Accurate methods for computing rotation-dominated flows

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March 10, 2015

Abstract

abstract...

Contents

Chapter 1

Introduction

The shallow water equations (SWEs) are a simplified but very effective model for incompressible fluid flow. Despite the name, this model has been particularly successful in modelling large-scale geophysical flows of atmospheric and oceanic currents — "shallow" refers to the relative vertical and horizontal length scales. Considering flows on the scale of the Earth, which can easily extend across several thousand kilometres, the roughly 4 kilometre deep ocean is indeed comparably shallow.

In the context of these applications, two effects are particularly important to model: the Coriolis force, due to the rotation of the Earth, as well as varying topography[1]. This report is concerned with simulating such models using *finite volume methods*. These will usually include the above effects as source terms. Computational difficulties arise in steady flow when these very large forces balance pressure gradients exactly. This is referred to as a geostrophic balance, and large-scale real-world currents are close to this balance at all times. Therefore, the interest of this report lies in numerical methods which work around these computational problems such that they preserve geostrophic flows exactly and can reliably compute small perturbations of these steady states.

1.1 Literature Overview

The SWEs are a hyperbolic system of conservation laws. A lot of effort has been put into studying and solving these systems of equations numerically. Several textbooks about the numerical methods considered in this report exist, including LeVeque [1992], Toro [1999] and LeVeque [2002], as well as Toro [2001] which focuses on the SWEs in particular. In addition, there is a textbook on the rotating SWEs by Zeitlin [2007], Chapter 4 of which focuses on numerical methods.

Such hyperbolic systems can be solved using finite volume methods in which the domain is divided up into a (not necessarily regular) grid of control volumes, and the conserved quantities are discretised by assuming that they are constant across each such volume. Note

[1] Undersea topography is usually referred to as *bathymetry*. This term will be used predominantly throughout this report.

that "volume" is being used in a generalised sense here — for a two-dimensional system like the SWEs, the grid cells are actually areas. The most popular of these methods is due to Godunov [1959][2] and is simply known as Godunov's method — in fact, today a whole family of Godunov methods has been developed based on the concepts derived in this original paper.

A fairly recent review of Godunov-type methods was conducted by Toro and Garcia-Navarro [2007] and an older review can be found in Sweby [2001].

The piecewise-constant discretisation reduces the problem to a number of step functions at the cell boundaries. Such an initial-value problem for conservation laws is known as a Riemann problem (see Toro [1999], Section 2.2.2, LeVeque [2002], Section 3.8, or Toro and Garcia-Navarro [2007], Section 2.1) and can be solved exactly for many systems. By solving each of these Riemann problems, the discretised SWEs can be solved. The other insight of Godunov's method is that each hyperbolic system has a number of characteristic waves which only propagate in certain directions at finite speeds, which allows to simplify the computation by only looking at waves that can propagate *into* each cell. This technique is known as *upwinding*.

While it is possible to solve the Riemann problem exactly for many systems (including the SWEs), this dominates the computations required to solve each time step. Therefore, approximate solvers have been developed, the most popular one being due to Roe [1981]. See the textbooks mentioned above for an overview of exact and approximate Riemann solvers.

However, these methods generally have been developed for homogeneous systems. A simple approach to account for source terms like bathymetry and the Coriolis force is to compute these independently of the homogeneous system in a separate step. This leads to problems in steady or quasi-steady scenarios, such that the flux terms arising from the homogeneous system and the source terms are balanced at all times. According to Toro and Garcia-Navarro [2007], the first authors to recognise this were Glimm et al. [1984]. Preserving such a balance, requires that a time step in the homogeneous system is cancelled exactly by the corresponding time step for the source terms. Since these terms can in principle be very large, even for balanced systems, due to different methods being employed and numerical inaccuracies, this is practically impossible. Hence, equilibria cannot be modelled accurately, even the system is as simple as a still lake. The numerical errors create spurious oscillations which may even be amplified in future time steps. Furthermore, small perturbations away from equilibrium would be completely dominated by said numerical errors. These problems are particularly relevant for large-scale geophysical flows, which are usually very close to geostrophic balance — an instance of balanced flux and source terms — at all times.

