EXPECTED VALUE

- $E(X) = \sum x_k \cdot P_X(x_k)$
- $E(X) = \int_{\mathbb{R}} x \cdot f(x)\, dx$
- $E(aX) = a\, E(X)$
- $E(g(X) + h(X)) = E(g(X)) + E(h(X))$
- $P(X = c) = 1 \rightarrow E(X) = c$

VARIANCE

- $VAR(X) = E((X - E(X))^2)$, $STD(X) = \sqrt{VAR(X)}$
- $VAR(X) = \sum_k (x_k - \mu)^2 \cdot P_X(x_k)$
- $VAR(X) = \int (x - \mu)^2 \cdot f_X(x)\, dx$
- 2° MOMENTO CENTRALE
- 'MOMENTO D'INERZIA', DISPERSIONE
- $VAR(X) = 0 \longleftrightarrow P(X = c) = 1$
- $VAR(aX) = a^2\, VAR(X)$
- $VAR(X + b) = VAR(X)$
- $VAR(X) = E(X^2) - E(X)^2$

CHEBYCHEV

- $P(|X - E(X)| > \epsilon) \leq \dfrac{VAR(X)}{\epsilon^2}$
- $P(X > \epsilon) \leq \dfrac{E(X^n)}{\epsilon^n}$ (MARKOV)

STANDARDIZATION

- $E(X) = m$
- $VAR(X) = \sigma^2$

$F_X(t) = P(X \leq t) = F_Y\left(\dfrac{t - m}{\sigma}\right)$

- $Y = \dfrac{X - m}{\sigma}$
- IN GAUSSIANA STANDARDIZZO $X \sim N(\mu, \sigma^2)$

MOMENTS

- SE $|X|^n$ HA VALORE ATTESO $\rightarrow E(X^n)$ [RAW MOMENT, ABOUT 0]
  E' K-TH MOMENT
- SE V.A. SIMMETRICA MOMENTI DISPARI NULLI
- MOMENTI PARI $\rightarrow$ SEMPRE NON NEGATIVI
- $E(X^q) = \int X^q(w)\, P(w)\, dw \,\Big|\, \sum_i p_i x_i^q$
- MOMENTI CENTRALI ORDINE M

$E\{(X - \mu)^q\} = \int (x - \mu)^q f(x)\, dx$

- SUCCESSIONE MOMENTI IDENTIFICA V.A. UNIVOCAMENTE
- SE V.A. NO DENSITA' MA SI CDF:
  $\mu'_n = E[X^n] = \int x^n\, dF(x)$
- 0TH MOMENT $\longleftrightarrow$ PDF $\leadsto 1$
- NORMALIZED MOMENTS
  CENTRAL

$k = \dfrac{E[(X - \mu)^n]}{\sigma^n}$, DIMENSIONLESS

MGF

$M_X(t) = E(e^{tX})$

LAPLACE TRANSFORM DI DENSITÁ

$M_X(t) = \sum_i^{\infty} e^{t x_i} \cdot p_i$

$M_X(t) = \int e^{tx} f(x)\, dx$

MAY NOT EXIST

- TROVO MOMENTI (CENTRALI) DERIVANDO MGF A 0
- X CONVOLUZIONE TRASFORMATE

SE $Y = \sum_i X_i$ (INDIP.)

$M_Y(\theta) = \prod_i M_{X_i}(\theta)$

CHARACTERISTIC FUNCTION

~~INVERSE~~ FOURIER TRANSFORM OF PDF, ALWAYS EXISTS

FULLY SPECIFIES DIST OF R.V. AND VICEVERSA, BIJECTIVE

- $\varphi_X(t) = E[e^{itX}] = \int e^{itx} f_X(x)\, dx$
- SE X_1, X_2 INDIPENDENTI (NON IID)
  $Y = X_1 + X_2 \longrightarrow \varphi_Y(t) = \varphi_{X_1}(t) \cdot \varphi_{X_2}(t)$
- SE MOMENTO $E|X^q|$ ESISTE ED E' FINITO
  $\rightarrow$ ESISTE ED E' CONTINUA DERIVATA q DI CHARF.
- $f_X(x) = \dfrac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-itx} \cdot \varphi_X(t)\, dt$

