

Steiner Triple Systems

Existence, representation and construction

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Introduction

Outline

- Challenge on *combinatorial design*
- What is STS?:
 - ▶ existence or non-existence
 - ▶ representation
 - ▶ construction

What is Steiner Triple System

(Definition) Steiner Triple Systems (STS)

is an ordered pair (S, T) (a *design*) where S is a finite set of *point/symbol* and T is a set of subsets of 3-symbol in which all possible pair of S are contained **once and only once**.

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More formally:

- define S such that $|S| = v$
- then $T = \{\{a, b, c\} \in S \times S \times S\}$
such that $\forall a, b \in S, a \neq b$
$$\sum_{\{x,y,z\} \in T} (\mathbb{I}_{\{a,b\} \in \{x,y\}} + \mathbb{I}_{\{a,b\} \in \{y,z\}} + \mathbb{I}_{\{a,b\} \in \{z,x\}}) = 1$$

More compact way to define STS by define the *order* v of STS by $v = |S|$

Examples of STS

$$S = \{a\}, T = \emptyset$$
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$$\{\{a, b, c\}, \{c, d, e\}, \{e, f, a\}, \{f, b, d\}, \{a, g, d\}, \{e, g, b\}, \{c, g, f\}\}$$

...

Balanced incomplete blocks design

(Definition) (v, k, λ) – BIBD

v, k and λ be positive integers such that $v > k \geq 2$. A balanced incomplete block design is a *design* (S, T) such that satisfy these properties:

- 1 $|S| = v$
- 2 $\forall t \in T \quad |t| = k$
- 3 for all distinct pairs are contained in exactly λ blocks (t)

Why **balanced** and **incomplete**?

balanced they share the same property (2)

incomplete by reason of $v = |S| > k = |t| \quad \forall t \in T$

What is Steiner Triple System 2

λ blocks (t) of $(v, k, \lambda) - \text{BIBD}$ iff $\lambda = 1, k = 3$.

$(v, k, \lambda) - \text{BIBD}$

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- ③ $\forall s \in S$ is contained in exactly λ blocks (t)

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All theory from BIBD is shared too in STS

Representation of $\text{STS}(v)$

How to represent

- through display each 3-set of T and S ($\{a, b, c, d, e, f, g\}, \{\{a, b, c\}, \{b, d, e\}, \dots, \{d, f, g\}\}$)
- through a *complete graph*

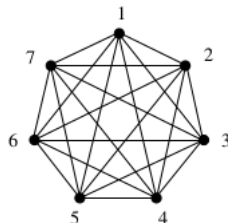


Figure: A complete graph of order $v = 7$

Example

Why a focus on representation?

- we talk about combinatorial design (display somehow somethings)
- help to design algorithm

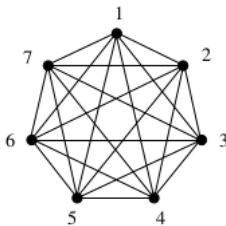


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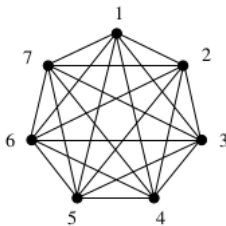


Figure: A complete graph of order $v = 7$

Focus on

How to choose a proper partition of the graph ?

Example

First non-dummy: STS of *order 7*

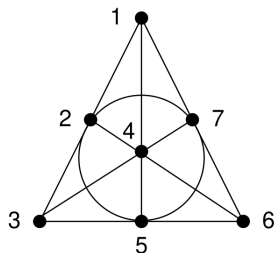


Figure: Fano plane

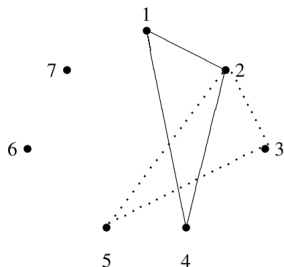


Figure: Building methods on STS(7)

[Kirkman, 1847]Existence proof

Theorem

A STS of order v **exists** if and only if $v \equiv 1, 3 \pmod{6}$

Proof.

