarios: the goal is to derive an expression for the XPM (Cross Phase Modulation) coefficient $X_{0,m,m}$ similar to the one in [2], identifying the similarities and the differences of the two approaches. In second place, the aim is to shed light on some mathematical aspects of the problem that may result not straightforward.

In section II a review of past work is given along with some comments on notation and methods that are useful for the rest of the article.

In section III we develop a new framework in which it is possible to generalize the analysis done in past works to scenarios in which wavelength and position dependent Raman gain and attenuation are present.

In order to compute the XPM coefficient of interest, in section IV the approximation developed in [2] is considered, and its limits are described. To circumvent these limitations, in section V the computation are done in the gaussian pulses hypotesis, which allow a simplification of original expression of the coefficient.

Appendices are given to clarify the mathematical aspects of the generalization.

II. Summary of previous results

In the model developed in [2], a single nonlinear Schrödinger Equation (NLSE) is adopted to model the field inside the fiber. Let us examine the structure of the equation.

A. Equation for the field and narrowband approximation

Let us consider the standard NLSE for a fiber with Raman amplification profile $g(z)$

$$\frac{\partial}{\partial z}A = -\frac{\alpha - g(z)}{2}A - \beta_1\frac{\partial}{\partial t}A - i\frac{\beta_2}{2}\frac{\partial^2}{\partial t^2}A + i\gamma|A|^2A \tag{1}$$

where t is the physical time. Recall that A is proportional to the electric field inside the fiber, in a way such that the dimension of A is $[A^2] = \text{W}$.

This equation holds for a narrowband field, such that $A(z,t)$ is a slowly varying function of t. In a WDM system, this approximation is assumed to be still true, as the usual channel spectral spacing is greater than 12.5GHz in third window (as defined in standard [3] for DWDM architectures). In the presence of tens of channels, the total field may still be assumed as narrowband.

However, while this assumption can be used for model dispersion in a simple way, using β_2 (second order derivative of the phase constant with respect to frequency), it is not sufficient in assuming constant attenuation-gain terms along all the wavelengths. In fact, a constant attenuation and Raman gain over the signal bandwidth may be an assumption too strong to make, especially when interested in multiple pump Raman amplification. This is the case of the study in [4], from which the present further analysis stemmed. So the model with a single NLSE equation holds for *perfect amplification* scenarios, but needs to be modified in the generalization.

Let us comment now a mathematical procedure that simplify the notation in the NLSE, in presence of attenuation.

B. Rescaling of fields

Let $\psi(z)$ be defined as a solution of the following differential equation

$$\frac{d}{dz}\psi(z) = -\frac{\alpha - g(z)}{2}\psi(z) \tag{2}$$

using such function, let $u(z,t)$ be defined as the *normalized field*

$$A(z,t) = \psi(z)u(z,t) \tag{3}$$

These definitions, when substituted in 1 give the following equation:

$$\frac{\partial}{\partial z}u = -i\frac{\beta_2}{2}\frac{\partial^2}{\partial t^2}u + i\gamma f(z)|u|^2 u \tag{4}$$

where $f(z) = \psi(z)^2$. The advantage in using the 4 is that the attenuation dynamics can be described with a space-dependent nonlinear term: $\gamma f(z)$.
In the model from [2], using the *perfect amplification* assumption, $\gamma f(z) = \gamma$, since $f(z) \equiv 1$.

III. Coupled NLS equations for WDM channels

By including attenuation and gain it is required to drop the narrowband approximation: in doing so a solution model is the one of the coupled NLSE.
Consider two WDM channels named A and B. The following hypotesis are made:

- channels A and B have a spectral separation of Ω (a multiple of the WDM spectral spacing),
- both channels have the same nonlinear coefficient (it depends from modal field distribution in the core which is assumed to be the same for all channels),
- the group velocity profile is approximately linear in the frequency (β_2 is constant) in the whole band of interest,
- attenuation and Raman gain depend on the channel choice, but are approximately constant within the band of a given channel.

