Generalization of NLIN model for WDM systems considering fiber attenuation and Raman gain

Abstract

Starting from [2], in which a model for NLIN (nonlinear Interference Noise) is provided in the assumption of perfect amplification, we develop a generalized model for this phenomenon in the presence of wavelength and position-dependent attenuation and Raman gain. This analysis aims to open the possibility to address the optimization problem with respect to noise performance in multi-pump Raman amplifiers, which was a work already

I. Introduction

nonlinear Interference Noise (NLIN) is a phenomenon which is a crucial limit to the performances of modern WDM (Wavelength Division Multiplexing) links. In systems that implement Raman amplification, along with Amplified Stimulated Emission noise (ASE), NLIN is a major subject of interest in determining the noise performance of amplifiers. A time-domain model for NLIN is proposed in [2]: the model proposes a coefficient of intrachannel interaction which can be used with general modulation schemes. However, in this work, the nonlinear interaction is modeled assuming a *perfect amplification*, which means that the coefficient of Raman amplification $g(z)$ is identically equal to α, and this relationship holds for *every* WDM channel. In order to drop this assumption, it is required to develop a slight modification of the model.

The aim of the article is twofold: in the first place is to generalize the model proposed in [2] to generic Raman gain and attenuation scenarios: the goal is to derive an expression for the XPM (Cross Phase Modulation) coefficient $X_{0,m,m}$ similar to the one in [2], identifying the similarities and the differences of the two approaches. In second place, the aim is to shed light on some mathematical aspects of the problem that may result not straightforward.

In section II a review of past work is given along with some comments on notation and methods that are useful for the rest of the article.

In section III we develop a new framework in which it is possible to generalize the analysis done in past works to scenarios in which wavelength and position dependent Raman gain and attenuation are present.

In section IV the new framework is compared to the old one and some linking formulas are given for clarity.

In order to compute the XPM coefficient of interest, in section V the approximation developed in [2] is considered, and its limits are described. To circumvent these limitations, in section VI the computation are done in the gaussian pulses hypotesis, which allow a simplification of original expression of the coefficient.

Appendices are given to clarify the mathematical aspects of the generalization.

II. Summary of previous results

In the model developed in [2], a single nonlinear Schrödinger Equation (NLSE) is adopted to model the field inside the fiber. Let us examine the structure of the equation.

A. Equation for the field and narrowband approximation

Let us consider the standard NLSE for a fiber with Raman amplification profile $g(z)$

$$\frac{\partial}{\partial z}A = -\frac{\alpha - g(z)}{2}A - \beta_1 \frac{\partial}{\partial t}A - i\frac{\beta_2}{2}\frac{\partial^2}{\partial t^2}A + i\gamma|A|^2 A \tag{1}$$

where t is the physical time. Recall that A is proportional to the electric field inside the fiber, in a way such that the dimension of A is $[A^2] = \text{W}$.

This equation holds for a narrowband field, such that $A(z,t)$ is a slowly varying function of t. In a WDM system, this approximation is assumed to be still true, as the usual channel spectral spacing is greater than 12.5GHz in third window (as defined in standard [3] for DWDM architectures). In the presence of tens of channels, the total field may still be assumed as narrowband.

However, while this assumption can be used for model dispersion in a simple way, using β_2 (second order derivative of the phase constant with respect to frequency), it is not sufficient in assuming constant attenuation-gain terms along all the wavelengths. In fact, a constant attenuation and Raman gain over the signal bandwidth may be an assumption too strong to make, especially when interested in multiple pump Raman amplification. This is the case of the study in [4], from which the present further analysis stemmed. So the model with a single NLSE equation holds for *perfect amplification* scenarios, but needs to be modified in the generalization.

Let us comment now a mathematical procedure that simplify the notation in the NLSE, in presence of attenuation.

B. Rescaling of fields

Let $\psi(z)$ be defined as a solution of the following differential equation

$$\frac{d}{dz}\psi(z) = -\frac{\alpha - g(z)}{2}\psi(z) \tag{2}$$

using such function, let $u(z,t)$ be defined as the *normalized field*

$$A(z,t) = \psi(z)u(z,t) \tag{3}$$

These definitions, when substituted in 1 give the following equation:

$$\frac{\partial}{\partial z}u = -i\frac{\beta_2}{2}\frac{\partial^2}{\partial t^2}u + i\gamma f(z)|u|^2 u \tag{4}$$

where $f(z) = \psi(z)^2$. The advantage in using the 4 is that the attenuation dynamics can be described with a space-dependent nonlinear term: $\gamma f(z)$.

