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On the derivations and automorphisms of the algebra $\Bbbk\langle x, y\rangle/(yx - xy - x^N)$

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Introduction

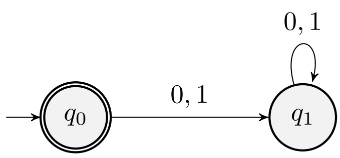

Figure 1: Only accept the empty string.

In this paper we fix a field $\Bbbk$ of characteristic zero and a non-negative integer N, and study the algebra A_N freely generated by two letters x and y subject to the relation

$$yx - xy = x^N.$$

with the objective of computing as explicitly as it is possible (to us!) some of its invariants of homological nature.

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For low values of N the algebra A_N is very well-known: when $N = 0$ it is the first Weyl algebra, which we can view as the algebra of regular differential operators on the affine line; when $N = 1$ it is the enveloping algebra of the non-abelian Lie algebra of dimension 2; and when $N = 2$ it is the so-called Jordan plane of non-commutative geometry [6, 53]. On the other hand, the family of algebras that we will study is contained in a larger one that has received a lot of attention: if for an arbitrary polynomial $h \in \Bbbk[x]$ we let A_h be the algebra freely generated by letters x and y subject to the relation $yx - xy = h(x)$, then of course our algebra A_N is A_{x^N}. One way to explain the interest of this larger family of algebras is by saying that it consists, up to isomorphism, of all skew-polynomial extensions of $\Bbbk[x]$ apart from the 1-parameter families of quantum planes and quantum Weyl algebras [1, Proposition 3.2]. The algebras of the form A_h have been studied in detail by G. Benkart, S. Lopez and M. Ondrus in the series of papers [9–11]. Our algebras A_N are, in many senses, the *worst* of the lot.

The main motivation for this work was the problem of giving an explicit description of the first Hochschild cohomology space $\mathrm{HH}^1(A_N)$ of A_N, which we view as the space of outer derivations of the algebra A_N, that is, the quotient $\mathrm{Der}(A_N)/\,\mathrm{InnDer}(A_N)$ of the Lie algebra $\mathrm{Der}(A_N)$ of all derivations of A_N by its ideal of inner derivations. This cohomology space and, in fact, the full Lie algebra $\mathrm{Der}(A_N)$ have been studied in detail before — J. Dixmier [21, 23] and R. Sridharan [52] for the Weyl algebra, E.N. Shirikov [47–49] for the case $N = 2$, A. Nowicki [42] and most notably G. Benkart, S. Lopez and M. Ondrus [10] for the general case of the algebras A_h and, building upon that, S. Lopez and A. Solotar [36] for the Lie module structure of the cohomology of A_h over $\mathrm{HH}^1(A_h)$ — and we can say that both $\mathrm{Der}(A_N)$ and $\mathrm{HH}^1(A_N)$ are very well understood. What we were after, though, was a description of the elements of $\mathrm{HH}^1(A_N)$ as cohomology classes of actual, explicit derivations, because we needed to do further calculations with them. In particular, while doing some calculations regarding the characteristic morphism $\mathrm{HH}^\bullet(A_N) \to \mathscr{Z}_{\mathrm{gr}}(D^b(A_N))$ for this algebra — which connects the Hochschild cohomology of the algebra with the graded center of the derived category of the category of modules of the algebra, for example as in [37] — certain rational numbers insistently appeared and required an explanation.

Let us describe the results we obtain. The algebra A_N can be endowed with a grading with respect to which the generators x and y are in degrees 1 and $N - 1$, respectively, and this grading induces others in many objects constructed from A_N. For example, the space $\mathrm{Der}(A_N)$ and, for each $p \geq 0$, the Hochschild cohomology space $\mathrm{HH}^p(A_N)$ are $\mathbb{Z}$-graded vector spaces. Our result about $\mathrm{HH}^1(A_N)$ is the following:

Theorem 1. Suppose that $N \geq 2$, let q be a variable, and let $(c_i(q))_{i\geq 0}$ be the sequence of polynomials in $\mathbb{Q}[q]$ such that

$$\sum_{j\geq 0} c_j(q)\frac{t^j}{j!} = \frac{t}{(1-qt)^{-1/q} - 1}. \tag{1}$$

(*i*) If l is a positive integer and i and j are the integers such that $l+1 = i + j(N-1)$ and $1 \leq i \leq N-1$, then there is a homogeneous derivation $\partial_l : A_N \to A_N$ of degree l such that

$$\begin{aligned}\partial_l(x) &= x^i y^j, \\ \partial_l(y) &= (N-i)x^{i-1}\frac{1}{j+1}\sum_{i=0}^{j}\binom{j+1}{i}c_i(N-1)x^{i(N-1)}y^{j+1-i} \\ &\qquad\qquad\qquad\qquad + \sum_{s+2+t=N}(s+1)x^{s+i}y^j x^t\end{aligned} \tag{2}$$

(*ii*) If l is an integer such that $-N+1 \leq i \leq 0$, then there is a unique homogeneous derivation $\partial_l : A_N \to A_N$ of degree l such that

$$\partial_l(x) = 0, \qquad \partial_l(y) = x^{l+N-1}.$$

There is moreover a homogeneous derivation $E : A_N \to A_N$ of degree 0 such that

$$E(x) = x, \qquad E(y) = (N-1)y.$$

(*iii*) The graded vector space $\mathrm{HH}^1(A_N)$ is locally finite and its Hilbert series is

$$h_{\mathrm{HH}^1(A_N)}(t) = 1 + \frac{t^{-N+1}}{1-t}.$$

If l is an non-zero integer such that $l \geq -N+1$, then the homogeneous component $\mathrm{HH}^1(A_N)_l$ of degree l in $\mathrm{HH}^1(A_N)$ is freely spanned by the cohomology class of the derivation ∂_l described above. On the other hand, the component $\mathrm{HH}^1(A_N)_0$ of degree 0 is freely spanned by the cohomology classes of the derivations ∂_0 and E.

The sequence of polynomials $(c_j(q))_{q\geq 0}$ that appears here — of which the first few are tabulated in Table 1 on page 28 — is a q-variant of the sequence of Bernoulli numbers, to which it degenerates as q goes to 0: indeed, the limit of the right hand side of the defining equality (1) as q approaches 0 is $t/(e^t-1)$, the exponential generating function of the Bernoulli numbers, and for all $j \in \mathbb{N}_0$ the constant term $c_j(0)$ is exactly the jth Bernoulli number. On the other hand, the leading coefficient of $c_j(q)$ is $(-1)^j j! G_j$, with G_j denoting the jth Gregory coefficient, certainly of much lesser fame.

It should be observed that, while the formula in (2) somehow indicates that the limiting case $q \rightsquigarrow 0$ corresponds to letting the integer N «converge» to 1 (which, of course, makes no sense), none of the claims of the theorem holds at the limit $N = 1$! Indeed, the theorem excludes the cases in which $N < 2$, and that is because they are rather different. When $N = 0$ we have, according to a calculation carried out originally by Dixmier in [23], that $\mathrm{HH}^1(A) = 0$, so there is nothing that needs being made explicit. When $N = 1$, on the other hand, the space $\mathrm{HH}^1(A)$ is one-dimensional and spanned by the cohomology class of a derivation $d_0 : A \to A$ such that $d_0(x) = 0$ and $d_0(y) = 1$, which is homogeneous of degree 0.

The way in which we prove this theorem is rather indirect — a direct proof of the first part of the statement would probably be quite unpleasant! We first compute $\mathrm{HH}^0(A)$ and $\mathrm{HH}^2(A)$ and, in particular, their Hilbert series, and using that and an argument involving Euler characteristics we deduce the Hilbert series of $\mathrm{HH}^1(A)$: this tells us of what degrees there are non-inner homogeneous derivations, and how many. We then show that such a derivation can be modified appropriately until it satisfies the conditions described in the theorem. We do this in Sections 3, 4 and 5, whose main results are the Propositions 3.4, 3.5, 5.2 and 5.6 that we subsumed in Theorem 1 above.

Once we have explicit derivations whose cohomology classes span $\mathrm{HH}^1(A_N)$, we can do things with them. The general qualitative structure of the Lie algebra $\mathrm{HH}^1(A)$ was described by G. Benkart, S. Lopez and M. Ondrus in [10]: its center is one-dimensional and a complement to the derived subalgebra $\mathrm{HH}^1(A_N)' \coloneqq [\mathrm{HH}^1(A_N), \mathrm{HH}^1(A_N)]$, this derived subalgebra has a unique maximal nilpotent ideal $\mathfrak{N}$ of nilpotency index N, and the quotient $\mathrm{HH}^1(A_N)'/\mathfrak{N}$ is isomorphic to the Lie algebra $\mathrm{Der}(\Bbbk[x])$ of derivations of $\Bbbk[x]$ or, equivalently, of regular vector fields on the affine line $\mathbb{A}^1_\Bbbk$, which is often called the Witt algebra. Starting from Theorem 1 we can prove — this is Corollary 6.2 in the text below — the following:

Theorem 2. *There is a sequence of derivations $(L_j)_{j\geq -N+1}$ of A_N whose cohomology classes freely span the derived subalgebra $\mathrm{HH}^1(A)'$ such that whenever l and m are integers with l, $m \geq -N+1$ we have*

$$[L_l, L_m] \sim \begin{cases} 0 & \text{if } i+u > N \text{ or } l+m < -N+1; \\ \dfrac{l(v+1)-m(j+1)}{N-1} L_{l+m} & \text{if } i+u \leq N, \end{cases}$$

with i, j, u and v the unique integers such that $l+1 = i+j(N-1)$, $m+1 = u+v(N-1)$, $1 \leq i, u \leq N-1$, and j, $v \geq -1$.

In fact, for each non-zero j the derivation L_j here is just a scalar multiple of the derivation ∂_j of Theorem 6.2, L_0 is a scalar multiple of the derivation E, and the center of $\mathrm{HH}^1(A)$ is the span of the class of ∂_0. In particular, the degree of L_j is j.

This shows that the derived subalgebra $\mathrm{HH}^1(A)'$ is, more or less, an infinitesimal deformation of order N of the Witt algebra $\mathrm{Der}(\Bbbk[x])$ — which is now realized as the subalgebra spanned by the sequence $(L_{(-N+1)j})_{j\geq -1}$. It would be very interesting to have *a priori* reasons for this.

The first Hochschild cohomology of an algebra plays — in principle, but usually not in reality — the role of the Lie algebra of the group of outer automorphism of the algebra. For our algebra this does not quite work, and since the units of A_N are all scalar and therefore central its usual outer automorphism group coincides with the plain automorphism group. There is an alternative notion of inner-automorphism that is useful in Lie theory, though, in which we call an automorphism inner if it is a composition of exponentials of locally ad-nilpotent elements, and in our situation it does do something, as the following result shows.

Theorem 3. *Let $\Bbbk[x] \bowtie \Bbbk^\times$ be the group whose underlying set is the cartesian product $\Bbbk[x] \times \Bbbk^\times$ and whose multiplication is such that*

$$(f,\lambda)\cdot(g,\mu) = (\mu^{N-1}f + g\cdot\lambda, \lambda\mu)$$

for all f, $g \in \Bbbk[x]$ and all λ, $\mu \in \Bbbk^\times$.

(*i*) *There is an isomorphism of groups*

$$\Phi : \Bbbk[x] \bowtie \Bbbk^\times \to \operatorname{Aut}(A_N)$$

such that $\Phi(f,\lambda)(x) = \lambda x$ and $\Phi(f,\lambda)(y) = \lambda^{N-1}y + f$ for all (f,λ) in $\Bbbk[x] \bowtie \Bbbk^\times$.

(*ii*) *The set of locally* ad*-nilpotent elements of A_N is $\Bbbk[x]$, and for each $f \in \Bbbk[x]$ we have $\exp\operatorname{ad}(f) = \phi_{x^N f,1}$. The subset $\operatorname{Exp}(A_N)$ of exponentials of locally* ad*-nilpotent elements is a normal subgroup of the automorphism group $\operatorname{Aut}(A_N)$, and the map Φ above induces an isomorphism*

$$\overline{\Phi} : \frac{\Bbbk[x]}{(x^N)} \bowtie \Bbbk^\times \to \frac{\operatorname{Aut}(A_N)}{\operatorname{Exp}(A_N)}.$$

In particular, the quotient $\operatorname{Aut}(A_N)/\operatorname{Exp}(A_N)$ has a natural structure of a Lie group over $\Bbbk$ of dimension $N+1$, solvable of class 2 and, in fact, an extension of $\Bbbk^\times$ by $\Bbbk^N$.

(*iii*) *For each $g \in \Bbbk[x]$ there is a derivation $d_g : A_N \to A_N$ such that $d_g(x) = 0$ and $d_g(y) = g$, and it is locally nilpotent. The map*

$$g \in \Bbbk[x] \mapsto d_g \in \operatorname{Der}(A_N)$$

is injective, and its image is the set of locally nilpotent derivations of A_N, which happens to be an abelian Lie subalgebra of $\operatorname{Der}(A_N)$. The set of exponentials of locally nilpotent derivations of A_N is the normal subgroup $\operatorname{Aut}_0(A_N) \coloneqq \{\phi_{g,1} : g \in \Bbbk[x]\}$, and it sits in an extension of groups

$$0 \longrightarrow \operatorname{Aut}_0(A) \hookrightarrow \operatorname{Aut}(A) \xrightarrow{\mathbf{det}} \Bbbk^\times \longrightarrow 1$$

in which $\mathbf{det}(\phi_{\lambda,f}) = \lambda$ for all $(\lambda,f) \in \Bbbk[x] \bowtie \Bbbk^\times$, and which is split by the morphism $\lambda \in \Bbbk^\times \mapsto \phi_{0,\lambda} \in \operatorname{Aut}(A)$.

We have collected in this statement the results of Propositions 2.4, 2.6 and 2.9, Corollaries 2.8 and 2.13. This information is useful, for example, when computing the action of $\operatorname{Aut}(A_N)$ on derived objects, like Hochschild cohomology or cyclic homology, on which exponentials of locally ad-nilpotent elements tend to act trivially. For now, let us say that such an explicit description of the automorphism group allows us to compute its center easily. It turns out to be significant in several ways:

Theorem 4. *There is a locally nilpotent derivation $\partial_0 : A_N \to A_N$ such that $\partial_0(x) = 0$ and $\partial_0(y) = x^{N-1}$ such that the map*

$$t \in \Bbbk \mapsto \sigma_t \coloneqq \exp t\partial_0 \in \operatorname{Aut}(A_N) \tag{3}$$

is an injective 1*-parameter subgroup of* $\mathrm{Aut}(A_N)$ *whose image is the center of* $\mathrm{Aut}(A_N)$*.*

(*i*) *The infinitesimal generator* ∂_0 *of this* 1*-parameter subgroup is not inner and its class in* $\mathrm{HH}^1(A)$ *spans the center of this Lie algebra.*

(*ii*) *The element* x *of* A *is normal, in that* $xA = Ax$*, and the automorphism* $\sigma_1 : A_N \to A_N$ *is the automorphism associated to it, so that* $ax = x\sigma_1(a)$ *for all* $a \in A_N$*.*

(*iii*) *The algebra* A_N *is twisted Calabi–Yau of dimension* 2*, and the automorphism* σ_1 *is its modular (or Nakayama) automorphism, so that in particular there is an automorphism of* A_N*-bimodules*

$$\mathrm{H}^2(A_N, A_N \otimes A_N) = \mathrm{Ext}_{A_N^e}(A_N, A_N \otimes A_N) \to {}_{\sigma_1}A_N.$$

(*iv*) *The derivation* ∂_0 *preserves the canonical «order» filtration on* A_N*, so it induces a derivation* $\overline{\partial}_0 : \mathrm{gr}\, A_N \to \mathrm{gr}\, A_N$ *on the corresponding associated graded algebra. If we endow* $\mathrm{gr}\, A$ *with its standard Poisson structure coming from the commutator of* A_N*, then* $\overline{\partial}_0$ *is the modular derivation of* A_N *in the sense of A. Weinstein* [56]*, and the corresponding modular flow*

$$\sigma : t \in \Bbbk \mapsto \exp t\overline{\partial}_0 \in \mathrm{Aut}(\mathrm{gr}\, A)$$

is exactly the 1*-parameter group of automorphisms induced by the flow* (3) *above.*

This theorem combines the results of Lemma 2.2, Corollaried 2.5 and 2.10, Proposition 2.3, and Remark 2.11. All the objects mentioned in this theorem are canonical. For example, as the units of A_N are central, the modular automorphism of A_N as a twisted Calabi–Yau algebra is well-determined. As we wrote in the theorem, the derivation ∂_0 is not inner, but it is «logarithmically inner», in that it coincides with the restriction to A_N of the derivation

$$a \in (A_N)_x \mapsto \frac{1}{x}[x, a] \in (A_N)_x$$

of the localization $(A_N)_x$ of A_N at its normal element x.

A natural thing to do at this point is to describe the finite subgroups of $\mathrm{Aut}(A_N)$, and that is easy since we know the group very well. Our Proposition 2.14 implies, among other things, the following:

Theorem 5. *Every finite subgroup of* $\mathrm{Aut}(A)$ *is cyclic, and conjugated to the subgroup generated by* $\phi_{0,\lambda}$*, with* λ *a root of unity in* $\Bbbk$*.*

Of course, with this result at hand the obvious next thing to do, following the classics, is to describe the invariant subalgebras corresponding to the finite subgroups of $\mathrm{Aut}(A_N)$. This appears to be fairly difficult, and we do not consider this problem here. We instead take a less classical direction and try to extend the result of the theorem and find all actions of finite dimensional Hopf algebras on A_N — that is, to put it in a colorful language, to find all *quantum finite groups* of automorphisms of A_N. Now, at that level of generality we do not know how to approach the problem, so we restrict ourselves to looking for all actions of generalized Taft Hopf algebras on A_N. We know that all finite groups of automorphisms are cyclic, and generalized Taft algebras can be viewed as «quantum thickenings» of cyclic groups, so this is a reasonable first step. What we find is the following result.

Theorem 6. *Let n and m be integers such that $1 < m$ and $m \mid n$, and let $\lambda \in \Bbbk^\times$ be a primitive mth root of unity in $\Bbbk$. There is no inner-faithful action of the generalized Taft algebra $T_n(\lambda, m)$ on A.*

We refer to Section 8 of the paper — which ends with a proof of this theorem, stated there as Proposition 8.5 — for the precise description of what we mean by generalized Taft algebra, and for the definition of inner-faithfulness, which is due to T. Banica and J. Bichon [8]. Let us just say here that inner-faithfulness intends to be to actions of Hopf algebras what faithfulness is to actions of groups.

The negative nature of this theorem is disappointing, but not unexpected. For example, this «no quantum finite symmetries» phenomenon occurs on Weyl algebras, algebras of differential operators, and more generally certain algebraic quantizations, as shown by J. Cuadra, P. Etingof and C. Walton in [17, 18] and by those last two authors in [24]. There are many other classes of algebras, though, which exhibit non-trivial quantum symmetries, and the study of this is an extremely interesting line of work. In this direction, one should mention the work of S. Montgomery and H.-J. Schneider [41] on finite dimensional algebras generated by one element, extended later by Z. Cline [15]; L. Centrone and F. Yasumura [14] on finite dimensional algebras; Y. Bahturin and S. Montgomery [7] on matrix algebras; R. Kinser and C. Walton [33] on path algebras; J. Gaddis, R. Won and D. Yee [27] on quantum planes and quantum Weyl algebras; Z. Cline and J. Gaddis [16] on quantum affine spaces, quantum matrices, and quantized Weyl algebras; and J. Gaddis and R. Won [26] in quantum generalized Weyl algebras.

The way we prove Theorem 6 is by studying the twisted derivations of our algebra A_N. This is a natural approach at this point, for it consists of fixing an automorphism ϕ of the algebra and computing $\mathrm{HH}^1(A_N, {}_\phi A_N)$, the first Hochschild cohomology space of the algebra A_N with values in the A_N-bimodule that can be obtained from A_N by twisted the left action by the automorphism ϕ, and we can more or less use the same ideas that we used to compute the regular Hochschild cohomology of A_N. We do this in Section 7. We do not describe here the precise results obtained therein because of their rather technical nature. We would like, nonetheless, to direct the reader's attention to our Remark 7.6, which explains the way we construct twisted derivations in terms of the Gerstenhaber algebra structure of Hochschild cohomology, and which is probably of more general applicability. Finally, it should be remarked that the study of twisted Hochschild cohomology should be useful to study actions of general finite dimensional Hopf algebras, since we know from the classification theory of A. Andruskiewitsch and H.-J. Schneider [5] that finite dimensional Hopf algebras are in many cases «built» from group-like and skew-primitive elements, which give rise to automorphisms and twisted derivations of the algebras upon which they act.

All the work described above involves the first Hochschild cohomology Lie algebra $\mathrm{HH}^1(A_N)$ and the automorphism group $\mathrm{Aut}(A_N)$. Our algebra also has a non-zero second Hochschild cohomology space $\mathrm{HH}^2(A_N)$, and according to the classical deformation theory of M. Gerstenhaber [28], its elements produce «by integration» formal deformations of the algebra A_N just as the elements of $\mathrm{HH}^1(A)$ produce (more or less…) automorphisms by exponentiation — indeed, as $\mathrm{HH}^3(A) = 0$ this procedure of integration of elements of $\mathrm{HH}^2(A)$ is unobstructed, so always possible. We leave for future work the explicit construction of these formal deformations, in the same vein as we constructed explicitly derivations spanning $\mathrm{HH}^1(A)$.

The natural thing to do, actually, is to study this problem for the whole family of algebras A_h of [9–11] to which the algebra A_N belongs at the same time: these algebras deform, under appropriate hypotheses, to one another, and they all appear as deformations of A_N. The algebra A_N is the most singular element of the class, so the geometry of the family at that point is the central point to elucidate. This was the main motivation for the study of the automorphism group of A_N described above, in fact. The description of the Lie action of $\mathrm{HH}^1(A_N)$ on $\mathrm{HH}^2(A_N)$ that can be read off from the work [36] of S. Lopes and A. Solotar should also be important to do this.

Let us finish this introduction with a question. The form of the derivations of A_N that we find in Section 5 and, in particular, the observations made at the end of Section 4 about the elements Φ_j, hint at the idea that the algebra A_1 that we get by putting $N = 1$ is not the correct «limit as N converges to 1 of the algebra A_N».

Question 7. Is there an algebra $\tilde{A}_1$ which better reflects the result of «taking the limit of A_N as N goes to 1» in that it has exterior derivations involving the Bernoulli polynomials that we found in equation (25) at the end of Section 4 and the Faulhaber formula?

We could say that the family $(A_N)_{N\geq 1}$ is, in a not very precise sense, flat with respect to N, but our calculation of $\mathrm{HH}^1(A_N)$ shows that this space is constant *except* at $N = 1$. An algebra $\tilde{A}_1$, obtained by something like «dimensional regularization» but with respect to N, would fix this. Another example of the anomalous status of the algebra A_1 in the family $(A_N)_{N\geq 1}$ is provided by Remark 1.2 below.

1 Preliminaries

We fix a field $\Bbbk$ of characteristic zero and an integer $N \geq 1$, and let A be the algebra freely generated by two letters x and y subject to the sole relation

$$yx - xy = x^N.$$

If $N = 0$, then this is a Weyl algebra, and if $N = 1$ what we get is isomorphic to the universal enveloping algebra of the non-abelian Lie algebra of dimension 2. In general, we can view this algebra as the skew polynomial algebra $\Bbbk[x][y;\theta]$, with $\theta : \Bbbk[x] \to \Bbbk[x]$ the derivation such that $\theta(x) = x^N$, and it follows from this that A is a noetherian domain, and that $\mathcal{B} = \{x^i y^j : i, j \geq 0\}$ is a basis for A. There is a grading on A that makes x and y homogeneous of degrees 1 and $N-1$, respectively. The homogeneous components of A are spanned by the monomials in the basis $\mathcal{B}$ that they contain, and they are all finite dimensional exactly when $N \geq 2$.

As $yx = xy + x^N = x(y + x^{N-1})$, it is easy to check that $Ax = xA$, that is, that the element x is normal — in Lemma 2.1 below we will see that in fact it generates, together with the non-zero scalars, the set of all the non-zero normal elements as a monoid. The set $S \coloneqq \{x^i : i \geq 0\}$ is then a left and right denominator set in A and we can consider the localization $A_x \coloneqq S^{-1}A$ at S. If $\Bbbk[x^{\pm 1}]$ is the algebra of Laurent polynomials and $\tilde{\theta} : \Bbbk[x^{\pm 1}] \to \Bbbk[x^{\pm 1}]$ is the extension of the derivation θ to $\Bbbk[x^{\pm 1}]$, then it is immediate that A_x can be viewed as the skew polynomial algebra $\Bbbk[x^{\pm 1}][y; \tilde{\theta}]$. The algebra A_x is graded, with x^{-1} of degree -1.

