Generalization of NLIN model for WDM systems

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Computation of integrals and survey on feasible approximations

- Numerical convergence of time integrals
- Numerical convergence of space integrals
- Insights on Papoulis approximation
- Correction term for Gaussian pulses space integrals

Integral of interest

$$X_{0,m,m} = \int_0^L dz f_B(z) \int_{-\infty}^{\infty} dt \, |g^{(0)}(z,t)|^2 \, |g^{(0)}(z, t - mT - \beta_2 \Omega z)|^2 \qquad (1)$$

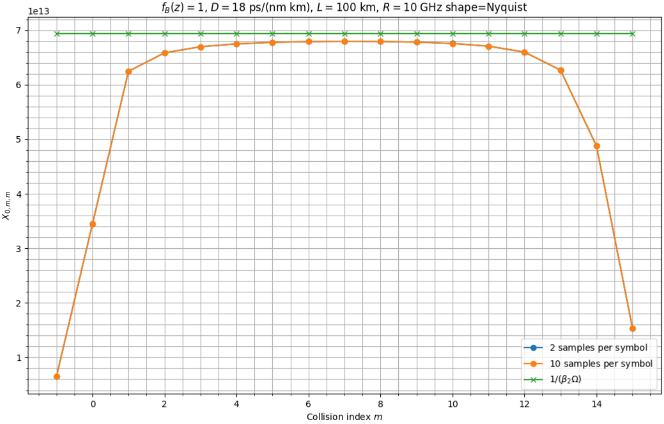
in the following the perfect amplification approximation is used

$$f_B(z) = 1 \qquad (2)$$

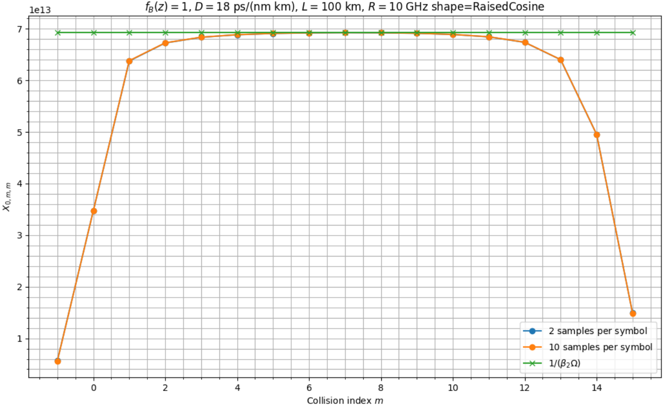
Convergence of *time* integrals for various pulse shapes

All computations are done using 40 points per collision in space integral

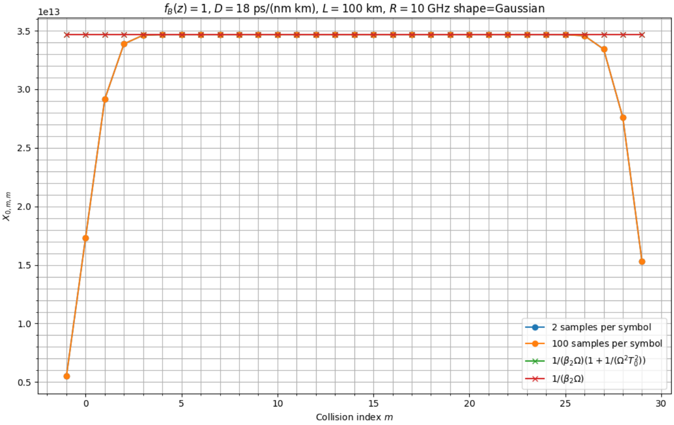
Nyquist (sinc) pulses - time steps evaluation



