LDPC Codes, Construction and Applications

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Abstract

In this report we study the construction of LDPC codes and implementation of a generic encoder and decoder for LDPC codes. We implement the progressive edge growth algorithm and algorithm for construction of LDPC codes based on Reed-Solomon Codes. We also implement an encoder for LDPC codes. For decoder belief propagation decoder is implemented.

1 Motivation

Coding Theory is one of the foundations of Communication Engineering. Search for better and better codes has always been an area of intense research. For a long time there was a significant gap between the theoretical capacity and practically achievable capacity. It is proved that LDPC codes introduced by Gallager, form a class of capacity approaching codes. The research in that area was dormant for a long time till they were rediscovered in the late 1990s[6]. It is shown that these codes decoded with iterative decoding algorithms like sum-product algorithm[6] or message passing algorithms[1] achieve good error performance.

GNU Radio provides a free and open-source platform for implementing practical communication systems using various compatible hardwares like USRP, and also provides a learning and simulation tool. The use of GNU Radio in research related to Software Defined Radio, Digital Signal Processing and Communication Engineering is becoming increasingly widespread. In this project we implement a generic encoder and decoder for LDPC codes in GNU Radio and also implement algorithms for construction of LDPC codes.

2 LDPC Codes

The LDPC codes, as the name suggests is characterized by a sparse parity check matrix $\mathbf{H}$. From parity check matrix we can construct the *Tanner graph* of the code. For example fig. 1 represents the Tanner graph of [7, 4, 3] Hamming code with

$$\mathbf{H} = \begin{pmatrix} 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 \end{pmatrix} \tag{1}$$

A codeword $\mathbf{x}$ from the codebook described by parity check matrix $\mathbf{H}$ should satisfy

$$\mathbf{Hx} = \mathbf{0} \tag{2}$$

In the Tanner graph of a (t_c, t_r) regular binary LDPC code, each variable node will have degree t_c and each check node will have degree t_r. Irregular LDPC codes can have different degrees for nodes. Let λ

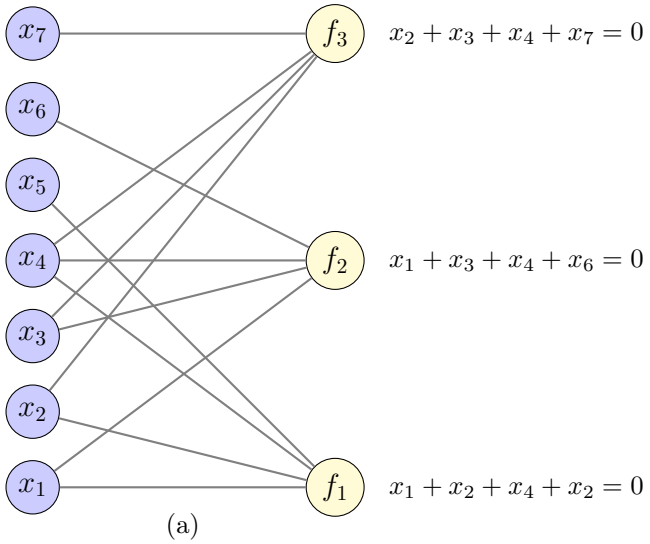


Figure 1: Tanner graph of [7, 4, 3] Hamming code with **H** given in eq. (11). The blue nodes on the left represents the components of a codeword. They are called variable nodes. The yellow nodes on the right represents each of the constraints in the system eq. (2). They are called check nodes. When $h_{i,j}$ is 1 there exists a link between j^{th} variable node and i^{th} check node.

and ρ be vectors such that their i^{th} component λ_i and ρ_i represent the fraction of edges connecting to a variable node of degree i and check node of degree i respectively. Then

$$\lambda(x) = \sum_i \lambda_i x^{i-1} \quad \text{and} \quad \rho(x) = \sum_i \rho_i x^{i-1}$$

are called variable and check degree distribution polynomials respectively. Standard ensemble LDPC(n, λ, ρ) of bipartite graphs is defined as the set of bipartite graphs with n variable nodes, variable degree distribution $\lambda(x)$ and check degree distribution $\rho(x)$.

