Generalization of NLIN model for WDM systems to wavelength-dependent Raman gain and attenuation scenarios

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***Abstract*—Starting from [Dar_2013] and [] we develop a generalized model for the phenomenon of NLIN (Non Linear Interference Noise) in the presence of wavelenght-dependent attenuation and Raman gain.**

I. Objective and summary of previous results

A. Equation for the field and narrowband approximation

Let us consider the standard NLSE for a fiber with Raman amplification profile $g(z)$

$$\frac{\partial}{\partial z}A = -\frac{\alpha - g(z)}{2}A - \beta_1\frac{\partial}{\partial t}A - i\frac{\beta_2}{2}\frac{\partial^2}{\partial t^2}A + i\gamma|A|^2A \quad (1)$$

where t is the physical time. Recall that A is proportional to the electric field inside the fiber, in a way such that the dimension of A is $[A^2] = \mathrm{W}$.

This equation holds for a narrowband field, such that $A(z,t)$ is a slowly varying function of t. In a WDM system, this approximation is assumed to be still true, as the usual channel spectral spacing is greater than 12.5GHz in third window (as defined in standard [**ITU-T**] for DWDM architectures). In the presence of hundreds of channels, the total field is still narrowband, as the percentage bandwidth is still low.

In the following, the theoretical framework proposed in [**Mecozzi_2012**] is reviewed, and the notation is kept as similar as possible. Then a generalization is developed in the case of two interacting WDM channel fields.

B. Rescaling of fields

Let us define $\psi(z)$ as

$$\frac{d}{dz}\psi(z) = -\frac{\alpha - g(z)}{2}\psi(z) \quad (2)$$

Using such function, define $u(z,t)$ as the normalized field

$$A(z,t) = \psi(z)u(z,t) \quad (3)$$

These definitions, when substituted in 1 give rise to a new equation

$$\frac{\partial}{\partial z}u = -i\frac{\beta_2}{2}\frac{\partial^2}{\partial t^2}u + i\gamma\psi(z)^2|u|^2u \quad (4)$$

where $f(z) = \psi(z)^2$. The resulting equation show a new, space-varying, nonlinear coefficien.

II. Coupled NLS equations for WDM channels

Consider two WDM channels named A and B. The following hypotesis are made:

- channels A and B have a spectral separation of Ω,
- both channels have the same nonlinear coefficient,
- the group velocity profile is approximatively linear in the frequency (β_2 is constant) in the whole band of interest
- attenuation and Raman gain depend on the channel choice, but are constant within the band of the same channel.

Following [**Agrawal**] p.

$$\begin{aligned}\frac{\partial}{\partial z}A_A = &-\frac{\alpha_A - g_A(z)}{2}A_A - \beta_1\frac{\partial}{\partial t}A_A - i\frac{\beta_2}{2}\frac{\partial^2}{\partial t^2}A_A \\ &+ i\gamma(|A_A|^2 + 2|A_B|^2)A_A\end{aligned} \quad (5)$$

$$\begin{aligned}\frac{\partial}{\partial z}A_B = &-\frac{\alpha_B - g_B(z)}{2}A_B - \beta_1\frac{\partial}{\partial t}A_B - i\frac{\beta_2}{2}\frac{\partial^2}{\partial t^2}A_B \\ &+ i\gamma(|A_B|^2 + 2|A_A|^2)A_B\end{aligned} \quad (6)$$

Let us consider the WDM channel A as the channel of interest.

We now proceed to normalize the fields A_A, A_B with the respective normalization functions ψ_A, ψ_B. In addition, a moving time reference frame is assumed, taking as a reference the time of arrival of the first pulse in channel A: $T = t - z/v_{gA} = t - \beta_{1A}z$. The resulting coupled equations are

$$\begin{aligned}\frac{\partial}{\partial z}u_A = &-i\frac{\beta_2}{2}\frac{\partial^2}{\partial t^2}u_A \\ &+ i\gamma\left(f_A(z)|u_A|^2 + 2\frac{f_A(z)}{f_B(z)}|u_B|^2\right)u_A\end{aligned} \quad (7)$$

$$\begin{aligned}\frac{\partial}{\partial z}u_B = &-\Delta\beta_1\frac{\partial}{\partial t}u_B - i\frac{\beta_2}{2}\frac{\partial^2}{\partial t^2}u_B \\ &+ i\gamma\left(f_B(z)|u_B|^2 + 2\frac{f_B(z)}{f_A(z)}|u_A|^2\right)u_B\end{aligned} \quad (8)$$

where $\Delta\beta_1 = \beta_{1B} - \beta_{1A} = \beta_2\Omega$.