(\Rightarrow) We know that all possible pairs are $\binom{v}{2}$, and by definition these pairs are partitioned (non-overlapping and union make all) into 3-element groups. Those groups are $|T| = \frac{\binom{v}{2}}{3} = \frac{v(v-1)}{6}$. Then for $\forall x \in S$ can be defined $T(x) = \{t \setminus \{x\} \mid x \in t \in T\}$. So if an $x \in S$ is fixed and then for every set t which contain x we remove the point x then we carry out $v-1$ point partitioned in 2-element set. As we can't make 2-element partition from a group of odd element, $v-1$ is even! So v is odd and it's equal to say $v \equiv 1, 3, 5 \pmod{6}$. The $\frac{v(v-1)}{6}$ is not an integer for every $v \equiv 5 \pmod{6}$. As a result $\text{STS} \Rightarrow v \equiv 1, 3 \pmod{6}$ □

Existence proof 2

$$(S, T) : |S| = v \wedge v \equiv 1, 3 \pmod{6} \Rightarrow STS(v)$$

In addition we suppose:

- each distinct pair of S belongs to *at least* one triple in T
- $|T| \leq \frac{v(v-1)}{6}$

Proof 1.

(Absurd) Assume the contrary and make a list L as follows: for every pair write down the triple with which it is associated. Then $|L| > \binom{v}{2}$ as there exists a pair with two triples. Now since each triple is counted by exactly three pairs so $|T| = |L|/3 > \frac{\binom{v}{2}}{3}$, a contradiction. □

Proof 2.

For each distinct pair of S belongs to at least one triple and if the number of triples is less than or equal to the right number of triples, then each pair of symbols in S belongs to exactly one triple in T . □

Proof 3.

We construct 2 methods to prove sufficient constraint to the Theorem by showing 2 methods:

- Bose construction
 - Skolem construction
-

Construction methods

How to create

- Bose method
- Skolem method
- $6n + 5$
- With quasigroups with holes
- Wilson
- and so many

Bose construction

We need first define:

idempotent commutative quasigroups of order $2n + 1$

But first: recap

(Definition) latin square of order n

is an $n \times n$ array where each row and column contains all symbols $\{1, \dots, n\}$ exactly one time.

1	3	2
2	1	3
3	2	1

Table: Latin square of order 3

(Definition) Quasigroup

A *quasigroup* of order n is an algebraic structure, a pair (Q, \circ) where $|Q| = n$ and $\circ : Q \times Q \rightarrow Q$.

$\forall a, b \in Q$ then $\exists! x, y$ (unique!) to the equations $a \circ x = b$ and $x \circ a = b$.

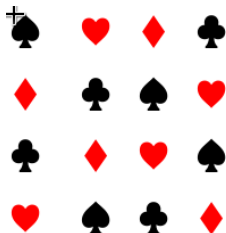
Examples of quasigroup are $(\mathbb{Z}_n, -), (\mathbb{Z}_n, +)$.

(Example) 1723 Latin square

♠A	♥K	♦Q	♣J
♦J	♣Q	♠K	♥A
♣K	♦A	♥J	♠Q
♥Q	♠J	♣A	♦K

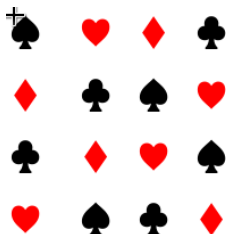
(Example) 1723 Latin square

♠A+	♥K	♦Q	♣J
♦J	♣Q	♠K	♥A
♣K	♦A	♥J	♠Q
♥Q	♠J	♣A	♦K



(Example) 1723 Latin square

♠A⁺ ♥K ♦Q ♣J
♦J ♣Q ♠K ♥A
♣K ♦A ♥J ♠Q
♥Q ♠J ♣A ♦K



A	K	Q	J
J	Q	K	A
K	A	J	Q
Q	J	A	K

Quasigroup and latin square

Theorem

The multiplication table of a quasigroup is a Latin square

A quasigroup (G, \circ) is a latin square of order $v = |G|$:

1	3	2
2	1	3
3	2	1

Table: Latin square of order 3

\circ	1	2	3
1	1	2	3
2	3	1	2
3	2	3	1

Table: Quasigroup of order 3

Quasigroup and latin square

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The multiplication table of a quasigroup is a Latin square

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1	3	2
2	1	3
3	2	1

Table: Latin square of order 3

\circ	1	2	3
1	1	2	3
2	3	1	2
3	2	3	1

Table: Quasigroup of order 3

A (Q, \circ) is said:

idempotent $\forall i : 1 \leq i \leq |G|$ the cell (i, i) contains α such that $\alpha \leq i$

commutative $\forall i, j : 1 \leq i < j \leq |G|$ the cell (i, j) contains the same of (j, i)

Commutative idempotent latin square

1	3	2
3	2	1
2	1	3

Table: C. I. latin square of order 3

Commutative idempotent latin square

1	3	2
3	2	1
2	1	3

Table: C. I. latin square of order 3

How can we create a *C.I. latinsquare/quasigroup of order v* ?

Theorem

*idempotent commutative quasigroups exist **if and only if** they have odd order.*

Great! We look at the half of all possible

Construction method of CI quasigroup

- 1 Let v be the order of quasigroup, take $(Z_v, +)$ where $+$ is the addition in Z_v .
- 2 For all element $i := i + 1$
- 3 Take the elements of main diagonal $\langle d_1, \dots, d_v \rangle$. Build a permutation $\sigma_v = \{(d_1, 1), (d_2, 2), \dots, (d_v, v)\}$.
- 4 Apply σ_v for all element of the *multiplication table*

As result you have a CI quasigroup.

Construction method of CI quasigroup of order 7

$Z_7, +$	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	1	2	3	4	5	6	0
2	2	3	4	5	6	0	1
3	3	4	5	6	0	1	2
4	4	5	6	0	1	2	3
5	5	6	0	1	2	3	4
6	6	0	1	2	3	4	5

	1	2	3	4	5	6	7
1	1	2	3	4	5	6	7
2	2	3	4	5	6	7	1
3	3	4	5	6	7	1	2
4	4	5	6	7	1	2	3
5	5	6	7	1	2	3	4
6	6	7	1	2	3	4	5
7	7	1	2	3	4	5	6

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Construction method of CI quasigroup of order 7

	1	2	3	4	5	6	7
1	1	2	3	4	5	6	7
2	2	3	4	5	6	7	1
3	3	4	5	6	7	1	2
4	4	5	6	7	1	2	3
5	5	6	7	1	2	3	4
6	6	7	1	2	3	4	5
7	7	1	2	3	4	5	6

$$\sigma_v = \{(1, 1), (3, 2), (5, 3), (7, 4), (2, 5), (4, 6), (6, 7)\}$$

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Construction method of CI quasigroup of order 7

	1	2	3	4	5	6	7
1	1	2	3	4	5	6	7
2	2	3	4	5	6	7	1
3	3	4	5	6	7	1	2
4	4	5	6	7	1	2	3
5	5	6	7	1	2	3	4
6	6	7	1	2	3	4	5
7	7	1	2	3	4	5	6

	1	2	3	4	5	6	7
1	1	5	2	6	3	7	4
2	5	2	6	3	7	4	1
3	2	6	3	7	4	1	5
4	6	3	7	4	1	5	2
5	3	7	4	1	5	2	6
6	7	4	1	5	2	6	3
7	4	1	5	2	6	3	7

We apply $\sigma_v = \{(1, 1), (3, 2), (5, 3), (7, 4), (2, 5), (4, 6), (6, 7)\}$ as result we have a idempotent commutative quasigroup of order v .

Bose construction($v \equiv 3 \bmod(6)$)

Let $v = 6n + 3$ and let (Q, \circ) be an idempotent commutative quasigroup of order $2n + 1$, where $Q = \{1, 2, 3, \dots, 2n + 1\}$. Let $S = Q \times \{1, 2, 3\}$ and define T to contain the following types of triples.