In the model from [2], using the *perfect amplification* assumption, $\gamma f(z) = \gamma$, since $f(z) \equiv 1$.

III. COUPLED NLS EQUATIONS FOR WDM CHANNELS

By including attenuation and gain it is required to drop the narrowband approximation: in doing so a solution model is the one of the coupled NLSE.

Consider two WDM channels named A and B. The following hypotesis are made:

- channels A and B have a spectral separation of Ω (a multiple of the WDM spectral spacing),
- both channels have the same nonlinear coefficient (it depends from modal field distribution in the core which is assumed to be the same for all channels),
- the group velocity profile is approximately linear in the frequency (β_2 is constant) in the whole band of interest,
- attenuation and Raman gain depend on the channel choice, but are approximately constant within the band of a given channel.

Following Agrawal [1, p.263], a system of *coupled NLSE* is given:

$$\frac{\partial}{\partial z}A_A = -\frac{\alpha_A - g_A(z)}{2}A_A - \beta_{1A}\frac{\partial}{\partial t}A_A - i\frac{\beta_2}{2}\frac{\partial^2}{\partial t^2}A_A + i\gamma(|A_A|^2 + 2|A_B|^2)A_A \tag{5}$$

$$\frac{\partial}{\partial z}A_B = -\frac{\alpha_B - g_B(z)}{2}A_B - \beta_{1B}\frac{\partial}{\partial t}A_B - i\frac{\beta_2}{2}\frac{\partial^2}{\partial t^2}A_B + i\gamma(|A_B|^2 + 2|A_A|^2)A_B \tag{6}$$

Let A be the WDM channel of interest, and B the interfering one.

Let us now proceed in normalizing the fields A_A, A_B with the respective normalization functions ψ_A, ψ_B, as described in the rescaling equation (II-B). In addition, a moving time reference frame is assumed, taking as a reference the time of arrival of the first pulse in channel A: *from this point, until the end of the article, the variable t indicates the normalized time.* These passages lead to:

$$\frac{\partial}{\partial z}u_A = -i\frac{\beta_2}{2}\frac{\partial^2}{\partial t^2}u_A + i\gamma\left(f_A(z)|u_A|^2 + 2\frac{f_A(z)}{f_B(z)}|u_B|^2\right)u_A \tag{7}$$

$$\frac{\partial}{\partial z}u_B = -\Delta\beta_1\frac{\partial}{\partial t}u_B - i\frac{\beta_2}{2}\frac{\partial^2}{\partial t^2}u_B + i\gamma\left(f_B(z)|u_B|^2 + 2\frac{f_B(z)}{f_A(z)}|u_A|^2\right)u_B \tag{8}$$

$$\tag{9}$$

where $\Delta\beta_1 = \beta_{1B} - \beta_{1A} = \beta_2\Omega$.

Following [2], a first order perturbation analysis is proposed for these equations.

A. Generalization of the 0th order term

Since the attenuation only affects the nonlinear term, the normalized field of the 0th order must be identical as the one derived in [2, eq. 1]. The only exception is due to the notation used: the total field in this case can not be expressed by a simple sum of terms $u_A^{(0)} + u_B^{(0)}$. There are in fact two notational caveats:

- $u_A^{(0)}$ and $u_B^{(0)}$ functions represent normalized fields with different normalization constants,
- the functions are derived from *complex amplitudes* (see for example [6, pp. 523-525]) of *different* carrier frequency signals

The explanation of the difference between field representations is postponed in section IV.

Let us consider the initial fields as sums of shifted impulses which encode a given message. Let T be the symbol period:

$$u_A(0,t) = \sum_k a_k g(0, t - kT)$$

$$u_B(0,t) = \sum_k b_k g(0, t - kT)$$

(10)

Let $g^{(0)}(z,t)$ be the linearly propagated field in channel A. The solution for the 0th order field is:

$$u_A^{(0)}(z,t) = \sum_k a_k g^{(0)}(z, t - kT)$$

$$u_B^{(0)}(z,t) = \sum_k b_k g^{(0)}(z, t - kT - \beta_2 \Omega z)$$

(11)

because of linearity, and definition of $g^{(0)}$.