Following Agrawal [1, p.263], a system of *coupled NLSE* is given:

$$\frac{\partial}{\partial z}A_A = -\frac{\alpha_A - g_A(z)}{2}A_A - \beta_{1A}\frac{\partial}{\partial t}A_A - i\frac{\beta_2}{2}\frac{\partial^2}{\partial t^2}A_A + i\gamma(|A_A|^2 + 2|A_B|^2)A_A \tag{5}$$

$$\frac{\partial}{\partial z}A_B = -\frac{\alpha_B - g_B(z)}{2}A_B - \beta_{1B}\frac{\partial}{\partial t}A_B - i\frac{\beta_2}{2}\frac{\partial^2}{\partial t^2}A_B + i\gamma(|A_B|^2 + 2|A_A|^2)A_B \tag{6}$$

Let A be the WDM channel of interest, and B the interfering one.

Let us now proceed in normalizing the fields A_A, A_B with the respective normalization functions ψ_A, ψ_B, as described in the rescaling equation (II-B). In addition, a moving time reference frame is assumed, taking as a reference the time of arrival of the first pulse in channel A: *from this point, until the end of the article, the variable t indicates the normalized time.*
These passages lead to:

$$\frac{\partial}{\partial z}u_A = \qquad -i\frac{\beta_2}{2}\frac{\partial^2}{\partial t^2}u_A + i\gamma\left(f_A(z)|u_A|^2 + 2\frac{f_A(z)}{f_B(z)}|u_B|^2\right)u_A \tag{7}$$

$$\frac{\partial}{\partial z}u_B = -\Delta\beta_1\frac{\partial}{\partial t}u_B - i\frac{\beta_2}{2}\frac{\partial^2}{\partial t^2}u_B + i\gamma\left(f_B(z)|u_B|^2 + 2\frac{f_B(z)}{f_A(z)}|u_A|^2\right)u_B \tag{8}$$

$$\tag{9}$$

where $\Delta\beta_1 = \beta_{1B} - \beta_{1A} = \beta_2\Omega$.
Following [2], a first order perturbation analysis is proposed for these equations.

A. Generalization of the 0^{th} order term

Since the attenuation only affects the nonlinear term, the normalized field of the 0^{th} order must be identical as the one derived in [2, eq. 1]. The only exception is due to the notation used: the total field in this case can not be expressed by a simple sum of terms $u_A^{(0)} + u_B^{(0)}$. There are in fact two notational caveats:

- $u_A^{(0)}$ and $u_B^{(0)}$ functions represent normalized fields with different normalization constants,
- the functions are derived from *complex amplitudes* (see for example [5, pp. 523-525]) of *different* carrier frequency signals

Let us consider the initial fields as sums of shifted impulses which encode a given message. Let T be the symbol period:

$$\begin{aligned} u_A(0,t) &= \sum_k a_k g(0, t - kT) \\ u_B(0,t) &= \sum_k b_k g(0, t - kT) \end{aligned} \tag{10}$$

Let $g^{(0)}(z,t)$ be the linearly propagated field in channel A. The solution for the 0^{th} order field is:

$$\begin{aligned} u_A^{(0)}(z,t) &= \sum_k a_k g^{(0)}(z,t-kT) \\ u_B^{(0)}(z,t) &= \sum_k b_k g^{(0)}(z,t-kT-\beta_2\Omega z) \end{aligned} \tag{11}$$

because of linearity, and definition of $g^{(0)}$.

As in [2], we define the operator of linear propagation for channel A

$$\mathbf{U}_A(z) = \exp\left[i\frac{\beta_2}{2}z\frac{\partial^2}{\partial t^2}\right]. \tag{12}$$

So the channel A propagated 0^{th} order impulse have this alternative expression:

$$g^{(0)}(z,t-kT) = \mathbf{U_A}(z)g(0,t-kT). \tag{13}$$

B. Generalization of first order perturbation theory

The splitting in two of the equation allow us to analyze separately the effects of SPM and XPM in a natural way. Let us apply the perturbation method to equation (7) for channel A:

$$\frac{\partial}{\partial z}u_A^{(1)} = -i\frac{\beta_2}{2}\frac{\partial^2}{\partial t^2}u_A^{(1)} + i\gamma\left(f_A(z)|u_A^{(0)}|^2 + 2\frac{f_A(z)}{f_B(z)}|u_B^{(0)}|^2\right)u_A^{(0)} \tag{14}$$

Notice that the normalized fields $u_A^{(0)}$ and $u_B^{(0)}$ can not be summed together because they are complex amplitudes with respect to different carriers. However, their squared absolute value is the power of the wave, so the summation of these terms makes physical sense.

