Still water and other hard-to-model systems

Martin Büttner
Supervisor: Prof. Ted Johnson

16 March, 2015

The Shallow Water Equations

- Simplified fluid model:
  - Incompressible
  - Inviscid
  - *Shallow* (horizontal length scale $\gg$ typical depth)
- Used to model oceans ($\sim$ 4km deep)
  and atmosphere ($\sim$ 10km deep)

$$h_t + (hu)_x = 0 \tag{1}$$

$$(hu)_t + \left(hu^2 + \frac{1}{2}h^2\right)_x = 0 \tag{2}$$

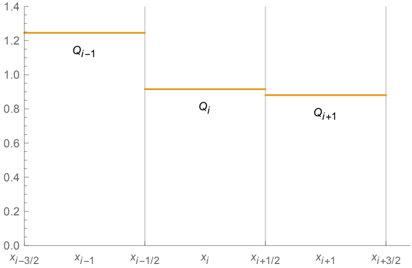
$$(hv)_t + (huv)_x = 0 \tag{3}$$

Finite Volume Methods

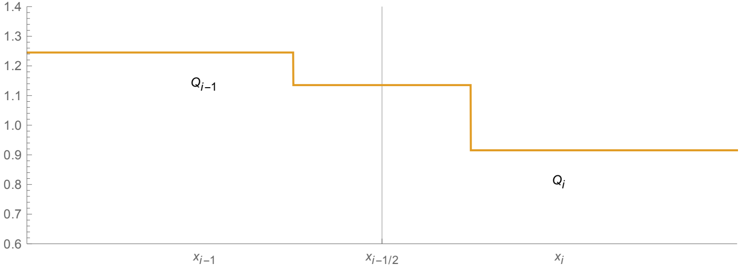
- Developed for *hyperbolic conservation laws*:

$$\mathbf{q}_t + \mathbf{f}(\mathbf{q})_x = 0 \tag{4}$$

- Discretisation:



Riemann Problems



- Initial value problem: step function
- Decompose into waves
- Finite speeds $\Rightarrow$ only affect neighbouring cells

Implementation: Clawpack

- "**C**onservation **law pack**age", by Randall J. LeVeque, 1994
- Implements "Godunov's method" in *wave-propagation form*
- Riemann solvers in Fortran 90
  - Take grid data, compute wave decomposition
- Configuration in Python 2
  - Grid setup
  - Initial conditions
  - Boundary conditions
  - Plotting instructions

Adding Source Terms

- Modelling additional effects:

$$h_t + (hu)_x = 0 \tag{5}$$

$$(hu)_t + \left(hu^2 + \frac{1}{2}h^2\right)_x = -hB_x + Khv \tag{6}$$

$$(hv)_t + (huv)_x = -Khu \tag{7}$$

- Full form of conservation law:

$$\mathbf{q}_t + \mathbf{f}(\mathbf{q})_x = \mathbf{s}(\mathbf{q}, x) \tag{8}$$

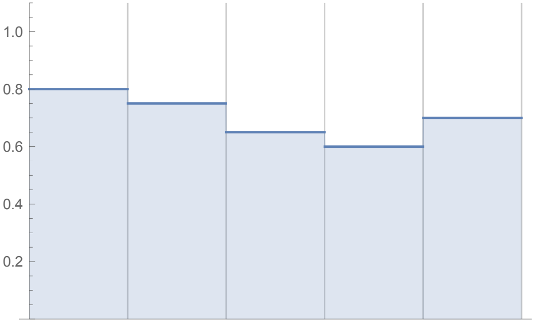
Adding Source Terms

- Split into two steps:

$$\mathbf{q}_t + \mathbf{f}(\mathbf{q})_x = 0 \tag{9}$$

$$\mathbf{q}_t = \mathbf{s}(\mathbf{q}, x) \tag{10}$$

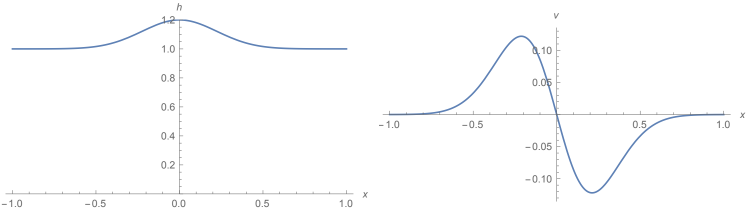
- The still water system:



Geostrophic Equilibria

- Coriolis force admits non-trivial equilibria when

$$u = 0, \quad v = \frac{h(h_s)_x}{K} \tag{11}$$

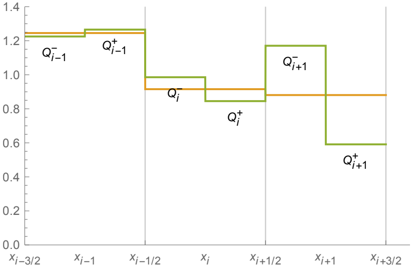


- Geophysical flows close to geostrophic equilibrium at all times

Well-Balanced Methods

LeVeque

- LeVeque, 1998, "Balancing Source Terms and Flux Gradients in High-Resolution Godunov Methods: The Quasi-Steady Wave-Propagation Algorithm"



$$\mathbf{f}(Q_i^+) - \mathbf{f}(Q_i^-) = \mathbf{s}(Q_i, x_i)\Delta x \tag{12}$$

Well-Balanced Methods

Rogers et al.