Therefore, a lot of research was conducted over the past two decades to develop so-

[2]The author could not obtain an English translation of this Russian paper, but the method developed by Godunov has been extensively reiterated in papers and textbooks. Therefore, the following explanation of the method is based on what the author was able to find in those secondary sources.

called *well-balanced* methods which are able to preserve these equilibria exactly. To the best of the author's knowledge, Greenberg and LeRoux [1996] were the first to use the term "well-balanced".

Subsequently, dozens of well-balanced methods have been developed, including LeVeque [1998], Garcia-Navarro and Vazquez-Cendon [2000], Hubbard and Garcia-Navarro [2000], Burguete and García-Navarro [2001], Gascón and Corberán [2001], Rogers et al. [2001], Bale et al. [2003], Rogers et al. [2003], Audusse et al. [2004], Chinnayya et al. [2004], Liang and Borthwick [2009], Liang and Marche [2009]. The most recent articles the author could find are by Zhang et al. [2014] and Chertock et al. [submitted 2014], the latter being of particular relevance here, as their assumptions align with those made in this report. Furthermore, Section 4.4 of Zeitlin [2007] presents a long list of other well-balanced methods applicable to the rotating SWEs and refers to the method discussed in Audusse et al. [2004] and related works as "the most classical [well-balanced] method". With Bouchut [2004], there is also a textbook focusing primarily on these methods.

Many of these methods deal with very specific models which are beyond the scope of this report. In particular, many address the use of geometric source terms to model the equations on an irregular grid. Especially more recent papers have largely focused on methods which are capable modelling dry states. Hence, two methods were chosen to be investigated in detail in this report. The method presented in LeVeque [1998], which balances the terms by introducing additional Riemann problems, as well as the method due to Rogers et al. [2003], which employs a change of variables.

Other methods were considered for closer investigation, in particular Hubbard and Garcia-Navarro [2000] and Chertock et al. [submitted 2014]. However, these essentially develop unsplit balanced methods, which require quite a different computational framework. Hence, the scope of this report is limited to the above two methods, both of which yield balanced but split methods.

1.2 Report Outline

The remainder of this report is structured as follows. Chapter 2 constitutes the main part of this report and introduces the theory behind these numerical methods and extends them to slightly different models where necessary. Subsequently, chapter 3 describes the test framework which was set up to evaluate these methods as well as implementation details of the methods themselves. Chapter 4 shows the results obtained from these implementations. Lastly, chapter 5 draws some conclusions and suggests ways in which further research could improve on the work presented here.

Chapter 2

Theory

This chapter introduces the system of equations used throughout this report and recapitulates the theory of hyperbolic conservation laws and Godunov method's. It then proceeds to introduce the balanced methods investigated and extends them to the relevant systems.

2.1 The Shallow Water Equations

The two-dimensional SWEs are a system of three partial differential equations in three conserved quantities: the water depth, h, and the two Cartesian components of the momentum, hu and hv (where u and v are the components of the velocity). The water depth can be viewed as the difference between the water surface, h_s and the bathymetry (or bed elevation), B, i.e. $h = h_s(x,t) - B(x)$. Throughout this report, $h_s > B$ will be assumed for all x and t. The PDEs can be written as:

$$h_t + (hu)_x + (hv)_y = 0 \tag{2.1a}$$

$$(hu)_t + \left(hu^2 + \frac{1}{2}gh^2\right)_x + (huv)_y = -ghB_x + fhv \tag{2.1b}$$

$$(hv)_t + (huv)_x + (hv^2 + \frac{1}{2}gh^2)_y = -ghB_y - fhu, \tag{2.1c}$$

where subscripts denote partial differentiation, g is acceleration due to gravity and f is the Coriolis coefficient. These can be obtained from the Navier–Stokes equations by assuming that the depth of the water is small compared to some significant horizontal length-scale and by depth-averaging the flow variables. For a full derivation see Dellar and Salmon [2005].

