Steiner Triple Systems

Existence,representation and construction

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Introduction

Outline

- Challenge on *combinatorial design*
- What is it?:
    - existence or non-existence
    - representation
    - construction

What is Steiner Triple System

(Definition) Steiner Triple Systems (STS)

is an ordered pair (S, T) (a *design*) where S is a finite set of *point*/*symbol* and T is a set of subsets of 3-symbol in which all possible pair of S are contained **once and only once**.

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More formally:

- define S such that $|S| = v$
- then $T = \{\forall\{a, b, c\} \in S \times S \times S\}$ such that $\forall a, b \in S \times S$ $a \neq b$ $\quad \sum_{\forall\{x,y,z\}\in T}(\mathbb{I}_{\{a,b\}\in\{x,y\}} + \mathbb{I}_{\{a,b\}\in\{y,z\}} + \mathbb{I}_{\{a,b\}\in\{z,x\}}) = 1$

More compact way to define STS by define the *order* v of STS by $v = |S|$

Examples of STS

$$S = \{a\},\ T = \emptyset$$
$$S = \{a, b\},\ T = \emptyset$$
$$S = \{a, b, c\},\ T = \{\{a, b, c\}\}$$
$$S = \{a, b, c, d\},\ T = \emptyset$$
$$S = \{a, b, c, d, e\},\ T = \emptyset$$
$$S = \{a, b, c, d, e, f\},\ T = \emptyset$$
$$S = \{a, b, c, d, e, f, g\},\ T =$$
$$\{\{a, b, c\}, \{c, d, e\}, \{e, f, a\}, \{f, b, d\}, \{a, g, d\}, \{e, g, b\}, \{c, g, f\}\}$$
$$\ldots$$

Balanced incomplete blocks design

(Definition) $(v, k, \lambda) - \mathrm{BIBD}$

v,k and λ be positive integers such that $v > k \geq 2$. A balanced incomplete block design is a *design* (S, T) such that satisfy these properties:

1. $|S| = v$
2. $\forall t \in T \quad |t| = k$
3. for all distinct pairs are contained in exactly λ blocks (t)

Why **balanced** and **incomplete**?

balanced they share the same property (2)

incomplete by reason of $v = |S| \quad > \quad k = |t| \ \forall t \in T$

What is Steiner Triple System 2

λ blocks (t) of $(v, k, \lambda) - \mathrm{BIBD}$ iff $\lambda = 1$, $k = 3$.

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3. $\forall s \in S$ is contained in exactly λ blocks (t)

All theory from BIBD is shared too in STS

[Kirkman, 1847]Existence proof

Theorem

A STS of order v **exists** *if and only if* $v \equiv 1, 3 \ mod(6)$

Proof.

$(\Rightarrow)$We know that all possible pairs are $\binom{v}{2}$, and by definition these pairs are partitioned (non-overlapping and union make all) into 3-element groups. Thoose groups are $|T| = \frac{\binom{v}{2}}{3} = \frac{v(v-1)}{6}$. Then for $\forall x \in S$ can be defined $T(x) = \{t \not\{x\} | x \in t \in T\}$. So if an $x \in S$ is fixed and then for every set t which contain x we remove the point x then we carry out $v - 1$ point partitioned in *2-element* set. As we can't make 2-element partition from a group of *odd* element, $v - 1$ is *even*! So v is *odd* and it's equal to say $v \equiv 1, 3, 5 mod(6)$. The $\frac{v(v-1)}{6}$ is not an integer for every $v \equiv 5 mod(6)$. As a result STS $\Rightarrow v \equiv 1, 3 \ mod(6)$ □

Existence proof 2

$(S, T) \ : |S| = v \ \wedge v \equiv 1, 3 \ mod(6) \Rightarrow STS(v)$

In addition we suppose:

- each dinstict pair of S belongs to *at least* one triple in T
- $|T| \leq \frac{v(v-1)}{6}$

Proof 1.

(Absurd) Assume the contrary and make a list L as follows: for every pair write down the triple with which it is associated. Then $|L| > \binom{v}{2}$ as there exists a pair with towo triples. Now since each triple is counted by exactly three pairs so $|T| = |L|/3 > \frac{\binom{v}{2}}{3}$, a contradiction. $\square$

Proof 2.