Raised cosine pulses - time steps evaluation



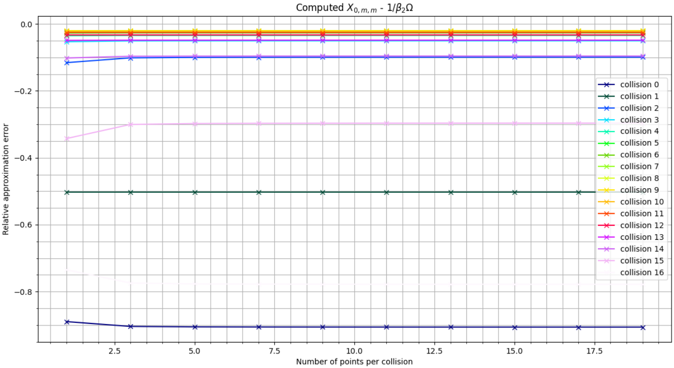
Gaussian pulses - time steps evaluation



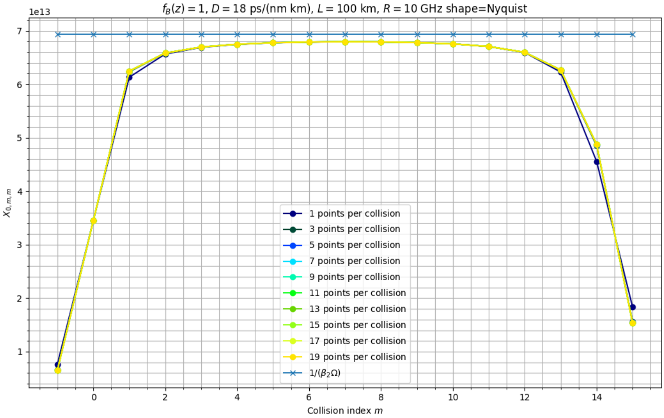
Convergence of *space* integrals for various pulse shapes

All computations are done using 10 samples per symbol in time integral

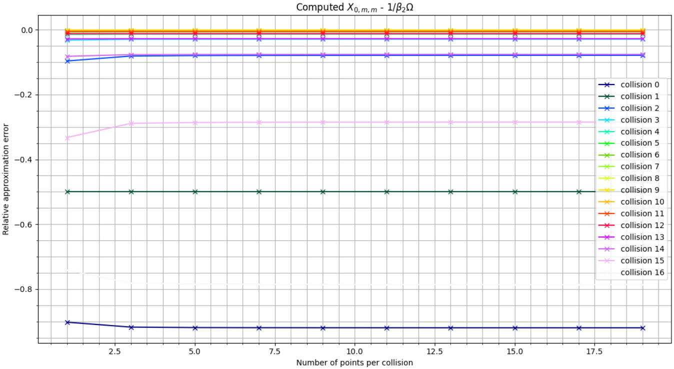
Nyquist (sinc) pulses - space steps evaluation



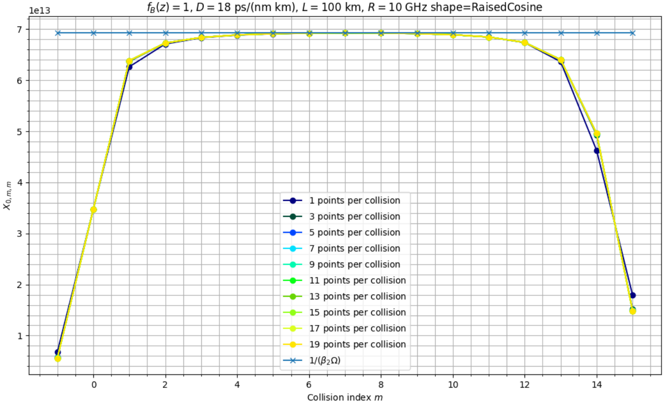
Nyquist (sinc) pulses - space steps evaluation



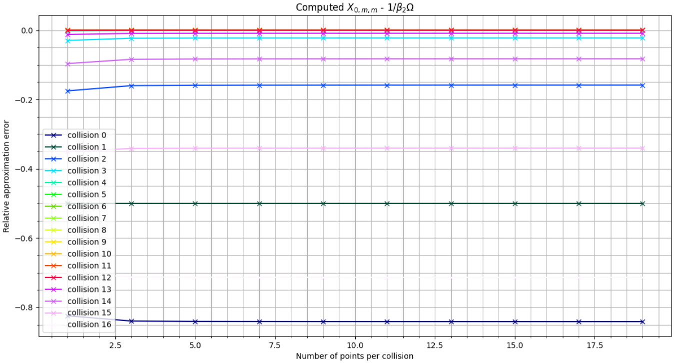
Raised cosine pulses - space steps evaluation