Numerically, two dimensional systems can be solved to a good approximation by applying a dimensional split. This refers to solving the equations on a grid along slices of constant y first, and solving them along slices of constant x. During each of those steps, variation along the orthogonal direction is completely ignored. This amounts to setting $\partial/\partial y = 0$ for solving the equations only in the x-direction (and vice versa). As this approx-

imate approach works very well, this report is only concerned with these x-split equations, which reduce the SWEs to a system in only one spatial dimension:

$$h_t + (hu)_x = 0 \tag{2.2a}$$

$$(hu)_t + \left(hu^2 + \frac{1}{2}gh^2\right)_x = -ghB_x + fhv \tag{2.2b}$$

$$(hv)_t + (huv)_x = -fhu. \tag{2.2c}$$

It is common practice in fluid dynamics to use dimensionless variables, in order to reduce the system to a minimal amount of free parameters — all additional parameters, like individual length scales, then merely give similarity solutions. To do so, we introduce a typical horizontal length scale L (the width of our domain), a vertical length scale H (the approximate depth of the water), a wave speed c and a time scale T. A reasonable choice for T is L/c, which is the time taken for a wave to travel across the domain. We can also define the wave speed as $c = \sqrt{gH}$, which is the speed of linear gravity waves at depth H. Using these parameters, we can rewrite the variables in terms of dimensionless quantities:

$$t = T\bar{t}, \quad x = L\bar{x}, \quad y = L\bar{y}, \quad h = H\bar{h}, \quad u = c\bar{u}, \quad v = c\bar{v}, \quad B = H\bar{B} \tag{2.3}$$

These can be substituted into each of the SWEs. Making use of $T = L/c$, the h-equation gives:

$$h_t + (hu)_x = 0 \tag{2.4}$$

$$\Rightarrow \frac{H}{T}\bar{h}_{\bar{t}} + \frac{Hc}{L}(\bar{h}\bar{u})_{\bar{x}} = 0 \tag{2.5}$$

$$\Rightarrow \bar{h}_{\bar{t}} + (\bar{h}\bar{u})_{\bar{x}} = 0, \tag{2.6}$$

so it remains unchanged. For the hu-equation, using both $T = L/c$ and $c^2 = gH$:

$$(hu)_t + \left(hu^2 + \frac{1}{2}gh^2\right)_x = -ghB_x + fhv \tag{2.7}$$

$$\Rightarrow \frac{Hc}{T}(\bar{h}\bar{u})_{\bar{t}} + \frac{1}{L}\left(Hc^2\bar{h}\bar{u}^2 + \frac{1}{2}gH^2\bar{h}^2\right)_{\bar{x}} = -\frac{gH^2}{L}\bar{h}\bar{B}_{\bar{x}} + fHc\bar{h}\bar{v} \tag{2.8}$$

$$\Rightarrow (\bar{h}\bar{u})_{\bar{t}} + \left(\bar{h}\bar{u}^2 + \frac{1}{2}\bar{h}^2\right)_{\bar{x}} = -\bar{h}\bar{B}_{\bar{x}} + \frac{fL}{c}\bar{h}\bar{v}. \tag{2.9}$$

This equation depends on a single parameter $K \equiv fL/c$. Similarly, for the hv-equation:

$$(hv)_t + (huv)_x = -fhu \tag{2.10}$$

$$\Rightarrow \frac{Hc}{T}(\bar{h}\bar{v})_{\bar{t}} + \frac{Hc^2}{L}(\bar{h}\bar{u}\bar{v})_{\bar{x}} = -fHc\bar{h}\bar{u} \tag{2.11}$$

$$\Rightarrow (\bar{h}\bar{v})_{\bar{t}} + (\bar{h}\bar{u}\bar{v})_{\bar{x}} = -\frac{fL}{c}\bar{h}\bar{u} = -K\bar{h}\bar{u} \tag{2.12}$$