SKEWNESS

- ASIMMETRY OF DISTRIBUTION ABOUT ITS MEAN
- NEGATIVE / POSITIVE / UNDEFINED
- NEGATIVE: MORE MASS ON RIGHT
- POSITIVE: " " " LEFT
- DOES NOT DISTINGUISH BETWEEN LONG / FAT TAILS
- $\gamma_1 = E\left[\left(\frac{X-\mu}{\sigma}\right)^3\right] = \frac{E[(X-\mu)^3]}{\sigma^3} = \frac{\mu_3}{\sigma_3}$

THIRD STANDARDIZED CENTRAL MOMENT

KURTOSIS

- PEAKEDNESS / HEAVYNESS OF TAILS (LACK OF SHOULDERS)
- OLD DEFINITION:
  4^{TH} STD. CENTRAL MOMENT
  $$\beta_2 = \frac{\mu_4}{\sigma_4^4}$$
- NEW: EXCESS KURTOSIS
  $$\gamma_4 = \frac{\mu_4}{\sigma^4} - 3 \quad \left[\frac{4^{TH}\text{ CUMULANT}}{(2^{ND}\text{ CUMULANT})^2}\right]$$
- -3 TO MAKE $\gamma_4(N \sim (0,1)) = 0$
- $\gamma_4 = 0$, MESOKURTIC, GAUSSIAN
- $\gamma_4 > 0$, LEPTOKURTIC, 'SLENDER', ACUTE PEAK, FATTER TAILS
- $\gamma_4 < 0$, PLATYKURTIC, LOWER PEAK, THINNER TAILS
  FLAT

PROBABILITY GENERATING FUNCTION

FOR DISCRETE DISTRIBUTIONS ONLY

- $G(z) = E[z^X] = \sum P(x) z^x$, $P(x)$ PMF
- $P(x) = \frac{G^{(x)}(0)}{n!}$

CUMULANT GENERATING FUNCTION

- $g(t) = \log \underline{E[e^{tX}]}$ MGF
- $g(t) = \sum k_n \frac{t^n}{n!}$

CUMULANTS

- QUANTITY SET ALTERNATIVE TO MOMENTS
  NONLINEAR COMBINATIONS OF MOMENTS

$$\begin{array}{ccc} \text{DISTR} & \leftrightarrow & \text{MOMENTS} \\ \updownarrow & \swarrow\!\!\nearrow & \\ & & \text{CUMULANTS} \end{array}$$

- $k_n = g^{(n)}(0)$ → DERIVATA
- $g_{X+Y}(t) = g_X(t) + g_Y(t)$: ~~SUM~~ CUMULANT OF SUM IS SUM OF CORRESPONDING CUMULANTS
- $g_{XY}(t) = g_X(g_Y(t))$
- $\mu_1' = k_1$
- $\mu_2' = k_2 + k_1^2$ } RAW MOMENTS
- FOR IID VARIABLES CUMULANTS ARE ADDITIVE

R.V. MULTIVARIATE

$X = \{X_1 \dots X_m\}^{(T)}$ $X_{1..m}$ R.V.

- $F_X = P(X_1 \le x_1, X_m \le x_m)$ JOINT
- $\lim_{x \to +\infty} F_{X,Y}(x,y) = P(Y \le y) = F_Y(y)$ MARGINAL (DISTRIBUTION) CDF
- JOINT → MARGINAL, MARGINAL $\not\to$ JOINT
- $F_X(x) = \sum p_X(y)$

  $F_X(x) = \int_{-\infty}^{x_1} \dots \int^{x_n} f_X(x_1 \dots x_n)\, dx_1 \dots dx_n$
- $f_{X,Y} = \frac{\partial^2 F_{XY}(x,y)}{\partial x \partial y}$ DENSITY