- Rogers, Borthwick, Taylor, 2003, "Mathematical balancing of flux gradient and source terms prior to using Roe's approximate Riemann solver"
- Change of variables: deviations from equilibrium

$$
\begin{aligned}
\mathbf{q}_t + \mathbf{f}(\mathbf{q})_x &= \mathbf{s}(\mathbf{q}, x) && (13)\\
\rightarrow (\mathbf{q} - \mathbf{q}_{\mathrm{eq}})_t + (\mathbf{f}(\mathbf{q}) - \mathbf{f}(\mathbf{q}_{\mathrm{eq}}))_x &= \mathbf{s}(\mathbf{q}, x) - \mathbf{s}(\mathbf{q}_{\mathrm{eq}}, x) && (14)\\
\text{or} \quad \mathbf{q}^*_t + \mathbf{f}^*_x &= \mathbf{s}^* && (15)
\end{aligned}
$$

- At equilibrium, all terms vanish
- Jacobian of $\mathbf{f}$ remains unchanged
  - easy to adapt existing solver

Well-Balanced Methods

Rogers et al.

- Derivation in original paper leads to

$$\mathbf{q}_t^* + \mathbf{f}'(\mathbf{q})\mathbf{q}_x^* = \mathbf{s}^* \tag{16}$$

- Implementation based on this failed away from equilibrium
- Derived alternative form:

$$\mathbf{q}_t^* + \mathbf{f}'(\mathbf{q})\mathbf{q}_x^* = \mathbf{s}^* - \frac{\partial \mathbf{f}^*}{\partial \mathbf{q}_{\mathrm{eq}}}(\mathbf{q}_{\mathrm{eq}})_x \tag{17}$$

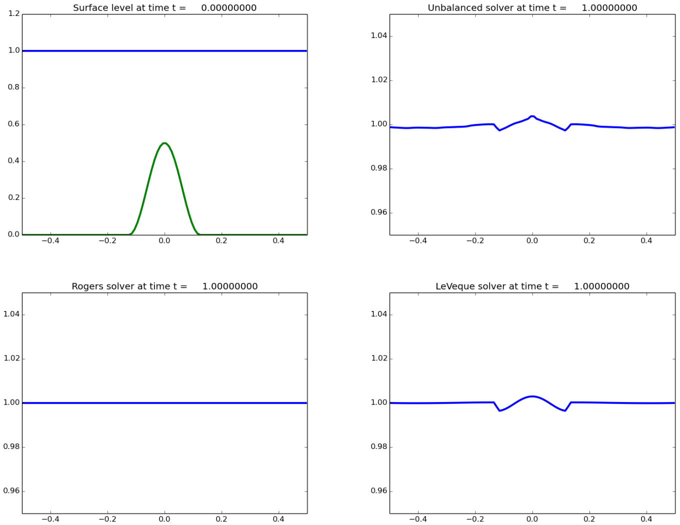
- This does work!

Project work

- Extend these methods the full SWEs with Coriolis force
- Also, derived a Rogers solver for geostrophic equilibria
  - Unfortunately, implementation is work in progress.
- Evaluation
  - Implemented evaluation framework in Clawpack (Python)
  - Supports 7 different bathymetries and 5 initial conditions
  - Implemented 4 Riemann solvers in Fortran (unbalanced, LeVeque, Rogers for still water, Rogers for geostrophic equilibrium)

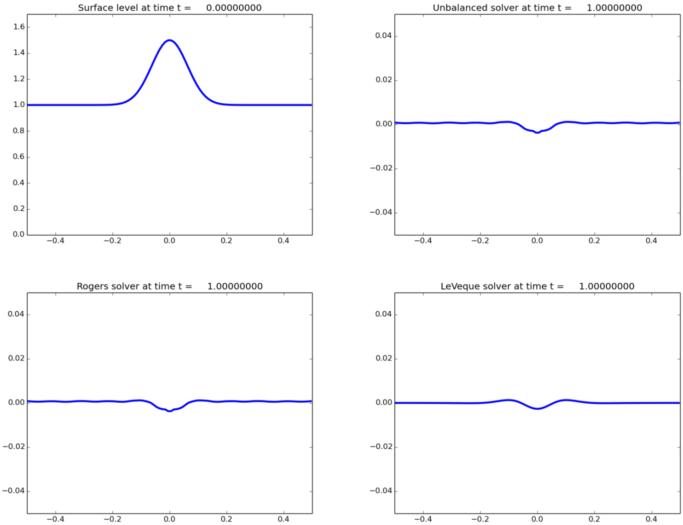
Results

Still water



Results

Geostrophic equilibrium



Conclusions

- LeVeque solver
  - Pro:
    - Well-balanced for all equilibria
  - Con:
    - Requires solving cubic for offset
    - Problematic away from equilibrium (transcritical flow)
- Rogers solver
  - Pro:
    - Works away from equilibrium
    - Fairly simple to derive and implement
  - Con:
    - Tailored to a specific equilibrium
    - Less accurate than unbalanced method away from equilibrium
- There is no free lunch!

Further work

- What's left?
  - Fix the bathymetry discretisation for LeVeque solver.
  - Get Rogers solver working for geostrophic equilibrium.
- Future research
  - Allow for dry states
  - Look at non-Godunov type methods (Chertock et al., 2014, "Well-Balanced Schemes for the Shallow Water Equations with Coriolis Forces")

Questions?

Godunov's Method

- Wave propagation form:

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}(\mathcal{A}^+ \Delta Q_{i-1/2} + \mathcal{A}^- \Delta Q_{i+1/2}) \tag{18}$$

$$\mathcal{A}^- \Delta Q_{i-1/2} = \sum_{p=1}^{M_w} (s^p_{i-1/2})^- \mathcal{W}^p_{i-1/2} \tag{19}$$

$$\mathcal{A}^+ \Delta Q_{i+1/2} = \sum_{p=1}^{M_w} (s^p_{i+1/2})^+ \mathcal{W}^p_{i+1/2} \tag{20}$$