Scientific Programming with Mathematica 2022

Problems 2

1. Prove that, with the natural choice of square roots

$$\sqrt{5}+\sqrt{22+2\sqrt{5}} \;=\; \sqrt{11+2\sqrt{29}}+\sqrt{16-2\sqrt{29}+2\sqrt{55-10\sqrt{29}}}\;.$$

   You may find the function MinimalPolynomial useful.

2. A common method of ensuring the integrity of documents and files is to compute a number that is very sensitive to changes in the underlying document. One such is Adler-32, this proceeds as follows. Given a string $c_1c_2\ldots c_n$, we compute two integers m an n from the character codes C_j of the c_j.

$$\begin{aligned} m &= 1+C_1+C_2+\cdots+C_n \quad \mod 65521 \\ n &= (1+C_1)+(1+C_1+C_2)+\cdots+(1+C_1+C_2\cdots+C_n) \quad \mod 65521\;. \end{aligned}$$

   we then form the checksum $m+65536n$. The integer 65536 is 2^{16} and 65521 is the largest prime less than this. Write a script that computes the checksum. You may find the function Accumulate useful.

   Compute the checksum for the following string (due to John Donne):

   No man is an Iland, intire of itselfe; every man is a peece of the Continent, a part of the maine; if a Clod bee washed away by the Sea, Europe is the lesse, as well as if a Promontorie were, as well as if a Manor of thy friends or of thine owne were; any mans death diminishes me, because I am involved in Mankinde; And therefore never send to know for whom the bell tolls; It tolls for thee.

   Check your answer by evaluating Hash[str, "Adler32"], for this string.

3. Conventions differ between references so we understand a modular form of weight k to satisfy
$$f(\gamma\tau) \;=\; (cz+d)^k f(\tau) \quad \text{for all} \quad \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)\;.$$
There is another relation, that is not a special case of the above if $N \neq 1$:
$$f\left(-\frac{1}{N\tau}\right) \;=\; \epsilon N^{k/2}\tau^k f(\tau)\;, \tag{$*$}$$
where $\epsilon = \pm 1$ is a sign that depends on the particular form f.

The particular form that we need is a weight 4 form for the group $\Gamma_0(14)$ and has the designation **14.4.a.a**. The LMFDB, http://www.lmfdb.org, provides, for this form, the expansion
$$f_{\mathbf{14.4.a.a}}(\tau) \;=\; q - 2q^2 + 8q^3 + 4q^4 - 14q^5 - 16q^6 - 7q^7 - 8q^8 + 37q^9 + \ldots\;; \quad q \;=\; \mathrm{e}^{2\pi\mathrm{i}\tau}$$
You can download the first 100 terms, which should suffice.

The L-function associated with f is given in terms of the Mellin transform g by
$$L(s) \;=\; \frac{(2\pi)^s}{\Gamma(s)}\,g(s) \quad \text{with} \quad g(s) \;=\; \int_0^\infty \mathrm{d}y\, f(\mathrm{i}y) y^{s-1}\;.$$
A standard manoeuvre with regard to the Mellin transform g consists of breaking the range of integration at $y = N^{-1/2}$ and using $(*)$ to rewrite the integral for the lower part of the range. In this way we have
$$g(s) \;=\; \int_{N^{-1/2}}^\infty \mathrm{d}y\, f(\mathrm{i}y)\left\{y^{s-1} + (-1)^{k/2}\epsilon\, N^{\frac{k}{2}-s} y^{k-1-s}\right\}\;,$$
for the particular form we are concerned with $\epsilon = +1$. The advantage of this representation is that, owing to the nonzero lower limit of integration, one can integrate the q-series and this yields good convergence.

Compute $L(2)$ to at least 50 figures.

The Dedekind eta function is defined, for $\operatorname{Im}\tau > 0$, by
$$\eta(\tau) \;=\; q^{1/24}\prod_{n=1}^\infty (1-q^n)\;; \quad q \;=\; \mathrm{e}^{2\pi\mathrm{i}\tau}$$
I have it on good authority that
$$f_{\mathbf{14.4.a.a}}(\tau) \;=\; \frac{(\eta_2\eta_7)^6}{(\eta_1\eta_{14})^2} - 4(\eta_1\eta_2\eta_7\eta_{14})^2 + \frac{(\eta_1\eta_{14})^6}{(\eta_2\eta_7)^2}\;,$$
where $\eta_j = \eta(j\tau)$.

Check the q-expansion of this expression to 100 terms.

4. A Calabi-Yau manifold rejoices in the designation AESZ34. This is a one parameter family of manifolds with parameter φ. The periods of the manifold satisfy a fourth order differential equation corresponding to an operator

$$\mathcal{L} \;=\; R_4\vartheta^4 + R_3\vartheta^3 + R_2\vartheta^2 + R_1\vartheta + R_0 \quad ; \quad \vartheta \;=\; \varphi\frac{\partial}{\partial\varphi}$$

and the functions R_j are given by

$$\begin{aligned} R_0 &= -900\varphi^3 + 285\varphi^2 - 5\varphi \\ R_1 &= -2700\varphi^3 + 1088\varphi^2 - 28\varphi \\ R_2 &= -2925\varphi^3 + 1580\varphi^2 - 63\varphi \\ R_3 &= -1350\varphi^3 + 1036\varphi^2 - 70\varphi \\ R_4 &= -225\varphi^3 + 259\varphi^2 - 35\varphi + 1 \;. \end{aligned}$$

Denote by ϖ_0 the power series solution to $\mathcal{L}\varpi = 0$, normalised such that $\varpi_0(0) = 1$. Find a recurrence relation for the coefficients of ϖ_0.

We wish to calculate $\varpi_0(-1/7)$. The value $\varphi = -1/7$ lies outside the radius of convergence of the series, which is $1/25$, as is seen by factoring R_4. In order to perform the calculation first calculate ϖ_0, and its first three derivatives, at $\varphi = -1/50$ to, say, 100 figures, then integrate the differential equation to reach $\varphi = -1/7$. A good set of options for NDSolve is
AccuracyGoal $\to$ 50,
WorkingPrecision $\to$ 55,
Method $\to$ {"ExplicitRungeKutta","DifferenceOrder" $\to$ 9 }.

Finally, compute

$$\frac{\pi^2\,\varpi_0(-\frac{1}{7})}{L(2)}\;,$$

where $L(s)$ is the function from the previous problem.