3 Marginalization Via Message Passing

Consider a function f with factorization:

$$f(x_1, x_2, x_3, x_4, x_5, x_6) = f_1(x_1, x_2, x_3) f_2(x_1, x_4, x_6) f_3(x_4) f_4(x_4, x_5) \tag{3}$$

Now, the marginal of function f w.r.t x_1 is:

$$f(x_1) = \sum_{x_2, x_3, x_4, x_5, x_6} f(x_1, x_2, x_3, x_4, x_5, x_6) \tag{4}$$

$$= \left[\sum_{x_2, x_3} f_1(x_1, x_2, x_3)\right] \left[\sum_{x_4} f_3(x_4) \left(\sum_{x_6} f_2(x_1, x_4, x_6)\right) \left(\sum_{x_5} f_4(x_4, x_5)\right)\right] \tag{5}$$

Factor graph associated with above factorization is given in Figure 2. Here the red circles represent the variables $(x_1, x_2, \cdots, x_6)$ and are called the *variable-nodes*. Blue squares represent the factor (f_1,f_2,f_3 and f_4) and referred as the *check-nodes*. In the factor graph, a variable-node is connected to a check-node iff the corresponding variable takes part in that factor. Notice that this factor graph is a tree (i.e. there is no cycle in the graph). In the case where the factor graph is a tree the marginalization problem can be broken into smaller tasks according to the structure of the tree. The algorithm is explained in the next paragraph

The algorithm proceeds by sending messages along the edges of the tree. Messages are functions on $\mathcal{X}$, or equivalently, vectors of length $|\mathcal{X}|$. The messages signify marginals of parts of the function and these

parts are combined to form the marginal of the whole function. Message passing originates at the leaf nodes. Messages are passed up the tree and as soon as a node has received messages from all its children, the incoming messages are processed and the result is passed up to the parent node. Figure 3 explains the computation to be performed at the nodes during marginalization.
To compute the marginal with respect to all the variables, one can consider for each variable, a tree rooted in this variable and execute single marginal message passing algorithm on each rooted tree. Here all the computations can be performed simultaneously on a single tree. Start at all leaf nodes and for every edge compute the outgoing message along this edge as soon as you have the incoming messages along all other edges that connect to a given node.

4 Decoding Via Message Passing

For memoryless channel without feedback, bitwise maximum a posteriori (MAP) decoding of a codeword is given by [1]:

$$\hat{x}_i(y) = \arg\max_{x_i} p_{X_i|Y}(x_i|y) \tag{6}$$

$$= \arg\max_{x_i} \sum_{\sim x_i} p_{X|Y}(x|y) \tag{7}$$

$$= \arg\max_{x_i} \sum_{\sim x_i} p_{Y|X}(y|x) p_X(x) \tag{8}$$

$$= \arg\max_{x_i} \sum_{\sim x_i} \prod_j p_{Y_j|X_j}(y_j|x_j) p_X(x) \tag{9}$$

The notation $\sum_{\sim x_i}$ indicates that sum is over all components of x, except x_i. We obtain (7) from (6) by total probability rule. On application of Bayes's rule, we obtain (8) from (7). Equation (7) has very high computational complexity. For a codeword of length n and rate R, approximate computational complexity is $\mathrm{O}(2^{nR})$. The computational complexity can be reduced by factorizing the sequence probability. To this end, in equation(9), we use the fact that the channel is memoryless, without feedback. Furthermore, since all the input codewords are equiprobable,

$$\hat{x}_i(y) = \arg\max_{x_i} \sum_{\sim x_i} \prod_j p_{Y_j|X_j}(y_j|x_j) \mathbf{1}_{\{x \in C\}} \tag{10}$$

