CS-49: Game Theory

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Problem 9.

Let x be the value of some particular impartial game. Then, as we saw in class, $x+x=0$, since, by symmetrizing, the sum of two copies of this game is in the class P whose value is 0 Since addition of games is commutative (and associative), the set of all possible values of impartial games should be an abelian group in which $x+x=0$ for all elements x.

Find an example of such a group, preferably one of countably infinite size.

The simplest group G in which every non-identity element x has order 2 is the commutative group of order 2,

$$C_2 = (\{0,1\}\,;\,;+,0) \cong \mathbb{Z}/2\mathbb{Z}.$$

We can generate higher-order groups by taking repeated direct products of C_2 with itself. For instance,

$$C_2 \times C_2 = (\{(0,0),(0,1),(1,0),(1,1)\}\,;+,(0,0))\,.$$

Through a little inspection, we can easily see that this group is isomorphic to the *Klein four* group, V_4, since each non-identity element has order 2 and $(1,1)=(0,1)+(1,0)$. To get a countably infinite group, consider the group generated by taking an infinite number of direct products of C_2 with itself. Such groups have similar structures to the corresponding vector spaces over C_2 — for example, V_4 is isomorphic to the vector space of dimension 2 over C_2:

$$\mathbf{V} = \left\{\begin{bmatrix}0\\0\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix},\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}1\\1\end{bmatrix}\right\}, \quad \mathbf{basis}(\mathbf{V}) = \left\{\begin{bmatrix}0\\1\end{bmatrix},\begin{bmatrix}1\\0\end{bmatrix}\right\}$$

Similarly, an infinite-dimensional group in which every element has order 2 is isomorphic to an infinite-dimensional vector space over $\mathbb{Z}/2\mathbb{Z}$. Since the basis (and therefore the vector space itself) can be ordered in a countable way, the corresponding group elements can also be ordered in a countable way.