As in [2], we define the differential operators

$$\mathbf{U}_A(z) = \exp\left[i\frac{\beta_2}{2}z\frac{\partial^2}{\partial t^2}\right]$$

(12)

$$\mathbf{U}_B(z) = \exp\left[i\frac{\beta_2}{2}z\frac{\partial^2}{\partial t^2} - i\Delta\beta_1 z\frac{\partial}{\partial t}\right]$$

(13)

So the propagated 0th order impulses have this alternative expression:

$$g^{(0)}(z, t - kT) = \mathbf{U}_j(z)g(0, t - kT)$$

(14)

with $j \in \{A, B\}$.

B. Generalization of first order perturbation theory

The separation of fields allow us to analyze separately the effects of SPM and XPM. Let us apply the perturbation method to 7

$$\frac{\partial}{\partial z}u_A^{(1)} = -i\frac{\beta_2}{2}\frac{\partial^2}{\partial t^2}u_A^{(1)} + i\gamma\left(f_A(z)|u_A^{(0)}|^2 + 2\frac{f_A(z)}{f_B(z)}|u_B^{(0)}|^2\right)u_A^{(0)}$$

(15)

Notice that the normalized fields $u_A^{(0)}$ and $u_B^{(0)}$ can not be summed together because they are complex amplitudes with respect to different carriers. However, their squared absolute value indicates only the power of the wave, so the summation of these terms makes physical sense. Writing the integral solution to the inhomogeneous linear equation above:

$$u_A^{(1)}(L,t) = i\gamma\int_0^L \mathbf{U}_A(L-z)\left(f_A(z)|u_A^{(0)}|^2 + 2\frac{f_A(z)}{f_B(z)}|u_B^{(0)}|^2\right)u_A^{(0)}dz$$

(16)

C. Generalization of estimation error

Using a matched filter receiver, with matching to the linearly propagated initial pulse waveform $g^{(0)}(L,T)$, we obtain the following equation for the estimation error on the first symbol Δa_0

$$\Delta a_0 = \int_{-\infty}^{\infty} u_A^{(1)}(L,t)g^{(0)*}(L,t)dt$$

(17)

Recall that $\mathbf{U}_A$ is unitary, so it holds

$$\mathbf{U}_A(L-z)g^{(0)*}(L,t) = g^{(0)*}(z,t)$$

(18)

Using this identity, the expression for the error can be written as:

$$\Delta a_0 = i\gamma\int_{z_0}^L dz \int_{-\infty}^{\infty} dt\, g^{(0)*}(z,t)\left(f_A(z)|u_A^{(0)}|^2 + 2f_B(z)|u_B^{(0)}|^2\right)u_A^{(0)}$$

(19)

By substituting the modulation of choice as in 11, we get an expression ready to be computed with respect to a given modulation format Notice that, being that the two fibers linear propagation terms are different, the substitution as done in 18 is valid only because matching is done by considering the propagated symbol waveform over channel A. Finally, substituting the modulation and using the same notation as [2, eq. 5, 6, 7], the resulting expression is

$$\Delta a_0 = i\gamma\sum_{h,k,m}(a_h a_k^* a_m S_{h,k,m} + 2a_h b_k^* b_m X_{h,k,m})$$

(20)

The (generalized) interaction terms are

$$S_{h,k,m} = \int_{z_0}^{L} dz f_A(z) \int_{-\infty}^{\infty} dt g^{(0)}(z, t-hT) g^{(0)*}(z, t-hT) g^{(0)}(z, t-kT) g^{(0)*}(z, t-mT) \tag{21}$$

for the SPM, and

$$X_{h,k,m} = \int_{z_0}^{L} dz f_B(z) \int_{-\infty}^{\infty} dt g^{(0)}(z, t-hT) g^{(0)*}(z, t-hT) g^{(0)}(z, t-kT) g^{(0)*}(z, t-mT) \tag{22}$$

for the XPM.