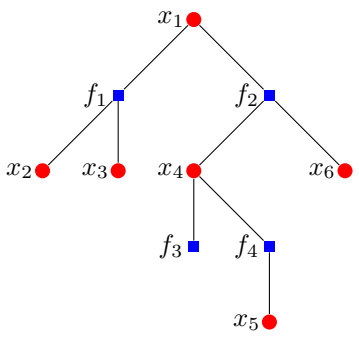


Figure 2: Factor Graph

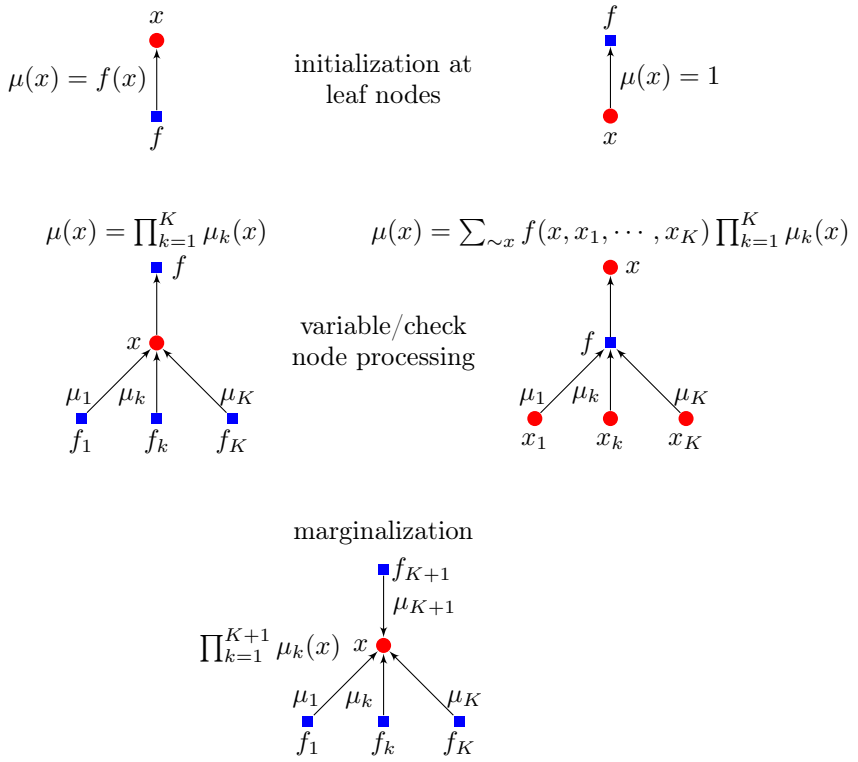


Figure 3: Message Passing Rules

Where $\mathbf{1}_{\{x \in C\}}$ is a code membership function for the codebook C. Code membership function $\mathbf{1}_{\{x \in C\}}$ has a factorized form, and this factorization is used in efficient decoding. Given the parity check matrix

$$\mathbf{H} = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{pmatrix} \tag{11}$$

the bitwise decoding rule for x_1 would be,

$$\hat{x}_i(y) = \arg\max_{x_i} \sum_{\sim x_i} \prod_j p_{Y_j|X_j}(y_j|x_j) \mathbf{1}_{\{x_1+x_4+x_6+x_7=0\}} \mathbf{1}_{\{x_2+x_4+x_5+x_6=0\}} \mathbf{1}_{\{x_3+x_5+x_6+x_7=0\}} \tag{12}$$

The corresponding factor graph would be,

4.1 Message Passing decoder for Binary Erasure Channel

By the standard message-passing rules the initial messages are $(\mu_j(0), \mu_j(1)) = (p_{Y_j|X_j}(y_j|0), (p_{Y_j|X_j}(y_j|1))$. In the case of binary erasure channel, the messages are either $(1-\epsilon, 0), (\epsilon, \epsilon)$, or $(0, 1-\epsilon)$ corresponding to 0, ? (erasure), or 1 respectively received from channel output. In this case we can normalize the message to (1, 0), (1, 1) and (0, 1) correspondingly. Now the message processing rules are discussed.

