

Math 71: Algebra

Groups of Small Order

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CONTENTS

1. Groups of Prime Order	3
References	5

1. GROUPS OF PRIME ORDER

Next, let's look at groups of prime order through some motivating theorems.

Theorem 1.1 (Lagrange). *If G is a finite group and H is a subgroup of G , then $|H|$ divides $|G|$ and the number of left cosets of H in G is equal to $\frac{|H|}{|G|}$. (1, p.89, Theorem 8)*

Proposition 1.2. *If G is a finite group of prime order p then every non-identity element of G has order p .*

Proof. Let g be a non-identity element of G , without loss of generality, such that the order of g is $x \in \mathbb{Z}_{\geq 0}$. Then $g^x = \varepsilon$. This immediately tells us that x cannot be 1, since g is clearly not the group identity. Let $\langle g \rangle$ be the set of all elements $g^i, i \in \mathbb{Z}$. Notice that $\langle g \rangle$ contains *at least* two elements: $g^x = \varepsilon$, and $g^1 = g$. Now, for arbitrary powers $n \in \mathbb{Z}$, let $ax + b = n$. Then:

$$g^n = g^{ax+b} = (g^x)^a \cdot g^b \varepsilon^a \cdot g^b = \varepsilon \cdot g^b = g^b$$

This tells us that $\langle g \rangle$ has at most x elements. In fact, $\langle g \rangle$ has exactly x elements since we took distinct powers of g and $g^x = \varepsilon$ is the smallest power that cycles back to 0. So;

- (a) The set $\langle g \rangle$ is finite (it has x elements).
- (b) The set $\langle g \rangle$ is closed under the group operation (since the equivalence of every power of g is in the set).
- (c) The set $\langle g \rangle$ contains the identity element.

Therefore, $\langle g \rangle$ is a group. Remember that we picked g from G , and G is itself a group that is closed under the group operation, so $\langle g \rangle \leq G$. The first part of Lagrange's theorem tells us that the order of any subgroup $S \leq G$ *must* divide the order of G . So x must divide p , but p is prime so $x = p$. Therefore, the order of the element g is p . \square

Proposition 1.3. *If a group of order n contains an element of order n then the group is cyclic.*

Proof. Let $g \in G$ such that the order of g is n . Then $g^n = \varepsilon$. Consider the set $\langle g \rangle$. As we saw in Proposition 1.2, $\langle g \rangle$ is a group of order n , and it is contained in G . Therefore, $\langle g \rangle = G$ since G has order n . Therefore, G is cyclic. \square

Proposition 1.4. *All cyclic groups of a fixed order n are isomorphic.*

Proof. Let G and H be cyclic groups of order n . Let $g \in G$ and $h \in H$ be generators. Then the map $\psi : G \rightarrow H$ defined by $\psi(g) = h$ is an isomorphism.

$$\begin{aligned} \psi(g) &= h \\ \psi(g^i) &= \psi\left(\prod_{j=0}^{i-1} g\right) = \prod_{j=0}^{i-1} \psi(g) = \prod_{j=0}^{i-1} h = h^i \\ \psi(g^i \cdot g^k) &= \psi(g^{i+k}) = h^{i+k} = h^i \cdot h^k = \psi(g^i) \cdot \psi(g^k) \end{aligned}$$

Therefore, the two groups are isomorphic. \square

Proposition 1.5. *If G is a cyclic group then G is abelian.*

Proof. Let G be a cyclic group of order p , with $g \in G$ as a generator. Then every element can be expressed as g^i for some $i \in \{0, 1, \dots, p-1\}$, with powers interpreted as repeated application of the group operation. Let $x \in G$ and $y \in G$ such that $x = g^a$ and $y = g^b$. Then:

$$xy = g^a \cdot g^b = g^{a+b} = g^b \cdot g^a = yx \quad (1.6)$$

Therefore, for any $x, y \in G$, $xy = yx$, and G is abelian. \square

Corollary 1.7. *If G is a finite group of prime order, then G is abelian.*

Proof. In combining propositions 1.2, 1.3, 1.4, and 1.5, we see that any group of prime order has a generating element, and by expressing every element in the group as a power of the generating element, we see that the group operation commutes for all elements. We also see that all groups of a given prime order p are isomorphic. Therefore, every group of prime order is abelian. \square

Definition 1.8. The cyclic group of order p is denoted as C_p (or Z_p , in parallel to the integers $\mathbb{Z} \pmod{p}$).

In summary, we have shown that every group of prime order is cyclic and abelian. This tells us that there is only a single structure for any groups of a given prime order p : the group C_p . For instance, the only groups of order 2, 3, 5, 7, 11, and 13 are the groups $C_2, C_3, C_5, C_7, C_{11}$, and C_{13} respectively.

Order	Group	Isomorphisms
2	C_2	$\mathbb{Z}/2\mathbb{Z}$
3	C_3	$\mathbb{Z}/3\mathbb{Z}$
5	C_5	$\mathbb{Z}/5\mathbb{Z}$
7	C_7	$\mathbb{Z}/7\mathbb{Z}$
11	C_{11}	$\mathbb{Z}/11\mathbb{Z}$
13	C_{13}	$\mathbb{Z}/13\mathbb{Z}$

TABLE 1. Groups of prime order $n \leq 15$

REFERENCES

1. D. S. Dummit, R. M. Foote, *Abstract Algebra* (John Wiley & Sons, ed. 3, 2003).