Generalization of NLIN model for WDM systems to wavelength-dependent Raman gain and attenuation scenarios

Step 2: generalization of $X_{0,m,m}$

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Comment

The calculation done in [Dar+13, eq. 1] are actually the correct version of the linearly propagated field (superposition of channel of interest and interferer) except for a sign. In the following derivation the calculations are made again.
In order to prove the equation let us recall the linear propagator operator (as defined in [Dar+13]):

$$\mathbf{U}[z] = \exp\left[i\frac{\beta_2}{2}z\frac{\partial^2}{\partial t^2}\right] \tag{1}$$

Then, let us focus on the interfering channel field at the *input*

$$u(0,t) = \sum_k b_k g(0, t - \tau k)e^{i\Omega t} \tag{2}$$

and apply the propagator.
This is best done in frequency domain, and, by linearity, it is possible to focus only on the symbol waveform g. Using frequency shifting property

$$g(0,t)e^{i\Omega t} \to \hat{g}(0, \omega - \Omega) \tag{3}$$

In frequency domain, we have the operator

$$\hat{\mathbf{U}}[z] = \exp\left[-i\frac{\beta_2}{2}z\omega^2\right] \tag{4}$$

Let us focus on the linear propagation of complex envelope *spectrum* of a single impulse

Computation of $u^{(0)}$

$$\hat{g}^{(0)}(z,\omega) = \exp\left[-i\frac{\beta_2}{2}z\omega^2\right]\hat{g}(0,\omega-\Omega) \tag{5}$$

considering the antitransform, with a square completion argument,

$$\frac{1}{2\pi}\int_{\mathbb{R}} \exp\left[-i\frac{\beta_2}{2}z\omega^2\right]\hat{g}(z,\omega-\Omega)\exp\left[i\omega t\right]d\omega = \tag{6}$$

$$= \frac{1}{2\pi}\int_{\mathbb{R}} \exp\left[-i\frac{\beta_2}{2}z(\omega-\Omega)^2\right]\hat{g}(z,\omega-\Omega)\exp\left[i(\omega-\Omega)t\right]$$
$$\underbrace{\exp\left[i\Omega t\right]}_{\text{frequency shifting}}\;\underbrace{\exp\left[-i\beta_2 z\omega\Omega\right]}_{\text{time delay}}\underbrace{\exp\left[i\frac{\beta_2}{2}z\Omega^2\right]}_{\text{constant}}d\omega \tag{7}$$

in the notation of [Dar+13], $g^{(0)}(z,t) = \mathbf{U}(z)g(0,t)$ is the pulse propagated as in the channel of interest, so we have the following antitransform relation

$$g^{(0)}(z,t) = \frac{1}{2\pi}\int_{\mathbb{R}} \exp\left[-i\frac{\beta_2}{2}z\omega^2\right]\hat{g}(z,\omega)\exp\left[i\omega t\right]d\omega \tag{8}$$

Computation of $u^{(0)}$

In conclusion, by using a simple change of variables and the time shifting property:

$$\mathcal{F}^{-1}\left[\exp[-i\omega t_0]\hat{x}(\omega)\right](t) = x(t - t_0) \tag{9}$$

The linearly propagated single impulse of the *interfering* channel is

$$\exp\left[i\Omega t\right]\exp\left[i\frac{\beta_2}{2}\Omega^2 z\right] g^{(0)}(z, t - \beta_2 \Omega z) \tag{10}$$

notice that the frequency component $\exp[i\Omega t]$ has opposite sign with respect to [Dar+13]. All the other terms are exactly the same. This may be due to a sign error in the usage of the frequency shifting property, which is

$$\mathcal{F}\left[\exp[i\Omega t]x(t)\right](\omega) = \hat{x}(\omega - \Omega) \tag{11}$$

There is still a point to be discussed, regarding the definition of the propagator.

Caveat on $\mathbf{U}(z)$ and proposed solution

The linear equation to be solved is

$$\frac{\partial}{\partial z} g^{(0)}(z,t) = -i\frac{\beta_2}{2}\frac{\partial^2}{\partial t^2} g^{(0)}(z,t) \tag{12}$$

By the shift theorem [Wie26] it is possible to write it in symbolic form

$$\exp\left[h\frac{\partial}{\partial z}\right] g^{(0)}(z,t) = g^{(0)}(z+h,t) = \exp\left[-i\frac{\beta_2}{2}h\frac{\partial^2}{\partial t^2}\right] g^{(0)}(0,t) \tag{13}$$

In this way we notice that the propagator operator may be defined as

$$\mathbf{U}(h) = \exp\left[-i\frac{\beta_2}{2}h\frac{\partial^2}{\partial t^2}\right] \tag{14}$$

which is in contradiction with respect to [Dar+13] in which the sign of the argument is inverted. By calculating again the propagated impulse, the result is

$$\exp\left[i\Omega t - i\frac{\beta_2}{2}\Omega^2 z\right] g^{(0)}(z, t+\beta_2\Omega z) \tag{15}$$

which shows inverted sign on the terms which involve β_2. This aspect will require further investigation, to be justified with physical arguments and to be matched with [Dar+13] and [ME12, eq. 23].