There is, on the other hand, an algebra filtration $(F_k)_{k \geq -1}$ on A that has, for each $k \geq -1$, its kth layer F_k spanned by the set of monomials $\{x^i y^j : i \geq 0, j \leq k\}$, and whose associated graded algebra $\operatorname{gr} A$ is freely generated as a commutative algebra by the principal symbols $\overline{x} \coloneqq x + F_{-1} \in F_0/F_1$ and $\overline{y} \coloneqq y + F_0 \in F_1/F_0$ of x and y, and these have degree 0 and 1, respectively. In particular — and we will use this fact all the time without mentioning it — for all i, $j \in \mathbb{N}_0$ we have that

$$y^i x^j \equiv x^i y^j \mod F_{i-1}.$$

As it is well-known, the non-commutativity of the algebra A gives rise to a Poisson algebra structure on $\operatorname{gr} A$: this is the biderivation $\{-,-\} : \operatorname{gr} A \times \operatorname{gr} A \to \operatorname{gr} A$ uniquely determined by the condition that

$$\{\overline{y}, \overline{x}\} = \overline{x}^N.$$

Computing in A usually involves a significant amount of reordering products, and the following lemma gives two important special cases of this that are moreover related by a pleasing symmetry. If k and x are elements of a commutative ring Λ and $i \in \mathbb{N}_0$, we define, following Rafael Díaz and Eddy Pariguan [20], the ***Pochhammer k-symbol*** to be

$$(x)_{k,i} \coloneqq x(x+k)(x+2k)\cdots(x+(i-1)k).$$

We will use a few times the equality

$$\sum_{j \geq 0} (a)_{k,j} \frac{t^j}{j!} = (1-kt)^{-a/k},$$

valid in the algebra $\Lambda[[t]]$, provided, so that it actually makes sense, that Λ contains $\mathbb{Q}$. It can be proved by noticing that the two sides are in the kernel of the differential operator $(1-kt)\frac{\mathrm{d}}{\mathrm{d}t} - a$ and have the same constant term.

Lemma 1.1. For each i, $j \geq 0$ we have that

$$y^j x^i = \sum_{t=0}^{j} (i)_{N-1,j-t} \binom{j}{t} x^{i+(j-t)(N-1)} y^t \tag{4}$$

and

$$x^i y^j = \sum_{t=0}^{j} (-i)_{N-1,j-t} \binom{j}{t} x^{(j-t)(N-1)} y^t x^i.$$

These two equalities tell us how to move x^i from one side of y^j to the other.

Proof. Let us start by proving an identity involving Pochhammer k-symbols that is a version of the well-known Zhu–Vandermonde identity. We fix $k \in \Bbbk$ and for each a in $\Bbbk$ consider, as above, the formal series $f_a \coloneqq \sum_{i\geq 0} (a)_{k,i} t^i/i! \in \Bbbk[[t]]$. We have that $f_a f_b = f_{a+b}$ for all choices of a and b in $\Bbbk$: this can be checked by showing that both sides of the equality are annihilated by the operator $(1-kt)\frac{\mathrm{d}}{\mathrm{d}t} - (a+b)$ and have the same constant term. Writing this equality in terms of coefficients we see that for all $j \geq 0$ we have that

$$\sum_{i=0}^{j} \frac{(a)_{k,j-i}}{(j-i)!} \frac{(b)_{i,k}}{i!} = \frac{(a+b)_{k,j}}{j!}.$$

Let us now prove the lemma. An obvious induction proves that

$$y^j x = \sum_{t=0}^{j} (1)_{N-1,j-t} \binom{j}{t} x^{1+(j-t)(N-1)} y^t \tag{5}$$

for all $j \geq 0$, and this is the special case of the identity (4) of the lemma in which $i = 1$. Let us now fix $i \geq 0$ and assume inductively that the identity (4) holds. Then

$$y^j x^{i+1} = \sum_{t=0}^{j} (i)_{N-1,j-t} \binom{j}{t} x^{i+(j-t)(N-1)} y^t x$$

and using (5) we see this is

$$\begin{aligned}
&= \sum_{t=0}^{j} \sum_{s=0}^{t} (i)_{N-1,j-t} \binom{j}{t} (1)_{N-1,t-s} \binom{t}{s} x^{i+1+(j-s)(N-1)} y^s \\
&= \sum_{s=0}^{j} \frac{j!}{s!} \left(\sum_{t=0}^{j-s} \frac{(i)_{N-1,j-s-t}}{(j-s-t)!} \frac{(1)_{N-1,t}}{t!} \right) x^{i+1+(j-s)(N-1)} y^s \\
&= \sum_{s=0}^{j} (i+1)_{N-1,j-s} \binom{j}{s} x^{i+1+(j-s)(N-1)} y^s.
\end{aligned}$$

This proves (4) for all values of j. To prove the remaining identity, we invert the one we already have: we fix i and j in $\mathbb{N}_0$ and compute that

$$\begin{aligned}
&\sum_{t=0}^{j} (-i)_{N-1,j-t} \binom{j}{t} x^{(j-t)(N-1)} y^t x^i \\
&\qquad = \sum_{t=0}^{j} \sum_{s=0}^{t} (-i)_{N-1,j-t} (i)_{N-1,t-s} \binom{j}{t} \binom{t}{s} x^{i+(j-s)(N-1)} y^s
\end{aligned}$$

$$= \sum_{s=0}^{j} \frac{j!}{s!} \left(\sum_{t=0}^{j-s} \frac{(-i)_{N-1,j-t-s}}{(j-t-s)!} \frac{(i)_{N-1,t}}{t!} \right) x^{i+(j-s)(N-1)} y^s = x^i y^j,$$

which is what we want. □

Remark 1.2. When $N = 1$ the identity (4) tells us that $y^j x^i = x^i (y+i)^j$, and we can write it, passing to the localization A_x, in the form

$$x^{-i} y^j x^i = (y+i)^j. \tag{6}$$

Let us suppose now that $N \neq 1$ and show how to view the identity (4) in a similar way also in this case. In the localization A_x we have that

$$x^{N-1}(x^{-N+1}y - yx^{-N+1})x^{N-1} = yx^{N-1} - x^{N-1}y = (N-1)x^{2N-2},$$

so that

$$\left[\frac{x^{-N+1}}{N-1}, y\right] = 1.$$

The subalgebra of A_x generated by $z := x^{-N+1}/(N-1)$ and y is thus isomorphic to the first Weyl algebra. In this subalgebra we can consider normal-order products — well-known in quantum field theory, for example — and powers: we consider the symbol $:zy:$ with the property that $(:zy:)^i = z^i y^i$ for all $i \geq 0$. For each choice of $\alpha \in \mathbb{C}$ and $j \in \mathbb{N}_0$ we let $L_j^{(\alpha)} \in \mathbb{C}[t]$ be the jth generalized Laguerre polynomial with parameter α. This is the unique solution of the Laguerre differential equation

$$tu'' + (\alpha + 1 - t)u' + ju = 0$$

that is a polynomial with leading coefficient $(-1)^j/j!$. It has degree j and can be found to be given by the formula

$$L_j^{(\alpha)}(t) = \sum_{i=0}^{j} (-1)^i \binom{j+\alpha}{j-i} \frac{t^i}{i!}. \tag{7}$$

When $\alpha > -1$, the sequence $(L_j^{(\alpha)})_{j\geq 0}$ is obtained from $(t^j)_{j\geq 1}$ by Gram–Schmidt orthogonalization over the interval $[0, +\infty)$ with respect to the weighting function $t^\alpha e^{-t}$ associated to the gamma distribution. We refer to [4, §6.3] for more information on these polynomials.

With this setup, we claim now that for all i, $j \in \mathbb{N}_0$ have that

$$x^{-i}(:zy:)^j x^i = (-1)^j j!\, L_j^{(-i/(N-1)-j)}(:zy:). \tag{8}$$

We view this as a reasonable generalization of (6) for N greater than 1 — in any case, the fact that the commutation relations of the algebra can be written in terms of hypergeometric functions is interesting! Notice that the polynomial $(-1)^j j! L_j^{(-i/(N-1)-j)}$ is monic, so the coefficient $(-1)^j j!$ appears here just as a consequence of the standard normalization of Laguerre polynomials. We can prove the equality (8) by evaluating its right hand side using the explicit formula (7) for the Laguerre polynomials and then using the first identity of Lemma 1.1. ◊

Another very useful observation that we will use many times is the following one.

Lemma 1.3. The centralizer of x in A is $\Bbbk[x]$.

Proof. If u is an element of $A \setminus \Bbbk[x]$ that commutes with x, then there exist $l \geq 1$ and $a_0, \dots, a_l \in \Bbbk[x]$ such that $u = \sum_{i=0}^{l} a_i y^i$ and $a_l \neq 0$, and we have that

$$0 = [u, x] = \sum_{i=0}^{l} a_l [y^i, x] \equiv l a_l x^N y^{l-1} \mod F_{l-2},$$

which is absurd. As every element of $\Bbbk[x]$ obviously commutes with x, this proves the lemma. □

Going a bit further, we can describe the center of A:

Proposition 1.4. The center of A is $\Bbbk$.

Proof. If u is a central element of A then in particular u commutes with x and we know from the lemma that this implies that $u \in \Bbbk[x]$. As it also commutes with y, we also have that $0 = [y, x] = x^N u'$, so that u is in fact in $\Bbbk$. □

We will need further on the following related result, which has an entirely similar proof.

Proposition 1.5. The center of the localization A_x is $\Bbbk$.

Proof. Let z be a central element of A_x. There is a positive integer $l \in \mathbb{N}$ such that $x^l z$ is in A, and, as this product commutes with x, we know that it belongs to $\Bbbk[x]$. We thus see that z is in $\Bbbk[x^{\pm 1}]$, and therefore that $0 = [y, z] = x^n z'$: it is in fact a scalar. □

2 The automorphism group

After the preliminaries of the previous section, our first objective is to compute the group of automorphisms of the algebra A, and to do that we will start by finding the ***normal*** elements of A, that is, those elements u of A such that $uA = Au$.

Lemma 2.1. The set of non-zero normal elements in A is $\mathbb{N} := \{\lambda x^i : \lambda \in \Bbbk^\times, i \geq 0\}$.

Proof. The element x is normal in A, since $yx = x(y + x^{N-1})$, and then it is clear that all the elements of the set $\mathbb{N}$ are normal in A. Conversely, let u be a non-zero normal element in A, and let $l \geq 0$ and $a_0, \dots, a_l \in \Bbbk[x]$ be such that $u = \sum_{i=0}^{l} a_i y^i$ and $a_l \neq 0$. As u is normal, the right ideal uA is a bilateral ideal and therefore $[u, x] \in uA$: there exists a $b \in A$ such that $[u, x] = ub$. As $[u, x] = \sum_{i=0}^{l} a_i [y^i, x] \in F_{l-1}$, this is only possible if in fact $[u, x] = 0$, so that $u \in \Bbbk[x]$. We also have that $x^N u' = [y, u] \in uA$ and, since of course $x^N u'$ is also in $\Bbbk[x]$, we have that u divides $x^N u'$: this implies that u is λx^i for some $\lambda \in \Bbbk$ and some integer $i \geq 0$, so that u belongs to the set $\mathbb{N}$. $\square$

In view of this lemma, the element x generates the monoid of normal elements of A up to non-zero scalars. Since A is a domain, each non-zero normal element u in A determines uniquely an automorphism of algebras $\nu_u : A \to A$ such that $au = u\nu_u(a)$ for all $a \in A$. Let us record for future use the description of the automorphism associated to x, which we will write σ_1 for reasons that will be clear later.

Lemma 2.2. The automorphism $\sigma_1 : A \to A$ such that $ax = x\sigma_1(a)$ for all $a \in A$ has $\sigma_1(x) = x$ and $\sigma_1(y) = y + x^{N-1}$. $\square$

This humble origin of σ_1 as the automorphism associated to the normal element x can actually be painted in a somewhat more impressive way:

Proposition 2.3. The algebra A_N is twisted Calabi–Yau algebra of dimension 2, and its modular automorphism is precisely the automorphism $\sigma_1 : A_N \to A_N$ of Lemma 2.2.

What we call here modular automorphism of a twisted Calabi–Yau algebra is often called the Nakayama automorphism of the algebra.

Proof. That A_N is twisted Calabi–Yau of dimension 2 is a consequence of the fact that it is an Ore extension of $\Bbbk[x]$, which is a Calabi–Yau algebra of dimension 1: this follows from a theorem of L.-Y. Liu, S. Wang and Q.-S. Wu proved in [35]. That the modular automorphism of A_N is σ_1 is also a consequence of their result. $\square$

There is a right action of the multiplicative group $\Bbbk^\times$ on $\Bbbk[x]$ by algebra automorphisms such that $x \cdot \lambda = \lambda x$ for all $\lambda \in \Bbbk^\times$. We denote $\Bbbk[x] \bowtie \Bbbk^\times$ the group whose underlying set is the cartesian product $\Bbbk[x] \times \Bbbk^\times$ and whose product is such that

$$(f, \lambda) \cdot (g, \mu) = (\mu^{N-1} f + g \cdot \lambda, \lambda\mu)$$

for all f, $g \in \Bbbk[x]$ and all λ, $\mu \in \Bbbk^\times$. This group[1] shows up in the following result.

Proposition 2.4. For each choice of $f \in \Bbbk[x]$ and $\lambda \in \Bbbk^\times$ there is unique automorphism $\phi_{f,\lambda} : A \to A$ such that

$$\phi_{f,\lambda}(x) = \lambda x, \qquad \phi_{f,\lambda}(y) = \lambda^{N-1} y + f.$$

The function $\Phi : (f, \lambda) \in \Bbbk[x] \bowtie \Bbbk^\times \mapsto \phi_{f,\lambda} \in \mathrm{Aut}(A)$ is an isomorphism of groups.

Recall that we have the standing hypothesis that $N \geq 1$: when $N = 0$, the algebra A is the first

[1] In general, if G and H are two groups, and $\triangleright$ and $\triangleleft$ are a left action and a right action of H on G by group automorphisms such that $h \triangleright (g \triangleleft h') = (h \triangleright g) \triangleleft h'$, then we can construct a group $G \bowtie H$ with underlying set $G \times H$ and multiplication such that $(g, h) \cdot (g', h') = ((g \triangleleft h')(h \triangleright g'), hh')$. One can see that this is isomorphic to a direct product of G and H with respect to an action of G on H constructed from $\triangleright$ and $\triangleleft$, but surprisingly we have not found this direct construction in the literature.

Weyl algebra and its automorphism group, which was computed by Jacques Dixmier [22] and Leonid Makar-Limanov [38], is both significantly larger and much less abelian. The case in which $N = 1$ was treated by Martha Smith in [51], and much later the case of a general Ore extension A_h by Jeffrey Bergen in [12]. Our small class of extensions allows for a less complicated argument, though.

Proof. An easy calculation shows that for each choice of $f \in \Bbbk[x]$ and $\lambda \in \Bbbk^\times$ there is indeed an algebra endomorphism $\phi_{f,\lambda} : A \to A$ mapping x and y to λx and $\lambda^{N-1}y + f$, respectively: we thus have a function $(f, \lambda) \in \Bbbk[x] \rtimes \Bbbk^\times \mapsto \phi_{f,\lambda} \in \operatorname{End}_{\mathrm{Alg}}(A)$. This is the map Φ described in the statement of the proposition. A direct calculation shows this map is a morphism of monoids, so it actually takes values in $\operatorname{Aut}(A)$. It is obvious that it is injective, and we will now show that it is also surjective.
Let $\phi : A \to A$ be an automorphism of algebras and, as before, let us write $\mathbb{N}$ for the set of non-zero normal elements of A, which is a monoid with respect to multiplication. Of course, we have that $\phi(\mathbb{N}) = \mathbb{N}$ and that the restriction $\phi|_{\mathbb{N}} : \mathbb{N} \to \mathbb{N}$ is an automorphism of monoids which is the identity on the subset $\Bbbk^\times$ of $\mathbb{N}$: it follows immediately from this that there is a scalar $\lambda \in \Bbbk^\times$ such that $\phi(x) = \lambda x$. Moreover, since $0 = \phi([y, x] - x^N) = [\phi(y), \lambda x] - \lambda^N x^N$ and $[\lambda^{N-1}y, \lambda x] = \lambda^N x^N$, we see that $[\phi(y) - \lambda^{N-1}y, x] = 0$ and, therefore, that $\phi(y) - \lambda^{N-1}y \in \Bbbk[x]$. This tells us that ϕ is in the image of the map Φ, so that this mar is surjective. □

With the very explicit description of the group $\operatorname{Aut}(A)$ that this proposition gives we can immediately compute its center:

Corollary 2.5. For each $t \in \Bbbk^\times$ there is a unique automorphism $\sigma_t : A \to A$ such that $\sigma_t(x) = x$ and $\sigma_t(y) = y + tx^{N-1}$, and it is central in $\operatorname{Aut}(A)$. The function

$$t \in \Bbbk \mapsto \sigma_t \in \operatorname{Aut}(A)$$

is an injective morphism of groups whose image is precisely the center of $\operatorname{Aut}(A)$. □

The center of $\operatorname{Aut}(A)$ is therefore a 1-parameter subgroup which goes through the automorphism σ_1 of Lemma 2.2. We will find later an infinitesimal generator for this 1-parameter subgroup as a class in the first Hochschild cohomology of the algebra.

Let Λ be an algebra. If $u \in \Lambda$, then, as usual, the ***inner derivation*** corresponding to u is

$$\operatorname{ad}(u) : a \in \Lambda \mapsto [u, a] \in \Lambda,$$

and we say that u is ***locally*** ad***-nilpotent*** if the derivation $\operatorname{ad}(u)$ is locally nilpotent. When that is the case we can consider the exponential of $\operatorname{ad}(u)$, namely the automorphism

$$\exp \operatorname{ad}(u) : a \in \Lambda \mapsto \sum_{i \geq 0} \frac{\operatorname{ad}(u)^i(a)}{i!} \in \Lambda,$$

because for each a in Λ the series appearing here is in fact a finite sum. We put

$$\operatorname{Exp}(\Lambda) \coloneqq \{\exp \operatorname{ad}(u) : u \in \Lambda \text{ is locally ad-nilpotent}\}.$$

This is a conjugation-invariant subset of $\operatorname{Aut}(\Lambda)$ but, in general, not a subgroup. We refer to G. Freudenburg's book [25] for general information about locally nilpotent derivations, albeit in a commutative setting.

Proposition 2.6. The set of ad-locally nilpotent elements in A is $\Bbbk[x]$ and for each $f \in \Bbbk[x]$ we have that

$$\exp \operatorname{ad}(f) = \phi_{x^N f',1}.$$

The set $\operatorname{Exp}(A)$ coincides with $\{\phi_{f,1} : f \in x^N \Bbbk[x]\}$ and is a normal subgroup of $\operatorname{Aut}(A)$.

The locally ad-nilpotent elements of the first Weyl algebra were described by Dixmier in [22, Théorème 9.1]: when viewing the algebra as that of differential operators on the line, they are the elements that belong to the orbits of constant coefficient differential operators under the action of the automorphism group of the algebra. Here one could use the same description, but it is much less interesting: the automorphism group of a Weyl algebra is much larger.

Proof. For each $a \in \Bbbk[x] \setminus 0$ let us write $\nu(a) \coloneqq \mathbf{max}\{t \in \mathbb{N}_0 : a \in x^t \Bbbk[x]\}$. On the other hand, if u is an element of A that is not in $\Bbbk[x]$, then there exist $l \geq 1$ and $a_0, \ldots, a_l \in \Bbbk[x]$ such that $u = \sum_{i=0}^{l} a_i y^i$ and $a_l \neq 0$, and we will call the rational number $\operatorname{sl}(u) \coloneqq \nu(a_l)/l$ the ***slope*** of u. We will start by showing that

Claim 2.7. If u and v are elements of $A \setminus \Bbbk[x]$ and $\operatorname{sl}(v) > \operatorname{sl}(u)$, then $[u, v] \in A \setminus \Bbbk[x]$ and $\operatorname{sl}([u, v]) > \operatorname{sl}(u)$.

To do so, let u, $v \in A \setminus \Bbbk[x]$, so that there are l, $m \geq 1$, $a_0, \ldots, a_l$, $b_0, \ldots, b_m \in \Bbbk[x]$ such that $u = \sum_{i=0}^{l} a_i y^i$, $v = \sum_{j=0}^{m} b_j y^j$, $a_l \neq 0$ and $b_m \neq 0$, and let us suppose that $\operatorname{sl}(v) > \operatorname{sl}(u)$, so that

$$\nu(b_m)/m > \nu(a_l)/l. \tag{9}$$

We have that

$$\begin{aligned} [u, v] = \sum_{i=0}^{l} \sum_{j=0}^{m} \Bigl(a_i [y^i, b_j] y^j + b_j [a_i, y^j] y^i\Bigr) &\equiv a_l [y^l, b_m] y^m + b_m [a_l, y^m] y^l \\ &\equiv x^N (l a_l b_m' - m b_m a_l') y^{l+m-1} \mod F_{l+m-2}. \end{aligned} \tag{10}$$

There are c, $d \in \Bbbk[x]$ not divisible by x such that $a_l = x^{\nu(a_l)} c$ and $b_m = x^{\nu(b_m)} d$, and

$$l a_l b_m' - m b_m a_l' = x^{\nu(a_l)+\nu(b_m)-1} \Bigl(\bigl(l\nu(b_m) - m\nu(a_l)\bigr) cd + x\bigl(lcd' - mdc'\bigr)\Bigr). \tag{11}$$

If the left hand side of this equality is 0, then — since x does not divide the product cd — we have that $l\nu(b_m) - m\nu(a_l) = 0$: this is absurd, since we are assuming that the inequality (9) holds. Going back to (10), we see that $[u, v] \in F_{l+m-1} \setminus F_{l+m-2}$ and, in particular, that $[u, v] \in A \setminus \Bbbk[x]$, since $l + m - 1 \geq 1$.

Moreover, in view of (10) and (11) and the fact that x does not divide the product cd, we have

$$\mathrm{sl}([u,v]) = \frac{\nu(a_l)+\nu(b_m)+N-1}{l+m-1}$$

and this is easily seen to be

$$> \frac{\nu(a_l)}{l} = \mathrm{sl}(u)$$

using (9) and the fact that $N \geq 1$. This proves the claim (2.7) above.

Suppose now that u is an element of $A \setminus \Bbbk[x]$, and let m be an integer such that $m > \mathrm{sl}(u)$. If we put $v_i := \mathrm{ad}(u)^i(x^m y)$ for each non-negative integer i, then an obvious induction using (2.7), starting with the observation that $v_0 \in A \setminus \Bbbk[x]$ and $\mathrm{sl}(v_0) = m > \mathrm{sl}(u)$, shows that $v_i \neq 0$ for all $i \in \mathbb{N}_0$. This proves that the element u is not locally ad-nilpotent and, therefore, that the set of locally ad-nilpotent element of A is contained in $\Bbbk[x]$.

Conversely, if f is an element of $\Bbbk[x]$ then we have that $\mathrm{ad}(f)(F_i) \subseteq F_{i-1}$ for all $i \in \mathbb{N}_0$, so that $\mathrm{ad}(f)^{i+1}(F^i) = 0$ for all $i \in \mathbb{N}_0$: this shows that f is locally ad-nilpotent and, with that, the first claim of the proposition. Moreover, a direct calculation show that $(\exp \mathrm{ad}(f))(x) = x$ and $(\exp \mathrm{ad}(f))(y) = y + x^N f'$, so that $\exp \mathrm{ad}(f)$ is the automorphism $\phi_{x^N f',1}$. That $\mathrm{Exp}(A)$ is $\{\phi_{f,1} : f \in x^N \Bbbk[x]\}$ is now clear, that it is a subgroup of $\mathrm{Aut}(A)$ follows another simple calculation, and its normality is obvious. □

The subgroup of $\mathrm{Aut}(A)$ generated by the exponentials of the locally ad-nilpotent elements of A, which we have just shown to be precisely $\mathrm{Exp}(A)$, is cognate to the group of inner automorphisms of a Lie algebra — as given by Roger Carter in [13, §3.2], for example — so it makes sense to view the quotient $\mathrm{Aut}(A)/\mathrm{Exp}(A)$ as a Lie-theoretic version of the ***outer automorphism group*** of A. We will describe this quotient below. For comparison, the *usual* inner automorphism group of A, in the sense of associative algebras, is trivial, as the units of A are central, so that the usual outer automorphism group of A is just $\mathrm{Aut}(A)$. On the other hand, the X-inner automorphism group of A — that is, the group of automorphisms of A that are restrictions of inner automorphisms of the Martindale left quotient ring of A, considered originally by Vladislav K. Harčenko in [29, 30] — is isomorphic to $\mathbb{Z}$: this was computed by Jeffrey Bergen in [12, Theorem 2.6].