IV. CONNECTION TO PREVIOUS RESULTS

Suppose that the fields are represented as complex amplitudes with respect to the channel A center frequency. The equivalent linearly propagated total field is computed as

$$\begin{aligned} A_{tot}(z,t) &= A_A(z,t) + A_B(z,t) \\ &= \psi_A(z) u_A(z,t) + \psi_B(z) u_B(z,t) \exp[-i\Omega t] \\ &= \psi_A(z) \sum_k a_k g^{(0)}(z,t) + \psi_B(z) \exp[-i\Omega t] \sum_k b_k g^{(0)}(z,t) \end{aligned} \tag{23}$$

neglecting the attenuation-gain coefficients, this expression can be written as

$$u_{tot}(z,t) = \sum_k a_k g^{(0)}(z,t) + \exp[-i\Omega t] \sum_k b_k g^{(0)}(z,t) \tag{24}$$

so this appears to be in contradiction with respect to [2, eq. 1], which is

$$u_{tot}(z,t) = \sum_k a_k g^{(0)}(z,t) + \exp\left[-i\Omega t + i\frac{\beta_2 \Omega^2}{2} z\right] \sum_k b_k g^{(0)}(z,t) \tag{25}$$

This inconsistency is only apparent, because even the *propagated* fields $g^{(0)}(z,t)$ differ in their definition, as the propagation equation for u_B contains the walkoff term $-\Delta\beta_1 \frac{\partial}{\partial t} u_B$. In the following, the calculations are made again. In order to prove the equation let us recall the linear propagator operator (as defined in [2]):

$$\mathbf{U}[z] = \exp\left[i\frac{\beta_2}{2} z \frac{\partial^2}{\partial t^2}\right] \tag{26}$$

Then, let us focus on the interfering channel field at the *input*

$$u(0,t) = \sum_k b_k g(0, t-Tk) e^{i\Omega t} \tag{27}$$

and apply the propagator. This is best done in frequency domain, and, by linearity, it is possible to focus only on the symbol waveform g. Using frequency shifting property

$$g(0,t) e^{i\Omega t} \to \hat{g}(0, \omega - \Omega) \tag{28}$$

In frequency domain, we have the operator

$$\hat{\mathbf{U}}[z] = \exp\left[-i\frac{\beta_2}{2} z \omega^2\right] \tag{29}$$

Let us focus on the linear propagation of complex envelope *spectrum* of a single impulse

$$\hat{g}^{(0)}(z,\omega) = \exp\left[-i\frac{\beta_2}{2} z \omega^2\right] \hat{g}(0, \omega - \Omega) \tag{30}$$

considering the antitransform, with a square completion argument,

$$\frac{1}{2\pi} \int_{\mathbb{R}} \exp\left[-i\frac{\beta_2}{2} z \omega^2\right] \hat{g}(z, \omega - \Omega) \exp[i\omega t] d\omega = \tag{31}$$

$$= \frac{1}{2\pi} \int_{\mathbb{R}} \exp\left[-i\frac{\beta_2}{2} z (\omega - \Omega)^2\right] \hat{g}(z, \omega - \Omega) \exp[i(\omega - \Omega)t] \underbrace{\exp[i\Omega t]}_{\text{frequency shifting}} \underbrace{\exp[-i\beta_2 z \omega \Omega]}_{\text{time delay}} \underbrace{\exp\left[i\frac{\beta_2}{2} z \Omega^2\right]}_{\text{constant}} d\omega \tag{32}$$

in the notation of [2], $g^{(0)}(z,t) = \mathbf{U}(z) g(0,t)$ is the pulse propagated as in the channel of interest, so we have the following antitransform relation

$$g^{(0)}(z,t) = \frac{1}{2\pi} \int_{\mathbb{R}} \exp\left[-i\frac{\beta_2}{2} z \omega^2\right] \hat{g}(z,\omega) \exp[i\omega t] d\omega \tag{33}$$

In conclusion, by using a simple change of variables and the time shifting property:

$$\mathcal{F}^{-1}\left[\exp[-i\omega t_0]\hat{x}(\omega)\right](t) = x(t - t_0) \tag{34}$$

The linearly propagated single impulse of the *interfering* channel is

$$\exp\left[i\Omega t\right]\exp\left[i\frac{\beta_2}{2}\Omega^2 z\right]g^{(0)}(z, t - \beta_2\Omega z) \tag{35}$$

notice that the frequency component $\exp[i\Omega t]$ has opposite sign with respect to [2]. All the other terms are exactly the same. This may be due to a sign convention in the usage of the frequency shifting property, which is

$$\mathcal{F}\left[\exp[i\Omega t]x(t)\right](\omega) = \hat{x}(\omega - \Omega) \tag{36}$$

There is still a point to be discussed, regarding the definition of the propagator.

