CS222 Homework 1

Algorithm Analysis & Deadline: 2020-09-21 Monday 24:00
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Exercises for Algorithm Design and Analysis by Li Jiang, 2020 Autumn Semester

1. Prove that $\log(\log n) = o(n^k)$, where k is a positive constant. (ps: $\log n$ refers to $\log_2 n$.)

Solution. $\lim_{x\to\infty} \dfrac{\log(\log n)}{n^k} = \lim_{x\to\infty} \dfrac{\frac{1}{\log n}\cdot\frac{1}{n}}{kn^{k-1}} = \lim_{x\to\infty} \dfrac{1}{kn^k \log n} = 0 \Rightarrow \log(\log n) = o(n^k), \forall k > 0.$

2. Prove that for any integer $n^2 - 1 > 3$, there is a prime p satisfying $n! > p > n$.

Solution. *We prove it by induction.*
(i) $n = 3 : \exists\ p = 5, n! > p > n$*;*
(ii) $n = k+1$*: by inductive assumption, exists a prime* p*,* $k < p < k!$*.*
case 1: $p > k+1$*, then we have* $n < p < k! < n!$ *for* $n = k+1$*.*
case 2: $p = k+1$*, we define* P *to be the set of primes less than or equal to* $k+1$*. Let* $q = 1 + \prod_{t\in[1,k+1]\cap P} t$*. Obviously,* $n = k+1 < 1 + 2(k+1) \le q \le 1 + \dfrac{(k+1)!}{4} < n!$ *for* $n = k+1$ *(Reason:* $p = k+1 \in P$*,* $4 \notin P$*). If there is no prime in the interval* (n, q)*, then* q *is a prime, because there is not a prime* p' *less than* q *such that* $p'|q$*. If there is a prime* q' *in the interval* (n, q)*, then we get it.*
From the two cases above, there exists a prime p *satisfying* $n < p < n!$ *for* $n = k+1$*.*

3. Assume that there is a recurrence formula as follows:

$$D(x) = \begin{cases} 1, & if\ x == 1 \\ 3D(x/4) + x - 2, & if\ x \ge 2 \end{cases}$$

Please deduce the non-recursive expression of $D(x)$ and point out its asymptotic complexity.

Solution. *By observing,* $D(x)$ *is approximate to* $4x$*. Let* $F(x) = D(x) - 4x$ *then we get* $F(x) = 3D\left(\dfrac{x}{4}\right) - 3x - 2 = 3F\left(\dfrac{x}{4}\right) - 2$*. Consider the function* $G(x)$ *with : (1)* $G(0) = F(1) = -3$*, (2)* $G(x) = 3G(x-1) - 2$*.* $G(x) - 1 = 3(G(x-1) - 1)$*, so* $G(x) = -4\cdot 3^x + 1$*. By observing, we can see* $F(x) = G(\log_4 x)$*, therefore* $D(x) = F(x) + 4x = 4x + 1 - 4\cdot 3^{\log_4 x}$*. We can check it easily.*
Obviously, $D(x) < 4x, \forall x \ge 1$*. And we can proof it easily by induction that* $x \le D(x), \forall x \ge 1$*. So* $D(x) = \Theta(x)$*.*

Update:

$$D(x) = \begin{cases} 1, & if\ \lfloor x \rfloor \le 1 \\ 3D(x/4) + x - 2, & if\ \lfloor x \rfloor > 1 \end{cases}$$

Solution. *By observing, $D(x)$ is possibly a piecewise function. We use python script to calculate the function values, and we found some law:*
$D(x)$ is linear in the intervals $[2,8), [8,32), [32,128), [128,512), \cdots$
The slope of $D(x)$ is $1, 1+0.75, 1+0.75+0.75^2, 1+0.75+0.75^2+0.75^3, \cdots$
And according to the solution of the original problem, we have: $D(x) = 1 + 4x - 6 \cdot 3^{\log_4 \frac{x}{2}}$, when $x = 2 \cdot 4^k$
So, we have $x \geq 2 \rightarrow D(x) = 1 + 4 \times (2 \cdot 4^{\lfloor \log_4 (x/2) \rfloor}) - 6 \cdot 3^{\lfloor \log_4 (x/2) \rfloor} + (4 - 3 \cdot (\frac{3}{4})^{\lfloor \log_4 (x/2) \rfloor})(x - 2 \cdot 4^{\lfloor \log_4 (x/2) \rfloor})$
Simplify it, we get :

$$D(x) = \begin{cases} 1, & \lfloor x \rfloor \leq 1 \\ 1 + 4x - 3x \cdot \left(\dfrac{3}{4}\right)^{\lfloor \log_4 (x/2) \rfloor}, & \lfloor x \rfloor > 1 \end{cases}$$

