Scientific Programming with Mathematica 2020

Problems 3

Please work through the entire problem set. Only the problems numbered (1) and (2) will be graded, though I will go though the entire sheet in the problem session.

Consider again the space AESZ34, from Problem set 2, for which we have the Picard-Fuchs equation

$$\mathcal{L}\varpi \;=\; 0 \quad \text{with} \quad \mathcal{L} \;=\; \sum_{j=0}^{4} R_j \vartheta^j \;;\quad \vartheta \;=\; \varphi\frac{\partial}{\partial\varphi}\;.$$

First, let us seek the four solutions to the differential equation in a neighborhood of $\varphi = 0$. Following Frobenius, we seek a solution of the form

$$\varpi \;=\; \sum_{n=0}^{\infty} A_n(\epsilon)\varphi^{n+\epsilon} \quad \text{with} \quad A_0(\epsilon) \;=\; 1\;.$$

Find a recurrence relation for the $A_n(\epsilon)$ and show that $\epsilon^4 = 0$. By expanding

$$\varpi = \varpi_0(\varphi) + \epsilon\varpi_1(\varphi) + \frac{1}{2!}\epsilon^2\varpi_2(\varphi) + \frac{1}{3!}\epsilon^3\varpi_3(\varphi) \quad \text{and} \quad A_n(\epsilon) = a_n + \epsilon b_n + \frac{1}{2!}\epsilon^2 c_n + \frac{1}{3!}\epsilon^3 d_n\;,$$

show that there are four solutions of the form

$$\begin{aligned}
\varpi_0 \;&=\; f_0\\
\varpi_1 \;&=\; f_0\log\varphi + f_1\\
\varpi_2 \;&=\; f_0\log^2\varphi + 2f_1\log\varphi + f_2\\
\varpi_3 \;&=\; f_0\log^3\varphi + 3f_1\log^2\varphi + 3f_2\log\varphi + f_3\;,
\end{aligned}$$

where the f_j are power series with coefficients a_n, b_n, c_n and d_n. The series f_0 is the series that was found in Problem Set 2.

The coefficients of the f_j are all real, so the f_j are real, for φ real and negative, which are the range of φ which will concern us. Note however that, owing to the logarithms, the ϖ_j, $j = 1, 2, 3$, are complex. We take the logarithms to be defined with the φ plane cut along the positive real axis, so, for φ real and negative, $\log\varphi \;=\; \log|\varphi| + \mathrm{i}\pi$.

Write routines that calculate the f_j, and their derivatives $\vartheta^k f_j$, with $0 \leqslant k \leqslant 3$, and so also the $\vartheta^k\varpi_j$, both as series to `nmax` terms and numerically to `nmax` terms. Check that $\mathcal{L}\varpi_3$ vanishes to the required order when $\mathsf{nmax} = 100$, say.

Now let $\varphi = \psi$ be a regular point of the differential equation, so that there are four solutions that are power series in a neighborhood of ψ. Let us choose a basis of solutions of the form

$$\begin{aligned}
\eta_0(\varphi,\psi) &= 1 + \mathcal{O}\left((\varphi{-}\psi)^4\right)\\
\eta_1(\varphi,\psi) &= (\varphi-\psi) + \mathcal{O}\left((\varphi{-}\psi)^4\right)\\
\eta_2(\varphi,\psi) &= (\varphi-\psi)^2 + \mathcal{O}\left((\varphi{-}\psi)^4\right)\\
\eta_3(\varphi,\psi) &= (\varphi-\psi)^3 + \mathcal{O}\left((\varphi{-}\psi)^4\right)
\end{aligned}$$

Find a recurrence relation for coefficients A_n, B_n, C_n, D_n, which will depend on ψ, such that $\eta_0 = \sum_{n=0}^{\infty} A_n(\varphi-\psi)^n$, $\eta_1 = \sum_{n=0}^{\infty} B_n(\varphi-\psi)^n$ etc.. Note that the $A_n, .., D_n$ all satisfy the same recurrence relation. The difference is in the initial conditions.