For each distinct pair of S belongs to at least one triple and if the number of triples is less than or equal to the right number of triples, then each pair of sumbols in S belongs to exactly one triple in T. □

Proof 3.

We costruct 2 methods to prove sufficient costraint to the Theorem by showing 2 methods:

- Bose construction
- Skolem construction

□

Representation

How to represent

- through display each 3-set of T $(\{\{a, b, c\}, \{b, d, e\}, ..., \{d, f, g\}\})$
- through a *complete graph*

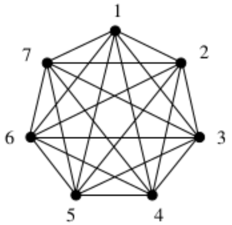

Figure: A complete graph of order $v = 7$

Example

Why a focus on representation?

- we talk about combinatorial design (display somehow somethings)
- help to design algorithm

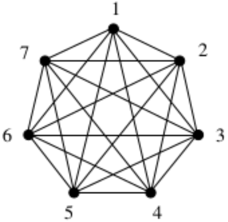

Figure: A complete graph of order $v = 7$

Focus on

How to choose a proper partition of the graph ?

Example

First non-dummy: STS of *order* 7

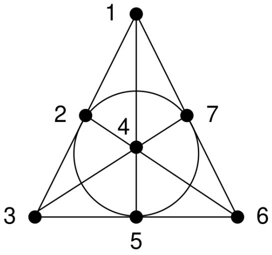

Figure: Fano plane

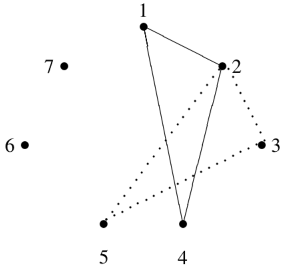

Figure: Building methods on STS(7)

Construction methods

How to create

- Bose method
- Skolem
- $6n + 5$
- With quasigroups with holes
- Wilson
- $2n + 1$
- $2n + 7$
- Even-Odd

Bose construction

We need first define:

idempotent commutative quasigroups of order $2n+1$

But first: recap

(Definition) latin square of order n

is an $n \times n$ array where each row and column contains all symbols $\{1, ..., n\}$ exactly one time.

1	3	2
2	1	3
3	2	1

Table: Latin square of order 3

(Definition) Quasigroup

A *quasigroup* of *order* n is an algebric structure, a pair $(Q, \circ)$ where $|Q| = n$ and $\circ : Q \times Q \to Q$.
$\forall a, b \in Q$ then $\exists! x, y$ (unique!) to the equations $a \circ x = b$ and $x \circ a = b$.

An examples of quasigroup are $(\mathrm{Z_n}, -)$,$(\mathrm{Z_n}, +)$.

(Example) 1723 Latin square

♠A	♥K	♦Q	♣J
♦J	♣Q	♠K	♥A
♣K	♦A	♥J	♠Q
♥Q	♠J	♣A	♦K

(Example) 1723 Latin square

♠A	♥K	♦Q	♣J
♦J	♣Q	♠K	♥A
♣K	♦A	♥J	♠Q
♥Q	♠J	♣A	♦K

♠	♥	♦	♣
♦	♣	♠	♥
♣	♦	♥	♠
♥	♠	♣	♦

(Example) 1723 Latin square

♠A	♥K	♦Q	♣J
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♠	♥	♦	♣
♦	♣	♠	♥
♣	♦	♥	♠
♥	♠	♣	♦

A	K	Q	J
J	Q	K	A
K	A	J	Q
Q	J	A	K

Quasigroup and latin square

Theorem

The multiplication table of a quasigroup is a Latin square

A quasigroup $(G, \circ)$ is a latin square of order $v = |G|$:

1	3	2
2	1	3
3	2	1

Table: Latin square of order 3

$\circ$	1	2	3
1	1	2	3
2	3	1	2
3	2	3	1

Table: Quasigroup of order 3

Quasigroup and latin square

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1	3	2
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3	2	1

