CS-49: Game Theory

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04/03/2023

Problem 3.

Your utility for owning $\$x$ is $\log x$. (The base of the logarithm doesn't matter; for this problem I recommend the natural logarithm.) You have $\$a$ and are permitted to make one bet on the flip of a coin that comes up "heads" with probability p which is known to you and greater than $1/2$. If you bet $\$b$ and win, you win $\$b$; otherwise you lose the amount bet. What fraction of $\$a$ should you bet, to maximize your expected utility?

$$\text{Current utility} = \log a$$

$$\text{Utility if you bet } \$b \text{ and win} = \log (a+b) = \log \frac{y+x}{y} a$$

$$\text{Utility if you bet } \$b \text{ and lose} = \log (a-b)$$

$$\text{Expected utility} = p \cdot \log (a+b) + (1-p) \cdot \log (a-b)$$

Optimally, if we fix the probability p then we want to optimize the amount bet b such that the expected utility is maximized.

$$\text{Expected utility} \geq \text{Current utility}$$

$$p \cdot \log (a+b) + (1-p) \cdot \log (a-b) \geq \log a$$

$$p \left(\log (a+b) - \log (a-b) \right) + \log (a-b) \geq \log a$$

$$p \left(\log (a+b) - \log (a-b) \right) \geq \log a - \log (a-b)$$

Given b is a fraction x/y of a;

$$p\left(\log \frac{y+x}{y} - \log \frac{y-x}{y}\right) \geq \log a - \log \frac{y-x}{y}$$