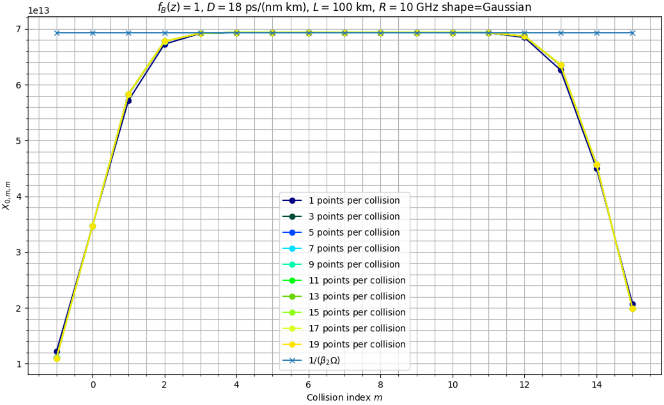
Raised cosine pulses - space steps evaluation



Gaussian pulses - space steps evaluation



Gaussian pulses - space steps evaluation



Approximation using Fresnel transform (Papoulis)

The approximation from [1] relates the propagated pulse shape with its Fourier transform

$$g^{(0)}(z,t) = \sqrt{\frac{i}{2\pi\beta_2 z}}\exp\left[-\frac{it^2}{2\beta_2 z}\right]\hat{g}\left(0, \frac{t}{\beta_2 z}\right) \tag{3}$$

This is derived using the system function properties of the fiber linear propagator operator. In spectral domain the operator reads

$$\hat{U}(z) = \exp\left[-i\frac{\beta_2}{2}z\omega^2\right] \tag{4}$$

and using the Gaussian transform relation

$$\exp\left[-st^2\right] \longleftrightarrow \sqrt{\frac{\pi}{s}}\exp\left[-\frac{\omega^2}{4s}\right] \tag{5}$$

we can find a convolution kernel

$$\hat{U}(z) = \exp\left[-i\frac{\beta_2}{2}z\omega^2\right] \longleftrightarrow U(z,t) = \sqrt{\frac{-i}{2\beta_2 z\pi}}\exp\left[i\frac{t^2}{2\beta_2 z}\right] \tag{6}$$

Quadratic Phase Filter

The linear propagation corresponds to a convolution operation by this kernel. Since the frequency response denotes a quadratic phase term, the corresponding filter is called Quadratic Phase Filter. The convolution is actually well-known, and it assumes the shape of a Fresnel transform.

$$\bar{f}(t) = \sqrt{\frac{\beta}{i\pi}} \int_{\mathbb{R}} f(\tau) \exp\left[-i\beta(t-\tau)^2\right] d\tau \tag{7}$$

Starting from the Fresnel transform, a *square completion argument* links the propagated field to its Fourier transform.
A comprehensive derivation presentation will soon be added.

Gaussian pulses

Substituting the Gaussian pulse shape in the XPM coefficient definition we obtain, using a change of *time* variable

$$X_{0,m,m} = \frac{1}{\sqrt{2}} \int_0^L dz \frac{U_0^4}{1 + D^2(z)} \exp\left[-\frac{(mT + \beta_2 \Omega z)^2}{2T_0^2\,(1 + D(z)^2)}\right] \int_{\mathbb{R}} d\eta \exp\left[-\frac{\eta^2}{T_0^2\,(1 + D(z)^2)}\right] \tag{8}$$

where $D = \dfrac{z}{L_D} = z\dfrac{\beta_2}{T_0^2}$. The time integral admits an analytic solution

$$X_{0,m,m} = \frac{1}{\sqrt{2}} \int_0^L dz \frac{\sqrt{\pi} U_0^4 T_0 \sqrt{1 + D(z)^2}}{1 + D(z)} \exp\left[-\frac{(mT + \beta_2 \Omega z)^2}{2T_0^2\,(1 + D(z)^2)}\right] \tag{9}$$