Corollary 2.8. Let ξ be the class of x in the quotient $Q := \Bbbk[x]/(x^N)$, let $\Bbbk^\times$ act on the right on this quotient by algebra automorphisms in such a way that $\xi \cdot \lambda = \lambda\xi$ for all $\lambda \in \Bbbk^\times$, and let $Q \bowtie \Bbbk^\times$ be the group that as a set coincides with $Q \times \Bbbk^\times$ and whose product is such that

$$(p,\lambda)\cdot(q,\mu) = (\mu^{N-1}p + q\cdot\lambda, \lambda\mu)$$

whenever (p,λ) and (q,μ) are two elements of $Q \times \Bbbk^\times$. The function

$$Q \bowtie \Bbbk^\times \to \frac{\mathrm{Aut}(A)}{\mathrm{Exp}(A)}$$

that maps each pair $(f + (x^N), \lambda)$ to the class of the automorphism $\phi_{f,\lambda}$ is an isomorphism of groups. The quotient $\operatorname{Aut}(A)/\operatorname{Exp}(A)$ has therefore a natural structure of Lie group over $\Bbbk$ of dimension $N+1$, solvable of class 2 and, in fact, an extension of $\Bbbk^N$ by $\Bbbk^\times$. $\square$

The most interesting part of this is, of course, that dividing by the group of inner automorphisms allows us to go from the infinite-dimensional group $\operatorname{Aut}(A)$ to a finite-dimensional one. As elements of $\operatorname{Exp}(A)$ has a tendency to act trivially on objects associated to A, this is useful. The proof of the corollary is immediate given the description of have of $\operatorname{Aut}(A)$ and of $\operatorname{Exp}(A)$, and we omit it.

Let us determine now the locally nilpotent derivations of our algebra — we already know those which are inner.

Proposition 2.9.

(*i*) If $g \in \Bbbk[x]$, then there is a unique derivation $d_g : A \to A$ such that $d_g(x) = 0$ and $d_g(y) = g$, it is locally nilpotent, it is inner exactly when $g \in x^N\Bbbk[x]$, and for all $t \in \Bbbk$ we have that $\exp(td_g) = \phi_{tg,1}$.

(*ii*) If $d : A \to A$ is a locally nilpotent, there is exactly one polynomial $g \in \Bbbk[x]$ such that $d = d_g$.

Proof. Let g be an element of $\Bbbk[x]$. That there is indeed a derivation $d_g : A \to A$ such that $d_g(x) = 0$ and $d_g(y) = g$ follows by a trivial calculation using the presentation of A. Since it is a derivation, for all $i, j \in \mathbb{N}_0$ we have that

$$d_g(x^iy^j) = \sum_{s+1+t=j} x^iy^sgy^t \equiv jgx^iy^{j-1} \mod F_{j-2}.$$

This implies that $d_g(F_j) \subseteq F_{j-1}$ for all $j \in \mathbb{N}_0$ and, in particular, that d_g is locally nilpotent. If $t \in \Bbbk$, then $\exp(td_g)(x) = x$ because $d_g(x) = 0$ and $\exp(td_g)(y) = y + td_g(y) = y + tg$ because $d_g^2(y) = 0$, and this tells us that $\exp(t\partial_0) = \phi_{tg,1}$.

If there is an element $a \in A$ such that $d_g = \operatorname{ad}(a)$, then $0 = d_g(x) = [a, x]$, so that $a \in \Bbbk[x]$ because of Lemma 1.3, and therefore $g = d_g(y) = [a, y] \in x^NA$. Conversely, if $g = x^Nh$ for some $h \in \Bbbk[x]$ and we chose $k \in \Bbbk[x]$ such that $k' = -h$, then $d_g = \operatorname{ad}(k)$, so that d_g is inner. With this we have proved all the claims in part (*i*) of the proposition.

Next, let $d : A \to A$ be a locally nilpotent derivation of A, so that, in particular, there are two non-negative integers l and m such that $d^{l+1}(x) = 0$ and $d^{m+1}(y) = 0$. As d is locally nilpotent, for each $t \in \Bbbk$ we can consider the automorphism $\exp(td) : A \to A$. In view of Proposition 2.4, for each $t \in \Bbbk$ there exist a non-zero scalar $\lambda_t \in \Bbbk^\times$ and a polynomial $f_t \in \Bbbk[x]$ such that $\exp(td) = \phi_{f_t,\lambda_t}$ and therefore

$$\sum_{i=0}^{l} t^i\frac{d^i(x)}{i!} = \lambda_t x, \qquad \sum_{i=0}^{m} t^i\frac{d^i(y)}{i!} = \lambda_t^{N-1}y + f_t. \tag{12}$$

Let $t_0, \ldots, t_l$ be l pairwise different elements of $\Bbbk$. The Vandermonde matrix built out of those $l+1$ scalars is invertible: it follows that from the $l+1$ equalities that we get from the first one in (12) by replacing t by each of $t_0, \ldots, t_l$ we can solve for $d(x)$ and find that there is a scalar α such that

$$d(x) = \alpha x. \tag{13}$$

Proceeding similarly with the second equation in (12) we see that there are a scalar $\beta \in \Bbbk$ and a polynomial $g \in \Bbbk[x]$ such that

$$d(y) = \beta y + g. \tag{14}$$

Of course, we must have that $d(yx - xy - x^N) = 0$, and writing this out we find that $\beta = (N-1)\alpha$. On the other hand, from (13) and (14) we can see immediately that there is a sequence of polynomial $(g_n)_{n\geq 0}$ in $\Bbbk[x]$ such that

$$d^n(y) = (N-1)^n \alpha^n y + g_n$$

for all $n \in \mathbb{N}_0$. As $d^m(y) = 0$, this implies that $\alpha = 0$. As g is uniquely determined by the derivation d, this proves the claim (ii) of the proposition. □

This proposition allows us to give a particularly important example of a locally nilpotent derivation:

Corollary 2.10. There is a derivation $\partial_0 : A \to A$ such that $\partial_0(x) = 0$ and $\partial_0(y) = x^{N-1}$, it is locally nilpotent, not inner, and

$$\exp(t\partial_0) = \sigma_t.$$

for all t. □

Here σ_t is the central automorphism of A that we described in Corollary 2.5, so the derivation ∂_0 is an infinitesimal generator for the 1-parameter subgroup of $\operatorname{Aut}(A)$ that is its center. We will show later that the class of the derivation ∂_0 generates the center of $\mathrm{HH}^1(A)$. The point of the following remark is that this derivation is also of Poisson-theoretic interest.

Remark 2.11. In proving this proposition we have noted that $\partial_0(F_j) \subseteq F_{j-1}$ for all $j \in \mathbb{N}_0$, and this implies immediately that the derivation ∂_0 induces a derivation $\overline{\partial}_0 : \operatorname{gr} A \to \operatorname{gr} A$ on the associated graded algebra $\operatorname{gr} A$ of A such that

$$\overline{\partial}_0(\overline{x}) = 0, \qquad \overline{\partial}_0(\overline{y}) = \overline{x}^{N-1}.$$

This is a Poisson derivation of $\operatorname{gr} F$ and, in fact, it is the ***modular derivation*** of that Poisson algebra, in the sense of Alan Weinstein in [56], that is, the map

$$f \in \operatorname{gr} A \mapsto \div H_f \in \operatorname{gr} A. \tag{15}$$

As a consequence of this, the morphism in Corollary 2.5 that, according to Corollary 2.10, arises by exponentiation from ∂_0, is precisely the ***modular flow*** of the Poisson algebra $\operatorname{gr} A$. In (15) we have written $H_f \coloneqq \{f, -\}$ for the Hamiltonian derivation corresponding to an element f of $\operatorname{gr} A$, and $\div$ for the divergence operator, so that

$$X \triangleright \mathrm{d}\overline{x} \wedge \mathrm{d}\overline{y} = \div L \cdot \mathrm{d}\overline{x} \wedge \mathrm{d}\overline{y}$$

for each $X \in \operatorname{Der}(\operatorname{gr} A)$, with $\triangleright$ the action of vector fields on differential forms. $\Diamond$

Remark 2.12. The derivation ∂_0 is not inner, but it is a «logarithmic derivation» in that

$$\partial_0(a) = \frac{1}{x}[x, a]$$

for all $a \in A$. Here the right hand side of the equality is, in principle, an element of the localization A_x of A at x, but it turns out to be in A. $\Diamond$

Knowing the locally nilpotent derivations we obtain another nice subgroup of $\operatorname{Aut}(A)$.

Corollary 2.13.

(*i*) The set $\operatorname{LND}(A)$ of all locally nilpotent derivations of A is an abelian subalgebra of the Lie algebra $\operatorname{Der}(A)$ of all derivations of A.

(*ii*) The set of the exponentials of the elements of $\operatorname{LND}(A)$ is the normal abelian subgroup

$$\operatorname{Aut}_0(A) \coloneqq \{\phi_{g,1} : g \in \Bbbk[x]\}$$

of $\operatorname{Aut}(A)$.

(*iii*) The function $\mathbf{det} : \operatorname{Aut}(A) \to \Bbbk^\times$ such that $\mathbf{det}(\phi_{f,\lambda}) = \lambda$ whenever $f \in \Bbbk[x]$ and $\lambda \in \Bbbk^\times$ is a morphism of groups. The sequence

$$0 \longrightarrow \operatorname{Aut}_0(A) \hookrightarrow \operatorname{Aut}(A) \xrightarrow{\mathbf{det}} \Bbbk^\times \longrightarrow 1$$

is an extension of groups that is split by the morphism $\lambda \in \Bbbk^\times \mapsto \phi_{0,\lambda} \in \operatorname{Aut}(A)$. $\square$

All this follows immediately from Proposition 2.9. This corollary describes the algebraic actions of the additive group $\mathbb{G}_a$ on the algebra A. Indeed, such a thing is the same thing as a right algebra-comodule structure on A over the coordinate Hopf algebra $\Bbbk[t]$ of $\mathbb{G}_a$, which is a coassociative morphism of algebras $\phi : A \to A \otimes \Bbbk[t]$. In fact, a linear map $A \to A \otimes \Bbbk[t]$ is determined by a sequence $(\phi_i)_{i\geq 0}$ of linear maps $A \to A$ such that for each $a \in A$ the sequence $(\phi_i(a))_{i\geq 0}$ has almost all its components equal to zero and $\phi(a) = \sum_{i=0}^{\infty} \phi_i(a)$, and it is a algebra-comodule structure on A over $\Bbbk[x]$ exactly when $\phi_1 : A \to A$ is a locally nilpotent derivation and $\phi_i = \phi_1^i/i!$ for all $i \in \mathbb{N}_0$.

It should be remarked that the space of locally nilpotent derivations of an algebra is most often not a subalgebra of the Lie algebra of derivations of the algebra.

On the end of the spectrum opposite to where the $\mathbb{G}_a$-actions lie are the actions of finite groups. With the description of the automorphism group of our algebra that we have we can easily find these. Later, in Section 8, we will consider more generally coactions of some finite-dimensional Hopf algebras — and that will require considerably more work.

Proposition 2.14.

(i) If ϕ is an element of $\operatorname{Aut}(A)$ of finite order m, then exists a unique $\lambda \in \Bbbk^\times$ of order exactly m such that ϕ is conjugated in $\operatorname{Aut}(A)$ to $\phi_{0,\lambda}$.

(ii) If λ is an element of order m in $\Bbbk^\times$, then the automorphism $\phi_{0,\lambda}$ has order m, and if $G \coloneqq \langle \phi_{0,\lambda} \rangle$ is the subgroup of $\operatorname{Aut}(A)$ generated by it, then the subalgebra A^G of invariants under G is the mth Veronese subalgebra $A^{(m)} = \bigoplus_{i \geq 0} A_{mi}$.

(iii) Every finite subgroup of $\operatorname{Aut}(A)$ is cyclic, and conjugated to the subgroup generated by $\phi_{0,\lambda}$, with λ a root of unity in $\Bbbk$.

The first two parts of the proposition imply that for all $m \in \mathbb{N}$ the number of conjugacy classes of $\operatorname{Aut}(A)$ of elements of order m coincides with the number of elements of order m in $\Bbbk^\times$. On the other hand, the third part tells us that set of conjugacy classes of finite subgroups of $\operatorname{Aut}(A)$ are in bijection with the set of integers m such that there is a primitive mth root of unity in $\Bbbk$, the bijection being given by taking the order.

Proof. Let $(f, \lambda) \in \Bbbk[x] \bowtie \Bbbk^\times$, let $k \in \mathbb{N}$ and let us suppose that the automorphism $\phi_{f,\lambda}$ has order k. Since

$$\phi_{f,\lambda}^k(x) = \lambda^k x, \qquad \phi_{f,\lambda}^k(y) = \lambda^{k(N-1)} y + \sum_{i=0}^{k-1} \lambda^{i(N-1)} f(\lambda^{k-1-i} x),$$

we see that $\lambda^k = 1$, so that λ has finite order in $\Bbbk^\times$, and that

$$\sum_{i=0}^{k-1} \lambda^{i(N-1)} f(\lambda^{k-1-i} x) = 0. \tag{16}$$

Let m be the order of λ in $\Bbbk^\times$, so that m divides k. If $h \in \Bbbk[x]$, then

$$\begin{aligned} (\phi_{h,1} \circ \phi_{f,\lambda} \circ \phi_{h,1}^{-1})(x) &= \lambda x, \\ (\phi_{h,1} \circ \phi_{f,\lambda} \circ \phi_{h,1}^{-1})(y) &= \lambda^{N-1} y + f(x) + \lambda^{N-1} h(x) - h(\lambda x) \end{aligned}$$

This tells us that replacing f by $f + \lambda^{N-1} h(x) - h(\lambda x)$ does not change the conjugacy class of $\phi_{f,\lambda}$ in $\operatorname{Aut}(A)$ and, as the subspace $\left\langle \lambda^{N-1} h(x) - h(\lambda x) : h \in \Bbbk[x] \right\rangle$ of $\Bbbk[x]$ is spanned by the monomials x^d with $d \not\equiv N-1 \mod m$, that up to conjugacy we can suppose that the polynomial f is a linear combination of monomials x^d with $d \equiv N-1 \mod m$. It is easy to check now that when that is the case the left hand side of the equality (16) is equal to $kf(\lambda^{k-1} x)$, and therefore that equality implies that $f = 0$, since our ground field has characteristic zero. This shows that every element of $\operatorname{Aut}(A)$ of finite order is conjugated to one of the form $\phi_{0,\lambda}$ with λ an element of finite order in $\Bbbk^\times$, and that of course the order of $\phi_{0,\lambda}$ is equal to the order of λ. To complete the proof of the first part of the proposition we need only notice that if λ and μ are two elements of $\Bbbk^\times$ the automorphisms $\phi_{0,\lambda}$ and $\phi_{0,\mu}$ are conjugated in $\operatorname{Aut}(A)$ exactly when $\lambda = \mu$: this is an immediate consequence of the fact that the map

$$\pi : \phi_{f,\lambda} \in \operatorname{Aut}(A) \mapsto \lambda \in \Bbbk^\times \tag{17}$$

is a morphism of groups.

If $m \in \mathbb{N}$ and λ is an element of order m in $\Bbbk^\times$, then $\phi_{0,\lambda}(x^i y^j) = \lambda^{i+j(N-1)} x^i y^j$. We see immediately from this that the subalgebra fixed by $\phi_{0,\lambda}$, and by the cyclic group it generates, is spanned by the monomials $x^i y^j$ with $m \mid i + j(N-1)$, that is, whose degree is divisible by m. The claim (ii) of the proposition follows at once from this.

Suppose now that G is a finite subgroup of $\operatorname{Aut}(A)$. If f and g are two elements of $\Bbbk[x]$ and λ one of $\Bbbk^\times$ such that $\phi_{f,\lambda}$ and $\phi_{g,\lambda}$ are both in G, then the composition $\phi_{f,\lambda} \circ \phi_{g,\lambda}$ has finite order and maps x and y to x and to $y + \lambda^{-N+1}(f-g)$, respectively: it follows immediately from this that $f = g$. This means that the map π from (17) is injective when restricted to G and therefore G is isomorphic to a finite subgroup of $\Bbbk^\times$ and, in particular, cyclic. If ϕ is a generator of G and m is the order of G, we have seen above that there is a primitive mth root of unity λ in $\Bbbk$ such that ϕ is conjugated to $\phi_{0,\lambda}$ and, therefore, G is conjugated to $\langle \phi_{0,\lambda} \rangle$. $\square$

Proposition 2.14 gives us a classification of the finite subgroups of $\operatorname{Aut}(A)$ and their corresponding invariant subalgebras, and it is natural to consider the classical problem of classifying these up to isomorphism. This seems to be a rather complicated problem, as even giving presentations for them is not easy. We will content ourselves with describing two «extreme» instances of this problem.

Let λ be a root of unity in $\Bbbk$, let m be its order, and let us suppose that $m > 1$. We let G be the subgroup generated by $\phi_{0,\lambda}$ in $\operatorname{Aut}(A)$, which has order m, so that the invariant subalgebra A^G is the mth Veronese subalgebra $A^{(m)}$ of A.

(i) Suppose first that m divides $N-1$, and let k be the integer such that $N - 1 = km$. The invariant subalgebra A^G is easily seen to be generated by x^n and y and, since

$$yx^n - x^n y = nx^{n+N-1} = n(x^n)^k,$$

isomorphic to $A_k = \Bbbk\langle x, y\rangle/(yx - xy - x^k)$. We will see below, in Corollary 6.5, that as m varies among the $\phi(N-1)$ positive divisors of $N-1$ the algebras that we obtain in this way are pairwise non-isomorphic. These algebras can obviously be generated by two elements and not by one.

(ii) Suppose now that $N-1$ divides properly m, and let l be the integer such that $m = l(N-1)$, which is at least 2. In this situation the subalgebra A^G is generated by the mth homogeneous component of A, which has dimension $l+1$, so that that A^G can be generated by $l+1$ elements. On the other hand, A^G cannot be generated by l elements: indeed, if there existed l elements $a_1, \dots, a_l$ in A^G that generate it as an algebra, the homogeneous components of those elements of degree m would generate the mth homogeneous component of A, and this is absurd. We thus see that A^G is minimally generated by $l+1$ elements and that therefore the subalgebras that we obtain in this way are pairwise non-isomorphic and, since $l \geq 2$, also non-isomorphic to any of the algebras that we found in (i).

The smallest case not covered by these considerations is that in which $N = 3$ and $m = 3$.

3 Hochschild cohomology

In this section we present a rather effortless calculation of the Hochschild cohomology of the algebra A. Our approach is directly targeted at obtaining completely explicit representatives of cohomology classes. As before, we suppose throughout that $N \geq 1$. The Hochschild cohomology of the first Weyl algebra, the algebra we get when $N = 0$, is well-know to be the same as that of the ground field — this was computed originally by Ramaiyengar Sridharan in [52].

Let V be the subspace of A spanned by x and y, let V^* be its dual vector space, and let $(\hat{x}, \hat{y})$ be the ordered basis of V^* dual to (x, y). There is a projective resolution P_* of A as an A-bimodule of the form

$$0 \longrightarrow A \otimes \Lambda^2 V \otimes A \xrightarrow{d_2} A \otimes V \otimes A \xrightarrow{d_1} A \otimes A \tag{18}$$

with differentials such that

$$\begin{aligned}
d_1(1 \otimes x \otimes 1) &= x \otimes 1 - 1 \otimes x, \\
d_1(1 \otimes y \otimes 1) &= y \otimes 1 - 1 \otimes y, \\
d_2(1 \otimes x \wedge y \otimes 1) &= y \otimes x \otimes 1 + 1 \otimes y \otimes x - x \otimes y \otimes 1 - 1 \otimes x \otimes y \\
&\qquad - \sum_{s+1+t=N} x^s \otimes x \otimes x^t
\end{aligned}$$

and augmentation $\varepsilon : A \otimes A \to A$ given by the multiplication of A. Clearly, V is a homogeneous subspace of A, and if we endow it with the induced grading, and in turn $\Lambda^2 V$ with the grading induced by that of V and each component of the complex (18) with the obvious tensor product gradings, that complex becomes a complex of graded A-bimodules. Applying to it the functor $\operatorname{Hom}_{A^e}(-, A)$ we obtain, up to standard identifications, the cochain complex

$$A \xrightarrow{\delta_0} A \otimes V^* \xrightarrow{\delta_1} A \otimes \Lambda^2 V^* \longrightarrow 0 \tag{19}$$

with differentials such that

$$\begin{aligned}
\delta_0(a) &= [x, a] \otimes \hat{x} + [y, a] \otimes \hat{y}, \\
\delta_1(b \otimes \hat{x} + c \otimes \hat{y}) &= \left([y, b] + [c, x] - \sum_{s+1+t=N} x^s b x^t \right) \otimes \hat{x} \wedge \hat{y}
\end{aligned}$$

for all choices of a, b and c in A. The cohomology of this complex is canonically isomorphic to the Hochschild cohomology $\mathrm{HH}^\bullet(A)$ of A, and we *identify* the two. If we grade V^* so that $\hat{x}$ and $\hat{y}$ have degrees -1 and $-(N-1)$, respectively, then the complex (19) is one of graded vector spaces and homogeneous maps of degree 0, and, consequently, its cohomology $\mathrm{HH}^\bullet(A)$ is also graded. This grading on Hochschild cohomology coincides with the canonical grading it gets from the fact that the algebra A is graded and has a free resolution as a graded bimodule over itself by finitely generated free bimodules — we will not belabor this point, but it is important that the grading we found is *the* grading on $\mathrm{HH}^\bullet(A)$.

The hardest part of the calculation is that of $\mathrm{HH}^1(A)$, and we will leave it for the end, as we will do it in a rather indirect way. As for that of $\mathrm{HH}^0(A)$, we have actually already done it:

Lemma 3.1. The 0th cohomology space $\mathrm{HH}^0(A)$, the kernel of the map δ_0, is $\Bbbk$, and its Hilbert series is therefore $h_{\mathrm{HH}^0(A)}(t) = 1$.

Proof. We established in Proposition 1.4 that the center of A, which coincides with the kernel of the map δ_0, is $\Bbbk$ and then that its Hilbert series is 1 is clear. □

We know from Lemma 2.1 that x^N is a normal element in A, so that the right ideal $x^N A$ is a bilateral ideal. Moreover, that ideal is related to commutators in the following way:

Lemma 3.2. We have $[A, x] = [A, A] = x^N A$.

Proof. If i, $j \geq 0$, then

$$[x^i y^j, x] = \sum_{s+1+t=j} x^i y^s [y, x] y^t = \sum_{s+1+t=j} x^i y^s x^N y^t \in x^N A$$

because $x^N A$ is an ideal, and this implies that $[A, x] \subseteq x^N A$.
To prove the reverse inclusion, we will show that $x^{i+N} y^j \in [A, x]$ for all i, $j \geq 0$ by induction on j. If $j \geq 0$ and $x^{i+N} y^k \in [A, x]$ for all $i \geq 0$ and all $k \in \{0, \ldots, j-1\}$ — this hypothesis is vacuous when $j = 0$, and this starts the induction — then there is an $u \in F_{j-1}$ such that $[x^i y^{j+1}, x] = (j+1) x^{i+N} y^j + u$ and, because of the hypothesis, a $v \in A$ such that $u = [v, x]$: we then have that $x^{i+N} y^j = [(j+1)^{-1} x^i y^{j+1} - v, x] \in [A, x]$. The induction is thus complete.
If i, j, k, $l \geq 0$, then we have

$$[x^i y^j, x^k y^l] = x^i [y^j, x^k] y^l + x^k [x^i, y^l] y^j,$$

so to prove that $[A, A]$ is contained in $x^N A$, and thus the rest of the equalities asserted by the lemma, it is enough to compute that

$$[y^j, x^k] = \sum_{\substack{s+1+t=j \\ u+1+v=k}} y^s x^u [y, x] x^v y^t \in x^N A$$

because $[y, x] = x^N$ and $x^N A$ is a bilateral ideal. □

The description of $[A, A]$ that we have now allows us to compute $\mathrm{HH}^2(A)$ easily.