• COVARIANCE

$COV(X_1, X_2) = E[(X_1 - E[X_1])(X_2 - E[X_2])]$

$\rho_{X_1 X_2} = \frac{COV(X_1, X_2)}{\sqrt{VAR(X_1)\, VAR(X_2)}}$ MOMENTO CONGIUNTO CENTRALE ORDINE 2

INDICATE LINEAR RELATIONSHIP

- $COV(X_1, X_2) = COV(X_2, X_1)$
- $COV(aX_1, X_2) = a\, COV(X_1, X_2)$
- $COV(X_1 + X_2, X_3) = COV(X_1, X_3) + COV(X_2, X_3)$
- $COV(X_1, X_2) = E(X_1 X_2) - E(X_1)E(X_2)$
- SE INDIP → $COV(X_1, X_2) = 0$

• COVARIANCE MATRIX

- $X = \{X_1 \dots X_N\}$

$C_X = \begin{bmatrix} VAR(X_1) & COV(X_1,X_2) & \dots & COV(X_1,X_N) \\ COV(X_2,X_1) & VAR(X2) & \dots & COV(X_2,X_N) \\ \vdots & \vdots & & \vdots \\ COV(X_N,X_1) & COV(X_N,X_2) & \dots & VAR(X_N) \end{bmatrix}$

- SIMMETRICA, SEMIDEFINITA POSITIVA
- $\begin{cases} C_Y = A \cdot C_X \cdot A^T \\ E[Y] = A\mu + b \end{cases}$ $\begin{cases} X \to C_X \\ Y = AX + b \end{cases}$

$C_X = D_X \cdot R_X \cdot D_X$

D_X = MAT DIAG STDDEV

R_X = MAT COEFF. CORRELAZIONE

• INDIPENDENZA

IFF $F_X = F_{X_1} \cdot F_{X_2} \cdot F_{X_N}$

$\Updownarrow$

$f_X = f_{X_1} \cdot f_{X_2} \cdots f_{X_N}$

- SOMMA POISSON INDIP:

$P(X_1 + X_2 = k) = e^{-(\lambda_1+\lambda_2)} \cdot \frac{(\lambda_1+\lambda_2)^k}{k!}$

- SOMMA GAUSSIANE INDIP:

$Z_1 + Z_2 \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$

- SOMMA EXP. INDIP:

$f_{X_1 .. X_n} = \frac{\mu^n}{(n-1)!} x^{n-1} e^{-\mu x} = \Gamma(n, \mu)$

• AVG

$E(X_1 + X_2) = E(X_1) + E(X_2)$

$E(X_1 \cdot X_2) = E(X_1) \cdot E(X_2)$

• VAR

$VAR(X_1 + X_2) = VAR(X_1) + VAR(X_2) + 2E[(X_1 - E(X_1))(X_2 - E(X_2))]$ COV

INDIP: $VAR(X_1 + X_2) = VAR(X_1) + VAR(X_2)$

• MOMENTS

JOINT MOMENT ORDINE p+q

$E[X^p Y^q] = \iint x^p y^q \cancel{(x-\mu_x)^p (y-\mu_y)^q} f_{XY}(x,y)\, dx\, dy$

CENTRALE

- SE INDIP → $E[X^p Y^q] = E[X^p] \cdot E[Y^q]$

CONDITIONALS

$f(x_2 | x_1) = \frac{f_X(x_1, x_2)}{\int f_X(x_1, x_2)\, dx_2}$

• MULTIVARIATE CHANGE OF VARIABLES

- $f_Y(y) = f_X(x) \left| \frac{dx}{dy} \right|$ 1D
- IS OF REPARAMETRIZATION TRICK
- $J = \frac{\partial(y_1 \dots y_N)}{\partial(x_1 \dots x_N)} = \begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \dots & \frac{\partial y_1}{\partial x_N} \\ \vdots & & \\ \frac{\partial y_N}{\partial x_1} & \dots & \frac{\partial y_N}{\partial x_N} \end{bmatrix}$
- $f_Y(y) = f_X(x) \cdot |\det J|$

R.V. Multivariate Gaussiane

STANDARD

$Z_N = \{Z_1 \dots Z_N\}^T$ IFF $Z_i \sim N(0,1)$ E INDIP

$$f_Z(z_1 \dots z_N) = \frac{1}{(2\pi)^{N/2}} \cdot e^{-\frac{1}{2}\sum_k^N z_k^2}$$

