# Steiner Triple Systems

## Existence, representation and construction

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# Introduction

# Outline

- Challenge on *combinatorial design*
- What is STS?:
  - existence or non-existence
  - representation
  - construction

# What is Steiner Triple System

**(Definition) Steiner Triple Systems (STS)**

is an ordered pair $(S, T)$ (a *design*) where $S$ is a finite set of *point*/*symbol* and $T$ is a set of subsets of 3-symbol in which all possible pair of $S$ are contained **once and only once**.

# What is Steiner Triple System

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More formally:

- define $S$ such that $|S| = v$
- then $T = \{\forall\{a, b, c\} \in S \times S \times S\}$ such that $\forall a, b \in S \times S$ $a \neq b$ $\sum_{\forall\{x,y,z\}\in T}(\mathbb{I}_{\{a,b\}\in\{x,y\}} + \mathbb{I}_{\{a,b\}\in\{y,z\}} + \mathbb{I}_{\{a,b\}\in\{z,x\}}) = 1$

More compact way to define STS by define the *order* $v$ of STS by $v = |S|$

# Examples of STS

$$S = \{a\}, T = \emptyset$$
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$$S = \{a, b, c, d\}, T = \emptyset$$
$$S = \{a, b, c, d, e\}, T = \emptyset$$
$$S = \{a, b, c, d, e, f\}, T = \emptyset$$

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$$S = \{a, b, c, d, e, f\},\ T = \emptyset$$
$$S = \{a, b, c, d, e, f, g\},\ T =$$
$$\{\{a, b, c\}, \{c, d, e\}, \{e, f, a\}, \{f, b, d\}, \{a, g, d\}, \{e, g, b\}, \{c, g, f\}\}$$
$$\ldots$$

# Balanced incomplete blocks design

**(Definition) $(v, k, \lambda) - \mathrm{BIBD}$**

$v$,$k$ and $\lambda$ be positive integers such that $v > k \geq 2$. A balanced incomplete block design is a *design* $(S, T)$ such that satisfy these properties:

1. $|S| = v$
2. $\forall t \in T \quad |t| = k$
3. for all distinct pairs are contained in exactly $\lambda$ blocks $(t)$

Why **balanced** and **incomplete**?

balanced they share the same property (2)

incomplete by reason of $v = |S| \quad > \quad k = |t| \; \forall t \in T$

# What is Steiner Triple System 2

$\lambda$ blocks $(t)$ of $(v, k, \lambda) - \mathrm{BIBD}$ iff $\lambda = 1$, $k = 3$.

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$v$,$k$ and $\lambda$ be positive integers such that $v > k \geq 2$. A balanced incomplete block design is a *design* $(S, T)$ such that satisfy these properties:

1. $|S| = v$
2. $\forall t \in T \quad |t| = k$
3. $\forall s \in S$ is contained in exactly $\lambda$ blocks $(t)$

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1. $|S| = v$
2. $\forall t \in T \quad |t| = k$
3. $\forall s \in S$ is contained in exactly $\lambda$ blocks $(t)$

All theory from BIBD is shared too in STS

# Representation

of STS(v)

# How to represent

- through display each 3-set of $T$ and $S$ $(\{a, b, c, d, e, f, g\},\{ \{a, b, c\}, \{b, d, e\}, ..., \{d, f, g\}\})$
- through a *complete graph*

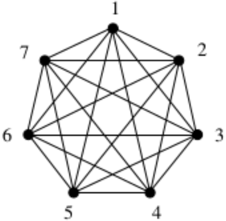
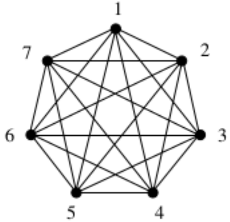



Figure: A complete graph of order $v = 7$

# Example

Why a focus on representation?

- we talk about combinatorial design (display somehow somethings)
- help to design algorithm

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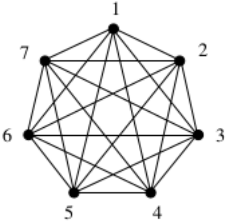


Figure: A complete graph of order $v = 7$

**Focus on**

How to choose a proper partition of the graph ?