Table: Latin square of order 3

$\circ$	1	2	3
1	1	2	3
2	3	1	2
3	2	3	1

Table: Quasigroup of order 3

A $(Q, \circ)$ is said:

idempotent $\forall i : 1 \leq i \leq |G|$ the cell (i, i) contains α such that $\alpha \leq i$

commutative $\forall i, j : 1 \leq i < j \leq |G|$ the cell (i, j) contains the same of (j, i)

Commutative idempotent latin square

1	3	2
3	2	1
2	1	3

Table: C. I. latin square of order 3

Commutative idempotent latin square

1	3	2
3	2	1
2	1	3

Table: C. I. latin square of order 3

How can we create a *C.I. latinsquare/quasigroup of order v* ?

contenuto...

Theorem

idempotent commutative quasigroups exist **if and only if** *they have odd order.*

Great! We look at the half of all possible

Construction method of CI quasigroup

1. Let v be the order of quasigroup, take $(Z_v, +)$ where $+$ is the addition in Z_v.
2. For all element $i := i + 1$
3. Take the elements of main diagonal $< d_1, ..., d_v >$. Build a permutation $\sigma_v = \{(d_1, 1), (d_2, 2), ..., (d_v, v)\}$.
4. Apply σ_v for all element of the *multiplication table*

As result you have a CI quasigroup.

Construction method of CI quasigroup of order 7

$Z_7,+$	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	1	2	3	4	5	6	0
2	2	3	4	5	6	0	1
3	3	4	5	6	0	1	2
4	4	5	6	0	1	2	3
5	5	6	0	1	2	3	4
6	6	0	1	2	3	4	5

	1	2	3	4	5	6	7
1	1	2	3	4	5	6	7
2	2	3	4	5	6	7	1
3	3	4	5	6	7	1	2
4	4	5	6	7	1	2	3
5	5	6	7	1	2	3	4
6	6	7	1	2	3	4	5
7	7	1	2	3	4	5	6

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Construction method of CI quasigroup of order 7

	1	2	3	4	5	6	7
1	1	2	3	4	5	6	7
2	2	3	4	5	6	7	1
3	3	4	5	6	7	1	2
4	4	5	6	7	1	2	3
5	5	6	7	1	2	3	4
6	6	7	1	2	3	4	5
7	7	1	2	3	4	5	6

$$\sigma_v = \{(1,1),(3,2),(5,3),(7,4),(2,5),(4,6),(6,7)\}$$

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Construction method of CI quasigroup of order 7

	1	2	3	4	5	6	7
1	1	2	3	4	5	6	7
2	2	3	4	5	6	7	1
3	3	4	5	6	7	1	2
4	4	5	6	7	1	2	3
5	5	6	7	1	2	3	4
6	6	7	1	2	3	4	5
7	7	1	2	3	4	5	6

	1	2	3	4	5	6	7
1	1	5	2	6	3	7	4
5	5	2	6	3	7	4	1
2	2	6	3	7	4	1	5
6	6	3	7	4	1	5	2
3	3	7	4	1	5	2	6
7	7	4	1	5	2	6	3
4	4	1	5	2	6	3	7

We apply $\sigma_v = \{(1,1),(3,2),(5,3),(7,4),(2,5),(4,6),(6,7)\}$ as result we have a idempotent commutative quasigroup of order v.

Bose construction($v \equiv 3 mod(6)$)

Let $v = 6n + 3$ and let $(Q, \circ)$ be an idempotent commutative quasigroup of order $2n + 1$, where $Q = \{1, 2, 3, ..., 2n + 1\}$. Let $S = Q \times \{1, 2, 3\}$ and define T to contain the following types of triples.

Type 1: For $1 \leq i \leq 2n + 1$, $\{(i, 1), (i, 2), (i, 3)\} \in T$

Type 2: For $1 \leq i < j \leq 2n + 1$, $\{\{(i, 1), (j, 1), (i \circ j, 2)\}, \{(i, 2), (j, 2), (i \circ j, 3)\}\{(i, 3), (j, 3), (i \circ j, 1)\}\} \in T$

Then (S, T) is a Steiner triple system of order $6n + 3$.