$E[Z_N] = [0] \qquad COV[Z_N] = I_N$

GENERIC

$X = AZ + \mu$

DOVE Z E' VETTORE GAUSSIANO STANDARD $Z \sim (0, I)$

$E[X] = \mu$

$C_X[X] = A \cdot \overbrace{I}^{C_X[Z]} \cdot A^T = A \cdot A^T$

• OK IFF $C = A \cdot A^T$ NON SINGOLARE

$$f_X(x) = \frac{1}{\sqrt{(2\pi)^n \det(C)}} \cdot e^{-\frac{1}{2}(x-\mu)^T C^{-1}(x-\mu)}$$

ALT: $$N(x|\mu, \Sigma) = \frac{1}{(2\pi)^{D/2}|\Sigma|^{1/2}} \cdot e^{-\frac{1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu)}$$

PRECISION MATRIX! $\Lambda = \Sigma^{-1}$

CONCENTRATION

Central Limit Theorem

$S_N = X_1 + \dots + X_N$, X_i INDIP & IID , μ, σ^2

• WEAK LARGE NUMBERS LAW

$$\lim_{n\to\infty} P\left(\left|\frac{S_n}{n} - \mu\right| > \epsilon\right) = 0$$

$\frac{S_n}{n}$ = SAMPLE MEAN = $\overline{X_N}$

DIM

X' IID

$VAR(S_N) = n\, VAR(X_1) = n\sigma^2$

$VAR\left(\frac{S_N}{N}\right) = \frac{\sigma^2}{n}$

$E\left(\frac{S_N}{N}\right) = \mu$

X' CHEBYCHEV

$P\left(\left|\frac{S_N}{n} - \mu\right| > \epsilon\right) \le \frac{\sigma^2}{n\epsilon^2} \rightarrow \lim_{x\to\infty}$

• STRONG L.N.L.

$$P\left(\{\omega : \lim_{n\to\infty} \overline{X_n} = \mu\}\right) = 1$$

• CLT

STD. SAMPLE MEAN: $\frac{\overbrace{\cancel{}(\overline{X_n} - \mu)}^{V.A.}}{\sigma/\sqrt{N}} \sim N(0,1)$

$$\lim_{n\to+\infty} P\left(\frac{\cancel{\sqrt{n}}(\overline{X_n} - \mu)}{\sigma/\sqrt{n}} \le x\right) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} du = \Phi(x)$$

SAMPLE MEAN OF N IID R.V. (μ, σ^2) HA FDR $\approx N\left(\mu, \frac{\sigma^2}{n}\right)$

• $\overline{X_N} \approx N\left(\cancel{a}\mu, \frac{\sigma^2}{n}\right)$, $P(\overline{X_N} \le t) \approx \Phi\left(\frac{\cancel{}\, t-\mu}{\sigma/\sqrt{n}}\right)$

PROOF BY CHAR. FCN

$Y(0,1) \rightarrow \varphi_Y(t) = 1 - \frac{t^2}{2} + o(t^2)$ [TAYLOR]

$\overline{X} = \sum_i \frac{Y_i}{\sqrt{n}}$, $Y_i = \frac{X_i - \mu}{\sigma}$

$$\varphi_{Z_N} = \varphi_{\sum_i \frac{Y_i(t)}{\sqrt{N}}} = \varphi_{Y_1}\left(\frac{t}{\sqrt{N}}\right) \cdot \varphi_{Y_2}\left(\frac{t}{\sqrt{N}}\right) \cdots \varphi_{Y_N}\left(\frac{t}{\sqrt{N}}\right) = \varphi_Y\left(\frac{t}{\sqrt{n}}\right)^n$$

APPROX TAYLOR $= \left[1 - \frac{t^2}{2n} + o\left(\frac{t^2}{n}\right)\right]^n \xrightarrow[n\to\infty]{} e^{-t/2}$ IS CHAR FCN OF $N(0,1)$!!!