From here on, the bars will be omitted, as the dimensionless quantities will be used throughout the report. To obtain dimensional results from the dimensionless quantities, the equations 2.3 can be used. In summary, the dimensionless SWEs are

$$h_t + (hu)_x = 0 \tag{2.13a}$$

$$(hu)_t + \left(hu^2 + \frac{1}{2}h^2\right)_x = -hB_x + Khv \tag{2.13b}$$

$$(hv)_t + (huv)_x = -Khu \tag{2.13c}$$

The development of well-balanced methods is motivated by the desire to model states which are either in equilibrium or are small perturbations about equilibrium. Hence, it is worth examining which equilibrium states exist for Eqs. 2.13. By definition, all time derivatives of an equilibrium state are zero, such that the equations reduce to

$$(hu)_x = 0 \tag{2.14a}$$

$$\left(hu^2 + \frac{1}{2}h^2\right)_x = -hB_x + Khv \tag{2.14b}$$

$$(huv)_x = -Khu \tag{2.14c}$$

The simplest equilibrium, which is exists regardless of the value of K is the so called *still water* or *still lake* equilibrium, defined by $u = v = 0$ and h_s being a constant. For simplicity, we will assume that $h_s = 1$, such that the dimension depth is H. In this case $h = 1 - B(x)$ and hence $h_x = -B_x$, which fulfils Eq. 2.14a. All other terms in the equations are zero. This is the equilibrium addressed in most papers, including LeVeque [1998] and Rogers et al. [2003] which are the focus of this report.

For non-zero K, there exists a less trivial, and geophysically much more relevant equilibrium state. Given the right velocity profile, any arbitrary (continuously differentiable) water surface profile can be maintained. This is called a geostrophic equilibrium, and most large-scale flows on Earth are close to such an equilibrium at all times. The condition on v for a given profile can easily be derived from Eqs. 2.14. We assume that $u = 0$ and $h = h_s(x) - B(x)$. Then only the x-momentum equation is non-zero and gives:

$$\left(\frac{1}{2}h^2\right)_x = -hB_x + Khv \tag{2.15}$$

$$\Rightarrow hh_x = -hB_x + Khv \tag{2.16}$$

$$\Rightarrow h(h_s - B)_x = -hB_x + Khv \tag{2.17}$$

$$\Rightarrow v = \frac{(h_s)_x}{K} \tag{2.18}$$

There are other steady states, in particular those which involve non-zero u, but these depend on the given bathymetry and are beyond the scope of this report. See Esler et al. [2005] for an analysis of the phase space of flow over a ridge.

Nevertheless, we are interested in setting up systems with a uniform background flow, in order to test the numericals for states which are not the known equilibria. In these cases, the Coriolis term requires some practical considerations. Let $B = 0$ for now and consider uniform flow with $u = U$, $v = 0$ and $h_s = 1$ at $t = 0$. In this case, the momentum equations of 2.13 become

$$(u)_t = Kv \tag{2.19}$$

$$(v)_t = -Ku \tag{2.20}$$

The solution to this system is circular motion with constant speed U. Hence, even without complicated bathymetry or an initial surface profile, this system cannot maintain uniform flow. This can be alleviated by introducing a transversal pressure gradient, $+KhU$ which balances the Coriolis force due to the background flow. From a practical point of view, this is equivalent to having the water at rest (in the rotating frame, i.e. in solid body rotation),towing an obstacle through the water and changing into the rest frame of the obstacle. The full equations for uniform background flow are thus

$$h_t + (hu)_x = 0 \tag{2.21a}$$

$$(hu)_t + \left(hu^2 + \frac{1}{2}h^2\right)_x = -hB_x + Khv \tag{2.21b}$$

$$(hv)_t + (huv)_x = KhU - Khu \tag{2.21c}$$

2.2 Hyperbolic Systems

This section reviews the relevant theory of hyperbolic conservation laws and Godunov methods.

Conservation laws are systems of partial differential equations (PDEs) which, in one dimension, can be written in the form:

$$\mathbf{q}_t + \mathbf{f}(\mathbf{q})_x = \mathbf{s}(\mathbf{q}). \tag{2.22}$$

Here, $\mathbf{q}$ is a vector of density functions of conserved quantities, $\mathbf{f}$ is a *flux vector*, while $\mathbf{s}$ stands for a number of *source terms*. For the components q_i to be conserved means that the integral $\int_{-\infty}^{\infty}(q_i - s_i)\,\mathrm{d}x$ is independent of time. The flux terms describe how the quantities $\mathbf{q}$ are transported through the domain. Apart from actual sources or sinks the source terms $\mathbf{s}$ may be used to model a variety of physical and geometric effects.