The linear equation to be solved is

$$\frac{\partial}{\partial z}g^{(0)}(z, t) = -i\frac{\beta_2}{2}\frac{\partial^2}{\partial t^2}g^{(0)}(z, t) \tag{37}$$

By the shift theorem it is possible to write it in symbolic form

$$\exp\left[h\frac{\partial}{\partial z}\right]g^{(0)}(z, t) = g^{(0)}(z + h, t) = \exp\left[-i\frac{\beta_2}{2}h\frac{\partial^2}{\partial t^2}\right]g^{(0)}(0, t) \tag{38}$$

In this way we notice that the propagator operator may be defined as

$$\mathbf{U}(h) = \exp\left[-i\frac{\beta_2}{2}h\frac{\partial^2}{\partial t^2}\right] \tag{39}$$

which is in contradiction with respect to [2] in which the sign of the argument is inverted. By calculating again the propagated impulse, the result is

$$\exp\left[i\Omega t - i\frac{\beta_2}{2}\Omega^2 z\right]g^{(0)}(z, t + \beta_2\Omega z) \tag{40}$$

which shows inverted sign on the terms which involve β_2. This aspect will require further investigation, to be justified with physical arguments and to be matched with [2] and [5, eq. 23].

V. COMPUTATION OF $X_{0,m,m}$

A. High dispersion approximation

Let us first comment the calculations done in [2, eq. 11, 12]. Starting from the highly-dispersed pulse approximation, we get

$$g^{(0)}(z, t) \approx \sqrt{\frac{i}{2\pi\beta_2 z}}\exp\left[-\frac{it}{2\beta_2 z}\right]\hat{g}\left(0, \frac{t}{\beta_2 z}\right) \tag{41}$$

Now, it is possible to compute the coefficient $X_{0,m,m}$ through energy integral in Fourier space by defining $\nu = t/\beta_2 z$

$$X_{0,m,m} = \int_{z_0}^{L}dz f(z)\int_{\mathbb{R}}\frac{d\nu}{4\pi^2\beta_2 z}|\hat{g}(0, \nu)|^2\left|\hat{g}\left(0, \nu - \Omega - \frac{mT}{\beta_2 z}\right)\right|^2 \tag{42}$$

The approximation is that the strongest overlap happens at $z_m = -\frac{mT}{\beta_2\Omega}$ By the *perfect amplification* hypotesis, it is assumed that $f(z) \equiv 1$. A further assumption is made claiming that most of the overlap happens at $z = z_m$, so the integrand is non negligible only when $z_m/z \approx 1$. The integral becomes

$$X_{0,m,m} = \int_{\mathbb{R}}\frac{d\nu}{2\pi}|\hat{g}(0, \nu)|^2\int_{\mathbb{R}}dz\frac{z_m f(z_m)}{4\pi^2\beta_2 z^2}\left|\hat{g}\left(0, \nu - \Omega - \frac{mT}{\beta_2 z}\right)\right|^2 \tag{43}$$

Notice that the integration along all the space allow us to recall an important property of the impulses: they are of unit energy. Using Parseval identity it is possible to eliminate the impulse waveform in the following way. Let us adopt this change of variables:

$$y := -\frac{mT}{\beta_2 z} \quad\Longrightarrow\quad dy = \frac{mT}{\beta_2 z^2} \tag{44}$$