Lemma 3.3. The image of the map δ_1 is $x^{N-1} A \otimes \hat{x} \wedge \hat{y}$, and therefore

$$\mathrm{HH}^2(A) \cong A / x^{N-1} A \otimes \hat{x} \wedge \hat{y}.$$

If $N = 1$ then this is of course 0, and if $N \geq 2$, so that $\mathrm{HH}^2(A)$ is graded with finite-dimensional homogeneous components, then the Hilbert series if this space is

$$h_{\mathrm{HH}^2(A)}(t) = \frac{t^{-N}}{1-t}.$$

Proof. For all $c \in A$ we have $\delta_1(c \otimes \hat{y}) = [c, x] \otimes \hat{x} \wedge \hat{y}$, and this, together with Lemma 3.2, tells us that $x^N A \otimes \hat{x} \wedge \hat{y} = \delta_1(A \otimes \hat{y}) \subseteq \operatorname{img} \delta_1$. On the other hand, if i, $j \geq 0$, we have that

$$\begin{aligned}
\delta_1(x^i y^j \otimes \hat{x}) &= \left([y, x^i y^j] - \sum_{s+1+t=N} x^{s+i} y^j x^t \right) \otimes \hat{x} \wedge \hat{y} \\
&= \left(i x^{i+N-1} y^j - \sum_{s+1+t=N} x^{s+i} y^j x^t \right) \otimes \hat{x} \wedge \hat{y} \\
&= \left((i-N) x^{i+N-1} y^j - \sum_{s+1+t=N} x^{s+i} [y^j, x^t] \right) \otimes \hat{x} \wedge \hat{y}.
\end{aligned}$$

According to Lemma 3.2, the sum appearing in this last expression is an element of $x^N A$: it follows from this that $\delta_1(x^i y^j \otimes \hat{x}) \subseteq x^{N-1} A$, and then that

$$x^N A \subseteq \operatorname{img} \delta_1 = \delta_1(A \otimes \hat{x}) + \delta_1(A \otimes \hat{y}) \subseteq x^{N-1} A + x^N A = x^{N-1} A.$$

Moreover, for each $j \geq 0$ we have that

$$\delta_1(y^j \otimes \hat{x}) = \left(-N x^{N-1} y^j - \sum_{s+1+t=N} x^s [y^j, x^y] \right) \otimes \hat{x} \wedge \hat{y}.$$

Since the sum appearing here is in $x^N A$ it is equal to $[b, x]$ for some $b \in A$ and therefore

$$-N x^{N-1} y^j \otimes \hat{x} \wedge \hat{y} = \delta_1(y^j \otimes \hat{x} + b \otimes \hat{y}) \in \operatorname{img} \delta_1.$$

Putting everything together we conclude that $\operatorname{img} \delta_1 = x^{N-1} A$, as we want. Finally, the quotient $A/x^{N-1}A$ is freely spanned by the classes of the monomials $x^i y^j$ with $0 \leq i < N-1$ and $j \geq 0$, and there is exactly one such monomial of each degree in $\mathbb{N}_0$: the Hilbert series of $A/x^{N-1}A$ is thus $(1-t)^{-1}$, and the Hilbert series of $\mathrm{HH}^2(A)$ is as described in the lemma, because of the factor $\hat{x} \wedge \hat{y}$. □

At this point we know the Hilbert series of $\mathrm{HH}^0(A)$ and of $\mathrm{HH}^2(A)$, and we can use the invariance of the Euler characteristic of a complex under the operation of taking cohomology to determine the Hilbert series of $\mathrm{HH}^1(A)$.

Proposition 3.4. If $N \geq 2$, then the Hilbert series of $\mathrm{HH}^1(A)$ is

$$h_{\mathrm{HH}^1(A)}(t) = 1 + \frac{t^{-N+1}}{1-t},$$

so that for all integers l we have that

$$\mathbf{dim}\,\mathrm{HH}^1(A)_l = \begin{cases} 2 & \text{if } l = 0; \\ 1 & \text{if } l \geq -N+1 \text{ and } l \neq 0; \\ 0 & \text{if } l \leq -N. \end{cases}$$

Proof. The Hilbert series of A is

$$h_A(t) = \frac{1}{(1-t)(1-t^{N-1})},$$

and then the Euler characteristic of the complex (19) is

$$h_A(t) - (t^{-1} + t^{-(N-1)})h_A(t) + t^{-N}h_A(t) = t^{-N}.$$

The invariance of the Euler characteristic when passing to cohomology implies now that

$$t^{-N} = h_{\mathrm{HH}^0(A)}(t) - h_{\mathrm{HH}^1(A)}(t) + h_{\mathrm{HH}^2(A)}(t),$$

and one can compute $h_{\mathrm{HH}^1(A)}(t)$ from this equality, since we know the values of both $h_{\mathrm{HH}^0(A)}(t)$ and $h_{\mathrm{HH}^2(A)}(t)$, finding the formula given in the proposition. $\square$

In Proposition 3.4 we excluded the case in which $N = 1$, which is special — for one thing, the graded algebra A_N is not locally finite-dimensional in that case, so we cannot even talk about its Hilbert series! Let us deal with it now for the sake of completeness.

Proposition 3.5. If $N = 1$, then there is a derivation $d_0 : A \to A$ such that $d_0(x) = 0$ and $d_0(y) = 1$. It is homogeneous of degree 0, and its cohomology class freely spans the vector space $\mathrm{HH}^1(A)$, which is thus 1-dimensional and concentrated in degree 0.

When $N = 1$ the map $\mathrm{ad}(y) : u \in A \mapsto [y, u] \in A$ acts on elements of each degree $l \in \mathbb{Z}$ by multiplication by l. We will use this fact in the proof.

Proof. A trivial calculation shows that there is indeed a derivation $d_0 : A \to A$ with $d_0(x) = 0$ and $d_0(y) = 1$, and it is clearly homogeneous of degree 0. Were it inner, we would have an element c such that $[x, c] = 0$ and $[y, c] = 1$: the first equality implies that $c \in \Bbbk[x]$, and then the second one that $x^N c' = 1$, which is absurd.

Let now $d : A \to A$ be an arbitrary homogeneous derivation and let l be its degree. If $a := d(x)$ and $b := d(y)$, then a and b are homogeneous of A of degree $l+1$ and l, respectively. As d is a derivation, we have that

$$0 = d([y, x] - x) = [b, x] + [y, a] - a = [b, x] + la.$$

If $l \neq 0$ this tells us that $a = -\frac{1}{l}[b, x]$, and using this we see at once that d is the inner derivation $\frac{1}{l}\,\mathrm{ad}(b)$. Let us then suppose that $l = 0$. Now the equality above tells us that b commutes with x, so that is an homogeneous element of $\Bbbk[x]$ of degree 0: in other words, we have $b \in \Bbbk$. On the other hand, since a has degree 1, there is an $f \in \Bbbk[y]$ such that $a = xf$. There exists a $g \in \Bbbk[y]$ such that $f = g(y+1) - g(y)$, and then $[g, x] = a$. The derivation $d' := d - \operatorname{ad}(g)$ is therefore homogeneous of degree 0, cohomologous to d, and has $d'(x) = 0$ and $d'(y) = b$. We can thus conclude that every homogeneous derivation is cohomologous lo a multiple of the derivation d_0. This proves the proposition. □

If we are asked for a reason explaining the difference between the case in which $N = 1$ and that in which $N \geq 2$ exhibited by the last two results, a good candidate for an answer is the following. Whatever the value of N, the algebra A has a grading, $A = \bigoplus_{j \geq 0} A_j$, and there is therefore a derivation $E : A \to A$ such that for all $j \geq 0$ the component A_j is invariant under E and the restriction $E|_{A_j} : A_j \to A_j$ is simply multiplication by the scalar j. This is a diagonalizable derivation and the homogeneous components of A are precisely its eigenspaces. Now a difference arises: if $N = 1$, the derivation E is inner, since it coincides with $\operatorname{ad}(y)$, while if $N \geq 2$ the derivation not inner. This is behind the collapse of $\mathrm{HH}^1(A)$ that occurs when $N = 1$.

4 A sequence of special elements in our algebra

In the next section we will exhibit explicit derivations that freely span the first cohomology vector space of our algebra, $\mathrm{HH}^1(A)$, and to do that we will need some calculations that we carry out in this one. Despite the fact that we will use these results there only when $N \geq 2$, in this section we work with an arbitrary N in $\mathbb{N}$, because when $N = 1$ something interesting happens.

Lemma 4.1. For each $j \geq 1$ there exists a unique element Φ_j in Ay homogeneous of degree $j(N-1)$ such that $[\Phi_j, x] = x^N y^{j-1}$, and it is such that

$$\Phi_j \equiv \frac{1}{j} y^j - \frac{N}{2} x^{N-1} y^{j-1} \mod F_{j-2}$$

when $j \geq 2$ and $\Phi_1 = y$ when $j = 1$.

Proof. The existence and uniqueness of the elements Φ_j is an immediate consequence of the exactness of the sequence of graded vector spaces

$$0 \longrightarrow \Bbbk[x] \hookrightarrow A \xrightarrow{\operatorname{ad}(x)} x^N A \longrightarrow 0$$

together with the fact that Ay is a complement for $\Bbbk[x]$ in A. It is clear that $\Phi_1 = y$. Suppose now that $j \geq 2$. Since Φ_j is homogeneous of degree $k(N-1)$ there are scalars $a_0, a_1, \ldots, a_j$ in $\Bbbk$ such that $\Phi_j = \sum_{i=0}^{j} a_i x^{i(N-1)} y^{j-i}$, and then working modulo F_{j-3} we have that

$$\begin{aligned} x^N y^{j-1} = [\Phi_j, x] = \sum_{i=0}^{j-1} a_i x^{i(N-1)} [y^{j-i}, x] &\equiv a_0 [y^j, x] + a_1 x^{N-1} [y^{j-1}, x] \\ &\equiv j a_0 x^N y^{j-1} + (j-1) \left(\frac{a_0 N j}{2} + a_1 \right) x^{2N-1} y^{j-2} \end{aligned}$$

i	$c_i(q)$
0	1
1	$-\frac{1}{2}q - \frac{1}{2}$
2	$-\frac{1}{6}q^2 + \frac{1}{6}$
3	$-\frac{1}{4}q^3 + \frac{1}{4}q$
4	$-\frac{19}{30}q^4 + \frac{2}{3}q^2 - \frac{1}{30}$
5	$-\frac{9}{4}q^5 + \frac{5}{2}q^3 - \frac{1}{4}q$
6	$-\frac{863}{84}q^6 + 12q^4 - \frac{7}{4}q^2 + \frac{1}{42}$
7	$-\frac{1375}{24}q^7 + 70q^5 - \frac{105}{8}q^3 + \frac{5}{12}q$
8	$-\frac{33953}{90}q^8 + 480q^6 - \frac{1624}{15}q^4 + \frac{50}{9}q^2 - \frac{1}{30}$
9	$-\frac{57281}{20}q^9 + 3780q^7 - \frac{9849}{10}q^5 + 70q^3 - \frac{21}{20}q$
10	$-\frac{3250433}{132}q^{10} + 33600q^8 - \frac{29531}{3}q^6 + \frac{5345}{6}q^4 - \frac{91}{4}q^2 + \frac{5}{66}$
11	$-\frac{1891755}{8}q^{11} + 332640q^9 - \frac{214995}{2}q^7 + \frac{47025}{4}q^5 - \frac{3465}{8}q^3 + \frac{15}{4}q$

Table 1: The polynomials $c_i(q)$ for small values of i.

so that $\alpha_0 = 1/j$ and $a_1 = -N/2$. The last claim of the lemma follows from this. □

Let us consider a variable q and the sequence $(c_i)_{i\geq 0}$ of elements of $\mathbb{Q}[q]$ such that

$$\sum_{i=0}^{j} \frac{(1)_{q,j+1-i}}{(j+1-i)!}\frac{c_i(q)}{i!} = \delta_{j,0} \tag{20}$$

for all $j \in \mathbb{N}_0$. This condition recursively determines the sequence starting with $c_0 = 1$.

We want to give a few properties of these polynomials, and in order to do that we need two classical sequences of rational numbers: that of the Bernoulli numbers $(B_j)_{j\geq 0}$ and of the Gregory coefficients $(G_i)_{i\geq 1}$. The first one is uniquely characterized by the equalities

$$\sum_{i=0}^{j} \binom{j+1}{i} B_i = \delta_{j,0}, \qquad \forall\, j \geq 0$$

and the second one by the equalities

$$G_1 = \frac{1}{2}, \qquad \sum_{i=1}^{j} (-1)^{i+1} \frac{G_i}{j+1-i} = \frac{1}{j+1}, \qquad \forall\, j \geq 2.$$

For convenience we put additionally $G_0 = 1$. The second sequence is not particularly famous, but quite a bit of information about it can be found in the article [39] that deals with the sequence of Cauchy numbers

n	0	1	2	3	4	5	6	7	8	9	10
B_n	1	$-\frac{1}{2}$	$\frac{1}{6}$	0	$-\frac{1}{30}$	0	$\frac{1}{42}$	0	$-\frac{1}{30}$	0	$\frac{5}{66}$
G_n	1	$\frac{1}{2}$	$-\frac{1}{12}$	$\frac{1}{24}$	$-\frac{19}{720}$	$\frac{3}{160}$	$-\frac{863}{60480}$	$\frac{275}{24192}$	$-\frac{33953}{3628800}$	$\frac{8183}{1036800}$	$-\frac{3250433}{479001600}$

Table 2: The first Bernoulli numbers and Gregory coefficients.

$(C_j)_{j\geq 0}$, which has $C_j = G_j/j!$ for all $j \geq 0$. The numbers in our sequence appear in the formula for approximate integration discovered by James Gregory in 1668, and that is why they are named after him — there is also a crater in the Moon (located at N2°12′0″ E127°12′0″) that carries his name: his work was mostly on Astronomy. One of the ways these numbers enter the theory of approximate integration is through the fact for all non-negative integers j we have

$$G_j = \int_0^1 \binom{x}{j}\,\mathrm{d}x.$$

The exponential generating function of the Bernoulli numbers and the ordinary generating function of the Gregory coefficients are, respectively,

$$\sum_{j\geq 0} B_j \frac{t^j}{j!} = \frac{t}{e^t - 1}, \qquad \sum_{j\geq 0} G_j t^j = \frac{z}{\ln(1+z)}. \tag{21}$$

It can be checked that none of the Gregory coefficients vanish.

Lemma 4.2. For each $j \geq 0$ the polynomial $c_j(q)$ has degree j, leading coefficient $(-1)^j j! G_j$ and constant term B_j. The exponential generating series of the sequence $(c_j(q))_{j\geq 0}$ is

$$\sum_{j\geq 0} c_j(q)\frac{t^j}{j!} = \frac{t}{(1-qt)^{-1/q} - 1}. \tag{22}$$

The limit as q approaches 0 (taken in $\mathbb{C}$, of course) of the function that appears in the right hand side of this last equality is the exponential generating function for the Bernoulli numbers that we wrote in (21).

Proof. Setting q to 0 in (20) we see that for all $j \in \mathbb{N}_0$ we have that

$$\sum_{i=0}^{j} c_i(0)\binom{j+1}{i} = \delta_{j,0},$$

and comparing this with the defining recurrence equation for the Bernoulli numbers shows that for all $j \in \mathbb{N}_0$ the constant term of c_j is B_j. As $c_0(q) = 1$ and $c_1(q)$ is the constant polynomial 1, it is clear that its degree and its leading coefficient are 0 and $(-1)^0 0! G_0$, respectively. On the other hand, if $j > 0$, then according to (20) we have that

$$\frac{c_j(q)}{j!} = -\sum_{i=0}^{j-1} \frac{(1)_{q,j+1-i}}{(j+1-i)!} \frac{c_i(q)}{i!}$$

and, since $(1)_{q,j+1-i}$ is a polynomial of degree $j-i$ and leading coefficient $(j-i)!$, the first claim of the lemma follows by induction from the definition of the Gregory coefficients and the fact that they are all non-zero.

Finally, the left hand side of the defining equation (20) is the coefficient of t^j in the product

$$\sum_{j\geq 0} c_j(q)\frac{t^j}{j!} \cdot \sum_{j\geq 0} \frac{(1)_{q,j+1}}{(j+1)!} t^j,$$

whose second factor sums to $((1-qt)^{-1/q}-1)/t$: the equality (22) follows from this. □

Using the polynomials $c_j(q)$ we are able to write down explicitly the elements Φ_j from Lemma 4.1.

Proposition 4.3. For each $j \geq 1$ we have

$$\Phi_j = \frac{1}{j}\sum_{i=0}^{j-1} \binom{j}{i} c_i(N-1) x^{i(N-1)} y^{j-i}. \tag{23}$$

Proof. Let us work in the algebra $A[[t]]$ of formal power series with coefficients in A. An easy induction shows that $\operatorname{ad}(y)^j(x) = (1)_{N-1,j} x^{1+j(N-1)}$ for all $j \in \mathbb{N}_0$, so that

$$e^{yt} x e^{-yt} = \sum_{j\geq 0} \operatorname{ad}(y)^j(x) \frac{t^j}{j!} = \sum_{j\geq 0} (1)_{N-1,j} x^{1+j(N-1)} t^j = x\left(1-(N-1)x^{N-1}t\right)^{-\frac{1}{N-1}}$$

and

$$[e^{yt}, x] = \left(e^{yt}xe^{-yt} - x\right)e^{yt} = x\left(\left(1-(N-1)x^{N-1}t\right)^{-\frac{1}{N-1}} - 1\right)e^{yt}. \tag{24}$$

If we write Ψ_j the right hand side of the equality (23) that we want to prove, we have that

$$\begin{aligned}\sum_{j\geq 1} \Psi_j \frac{t^{j-1}}{(j-1)!} &= \sum_{j\geq 0} c_j(N-1)\frac{x^{j(N-1)}t^j}{j!} \cdot \sum_{j\geq 1} \frac{y^j t^j}{j!} \\ &= \frac{x^{N-1}t}{(1-(N-1)x^{N-1}t)^{-1/(N-1)} - 1} \cdot \frac{e^{yt}-1}{t},\end{aligned}$$

so that

$$\sum_{j\geq 1} [\Psi_j, x] \frac{t^{j-1}}{(j-1)!} = \frac{x^{N-1}t}{(1-(N-1)x^{N-1}t)^{-1/(N-1)} - 1} \cdot \left[\frac{e^{yt}-1}{t}, x\right] = x^N e^{yt},$$

using (24) to obtain the last equality. Looking at the coefficients in these series, we see that $[\Psi_j, x] = x^N y^{j-1}$ for all $j \in \mathbb{N}$. Since Ψ_j is homogeneous of degree $j(N-1)$ and belongs to Ay, we can conclude that $\Psi_i = \Phi_j$, as the proposition claims. □

Let us single out three special cases of this proposition.

- If $N = 1$, then for all $j \geq 1$ we have

$$\Phi_j = \frac{1}{j}\sum_{i=0}^{j-1}\binom{j}{i}B_i y^{j-i} = \frac{B_j(y) - B_j}{j}, \tag{25}$$

  with $B_j(t) \coloneqq \sum_{i=0}^{j}\binom{j}{i}B_i t^{j-i}\mathbb{Q}[t]$, the usual jth Bernoulli polynomial. The last member of this equality appears in the famous Faulhaber formula for the sums of powers of the first integers: for any n, $p \in \mathbb{N}$ we have

$$\frac{B_p(n) - B_p}{p} = \sum_{k=0}^{n-1} k^{p-1}.$$

  It would be interesting to know what is behind this.

- If $N = 2$, then the right hand side of the equality (22) that appears in Lemma 4.2 with $q = N-1 = 1$ is simply $1 - t$, so that

$$c_0(1) = 1, \qquad c_1(1) = -1, \qquad c_j(1) = 0 \quad \text{if } j \geq 2.$$

  It follows from this and the proposition that for all $j \in \mathbb{N}$ we have

$$\Phi_j = \frac{y^j - jxy^{j-1}}{j}.$$

- If $N = 3$, then the right hand side of (22) with $q = N - 1 = 2$ simplifies to

$$\frac{\sqrt{1-2t} - (1-2t)}{2},$$

  and we obtain its Taylor series at once from Newton's binomial series, getting

$$c_0(2) = 1, \qquad c_1(2) = -\frac{3}{2}, \qquad c_j(2) = -\frac{(2j-3)!!}{2} \quad \text{if } j \geq 2.$$

  From this we can get explicit yet unenlightening descriptions of the elements Φ_j.

If $N \geq 4$ and we put $q = N - 1$ in (22), then the coefficients of the series do not seem to have a simple form — they do not appear in the OEIS [43].

5 Explicit representatives for classes in $\mathrm{HH}^1(A)$

In this section, as promised, we will give explicit derivations representing the elements of $\mathrm{HH}^1(A)$ when $N \geq 2$. We start with a simple observation that allows us to show that some derivations are not inner.

Lemma 5.1. Let $d : A \to A$ be a derivation.

(i) If $d(x) = 0$, then $d(y) \in \Bbbk[x]$.

(ii) If d is inner, then $d(x) \in x^N A$. If additionally $d(x) = 0$, then also $d(y) \in x^N \Bbbk[x]$.

Proof. (*i*) If $d(x) = 0$, then $0 = d([y, x] - x^N) = [d(y), x]$, and $d(y)$ is in the centralizer of x in A, which we know to be equal to $\Bbbk[x]$.

(*i*) If d is inner, then there is an $u \in A$ such that $d(a) = [u, a]$ for all $a \in A$ and, in particular, $d(x) = [u, x] \in [A, x] = x^N A$. If additionally $d(x) = 0$, then $[u, x] = 0$ and therefore $u \in \Bbbk[x]$: it follows from this that $d(y) = [u, y] = x^N u' \in x^N \Bbbk[x]$. □

The simplest derivations of our algebra are those of non-positive degree. We describe them in the following result.

Proposition 5.2. Suppose that $N \geq 2$. For each $l \in [\![-N+1, 0]\!]$ there is a derivation $\partial_l : A \to A$ of degree l such that

$$\partial_l(x) = 0, \qquad \partial_l(y) = x^{l+N-1},$$

and there is a derivation $E : A \to A$ of degree 0 such that

$$E(x) = x, \qquad E(y) = (N-1)y.$$

For each $l \in [\![-N+1, -1]\!]$ the space $\mathrm{HH}^1(A)_l$ is freely spanned by the class of ∂_l, and $\mathrm{HH}^1(A)_0$ is freely spanned by ∂_0 and E.

The derivation ∂_0 is the same one we encountered in Corollary 2.10.

Proof. A very simple calculation shows that there are indeed derivations of A as described in the statement of the proposition, and they manifestly have the degrees given there. None of the derivations $\partial_{-N+1}, \ldots, \partial_0$ and E is inner — this follows immediately from the second part of Lemma 5.1, because $N \geq 2$ — and then, in view of Proposition 3.4, we see that second claim of the proposition holds. □

To deal with classes in $\mathrm{HH}^1(A)$ of positive degree, we start by showing that they have a representative with a useful normalization:

Lemma 5.3. Suppose that $N \geq 2$. Let l be a positive integer, and let i and j be the integers such that $l + 1 = i + j(N-1)$, $j \geq 0$, and $1 \leq i \leq N-1$. There exists a derivation $d : A \to A$ that is homogeneous of degree l, is not inner, and has $d(x) = x^i y^j$.

Proof. Let l be a positive integer. The vector space $\mathrm{HH}^1(A)_l$ has dimension 1. We pick a derivation $d : A \to A$ homogeneous of degree l whose cohomology class spans that vector space. The element $d(x)$ of A is then homogeneous of degree $l+1$, and we can write it in the form $u + x^N v$ with u an homogeneous element of $P := \sum_{k=0}^{N-1} x^k \Bbbk[y]$ of degree $l+1$, and v a homogeneous element of A of degree $l+1-N$. There is a $w \in A$ such that $[w, x] = x^N v$, and it can be taken to be homogeneous of degree l: the derivation $d' := d - \mathrm{ad}(w)$ is then homogeneous of degree l, cohomologous to d, and has $d'(x) \in P$. The upshot of all this is that we could have simply chosen our original derivation d so that

$d(x)$ is in P from the start, and we do so now.

An homogeneous element of degree $l+1$ in P, such as $d(x)$, is a linear combination of the monomials of the form $x^r y^s$ with $r + s(N-1) = l+1$ and $0 \leq r < N$. We consider now two cases.