# Example

First non-dummy: STS of *order* 7

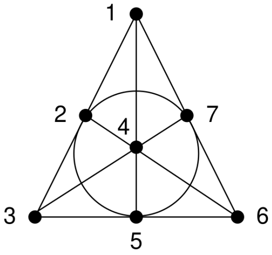


Figure: Fano plane

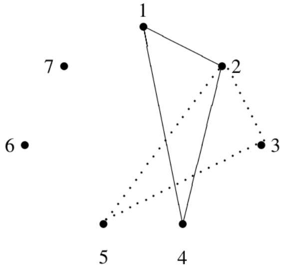


Figure: Building methods on STS(7)

# [Kirkman, 1847]Existence proof

**Theorem**

*A STS of order v* **exists** *if and only if* $v \equiv 1, 3 \; mod(6)$

**Proof.**

$(\Rightarrow)$We know that all possible pairs are $\binom{v}{2}$, and by definition these pairs are partitioned (non-overlapping and union make all) into 3-element groups. Those groups are $|T| = \frac{\binom{v}{2}}{3} = \frac{v(v-1)}{6}$. Then for $\forall x \in S$ can be defined $T(x) = \{t \setminus \{x\} | x \in t \in T\}$. So if an $x \in S$ is fixed and then for every set $t$ which contain $x$ we remove the point $x$ then we carry out $v - 1$ point partitioned in *2-element* set. As we can't make 2-element partition from a group of *odd* element, $v - 1$ is *even*! So $v$ is *odd* and it's equal to say $v \equiv 1, 3, 5 mod(6)$. The $\frac{v(v-1)}{6}$ is not an integer for every $v \equiv 5 mod(6)$. As a result STS $\Rightarrow v \equiv 1, 3 \; mod(6)$ □

# Existence proof 2

$(S, T)\ : |S| = v\ \wedge v \equiv 1, 3\ mod(6) \Rightarrow STS(v)$

In addition we suppose:

- each dinstict pair of $S$ belongs to *at least* one triple in $T$
- $|T| \leq \frac{v(v-1)}{6}$

## Proof 1.

(Absurd) Assume the contrary and make a list $L$ as follows: for every pair write down the triple with which it is associated. Then $|L| > \binom{v}{2}$ as there exists a pair with tow triples. Now since each triple is counted by exactly three pairs so $|T| = |L|/3 > \frac{\binom{v}{2}}{3}$, a contradiction. $\square$

## Proof 2.

For each distinct pair of $S$ belongs to at least one triple and if the number of triples is less than or equal to the right number of triples, then each pair of sumbols in $S$ belongs to exactly one triple in $T$. □

## Proof 3.

We costruct 2 methods to prove sufficient costraint to the Theorem by showing 2 methods:

- Bose construction
- Skolem construction

□

# Construction methods

# How to create

- Bose method
- Skolem method
- $6n + 5$
- With quasigroups with holes
- Wilson
- and so many

# Bose construction

We need first define:

**idempotent commutative quasigroups of order** $2n + 1$

# But first: recap

**(Definition) latin square of order $n$**

is an $n \times n$ array where each row and column contains all symbols $\{1, ..., n\}$ exactly one time.

| | | |
|---|---|---|
| 1 | 3 | 2 |
| 2 | 1 | 3 |
| 3 | 2 | 1 |

Table: Latin square of order 3

**(Definition) Quasigroup**

A *quasigroup* of *order* $n$ is an algebric structure, a pair $(Q, \circ)$ where $|Q| = n$ and $\circ : Q \times Q \to Q$.
$\forall a, b \in Q$ then $\exists! x, y$ (unique!) to the equations $a \circ x = b$ and $x \circ a = b$.

An examples of quasigroup are $(\mathrm{Z_n}, -)$,$(\mathrm{Z_n}, +)$.

# (Example) 1723 Latin square

| | | | |
|---|---|---|---|
| ♠A | ♥K | ♦Q | ♣J |
| ♦J | ♣Q | ♠K | ♥A |
| ♣K | ♦A | ♥J | ♠Q |
| ♥Q | ♠J | ♣A | ♦K |

# (Example) 1723 Latin square

| | | | |
|---|---|---|---|
| ♠A | ♥K | ♦Q | ♣J |
| ♦J | ♣Q | ♠K | ♥A |
| ♣K | ♦A | ♥J | ♠Q |
| ♥Q | ♠J | ♣A | ♦K |

| | | | |
|---|---|---|---|
| ♠ | ♥ | ♦ | ♣ |
| ♦ | ♣ | ♠ | ♥ |
| ♣ | ♦ | ♥ | ♠ |
| ♥ | ♠ | ♣ | ♦ |