Writing the integral solution to the inhomogeneous linear equation above gives

$$u_A^{(1)}(L,t) = i\gamma\int_0^L \mathbf{U}_A(L-z)\left(f_A(z)|u_A^{(0)}|^2 + 2\frac{f_A(z)}{f_B(z)}|u_B^{(0)}|^2\right)u_A^{(0)}dz. \tag{15}$$

Using this result it is possible to obtain the estimation error of a receiver.

C. Generalization of estimation error

Using a matched filter receiver, with matching to the linearly propagated initial pulse waveform $g^{(0)}(L,T)$, we obtain the following equation for the estimation error on the first symbol Δa_0

$$\Delta a_0 = \int_{-\infty}^{\infty} u_A^{(1)}(L,t)g^{(0)*}(L,t)dt \tag{16}$$

Recall that $\mathbf{U}_A$ is unitary, so it holds

$$\mathbf{U}_A(L-z)g^{(0)*}(L,t) = g^{(0)*}(z,t) \tag{17}$$

Using this identity, the expression for the error can be written as:

$$\Delta a_0 = i\gamma\int_{z_0}^L dz\int_{-\infty}^{\infty} dt g^{(0)*}(z,t)\left(f_A(z)|u_A^{(0)}|^2 + 2f_B(z)|u_B^{(0)}|^2\right)u_A^{(0)} \tag{18}$$

By substituting the modulation of choice as in 11, we get an expression ready to be computed with respect to a given modulation format Notice that, being that the two fibers linear propagation terms are different, the substitution described in equation (17) is only valid because matching is done considering the propagated symbol waveform over channel A. Finally, substituting the modulation and using the same notation as [2, eq. 5, 6, 7], the resulting expression is

$$\Delta a_0 = i\gamma\sum_{h,k,m}\left(a_h a_k^* a_m S_{h,k,m} + 2a_h b_k^* b_m X_{h,k,m}\right) \tag{19}$$

In conclusion the terms that generalize the model, including attenuation and gain, are written in the interaction terms, as follows

$$S_{h,k,m} = \int_{z_0}^L dz f_A(z)\int_{-\infty}^{\infty} dt g^{(0)}(z,t)g^{(0)*}(z,t-hT)g^{(0)}(z,t-kT)g^{(0)*}(z,t-mT) \tag{20}$$

for the SPM, and

$$X_{h,k,m} = \int_{z_0}^L dz f_B(z)\int_{-\infty}^{\infty} dt g^{(0)}(z,t)g^{(0)*}(z,t-hT)g^{(0)}(z,t-kT-\beta_2\Omega z)g^{(0)*}(z,t-mT-\beta_2\Omega z) \tag{21}$$

for the XPM.

This argument proves that the only modification to the original model is to include the terms $f_A(z)$, $f_B(z)$, which represent the power exchanged with the medium, into nonlinear interaction terms $S_{h,k,m}$, $X_{h,k,m}$.