Type of partitions

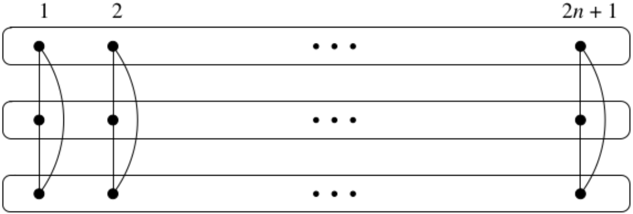

Figure: Type 1

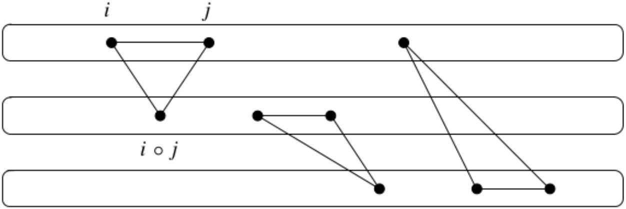

Figure: Type 2

Proof

$|T|$ is made up with 2 type:

- *Type 1*: $2n+1$ triples
- *Type 2*: $\binom{2n+1}{2}$ choices for i and j, for all of them there are 3 another type.

Then $|T| = (2n+1) + 3\frac{(2n+1)2n}{2} = \frac{(2n+1)(6n+2)}{2} = v(v-1)/6$ have the right number of triple.
To show that every pairs is contained in at least 1 triple, think about 2 possible pair of point $(a,b),(c,d) \in Q \times \{1,2,3\}$:

Cont..

$\forall (a, b), (c, d) \in Q \times \{1, 2, 3\}$

- $a = c \wedge b = d$ impossible
- if $a = c$ (so $b \neq d$) is contained in at least 1 triple of *type 1* $\{(a, 1), (a, 2), (a, 3)\}$
- if $b = d \wedge a \neq c$ is contained in at least 1 triple of *type 2* $\{\{(a, b), (c, b), (a \circ c, b + 1 mod(3)\ )\}, \{(x, 1), (a, 1), (b, 2)\}, ...\}$
- $a \neq c \wedge b \neq d$.Assume $b = 1$ and $d = 2$. Since $(Q, \circ)$ is a quasigroup $a \circ i = c$ and $j \circ a = c$ for some i, j. Because the *commutative* $i = j$ and because *idempotent* only $a \circ a = a$, all the others we are sure that $i \neq a$. So $\{(a, 1), (i, 1), (a \circ i = c, 2)\}$.

All possible point have been shown that are in T.

Skolem construction ($v \equiv 3 mod(6)$)

Let $v = 6n + 1$ and let $(Q, \circ)$ be a half-idempotent commutative quasigroup of order $2n$, where ....

Half-idempotent commutative latin square

(Definition)Half-idempotent commutative latin square

A latin square (multiplication table of quasigroup of the same order) L of order $2n$ is *half-idempotent* if the cells (i, i) contain the same symbol i of the cell $(n+i, n+i)$ $\quad \forall 1 \leq i \leq n$

1	3	2	4
3	2	4	1
2	4	1	3
4	1	3	2

Half-idempotent latin square of order 4 $(n = 2)$

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2	4	1	3
4	1	3	2

Half-idempotent latin square of order 4 $(n = 2)$

Commutative half-idempotent latin squares exist for all *even* order.

Example half-idempotent quasigroup

1	3	2	4
3	2	4	1
2	4	1	3
4	1	3	2

1	4	2	5	3	6
4	2	5	3	6	1
2	5	3	6	1	4
5	3	6	1	4	2
3	6	1	4	2	5
6	1	4	2	5	3

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2	5	3	6	1	4
5	3	6	1	4	2
3	6	1	4	2	5
6	1	4	2	5	3

How can we algorithmically build a H-I Latin Square?