Type 1: For $1 \leq i \leq 2n + 1$, $\{(i, 1), (i, 2), (i, 3)\} \in T$

Type 2: For $1 \leq i < j \leq 2n + 1$, $\{(i, 1), (j, 1), (i \circ j, 2)\}, \{(i, 2), (j, 2), (i \circ j, 3)\}, \{(i, 3), (j, 3), (i \circ j, 1)\} \in T$

Then (S, T) is a Steiner triple system of order $6n + 3$.

Type of partitions

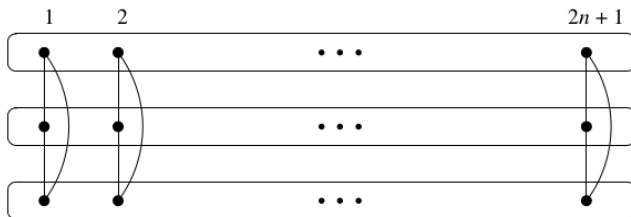


Figure: Type 1

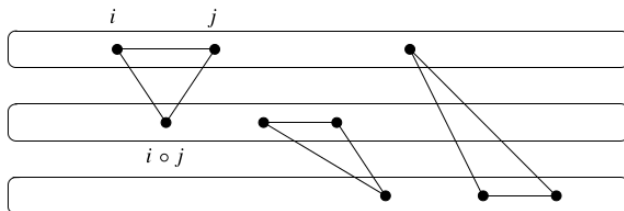


Figure: Type 2

We need to prove that all pairs are $v(v-1)/6$ then, as result of all pair is contained only one, we have proved the corectness.

Proof right number

$|T|$ is made up with 2 type:

- *Type 1*: $2n + 1$ triples
- *Type 2*: $\binom{2n+1}{2}$ choices for i and j , for all of them there are 3 another type.

Then $|T| = (2n + 1) + 3 \frac{(2n+1)2n}{2} = \frac{(2n+1)(6n+2)}{2} = v(v - 1)/6$ have the right number of triple.

To show that every pair is contained in at least 1 triple, think about 2 possible pairs of point $(a, b), (c, d) \in Q \times \{1, 2, 3\}$:

Cont..

$$\forall (a, b), (c, d) \in Q \times \{1, 2, 3\}$$

- $a = c \wedge b = d$ impossible
- if $a = c$ (so $b \neq d$) is contained in at least 1 triple of *type 1*
 $\{(a, 1), (a, 2), (a, 3)\}$
- if $b = d \wedge a \neq c$ is contained in at least 1 triple of *type 2*
 $\{((a, b), (c, b), (a \circ c, b + 1 \bmod(3)))\}, \{(x, 1), (a, 1), (b, 2)\}, \dots\}$
- $a \neq c \wedge b \neq d$. Assume $b = 1$ and $d = 2$. Since (Q, \circ) is a quasigroup $a \circ i = c$ and $j \circ a = c$ for some i, j . Because the *commutative* $i = j$ and because *idempotent* only $a \circ a = a$, all the others we are sure that $i \neq a$. So $\{(a, 1), (i, 1), (a \circ i = c, 2)\}$.

All possible points have been shown that are in T .

Skolem construction ($v \equiv 3 \pmod{6}$)

Let $v = 6n + 1$ and let (Q, \circ) be a half-idempotent commutative quasigroup of order $2n$, where

Half-idempotent commutative latin square

(Definition) Half-idempotent commutative latin square

A latin square (multiplication table of quasigroup of the same order) L of order $2n$ is *half-idempotent* if the cells (i, i) contain the same symbol i of the cell $(n + i, n + i) \quad \forall 1 \leq i \leq n$

1	3	2	4
3	2	4	1
2	4	1	3
4	1	3	2

Half-idempotent latin square
of order 4 ($n = 2$)

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Half-idempotent latin square
of order 4 ($n = 2$)

Commutative half-idempotent latin squares exist for all even order.