- Suppose first that $N-1 \nmid l+1$, so that in each such monomial we have in fact that $1 \leq r < N-1$, and therefore that r and s are necessarily equal to i and j, respectively: in particular, there is exactly one such monomial, $x^i y^j$, and there is thus a scalar α such that $d(x) = \alpha x^i y^j$.

- Suppose next that $N-1 \mid l+1$, so that $i = N-1$. There are now two monomials of degree $l+1$ in P, namely $x^{N-1}y^j$ and y^{j+1}, and there are scalars α and β such that $d(x) = \alpha x^{N-1}y^j + \beta y^{j+1}$. As

$$
\begin{aligned}
0 = d([y,x] - x^N) &= [d(y), x] + [y, d(x)] - \sum_{s+1+t=N} x^s d(x) x^t \\
&= [d(y), x] + \alpha(N-1)x^{2N-2}y^j - \alpha \sum_{s+1+t=N} x^{s+N-1}y^j x^t - \beta \sum_{s+1+t=N} x^s y^{j+1} x^t \\
&\equiv -\beta N x^{N-1} y^{j+1} \mod (F_j + x^N A),
\end{aligned}
$$

  the scalar β is actually 0.

The end result of all this is that in any case there is a scalar α such that $d(x) = \alpha x^i y^j$. If α is 0, then the first part of Lemma 5.1 tells us that $d(y) \in \Bbbk[x]$ and, since $d(y)$ is a homogeneous element of degree $l+N-1$, that in fact there is a scalar γ such that $d(y) = \gamma x^{l+N-1}$: this is impossible, since in that case we have $d = \gamma \operatorname{ad}(x^l)$, and the derivation d is not inner. The derivation $\alpha^{-1}d$ thus satisfies the condition we want. □

With the same idea that we used in the end of this proof we can also obtain the following criterion that allows us to prove a derivation is inner:

Lemma 5.4. Suppose that $N \geq 2$. Let l be a positive integer, and let i and j be the integers such that $l+1 = i + j(N-1)$, $j \geq 0$, and $1 \leq i \leq N-1$. A homogeneous derivation $d : A \to A$ of degree l such that $d(x) \in F_{j-1}$ is inner.

Proof. Let $d : A \to A$ be a homogeneous derivation of degree l such that $d(x)$ belongs to F_{j-1}. There are then scalars $a_1, \dots, a_j$ in $\Bbbk$ such that $d(x) = \sum_{k=1}^{j} a_k x^{i+k(N-1)} y^{j-k}$ and therefore $d(x) \in x^N A$. As we know, this implies that there is an element u in A, which can be chosen of degree l, such that $d(x) = [u, x]$, and therefore the derivation $d' := d - \operatorname{ad}(u)$ is homogeneous of degree l and vanishes on x. According to the first part of Lemma 5.1, we have $d'(y) \in k[x]$, so that in view of the homogeneity of d' there is a scalar b in $\Bbbk$ such that $d'(y) = bx^{l+N-1} = -\frac{b}{l}[x^l, y]$. It follows from this that $d = \operatorname{ad}(u) - \frac{b}{l}\operatorname{ad}(x^l)$, and this proves the lemma. □

We need yet one more commutation identity. We promise it is the last one.

Lemma 5.5. For each element u of A we have that

$$\sum_{s+1+t=N} x^s u x^t - N x^{N-1} u = \left[\sum_{s+2+t=N} (s+1) x^s u x^t, x \right].$$

Proof. The identity can be proved by expanding the commutators appearing in its right hand side and simplifying. □

Using all these results, we can finally exhibit representatives for classes in $\mathrm{HH}^1(A)$:

Proposition 5.6. Suppose that $N \geq 2$. Let l be a positive integer, and let i and j be the integers such that $l+1 = i + j(N-1)$, $j \geq 0$, and $1 \leq i \leq N-1$. The vector space $\mathrm{HH}^1(A)_l$ is spanned by a derivation $\partial_l : A \to A$ that is homogeneous of degree l and such that

$$\partial_l(x) = x^i y^j, \qquad \partial_l(y) = \sum_{s+2+t=N} (s+1) x^{s+i} y^j x^t + (N-i) x^{i-1} \Phi_{j+1}.$$

The element Φ_j appearing here is the one from Proposition 4.3 of Section 4. In this proposition the integer l is assumed to be positive: we can observe that if we allow it to be 0, then the formulas in the statement actually do also produce a derivation of degree 0 that maps x and y to x and to $N(N-1)x^{N+1}/2 + (N-1)y$, and that this is the derivation $E + N(N-1)\partial_0/2$ that we have already found in Proposition 5.2.

Proof. Let $d : A \to A$ be a derivation that is homogeneous of degree l and not inner, and that satisfies the condition of Lemma 5.3, so that $d(x) = x^i y^j$. As $[y,x] - x^N = 0$ in A, we have, using the result of Lemma 5.5, that

$$\begin{aligned} [d(y), x] &= \sum_{s+1+t=N} x^s d(x) x^t - [y, d(x)] \\ &= \left[\sum_{s+2+t=N} (s+1) x^{s+i} y^j x^t, x \right] + (N-i) x^{i+N-1} y^j, \end{aligned}$$

and therefore that

$$\left[d(y) - \sum_{s+2+t=N} (s+1) x^{s+i} y^j x^t - (N-i) x^{i-1} \Phi_{j+1}, x \right] = 0.$$

If follows from this that there exists an $f \in \Bbbk[x]$ such that

$$d(y) = \sum_{s+2+t=N} (s+1) x^{s+i} y^j x^t + (N-i) x^{i-1} \Phi_{j+1} + f.$$

Clearly f has to be homogeneous of degree $l+N-1$, so equal to $(l+N-1)\lambda x^{l+N-1}$ for some $\lambda \in \Bbbk$. The derivation $\partial_l := d - \lambda \operatorname{ad}(x^{l-1} y)$, which is cohomologous to d, is as in the statement of the theorem. □

We remark the nice fact that we were able exhibit the derivation in Proposition 5.6 without needing to prove that it actually is a derivation — which is probably a very unpleasant calculation!

6 The Lie algebra structure on $\mathrm{HH}^1(A)$

Now that we have explicit derivations whose classes freely span $\mathrm{HH}^1(A)$ we can compute the canonical Lie algebra structure of this vector space. In what follows we will write $\sim$ for the relation of cohomology between derivations of A.

Lemma 6.1. Let l and m be two integers such that $l \geq -N+1$, $m \geq -N+1$, and let i, j, u and v be the unique integers such that $l+1 = i + j(N-1)$, $m+1 = u + v(N-1)$, i, $u \in [\![1, N-1]\!]$ and j, $v \geq -1$. Suppose that, moreover, $l \leq m$.

(*i*) If $i + u > N$, then $[\partial_l, \partial_m] \sim 0$.

(*ii*) If $i + u \leq N$, then $[\partial_l, \partial_m]$ is cohomologous to

$$\begin{cases} \left(u - i + \dfrac{N-i}{j+1}v - \dfrac{N-u}{v+1}j\right)\partial_{l+m} & \text{if } l \geq 1 \text{ and } m \geq 1; \\ 0 & \text{if } l \leq -1 \text{ and } m \leq -1 \text{ or } lm = 0 \\ v\partial_{l+m} & \text{if } lm < 0 \text{ and } l+m \geq 1; \\ -(l+m)\partial_{l+m} & \text{if } lm < 0 \text{ and } l+m \leq -1; \\ E & \text{if } l = -N+1 \text{ and } m = N-1. \end{cases}$$

Here we cannot have that $lm < 0$, $l + m = 0$ and $l > -N+1$.

(*iii*) Finally, $[E, \partial_m] = m\partial_m$.

Proof. The third claim of the lemma can be proved by a very simple direct calculation that we omit. We will split the calculation that proves the first two claims in several parts, and the following table describes in which part we consider each particular combination of indices l and m.

	$l \leq -1$	$l = 0$	$l \geq 1$
$m \leq -1$	Second	Third	Fourth
$m = 0$	Third	Third	Third
$m \geq 1$	Fourth	Third	First

First part. Let us suppose first that $l \geq 1$ and $m \geq 1$, so that $j \geq 0$ and $v \geq 0$. If $j \geq 1$, then in view of Proposition 5.6 and Lemma 4.1 we have that modulo F_{j-1}

$$\partial_l(x) \equiv x^i y^j$$

and

$$\partial_l(y) = \sum_{s+2+t=N} (s+1)x^{s+i}y^j x^t + (N-i)x^{i-1}\Phi_{j+1}$$

$$\equiv \frac{N(N-1)}{2}x^{i+N-2}y^j + (N-i)x^{i-1}\left(\frac{1}{j+1}y^{j+1} - \frac{N}{2}x^{N-1}y^j\right)$$
$$= \frac{N-i}{j+1}x^{i-1}y^{j+1} + \frac{N(i-1)}{2}x^{i+N-2}y^j,$$

and using this we can see that

$$\partial_l(\partial_m(x)) = \partial_l(x^u y^v) \equiv \left(u + \frac{N-i}{j+1}v\right)x^{i+u-1}y^{j+v} \mod F_{j+v-1}. \tag{26}$$

If instead $j = 0$, then Lemma 4.1 gives us a slightly different formula for Φ_{j+1} and what we have is that $\partial_l(x) = x^i$ and

$$\partial_l(y) = \sum_{s+2+t=N}(s+1)x^{s+i+t} + (N-i)x^{i-1}\Phi_1 = \frac{N(N-1)}{2}x^{i+N-2} + (N-i)x^{i-1}y$$

and using this we can see that the congruence (26) also holds when $j = 0$.
By symmetry we get from (26) a formula for $\partial_m(\partial_l(x))$, and finally conclude that

$$[\partial_l, \partial_m](x) \equiv \left(u - i + \frac{N-i}{j+1}v - \frac{N-u}{v+1}j\right)x^{i+u-1}y^{j+v} \mod F_{j+v-1}. \tag{27}$$

There are now two cases.

- If $i + u > N$, then we have

  $$l + m + 1 = (i + u - N) + (j + v + 1)(N - 1), \qquad 1 \le i + u - N \le N - 1,$$

  and the above formula (27) tells us that the derivation $[\partial_l, \partial_m]$, which is homogeneous of degree $l+m$, maps x into F_{j+v}: it follows from Lemma 5.4 that $[\partial_l, \partial_m]$ is inner in this situation.

- Suppose now that $i + u \le N$, so that

  $$l + m + 1 = (i + u - 1) + (j + v)(N - 1), \qquad 1 \le i + u - 1 \le N - 1,$$

  and let α be the scalar that appears between parentheses in the right hand side of the congruence (27). We then have that the derivation $[\partial_l, \partial_m] - \alpha\partial_{l+m}$, which is homogeneous of degree $l + m$, maps x into F_{j+v-1}: again using Lemma 5.4 we see that that difference is inner. We thus have that $[\partial_l, \partial_m] \sim \alpha\partial_{l+m}$ in this situation.

Second part. If $l \le -1$ and $m \le -1$, then it follows immediately from the description of ∂_l and ∂_m given in Proposition 5.2 that $[\partial_l, \partial_m] = 0$, independently of whether the inequality $i + u \ge N$ holds or not.

Third part. We now want to prove that the derivation $[\partial_l, \partial_m]$ is inner if one of l or m is zero and, of course, we can suppose that it is l that is zero. Let us notice that in this situation we have that $i = 1$, $j = 0$ and $i + u \le N$.
If $m \ge 1$, then $\partial_m(\partial_0(x)) = 0$ and

$$\partial_0(\partial_m(x)) = \partial_0(x^u y^v) = \sum_{s+1+t=v} x^u y^s x^{N-1} y^t \in F_{v-1},$$

so that $[\partial_0, \partial_m]$ is a homogeneous derivation of degree m that maps x into F_{v-1}: as $m+1 = u + v(N-1)$ and $1 \leq u \leq N-1$, we know from Lemma 5.4 that that commutator is inner. If instead $m \leq 0$, then using the description of ∂_m given in Proposition 5.2 can compute directly that $[\partial_0, \partial_m] = 0$.

FOURTH PART. Finally, let us consider the case in which $lm < 0$ and, without any loss of generality thanks to anti-symmetry, $l \leq -1$ and $m \geq 1$. We have that

$$\begin{gathered} j = -1, \qquad i = l + N, \qquad v \geq 0, \\ l + m + 1 = l + u + v(N-1), \qquad -N+2 \leq l+u \leq N-2, \end{gathered} \tag{28}$$

and that, as $\partial_m(\partial_l(x)) = 0$,

$$[\partial_l, \partial_m](x) = \partial_l(\partial_m(x)) = \partial_l(x^u y^v) = \sum_{s+1+t=v} x^u y^s x^{l+N-1} y^t. \tag{29}$$

Let us suppose first that $i + u > N$.

- If $v = 0$ and $i + u > N + 1$, then $l + m = i - N + u - 1 \geq 1$, so that the degree of $[\partial_l, \partial_m]$ is positive, and, since $1 \leq i - N + u = l + u \leq N - 2$, it follows from the first equality in (28), the formula (29) tells us that $[\partial_l, \partial_m](x) = 0 \in F_{v-1}$, and Lemma 5.4, that $[\partial_l, \partial_m] \sim 0$.
- If $v = 0$ and $i + u = N + 1$, then the formula (29) tells us again that $[\partial_l, \partial_m](x) = 0$, and using the definitions of ∂_l and ∂_m we see that

  $$\partial_l(\partial_m(y)) = \partial_l\left(\frac{N(N-1)}{2} x^{u+N-2} + (N-u)x^{u-1}y\right) = (N-u)x^{N-1}$$

  and

  $$\partial_m(\partial_l(y)) = \partial_m(x^{l+N-1}) = (l+N-1)x^{N-1} = (i-1)x^{N-1}$$

  so that $[\partial_l, \partial_m](y) = (N - u - i + 1)x^{N-1} = 0$ and, therefore, $[\partial_l, \partial_m] = 0$.
- If $v \geq 1$, then $l+m = i-N+u+v(N-1)-1 > N-2 \geq 0$, so that the degree of the derivation $[\partial_l, \partial_m]$ is positive, and since $l+m+1 = i-N+u+v(N-1)$, $1 \leq i-N+u \leq N-2$, and $[\partial_l, \partial_m](x) \in F_{v-1}$ because of (29), Lemma 5.4 tells us that $[\partial_l, \partial_m] \sim 0$.

Let us suppose, finally, that $i + u \leq N$. We then have that

$$l + m + 1 = (i + u - 1) + (v - 1)(N - 1), \qquad 1 \leq i + u - 1 \leq N - 1, \tag{30}$$

and, as before, we consider several cases.

- If $l + m \geq 1$, then (30) implies that $\partial_{l+m}(x) = x^{i+u-1}y^{v-1}$ while (29) implies that $[\partial_l, \partial_m](x) \equiv vx^{i+u-1}y^{v-1} \mod F_{v-1}$: it follows from this that $[\partial_l, \partial_m] - v\partial_{l+m}$, a homogeneous derivation of degree $l + m$, maps x into F_{v-2}, so that Lemma 5.4 and (30) let us conclude that it is inner and, therefore, that $[\partial_l, \partial_m] \sim v\partial_{l+m}$.
- Suppose now that $l + m \leq -1$. We have that $v = 0$, for otherwise $v \geq 1$ and

  $$0 \geq l + m + 1 = i - N + u + v(N-1) \geq i - N + u + N - 1 = i + u - 1 \geq 1,$$

which is absurd. From (29) we see then that $[\partial_l, \partial_m](x) = 0$ and we can compute directly that

$$\partial_l(\partial_m(y)) = \partial_l\left(\frac{N(N-1)}{2}x^{u+N-2} + (N-u)x^{u-1}y\right) = (N-u)x^{l+m+N-1}$$

and

$$\partial_m(\partial_l(y)) = \partial_m(x^{l+N-1}) = (l+N-1)x^{l+m+N-1},$$

so that $[\partial_l, \partial_m](y) = -(l+m)x^{l+m+N-1}$, and therefore $[\partial_l, \partial_m] = -(l+m)\partial_{l+m}$.

- It is easy to check that we cannot have $l + m = 0$ and $l > -N + 1$.
- We have one last case to consider: that in which $l + m = 0$ and $l = -N + 1$. We then have that $m = N - 1$, $u = 1$ and $v = 1$, and computing directly we see that $[\partial_l, \partial_m]$ maps x and y to x and $(N - 1)y$, so that it is equal to E.

We have proved all the claims in the lemma. □

It will be convenient to work with a different basis of $\mathrm{HH}^1(A)$ with respect to which the structure constants of the Lie bracket are simpler. If l is an integer such that $l \geq -N+1$, there is a unique way of choosing integers i and j such that $i \in [\![1, N-1]\!]$, $j \geq -1$ and $l + 1 = i + j(N - 1)$, and we define

$$L_l := \begin{cases} -\dfrac{j+1}{N-1}\partial_l & \text{if } l \geq 1; \\ -\dfrac{1}{N-1}E & \text{if } l = 0; \\ \dfrac{l}{N-1}\partial_l & \text{if } l \leq -1. \end{cases}$$

Clearly the classes of ∂_0 and the derivations L_l with $l \geq -N + 1$ freely span $\mathrm{HH}^1(A)$.

Corollary 6.2. Let l and m be two integers such that $l \geq -N+1$, $m \geq -N+1$, and let i, j, u and v be the unique integers such that $l+1 = i+j(N-1)$, $m+1 = u+v(N-1)$, i, $u \in [\![1, N-1]\!]$ and j, $v \geq -1$. We have that

$$[L_l, L_m] \sim \begin{cases} 0 & \text{if } i+u > N \text{ or } l+m < -N+1; \\ \dfrac{l(v+1) - m(j+1)}{N-1}L_{l+m} & \text{if } i+u \leq N. \end{cases} \tag{31}$$

Proof. Let us consider first the situation in which $i + u > N$. Neither i nor u is equal to 1: were $i = 1$, for example, we would have that $i+u \leq 1+(N-1) = N$. It follows then, in particular, that $l \neq 0$ and $m \neq 0$, so there are scalars a and b such that $L_l = a\partial_l$ and $L_m = b\partial_m$, and therefore that $[L_l, L_m] = ab[\partial_l, \partial_m] \sim 0$ because of the first part of Lemma 6.1. This proves the first line in (31).

In order to prove the second line let us suppose from now on that $i + u \leq N$ and, as if bound by a hex cast upon us, handle each of the several following special cases separately.

- Suppose first that $l \leq -1$ and $m \leq -1$. As before, there are scalars a and b such that $L_l = a\partial_l$ and $L_m = b\partial_m$, and therefore $[L_l, L_m] = ab[\partial_l, \partial_m] \sim 0$ according to Lemma 6.1. On the other hand, we have that $j = -1$ and $v = -1$, so that the numerator of the fraction appearing in (31) is 0 and that cohomology holds in this case.

- Suppose now that $l \geq 1$ and $m \geq 1$. Using Lemma 6.1 we see that $[L_l, L_m]$ is

$$\frac{(j+1)(v+1)}{(N-1)^2}[\partial_l, \partial_m] \sim \frac{(j+1)(v+1)}{(N-1)^2}\left(u - i + \frac{N-i}{j+1}v - \frac{N-u}{v+1}j\right)\partial_{l+m},$$

and a little calculation shows that this is equal to

$$\frac{l(v+1) - m(j+1)}{N-1}L_{l+m}.$$

- Suppose next that $lm = 0$, so that one of l or m is zero — and by anti-symmetry we can suppose additionally that l is, so that $j = 0$. If $m \geq 1$, then

$$[L_l, L_m] = \left[-\frac{1}{N-1}E, -\frac{v+1}{N-1}\partial_m\right] \sim m\frac{v+1}{(N-1)^2}\partial_m = -\frac{m}{N-1}L_m,$$

if $m = 0$, then $[L_l, L_m] = 0$, and if $m \leq -1$, then

$$[L_l, L_m] = \left[-\frac{1}{N-1}E, \frac{m}{N-1}\partial_m\right] \sim -\frac{m^2}{(N-1)^2}\partial_m = -\frac{m}{N-1}L_m$$

In any case, we see that (31) holds.

At this point we are left with considering the case in which $lm < 0$ and, thanks to anti-symmetry, we can further suppose that in fact $l \leq -1$ and $m \geq 1$.

- If $l \leq -1$, $m \geq 1$ and $l + m \geq 1$, then $j = -1$, $1 \leq i + u - 1 \leq N - 1$, and $l + m + 1 = (i + u - 1) + (v - 1)(N - 1)$, so that $L_{l+m} = -v\partial_{l+m}/(N-1)$ and

$$[L_l, L_m] = \left[\frac{l}{N-1}\partial_l, -\frac{v+1}{N-1}\partial_m\right] = -\frac{l(v+1)v}{(N-1)^2}\partial_{l+m} = \frac{l(v+1)}{N-1}L_{l+m}.$$

- Suppose now that $l \leq -1$, $m \geq 1$ and $l + m = 0$. We then have that $j = -1$, that $0 = l + m = (i + u - 2) + (v - 1)(N - 1)$, and that $0 \leq i + u - 2 \leq N - 2$, so that in fact $i = 1$, $u = 1$ and $v = 1$: this tells us that $l = -N + 1$ and $m = N - 1$, so that

$$[L_l, L_m] = \left[\frac{-N+1}{N-1}\partial_{-N+1}, -\frac{2}{N-1}\partial_{N-1}\right] = \frac{2}{N-1}E = -2L_0$$

- Finally, if $l \leq -1$, $m \geq 1$ and $l + m \leq -1$, then

$$[L_l, L_m] = \left[\frac{l}{N-1}\partial_l, -\frac{v+1}{N-1}\partial_m\right] = \frac{l(v+1)(l+m)}{(N-1)^2}\partial_{l+m} = \frac{l(v+1)}{N-1}L_{l+m}.$$

In all cases what we have found is a specialization of the formula that appears in (31). □

For each integer l let us write $\rho(l)$ for the element of $[\![0, N-2]\!]$ that is the remainder of the division of l by $N - 1$. Above we have used many times the fact that l determines uniquely integers i and j such that $l + 1 = i + j(N - 1)$: we have $\rho(l) = i - 1$. Clearly, whenever l and m are integers we have that

Claim 6.3. $\rho(l + m) = \rho(l) + \rho(m)$ if $\rho(l) + \rho(m) \leq N - 2$.

From now on we will, in contexts where this does not lead to confusion, take the liberty of not making an explicit difference between a derivation of A and its class in $\mathrm{HH}^1(A)$.

Let us recall that the Lie algebra $\mathrm{Der}(\Bbbk[t])$ of derivations of the polynomial algebra $\Bbbk[t]$ is freely spanned as a vector space by the derivations

$$-t^{j+1}\frac{\mathrm{d}}{\mathrm{d}t}, \qquad j \in \mathbb{Z},\ j \geq -1,$$

which are such that

$$\left[-t^{j+1}\frac{\mathrm{d}}{\mathrm{d}t}, -t^{v+1}\frac{\mathrm{d}}{\mathrm{d}t}\right] = (j-v)L_{j+v} \tag{32}$$

whenever j and v are integers such that j, $v \geq -1$. This Lie algebra is often called the ***Witt algebra***, although the same name is more commonly applied to the Lie algebra $\mathrm{Der}(\Bbbk[t^{\pm 1}])$, which is different — there is a *third* Lie algebra called the Witt algebra, but it only occurs in positive characteristic. The Lie algebra $\mathrm{Der}(\Bbbk[x])$ is simple — in fact, David Jordan proved in [31] that the Lie algebra of derivations on an affine variety is simple exactly when the variety is smooth, and that applies here (see also Thomas Siebert's [50]), but one can prove this particular case very easily «by hand».

Proposition 6.4. For each $r \in \mathbb{N}_0$ let $\mathfrak{N}_r$ be the span of $\{L_l : \rho(l) \geq r\}$ in $\mathrm{HH}^1(A)$, so that, in particular, $\mathfrak{N}_r = 0$ when $r \geq N-2$ and there is a descending chain of subspaces

$$\mathfrak{N}_0 \supsetneq \underbrace{\mathfrak{N}_1 \supsetneq \cdots \supsetneq \mathfrak{N}_{N-2} \supsetneq \mathfrak{N}_{N-1} = 0}. \tag{33}$$

(*i*) For each r, $s \in [\![0, N-1]\!]$ we have that $[\mathfrak{N}_r, \mathfrak{N}_s] \subseteq \mathfrak{N}_{r+s}$.

(*ii*) For each $r \in [\![1, N-2]\!]$ we have that $\mathfrak{N}_{r+1} \subseteq [L_{-N+2}, \mathfrak{N}_r] \subseteq [\mathfrak{N}_1, \mathfrak{N}_r]$.