The multiplication by z_m/z creates the term dy along with the other constants:

$$X_{0,m,m} = \int_{\mathbb{R}} \frac{d\nu}{2\pi} |\hat{g}(0,\nu)|^2 \int_{\mathbb{R}} \frac{f(z_m)}{2\pi\beta_2\Omega} \Big(\underbrace{-\frac{mT}{\beta_2 z^2} dz}_{dy} \Big) \Big| \hat{g}\Big(0, \nu - \Omega - \frac{mT}{\beta_2 z}\Big) \Big|^2 \tag{45}$$

$$= \frac{f(z_m)}{\beta_2\Omega} \int_{\mathbb{R}} \frac{d\nu}{2\pi} |\hat{g}(0,\nu)|^2 \int_{\mathbb{R}} -\frac{dy}{2\pi} |\hat{g}(0, \nu - \Omega + y)|^2 \tag{46}$$

If $f(z_m)$ is assumed to be 1 in perfect amplification scenario, the integrals simplify to

$$= \frac{1}{\beta_2\Omega} \int_{\mathbb{R}} \frac{d\nu}{2\pi} |\hat{g}(0,\nu)|^2 \int_{\mathbb{R}} \frac{dy}{2\pi} |\hat{g}(0, \nu - \Omega + y)|^2 \tag{47}$$

finally, both integrals, by Parseval, sum to 1, so

$$X_{0,m,m} = \frac{1}{\beta_2\Omega} \tag{48}$$

when z_m falls inside the fiber and in the region of high dispersion, 0 otherwise.

B. Criticalities of high dispersion approximation

The *generalization* of the above calculation has one major difficulty: in the overlap region the cumulative pulse attenuation f is assumed to be constant. If this assumption holds true, the expression generalizes naturally

$$X_{0,m,m} = \frac{f_B(z_m)}{\beta_2\Omega} \tag{49}$$

Otherwise, we may be interested in cases in which interaction can not be assumed as local. Furthermore, the approximation used for highly-dispersed pulses (41) is valid for lengths much higher than the dispersion length, and in the region of interest for Raman amplifiers the approximation may not suffice in calculating the XPM coefficient.

Dropping the locality assumption (common Raman amplifiers have complex amplification profiles over hundreds of km), the integral to consider is the one stemming from time-domain analysis 22:

$$X_{0,m,m} = \int_{z_0}^{L} dz f_B(z) \int_{-\infty}^{\infty} dt |g^{(0)}(z,t)|^2 |g^{(0)}(z, t - mT)|^2 \tag{50}$$

in order to find a suitable computational simplification, let us assume that the pulses are gaussian.

VI. GAUSSIAN PULSES

The effect of linear propagation for gaussian pulses has a closed form expression:

$$g(z,t) = \frac{U_0 \exp[\frac{i}{2} \arctan(D(z))]}{(1 + D^2(z))^{1/4}} \exp\left[-\frac{t^2}{2T_0^2} \frac{1 + iD(z)}{1 + D^2(z)} \right] \tag{51}$$

where $D(z) = z\beta_2/T_0^2$. Some comments on this representation are given in Appendix A.

Assuming pulse energy normalization to 1, amplitude and width parameters need to satisfy the following equation

$$U_0^2 T_0 \sqrt{\pi} = 1 \tag{52}$$

considering

$$|g^{(0)}(z,t)|^2 = \frac{U_0^2}{(1 + D^2(z))^{1/2}} \exp\left[-\frac{t^2}{T_0^2} \frac{1}{1 + D^2(z)} \right] \tag{53}$$

and then substituting the relative expressions in 50, the following expression is given

$$X_{0,m,m} = \int_0^L dz f_B(z) \int_{\mathbb{R}} dt \frac{U_0^4}{1 + D^2(z)} \exp\left[-\frac{1}{T_0^2(1 + D^2(z))} \underbrace{\left(t^2 + (t - mT - \beta_2\Omega z)^2 \right)}_{\varphi} \right] \tag{54}$$

Let s be defined as

$$s := mT + \beta_2\Omega z \tag{55}$$

then, the term φ can be rewritten as:

$$\varphi = 2t^2 - 2ts + s^2 \tag{56}$$

$$= \left(\sqrt{2}t - \frac{s}{\sqrt{2}}\right)^2 + \frac{s^2}{2} \tag{57}$$

let us use a change of variable of integration: $\eta := \sqrt{2}t - \frac{s}{\sqrt{2}}$, from which $dz\,dt = dz\,d\eta\frac{1}{\sqrt{2}}$. So we have the expression $\varphi = \eta^2 + \frac{s^2}{2}$, and the integral admits this expression

$$X_{0,m,m} = \int_0^L dz f_B(z) \int_{\mathbb{R}} \frac{d\eta}{\sqrt{2}} \frac{U_0^4}{1 + D^2(z)} \exp\left[-\frac{\eta^2}{T_0^2(1 + D^2(z))}\right] \exp\left[-\frac{s^2}{2T_0^2(1 + D^2(z))}\right] \tag{58}$$