Example half-idempotent quasigroup

1	3	2	4
3	2	4	1
2	4	1	3
4	1	3	2

1	4	2	5	3	6
4	2	5	3	6	1
2	5	3	6	1	4
5	3	6	1	4	2
3	6	1	4	2	5
6	1	4	2	5	3

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1	4	2	5	3	6
4	2	5	3	6	1
2	5	3	6	1	4
5	3	6	1	4	2
3	6	1	4	2	5
6	1	4	2	5	3

How can we algorithmically build a H-I Latin Square?

How to construct H-I latin square/quasigroup

0	1	2	3	4	5
1	2	3	4	5	0
2	3	4	5	0	1
3	4	5	0	1	2
4	5	0	1	2	3
5	0	1	2	3	4

Table: Quasigroup $(\mathbb{Z}_6, + \text{mod}(6))$

How to construct H-I latin square/quasigroup

0	1	2	3	4	5
1	2	3	4	5	0
2	3	4	5	0	1
3	4	5	0	1	2
4	5	0	1	2	3
5	0	1	2	3	4

Table: Quasigroup $(\mathbb{Z}_6, + \text{mod}(6))$

The bijection σ is built increasing all value by 1(or by taking the next element). Then by taking the main diagonal from the left grid ($\langle 1, 3, 5, \dots \rangle$) and assign the right number ($\langle 1, 2, 3, \dots \rangle$)

How to construct H-I latin square/quasigroup

0	1	2	3	4	5
1	2	3	4	5	0
2	3	4	5	0	1
3	4	5	0	1	2
4	5	0	1	2	3
5	0	1	2	3	4

Table: Quasigroup $(\mathbb{Z}_6, + \text{mod}(6))$

1	4	2	5	3	6
4	2	5	3	6	1
2	5	3	6	1	4
5	3	6	1	4	2
3	6	1	4	2	5
6	1	4	2	5	3

Table: Applied func σ on the quasigroup

The bijection σ is built increasing all value by 1 (or by taking the next element). Then by taking the main diagonal from the left grid ($\langle 1, 3, 5, \dots \rangle$) and assign the right number ($\langle 1, 2, 3, \dots \rangle$)

Example Skolem construction

Let $v = 6n + 1$ and let (Q, \circ) be a half-idempotent commutative quasigroup of order $2n$, where $Q = \{1, 2, 3, \dots, 2n\}$. Let $S = \{\infty\} \cup (Q \times \{1, 2, 3\})$. We define T as follow:

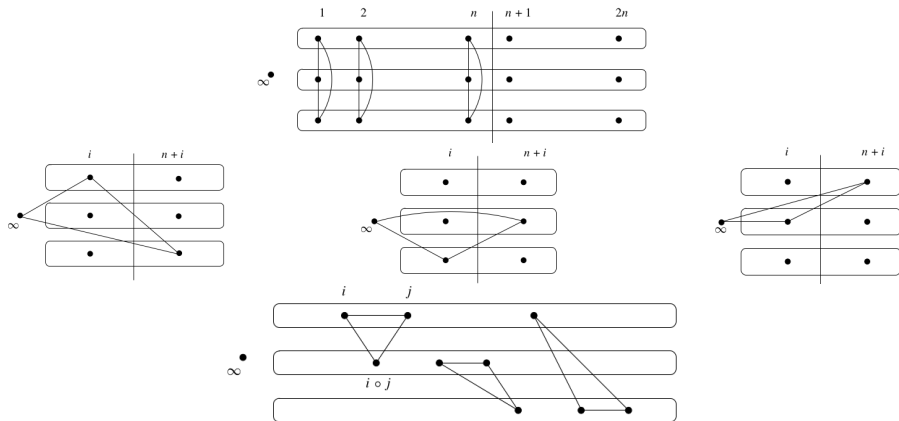
Type 1 : for $1 \leq i \leq n$, $\{(i, 1), (i, 2), (i, 3)\} \in T$

Type 2 : for $1 \leq i \leq n$, $\{\infty, (n+i, 1), (i, 2)\}, \{\infty, (n+i, 2), (n+i, 3)\}, \{\infty, (n+i, 3), (i, 1)\} \in T$

Type 3 : for $1 \leq i < j \leq 2n$, $\{(i, 1), (j, 1), (i \circ j, 2)\}, \{(i, 2), (j, 2), (i \circ j, 3)\}, \{(i, 3), (j, 3), (i \circ j, 1)\} \in T$

Then (S, T) is STS($6n + 1$).