IV. Computation of $X_{0,m,m}$

A. *High dispersion approximation*

Let us first comment the calculations done in [2, eq. 11, 12]. Starting from the highly-dispersed pulse approximation, we get

$$g^{(0)}(z,t) \approx \sqrt{\frac{i}{2\pi\beta_2 z}} \exp\left[-\frac{it}{2\beta_2 z}\right] \hat{g}\left(0, \frac{t}{\beta_2 z}\right) \tag{22}$$

Now, it is possible to compute the coefficient $X_{0,m,m}$ through energy integral in Fourier space by defining $\nu = t/\beta_2 z$

$$X_{0,m,m} = \int_{z_0}^{L} dz f(z) \int_{\mathbb{R}} \frac{d\nu}{4\pi^2\beta_2 z} |\hat{g}(0,\nu)|^2 \left|\hat{g}\left(0, \nu - \Omega - \frac{mT}{\beta_2 z}\right)\right|^2 \tag{23}$$

The approximation is that the strongest overlap happens at $z_m = -\frac{mT}{\beta_2\Omega}$ By the *perfect amplification* hypotesis, it is assumed that $f(z) \equiv 1$. A further assumption is made claiming that most of the overlap happens at $z = z_m$, so the integrand is non negligible only when $z_m/z \approx 1$. The integral becomes

$$X_{0,m,m} = \int_{\mathbb{R}} \frac{d\nu}{2\pi} |\hat{g}(0,\nu)|^2 \int_{\mathbb{R}} dz \frac{z_m f(z_m)}{4\pi^2\beta_2 z^2} \left|\hat{g}\left(0, \nu - \Omega - \frac{mT}{\beta_2 z}\right)\right|^2 \tag{24}$$

Notice that the integration along all the space allow us to recall an important property of the impulses: they are of unit energy. Using Parseval identity it is possible to eliminate the impulse waveform in the following way. Let us adopt this change of variables:

$$y := -\frac{mT}{\beta_2 z} \qquad \Longrightarrow \qquad dy = \frac{mT}{\beta_2 z^2} \tag{25}$$

The multiplication by z_m/z creates the term dy along with the other constants:

$$X_{0,m,m} = \int_{\mathbb{R}} \frac{d\nu}{2\pi} |\hat{g}(0,\nu)|^2 \int_{\mathbb{R}} \frac{f(z_m)}{2\pi\beta_2\Omega} \Big(\underbrace{-\frac{mT}{\beta_2 z^2} dz}_{dy} \Big) \left|\hat{g}\left(0, \nu - \Omega - \frac{mT}{\beta_2 z}\right)\right|^2 \tag{26}$$

$$= \frac{f(z_m)}{\beta_2\Omega} \int_{\mathbb{R}} \frac{d\nu}{2\pi} |\hat{g}(0,\nu)|^2 \int_{\mathbb{R}} -\frac{dy}{2\pi} \left|\hat{g}\left(0, \nu - \Omega + y\right)\right|^2 \tag{27}$$

If $f(z_m)$ is assumed to be 1 in perfect amplification scenario, the integrals simplify to

$$= \frac{1}{\beta_2\Omega} \int_{\mathbb{R}} \frac{d\nu}{2\pi} |\hat{g}(0,\nu)|^2 \int_{\mathbb{R}} \frac{dy}{2\pi} \left|\hat{g}\left(0, \nu - \Omega + y\right)\right|^2 \tag{28}$$

finally, both integrals, by Parseval theorem, sum to 1, so

$$X_{0,m,m} = \frac{1}{\beta_2\Omega} \tag{29}$$

when z_m falls inside the fiber and in the region of high dispersion, 0 otherwise.

B. *High dispersion approximation is unfit for generalized scenario*

The *generalization* of the above calculation has one major difficulty: in the overlap region the cumulative pulse attenuation f is assumed to be constant. If this assumption holds true, the expression generalizes naturally

$$X_{0,m,m} = \frac{f_B(z_m)}{\beta_2\Omega} \tag{30}$$

Otherwise, we may be interested in cases in which interaction can not be assumed as local. Furthermore, the approximation used for highly-dispersed pulses (22) is valid for lengths higher than the dispersion length, and in the region of interest for Raman amplifiers the approximation may not suffice in calculating the XPM coefficient, as they employ relatively short fibers.

Dropping the locality assumption (common Raman amplifiers have fast varying amplification profiles over hundreds of km), the integral to consider is the one stemming from time domain analysis (21):

$$X_{0,m,m} = \int_0^L dz f_B(z) \int_{-\infty}^{\infty} dt |g^{(0)}(z,t)|^2 |g^{(0)}(z, t - mT - \beta_2\Omega z)|^2. \tag{31}$$

In order to find a suitable computational simplification, let us assume that the pulses are gaussian.