Claim : For BEC general message passing rules shown in Figure 3 simplify to the following. At a variable node the outgoing message is an erasure if all incoming messages are erasures. Otherwise, since the channel never introduces errors, all non-erasure messages must agree and either be 0 or 1. In this case the outgoing message is equal to this common value. At a check node the outgoing message is an erasure if any of the incoming message is an erasure. Otherwise, if all the incoming messages are either 0 or 1, then the outgoing message is the mod-2 sum of the incoming messages

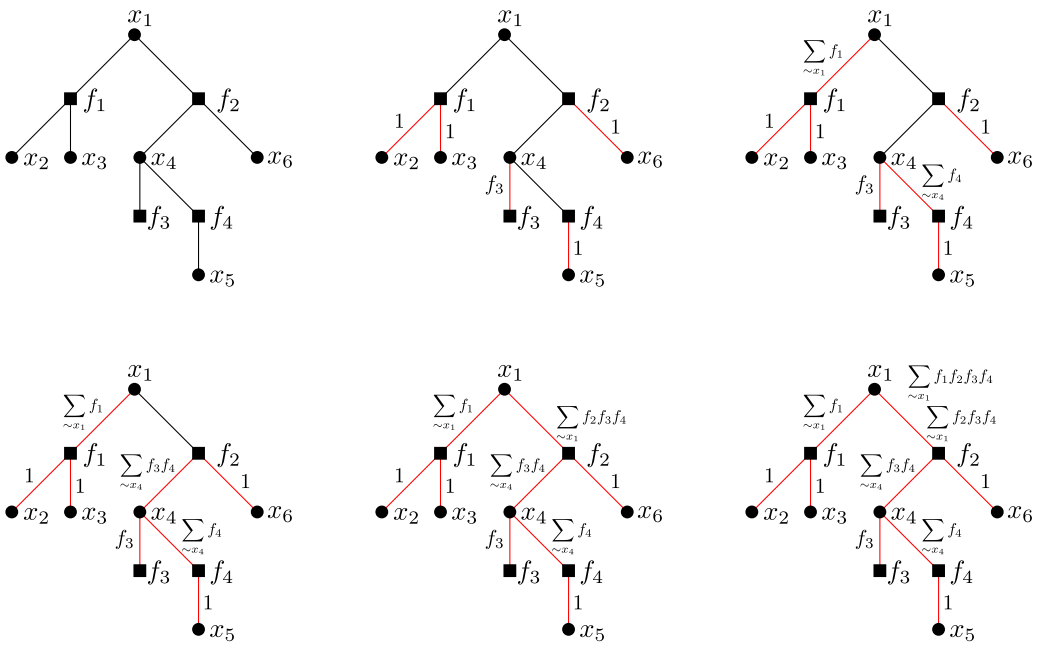
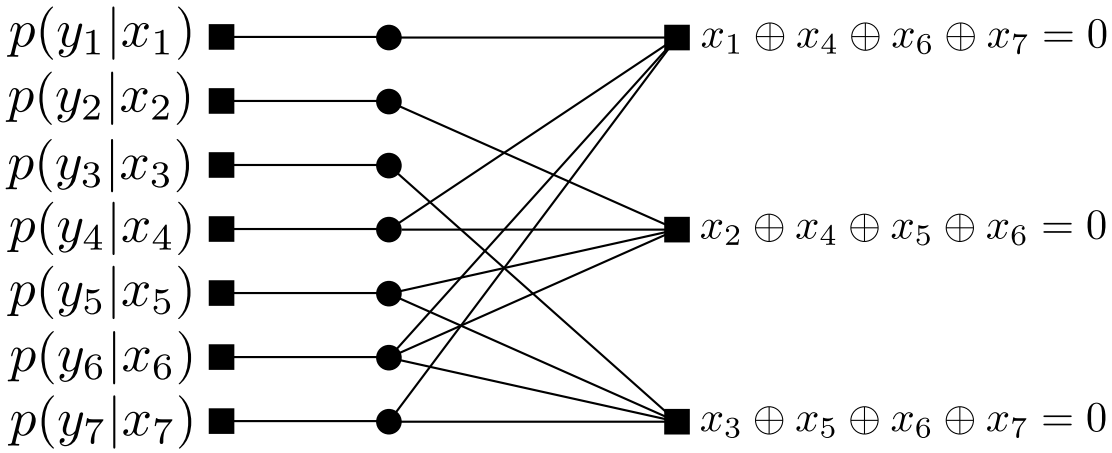