For the purpose of this project only the above bathymetry and Coriolis source terms will be considered, but more advanced treatment of shallow water systems might include further terms to model other physical effects. Examples include bed friction, surface tension and eddy viscosity. If the SWEs are discretised on an irregular grid, geometric source terms might also be used which represent properties of the grid cells.

Such a system is called *hyperbolic* if the Jacobian matrix $\partial\mathbf{f}/\partial\mathbf{q}$ has real eigenvalues.

Associating the dimensionless SWEs (Eq.2.13) with Eq. 2.22, the SWEs can be written in vector form using

$$\mathbf{q} = \begin{pmatrix} h \\ hu \\ hv \end{pmatrix} \equiv \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix},\ \mathbf{f} = \begin{pmatrix} hu \\ hu^2 + \frac{1}{2}h^2 \\ huv \end{pmatrix} = \begin{pmatrix} q_2 \\ q_2^2/q_1 + \frac{1}{2}q_1^2 \\ q_2 q_3/q_1 \end{pmatrix},\ \mathbf{s} = \begin{pmatrix} 0 \\ -hB_x + Khv \\ -Khu \end{pmatrix}.$$

The Jacobian of this matrix is

$$\mathbf{A} \equiv \frac{\partial \mathbf{f}}{\partial \mathbf{q}} = \begin{pmatrix} 0 & 1 & 0 \\ -(q_2/q_1)^2 + q_1 & 2q_2/q_1 & 0 \\ -q_2 q_3/q_1^2 & q_3/q_1 & q_2/q_1 \end{pmatrix} \tag{2.23}$$

$$= \begin{pmatrix} 0 & 1 & 0 \\ c^2 - u^2 & 2u & 0 \\ -uv & b & u \end{pmatrix}, \tag{2.24}$$

where $c = \sqrt{h}$ is the wave speed (for reasons that will become apparent further down). Note that a dimensional wave speed can be recovered by multiplying by $\sqrt{gH}$, giving the familiar result $c = \sqrt{gHh}$. This Jacobian has eigenvalues

$$\lambda_1 = u - c, \quad \lambda_2 = u, \quad \lambda_3 = u + c \tag{2.25}$$

All of these are real, and hence the SWEs are indeed a hyperbolic system of conservation laws. For completeness and future reference, the corresponding right eigenvectors are given by

$$r_1 = \begin{pmatrix} 1 \\ u-c \\ v \end{pmatrix}, \quad r_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad r_3 = \begin{pmatrix} 1 \\ u+c \\ v \end{pmatrix} \tag{2.26}$$

Godunov's method has been studied thoroughly for homogeneous hyperbolic conservation laws, where $\mathbf{s} = 0$, and is based on the integral form of such systems:

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{x_{i-1/2}}^{x_{i+1/2}} \mathbf{q}(x)\mathrm{d}x = \mathbf{f}(\mathbf{q}(x_{i-1/2}, t)) - \mathbf{f}(\mathbf{q}(x_{i+1/2}, t)), \tag{2.27}$$

where $x_{i-1/2}$ and $x_{i+1/2}$ are the boundaries of a control volume centred at x_i. As opposed to the differential form, this integral form admits discontinuities, like hydraulic jumps.

Note that special care has to be taken with dry states in systems where negative values for one or more conserved quantities are unphysical. The SWEs are an example of such a system, as the water depth, h, is strictly non-negative. In fact, the SWEs do not even hold for regions where the depth is zero. Therefore, wet/dry fronts have to be dealt with differently than other cell boundaries. Modelling these is particularly important in certain geophysical flows, where they appear on beaches of protruding topography, or during outcropping of stratified flow. See Toro [2001], Chapter 6, for exact Riemann solvers in the presence of dry states. For simplicity, this report assumes that the water depth is positive for all x and t, such that dry states need not be accounted for.

The integral form 2.27 is discretised on a regular grid, with cells of width Δx, centred at x_i. The cell edges are located at $x_{i\pm1/2} \equiv x_i \pm \Delta x/2$. To retain the conservation properties of the equations, the variables replaced by a piecewise constant approximation, where the value $\mathbf{Q}_i$ in each of the cells is equal to the true cell average. Then this cell average can be updated over a finite time step with a numerical method of the form

$$\mathbf{Q}_i^{n+1} = \mathbf{Q}_i^n - \frac{\Delta t}{\Delta x}(\mathbf{F}_{i+1/2}^n - \mathbf{F}_{i-1/2}^n), \tag{2.28}$$

where the superscripts denote the time level and $\mathbf{F}_{i+1/2}^n$ is a suitable approximation to the time integral of the flux through that boundary.

