Steiner Triple Systems

Existence,representation and construction

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Introduction

Outline

- Challenge on *combinatorial design*
- What is it?:
  - existence or non-existence
  - representation
  - construction

What is Steiner Triple System

(Definition) Steiner Triple Systems (STS)

is an ordered pair (S, T) (a *design*) where S is a finite set of *point*/*symbol* and T is a set of subsets of 3-symbol in which all possible pair of S are contained **once and only once**.

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More formally:

- define S such that $|S| = v$
- then $T = \{\forall\{a, b, c\} \in S \times S \times S\}$
  such that
  $\forall a, b \in S \times S \ a \neq b \quad \sum_{\forall\{x,y,z\}\in T}(\mathbb{I}_{\{a,b\}\in\{x,y\}\wedge\{y,z\}\wedge\{z,x\}}) = 1$

More compact way to define STS by define the *order* v of STS by $v = |S|$

Examples of STS

$$S = \{a\},\ T = \emptyset$$
$$S = \{a, b\},\ T = \emptyset$$
$$S = \{a, b, c\},\ T = \{\{a, b, c\}\}$$
$$S = \{a, b, c, d\},\ T = \emptyset$$
$$S = \{a, b, c, d, e\},\ T = \emptyset$$
$$S = \{a, b, c, d, e, f\},\ T = \emptyset$$
$$S = \{a, b, c, d, e, f, g\},\ T =$$
$$\{\{a, b, c\}, \{c, d, e\}, \{e, f, a\}, \{f, b, d\}, \{a, g, d\}, \{e, g, b\}, \{c, g, f\}\}$$
$$\dots$$

Balanced incomplete blocks design

(Definition) $(v, k, \lambda) - \mathrm{BIBD}$

v,k and λ be positive integers such that $v > k \geq 2$. A balanced incomplete block design is a *design* (S, T) such that satisfy these properties:

1. $|S| = v$
2. $\forall t \in T \quad |t| = k$
3. for all distinct pairs are contained in exactly λ blocks (t)

Why **balanced** and **incomplete**?

balanced they share the same property (2)

incomplete by reason of $v = |S| \quad > \quad k = |t| \; \forall t \in T$

What is Steiner Triple System 2

λ blocks (t) of $(v, k, \lambda) - \mathrm{BIBD}$ iff $\lambda = 1$, $k = 3$.

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Requirement **1** satisfied by definition of STS on the design (S, T).
Requirement **2** satisfied by definition of STS.
Requirement **3** for all distinct pairs are contained in exactly

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All theory from BIBD is shared too in STS

Representation

How to represent

- through display each 3-set of T $(\{\{a, b, c\}, \{b, d, e\}, ..., \{d, f, g\}\})$
- through a *complete graph*

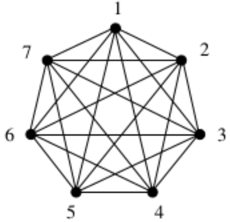
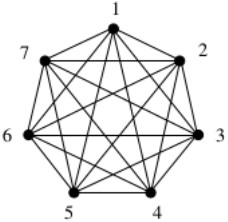


Figure: A complete graph of order $v = 7$

Example

Why a focus on representation?

- we talk about combinatorial design (display somehow somethings)
- help to design algorithm



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Focus on

How to choose a proper partition of the graph ?

Example

First non-dummy: STS of *order* 7

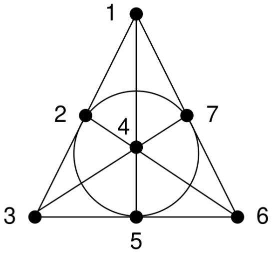

Figure: Fano plane

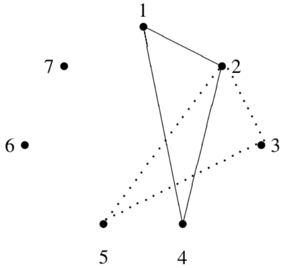

Figure: Building methods on STS(7)

How to create

- rotation of 1-left incidence triangle
- Bose method
- Skolem
- $6n + 5$
- With quasigroups with holes
- Wilson
- $2n + 1$
- $2n + 7$
- Even-Odd

Practical example

Cyclic Steiner triple system

STS $\Rightarrow$ Idempotent totally symmetric quasigroup construction

Kirkamn triple systems

Intersections of Steiner Triple Systems

Teirlink's Algorithm

Embedding partial of Steiner Triple Systems

Teirlink's Algorithm

References I