# (Example) 1723 Latin square

| | | | |
|---|---|---|---|
| ♠A | ♥K | ♦Q | ♣J |
| ♦J | ♣Q | ♠K | ♥A |
| ♣K | ♦A | ♥J | ♠Q |
| ♥Q | ♠J | ♣A | ♦K |

| | | | |
|---|---|---|---|
| ♠ | ♥ | ♦ | ♣ |
| ♦ | ♣ | ♠ | ♥ |
| ♣ | ♦ | ♥ | ♠ |
| ♥ | ♠ | ♣ | ♦ |

| | | | |
|---|---|---|---|
| A | K | Q | J |
| J | Q | K | A |
| K | A | J | Q |
| Q | J | A | K |

# Quasigroup and latin square

**Theorem**

*The multiplication table of a quasigroup is a Latin square*

A quasigroup $(G, \circ)$ is a latin square of order $v = |G|$:

| | | |
|---|---|---|
| 1 | 3 | 2 |
| 2 | 1 | 3 |
| 3 | 2 | 1 |

Table: Latin square of order 3

| $\circ$ | 1 | 2 | 3 |
|---|---|---|---|
| 1 | 1 | 2 | 3 |
| 2 | 3 | 1 | 2 |
| 3 | 2 | 3 | 1 |

Table: Quasigroup of order 3

# Quasigroup and latin square

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| $\circ$ | 1 | 2 | 3 |
|---|---|---|---|
| 1 | 1 | 2 | 3 |
| 2 | 3 | 1 | 2 |
| 3 | 2 | 3 | 1 |

Table: Quasigroup of order 3

A $(Q, \circ)$ is said:

idempotent $\forall i : 1 \leq i \leq |G|$ the cell $(i, i)$ contains $\alpha$ such that $\alpha \leq i$

commutative $\forall i, j : 1 \leq i < j \leq |G|$ the cell $(i, j)$ contains the same of $(j, i)$

# Commutative idempotent latin square

| | | |
|---|---|---|
| 1 | 3 | 2 |
| 3 | 2 | 1 |
| 2 | 1 | 3 |

Table: C. I. latin square of order 3

# Commutative idempotent latin square

| | | |
|---|---|---|
| 1 | 3 | 2 |
| 3 | 2 | 1 |
| 2 | 1 | 3 |

Table: C. I. latin square of order 3

How can we create a *C.I. latinsquare/quasigroup of order v* ?

**Theorem**

*idempotent commutative quasigroups exist* **if and only if** *they have odd order.*

Great! We look at the half of all possible

# Construction method of CI quasigroup

1. Let $v$ be the order of quasigroup, take $(Z_v, +)$ where $+$ is the addition in $Z_v$.
2. For all element $i := i + 1$
3. Take the elements of main diagonal $< d_1, ..., d_v >$. Build a permutation $\sigma_v = \{(d_1, 1), (d_2, 2), ..., (d_v, v)\}$.
4. Apply $\sigma_v$ for all element of the *multiplication table*

As result you have a CI quasigroup.

# Construction method of CI quasigroup of order 7

| $Z_7,+$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| 1 | 1 | 2 | 3 | 4 | 5 | 6 | 0 |
| 2 | 2 | 3 | 4 | 5 | 6 | 0 | 1 |
| 3 | 3 | 4 | 5 | 6 | 0 | 1 | 2 |
| 4 | 4 | 5 | 6 | 0 | 1 | 2 | 3 |
| 5 | 5 | 6 | 0 | 1 | 2 | 3 | 4 |
| 6 | 6 | 0 | 1 | 2 | 3 | 4 | 5 |

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| 1 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| 2 | 2 | 3 | 4 | 5 | 6 | 7 | 1 |
| 3 | 3 | 4 | 5 | 6 | 7 | 1 | 2 |
| 4 | 4 | 5 | 6 | 7 | 1 | 2 | 3 |
| 5 | 5 | 6 | 7 | 1 | 2 | 3 | 4 |
| 6 | 6 | 7 | 1 | 2 | 3 | 4 | 5 |
| 7 | 7 | 1 | 2 | 3 | 4 | 5 | 6 |