ADD GODUNOV'S METHOD AND WAVE PROPAGATION FORM HERE.

The previous discussion assumed homogeneous systems. However, the interest of this project does not lie in homogeneous systems, but in conservation laws with source terms. Similar to how a dimensional splitting can be applied, traditionally, hyperbolic systems with source terms were solved by splitting the system into two parts. The homogeneous hyperbolic PDEs:

$$\mathbf{q}_t + \mathbf{f}(\mathbf{q})_x = 0.$$

And a set of ordinary different equations (ODEs) for the source terms:

$$\mathbf{q}_t = \mathbf{s}(\mathbf{q}).$$

This way, the homogeneous system can be solved using well-studied Godunov-type methods, and the source terms can be solved independently by a simple integration in time, also using established methods like Runge-Kutta (originally developed by Runge [1895] and Kutta [1901]; see Kaw et al. [2009], Sections 8.3 and 8.4 for a modern account). See Toro [2001], Section 12.2.2 or LeVeque [2002], Sections 17.2.2 to 17.5, for instance.

2.3 The Approximate Roe Solver

The SWEs are a non-linear system, for which obtaining the full solution to each Riemann problem can be computationally very expensive. A common approach is to linearise the problem at each cell boundary in the form

$$\mathbf{q}_t + \hat{\mathbf{A}}_{i-1/2}\mathbf{q}_x = 0, \tag{2.29}$$

where $\hat{\mathbf{A}}_{i-1/2}$ is an approximation to the true flux Jacobian $\partial\mathbf{f}/\partial\mathbf{q}$ evaluated at $x_{i-1/2}$. One of the most popular approximations is the solver due to Roe [1981]. The following derivation and notation follows closely section 15.3 of LeVeque [2002]. The basic idea is to perform an invertible change of variables $\mathbf{z} = \mathbf{z}(\mathbf{q})$, and parametrise this variable between the cell values surrounding the boundary in question:

$$\mathbf{z}(\xi) = \mathbf{Z}_{i-1} + (\mathbf{Z}_i - \mathbf{Z}_{i-1})\xi \tag{2.30}$$

Then one can obtain two matrices from the integrals:

$$\hat{\mathbf{B}}_{i-1/2} = \int_0^1 \frac{\mathrm{d}\mathbf{q}(\mathbf{z}(\xi))}{\mathrm{d}\mathbf{z}}\mathrm{d}\xi \tag{2.31a}$$

$$\hat{\mathbf{C}}_{i-1/2} = \int_0^1 \frac{\mathrm{d}\mathbf{f}(\mathbf{z}(\xi))}{\mathrm{d}\mathbf{z}}\mathrm{d}\xi. \tag{2.31b}$$

The approximate flux Jacobian is then:

$$\hat{\mathbf{A}}_{i-1/2} = \hat{\mathbf{C}}_{i-1/2}\hat{\mathbf{B}}^{-1}_{i-1/2} \tag{2.32}$$

The purpose of the change of variables is to make the integrals more easily solvable. If one tried to parametrise $\mathbf{Q}$ and integrate the flux Jacobian directly, the integrand would contain rational functions of ξ. With a suitable choice for $\mathbf{z}(\mathbf{q})$, one can simplify the integrands to polynomials.

