CS-49: Game Theory

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Problem 22.

Let S be a board of size $2n$ for a selection game G in which, prior to each move, the next card is drawn from a well-shuffled deck of n blue cards and n red cards. The next move is then taken by Louise if the card is blue, Richard if red. The payoff to Louise is $f(S_1)$, where f is a function from subsets of S of size n to the real numbers, and the payoff to Richard is $-f(S_1)$, where S_1 is the set Louise ends up with. Show that the (expected) value to Louise of this game is the expected value of $f(R)$, where R is a uniformly random subset of S of size n.

In each turn, the payoff is determined by the set S and not the who makes the move. The expected value to Louise is the averaged value of all such sets of size n. The number of such possible sets of size n is $\binom{2n}{n}$. Therefore, the probability of any one subset of 2^S of size n being selected is $1/\binom{2n}{n}$. The expected value of the game to Louise is

$$\mathbf{value}(G) = \sum_{i=1}^{\binom{2n}{n}} \frac{1}{\binom{2n}{n}} f(S_i)$$

Now, consider if a uniformly random $R \subset S$ of size n is selected. There are $2n$ possible candidates for the first element of the subset, $2n-1$ for the second, and so on up to the n'th. In total, there are $\binom{2n}{n}$ possible configurations of subsets of size n. The probability of R being any one specific subset is $1/\binom{2n}{n}$. Therefore, the expected value of R is:

$$f(R) = \sum_{i=1}^{\binom{2n}{n}} \frac{1}{\binom{2n}{n}} f(S_i)$$

Therefore,

$$\mathbf{value}(G) = f(R).$$

References

(i) For random turn games, see https://arxiv.org/pdf/math/0508580.pdf.

(ii) For Knuth's Surreal Numbers, see https://people.math.harvard.edu/~knill/teaching/mathe320_2015_fall/blog15/surreal1.pdf.