Space integral

The space integral can be simplified to express a space-dependent Gaussian width $\sigma = T_0\sqrt{1+D^2}$. Extending to the whole space,

$$X_{0,m,m} = \sqrt{\pi}U_0^4T_0^2 \int_{\mathbb{R}} dz \frac{1}{\sqrt{2}T_0\sqrt{1+D(z)^2}} \exp\left[-\frac{(mT+\beta_2\Omega z)^2}{2T_0^2\left(1+D(z)^2\right)}\right] \tag{10}$$

and adopting a change of variable

$$X_{0,m,m} = \frac{\sqrt{\pi}U_0^4T_0^2}{\beta_2\Omega} \int_{\mathbb{R}} d\zeta \frac{1}{\sqrt{2}T_0\sqrt{1+D^2}} \exp\left[-\frac{(\zeta+mT)^2}{2T_0^2\left(1+D^2\right)}\right] \tag{11}$$

in the last integral, with a slight abuse of notation $D = D(\zeta)$.
Remember the normalization condition

$$\pi U_0^4 T_0^2 = 1 \tag{12}$$

Local interaction (0^{th}-order) approximation

As proven in the previous computation, a local interaction approximation is useful when $D(z) \approx D_m := D(z_m)$, with $z_m = -\frac{mT}{\beta_2 \Omega}$. In this case, the time integral is a simple Gaussian integral, and it can be evaluated as

$$X_{0,m,m} = \frac{1}{\beta_2 \Omega} \tag{13}$$

this approximation is not useful for Nyquist and raised cosine pulses, and a slight improvement is available for Gaussian integrals.

Correction to the 0^{th}-order approximation

By expanding the dependence on D to the second order, around D_m, we obtain a correction which accounts for the dispersion phenomenon *inside* the relevant collision region

$$X_{0,m,m} \approx X^{(0)}_{0,m,m} + \Delta X^{(2)}_{0,m,m} \tag{14}$$

where

$$\Delta X^{(2)}_{0,m,m} = \frac{1}{\beta_2 \Omega} \frac{1}{\Omega^2 T_0^2} \tag{15}$$

the corrected coefficient reads

$$X_{0,m,m} \approx \frac{1}{\beta_2 \Omega} \left(1 + \frac{1}{\Omega^2 T_0^2} \right) \tag{16}$$

this correction is well verified by numerical computation of the integral.

First order correction

In order to simplify the computation we use a change of variable

$$\frac{d\sigma}{dD} = \left(-T_0 \frac{D_m}{\sqrt{1 + D_m^2}} \right) \tag{17}$$

By using the rule for computing Gaussian moments we have

$$\Delta X_{0,mm}^{(1)} = \frac{\sqrt{\pi} U_0^4 T_0^2}{\beta_2 \Omega} \int_{\mathbb{R}} d\zeta \frac{\partial C}{\partial D} (D_m, \zeta) \cdot (D - D_m) = \tag{18}$$

$$= \frac{\sqrt{\pi} U_0^4 T_0^2}{p_2 \Omega} \int_{\mathbb{R}} d\zeta \frac{1}{\sqrt{2}\sigma_m} \cdot \exp\left[-\frac{(\zeta + mT)^2}{2\sigma_m^2} \right] \left[-\frac{1}{\sigma_m} + \frac{(\zeta + mT)^2}{\sigma_m^3} \right] \times \tag{19}$$

$$\times \left[-T_0 \frac{D_m}{\sqrt{1 + D_m^2}} \right] \cdot \left(\frac{\zeta + mT}{\Omega T_0^2} \right) = 0 \tag{20}$$

as the only moments occuring are mean and kurtosis.

Second order correction (1)

Let us use the chain rule twice

$$\frac{\partial^2 C}{\partial D^2} = \frac{d^2\sigma}{dD^2} \cdot \frac{\partial C}{\partial \sigma} + \left(\frac{d\sigma}{dD}\right)^2 \frac{\partial^2 C}{\partial \sigma^2} \tag{21}$$

By integrating the first term of the sum in eq (21) we obtain

$$\frac{\sqrt{\pi} U_0^4 T_0^2}{\beta_2 \Omega} \int_{\mathbb{R}} d\zeta \frac{1}{\sqrt{2}\sigma_m} \exp\left[-\frac{(\zeta + mT)^2}{2\sigma_m^2}\right] \left[-\frac{1}{\sigma_m} + \frac{(\zeta + mT)^2}{\sigma_m^3}\right] \times \tag{22}$$

$$\tag{23}$$

$$\times \left[\frac{T_0}{(1 + D_m^2)^{3/2}}\right] \frac{1}{2} \frac{(\zeta + mT)^2}{\Omega^2 T_0^4} \tag{24}$$

so we have a variance, and a central moment of order 4.