Now let $\varpi = (\varpi_j)$ and $\eta = (\eta_j)$, that is we have defined vectors of solutions. If $|\psi| < \frac{1}{25}$, and φ is such that both ϖ and η converge then there will be a matrix M, independent of φ, such that $\eta = M\varpi$. Let W_ϖ and W_η be the Wronskian matrices

$$W_{\varpi\, jk} = \vartheta^k \varpi_j \quad \text{and} \quad W_{\eta\, jk} = \vartheta^k \eta_j \ .$$

Note that

$$W_\eta = M W_\varpi \quad \text{so that} \quad M = W_\eta W_\varpi^{-1} \ .$$

For the case $\psi = \frac{1}{50}$, calculate M numerically, to machine precision.

A little more analytical work: Let y denote any vector of solutions to the differential equation and let W_y be the corresponding Wronskian. By considering the columns of ϑW_y show that

$$W_y^{-1}\vartheta W_y = \begin{pmatrix} 0 & 0 & 0 & -R_0/R_4\\ 1 & 0 & 0 & -R_1/R_4\\ 0 & 1 & 0 & -R_2/R_4\\ 0 & 0 & 1 & -R_3/R_4 \end{pmatrix} .$$

Now by means of the formulas, valid for a generic matrix A,

$$\det A = \exp(\operatorname{Tr} \log A) \quad \text{and its consequence} \quad \vartheta \det A = (\det A)\operatorname{Tr}\left(A^{-1}\vartheta A\right) ,$$

you can now show that

$$\det W_y = \exp\left(-\int \frac{d\varphi}{\varphi}\frac{R_3}{R_4}\right) = \frac{w_y}{R_4^2(\varphi)} ,$$

where w_y is a constant that depends on the basis y.

Compute w_ϖ and w_η. Note that one can use this relation for $\det W$ to check on the accuracy of a numerical computation of W. Write a function that computes W to a given accuracy, using this check to increase `nmax` as necessary.

1. Now given a sequence of φ values $\{\psi_0 = 0, \psi_1, \psi_2, \ldots, \psi_f\}$ such that each ψ_{i+1} lies in a disk such that $\eta(\varphi, \psi_i)$ converges for $\varphi = \psi_{i+1}$. How is $W_\varpi(\psi_f)$ related to $W_{\eta(\psi_{f-1})}(\psi_f)$? Write a code that evaluates $W_\varpi(\psi_f)$ to a specified accuracy.

   One way to choose the ψ_j proceeds as follows. Suppose $\psi_0 = 0$ and that the initial radius of convergence is r $(= 1/25)$ and that there are no singularities in $\operatorname{Re}\varphi \leqslant 0$. Choose $\psi_1 = -r/2$. The new radius of convergence runs as far as the singularity at $\varphi = 0$, so has radius $r/2$. Choose ψ_2 half-way between ψ_1 and the edge of the region of convergence so $\psi_2 = \frac{3\psi_1}{2}$, and so on. In this way we get a sequence of circles, that all pass through $\varphi = 0$ and have radius $(\frac{3}{2})^{n-1}\frac{r}{2}$, and we can take

$$\psi_n \;=\; \left(\frac{3}{2}\right)^{n-1} \psi_1 \;.$$

   A figure will be given in Lecture 6.

2. Evaluate $\varpi(-1/7)$ to 100 figures. Evaluate also $L(1)$ and $L(2)$ to the same accuracy, where $L(s)$ is the function introduced in Problem sheet 2, and seek linear relations between each of the quantities $\varpi_j(-1/7)/\pi^j$, $0 \le j \le 2$ and the quantities $L(k)/\pi^k$, $k = 1, 2$, using the function FindIntegerNullVector. The quantity ϖ_3 is special, and you should seek a linear relation between $\varpi_3(-1/7)/\pi^3$, $\zeta(3)\varpi_0(-1/7)/\pi^3$ and the $L(k)/\pi^k$.