Figure 4: Marginalizaion by message passing



Proof : If all messages entering a variable node are from the set $\{(1,0),(1,1)\}$, then the outgoing message, which is equal to the component-wise product of the incoming messages according to the general message-passing rules, is also from this set. Further, it is equal to (1, 1) only if all incoming messages are of the form (1, 1) i.e, erasures. The equivalent statement is true if all incoming messages are from the set $\{(1,0),(1,1)\}$. Since channel never introduces errors, we only need to consider these two cases. Next we consider the claim concerning the message-passing rule at a check node: We consider only a check node of degree 3 with two incoming messages, since check node of higher degree can be modeled as the cascade of several check nodes, each of which has 2 inputs. Let $(\mu_1(0),\mu_1(1))$, and $(\mu_2(0),\mu_2(1))$ denote the incoming messages. By the message processing rules the outgoing message is

$$\begin{aligned}(\mu(0),\mu(1)) &= \left(\sum_{x_1,x_2}\mathbf{1}_{\{x_1+x_2=0\}}\mu_1(x_1)\mu_2(x_2),\sum_{x_1,x_2}\mathbf{1}_{\{x_1+x_2=1\}}\mu_1(x_1)\mu_2(x_2)\right) &(13)\\ &= (\mu_1(0)\mu_2(0)+\mu_1(1)\mu_2(1),\mu_1(0)\mu_2(1)+\mu_1(1)\mu_2(0)) &(14)\end{aligned}$$

If $(\mu_j(0),\mu_j(1))=(1,1), j\in\{1,2\}$, then after normalization, $(\mu(0),\mu(1))=(1,1)$. Thus if any of the input is an erasure then output is an erasure. Also if both messages are known, then by explicit check, we can see that the output corresponds to mod-2 sum. Figure 5 shows an example of this decoder on (7, 4, 3) Hamming code with parity check matrix given in Equation(11)

4.2 Message Passing Decoder for Binary Input AWGN Channel

Given in Figure 8 is a mathematical model of AWGN channel. We think that the channel input $x_i\in\{-1,+1\}$. Here again a message can be thought of as a real-valued vector of length 2, $(\mu(-1),\mu(1))$. We initialize the message passing iterations by sending from i^{th} variable node the vector $(p_{Y_i|X_i}(y_i|-1),p_{Y_i|X_i}(y_i|1))$. In case of binary input channels, it suffices to send the log-likelihood ratios.

$$l=\log\left(\frac{\mu(-1)}{\mu(1)}\right)$$

On using log-likelihood ratios, the message passing rules simplify to the following. At variable node, simply add the incoming messages and at a check node of degree $J+1$, use the following computation.

$$l=2\,\mathrm{arc\,tanh}\left(\prod_{j=1}^{J}\tanh\left(l_j/2\right)\right) \tag{15}$$