How costruct H-I latin square/quasigroup

0	1	2	3	4	5
1	2	3	4	5	0
2	3	4	5	0	1
3	4	5	0	1	2
4	5	0	1	2	3
5	0	1	2	3	4

Table: Quasigroup $(\mathbb{Z}_6, +mod(6))$

How costruct H-I latin square/quasigroup

0	1	2	3	4	5
1	2	3	4	5	0
2	3	4	5	0	1
3	4	5	0	1	2
4	5	0	1	2	3
5	0	1	2	3	4

Table: Quasigroup $(\mathbb{Z}_6, +mod(6))$

The bijection σ is built increasing all value by 1(or by taking the next element). Then by taking the main diagonal from the left grid $(< 1, 3, 5, ... >)$ and assign the right number $(< 1, 2, 3, ... >)$

How costruct H-I latin square/quasigroup

0	1	2	3	4	5
1	2	3	4	5	0
2	3	4	5	0	1
3	4	5	0	1	2
4	5	0	1	2	3
5	0	1	2	3	4

Table: Quasigroup $(\mathbb{Z}_6, +mod(6))$

1	4	2	5	3	6
4	2	5	3	6	1
2	5	3	6	1	4
5	3	6	1	4	2
3	6	1	4	2	5
6	1	4	2	5	3

Table: Applied func σ on the quasigroup

The bijection σ is built increasing all value by 1(or by taking the next element). Then by taking the main diagonal from the left grid $(< 1, 3, 5, ... >)$ and assign the right number $(< 1, 2, 3, ... >)$

Example Skolem construction

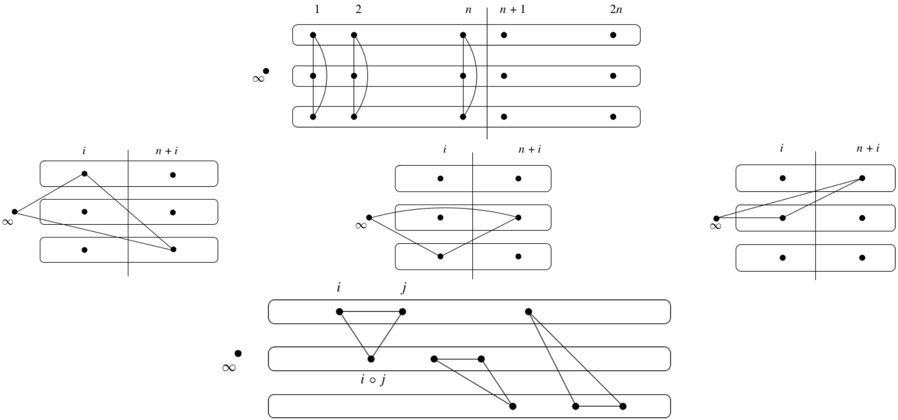
Let $v = 6n + 1$ and let $(Q, \circ)$ be a half-idempotent commutative quasigroup of order $2n$, where $Q = \{1, 2, 3, ..., 2n\}$. Let $S = \{\infty\} \cup (Q \times \{1, 2, 3\})$. We define T as follow:

Type 1 : for $1 \leq i \leq n$, $\{(i, 1), (i, 2), (i, 3)\} \in T$

Type 2 : for $1 \leq i \leq n$, $\{\infty, (n + i, 3), (i, 2)\}, \{\infty, (n + i, 2), (n + i, 3)\}, \{\infty, (n + i, 3), (i, 1)\} \in T$

Type 3 : for $1 \leq i < j \leq 2n$,$\{(i, 1), (j, 1), (i \circ j, 2)\}, \{(i, 2), (j, 2), (i \circ j, 3)\}, \{\} \in T$

Example Skolem construction



Proof

We have to prove (in a similar way of Bose) there are the right number of $t \in T$ and every pair (from $\binom{6n+1}{2}$)) is contained almost 1.

1: Right number of $|T|$.

We sum up 3 different element:

- *type 1*: for $1 \leq i \leq n$, $\{(i,1),(i,2),(i,3)\} \in T$ are n
- *type 2*: for $1 \leq i \leq n$, $\{\infty,(n+i,3),(i,2)\},\{\infty,(n+i,2),(n+i,3)\},\{\infty,(n+i,3),(i,1)\} \in T$ are $3n$
- *type 3*: for $1 \leq i < j \leq 2n$,$\{(i,1),(j,1),(i \circ j,2)\},\{(i,2),(j,2),(i \circ j,3)\},\{\} \in T$ are $3\binom{2n}{2}$

We have to prove $|T| = \frac{v(v-1)}{6} = \frac{(6n+1)(6n)}{6} = n + n + 3\binom{2n}{2}$.
Easily
$n + 3n + 3\binom{2n}{2} = \frac{2n*4+2n*(6n-3)}{2} = \frac{2n(6n+1)}{2} = \frac{3}{3}\frac{2n(6n+1)}{2} = \frac{6n(6n+1)}{2} = |T|$ $\square$

2: every pair of point $\in T$.