(*iii*) $\mathfrak{N}_1$ is a nilpotent ideal in $\mathrm{HH}^1(A)$, the chain underlined in (33) is its lower central series and, in particular, its nilpotency index is exactly $N-2$.

(*iv*) The center of $\mathrm{HH}^1(A)$ is the subspace $\Bbbk\partial_0$ spanned by ∂_0, the derived subalgebra of $\mathrm{HH}^1(A)$ is $\mathfrak{N}_0$, which is a perfect subalgebra, and

$$\mathrm{HH}^1(A) = Z \oplus \mathfrak{N}_0.$$

(*v*) There is an injective morphism of Lie algebras $\Phi : \mathrm{Der}(\Bbbk[t]) \to \mathfrak{N}_0$ that for all integers j such that $j \geq -1$ has

$$\Phi\left(-t^{j+1}\frac{\mathrm{d}}{\mathrm{d}t}\right) = L_{j(N-1)}.$$

(*vi*) $\mathfrak{N}_0 = \Phi(\mathrm{Der}(\Bbbk[t])) \oplus \mathfrak{N}_1$ and $\mathfrak{N}_1$ is the unique maximal ideal of $\mathfrak{N}_0$.

Proof. (*i*) Let r and s be elements of $[\![0, N-1]\!]$, let l and m be integers such that $l \geq -N+1$, $m \geq -N+1$, $\rho(l) \geq r$ and $\rho(m) \geq s$, and let i, j, u, v be the integers such that i, $u \in [\![1, N-1]\!]$, j, $v \geq -1$, $l+1 = i + j(N-1)$ and $m+1 = u + v(N-1)$, so that $i = \rho(l)+1$ and $u = \rho(m)+1$. If $\rho(l)+\rho(m) > N-2$, then we know from Corollary 6.2 that $[L_l, L_m] = 0 \in \mathfrak{N}_{r+s}$. If instead $\rho(l)+\rho(m) \leq N-2$, then $\rho(l+m) = \rho(l)+\rho(m) \geq r+s$ and that corollary tells us now that there is a scalar $\lambda \in \Bbbk$ such that $[L_l, L_m] = \lambda L_{l+m} \in \mathfrak{N}_{r+s}$. This shows that $[\mathfrak{N}_r, \mathfrak{N}_s] \subseteq \mathfrak{N}_{r+s}$.

(ii) Let r be an element of $[\![1, N-2]\!]$, so that in particular we have $N \geq 3$, and let m be an integer such that $m \geq -N+1$ and $\rho(m) \geq r+1$. There is an integer v such that $v \geq -1$ and $m+1 = \rho(m)+1+v(N-1)$, and we set $m' := \rho(m) - 1 + (v+1)(N-1)$. Since $2 \leq r+1 \leq \rho(m) \leq N-2$, we have that $\rho(m') = \rho(m) - 1 \geq r$ and, therefore, that $L_{m'} \in \mathfrak{N}_{r+1}$. Since $\rho(m) \geq 2$ and $v \geq -1$, we have $m' \geq 1$. Since $(-N+2) + m' = m \neq 0$ because $\rho(m) \geq r+1 > 0$, it follows from Lemma 6.1 that

$$[L_{-N+2}, L_{m'}] = -\frac{N-2}{N-1}(v+2)L_m$$

and, in any case, that $L_m \in [L_{-N+2}, \mathfrak{N}_r]$. We have proved that $\mathfrak{N}_{r+1} \subseteq [L_{-N+2}, \mathfrak{N}_r]$.

Lemma 6.1 tells us that the class of the derivation ∂_0 is central in $\mathrm{HH}^1(A)$. On the other hand, a central element of $\mathrm{HH}^1(A)$ has to commute with E, so has degree 0: as E is not central, we see that the center of $\mathrm{HH}^1(A)$ es precisely $\Bbbk\partial_0$. Clearly $\mathrm{HH}^1(A) = \Bbbk\partial_0 \oplus \mathfrak{N}_0$ and the derived subalgebra of $\mathrm{HH}^1(A)$ is $[\mathfrak{N}_0, \mathfrak{N}_0]$.

The map Φ described in the lemma is immediately seen to be an injective morphism of Lie algebras — thanks to Corollary 6.2 and the formulas in (32) — and clearly $\mathfrak{N}_0 = \Phi(\mathrm{Der}(\Bbbk[t])) \oplus \mathfrak{N}_1$. As the algebra $\mathrm{Der}(\Bbbk[x])$ is simple, it is perfect, and thus $\Phi(\mathrm{Der}(\Bbbk[t]))$ is contained in $[\mathfrak{N}_0, \mathfrak{N}_0]$. As also $\mathfrak{N}_1 = [M_0, \mathfrak{N}_1] \subseteq [\mathfrak{N}_0, \mathfrak{N}_0]$, we see that $\mathfrak{N}_0$ is perfect and equal to the derived subalgebra of $\mathrm{HH}^1(A)$.

From (i) we have that $[\mathfrak{N}_0, \mathfrak{N}_1] \subseteq \mathfrak{N}_1$, so that the subspace $\mathfrak{N}_1$ is an ideal of $\mathfrak{N}_0$ and in $\mathrm{HH}^1(A)$. From (i) and (ii) we see that for each $r \in [\![1, N-2]\!]$ we have $[\mathfrak{N}_1, \mathfrak{N}_r] = \mathfrak{N}_{r+1}$, so that $\mathfrak{N}_1 \supsetneq \mathfrak{N}_2 \supsetneq \cdots \supsetneq \mathfrak{N}_{N-2}$ is the beginning of the lower central series of $\mathfrak{N}_1$. As $\mathfrak{N}_{N-2} = 0$ and $\mathfrak{N}_{N-1} \neq 0$, we see that $\mathfrak{N}_1$ is nilpotent of index exactly $N-2$. As $\mathfrak{N}_0/\mathfrak{N}_1 \cong \mathrm{Der}(\Bbbk[t]))$, a simple Lie algebra, $\mathfrak{N}_1$ is a maximal ideal in $\mathfrak{N}_0$.

Suppose, finally, that I is an ideal of $\mathfrak{N}_0$: as I is $\mathrm{ad}(E)$-invariant and $\mathrm{ad}(E) : \mathfrak{N}_0 \to \mathfrak{N}_0$ is diagonalizable and its eigenspaces are the homogeneous components of $\mathfrak{N}_0$, we see that I is a *homogeneous* ideal of $\mathfrak{N}_0$. The intersection $I \cap \Phi(\mathrm{Der}(\Bbbk[t])$ is zero, because it is an ideal in $\Phi(\mathrm{Der}(\Bbbk[t]))$, and this tells us that I is generated by homogeneous elements of degree not divisible by $N-1$ and thus that it is contained in $\mathfrak{N}_1$. This proves that $\mathfrak{N}_1$ is the unique maximal ideal of $\mathfrak{N}_0$. $\square$

An immediate consequence of this is that the number N is a derived invariant of the algebra A:

Corollary 6.5. If N and N' are two non-negative integers such that the algebras A_N and $A_{N'}$ are derived equivalent, then $N = N'$.

Proof. Let N and N' be two non-negative integers such that $N \leq N'$ and the algebras A_N and $A_{N'}$ are derived equivalent. According to a theorem of Bernhard Keller in [32] we then have that the Lie algebras $\mathrm{HH}^1(A_N)$ and $\mathrm{HH}^1(A_{N'})$ are isomorphic as Lie algebras. If $N \geq 2$, then Proposition 6.4 tells us that we can compute the number $N-2$ from the Lie algebra $\mathrm{HH}^1(A_N)$ as the nilpotency index of the unique maximal ideal of the quotient of $\mathrm{HH}^1(A_N)$ by its center is $N-2$, and we therefore have that $N = N'$.

It is easy to deal with the remaining possibilities. If $N = 0$, then A_N is the Weyl algebra and $\mathrm{HH}^1(A) = 0$: from Propositions 3.4 and 3.5 we see that then $N' = 0$. If $N = 1$, then we cannot have $N' > 1$, for in that case $\mathbf{dim}\,\mathrm{HH}^1(A_N) = 1 < \infty = \mathbf{dim}\,\mathrm{HH}^1(A_{N'})$, so again $N = N'$. □

The next thing we want to do is to obtain a computational criterion for a derivation of A to be the restriction of an inner derivation of the algebra A_x obtained by localizing at x. This is possible because A_x can also be obtained as a localization of a Weyl algebra, with which we can work with ease. The result we need is the following:

Proposition 6.6. Let $\mathscr{W}$ be the algebra freely generated by letters p and q subject to the relation

$$pq - qp = 1,$$

and let $\mathscr{W}_q$ be the localization of $\mathscr{W}$ at q. The first Hochschild cohomology space $\mathrm{HH}^1(\mathscr{W}_q)$ is one-dimensional and is spanned by the cohomology class of the derivation $\partial : \mathscr{W}_q \to \mathscr{W}_q$ such that

$$\partial(p) = q^{-1}, \qquad \partial(q) = 0.$$

There is a representation of $\mathscr{W}_q$ on the algebra $\Bbbk[t^{\pm 1}]$ of Laurent polynomials such that

$$p \triangleright f(t) = \frac{\mathrm{d}}{\mathrm{d}t} f(t), \qquad q \triangleright f(t) = t f(t)$$

for all $f \in \Bbbk[t^{\pm 1}]$. A derivation $\delta : \mathscr{W}_q \to \mathscr{W}_q$ is inner if and only if

$$\mathrm{Res}_0 \left(\delta(pq) \triangleright \frac{1}{t} \right) = 0. \tag{34}$$

Here $\mathrm{Res}_0 : \Bbbk[t^{\pm 1}] \to \Bbbk$ is the usual residue map that picks the coefficient of t^{-1} in its argument. Just as the Weyl algebra $\mathscr{W}$ is the algebra of regular differential operators on the affine line $\mathbb{A}^1$, the localization $\mathscr{W}_q$ is the algebra of regular differential operators on the punctured affine line $\mathbb{A}^1 \setminus \{0\}$ — and this is the representation $\triangleright$ that appears in the proposition. As this is a smooth affine scheme and our ground field has characteristic zero, we know that the Hochschild cohomology $\mathrm{HH}^*(\mathscr{W}_q)$ is isomorphic to the algebraic De Rham cohomology $\mathrm{H}^*(\mathbb{A}^1 \setminus \{0\})$. In particular, cohomology classes of derivations of $\mathscr{W}_q$ can be viewed as cohomology classes of 1-forms on $\mathbb{A}^1 \setminus \{0\}$: the condition (34) above for a derivation to be inner corresponds to the condition that a 1-form «integrate to zero along a curve around the puncture» for it to be a exact, so it is a version of Poincaré duality in our context. We will prove the proposition in a purely algebraic way, but geometry and various comparison maps could be used instead.

Proof. Let W be the subspace of $\mathscr{W}$ spanned by p and q, let $\hat{W}$ be its dual space, and let $(\hat{p}, \hat{q})$ be the ordered basis of $\hat{W}$ dual to (p, q). There is a projective resolution of $\mathscr{W}$ as a $\mathscr{W}$-bimodule of the form

$$\mathscr{W} \otimes \Lambda^2 W \otimes \mathscr{W} \xrightarrow{d} \mathscr{W} \otimes W \otimes \mathscr{W} \xrightarrow{d} \mathscr{W} \otimes \mathscr{W} \overset{\varepsilon}{\dashrightarrow} \mathscr{W}$$

with differentials such that

$$
\begin{aligned}
&d(1\otimes 1) = 1,\\
&d(1\otimes w\otimes 1) = w\otimes 1 - 1\otimes w, \qquad \forall\, w\in W,\\
&d(1\otimes p\wedge w\otimes 1) = p\otimes q\otimes 1 + 1\otimes p\otimes q - q\otimes p\otimes 1 - 1\otimes q\otimes p,
\end{aligned}
$$

and augmentation $\varepsilon : \mathscr{W}\otimes\mathscr{W}\to\mathscr{W}$ given by the multiplication of the algebra $\mathscr{W}$. The localization $\mathscr{W}_q$ of $\mathscr{W}$ at q is a $\mathscr{W}$-bimodule that is flat on both sides, so applying the functor $\mathscr{W}_q\otimes_{\mathscr{W}}(-)\otimes_{\mathscr{W}}\mathscr{W}_q$ to the resolution above gives a projective resolution

$$
\mathscr{W}_q\otimes\Lambda^2W\otimes\mathscr{W}_q \xrightarrow{d} \mathscr{W}_q\otimes W\otimes\mathscr{W}_q \xrightarrow{d} \mathscr{W}_q\otimes\mathscr{W}_q \overset{\varepsilon}{\dashrightarrow} \mathscr{W}_q\otimes_{\mathscr{W}}\mathscr{W}_q
$$

of the $\mathscr{W}_q$-bimodule $\mathscr{W}_q\otimes_{\mathscr{W}}\mathscr{W}_q$. As the map $\mathscr{W}_q\otimes_{\mathscr{W}}\mathscr{W}_q\to\mathscr{W}_q$ induced by the multiplication of $\mathscr{W}_q$ is an isomorphism of $\mathscr{W}_q$-bimodules, what we have is a projective resolution of $\mathscr{W}_q$ as a bimodule over itself. Applying to it the functor $\operatorname{Hom}_{\mathscr{W}_q^e}(-,\mathscr{W}_q)$ and doing standard identifications we obtain the complex

$$
\mathscr{W}_q \underset{s_1}{\overset{\delta}{\rightleftarrows}} \mathscr{W}_q\otimes W \underset{s_2}{\overset{\delta}{\rightleftarrows}} \mathscr{W}_q\otimes\Lambda^2\hat{W} \tag{35}
$$

with differentials such that for all a and b in $\mathscr{W}_q$ have

$$
\delta(a) = [p,a]\otimes\hat{p} + [q,a]\otimes\hat{q}, \qquad \delta(a\otimes\hat{p}+b\otimes\hat{q}) = \big([p,b]-[q,a]\big)\otimes\hat{p}\wedge\hat{q}.
$$

Let us write $\mathscr{X}$ for this complex of vector spaces, whose cohomology is $\mathrm{HH}^*(\mathscr{W}_q)$, the Hochschild cohomology we are trying to compute, up to canonical isomorphisms.

There is a $\mathbb{Z}$-grading on the algebra $\mathscr{W}_q$ that assigns to p and q degrees -1 and 1, respectively. On the other hand, the inner derivation $\operatorname{ad}(pq):\mathscr{W}_q\to\mathscr{W}_q$ is diagonalizable with spectrum $\mathbb{Z}$, and for each $n\in\mathbb{Z}$ the eigenspace of $\operatorname{ad}(pq)$ corresponding to the eigenvalue n is precisely the homogeneous component of $\mathscr{W}_q$ of degree n — this follows at once from the facts that $\operatorname{ad}(pq)(p)=-p$ and $\operatorname{ad}(pq)=q$, on one hand, and, on the other, that the set $\{p^iq^j : i\in\mathbb{N}_0, j\in\mathbb{Z}\}$ is a basis of $\mathscr{W}_q$.

We consider on the complex $\mathscr{X}$ the maps s_1 and s_2 indicated with dashed arrows in the diagram (35) above that are given by

$$
s_1(a\otimes\hat{p}+b\otimes\hat{q}) = aq+pb, \qquad s_2(a\otimes\hat{p}\wedge\hat{q}) = -pa\otimes\hat{p}+aq\otimes\hat{q}
$$

for all a and b in A, and define maps

$$
\begin{aligned}
&r_0 \coloneqq s_1\circ\delta : \mathscr{W}_q\to\mathscr{W}_q,\\
&r_1 \coloneqq \delta\circ s_1 + s_2\circ\delta : \mathscr{W}_q\otimes W\to\mathscr{W}_q\otimes W,\\
&r_2 \coloneqq \delta\circ s_1 : \mathscr{W}_q\otimes\Lambda^2W\to\mathscr{W}_q\otimes\Lambda^2W.
\end{aligned}
$$

Of course, in this way we obtain an endomorphism $r_* : \mathscr{X}\to\mathscr{X}$ of the complex $\mathscr{X}$ that is homotopic to zero. A simple calculation shows that

$$
r_0(a) = \operatorname{ad}(pq)(a),
$$

$$r_1(a\otimes\hat{p}+b\otimes\hat{q}) = (\mathrm{ad}(pq)(a)-a)\otimes\hat{p}+(\mathrm{ad}(pq)(b)+b)\otimes\hat{q},$$
$$r_2(a\otimes\hat{p}\wedge\otimes\hat{q}) = \mathrm{ad}(pq)(q)\otimes\hat{p}\otimes\hat{q}$$

for all choices of a and b in $\mathscr{W}_q$, and it follows from this that the endomorphism r_* is diagonalizable, since $\mathrm{ad}(pq)$ is. As a consequence of all this, the subcomplex $\mathscr{X}'$ of $\mathscr{X}$ spanned by all eigenvectors of the map r_* corresponding to non-zero eigenvalues is exact and a complement to $\ker r_*$, so that

$$\mathscr{X} = \mathscr{X}'\oplus\ker r_*. \tag{36}$$

In particular, the inclusion $\ker r_*\hookrightarrow\mathscr{X}$ is a quasi-isomorphism.
The kernel $\ker r_*$ is easily computed to be the complex

$$\Bbbk[pq] \xrightarrow{\delta} \big(\Bbbk[pq]q^{-1}\otimes\Bbbk\hat{p}\big)\oplus\big(\Bbbk[pq]q\otimes\Bbbk\hat{q}\big) \xrightarrow{\delta} \Bbbk[pq]\otimes\Bbbk\hat{p}\wedge\hat{q} \tag{37}$$

with differentials such that

$$\delta(p^iq^i) = ip^iq^{i-1}\otimes\hat{p} - ip^{i-1}q^i\otimes\hat{q},$$
$$\delta(p^iq^{i-1}\otimes\hat{p}) = ip^{i-1}q^{i-1}\otimes\hat{p}\wedge\hat{q},$$
$$\delta(p^iq^{i+1}\otimes\hat{q}) = (i+1)p^iq^i\otimes\hat{p}\wedge\hat{q}$$

for all $i\geq 0$. In writing this we have used the fact that the set $\{p^iq^i : i\in\mathbb{N}_0\}$ is a basis for the subalgebra $\Bbbk[pq]$ of $\mathscr{W}_q$. A standard calculation now shows that the cohomology of the complex (37) is of total dimension 2: in degree 0 it is spanned by the cohomology class of $1\in\Bbbk[pq]$, and in degree 1 by the cohomology class of $q^{-1}\otimes\hat{p}$. The class of this last cocycle thus freely spans $\mathrm{HH}^1(\mathscr{W}_q)$, and translating back we see that it corresponds to the derivation $\partial:\mathscr{W}_q\to\mathscr{W}_q$ such that $\partial(p)=q^{-1}$ and $\partial(q)=0$.
If $i\in\mathbb{N}_0$ and $j\in\mathbb{Z}$, then one can easily check that

$$\mathrm{Res}_0\left(p^iq^j\triangleright\frac{1}{t}\right) = \begin{cases}1 & \text{if } i=j=0;\\ 0 & \text{in any other case.}\end{cases} \tag{38}$$

Let $\delta:\mathscr{W}_q\to\mathscr{W}_q$ be a derivation. There is a sequence of derivations $(\delta_l)_{l\in\mathbb{Z}}$, all of which apart from a finite number are zero, such that $\delta=\sum_{l\in\mathbb{Z}}\delta_l$ and for each $l\in\mathbb{Z}$ the derivation $\delta_l:\mathscr{W}_q\to\mathscr{W}_q$ is homogeneous of degree l. Since pq has degree 0, for each $l\in\mathbb{Z}\setminus 0$ the element $\delta_l(pq)$ is a linear combination of monomials of the form p^iq^j with $i\in\mathbb{N}_0$, $j\in\mathbb{Z}$ and $j-i\neq 0$: in view of (38), we then have that

$$\mathrm{Res}_0\left(\delta(pq)\triangleright\frac{1}{t}\right) = \mathrm{Res}_0\left(\delta_0(pq)\triangleright\frac{1}{t}\right).$$

According to the decomposition (36), the derivation δ is inner if and only if the homogeneous derivation δ_0 is inner. Now, if δ_0 is inner, then there is a $u\in\mathscr{W}_q$ of degree 0 such that $\delta_0=\mathrm{ad}(u)$ and, in particular, $\delta_0(pq)=\mathrm{ad}(u)(pq)=[u,pq]=0$. Putting everything together, we see the map

$$\delta\in\mathrm{Der}(\mathscr{W}_q)\mapsto\mathrm{Res}_0\left(\delta(pq)\triangleright\frac{1}{t}\right)\in\Bbbk \tag{39}$$

vanishes on the subspace $\mathrm{InnDer}(\mathscr{W}_q)$ of all inner derivations. Our calculation of $\mathrm{HH}^1(\mathscr{W}_q)$ implies that $\mathrm{Der}(A) = \Bbbk\partial \oplus \mathrm{InnDer}(\mathscr{W}_q)$: as the map (39) takes the value 1 on the derivation ∂, we conclude that its kernel is exactly $\mathrm{InnDer}(\mathscr{W}_q)$. □

Going back to our algebra A is just a matter of changing coordinates.

Corollary 6.7. There is a representation of A on the algebra $\Bbbk[t^{\pm 1}]$ of Laurent polynomials such that

$$y \triangleright f(t) = t^N \frac{\mathrm{d}}{\mathrm{d}t} f(t), \qquad x \triangleright f(t) = t f(t)$$

for all $f \in \Bbbk[t^{\pm 1}]$. If $\delta : A \to A$ is a derivation and $\tilde{\delta} : A_x \to A_x$ is its extension to A_x, then the derivation δ is A_x-inner if and only if

$$\mathrm{Res}_0 \left(\tilde{\delta}(x^{-N+1} y) \triangleright \frac{1}{t} \right) = 0.$$

Proof. There is an injective algebra map $\iota : A \to \mathscr{W}$ such that $\iota(x) = q$ and $\iota(y) = q^N p$ — this is easy to check and is done in detail in [11, Section 3] — and clearly this map extends to one on localizations $\tilde{\iota} : A_x \to \mathscr{W}_q$ that is an isomorphism. The representation of A on $\Bbbk[t^{\pm 1}]$ that is mentioned in the corollary is the restriction to A along $\tilde{\iota}$ of the representation of $\mathscr{W}_q$ given in Proposition 6.6. Let $\delta : A \to A$ be a derivation of A. It extends uniquely to a derivation $\tilde{\delta} : A_x \to A_x$ and, conjugating by $\tilde{\iota}$, gives a derivation on $\mathscr{W}_q$: the claim of the corollary thus follows immediately from the last part of Proposition 6.6. □

7 Twisted cohomology

In Proposition 2.14 we described the finite subgroups of $\mathrm{Aut}(A)$ or, equivalently, the finite groups that act faithfully on our algebra A by algebra automorphisms, and showed that they are all cyclic. A natural next step is to describe the finite dimensional Hopf algebras that act on A faithfully in the appropriate sense. In that generality this is a problem that we do not know how to approach, so we will settle for the much smaller problem of describing the faithful actions of Taft Hopf algebras, motivated by the fact that these Hopf algebras can be viewed as «quantum thickenings» of cyclic groups and are therefore very close to them. We will do this in the next section.

We will recall the details about Taft algebras later. For now, let us mention simply that an action of a Taft Hopf algebra on our algebra A is determined by an algebra automorphism $\phi : A \to A$ of finite order and a ϕ-twisted derivation of A. We know the automorphisms very well already, and our task in the present section will be to obtain a description of the twisted derivations.

Let $\omega \in \Bbbk$ be different from 0 and 1, and let $\phi = \phi_{0,\omega} : A \to A$ be the algebra automorphism such that $\phi(x) = \omega x$ and $\phi(y) = \omega^{N-1} y$; clearly ϕ has finite order exactly when ω has finite order, but we do not need to impose that restriction yet. We write ${}_\phi A$ for the A-bimodule which coincides with A as a right A-module and whose left action is such that $a \triangleright x = \phi(a)x$ for all $a \in A$ and all $x \in {}_\phi A$. As the automorphism ϕ is homogeneous, the A-bimodule ${}_\phi A$ is a graded one.

We are interested in computing the space of ***ϕ-twisted derivations***: such a map is a linear function $\partial : A \to A$ such that

$$\partial(ab) = \partial(a) \cdot b + \phi(a) \cdot \partial(b)$$

for all a and b in A. It is immediate that a ϕ-twisted derivation $A \to A$ is exactly the same thing as a derivation $A \to {}_\phi A$ into the bimodule ${}_\phi A$, and because of this we will write $\operatorname{Der}(A, {}_\phi A)$ for the space of all ϕ-twisted derivations and conflate the two notions.