V. Gaussian pulses

The effect of linear propagation for gaussian pulses has a closed form expression:

$$g(z,t) = \frac{U_0 \exp[\frac{i}{2}\arctan(D(z))]}{(1+D^2(z))^{1/4}} \exp\left[-\frac{t^2}{2T_0^2}\frac{1+iD(z)}{1+D^2(z)}\right] \tag{32}$$

where $D(z) = z\beta_2/T_0^2$. Some comments on this representation are given in Appendix A.

Assuming pulse energy normalization to 1, amplitude and width parameters need to satisfy the following equation

$$U_0^2 T_0 \sqrt{\pi} = 1 \tag{33}$$

considering

$$|g^{(0)}(z,t)|^2 = \frac{U_0^2}{(1+D^2(z))^{1/2}} \exp\left[-\frac{t^2}{T_0^2}\frac{1}{1+D^2(z)}\right] \tag{34}$$

and then substituting the relative expressions in 31, the following expression is given

$$X_{0,m,m} = \int_0^L dz f_B(z) \int_{\mathbb{R}} dt \frac{U_0^4}{1+D^2(z)} \exp\left[-\frac{1}{T_0^2(1+D^2(z))} \underbrace{\left(t^2 + (t - mT - \beta_2\Omega z)^2\right)}_{\varphi}\right] \tag{35}$$

Let s be defined as

$$s := mT + \beta_2\Omega z \tag{36}$$

then, the term φ can be rewritten as:

$$\varphi = 2t^2 - 2ts + s^2 \tag{37}$$

$$= \left(\sqrt{2}t - \frac{s}{\sqrt{2}}\right)^2 + \frac{s^2}{2} \tag{38}$$

let us use a change of variable of integration: $\eta := \sqrt{2}t - \frac{s}{\sqrt{2}}$, from which $dz\,dt = dz\,d\eta\frac{1}{\sqrt{2}}$. So we have the expression $\varphi = \eta^2 + \frac{s^2}{2}$, and the integral admits this expression

$$X_{0,m,m} = \int_0^L dz f_B(z) \int_{\mathbb{R}} \frac{d\eta}{\sqrt{2}} \frac{U_0^4}{1+D^2(z)} \exp\left[-\frac{\eta^2}{T_0^2(1+D^2(z))}\right] \exp\left[-\frac{s^2}{2T_0^2(1+D^2(z))}\right] \tag{39}$$

The last step is the assumption of local interaction for $z = z_m = -mT/\beta_2\Omega$. This means that the functions $f_B(z)$ and $D(z)$ can be substituted with constants $f_B(z_m)$ and $D(z_m)$, respectively. This is justified also by numerical simulations with usual Raman amplifiers parameters: they show that this approximation usually gives integrals with relative error $< 1\%$. We can also extend integration to $\mathbb{R}$, for every m such that $z_m \in [0, L]$, if we assume to neglect border effects. Than it is possible to simplify the integral

$$X_{0,m,m} = f_B(z_m) \frac{U_0^4}{1+D^2(z_m)} \frac{1}{\sqrt{2}} \int_{\mathbb{R}} dz \int_{\mathbb{R}} d\eta \exp\left[-\frac{\eta^2}{T_0^2(1+D^2(z))}\right] \exp\left[-\frac{s^2}{2T_0^2(1+D^2(z))}\right]. \tag{40}$$

The remaining equation contains two gaussian integrals

$$\int_{\mathbb{R}} dt \exp\left[-\frac{t^2}{\alpha}\right] = \sqrt{\alpha\pi} \tag{41}$$

integrating with respect to η we obtain

$$f_B(z_m) \frac{U_0^4}{1+D^2(z_m)} \frac{1}{\sqrt{2}} (T_0^2(1+D^2(z_m)))^{\frac{1}{2}} \sqrt{\pi} \int_{\mathbb{R}} dz \exp\left[-\frac{s^2}{2T_0^2(1+D^2(z))}\right]. \tag{42}$$