Example Skolem construction



Proof

We have to prove (in a similar way of Bose) there are the right number of $t \in T$ and every pair (from $\binom{6n+1}{2}$) is contained almost 1.

1: Right number of $|T|$.

We sum up 3 different element:

- *type 1*: for $1 \leq i \leq n$, $\{(i, 1), (i, 2), (i, 3)\} \in T$ are n
- *type 2*: for $1 \leq i \leq n$,
 $\{\infty, (n+i, 3), (i, 2)\}, \{\infty, (n+i, 2), (n+i, 3)\}, \{\infty, (n+i, 3), (i, 1)\} \in T$
are $3n$
- *type 3*: for
 $1 \leq i < j \leq 2n, \{(i, 1), (j, 1), (i \circ j, 2)\}, \{(i, 2), (j, 2), (i \circ j, 3)\}, \{\} \in T$ are
 $3\binom{2n}{2}$

We have to prove $|T| = \frac{v(v-1)}{6} = \frac{(6n+1)(6n)}{6} = n + n + 3\binom{2n}{2}$.

Easily

$$n + 3n + 3\binom{2n}{2} = \frac{2n*4 + 2n*(6n-3)}{2} = \frac{2n(6n+1)}{2} = \frac{3}{3} \frac{2n(6n+1)}{2} = \frac{6n(6n+1)}{2} = |T|$$



2: every pair of point $\in T$.

We have to prove all possible pair of point (a, b) and (c, d) :

- $a = c = \infty \wedge b \neq d$
- $a = c \neq \infty \wedge b \neq d$
- $a = \infty \neq b \wedge b = d$
- $a \neq c \wedge b \neq d$



2: every pair of point $\in T$.

We have to prove all possible pair of point (a, b) and (c, d) :

- $a = c = \infty \wedge b \neq d$
- $a = c \neq \infty \wedge b \neq d$
- $a = \infty \neq b \wedge b = d$
- $a \neq c \wedge b \neq d$



All covered by Skolem construction $\wedge |T|$ is the right number \Rightarrow is a STS(2n)

Practical example

[1850] The Lady's and Gentleman's Diary/Kirkman's schoolgirl problem

A teacher would like to take 15 schoolgirls out for a walk, the girls being arranged in 5 rows of three. The teacher would like to ensure equal chances of friendship between any two girls. Hence it is desirable to find different row arrangements for the 7 days of the week such that any pair of girls walk in the same row exactly one day of the week.

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A teacher would like to take 15 schoolgirls out for a walk, the girls being arranged in 5 rows of three. The teacher would like to ensure equal chances of friendship between any two girls. Hence it is desirable to find different row arrangements for the 7 days of the week such that any pair of girls walk in the same row exactly one day of the week.

Solution

[1971 (Ray-Chaudhuri and Wilson)] it asks for a Steiner Triple System on $6t + 3$ varieties whose blocks can be partitioned into $3t + 1$ sets so that any variety appears only once in a set.

Important remark

Theorem

If a (v, k, λ) - BIBD exist, then $\lambda(v - 1) \equiv 0 \pmod{(k - 1)}$ and $\lambda v(v - 1) \equiv 0 \pmod{k(k - 1)}$.

Theorem

A STS of order v **exists** if and only if $v \equiv 1, 3 \pmod{6}$

Theorem

If a (v, k, λ) - BIBD exist, then $\lambda(v - 1) \equiv 0 \pmod{(k - 1)}$ and $\lambda v(v - 1) \equiv 0 \pmod{k(k - 1)}$.

Only **necessary**.

Theorem

A STS of order v **exists** if and only if $v \equiv 1, 3 \pmod{6}$

Necessary and **sufficient**.