Now, if m is an element of A, then the map $\operatorname{ad}_\phi(m) : a \in A \mapsto ma - \phi(a)m \in A$ is an element of $\operatorname{Der}(A, {}_\phi A)$: we say that such a ϕ-twisted derivation is an ***inner*** one. We can thus split the calculation of $\operatorname{Der}(A, {}_\Phi A)$ in two steps: compute the ϕ-twisted derivations modulo inner ones, and then glue back the inner ones into the result. The first step amounts, precisely, to the calculation of the first Hochschild cohomology space $\mathrm{H}^1(A, {}_\phi A)$, and this is what we will do — and just as in Section 3 we will do this by first computing the spaces $\mathrm{H}^0(A, {}_\phi A)$ and $\mathrm{H}^2(A, {}_\phi A)$.

The Hochschild cohomology $\mathrm{H}^*(A, {}_\phi A)$ can be identified with the cohomology of the complex $\operatorname{Hom}_{A^e}(P_*, {}_\phi A)$ with P_* the projective resolution described in Section 3, and this complex, in turn — just as we built the complex (19) in that section — is naturally isomorphic to the complex

$$A \xrightarrow{\delta_0} A \otimes V^* \xrightarrow{\delta_1} A \otimes \Lambda^2 V^* \longrightarrow 0 \tag{40}$$

with differentials such that

$$\delta_0(a) = (\omega x a - a x) \otimes \hat{x} + (\omega^{N-1} y a - a y) \otimes \hat{y}, \tag{41}$$

$$\delta_1(b \otimes \hat{x} + c \otimes \hat{y}) = \left((\omega^{N-1} y b - b y) + (c x - \omega x c) - \sum_{s+1+t=N} \omega^s x^s b x^t \right) \otimes \hat{x} \wedge \hat{y} \tag{42}$$

for all choices of a, b and c in A. We remark that we are writing «A» here and in (40) and not «${}_\phi A$»: we will write the twisted left action explicitly, as in the formulas above for the differentials — we hope this is less confusing than the alternative.

Just as when we dealt with the «untwisted» Hochschild cohomology, we start our calculation with a lemma that tells us how some specific commutators — twisted commutators now[2] — behave. In its statement and in the rest of this section we will use the convention that if *snark* is an object related to A then we will write $\widetilde{snark}$ the 'corresponding' object related to the localization A_x.

Lemma 7.1. The map $\alpha : a \in A \mapsto \omega x a - a x \in A$ is injective and has image equal to xA, while the map $\tilde{\alpha} : a \in A_x \mapsto \omega x a - a x \in A_x$ is bijective.

[2]So we have not broken our promise not to do any more commutation formulas!

Proof. Let a be a non-zero element of A. There are $d \in \mathbb{N}_0$ and $a_0, \dots, a_d \in \Bbbk[x]$ such that $a_d \neq 0$ and $a = \sum_{i=0}^d a_i y^i$, and we have that

$$\alpha(a) = \sum_{i=0}^{d} a_i(\omega x y^i - y^i x) \equiv (\omega - 1)a_d x y^d \mod F_{d-1}. \tag{43}$$

Since $\omega \neq 1$, it follows immediately from this that $\alpha(a) \neq 0$, so that $\ker \alpha = 0$.

Let us now show that xa is in the image of α: we will proceed by induction with respect to the integer d. If $d = 0$, then $a = a_0 \in \Bbbk[x]$ and we have that $xa = \alpha(a/(\omega - 1)) \in \operatorname{img} \alpha$. If instead $d \geq 1$, then we have, according to (43), that

$$xa - \alpha\left(\frac{a_d y^d}{\omega - 1}\right) \in F_{d-1}. \tag{44}$$

As $xA = Ax$, the image of α is certainly contained in xA, and therefore the difference in (44) is both in F_{d-1} and in xA: the inductive hypothesis therefore implies that it is in the image of α, and so is xa, as we wanted. With this we have proved the claims of the lemma about the map α.

Let now a be an element of A_x. There is a non-negative integer $l \in \mathbb{N}_0$ such that $x^l a \in A$. If $\tilde{\alpha}(a) = 0$, then $\alpha(x^l a) = x^l \tilde{\alpha}(a) = 0$ and what we have already proved implies that $x^l a = 0$, so that $a = 0$: this shows that the map $\tilde{\alpha}$ is injective. On the other hand, as $x^{l+1} a \in xA$, we already know there is a $b \in A$ such that $\alpha(b) = x^{l+1} a$ and then $\tilde{\alpha}(x^{-l-1} b) = a$: the map $\tilde{\alpha}$ is also surjective. □

With this lemma we can now easily determine the twisted cohomology $\mathrm{HH}^*(A, {}_\phi A)$ in degrees 0 and 2. Recall that when $N \geq 2$ the homogeneous components of A are finite-dimensional, so we can work with Hilbert series in that case.

Proposition 7.2. We have that $\mathrm{HH}^0(A, {}_\phi A) = 0$. If $N \geq 2$, then the Hilbert series of $\mathrm{HH}^2(A, {}_\phi A)$ is

$$h_{\mathrm{HH}^2(A, {}_\phi A)} = \begin{cases} t^{-N} & \text{if } \omega^{N-1} \neq 1; \\ \dfrac{t^{-N}}{1 - t^{N-1}} & \text{if } \omega^{N-1} = 1. \end{cases}$$

If $N = 1$, then $\mathrm{HH}^2(A, {}_\phi A) = 0$.

In the proof of the corollary we will exhibit concrete representatives for all cohomology classes in $\mathrm{HH}^2(A, {}_\phi A)$.

Proof. Since the map α of the lemma is injective, it follows from the formula (41) for the differential δ_0 that the latter is itself injective and, therefore, that $\mathrm{HH}_0(A, {}_\phi A) = 0$.

Let us suppose that $N \geq 2$ and that $\omega^{N-1} \neq 1$. If $f \in \Bbbk[y]$ and $v \in A$, then there is $c \in A$ such that

$$\omega x c - c x = \alpha(c) = -(\omega^{N-1} - 1) x v - \sum_{s+1+t=N} \omega^s x^s f x^t,$$

because the right hand side of this equality is in xA — recall that $xA = Ax$ — and therefore

$$\delta_1(f \otimes \hat{x} + c \otimes y) = \left((\omega^{N-1} y f - f y) + (c x - \omega x c) - \sum_{s+1+t=N} \omega^s x^s f x^t \right) \otimes \hat{x} \wedge \hat{y}$$

$$= (\omega^{N-1} - 1)(fy + xv) \otimes \hat{x} \wedge \hat{y}.$$

This tells us that

$$(\Bbbk[y]y + xA) \otimes \hat{x} \wedge \hat{y} \subseteq \operatorname{img} \delta_1. \tag{45}$$

The subspace $\Bbbk[y]y + xA$ that appears here is the bilateral ideal I of A generated by x and y. It is clear from the formula (42) and our hypothesis that $N \geq 2$ that the image of δ_1 is contained in $I \otimes \hat{x} \wedge \hat{y}$, so in (45) we actually have an equality. It follows from this that $\mathrm{HH}^2(A, {}_\phi A)$ can be identified with $A/I \otimes \hat{x} \wedge \hat{y}$, so it has dimension 1, and is generated by the class of the element $1 \otimes \hat{x} \wedge \hat{y}$, which is homogeneous of degree $-N$. We thus have that $h_{\mathrm{HH}^2(A, {}_\phi A)} = t^{-N}$.

Next, we suppose that $N \geq 2$ and that $\omega^{N-1} = 1$. According to (42), if b and c are in A we have now

$$\delta_1(b \otimes \hat{x} + c \otimes \hat{y}) = \left([y, b] + (cx - \omega xc) - \sum_{s+1+t=N} \omega^s x^s b x^t \right) \otimes \hat{x} \wedge \hat{y}.$$

The three terms in the sum wrapped with big parentheses are clearly in xA, so

$$\operatorname{img} \delta_1 \subseteq xA \otimes \hat{x} \wedge \otimes \hat{y}. \tag{46}$$

As $\delta_1(c \otimes 1) = -\alpha(c) \otimes \hat{x} \wedge \hat{y}$ for each $c \in A$, Lemma 7.1 implies us that in fact we have an equality in (46) and therefore that $\mathrm{HH}^2(A, {}_\phi A)$ is in this case isomorphic to $A/xA \otimes \hat{x} \wedge \hat{y}$. This is isomorphic as a graded vector space to $\Bbbk[y] \otimes \hat{x} \wedge \hat{y}$, and therefore in this case the Hilbert series we are after is

$$h_{\mathrm{HH}^2(A, {}_\phi A)} = \frac{t^{-N}}{1 - t^{N-1}}.$$

Finally, let us suppose that $N = 1$. For all b and c in A we have now that

$$\delta_1(b \otimes \hat{x} + c \otimes \hat{y}) = \Big([y, b] + (cx - \omega xc) - b \Big) \otimes \hat{x} \wedge \hat{y}.$$

If u is an element of A, then there are $f \in \Bbbk[y]$ and $v \in A$ such that $u = f + xv$, and there is a $g \in A$ such that $gx - \omega xg = -\alpha(g) = xv$, so that $\delta_1(-f \otimes \hat{x} + g \otimes \hat{y}) = u \otimes \hat{x} \wedge \hat{y}$. This tells us that the map δ_1 is in this case surjective, so that we have $\mathrm{HH}^2(A, {}_\phi A) = 0$. $\square$

We can now compute $\mathrm{HH}^1(A, {}_\phi A)$ using the same strategy that we used for the regular cohomology in Section 3.

Proposition 7.3. If $N \geq 2$, then the Hilbert series of $\mathrm{HH}^1(A, {}_\phi A)$ is

$$h_{\mathrm{HH}^1(A, {}_\phi A)}(t) = \begin{cases} 0 & \text{if } \omega^{N-1} \neq 1; \\ \dfrac{1}{t(1 - t^{N-1})} & \text{if } \omega^{N-1} = 1. \end{cases}$$

If $N = 1$, the $\mathrm{HH}^1(A, {}_\phi A) = 0$.

Proof. Let us suppose first that $N \geq 2$. The Hilbert series of our algebra A, as we saw before, is $h_A(t) = (1-t)^{-1}(1-t^{N-1})^{-1}$, and the Euler characteristic of the complex (40) is

$$h_A(t) - (t^{-1} + t^{-(N-1)})h_A(t) + t^{-N}h_A(t) = t^{-N},$$

so the invariance of the Euler characteristic when passing to cohomology implies that

$$h_{\mathrm{HH}^0(A,{}_\phi A)}(t) - h_{\mathrm{HH}^1(A,{}_\phi A)}(t) + h_{\mathrm{HH}^2(A,{}_\phi A)}(t) = t^{-N}.$$

The Hilbert series $h_{\mathrm{HH}^1(A,{}_\phi A)}(t)$ can be computed immediately from this and the preceding proposition, and turns out to be what is described in the statement.

Let us now suppose that $N = 1$ and let $\eta = b \otimes \hat{x} + c \otimes \hat{y}$, with b and c in A, be a 1-cocycle in the complex (40). If $f \in A$, then

$$\eta + \delta_0(f) = (b + \omega x f - x f) \otimes \hat{x} + (\text{something}) \otimes \hat{y},$$

and Lemma 7.1 implies that we can choose f so that $b + \omega x f - x f$ is in $\Bbbk[y]$. This means that, up to replacing η by a cohomologous 1-cocycle, we can suppose that b itself is in $\Bbbk[y]$. In that case we have, according to (42), that

$$0 = \delta_1(\eta) = (cx - \omega x c - b) \otimes \hat{x} \wedge \hat{y}.$$

Since $cx - \omega x c \in xA$ and $b \in \Bbbk[y]$, this implies that $cx - \omega x c = b = 0$ and the injectivity of the map α that, in fact, also $c = 0$. This shows that every 1-cocycle in our complex (40) is cohomologous to zero and therefore that $\mathrm{HH}^1(A, {}_\phi A) = 0$. □

The proposition we have just proved shows that all ϕ-twisted derivations $\mathrm{Der}(A, {}_\phi A)$ are in fact inner except in the special case in which $N \geq 2$ and $\omega^{N-1} = 1$. In this special case we can make the following observations. We will work with the localization A_x of A at the powers of the normal element x, in which we have the following commutation rule:

$$yx^{-1} - x^{-1}y = -x^{N-2}.$$

The canonical map $A \to A_x$ is injective, so we view A as a subalgebra of A_x. The automorphism $\phi : A \to A$ such that $\phi(x) = \omega x$ and $\phi(y) = \omega^{N-1}y$ clearly extends to an automorphism $\tilde{\phi} : A_x \to A_x$, and every ϕ-twisted derivation $\partial : A \to A$ extends uniquely to a $\tilde{\phi}$-twisted derivation $\tilde{\partial} : A_x \to A_x$ that has $\tilde{\partial}(x^{-1}) = -\omega^{-1}x^{-1}\partial(x)x^{-1}$.

The following technical result will allow us to construct twisted derivations.

Proposition 7.4. Suppose that $N \geq 2$ and $\omega^{N-1} = 1$.

(*i*) There is a ϕ-twisted derivation $\partial_0^\phi : A \to A$ such that

$$\partial_0^\phi(x) = 1 - \omega, \qquad \partial_0^\phi(y) = x^{N-2},$$

and it is homogeneous of degree -1 and not inner. The inner $\tilde{\phi}$-twisted derivation $\operatorname{ad}_{\tilde{\phi}}(x^{-1}) : A_x \to A_x$ of A_x corresponding to x^{-1} preserves the subalgebra A and, in fact, the map ∂_0^ϕ is the restriction of $\operatorname{ad}_{\tilde{\phi}}(x^{-1})$ to A.

(*ii*) There is a derivation $\partial_{N-1} : A \to A$ such that

$$\partial_{N-1}(x) = xy, \qquad \partial_{N-1}(y) = \sum_{s+2+t=N} (s+1)x^{s+1}yx^t + \frac{N-1}{2}\left(y^2 - Nx^{N-1}y\right).$$

This derivation commutes with $\phi : A \to A$.

(*iii*) If $\partial : A \to {}_\phi A$ is a twisted derivation that is homogeneous of some degree l, then the map

$$\partial_{N-1} \circ \partial - \partial \circ \partial_{N-1} : A \to A$$

is also a ϕ-twisted derivation that is homogeneous, now of degree $l + N - 1$. There is therefore a map

$$\operatorname{ad}(\partial_{N-1}) : \partial \in \operatorname{Der}(A, {}_\phi A) \longmapsto \partial_{N-1} \circ \partial - \partial \circ \partial_{N-1} \in \operatorname{Der}(A, {}_\phi A)$$

that is homogeneous of degree $N - 1$.

(*iv*) If u is an element of A_x such that the inner $\tilde{\phi}$-derivation $\operatorname{ad}_{\tilde{\phi}}(u) : A_x \to A_x$ preserves the subalgebra A, so that the restriction $\operatorname{ad}_{\tilde{\phi}}(u)|_A : A \to A$ is an element of $\operatorname{Der}(A, {}_\phi A)$, then the map $\operatorname{ad}_{\tilde{\phi}}(\tilde{\partial}_{N-1}(u)) : A_x \to A_x$ also preserves A and

$$\operatorname{ad}(\partial_{N-1})\Big(\operatorname{ad}_{\tilde{\phi}}(u)|_A\Big) = \operatorname{ad}_{\tilde{\phi}}(\tilde{\partial}_{N-1}(u))|_A. \tag{47}$$

Here $\tilde{\partial}_{N-1}(u)$ denotes the value at u of the extension $\tilde{\partial}_{N-1}$ of the derivation ∂_{N-1} of (*ii*) to A_x.

Proof. (*i*) Let $\Bbbk\langle X, Y\rangle$ be the free algebra on two generators X and Y. There is an automorphism $\hat{\phi} : \Bbbk\langle X, Y\rangle \to \Bbbk\langle X, Y\rangle$ such that $\hat{\phi}(X) = \omega X$ and $\hat{\phi}(Y) = Y$, and it preserves the bilateral ideal $I = (YX - XY - X^N)$, since $\omega^{N-1} = 1$. The quotient $\Bbbk\langle X, Y\rangle/I$ is of course isomorphic to A, and we may thus view A as a $\Bbbk\langle X, Y\rangle$-bimodule. Since the algebra $\Bbbk\langle X, Y\rangle$ is free, there is a unique derivation $\hat{\partial} : \Bbbk\langle X, Y\rangle \to {}_{\hat{\phi}}A$ such that $\hat{\partial}(X) = 1 - \omega$ and $\hat{\partial}(Y) = x^{N-2}$. This derivation vanishes on the ideal I: to check this, it is sufficient to show that it vanishes on the generator $YX - XY - X^N$ of that ideal, and that is easy. One need only keep in mind that $\hat{\partial}$ is a $\hat{\phi}$-*twisted* derivation, so that

$$\hat{\partial}(X^N) = (1 + \omega + \cdots + \omega^{N-1})(1 - \omega)x^{N-1} = (1 - \omega)x^{N-1}.$$

It follows from this that $\hat{\partial}$ induces a derivation $\partial_0^\phi : A \to {}_\phi A$ such that $\partial_0^\phi(x) = 1 - \omega$ and $\partial_0^\phi(y) = x^{N-2}$. It is manifestly homogeneous of degree -1, and it is not inner because there are no elements of negative degree in A.

Now, we certainly have an inner $\tilde{\phi}$-twisted derivation $\operatorname{ad}_{\tilde{\phi}}(x^{-1}) : A_x \to A_x$, and we can compute that

$$\operatorname{ad}_{\tilde{\phi}}(x^{-1})(x) = x^{-1}x - \tilde{\phi}(x)x^{-1} = 1 - \omega,$$
$$\operatorname{ad}_{\tilde{\phi}}(x^{-1})(y) = x^{-1}y - \tilde{\phi}(y)x^{-1} = x^{N-2}.$$

Since $\tilde{\phi}(A) = A$, the facts that x and y generate A and that $\mathrm{ad}_{\tilde{\phi}}(x^{-1})$ is a $\tilde{\phi}$-twisted derivation imply together that $\mathrm{ad}_{\tilde{\phi}}(x^{-1})$ maps the subalgebra A into itself. Moreover, its restriction to A is a ϕ-twisted derivation that coincides with ∂_0^ϕ at x and at y, so we have that $\mathrm{ad}_{\tilde{\phi}}(x^{-1})|_A = \partial_0^\phi$.

(ii) The derivation ∂_{N-1} described here is simply the one given by Proposition 5.6 when $l = N-1$, which we know to be homogeneous of degree $N-1$ and not inner. To make this explicit we have used the fact that $\Phi_2 = \frac{1}{2}(y^2 - Nx^{N-1}y)$. A simple direct calculation shows that ∂_{N-1} commutes with ϕ.

(iii) This follows from an easy direct calculation. They key fact that makes things work is that the derivation $\partial_{N-1} : A \to A$ commutes with the automorphism $\phi : A \to A$.

(iv) A direct calculation shows that we have an equality of maps $A_x \to A_x$

$$\mathrm{ad}(\tilde{\partial}_{N-1})\Big(\mathrm{ad}_{\tilde{\phi}}(u)\Big) = \mathrm{ad}_{\tilde{\phi}}\Big(\tilde{\partial}_{N-1}(u)\Big)$$

The left hand side preserves A because $\mathrm{ad}_{\tilde{\phi}}(u)$ and $\tilde{\partial}_{N-1}$ do, so the right hand side also preserves it. The equality (47) that appears in the statement of the proposition is now immediate. □

Combining the different parts of the proposition we have just proved we can construct ϕ-twisted derivations $A \to {}_\phi A$ of all degrees of the form $-1 + l(N-1)$ with $l \in \mathbb{N}_0$.

Corollary 7.5. Suppose that $N \geq 2$ and $\omega^{N-1} = 1$, and let $\phi : A \to A$ be the automorphism such that $\phi(x) = \omega x$ and $\phi(y) = y$. Let $\tilde{\phi} : A_x \to A_x$ and $\tilde{\partial}_{N-1} : A_x \to A_x$ be the extensions of ϕ and of the ϕ-twisted derivation $\partial_{N-1} : A \to A$ of Proposition 7.4. For each $l \in \mathbb{N}_0$ the $\tilde{\phi}$-twisted derivation

$$\mathrm{ad}_{\tilde{\phi}}\big(\tilde{\partial}_{N-1}^l(x^{-1})\big) : A_x \to A_x$$

preserves A and its restriction to A is a map

$$\partial_l^\phi : A \to A$$

that is a homogeneous ϕ-twisted derivation of degree $-1 + l(N-1)$ and that coincides with the map

$$\mathrm{ad}(\partial_{N-1})^l(\partial_0^\phi) \in \mathrm{Der}(A, {}_\phi A).$$

Proof. This can be proved immediately by induction starting with (i) of the proposition and using part (iv) for the inductive step. □

Remark 7.6. There is a general-nonsense way of looking at our last few results. We will explain it, as it helps to make sense of them.

Suppose that $N \geq 2$ and that $\omega^{N-1} = 1$, and let ν be the order of ω in $\Bbbk^\times$, which is a divisor of $N-1$. General considerations imply that the automorphism ϕ acts in a canonical way on the Hochschild cohomology $\mathrm{HH}^*(A)$ of A, the invariant subspace $\mathrm{HH}^*(A)^\phi$ is a sub-Gerstenhaber algebra of $\mathrm{HH}^*(A)$, and there is a «Gerstenhaber module structure»

$$\mathrm{HH}^*(A)^\phi \otimes \mathrm{HH}^*(A, {}_\phi A) \to \mathrm{HH}^*(A, {}_\phi A)$$

that in particular restricts to a Lie module structure

$$\mathrm{HH}^1(A)^\phi \otimes \mathrm{HH}^1(A, {}_\phi A) \to \mathrm{HH}^1(A, {}_\phi A), \tag{48}$$

which itself is induced by the map

$$f \otimes g \in \mathrm{Der}(A) \otimes \mathrm{Der}(A, {}_\phi A) \mapsto \mathrm{ad}(f)(g) \in \mathrm{Der}(A, {}_\phi A), \tag{49}$$

using the notation of part (iii) of Proposition 7.4.
If we view $\mathrm{HH}^1(A)$ as $\mathrm{Der}(A)/\operatorname{InnDer}(A)$ then the action of ϕ on $\mathrm{HH}^1(A)$ is induced by the map

$$\phi^\sharp : d \in \mathrm{Der}(A) \mapsto \phi \circ d \circ \phi^{-1} \in \mathrm{Der}(A),$$

and a direct calculation proves that if $d : A \to A$ is a homogeneous derivation of degree l, then $\phi^\sharp(d) = \omega^l d$. It follows from this that the invariant subspace $\mathrm{HH}^1(A)^\phi$ is spanned by the classes of the derivations of the form $L_{j\nu}$, with $j \in \mathbb{Z}$ such that $j \geq -1$, and ∂_0, using the notation of Section 5. On the other hand, Proposition 7.3 tells us that the only homogeneous components of $\mathrm{HH}^1(A, {}_\phi A)$ that are non-zero are those of degrees l such that $l \geq -1$ and $l \equiv -1 \mod N-1$: as the action (48) is homogeneous, this implies that the only L_l in $\mathrm{HH}^1(A)^\phi$ that can act non-trivially on $\mathrm{HH}^1(A, {}_\phi A)$ are those with those with $l \equiv 0 \mod N-1$. What Corollary 7.5 does is, essentially, to describe the action of L_{N-1}, the simplest of those: in part (i) of Proposition 7.4 we exhibited an element ∂_0^ϕ in $\mathrm{Der}(A, {}_\phi A)$, and in Corollary 7.5 we produced many others by using the action (49). We will see below that in this way we obtain a basis for $\mathrm{HH}^1(A, {}_\phi A)$. ◊

We are just a hop, skip and jump away from the description of $\mathrm{H}^1(A, {}_\phi A)$ in the exceptional case in which it is non-zero — apart, alas, for one exception.

Proposition 7.7. Suppose that $N \geq 4$ and that $\omega^{N-1} = 1$, and let $\phi : A \to A$ be the automorphism of A such that $\phi(x) = \omega x$ and $\phi(y) = y$. The cohomology classes of the ϕ-twisted derivations ∂_l^ϕ with $l \in \mathbb{N}_0$ that we described in Corollary 7.5 freely span the vector space $\mathrm{H}^1(A, {}_\phi A)$.