Using s as a new integration variable

$$s = mT + \beta_2\Omega z \qquad dz = \frac{1}{\beta_2\Omega} ds \tag{43}$$

it is possible to solve even the last integral as

$$X_{0,m,m} = \frac{f_B(z_m)}{\beta_2\Omega} U_0^4 T_0^2 \pi. \tag{44}$$

Recall the normalization condition for the pulse energy in (33): the substitution cancels both parameters U_0 and T_0 from the final expression

$$X_{0,m,m} = \frac{f_B(z_m)}{\beta_2\Omega}\underbrace{U_0^4 T_0^2 \pi}_{=1} = \frac{f_B(z_m)}{\beta_2\Omega} \tag{45}$$

The procedure assumes that borders effect are neglected, which in Raman amplifiers may not be a reasonable assumption. Nonetheless, even with dropping this assumption, the necessary numerical integration is simplified as the time integral is computed analytically anyhow.

VI. Conclusion

In order to generalize the NLIN model, a coupled NLSE model was adopted, by redefining signal and interferent field. This formulation, when applied to the analysis of NLIN on a symbol, gives SPM and XPM coefficients of interaction which are very similar to the one in [2]. In order to compute the coefficient for XPM, the approximation proposed in [2] was studied, but it is unfit in the case of Raman amplifiers. So, in order to find a suitable simplification of time domain non approximated integrals, the assumption of gaussian pulses was examinated. This last analysis, along with local interaction approximation, lead to the direct analytical computation of one of the two integrals. The last integral can be computed numerically including border effects, or can be approximated by extending the domain of integration and gives an analytical expression similar to the one obtained after the approximation for high dispersion in [2].

References

[1] Govind P. Agrawal. *Nonlinear fiber optics*. Academic Press, 2013.

[2] Ronen Dar et al. "Properties of nonlinear noise in long, dispersion-uncompensated fiber links". In: *Optics Express* 21.22 (Oct. 2013), p. 25685. DOI: 10.1364/oe.21.025685.

[3] "G.694.1 (10/2020)". In: ().

[4] Gianluca Marcon et al. "Model-Aware Deep Learning Method for Raman Amplification in Few-Mode Fibers". In: *Journal of Lightwave Technology* 39.5 (Mar. 2021), pp. 1371–1380. DOI: 10.1109/jlt.2020.3034692.

[5] Carlo Giacomo Someda. *Electromagnetic waves*. CRC press, 2006.

Appendix A

The field expression in (32) contains a phase term in the form $\exp\left[\frac{i}{2}\arctan(D(z))\right]$. This term highlights the phase and modulus representation of the coefficient of the exponentials. However, it may not seem so straightforward. From Fourier antitransform, the field has the following expression

$$u(z,t) = U_0\left(\frac{1+iD(z)}{1+D^2(z)}\right)^{\frac{1}{2}}\exp\left[-\frac{t^2}{2T_0^2}\frac{1+iD(z)}{1+D^2(z)}\right] \tag{46}$$

the two field expression are connected by this calculations which uses fundamental goniometric identities

$$\exp\left[\frac{i}{2}\arctan(D(z))\right] = \exp\left[2i\arctan(D(z))\right]^{\frac{1}{4}} = \tag{47}$$

$$= \left[\cos(2\arctan(D(z))) + i\sin(2\arctan(D(z)))\right]^{\frac{1}{4}} = \tag{48}$$

$$= \left[\frac{1-t^2}{1+t^2} + i\frac{2t}{1+t^2}\right]^{\frac{1}{4}} = \qquad \text{where } t = \tan\left(\frac{\cancel{2}\arctan(D(z))}{\cancel{2}}\right) = D(z) \tag{49}$$

$$= \left[\frac{(1+iD(z))^2}{1+D^2(z)}\right]^{\frac{1}{4}} = \frac{(1+iD(z))^{\frac{1}{2}}}{(1+D^2(z))^{\frac{1}{4}}} \tag{50}$$