1. Let $v$ be the order of quasigroup, take $(Z_v, +)$ where $+$ is the addition in $Z_v$.
2. For all element $i := i + 1$

# Construction method of CI quasigroup of order 7

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| 1 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| 2 | 2 | 3 | 4 | 5 | 6 | 7 | 1 |
| 3 | 3 | 4 | 5 | 6 | 7 | 1 | 2 |
| 4 | 4 | 5 | 6 | 7 | 1 | 2 | 3 |
| 5 | 5 | 6 | 7 | 1 | 2 | 3 | 4 |
| 6 | 6 | 7 | 1 | 2 | 3 | 4 | 5 |
| 7 | 7 | 1 | 2 | 3 | 4 | 5 | 6 |

$$\sigma_v = \{(1,1),(3,2),(5,3),(7,4),(2,5),(4,6),(6,7)\}$$

1. Let $v$ be the order of quasigroup, take $(Z_v, +)$ where $+$ is the addition in $Z_v$.
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# Construction method of CI quasigroup of order 7

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| 1 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| 2 | 2 | 3 | 4 | 5 | 6 | 7 | 1 |
| 3 | 3 | 4 | 5 | 6 | 7 | 1 | 2 |
| 4 | 4 | 5 | 6 | 7 | 1 | 2 | 3 |
| 5 | 5 | 6 | 7 | 1 | 2 | 3 | 4 |
| 6 | 6 | 7 | 1 | 2 | 3 | 4 | 5 |
| 7 | 7 | 1 | 2 | 3 | 4 | 5 | 6 |

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| 1 | 1 | 5 | 2 | 6 | 3 | 7 | 4 |
| 2 | 5 | 2 | 6 | 3 | 7 | 4 | 1 |
| 3 | 2 | 6 | 3 | 7 | 4 | 1 | 5 |
| 4 | 6 | 3 | 7 | 4 | 1 | 5 | 2 |
| 5 | 3 | 7 | 4 | 1 | 5 | 2 | 6 |
| 6 | 7 | 4 | 1 | 5 | 2 | 6 | 3 |
| 7 | 4 | 1 | 5 | 2 | 6 | 3 | 7 |

We apply $\sigma_v = \{(1,1),(3,2),(5,3),(7,4),(2,5),(4,6),(6,7)\}$ as result we have a idempotent commutative quasigroup of order $v$.

# Bose construction($v \equiv 3 mod(6)$)

Let $v = 6n + 3$ and let $(Q, \circ)$ be an idempotent commutative quasigroup of order $2n + 1$, where $Q = \{1, 2, 3, ..., 2n + 1\}$. Let $S = Q \times \{1, 2, 3\}$ and define $T$ to contain the following types of triples.

Type 1: For $1 \leq i \leq 2n + 1$, $\{(i, 1), (i, 2), (i, 3)\} \in T$

Type 2: For $1 \leq i < j \leq 2n + 1$, $\{\{(i, 1), (j, 1), (i \circ j, 2)\}, \{(i, 2), (j, 2), (i \circ j, 3)\}\{(i, 3), (j, 3), (i \circ j, 1)\}\} \in T$

Then $(S, T)$ is a Steiner triple system of order $6n + 3$.

# Type of partitions

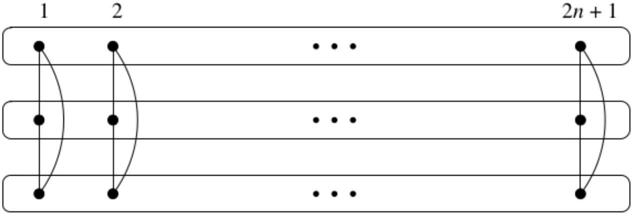



Figure: Type 1

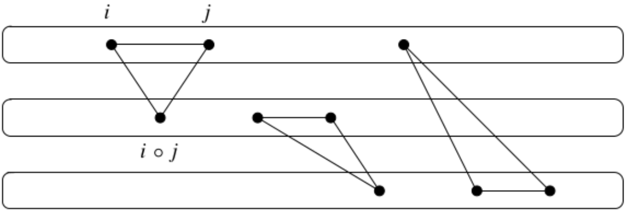



Figure: Type 2

We need to prove that all pairs are v(v-1)/6 then, as result of all pair is contained only one, we have proved the corectness.