We can check it easily.
Similar to the solution in the original problem, we can prove it by induction that $x \leq D(x) < 4x, \forall x > 0, x \in \mathbb{Z}$. And we can check it easily that $D(x)$ is continuous and linear in intervals $[k, k+1), \forall k \in \mathbb{N}_+$, so we can prove $\forall x > 0, x \leq D(x) < 4x$. Therefore, $D(x) = \Theta(x)$.

4. Use the minimal counterexample principle to prove that for any integer $n > 10$, there exist integers $i_n \geq 0$ and $j_n \geq 0$, such that $n = i_n \times 3 + j_n \times 4$.

Solution. *We assume that n $(n > 10)$ is the **smallest** integer greater than 10 which can **NOT** be expressed as $n = 3 \times i_n + 4 \times j_n$. Obviously $n > 13$, because $11 = 3 \times 1 + 4 \times 2,\ 12 = 3 \times 4 + 4 \times 0,\ 13 = 3 \times 3 + 4 \times 1$. Therefore, $n - 3 > 13 - 3 = 10$. We assert $10 < n - 3 = 3 \times i_{n-3} + 4 \times j_{n-3}$, because n is the smallest counterexample. Then we have $n = n - 3 + 3 = 3 \times (i_{n-3} + 1) + 4 \times j_{n-3}$. Contradict!*

5. Analyze the **average** time complexity of QuickSort in Alg. 1.

Algorithm 1: QuickSort

Input: An array $A[1, \cdots, n]$
Output: $A[1, \cdots, n]$ sorted nondecreasingly

$pivot \leftarrow A[n]$; $i \leftarrow 1$;
for $j \leftarrow 1$ **to** $n - 1$ **do**
  if $A[j] < pivot$ **then**
    swap $A[i]$ and $A[j]$;
    $i \leftarrow i + 1$;
  end
end
swap $A[i]$ and $A[n]$;
if $i > 1$ **then** QuickSort($A[1, \cdots, i-1]$);
if $i < n$ **then** QuickSort($A[i+1, \cdots, n]$);

Solution. $\mathbb{E}T(1) = 0; n \geq 2 \rightarrow \mathbb{E}T(n) = n + \dfrac{1}{n}\left(\sum_{i=1}^{n} [\mathbb{E}T(i-1) + \mathbb{E}T(n-i)]\right) = n + \dfrac{2}{n}\sum_{i=1}^{n-1} \mathbb{E}T(i)$.
When n is big enough,

$$\begin{aligned} n + \frac{2}{n}\sum_{i=1}^{n-1} 2i \ln i &= n + 2\sum_{i=1}^{n-1} \frac{2i}{n}\left(\ln \frac{i}{n} + \ln n\right) \\ &= n + 4n \sum_{i=1}^{n-1} \frac{i}{n^2} \ln \frac{i}{n} + \frac{4 \ln n}{n} \sum_{i=1}^{n-1} i \\ &\approx n + 4n \int_0^1 x \ln x \ \mathtt{d}x + \frac{4 \ln n}{n} \cdot \frac{n^2}{2} \\ &= n + 4n \cdot (-\frac{1}{4}) + 2n \ln n = 2n \ln n \end{aligned}$$

So, there is reason to believe that when n is big enough $\mathbb{E}T(n) \approx 2n \ln n$ and $\mathbb{E}T(n) = \Theta(n \log n)$
Discussing with classmates, I get a better solution:

$\mathbb{E}T(n) = n + \dfrac{2}{n}\sum_{i=1}^{n-1}\mathbb{E}T(i) \Rightarrow n\mathbb{E}T(n) = n^2 + 2\sum_{i=1}^{n-1}\mathbb{E}T(i)$ *(1)*
and $(n+1)\mathbb{E}T(n+1) = (n+1)^2 + 2\sum_{i=1}^{n}\mathbb{E}T(i)$ *(2). Therefore, (2)-(1):* $\dfrac{\mathbb{E}T(n+1)-1}{n+2} = \dfrac{\mathbb{E}T(n)-1}{n+1} + \dfrac{2}{n+2}$
Therefore, $\dfrac{\mathbb{E}T(n+1)-1}{n+2} \sim 2\ln(n+1) \Rightarrow \mathbb{E}T(n) \sim 2n\ln n \Rightarrow \mathbb{E}T(n) = \Theta(n\log n)$.