III. Generalization of the 0-th order term

The term of the 0-th order is identical as the one derived in [**Dar_2013**], and this is clearly a result of superposition property of linear equations. The only exception is due to the notation used: the total field in this case cannot be expressed by a simple sum of terms $u_A^{(0)} + u_B^{(0)}$. There are in fact two caveats:

- $u_A^{(0)}$ and $u_B^{(0)}$ functions represent fields with different normalization constants,
- the functions are derived from *complex amplitudes* of different carrier frequency signals

the equivalent field complex amplitude, with respect to the channel A carrier frequency, is actually

$$\begin{aligned} A_{tot}(z,T) &= A_A(z,T) + A_B(z,T) \\ &= \psi_A(z)u_A(z,T) + \psi_A(z)u_B(z,T)\exp[-i\Omega T] \end{aligned} \tag{9}$$

Let us consider the initial fields as sums of shifted impulses which codify a given message. Let τ be the symbol period:

$$\begin{aligned} u_A(0,T) &= \sum_k a_k g(0, T - k\tau) \\ u_B(0,T) &= \sum_k b_k g(0, T - k\tau) \end{aligned} \tag{10}$$

The solution for the 0-th order field is simply:

$$\begin{aligned} u_A^{(0)}(z,T) &= \sum_k a_k g^{(0)}(z, T - k\tau) \\ u_B^{(0)}(z,T) &= \sum_k b_k g^{(0)}(z, T - k\tau - \beta_2 \Omega z) \end{aligned} \tag{11}$$

because of linearity. As in [**Dar_2013**], we define the differential operators $\mathbf{U}_A(z) = \exp[i\frac{\beta_2}{2}z\frac{\partial^2}{\partial T^2}]$ and $\mathbf{U}_B(z) = \exp[i\frac{\beta_2}{2}z\frac{\partial^2}{\partial T^2} - i\Delta\beta_1 z\frac{\partial}{\partial T}]$. So the propagated 0-th order impulses are:

ci vuole il - davanti al termine di GVD??

$$g^{(0)}(z, T - k\tau) = \mathbf{U}_j(z) g(z, T - k\tau) \tag{12}$$

with $j \in \{A, B\}$.

IV. Generalization of first order perturbation theory

The separation of fields allow us to analyze separately the effects of SPM and XPM. Let us apply the perturbation method to 7

$$\begin{aligned} \frac{\partial}{\partial z}u_A^{(1)} &= -i\frac{\beta_2}{2}\frac{\partial^2}{\partial t^2}u_A^{(1)} \\ &+ i\gamma\left(f_A(z)|u_A^{(0)}|^2 + 2\frac{f_A(z)}{f_B(z)}|u_B^{(0)}|^2\right)u_A^{(0)} \end{aligned} \tag{13}$$

Writing the integral solution to the inhomogeneous linear equation above:

$$\begin{aligned} &u_A^{(1)}(L,T) = \\ &i\gamma\int_0^L \mathbf{U}_A(L-z)\left(f_A(z)|u_A^{(0)}|^2 + 2\frac{f_A(z)}{f_B(z)}|u_B^{(0)}|^2\right)u_A^{(0)}dz \end{aligned} \tag{14}$$

V. Generalization of estimation error

A. Generic expression under matched filter conditions

Using a matched filter receiver, with matching to the linearly propagated initial pulse waveform $g^{(0)}(z,T)$, we obtain the following equation for the estimation error on the first symbol Δa_0 by expanding the perturbation term

$$\begin{aligned} \Delta a_0 &= \int_{-\infty}^{\infty} u_A^{(1)}(L,T) g^{(0)*}(z,T) dT = \\ &= \end{aligned} \tag{15}$$

B. Expanded expression for given modulation format

$$\mathbf{U}[z] = \exp\left[i\frac{\beta_2}{2}z\frac{\partial^2}{\partial t^2}\right] \tag{16}$$

$$u(0,t) = \sum_k b_k g(0, t - \tau k)e^{i\Omega t} \tag{17}$$

$$g(0,t)e^{i\Omega t} \rightarrow \hat{g}(0, \omega - \Omega) \tag{18}$$

$$\hat{\mathbf{U}}[z] = \exp\left[-i\frac{\beta_2}{2}z\omega^2\right] \tag{19}$$

C. XPM term