## Proof right number

$|T|$ is made up with 2 type:

- *Type 1*: $2n+1$ triples
- *Type 2*: $\binom{2n+1}{2}$ choices for $i$ and $j$, for all of them there are 3 another type.

Then $|T| = (2n+1) + 3\frac{(2n+1)2n}{2} = \frac{(2n+1)(6n+2)}{2} = v(v-1)/6$ have the right number of triple.

To show that every pair is contained in at least 1 triple, think about 2 possible pairs of point $(a, b), (c, d) \in Q \times \{1, 2, 3\}$:

# Cont..

$\forall (a, b), (c, d) \in Q \times \{1, 2, 3\}$

- $a = c \wedge b = d$ impossible
- if $a = c$ (so $b \neq d$) is contained in at least 1 triple of *type 1* $\{(a, 1), (a, 2), (a, 3)\}$
- if $b = d \wedge a \neq c$ is contained in at least 1 triple of *type 2* $\{\{(a, b), (c, b), (a \circ c, b + 1 mod(3)\ )\}, \{(x, 1), (a, 1), (b, 2)\}, ...\}$
- $a \neq c \wedge b \neq d$.Assume $b = 1$ and $d = 2$. Since $(Q, \circ)$ is a quasigroup $a \circ i = c$ and $j \circ a = c$ for some $i, j$. Because the *commutative* $i = j$ and because *idempotent* only $a \circ a = a$, all the others we are sure that $i \neq a$. So $\{(a, 1), (i, 1), (a \circ i = c, 2)\}$.

All possible points have been shown that are in $T$.

# Skolem construction ($v \equiv 3 mod(6)$)

Let $v = 6n + 1$ and let $(Q, \circ)$ be a half-idempotent commutative quasigroup of order $2n$, where ....

# Half-idempotent commutative latin square

## (Definition)Half-idempotent commutative latin square

A latin square (multiplication table of quasigroup of the same order) $L$ of order $2n$ is *half-idempotent* if the cells $(i, i)$ contain the same symbol $i$ of the cell $(n+i, n+i)$ $\quad \forall 1 \leq i \leq n$

| | | | |
|---|---|---|---|
| 1 | 3 | 2 | 4 |
| 3 | 2 | 4 | 1 |
| 2 | 4 | 1 | 3 |
| 4 | 1 | 3 | 2 |

Half-idempotent latin square of order 4 $(n = 2)$

# Half-idempotent commutative latin square

## (Definition)Half-idempotent commutative latin square

A latin square (multiplication table of quasigroup of the same order) $L$ of order $2n$ is *half-idempotent* if the cells $(i, i)$ contain the same symbol $i$ of the cell $(n + i, n + i) \quad \forall 1 \leq i \leq n$

| | | | |
|---|---|---|---|
| 1 | 3 | 2 | 4 |
| 3 | 2 | 4 | 1 |
| 2 | 4 | 1 | 3 |
| 4 | 1 | 3 | 2 |

Half-idempotent latin square of order 4 $(n = 2)$

*Commutative half-idempotent latin squares* exist for all *even* order.

# Example half-idempotent quasigroup

| | | | |
|---|---|---|---|
| 1 | 3 | 2 | 4 |
| 3 | 2 | 4 | 1 |
| 2 | 4 | 1 | 3 |
| 4 | 1 | 3 | 2 |

| | | | | | |
|---|---|---|---|---|---|
| 1 | 4 | 2 | 5 | 3 | 6 |
| 4 | 2 | 5 | 3 | 6 | 1 |
| 2 | 5 | 3 | 6 | 1 | 4 |
| 5 | 3 | 6 | 1 | 4 | 2 |
| 3 | 6 | 1 | 4 | 2 | 5 |
| 6 | 1 | 4 | 2 | 5 | 3 |

# Example half-idempotent quasigroup

| | | | |
|---|---|---|---|
| 1 | 3 | 2 | 4 |
| 3 | 2 | 4 | 1 |
| 2 | 4 | 1 | 3 |
| 4 | 1 | 3 | 2 |

| | | | | | |
|---|---|---|---|---|---|
| 1 | 4 | 2 | 5 | 3 | 6 |
| 4 | 2 | 5 | 3 | 6 | 1 |
| 2 | 5 | 3 | 6 | 1 | 4 |
| 5 | 3 | 6 | 1 | 4 | 2 |
| 3 | 6 | 1 | 4 | 2 | 5 |
| 6 | 1 | 4 | 2 | 5 | 3 |

How can we algorithmically build a H-I Latin Square?