6. Rank the following functions by order of growth with explanations: that is, find an arrangement $g_1, g_2, \ldots, g_k$ of the functions $g_1 = \Omega(g_2), g_2 = \Omega(g_3), \ldots, g_{k-1} = \Omega(g_k)$. Partition your list into equivalence classes such that functions $f(n)$ and $g(n)$ are in the same class if and only if $f(n) = \Theta(g(n))$. Use symbols "=" and "$\prec$" to order these functions appropriately. (ps: $\log n$ refers to $\log_2 n$.)

$2^{\log n}$	$(\log n)^{\ln n}$	n^2	$n!$	$(n-1)!$
2^n	n^3	$\log^2 n$	e^n	2^{2^n}
$\log\log n$	$(n+1)\cdot 2^n$	n	$\log(n^2-n)$	$2^{\ln n}$

Solution.

$$\log\log n \prec \log(n^2-n) \prec \log^2 n \prec 2^{\ln n} \prec n = 2^{\log n} \prec n^2$$
$$\prec n^3 \prec (\log n)^{\ln n} \prec 2^n \prec (n+1)\cdot 2^n \prec e^n \prec (n-1)! \prec n! \prec 2^{2^n}$$

Explanations:

(a) $\lim_{x\to\infty}\dfrac{\log\log n}{\log(n^2-n)} = \lim_{x\to\infty}\dfrac{n-1}{(2n-1)\log n} = 0.$

(b) $\log(n^2-n) < 2\log n \prec \log^2 n$ *it is the same with* $n \prec n^2$

(c) $2^{\ln n} = 2^{\frac{\log n}{\log e}} = n^{\frac{1}{\log e}} \Rightarrow \log^2 n \prec n^{\frac{1}{\log e}} \prec n$ *because* $\lim_{x\to\infty}\dfrac{\log^2 n}{n^a} = \lim_{x\to\infty}\dfrac{2\log n}{an^a} = 0, a > 0.$

(d) $2^{\log n} = 2^{\log_2 n} = n$

(e) $n \prec n^2 \prec n^3$ *is obvious enough.*

(f) when $\log n > e^4$, $(\log n)^{\ln n} > (e^4)^{\ln n} = (e^4)^{\frac{\log_{e^4} n}{\log_{e^4} e}} = n^4 \succ n^3$

(g) $\ln n\log\log n \prec n \Rightarrow \forall c > 0, \ln n\log\log n < cn \Rightarrow \lim_{x\to\infty}\dfrac{(\log n)^{\ln n}}{2^n} = \lim_{x\to\infty}\dfrac{2^{\ln n\log\log n}}{2^n}$
$< \lim_{x\to\infty}\dfrac{1}{2^{0.9n}} = 0$

(h) $\lim_{x\to\infty}\dfrac{2^n}{(n+1)\cdot 2^n} = 0$

(i) $\lim_{x\to\infty}\dfrac{(n+1)\cdot 2^n}{e^n} = \lim_{x\to\infty}\dfrac{n+1}{(e/2)^n} = 0$

(j) $e^3 < 27$, *so take* $n > 28 > 27 > e^3, (n-1)! > e^{3(n-1-27)} = e^{3n-28} > e^{2n} \succ e^n$

(k) $\lim_{x\to\infty}\dfrac{(n-1)!}{n!} = 0$

(l) $\log(n!) < n\log n < n^2 \prec 2^n \Rightarrow \log(n!) < 0.1\cdot 2^n$ *when n is big enough.*
$\lim_{x\to\infty}\dfrac{n!}{2^{2^n}} = \lim_{x\to\infty}\dfrac{2^{\log(n!)}}{2^{2^n}} = \lim_{x\to\infty}\dfrac{1}{2^{2^n-\log(n!)}} < \lim_{x\to\infty}\dfrac{1}{2^{0.9\cdot 2^n}} = 0$

Remark: You need to upload your .pdf file.