CS-49: Game Theory

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Problem 15.

The convex closure $\mathbf{cc}(U)$ of a set $U \subseteq E_d$ is the intersection of all convex sets containing U. Show that $\mathbf{cc}(U)$ can also be described as the set of all points $c_1p_1 + c_2p_2 + \ldots + c_np_n$ where n is any positive integer, each p_i is any point (i.e., d-tuple) of U, and the c_i's are arbitrary nonnegative reals that sum to 1.

A convex set is a set such that for any two points in the set, every point on the line segment connecting them is also in the set. While U need not be convex, $\mathbf{cc}(U)$ is the smallest convex set containing U. This means that $\mathbf{cc}(U)$ is the set of all points in U and all points on the lines between any two points in U. Thus, every point in $\mathbf{cc}(U)$ is either:

(i) A member of U. In this case, we can write the point p as $1 \cdot p$.

(ii) The point is not a member of U, but is on the line segment between some two points in U. Let p_1 and p_2 be two points in U, and p_3 be any point on the line segment between p_1 and p_2, then we can write $p_3 = p_1 + \lambda(p_2 - p_1)$ for some $\lambda \in [0, 1]$. Refactoring, we see that $p_3 = (1-\lambda)p_1 + \lambda p_2$. Most importantly, $c_1 = 1 - \lambda$ and $c_2 = \lambda$ are nonnegative and sum to 1. Suppose $p_4 \in U$ is any other point, then every point on the line segment between p_3 and p_4 is also contained in $\mathbf{cc}(U)$. Using the same process as we did for p_3, if p_5 is any point on the line segment between p_3 and p_4, then we can write $p_5 = (1 - \mu)p_3 + \mu p_4$ for some $\mu \in [0, 1]$. Since $p_3 = (1 - \lambda)p_1 + \lambda p_2$, we can write

$$p_5 \quad = \quad (1-\mu)((1-\lambda)p_1 + \lambda p_2) + \mu p_4 \quad = \quad (1-\mu)(1-\lambda)p_1 + (1-\mu)\lambda p_2 + \mu p_5.$$

Since $(1 - \lambda) + \lambda = 1$,

$$(1-\mu)(1-\lambda) + (1-\mu)\lambda = (1-\mu)((1-\lambda)+\lambda) = 1-\mu.$$

On the other hand, $(1 - \mu) + \mu = 1$, so the constants in the equation for p_5 sum to 1. Using the same logic, we can extend the equation to all points between p_5 and any other point in U, since any such point on the line segment must be contained in $\mathbf{cc}(U)$. Therefore, every point in $\mathbf{cc}(U)$ can be written as such a combination of points in U with nonnegative coefficients that sum to 1.