This excludes the cases in which $2 \leq N \leq 3$. Now if $N = 2$ then the hypothesis on ω is that $\omega = 1$ and, since we are assuming throughout that $\omega \neq 1$, this case cannot actually occur. The case in which $N = 3$, on the other hand, is a real possibility that is excluded in this proposition — and for good reason, as we will see below.

Proof. It is enough that we show that the cohomology classes of those ϕ-twisted derivations are linearly independent in $\mathrm{H}^1(A, {}_\phi A)$, because if that is the case then they span the whole space: this follows immediately from our calculation of the Hilbert series of $\mathrm{H}^1(A, {}_\phi A)$ in Proposition 7.3 and the fact that for all $l \in \mathbb{N}_0$ the twisted derivation ∂_l^ϕ has degree $-1 + l(N-1)$.

Let us suppose that the cohomology classes of the twisted derivations in the statement are linearly dependent. There are then an integer $d \geq 0$ and scalars $a_0, \ldots, a_d$ in $\Bbbk$ such that $a_d \neq 0$ and the ϕ-twisted derivation

$$\delta := \sum_{i=0}^{d} a_i \partial_i^\phi : A \to A$$

is inner, so that there is an element u in A such that $\delta = \mathrm{ad}_\phi(u)$. We let P be the polynomial $\sum_{i=0}^d a_i T \in \Bbbk[T]$ and consider the $\tilde{\phi}$-twisted derivation

$$\mathrm{ad}_{\tilde{\phi}}\big(P(\tilde{\partial}_{N-1})(x^{-1})\big) : A_x \to A_x \tag{50}$$

It follows from Corollary 7.5 that this map preserves A and its restriction to A is precisely δ. In particular, the map in (50) and $\mathrm{ad}_{\tilde{\phi}}(u) : A_x \to A_x$ take the same value on x and y and therefore, since they are $\tilde{\phi}$-twisted derivations, they are in fact equal — in other words, we have that

$$\mathrm{ad}_{\tilde{\phi}}\Big(P(\tilde{\partial}_{N-1})(x^{-1}) - u\Big) = 0. \tag{51}$$

We know form Lemma 7.1 that the map $\tilde{\alpha} : a \in A_x \mapsto \omega x a - x a \in A_x$ is injective, and as a consequence of that

Claim 7.8. If $v \in A_x$ and the map $\mathrm{ad}_{\tilde{\phi}}(v) : A_x \to A_x$ vanishes at x, then $v = 0$.

From this and the equality (51) we can conclude that

$$P(\tilde{\partial}_{N-1})(x^{-1}) = u \in A.$$

Let us next check that for all $l \in \mathbb{N}_0$ we have that

$$\tilde{\partial}_{N-1}^l(x^{-1}) \equiv \prod_{i=0}^{l-1}\left(\frac{t(N-1)}{2} - 1\right) \cdot x^{-1} y^l \mod \tilde{F}_{l-1}. \tag{52}$$

This is clear when $l = 0$, and when $l = 1$ we have that

$$\tilde{\partial}_{N-1}(x^{-1}) = -x^{-1}\tilde{\partial}_{N-1}(x)x^{-1} = -yx^{-1} \equiv -x^{-1}y \mod \tilde{F}_0,$$

in accordance with the formula we want to prove. The general case follows by induction and the observation that for all $l \in \mathbb{N}_0$ we have

$$\begin{aligned}
\tilde{\partial}_{N-1}(x^{-1}y^l) &= \tilde{\partial}_{N-1}(x^{-1})y^l + x^{-1}\tilde{\partial}_{N-1}(y^l) = -yx^{-1}y^l + x^{-1}\sum_{i=0}^{l-1} y^i \tilde{\partial}_{N-1}(y) y^{l-1-i} \\
&= -yx^{-1}y^l + x^{-1}\sum_{i=0}^{l-1} y^i \left[\sum_{s+2+t=N} (s+1)x^{s+1}yx^t + \frac{N-1}{2}\left(y^2 - Nx^{N-1}y\right)\right] y^{l-1-i} \\
&\equiv \left(\frac{l(N-1)}{2} - 1\right) x^{-1}y^{l+1} \mod \tilde{F}_l.
\end{aligned}$$

Now, using the congruence (52) we have just proved we see that

$$P(\tilde\partial_{N-1})(x^{-1}) \equiv \underbrace{a_d \prod_{i=0}^{d-1}\left(\frac{t(N-1)}{2}-1\right)} \cdot x^{-1}y^d \mod \tilde F_{d-1}$$

As $N \geq 4$, the scalar marked with a brace is non-zero. If we call it γ for brevity, then what we have is that

$$A \ni P(\tilde\partial_{N-1})(x^{-1}) = \gamma \cdot x^{-1}y^d + w$$

for some $w \in \tilde F_d$, and this is absurd. This contradiction proves the proposition. $\square$

When $N = 3$, in the situation of Proposition 7.7 we can compute that

$$\tilde\partial_2(x) = xy, \qquad \tilde\partial_2(x^{-1}) = -x^{-1}y + x, \qquad \tilde\partial_2(y) = x^4 + y^2$$

and using that that

$$\tilde\partial_2^2(x^{-1}) = -x^3 \in A.$$

This tells us that the ϕ-twisted derivation $\partial_2^\phi : A \to A$ is inner, and implies that in fact the ϕ-twisted derivation $\partial_l^\phi : A \to A$ is inner whenever l is an integer such that $l \geq 2$. The conclusion of Proposition 7.7 is therefore very false when $N = 3$. We can fix it as follows:

Proposition 7.9. Suppose that $N = 3$ and that $\omega^{N-1} = 1$, so that $\omega = -1$, and let $\phi : A \to A$ be the automorphism of A such that $\phi(x) = -x$ and $\phi(y) = y$. The $\tilde\phi$-twisted derivations of A_x

$$\mathrm{ad}_{\tilde\phi}(x^{-1}), \qquad \mathrm{ad}_{\tilde\phi}(\tilde\partial_2(x^{-1})), \qquad \mathrm{ad}_{\tilde\phi}(\tilde\partial_2^l(x^{-1}y^2)), \quad l \geq 0$$

preserve the subalgebra A and the cohomology classes of their restrictions to A freely span the vector space $\mathrm{H}^1(A, {}_\phi A)$.

Proof. This can be proved in essentially the same way as the previous proposition, and we therefore omit the details. $\square$

With this we have concluded the calculation of the twisted Hochschild cohomology $\mathrm{H}^*(A, {}_\phi A)$. One nice observation we can immediately make with the result at hand is:

Proposition 7.10. Suppose that $N \geq 1$ and that $\omega \in \Bbbk^\times \setminus \{1\}$, and let $\phi : A \to A$ be the automorphism of A such that $\phi(x) = \omega x$ and $\phi(y) = \omega^{N-1}y$. Every ϕ-twisted derivation $A \to {}_\phi A$ is the restriction to A of an inner $\tilde\phi$-twisted derivation $A_x \to {}_{\tilde\phi}A_x$ that preserves A. $\square$

By analogy with the notion of X-inner automorphisms of Harčenko [29,30], we can rephrase this proposition saying that all ϕ-twisted derivations of A are X-inner.

When $N = 0$, so that the algebra A is the first Weyl algebra, the calculation of the twisted Hochschild cohomology $\mathrm{H}^*(A, {}_\phi A)$ for an automorphism $\phi : A \to A$ such that $\phi(x) = \omega x$ and $\phi(y) = \omega^{-1}y$ for some

$\omega \in \Bbbk^\times$ was done in [2] by Jacques Alev, Thierry Lambre, Marco Farinati and Andrea Solotar: the end result is that

$$\dim \mathrm{H}^p(A, {}_\phi A) \cong \begin{cases} 1 & \text{if } p = 0 \text{ and } \omega = 1, \text{ or if } p = 2 \text{ and } \omega \neq 1; \\ 0 & \text{in any other case.} \end{cases}$$

This is rather different from what we have found above for $N \geq 1$. It should be noticed that while the automorphism group of the first Weyl algebra is considerably larger than that of our algebras A with $N \geq 1$, but every one of its elements of finite order is conjugated to one of the form of the automorphism ϕ described here. On the other hand, there are non-cyclic finite groups of automorphisms of the first Weyl algebra.

8 Actions of Taft algebras

Let now n, $m \in \mathbb{N}$ be such that $1 < m$ and $m \mid n$, and let $\lambda \in \Bbbk^\times$ be a primitive mth root of unity. The ***generalized Taft algebra*** $T = T_n(\lambda, m)$ is the algebra freely generated by two letters g and ξ subject to the relations

$$g^n = 1, \qquad \xi^m = 0, \qquad g\xi = \lambda\xi g. \tag{53}$$

This algebra was originally studied by David Radford in [45]. It is finite dimensional of dimension nm, and the set $\{\xi^i g^j : 0 \leq i < m, 0 \leq j < n\}$ is one of its bases. It is a Hopf algebra with comultiplication $\Delta : T \to T \otimes T$ and augmentation $\varepsilon : T \to \Bbbk$ such that

$$\Delta(g) = g \otimes g, \qquad \Delta(\xi) = \xi \otimes 1 + g \otimes \xi, \qquad \varepsilon(g) = 1, \qquad \varepsilon(\xi) = 0.$$

When $n = m$ and $\theta = 0$ this is the classical Taft Hopf algebra constructed by Earl Taft in [55] with the purpose of exhibiting finite dimensional Hopf algebras with antipode of arbitrarily large order. We will use freely the Heyneman–Sweedler notation for the coproduct of T and even omit the sum: we will write $\Delta(a)$ in the form $a_1 \otimes a_2$.

We are interested in left T-module-algebra structures on our algebra A, that is, left T-module structures $\triangleright : T \otimes A \to A$ such that the multiplication and unit map $A \otimes A \to A$ and $\Bbbk \to A$ are both T-linear — this means that $h \triangleright 1_A = \varepsilon(h)1_A$ and $h \triangleright ab = (h_1 \triangleright a)(h_2 \triangleright b)$ for all choices of a and b in A and h in T. We refer the reader to [40, Chapter 4] for general information about module-algebras over Hopf algebras.

We will further restrict our attention to T-module algebra structures on A that are ***inner-faithful*** — a notion introduced by T. Banica and J. Bichon in [8] — as these correspond to faithful group actions on A: the condition is that there be no non-zero Hopf ideal I in T such that $I \triangleright A = 0$. According to [15, Corollary 3.7], in the specific case in which the Hopf algebra is our generalized Taft algebra T we have a handy criterion:

Claim 8.1. A T-module-algebra structure on A is inner-faithful if and only if the group $G(T) = \langle g \rangle$ of group-like elements of T acts faithfully on A and $\xi \triangleright A \neq 0$.

As the algebra T is generated by g and ξ subject to the relations in (53), giving a T-module structure on A is the same as giving the two maps $\phi : a \in A \mapsto g \triangleright a \in A$ and $\partial : a \in A \mapsto \xi \triangleright a \in A$ such that $\phi^n = \mathsf{id}_A$, $\partial^m = 0$ and $\phi\partial = \lambda\partial\phi$, and that structure will be a T-module-algebra structure exactly when ϕ is an automorphism of A and ∂ a ϕ-twisted derivation $A \to A$. The automorphism ϕ will have finite order: according to Proposition 2.14, up to conjugating the whole module-algebra structure by an algebra automorphism of A we can suppose then that there is a scalar $\omega \in \Bbbk$ such that $\phi(x) = \omega x$ and $\phi(y) = \omega^{N-1} y$, and then the group of group-like elements $G(T) = \langle g \rangle$ of T, which is cyclic of order n, will clearly act faithfully on A if and only if the scalar ω is a primitive nth root of unity.

We are left with the task of understanding the possibilities for the map ∂. Since it is a ϕ-twisted derivation $A \to A$, we now from Proposition 7.10 that the map ∂ will be, in fact, the restriction of an inner $\tilde{\phi}$-twisted derivation of the localization A_x that preserves A. The following lemma imposes significant restrictions on what can actually happen.

Lemma 8.2. Let ω be a primitive nth root of unity in $\Bbbk$, let $\phi : A \to A$ be the algebra automorphism such that $\phi(x) = \omega x$ and $\phi(y) = \omega^{N-1} y$, and let $\tilde{\phi} : A_x \to A_x$ be the unique extension of ϕ to the localization A_x. Let u be a non-zero element of A_x such that the inner $\tilde{\phi}$-twisted derivation $\mathrm{ad}_{\tilde{\phi}}(u) : A_x \to A_x$ preserves the subalgebra A and let

$$\partial := \mathrm{ad}_{\tilde{\phi}}(u)|_A : A \to A$$

be its restriction to A, which is a ϕ-twisted derivation.

(i) The map ∂ is non-zero.

(ii) If λ is a scalar, then we have that $\phi\partial = \lambda\partial\phi$ exactly when $\tilde{\phi}(u) = \lambda u$, and when that is the case we have that $\lambda^n = 1$.

(iii) If there are a scalar λ and a positive integer m such that $\tilde{\phi}(u) = \lambda u$ and $\partial^m = 0$, then $n \leq m$.

Proof. (i) We have $\partial(x) = ux - \omega xu = -\tilde{\alpha}(u) \neq 0$, according to Lemma 7.1.

(ii) If $\lambda \in \Bbbk$ is such that $\phi\partial = \lambda\partial\phi$, then

$$\begin{aligned}\tilde{\phi}(u)\omega x - \omega^2 x\tilde{\phi}(u) = \tilde{\phi}(ux - \omega xu) = \phi(ux - \omega xu) = \phi(\partial(x)) = \lambda\partial(\phi(x))\\ = \lambda\partial(\omega x) = \lambda(u\omega x - \omega^2 xu),\end{aligned}$$

and therefore

$$\tilde{\alpha}\big(\tilde{\phi}(u) - \lambda u\big) = \omega x\big(\tilde{\phi}(u) - \lambda u\big) - \big(\tilde{\phi}(u) - \lambda u\big)x = 0,$$

so that $\tilde{\phi}(u) = \lambda u$ because of Lemma 7.1. Conversely, if λ is a scalar such that $\tilde{\phi}(u) = \lambda u$, then a simple and direct calculation shows that we have that $\phi\partial = \lambda\partial\phi$. Moreover, since the automorphism $\tilde{\phi}$ is manifestly diagonalizable and all its eigenvalues are powers of ω, in that case we have that $\lambda^n = 1$.

(*iii*) Let us suppose that there are $\lambda \in \Bbbk$ and $m \in \mathbb{N}$ such that $\tilde{\phi}(u) = \lambda u$ and $\partial^m = 0$. Since $u \neq 0$, there is an integer $d \in \mathbb{N}_0$ such that $u \in \tilde{F}_d \setminus \tilde{F}_{d-1}$. An induction shows that if $l \in \mathbb{N}_0$, $a \in A$ and $t \in \Bbbk$ are such that $a \in F_l \setminus F_{l-1}$ and $\phi(a) = ta$, then for all $k \in \mathbb{N}_0$ we have that

$$\partial^k(a) \equiv \prod_{i=0}^{k-1}(1 - t\lambda^i) \cdot au^k \mod \tilde{F}_{l+kd-1}$$

and, in particular, since $\partial^m(a) = 0$, that

$$\prod_{i=0}^{m-1}(1 - t\lambda^i) \cdot au^m \in \tilde{F}_{l+md-1}.$$

As the graded algebra $\operatorname{gr} A_x$ for the filtration $(\tilde{F}_i)_{i \geq -1}$ is an integral domain, this tells us that, in fact,

$$\prod_{i=0}^{m-1}(1 - t\lambda^i) = 0$$

and, then, that $t \in \{\lambda^{-i} : 0 \leq i < m\}$. In particular, for each $j \in \{0, \dots, n-1\}$ we can take $a = x^j$, that has $\phi(a) = \omega^j a$, and conclude that the n pairwise different scalars $\omega^0, \dots, \omega^{n-1}$ all belong to the set $\{\lambda^{-i} : 0 \leq i < m\}$. This set has cardinal at most m, so $n \leq m$, as the lemma claims. □

The obvious question after having proved Lemma 8.2 is: when is the map ∂ appearing there such that $\partial^n = 0$? The answer is simple: never. With the objective of proving this, let us recall some standard notations from the theory of q-variants. If q is a variable, then for each $n \in \mathbb{N}_0$ we let $[n]_q \coloneqq 1 + q + \cdots + q^{n-1}$ and $[n]_q! \coloneqq [1]_q[1]_q \cdots [n]_q$, and consider for each choice of k and i in $\mathbb{N}_0$ such that $0 \leq i \leq k$ the ***q*-binomial** or ***Gaussian binomial coefficient***

$$\binom{k}{i}_q \coloneqq \frac{[k]_q!}{[i]_q! \cdot [k-i]_q!},$$

which is an element of $\mathbb{Z}[q]$.

Lemma 8.3. Let ω and λ be primitive nth roots of unity in $\Bbbk$, let $\phi : A \to A$ be the algebra automorphism such that $\phi(x) = \omega x$ and $\phi(y) = \omega^{N-1} y$, and let $\tilde{\phi} : A_x \to A_x$ be the unique extension of ϕ to the localization A_x. If u is a non-zero element of A_x such that $\tilde{\phi}(u) = \lambda u$ and the inner $\tilde{\phi}$-twisted derivation $\operatorname{ad}_{\tilde{\phi}}(u) : A_x \to A_x$ preserves the subalgebra A, then the restriction $\partial \coloneqq \operatorname{ad}_{\tilde{\phi}}(u)|_A : A \to A$, which is a ϕ-twisted derivation, has $\partial^n \neq 0$.

Proof. Let u be a non-zero element of A_x such that $\tilde{\phi}(u) = \lambda u$ and $\operatorname{ad}_{\tilde{\phi}}(u)(A) \subseteq A$, and let us consider the ϕ-twisted derivation $\partial \coloneqq \operatorname{ad}_{\tilde{\phi}}(u)|_A : A \to A$.
Let a in A and $t \in \Bbbk$ be such that $\phi(a) = ta$. For all $k \in \mathbb{N}_0$ we have that

$$\partial^k(a) = \sum_{i=0}^{k}(-1)^{k-i}\lambda^{\binom{i}{2}}\binom{k}{i}_\lambda t^i u^{k-i} a u^i, \tag{54}$$

as can be proved by an obvious induction using the well-known analogue of Pascal's identity,

$$\binom{k}{i}_q = \binom{k-1}{i}_q + q^{k-i}\binom{k-1}{i-1}_q,$$

valid in $\mathbb{Z}[q]$ whenever $0 < i < k$. In particular, taking $k = n$ in (54) we find that

$$\partial^n(a) = \sum_{i=0}^{n} (-1)^{n-i} \lambda^{\binom{i}{2}} \binom{n}{i}_\lambda t^i u^{n-i} a u^i. \tag{55}$$

Now, according to the Cauchy q-binomial theorem we have that

$$\prod_{i=0}^{n} (y + zq^i) = \sum_{i=0}^{n} q^{\binom{i}{2}} \binom{n}{i}_q y^{n-i} z^i$$

in the polynomial ring $\mathbb{Z}[q, y, z]$. Extending scalars to $\Bbbk$ and specializing z at -1 and q at λ, the left hand side of this equality becomes $y^n - 1$, so by looking at the coefficients of the powers of y in the right hand side we see that $\binom{n}{i}_\lambda = 0$ when $0 < i < n$. The equality (55) therefore violently simplifies to

$$\partial^n(a) = \lambda^{\binom{n}{2}} a u^i + (-1)^n u^n a,$$

since $t^n = 1$, as t is an eigenvalue of the automorphism ϕ and thus a power of ω. As λ is a primitive nth root of unity, we have that $\lambda^{\binom{n}{2}} = (-1)^{n+1}$ and, therefore, that

$$\partial^n(a) = (-1)^n [u^n, a].$$

This equality holds whenever a is an eigenvector of ϕ in A and, since the automorphism ϕ is diagonalizable, it then follows immediately that the equality holds in fact for all a in A. In particular, the map ∂^n is zero exactly when the element u^n of A_x commutes with all elements of A: this happens if and only if u^n is central in A_x, so an element of $\Bbbk$ according to Proposition 1.5, and clearly this can happen if and only if $u \in \Bbbk$. Now, as $\lambda \neq 1$ and $\phi(u) = \lambda u$, we certainly have that u is not in $\Bbbk$, so $\partial^n \neq 0$, as the lemma claims. □

Remark 8.4. The vanishing of the Gaussian binomial coefficients at well-chosen roots of unity that we used in this proof is a very special case of the q-Lucas theorem, and it was in fact in that way that we originally proceeded. The simpler recourse to the q-binomial theorem that we used above was suggested by a comment of Richard Stanley on MathOverflow [3]. Let us note that the q-Lucas theorem was first proved by Gloria Olive in [44], where the theorem appears as Equation (1.2.4). This beautiful result, which generalizes the classical Lucas theorem for binomial coefficients, has also been proved by Jacques Désarménien in [19], by Volker Strehl in [54], by Bruce Sagan in [46], and Donald E. Knuth and Herbert S. Wilf describe in [34] a factorization of the Gaussian binomial coefficient $\binom{n}{i}_q$ with $0 < i < n$ which certainly includes the nth cyclotomic polynomial among the factors. ◊

After all this work we can state and proof the following markedly disappointing result:

Proposition 8.5. Let n and m be integers such that $1 < m$ and $m \mid n$, and let $\lambda \in \Bbbk^\times$ be a primitive mth root of unity in $\Bbbk$. There is no inner-faithful action of the generalized Taft algebra $T_n(\lambda, m)$ on A.

Proof. Let us suppose, in order to reach a contradiction, that there is an inner-faithful module-algebra action of the Hopf algebra $= T_n(\lambda, m)$ on A, and let us consider the maps $\phi : a \in A \mapsto g \triangleright a \in A$ and $\partial : a \in A \mapsto \xi \triangleright a \in A$. The relations that define the algebra T imply that $\phi^n = \mathsf{id}_A$, $\partial^m = 0$ and $\phi\partial = \lambda\partial\phi$, and the fact that what we have is a module-algebra structure that ϕ is an automorphism and ∂ a ϕ-twisted derivation.

The criterion (8.1) for inner-faithfulness from Cline's paper [15] implies that the order of ϕ is exactly n, since g has order n in T, and then, according to Proposition 2.14, up to conjugating the action of T on A by an algebra automorphism we can suppose that there is a primitive nth root of unity ω in $\Bbbk$ such that $\phi(x) = \omega x$ and $\phi(y) = \omega^{N-1} y$. Proposition 7.10 now tells us that there is an element u in the localization A_x such that the $\tilde{\phi}$-twisted derivation $\mathrm{ad}_{\tilde{\phi}}(u) : A_x \to A_x$ preserves the subalgebra A of A_x and the map ∂ coincides with its restriction $\mathrm{ad}_{\tilde{\phi}}(u)|_A$ to A. The second part of Lemma 8.2 tells us tat $\tilde{\phi}(u) = \lambda u$ and, since $\partial^m = 0$, the third part of that lemma that $n \leq m$. Of course, since m divides n we have in fact that $n = m$, and we are therefore in the situation of Lemma 8.3: this is absurd, for the lemma tells us that $\partial^n \neq 0$. □

In [45] Radford considers a more general class of generalized Taft algebras: given two positive integers n, $m \in \mathbb{N}$ such that $1 < m$ and $m \mid n$, a primitive mth root of unity λ in $\Bbbk$, and an arbitrary scalar $\tau \in \Bbbk$, he lets $T_n(\lambda, m, \tau)$ be the algebra freely generated by two letters g and ξ subject to the relations

$$g^n = 1, \qquad \xi^m = \tau(g^m - 1), \qquad g\xi = \lambda\xi g,$$

which is a Hopf algebra with respect to the comultiplication and counit such that

$$\Delta(g) = g \otimes g, \qquad \Delta(\xi) = \xi \otimes 1 + g \otimes \xi, \qquad \varepsilon(g) = 1, \qquad \varepsilon(\xi) = 0.$$

When $\tau = 0$ we obtain the algebras we considered above, since $T_n(\lambda, m, 0) = T_n(\lambda, m)$, and the connection between the two families is that $T_n(\lambda, m, 0)$ is the graded Hopf algebra corresponding to the coradical filtration of the pointed Hopf algebra $T_n(\lambda, m, \tau)$ — we refer to Section 5.2 in Montgomery's book [40] for information of this. It is a natural guess, after our last proposition, that there is also no inner-faithful module-algebra action of these more general Hopf algebras on our algebra A, although a new idea is needed to verify this. For example, the twisted derivation ∂ that corresponds to the element ξ in an action of $T_n(\lambda, m, \tau)$ is locally finite — in the sense that every element of A is contained in a finite-dimensional subspace that is invariant under ∂ — so classifying locally-finite twisted derivations of A could well be helpful.

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