Sets

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Definition and Examples

Definition (Set)

A **set** is a collection of objects, which are called the **elements** of the set, without regards to order or repetition (so elements in a set are only counted once). We write $s \in S$ to mean that s is an element of the set S and $t \notin S$ to mean that t is not an element of S.

There are lots of different ways of describing a set, one way is just to write out the elements of the set in a list. For example the set $\{1, 3, 5\}$ is the set consisting of elements 1, 3 and 5.

However, this is often not a convenient way to describe a set. For example the set of all people who live in Denmark cannot easily be described in this way. We also can't describe a set like "the set of all real numbers between 0 and 1". To describe sets like this we use the following alternate notation:

Suppose X is a set and P is the property of some elements in X, we can write a set $\{x : x \in X, P(x)\}$ for all of the elements of X for which $P(x)$ is true. For example the set of all real numbers between 0 and 1 is conveniently described by the notation:

$$\{x : x \in R, 0 < x < 1\}.$$

Example

Here are some common sets and the symbols used to denote them

- $\mathbb{N} = \{1, 2, 3, ...\}$ is the natural numbers

- $\mathbb{N}_0 = \{0, 1, 2, ...\}$ is the natural numbers with 0
- $\mathbb{Z} = \{..., -2, -1, 0, 1, 2, ...\}$ is the integers
- $\mathbb{Q} = \{\frac{a}{b} : a, b \in \mathbb{Z}, b \neq 0\}$ is the rational numbers
- $\mathbb{R}$ is the real numbers
- $\mathbb{C}$ is the complex numbers
- $\emptyset$ is the empty set (set with no elements)

We call S a **finite** set if it only has a finite number of elements. If S has n elements, we write $|S| = n$. If a set is not finite, it is said to be an **infinite** set.

Equality of Sets and Subsets

Definition (Equality of Sets)

For two sets A and B. A is **equal** to B, written as $A = B$, if

$$(\forall x)\ x \in A \iff x \in B,$$

i.e. the two sets are equal if they have the same elements.

This basically just reiterates the point that elements of a set are only counted once. So $\{1, 2, 3, 3, 4\} = \{1, 2, 2, 3, 4\}$.

Definition (Subsets)

For two sets A and B. A is a **subset** of B, written $A \subseteq B$, if all elements of A are in B:

$$(\forall x)\ x \in A \implies x \in B.$$

Example

The subsets of $\{1, 2\}$ are

$$\{1, 2\}, \{1\}, \{2\}, \emptyset.$$

(By convention, $\emptyset$ is a subset of every set and every set is a subset of itself.)

Theorem

$$(A = B) \iff (A \subseteq B \text{ and } B \subseteq A)$$

Union, Intersection and Difference

Definition (Unions and Intersections)

Let A and B be sets. The **union** of A and B, written $A \cup B$, is the set consisting of all elements that lie in either A or B (or both). Symbolically,

$$A \cup B = \{x : x \in A \text{ or } x \in B\}.$$

The **intersection** of A and B, written $A \cap B$, is the set consisting of all elements that lie in both A and B; this

$$A \cap B = \{x : x \in A \text{ and } x \in B\}.$$

Note that two sets are said to be **disjoint** if their union is the empty set.

Proposition

Let A, B, C be sets. Then

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

Proof

$$\begin{aligned} x \in A \cap (B \cup C) &\iff x \in A \text{ and } x \in (B \text{ or } C) \\ &\iff (x \in A \text{ and } x \in B) \text{ or } (x \in A \text{ and } x \in C) \\ &\iff x \in (A \cap B) \cup (A \cap C). \end{aligned}$$

Hence, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

We sometimes use the more concise notation

$$A_1 \cup ... \cup A_n = \bigcup_{i=1}^{n} A_i$$

and

$$A_1 \cap ... \cap A_n = \bigcap_{i=1}^{n} A_i.$$

Definition (Set Difference)

If A and B are sets, their **difference** is define to be the set

$$A - B = \{x : x \in A \text{ and } x \notin B\}.$$

Example

For example $\{1,2,3,4\} - \{2,4\} = \{1,3\}$. And $\mathbb{R}^* = \mathbb{R} - \{0\}$, the set of non-zero real numbers.

Cartesian Products

Definition (Ordered Pair)

An **ordered pair** (a, b) is a pair of two items in which ordered matters. Formally, it is defined as $\{\{a\}, \{a, b\}\}$. We have $(a, b) = (a', b')$ if and only if $a = a'$ and $b = b'$.

Definition (Cartesian Product)

Given two sets A, B, the **Cartesian product** of A and B is $A \times B = \{(a, b) : a \in A, b \in B\}$. This can be extended to n products, e.g. $\mathbb{R}^3 = \mathbb{R} \times \mathbb{R} \times \mathbb{R} = \{(x, y, z) : x, y, z \in \mathbb{R}\}$ (which is technically $\{(x, (y, z)) : x, y, z \in \mathbb{R}\}$).

Example

For example if $A = \{1, 2\}$ and $B = \{1, 4, 5\}$, then $A \times B$ consists of the six ordered pairs

$$(1,1), (1,4), (1,5), (2,1), (2,4), (2,5).$$

The Inclusion-Exclusion Principle

Proposition

If A and B are finite sets, then

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

Proof

Let $|A \cap B| = k$, say $A \cap B = \{x_1, \dots, x_k\}$. These elements, and no others belong to both A and B, so we can write

$$A = \{x_1, \dots, x_k, a_1, \dots, a_l\},$$

$$B = \{x_1, \dots, x_k, b_1, \dots, b_m\},$$

where $|A| = k + l$ and $|B| = k + m$. Then

$$|A \cup B| = \{x_1, \dots, x_k, a_1, \dots, a_l, b_1, \dots, b_m\},$$

and all these elements are different, so

$$\begin{aligned} |A \cup B| = k + l + m \quad &= (k + l) + (k + m) - k \\ &= |A| + |B| - |A \cap B|. \end{aligned}$$

This can be generalized to any number of sets

Theorem (Inclusion-Exclusion Principle)

Let n be a positive integer, and let $A_1, \dots, A_n$ be finite sets. Then

$$|A_1 \cup \cdots \cup A_n| = c_1 - c_2 + c_3 - \cdots + (-1)^n c_n,$$

where for $1 \leqslant i \leqslant n$, the number c_i is the sum of the sizes of the intersections taken i at a time.

In case any clarification is needed, for $n = 3$ we have

$$\begin{aligned} c_1 \quad &= |A_1| + |A_2| + |A_3|, \\ c_2 \quad &= |A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3|, \\ c_3 \quad &= |A_1 \cap A_2 \cap A_3|. \end{aligned}$$

References