Steiner systems , sequences related to :

Steiner systems, quadruple (SQS's): [A051390*](#) [A124120](#) [A124119](#)

Steiner systems: [A001293*](#) ($S(5,8,24)$)

Steiner systems: [A187567](#) and [A187585](#) ($S(2,4,n)$)

Steiner triple systems (STS's): [A001201*](#), [A030128*](#), [A030129*](#), [A051390*](#), [A002885](#) (cyclic),
[A006181](#), [A006182](#), [A051391](#)

Isomorphisms between two *design*

Isomorphism

Two designs (X, A) and (Y, B) where $|X| = |Y|$ are *isomorphic* if there exists a bijection $\alpha : X \rightarrow Y$ such that:

$$\{\alpha(x) : x \in A\} = B$$

Then α is called isomorphism.

$$X = \{1, 2, 3, 4, 5, 6, 7\}, \quad \text{and}$$

$$A = \{123, 145, 167, 246, 257, 347, 356\};$$

$$Y = \{a, b, c, d, e, f, g\}, \quad \text{and}$$

$$B = \{abd, bce, cdf, deg, aef, bfg, acg\}.$$

Beyond Existence or non-existence

	t_1
1	1
2	1
3	1

Table: Incidence matrix of STS(3)

Incidence matrix

Fixed a design

$(S, T) \equiv (\{s_1, \dots, s_{|S|}\}, \{t_1, \dots, t_{|T|}\})$

let all elements $m_{i,j}$ of *incidence matrix* of a design (S, T) be:

$$m_{i,j} = \begin{cases} 1 & \text{if } s_i \in t_j \\ 0 & \text{if } s_i \notin t_j \end{cases}$$

2-isomorphic sts(7)

$$v = 7, \quad k = \binom{7}{2}/3$$

1	2		4			
	2	3		5		
		3	4		6	
			4	5		7
1				5	6	
	2				6	7
1		3				7

2-isomorphic sts(7)

$$v = 7, \quad k = \binom{7}{2}/3$$

1	2		4			
	2	3		5		
		3	4		6	
			4	5		7
1				5	6	
	2				6	7
1		3				7

				5	6	7
		3	4			7
1	2					7
	2		4		6	
1		3			6	
	2	3		5		
1			4	5		

The enumeration of non-isomorphic STS is complex and a open field.

The enumeration of non-isomorphic STS is complex and a open field.

(A030129) Number of nonisomorphic Steiner triple systems (STS's)
 $S(2, 3, n)$ on n points

< 1, 0, 1, 0, 0, 0, 1, 0, 1, 0, 0, 0, 2, 0, 80, 0, 0, 0, 11084874829 >

(A051390) Number of nonisomorphic Steiner quadruple systems
(SQS's) of order n

< 1, 1, 0, 1, 0, 0, 0, 1, 0, 1, 0, 0, 0, 4, 0, 1054163 >

[1974 Wilson] Upper bound to non-isomorphic STS(v)

A algebraic result:

$$F(v) \leq ((1 + o(1)) \frac{v}{e^2})^{\frac{n^2}{6}}$$

[1985 Stinson] Estimation of STS(19)

Was discovered 284457 non-isomorphic through different methods. They had discovered $N(19) \geq 2395687$ through a *non-deterministic hill-climbing algorithm*. Then for every STS(19) calculate 2 *invariants* (non-isomorphic give the property of different invariant.) First they conclude $N = 3.54 \times 10^8$, but the random seed was from a population of $10^9(11084874829)$ and some 2 isomorphic STS may have the same invariants.

They made another estimation by knowing the *right number* of *sub* – STS(9) the $N(19)$ by looking the ratio of *sub* – STS(9) found divided by the right times the number of nonisomorphic sts. So... they **miss by an order of magnitude** but really close!.

References

- Stinson, Douglas R. (Douglas Robert), 1956 Combinatorial designs : constructions and analysis*
- C.C. Lindner and C.A. Rodger, Design Theory, CRC Press, 2008*
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