In the case of Binary Input AWGN Channel, we can initialize the message passing iterations by sending

$$\begin{aligned}l_i &= \log\left(\frac{\mu(-1)}{\mu(1)}\right) &(16)\\ &= \log\left(\frac{p_{Y_i|X_i}(y_i|-1)}{p_{Y_i|X_i}(y_i|1)}\right) &(17)\\ &= \frac{-2y_i}{\sigma^2} &(18)\end{aligned}$$

from variable nodes to the check nodes. Further computations are carried out as explained above. One iteration consists of message passing from variable nodes to check nodes, computations at the check nodes and message passing from check nodes to variable nodes. After one such iteration, we find the probability ratios by the marginalization step shown in the Figure 3. Based on the probability ratios, we estimate the codeword, by the following rule. If the probability ratio is greater than 0, we decode that bit in favor of -1 and else in favor of 1. We check if the estimated codeword is an actual codeword by parity checks. If the estimated codeword is an actual codeword, we stop, else we continue with the next round of message passing iterations.

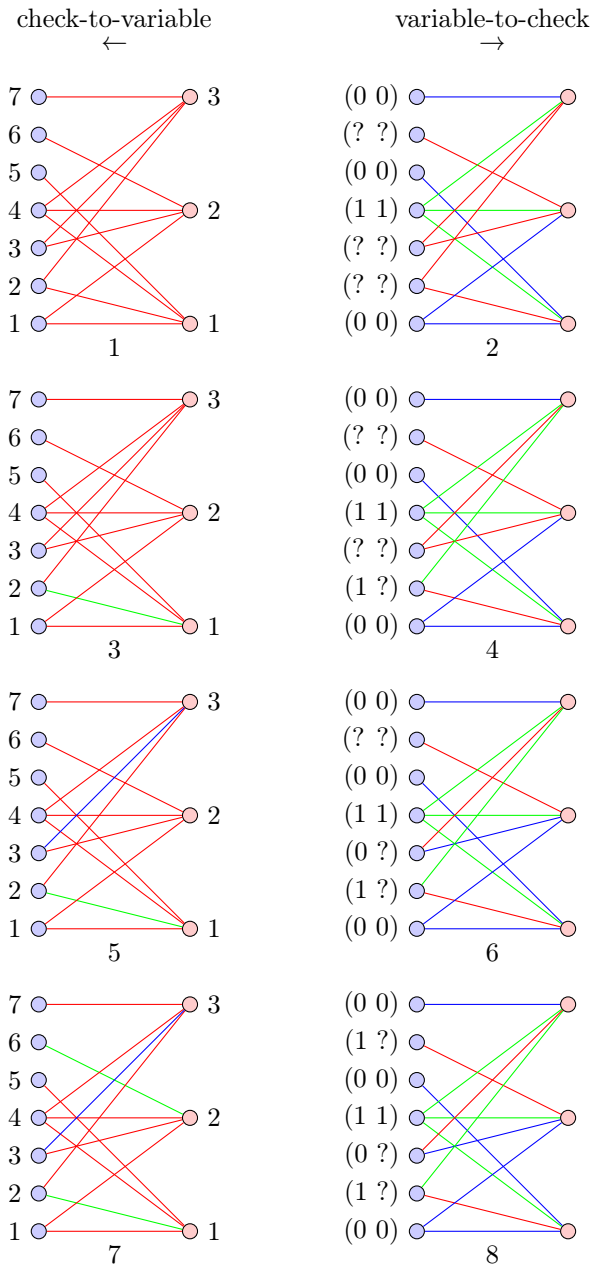


Figure 5: Message-passing decoding of the [7, 4, 3] Hamming code with the received code word y = (0, ?, ?, 1, 0, ?, 0). The left column is represents message passing from check to variable nodes. Right column represents message passing from variable to check nodes. Red links represents erasure messages, blue links represents messages 1 (corresponds to codebit 0) and green links represents message -1 (corresponds to codebit 1). The labels on the variable nodes in the right column represents the pair (estimated bit, received bit)

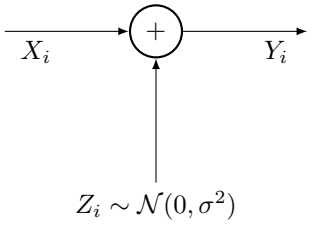


Figure 6: AWGN Channel

5 Construction of the parity check matrix

The parity check matrix of a LDPC code is a sparse matrix. Ideally the Bipartite graph constructed using the parity check matrix should not contain a cycle for optimal decoding. However, in practice it is hard to generate parity check matrices without cycles. Nevertheless, when constructing a parity check matrix for an LDPC code, we try to achieve a minimum girth. Explained in this section are two such methods, one using Reed-Solomon Codes and another using Progressive Edge Growth algorithm.