# How to construct H-I latin square/quasigroup

| | | | | | |
|---|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 4 | 5 |
| 1 | 2 | 3 | 4 | 5 | 0 |
| 2 | 3 | 4 | 5 | 0 | 1 |
| 3 | 4 | 5 | 0 | 1 | 2 |
| 4 | 5 | 0 | 1 | 2 | 3 |
| 5 | 0 | 1 | 2 | 3 | 4 |

Table: Quasigroup $(\mathbb{Z}_6, +mod(6))$

# How to construct H-I latin square/quasigroup

| | | | | | |
|---|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 4 | 5 |
| 1 | 2 | 3 | 4 | 5 | 0 |
| 2 | 3 | 4 | 5 | 0 | 1 |
| 3 | 4 | 5 | 0 | 1 | 2 |
| 4 | 5 | 0 | 1 | 2 | 3 |
| 5 | 0 | 1 | 2 | 3 | 4 |

Table: Quasigroup $(\mathbb{Z}_6, +mod(6))$

The bijection $\sigma$ is built increasing all value by 1(or by taking the next element). Then by taking the main diagonal from the left grid $(< 1, 3, 5, ... >)$ and assign the right number $(< 1, 2, 3, ... >)$

# How to construct H-I latin square/quasigroup

| | | | | | |
|---|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 4 | 5 |
| 1 | 2 | 3 | 4 | 5 | 0 |
| 2 | 3 | 4 | 5 | 0 | 1 |
| 3 | 4 | 5 | 0 | 1 | 2 |
| 4 | 5 | 0 | 1 | 2 | 3 |
| 5 | 0 | 1 | 2 | 3 | 4 |

Table: Quasigroup $(\mathbb{Z}_6, +mod(6))$

| | | | | | |
|---|---|---|---|---|---|
| 1 | 4 | 2 | 5 | 3 | 6 |
| 4 | 2 | 5 | 3 | 6 | 1 |
| 2 | 5 | 3 | 6 | 1 | 4 |
| 5 | 3 | 6 | 1 | 4 | 2 |
| 3 | 6 | 1 | 4 | 2 | 5 |
| 6 | 1 | 4 | 2 | 5 | 3 |

Table: Applied func $\sigma$ on the quasigroup

The bijection $\sigma$ is built increasing all value by 1(or by taking the next element). Then by taking the main diagonal from the left grid $(< 1, 3, 5, ... >)$ and assign the right number $(< 1, 2, 3, ... >)$

# Example Skolem construction

Let $v = 6n + 1$ and let $(Q, \circ)$ be a half-idempotent commutative quasigroup of order $2n$, where $Q = \{1, 2, 3, ...., 2n\}$. Let $S = \{\infty\} \cup (Q \times \{1, 2, 3\})$. We define $T$ as follow:

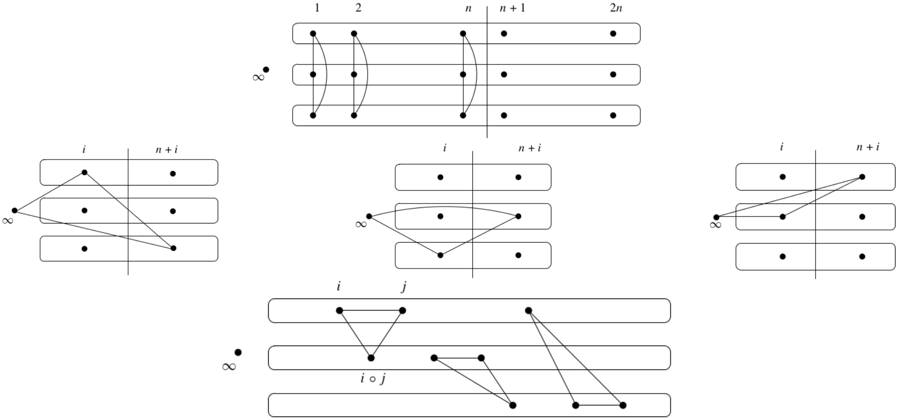
Type 1 : for $1 \leq i \leq n$, $\{(i, 1), (i, 2), (i, 3)\} \in T$