Second order correction (2)

As for the second term in 21 the integral reads

$$= \frac{\sqrt{\pi} U_0^4 T_0^2}{\beta_2 \Omega} \int_{\mathbb{R}} d\zeta \frac{1}{\sqrt{2}\sigma_m} \exp\left[-\frac{(\zeta + mT)^2}{2\sigma_m^2}\right] \left[\left(-\frac{1}{\sigma_m} + \frac{(\zeta + mT)^2}{\sigma_m^3}\right)^2 + \right. \tag{25}$$

$$\tag{26}$$

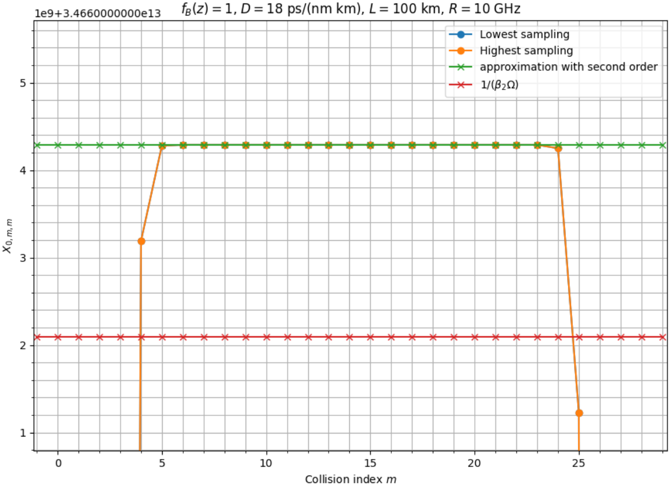
$$\left. + \left(\frac{1}{\sigma_m^2} - \frac{3(\zeta + mT)^2}{\sigma_m^4}\right)\right] \left(-T_0 \frac{D_m}{\sqrt{1 + D_m^2}}\right)^2 \frac{1}{2} \frac{(\zeta + mT)^2}{\Omega^2 T_0^4} \tag{27}$$

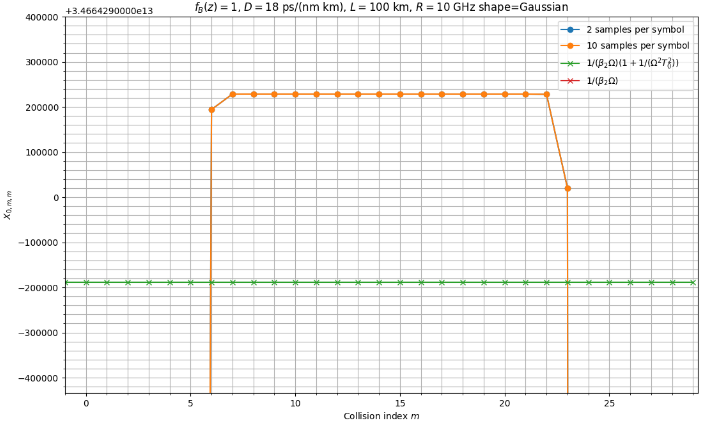
so we have a variance, and central moments of order 4 and 6. Using the rule

$$\mathbb{E}[(X - \mu)^n] = (n-1)!!\sigma^n \tag{28}$$

we obtain the integrals.
Summing the two contribution the dependence on D_m vanishes, and we are left with eq. (15)

Second order correction





$f_B(z) = 1$, $D = 18$ ps/(nm km), $L = 100$ km, $R = 10$ GHz shape=Gaussian
+3.4664290000e13
400000
300000
200000
100000
0
−100000
−200000
−300000
−400000
$X_{0,m,m}$
0
5
10
15
20
25
Collision index m
2 samples per symbol
10 samples per symbol
$1/(\beta_2\Omega)(1+1/(\Omega^2 T_0^2))$
$1/(\beta_2\Omega)$