Still water and other hard-to-model systems

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Supervisor: Prof. Ted Johnson

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The Shallow Water Equations

- Simplified fluid model:
  - Incompressible
  - Inviscid
  - *Shallow* (horizontal length scale $\gg$ typical depth)
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Finite Volume Methods

- Developed for *hyperbolic conservation laws*:

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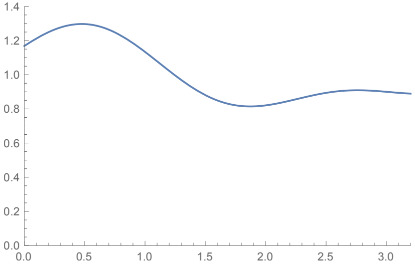
- Discretisation:



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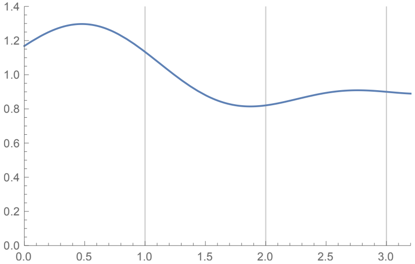

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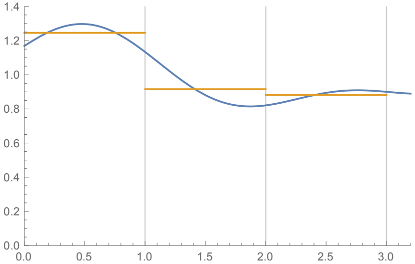

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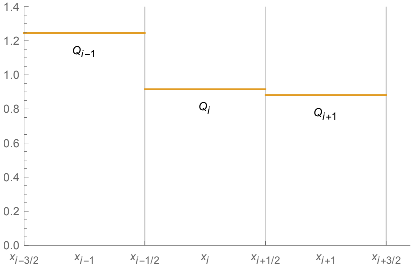

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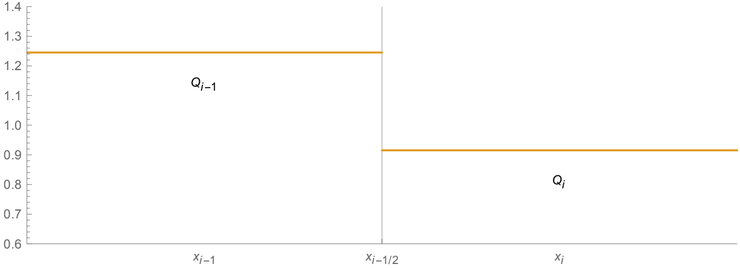

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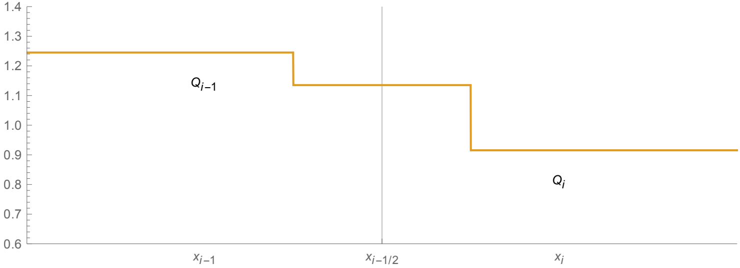

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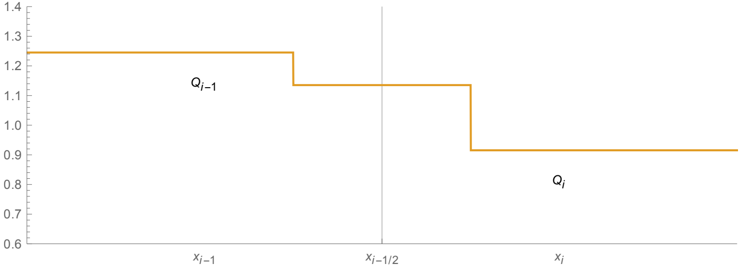

- Initial value problem: step function
- Decompose into waves
- Finite speeds $\Rightarrow$ only affect neighbouring cells

Godunov's Method

- Wave propagation form:

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}(\mathcal{A}^+ \Delta Q_{i-1/2} + \mathcal{A}^- \Delta Q_{i+1/2}) \qquad (5)$$

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$$\mathcal{A}^- \Delta Q_{i-1/2} = \sum_{p=1}^{M_w} (s_{i-1/2}^p)^- \mathcal{W}_{i-1/2}^p \qquad (6)$$

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Implementation: Clawpack

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  - Take grid data, compute wave decomposition
- Configuration in Python 2
  - Grid setup
  - Initial conditions
  - Boundary conditions
  - Plotting instructions

Adding Source Terms

- Modelling additional effects:

$$h_t + (hu)_x = 0 \tag{8}$$

$$(hu)_t + \left(hu^2 + \frac{1}{2}gh^2\right)_x = 0 \tag{9}$$

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- Full form of conservation law:

$$\mathbf{q}_t + \mathbf{f}(\mathbf{q})_x = \mathbf{s}(\mathbf{q}, x) \tag{11}$$

Adding Source Terms

- Split into two steps:

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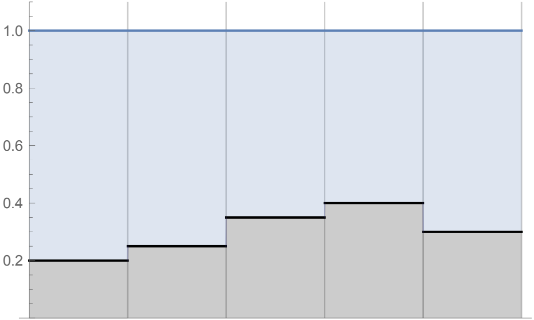

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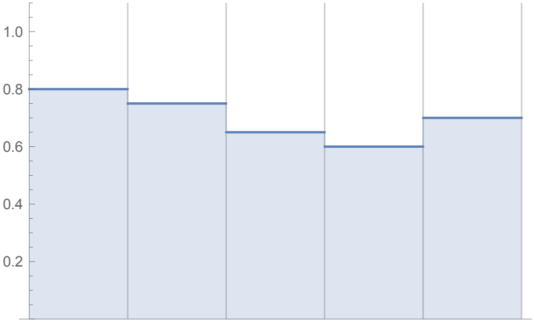
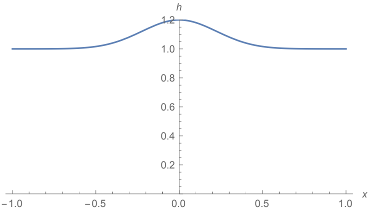
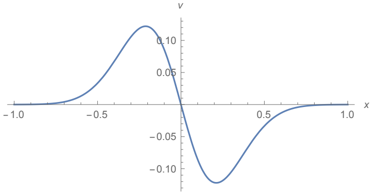


Geostrophic Equilibria

- Coriolis force admits non-trivial equilibria when

$$u = 0, \quad v = \frac{h(h_s)_x}{K} \tag{14}$$





- Geophysical flows close to geostrophic equilibrium at all times

Well-Balanced Methods

LeVeque

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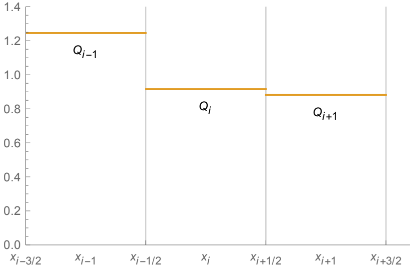

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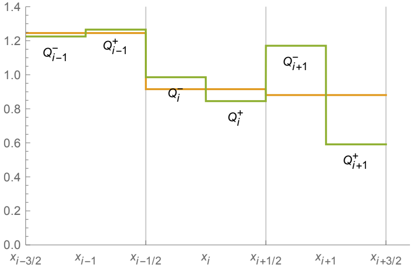

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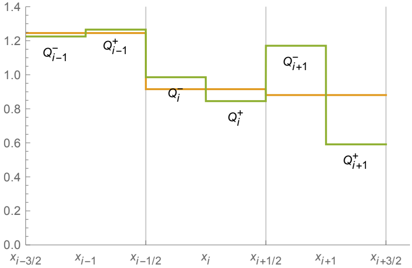

$$\mathbf{f}(Q_i^+) - \mathbf{f}(Q_i^-) = \mathbf{s}(Q_i, x_i)\Delta x \tag{15}$$

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- Pro:
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- Con:
  - Requires solving cubic for offset
  - Problematic away from equilibrium (transcritical flow)

Well-Balanced Methods

Rogers et al.

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$$\mathbf{q}_t + \mathbf{f}(\mathbf{q})_x = \mathbf{s}(\mathbf{q}, x) \tag{16}$$

$$\to (\mathbf{q} - \mathbf{q}_{\mathrm{eq}})_t + (\mathbf{f}(\mathbf{q}) - \mathbf{f}(\mathbf{q}_{\mathrm{eq}}))_x = \mathbf{s}(\mathbf{q}, x) - \mathbf{s}(\mathbf{q}_{\mathrm{eq}}, x) \tag{17}$$

$$\text{or} \quad \mathbf{q}_t^* + \mathbf{f}_x^* = \mathbf{s}^* \tag{18}$$

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- At equilibrium, all terms vanish
- Jacobian of $\mathbf{f}$ remains unchanged
  - easy to adapt existing solver

- Derivation in original paper leads to

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- Derived alternative form:

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- Also derived solver for geostrophic equilibria
  - Implementation does not work yet

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Rogers et al.

- Pro:
  - Works away from equilibrium
  - Fairly simple to derive and implement
- Con:
  - Tailored to a specific equilibrium
  - Need to know equilibrium a priori
  - Less accurate than unbalanced method away from equilibrium

Evaluation

- Implemented evaluation framework in Clawpack (Python)
- Supports 7 different bathymetries and 5 initial conditions
- Implemented 4 Riemann solvers in Fortran (unbalanced, LeVeque, Rogers for still water, Rogers for geostrophic equilibrium)