CS-49: Game Theory

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04/03/2023

Problem 3.

Your utility for owning $\$x$ is $\log x$. (The base of the logarithm doesn't matter; for this problem I recommend the natural logarithm.) You have $\$a$ and are permitted to make one bet on the flip of a coin that comes up "heads" with probability p which is known to you and greater than $1/2$. If you bet $\$b$ and win, you win $\$b$; otherwise you lose the amount bet. What fraction of $\$a$ should you bet, to maximize your expected utility?

Suppose you bet $\$b$ which is a fraction m of your total wealth $\$a$. Then:

$$\begin{aligned}
\text{Current utility} &= \log a \\
\text{Utility if your bet wins} &= \log(a+b) = \log((1+m)a) = \log(1+m) + \log a \\
\text{Utility if you bet \$}b\text{ and lose} &= \log(a-b) = \log((1-m)a) = \log(1-m) + \log a \\
\text{Expected utility} &= p(\log(1+m) + \log a) + (1-p)(\log(1-m) + \log a) \\
&= p\log a + (1-p)\log a + p\log(1+m) + (1-p)\log(1-m) \\
&= \log a + p\log(1+m) + (1-p)\log(1-m)
\end{aligned}$$

To improve utility, we want to maximize the change

$$p\log(1+m) + (1-p)\log(1-m).$$

Via a little guess-work (see plots below and reference (1)), it seems $m = 2p - 1$ is a good choice to mazimize the change in utility when the probability of a favorable outcome is known to be p.

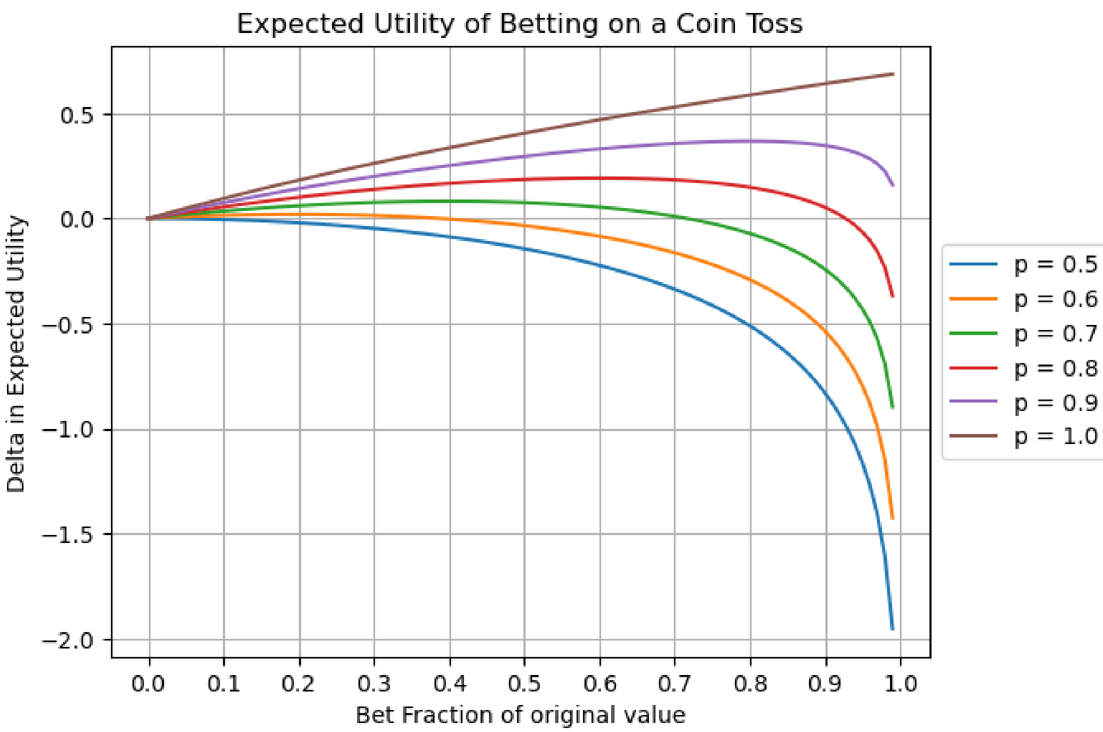

FIGURE 1. The change in utility with various p and various m.

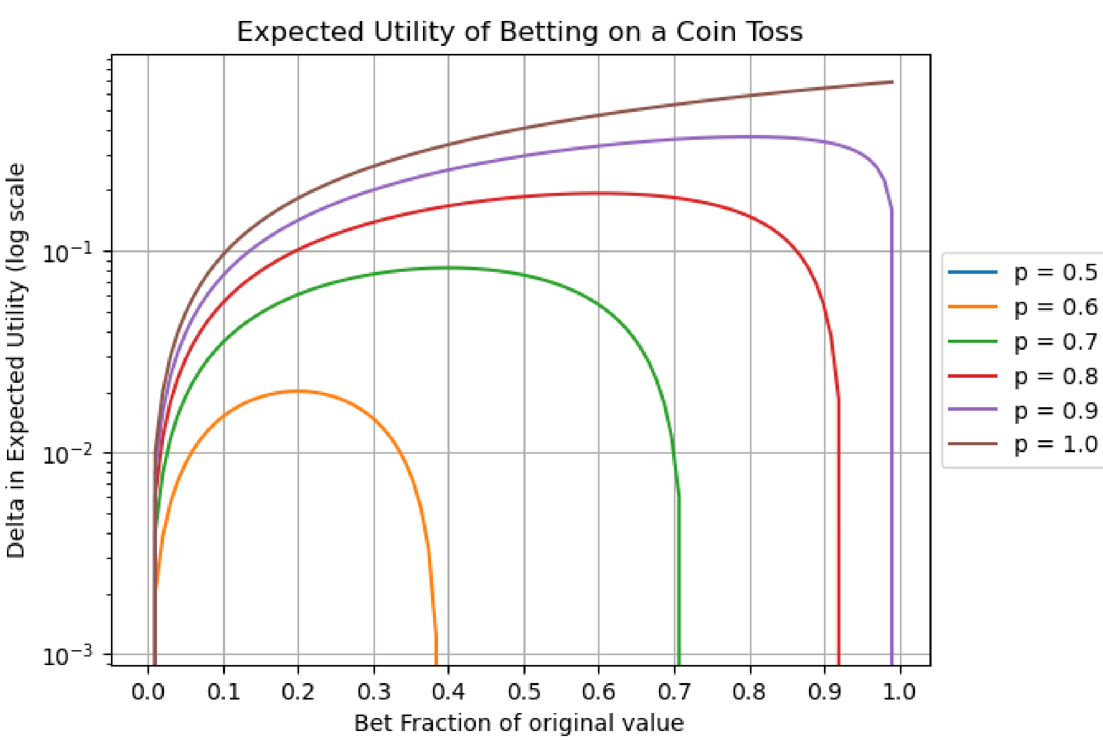

FIGURE 2. The change in utility with various p and various m (**log** scale).

References

1. A. Siavava, *Game Theory; Associated Code*, https://github.com/siavava/game-theory/blob/main/assignments/03/optimize.ipynb, 2023.