Type 2 : for $1 \leq i \leq n$ , $\{\infty, (n + i, 1), (i, 2)\}, \{\infty, (n + i, 2), (n + i, 3)\}, \{\infty, (n + i, 3), (i, 1)\} \in T$

Type 3 : for $1 \leq i < j \leq 2n$,$\{(i, 1), (j, 1), (i \circ j, 2)\}, \{(i, 2), (j, 2), (i \circ j, 3)\}, \{(i, 3), (j, 3), (i \circ j, 1)\} \in T$

Then $(S, T)$ is STS$(6n + 1)$.

# Example Skolem construction



# Proof

We have to prove (in a similar way of Bose) there are the right number of $t \in T$ and every pair (from $\binom{6n+1}{2}$) is contained almost 1.

## 1: Right number of $|T|$.

We sum up 3 different element:

- *type 1*: for $1 \leq i \leq n$, $\{(i,1),(i,2),(i,3)\} \in T$ are $n$
- *type 2*: for $1 \leq i \leq n$ , $\{\infty,(n+i,3),(i,2)\},\{\infty,(n+i,2),(n+i,3)\},\{\infty,(n+i,3),(i,1)\} \in T$ are $3n$
- *type 3*: for $1 \leq i < j \leq 2n$,$\{(i,1),(j,1),(i \circ j,2)\},\{(i,2),(j,2),(i \circ j,3)\},\{\} \in T$ are $3\binom{2n}{2}$

We have to prove $|T| = \frac{v(v-1)}{6} = \frac{(6n+1)(6n)}{6} = n + n + 3\binom{2n}{2}$.
Easily
$n + 3n + 3\binom{2n}{2} = \frac{2n*4+2n*(6n-3)}{2} = \frac{2n(6n+1)}{2} = \frac{3}{3}\frac{2n(6n+1)}{2} = \frac{6n(6n+1)}{2} = |T|$ □

## 2: every pair of point $\in T$.

We have to prove all possible pair of point $(a, b)$ and $(c, d)$:

- $a = c = \infty \wedge b \neq d$
- $a = c \neq \infty \wedge b \neq d$
- $a = \infty \neq b \wedge b = d$
- $a \neq c \wedge b \neq d$

□

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$\square$

All covered by Skolem construction $\wedge$ $|T|$ is the right number $\Rightarrow$ is a STS$(2n)$

# Practical example

# [1850] The Lady's and Gentleman's Diary/Kirkman's shoolgirl problem

*A teacher would like to take 15 schoolgirls out for a walk, the girls being arranged in 5 rows of three. The teacher would like to ensure equal chances of friendship between any two girls. Hence it is desirable to find different row arrangements for the 7 days of the week such that any pair of girls walk in the same row exactly one day of the week.*

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**Solution**

[1971 (Ray-Chaudhuri and Wilson)] it asks for a Steiner Triple System on $6t + 3$ varieties whose blocks can be partitioned into $3t + 1$ sets so that any variety appears only once in a set.

# Important remark

### Theorem

*If a* $(v, k, \lambda) - BIBD$ *exist, then* $\lambda(v-1) \equiv 0(mod(k-1))$ *and* $\lambda v(v-1) \equiv (modk(k-1))$.

### Theorem

*A STS of order v* **exists** *if and only if* $v \equiv 1, 3\ mod(6)$

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Only **necessary**.

**Theorem**

*A STS of order v* **exists** *if and only if* $v \equiv 1, 3\ mod(6)$

**Necessary** and **sufficient**.

# STS on OEIS

**Steiner systems , sequences related to :**

Steiner systems, quadruple (SQS's): A051390* A124120 A124119

Steiner systems: A001293* (S(5,8,24))

Steiner systems: A187567 and A187585 (S(2,4,n))

Steiner triple systems (STS's): A001201*, A030128*, A030129*, A051390*, A002885 (cyclic), A006181, A006182, A051391

# Isomorphisms between two *design*

## Isomorphism

Two designs $(X, \mathrm{A})$ and $(Y, \mathrm{B})$ where $|X| = |Y|$ are *isomorphic* if there *exists* a bijection $\alpha : X \to Y$ such that:

$$\{\alpha(x) : x \in A\} = \mathrm{B}$$

Then $\alpha$ is called isomorphism.