Computation of $X_{0,m,m}$

Let us derive the calculations done in [Dar+13, eq. 11, 12]. Starting from the highly-dispersed pulse approximation, we get

$$g^{(0)}(z,t) \approx \sqrt{\frac{i}{2\pi\beta_2 z}} \exp\left[-\frac{it}{2\beta_2 z}\right] \hat{g}\left(0, \frac{t}{\beta_2 z}\right) \tag{16}$$

Now, it is possible to compute the coefficient $X_{0,m,m}$ through energy integral in Fourier space by defining $\nu = t/\beta_2 z$

$$X_{0,m,m} = \int_{z_0}^{L} dz f(z) \int_{\mathbb{R}} \frac{d\nu}{4\pi^2\beta_2 z} |\hat{g}(0,\nu)|^2 \left|\hat{g}\left(0, \nu - \Omega - \frac{m\tau}{\beta_2 z}\right)\right|^2 \tag{17}$$

The approximation is that the strongest overlap happens at $z_m = -\frac{m\tau}{\beta_2 \Omega}$
If the pulse centered at z_m suffers approximately the same attenuation in all of its spatial positions, it is allowed to assume the f function constant and $f(z) = f(z_m)$. A further assumption is made as z_m/z is assumed to be unitary, as most of the overlap happens at $z = z_m$. So the integral becomes

$$X_{0,m,m} = \int_{\mathbb{R}} \frac{d\nu}{2\pi} |\hat{g}(0,\nu)|^2 \int_{\mathbb{R}} dz \frac{z_m f(z_m)}{4\pi^2\beta_2 z^2} \left|\hat{g}\left(0, \nu - \Omega - \frac{m\tau}{\beta_2 z}\right)\right|^2 \tag{18}$$

Computation of $X_{0,m,m}$

Notice that the integration along all the space allow us to recall an important property of the impulses: they are of unit energy. Using Parseval identity it is possible to eliminate the impulse waveform in the following way. Let us adopt this change of variables:

$$y := -\frac{m\tau}{\beta_2 z} \quad \Longrightarrow \quad dy = \frac{m\tau}{\beta_2 z^2} \tag{19}$$

The multiplication by z_m/z creates the term dy along with the other constants:

$$X_{0,m,m} = \int_{\mathbb{R}} \frac{d\nu}{2\pi} |\hat{g}(0,\nu)|^2 \int_{\mathbb{R}} \frac{f(z_m)}{2\pi\beta_2\Omega} \Big(- \underbrace{\frac{m\tau}{\beta_2 z^2} dz}_{dy} \Big) \left| \hat{g} \left(0, \nu - \Omega - \frac{m\tau}{\beta_2 z} \right) \right|^2 \tag{20}$$

$$= \frac{f(z_m)}{\beta_2\Omega} \int_{\mathbb{R}} \frac{d\nu}{2\pi} |\hat{g}(0,\nu)|^2 \int_{\mathbb{R}} -\frac{dy}{2\pi} \left| \hat{g} \left(0, \nu - \Omega + y \right) \right|^2 \tag{21}$$

Computation of $X_{0,m,m}$

If $f(z_m)$ is assumed to be 1 in perfect amplification scenario, the integrals simplify to

$$= \frac{1}{\beta_2 \Omega} \int_{\mathbb{R}} \frac{d\nu}{2\pi} |\hat{g}(0,\nu)|^2 \int_{\mathbb{R}} \frac{dy}{2\pi} \left|\hat{g}\left(0, \nu - \Omega + y\right)\right|^2 \tag{22}$$

finally, both integrals, by Parseval, sum to 1, so

$$X_{0,m,m} = \frac{1}{\beta_2 \Omega} \tag{23}$$

when z_m falls inside the fiber and in the region of high dispersion, 0 otherwise.

Generalization of $X_{0,m,m}$

The generalization of the above calculation has only one critical point:

- In the overlap region the cumulative pulse attenuation f is assumed to be constant.

If this assumption holds true, the expression generalizes naturally

$$X_{0,m,m} = \frac{f_B(z_m)}{\beta_2 \Omega} \tag{24}$$

Otherwise, we may be interested in cases in which this assumption do not hold:

1. walkoff near zero (interaction happens not only near z_m, but in a broader region)
2. very long fibers (pulses fully interact without border effects)

and in general when interaction may not be assumed local.

Generalization of $X_{0,m,m}$

In such cases, a possible way to compute $X_{0,m,m}$ is dependent only on

1. Pulse shape (modulation format)
2. Cumulative attenuation of interfering channel f_B

References

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