Following the derivation for the one-dimensional SWEs in section 15.3.3 of LeVeque [2002], a Roe solver can be derived for the x-split SWEs by the following choice for $\mathbf{z}$:

$$\mathbf{z} = \sqrt{h} \quad \Rightarrow \quad \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} \sqrt{h} \\ \sqrt{h}u \\ \sqrt{h}v \end{pmatrix} \tag{2.33}$$

Inverting this relation:

$$\mathbf{q} = \begin{pmatrix} z_1^2 \\ z_1 z_2 \\ z_1 z_3 \end{pmatrix} \quad \Rightarrow \quad \frac{\mathrm{d}q}{\mathrm{d}z} = \begin{pmatrix} 2z_1 & 0 & 0 \\ z_2 & z_1 & 0 \\ z_3 & 0 & z_1 \end{pmatrix} \tag{2.34}$$

Further, writing $\mathbf{f}$ as in terms of the components of $\mathbf{z}$, the Jacobian can be found:

$$\mathbf{f} = \begin{pmatrix} z_1 z_2 \\ z_2^2 + \frac{1}{2} z_1^4 \\ z_2 z_3 \end{pmatrix} \quad \Rightarrow \quad \frac{\mathrm{d}f}{\mathrm{d}z} = \begin{pmatrix} z_2 & z_1 & 0 \\ 2z_1^3 & 2z_2 & 0 \\ 0 & z_3 & z_2 \end{pmatrix} \tag{2.35}$$

Now, let $z_k = (Z_k)_{i-1} + ((Z_k)_i - (Z_k)_{i-1})\xi$ for $k = 1, 2, 3$ and perform the integrals in Eqs. 2.31. As for the one-dimensional SWEs, the linear terms become

$$\frac{1}{2}((Z_k)_{i-1} + (Z_k)_i) \equiv \bar{Z}_k \tag{2.36}$$

and the cubic term becomes

$$\frac{1}{2}((Z_1)_{i-1} + (Z_1)_i)\frac{1}{2}\left((Z_1)_{i-1}^2 + (Z_1)_i^2\right) \equiv \bar{Z}_1 \bar{h}. \tag{2.37}$$

Hence, the intermediate matrices are

$$\hat{\mathbf{B}}_{i-1/2} = \begin{pmatrix} 2\bar{Z}_1 & 0 & 0 \\ \bar{Z}_2 & \bar{Z}_1 & 0 \\ \bar{Z}_3 & 0 & Z_1 \end{pmatrix} \tag{2.38}$$

$$\hat{\mathbf{C}}_{i-1/2} = \begin{pmatrix} \bar{Z}_2 & \bar{Z}_1 & 0 \\ 2\bar{Z}_1\bar{h} & 2\bar{Z}_2 & 0 \\ 0 & \bar{Z}_3 & \bar{Z}_2 \end{pmatrix} \tag{2.39}$$

and using Eq. 2.32, the approximate flux Jacobian is found to be

$$\hat{\mathbf{A}}_{i-1/2} = \begin{pmatrix} 0 & 1 & 0 \\ \bar{h} - (\bar{Z}_2/\bar{Z}_1)^2 & 2\bar{Z}_2/\bar{Z}_1 & 0 \\ -\bar{Z}_2\bar{Z}_3/\bar{Z}_1^2 & \bar{Z}_3/\bar{Z}_1 & \bar{Z}_2/\bar{Z}_1 \end{pmatrix} \tag{2.40}$$

$$= \begin{pmatrix} 0 & 1 & 0 \\ \bar{h} - \hat{u}^2 & 2\hat{u} & 0 \\ -\hat{u}\hat{v} & \hat{v} & \hat{u} \end{pmatrix}, \tag{2.41}$$

where

$$\hat{u} = \frac{\sqrt{h_{i-1}}u_{i-1} + \sqrt{h_i}u_i}{\sqrt{h_{i-1}} + \sqrt{h_i}} \tag{2.42}$$

$$\hat{v} = \frac{\sqrt{h_{i-1}}v_{i-1} + \sqrt{h_i}v_i}{\sqrt{h_{i-1}} + \sqrt{h_i}} \tag{2.43}$$

are special weighted averages, called *Roe averages*. Note that, comparing this result with Eq. 2.24, just as in the one-dimensional case this is simply the flux Jacobian of the SWEs evaluated at this special Roe-averaged state, with average wave speed, $\hat{c} = \sqrt{\bar{h}}$.

- unbalanced method
- for each balanced method:
  - quick recap of the method in general
  - derivation of method for our equations

Chapter 3

Implementation

- introduce Clawpack
- describe test harness
- go through test models
- go through relevant implementation details for all methods

Chapter 4

Results

- show how unbalanced method fails
- go through different methods, showing where they work and where they fail

Chapter 5

Conclusion

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