The last step is the assumption of local interaction for $z = z_m = -mT/\beta_2\Omega$. This means that the functions $f_B(z)$ and $D(z)$ can be substituted with constants $f_B(z_m)$ and $D(z_m)$, respectively. We can also extend integration to $\mathbb{R}$, for every m such that $z_m \in [0, L]$. Than it is possible to simplify the integral

$$X_{0,m,m} = f_B(z_m)\frac{U_0^4}{1 + D^2(z_m)}\frac{1}{\sqrt{2}} \int_{\mathbb{R}} dz \int_{\mathbb{R}} d\eta \exp\left[-\frac{\eta^2}{T_0^2(1 + D^2(z))}\right] \exp\left[-\frac{s^2}{2T_0^2(1 + D^2(z))}\right] \tag{59}$$

The remaining equation contains two gaussian integrals

$$\int_{\mathbb{R}} dt \exp\left[-\frac{t^2}{\alpha}\right] = \sqrt{\alpha\pi} \tag{60}$$

integrating with respect to η we obtain

$$f_B(z_m)\frac{U_0^4}{1 + D^2(z_m)}\frac{1}{\sqrt{2}}(T_0^2(1 + D^2(z_m)))^{\frac{1}{2}}\sqrt{\pi} \int_{\mathbb{R}} dz \exp\left[-\frac{s^2}{2T_0^2(1 + D^2(z))}\right] \tag{61}$$

Finally, using s as a new integration variable

$$s = mT + \beta_2\Omega z \qquad dz = \frac{1}{\beta_2\Omega}ds \tag{62}$$

it is possible to solve even the last integral as

$$X_{0,m,m} = \frac{f_B(z_m)}{\beta_2\Omega}U_0^4 T_0^2 \pi \tag{63}$$

Recall the normalization condition for the pulse energy in (52): the substitution cancels both parameters U_0 and T_0 from the final expression

$$X_{0,m,m} = \frac{f_B(z_m)}{\beta_2\Omega} \underbrace{U_0^4 T_0^2 \pi}_{=1} = \frac{f_B(z_m)}{\beta_2\Omega} \tag{64}$$

VII. Conclusion

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Appendix A

The field expression in (51) contains a phase term in the form $\exp\left[\frac{i}{2}\arctan(D(z))\right]$. This term highlights the phase and modulus representation of the coefficient of the exponentials. However, it may not seem so straightforward. From Fourier antitransform, the field has the following expression

$$u(z,t) = U_0 \left(\frac{1 + iD(z)}{1 + D^2(z)}\right)^{\frac{1}{2}} \exp\left[-\frac{t^2}{2T_0^2}\frac{1 + iD(z)}{1 + D^2(z)}\right] \tag{65}$$

the two field expression are connected by this calculations which uses fundamental goniometric identities

$$\exp\left[\frac{i}{2}\arctan(D(z))\right] = \exp\left[2i\arctan(D(z))\right]^{\frac{1}{4}} = \tag{66}$$

$$= \left[\cos(2\arctan(D(z))) + i\sin(2\arctan(D(z)))\right]^{\frac{1}{4}} = \tag{67}$$

$$= \left[\frac{1-t^2}{1+t^2} + i\frac{2t}{1+t^2}\right]^{\frac{1}{4}} = \qquad \text{where } t = \tan\left(\frac{\cancel{2}\arctan(D(z))}{\cancel{2}}\right) = D(z) \tag{68}$$

$$= \left[\frac{(1+iD(z))^2}{1+D^2(z)}\right]^{\frac{1}{4}} = \frac{(1+iD(z))^{\frac{1}{2}}}{(1+D^2(z))^{\frac{1}{4}}} \tag{69}$$