$$X = \{1, 2, 3, 4, 5, 6, 7\}, \quad \text{and}$$
$$\mathcal{A} = \{123, 145, 167, 246, 257, 347, 356\};$$

$$Y = \{a, b, c, d, e, f, g\}, \quad \text{and}$$
$$\mathcal{B} = \{abd, bce, cdf, deg, aef, bfg, acg\}.$$

# Beyond Existence or non-existence

| | $t_1$ |
|---|---|
| 1 | 1 |
| 2 | 1 |
| 3 | 1 |

Table: Incidence matrix of STS(3)

## Incidence matrix

Fixed a design $(S, T) \equiv (\{s_1, ..., s_{|S|}\}, \{t_1, ..., t_{|T|}\})$ let all elements $m_{i,j}$ of *incidence matrix* of a design $(S, T)$ be:

$$m_{i,j} = \begin{cases} 1 & \text{if } s_i \in t_j \\ 0 & \text{if } s_i \notin t_j \end{cases}$$

# 2-isomorphic sts(7)

$v = 7, \quad k = \binom{7}{2}/3$

| | | | | | | |
|---|---|---|---|---|---|---|
| 1 | 2 | | 4 | | | |
| | 2 | 3 | | 5 | | |
| | | 3 | 4 | | 6 | |
| | | | 4 | 5 | | 7 |
| 1 | | | | 5 | 6 | |
| | 2 | | | | 6 | 7 |
| 1 | | 3 | | | | 7 |

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$v = 7, \quad k = \binom{7}{2}/3$

| | | | | | | |
|---|---|---|---|---|---|---|
| 1 | 2 | | 4 | | | |
| | 2 | 3 | | 5 | | |
| | | 3 | 4 | | 6 | |
| | | | 4 | 5 | | 7 |
| 1 | | | | 5 | 6 | |
| | 2 | | | | 6 | 7 |
| 1 | | 3 | | | | 7 |

| | | | | | | |
|---|---|---|---|---|---|---|
| | | | | 5 | 6 | 7 |
| | | 3 | 4 | | | 7 |
| 1 | 2 | | | | | 7 |
| | 2 | | 4 | | 6 | |
| 1 | | 3 | | | 6 | |
| | 2 | 3 | | 5 | | |
| 1 | | | 4 | 5 | | |

The enumeration of non-isomorphic STS is complex and a open field.

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(A030129) Number of nonisomorphic Steiner triple systems (STS's) $S(2,3,n)$ on $n$ points

$< 1, 0, 1, 0, 0, 0, 1, 0, 1, 0, 0, 0, 2, 0, 80, 0, 0, 0, 11084874829 >$

(A051390)Number of nonisomorphic Steiner quadruple systems (SQS's) of order $n$

$< 1, 1, 0, 1, 0, 0, 0, 1, 0, 1, 0, 0, 0, 4, 0, 1054163 >$

# [1974 Wilson] Upper bound to non-isomorphic STS(v)

A algebric result:

$$F(v) \leq ((1 + o(1)\frac{v}{e^2})^{\frac{n^2}{6}}$$

# [1985 Stinson] Estimation of STS(19)

Was discovered 284457 non-isomorphic through different methods. They had discovered $N(19) \geq 2395687$ through a *non-deterministic hill-climbing algorithm*. Then for every STS(19) calculate 2 *invariants* (non-isomorphic give the property of different invariant.) First they conclude $N = 3.54 \times 10^8$, but the random seed was from a population of $10^9$(11084874829) and some 2 isomorphic STS may have the same invariants.

They made another estimation by knowing the *right number* of $sub - STS(9)$ the $N(19)$ by looking the ratio of $sub - STS(9)$ fount divided by the right times the number of nonisomorphic sts. So... they miss by an order of magnitude but really close!.

# References

*Stinson, Douglas R. (Douglas Robert), 1956 Combinatorial designs : constructions and analysis*
*C.C. Lindner and C.A. Rodger, Design Theory, CRC Press, 2008*
*M. Wilson, Richard. (1974). Nonisomorphic Steiner triple systems. Mathematische Zeitschrift. 135. 303-313. 10.1007/BF01215371.*
*D. R. Stinson and H. Ferch (1985). 2000000 Steiner Triple Systems of Order 19. Mathematics of computation. 44. 533-555*