We have to prove all possible pair of point (a, b) and (c, d):

- $a = c = \infty \wedge b \neq d$
- $a = c \neq \infty \wedge b \neq d$
- $a = \infty \neq b \wedge b = d$
- $a \neq c \wedge b \neq d$

$\square$

2: every pair of point $\in T$.

We have to prove all possible pair of point (a, b) and (c, d):

- $a = c = \infty \wedge b \neq d$
- $a = c \neq \infty \wedge b \neq d$
- $a = \infty \neq b \wedge b = d$
- $a \neq c \wedge b \neq d$

$\square$

All covered by Skolem construction $\wedge$ $|T|$ is the right number $\Rightarrow$ is a STS$(2n)$

Practical example

[1850] The Lady's and Gentleman's Diary/Kirkman's shoolgirl problem

A teacher would like to take 15 schoolgirls out for a walk, the girls being arranged in 5 rows of three. The teacher would like to ensure equal chances of friendship between any two girls. Hence it is desirable to find different row arrangements for the 7 days of the week such that any pair of girls walk in the same row exactly one day of the week.

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Solution

[1971 (Ray-Chaudhuri and Wilson)] it asks for a Steiner Triple System on $6t+3$ varieties whose blocks can be partitioned into $3t+1$ sets so that any variety appears only once in a set.

Important remark

Theorem

If a $(v,k,\lambda)-BIBD$ *exist, then* $\lambda(v-1)\equiv 0(mod(k-1))$ *and* $\lambda v(v-1)\equiv(modk(k-1))$.

Theorem

A STS of order v **exists** *if and only if* $v \equiv 1, 3 \ mod(6)$

Theorem

If a $(v, k, \lambda) - BIBD$ *exist, then* $\lambda(v-1) \equiv 0(mod(k-1))$ *and* $\lambda v(v-1) \equiv (modk(k-1))$.

Only **necessary**.

Theorem

A STS of order v **exists** *if and only if* $v \equiv 1, 3 \ mod(6)$

Necessary and **sufficient**.

STS on OEIS

Steiner systems , sequences related to :

Steiner systems, quadruple (SQS's): A051390* A124120 A124119

Steiner systems: A001293* (S(5,8,24))

Steiner systems: A187567 and A187585 (S(2,4,n))

Steiner triple systems (STS's): A001201*, A030128*, A030129*, A051390*, A002885 (cyclic), A006181, A006182, A051391

Isomorphisms between two *design*

Isomorphism

Two designs (X, A) and (Y, B) where $|X| = |Y|$ are *isomorphic* if there *exists* a bijection $\alpha : X \to Y$ such that:

$$[\{\alpha(x) : x \in A\} : A\mathrm{A}] = \mathrm{B}$$

Then α is called isomorphism.

$$X = \{1, 2, 3, 4, 5, 6, 7\}, \quad \text{and}$$
$$\mathcal{A} = \{123, 145, 167, 246, 257, 347, 356\};$$

$$Y = \{a, b, c, d, e, f, g\}, \quad \text{and}$$
$$\mathcal{B} = \{abd, bce, cdf, deg, aef, bfg, acg\}.$$

Beyond Existence or non-existence

The existence of non-isomorphic STS is complex and a open field.

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(A030129) Number of nonisomorphic Steiner triple systems (STS's) $S(2,3,n)$ on n points

$< 1, 0, 1, 0, 0, 0, 1, 0, 1, 0, 0, 0, 2, 0, 80, 0, 0, 0, 11084874829 >$

(A051390)Number of nonisomorphic Steiner quadruple systems (SQS's) of order n

$< 1, 1, 0, 1, 0, 0, 0, 1, 0, 1, 0, 0, 0, 4, 0, 1054163 >$

References