5.1 Construction of LDPC Codes using Reed-Solomon code

Reed-Solomon codes

Consider the Galois field $GF(q)$ with q elements, where q is a positive integer power of a prime number. Let ρ be a positive integer such that $2 \leq \rho < q$. The generator polynomial of a (n, k, d_{min}) RS code $\mathcal{C}$ is given by:

$$\begin{aligned} \mathbf{g}(X) &= (X-\alpha)(X-\alpha^2)\cdots(X-\alpha^{\rho-2}) && (19)\\ &= g_0 + g_1 X + \cdots + X^{\rho-2} && (20) \end{aligned}$$

Notice that $n = q-1$, $k = q-\rho+1$, $g_i \in GF(q)$ and α is a primitive element of a field. The generator matrix of this code $\mathcal{C}$ is given by

$$\mathbf{G} = \begin{pmatrix} g_0 & g_1 & g_2 & . & . & . & 1 & 0 & . & 0 \\ 0 & g_0 & g_1 & g_2 & . & . & . & 1 & 0 & . \\ 0 & 0 & g_0 & g_1 & . & . & . & . & 1 & . \\ . & . & . & . & . & . & . & . & . & . \\ 0 & . & 0 & g_0 & g_1 & g_2 & . & . & . & 1 \end{pmatrix} \tag{21}$$

Now we shorten $\mathcal{C}$ by deleting the first $1-\rho-1$ information symbols from each codeword of $\mathcal{C}$. We obtain a $(\rho, 2, \rho-1)$ shortened RS code $\mathcal{C}_b$ with only 2 information symbols. The generator matrix of that is given by

$$\mathbf{G}_b = \begin{pmatrix} g_0 & g_1 & g_2 & . & . & . & 1 & 0 \\ 0 & g_0 & g_1 & g_2 & . & . & . & 1 \end{pmatrix} \tag{22}$$

Properties of $\mathcal{C}_b$

1. Since the length of codewords in $\mathcal{C}_b$ is ρ and the minimum distance between two codewords of $\mathcal{C}_b$ is $(\rho-1)$, two codewords in $\mathcal{C}_b$ only agree at most at one location.
2. Let c be a codeword of weight ρ. If we multiply c by $\forall\beta \in GF(q)$, we get set $\mathcal{C}_b^{(1)}$ of $(q-1)$ codewords of weight ρ. Now, length of every codeword of $\mathcal{C}_b^{(1)}$ is also ρ. So $\mathcal{C}_b^{(1)}$ is a MDS (Maximum Distance Separable) code.
3. Let us partition $\mathcal{C}_b$ into a q cosets $\mathcal{C}_b^{(1)}, \mathcal{C}_b^{(2)}, ..., \mathcal{C}_b^{(q)}$ based on $\mathcal{C}_b^{(1)}$. Notice that $\mathcal{C}_b^{(i)}$ is MDS code. Therefore two codewords in any coset $\mathcal{C}_b^{(i)}$ must differ in all the locations.

Construction

Let us now explain the explicit construction procedure for a LDPC code check matrix $\mathbf{H}$.

1. All q elements of $GF(q)$ can be expressed as some power of a primitive element α. Let us define the location vector of α^i as a q-tuple over $GF(2)$, $z(\alpha^i) = (0, 0, ..., 1, 0, ..., 0)$,, where i^{th} element of $z(\alpha^i)$ is 1 and all other elements are 0. Choose one codeword $b = (b_1, b_2, ..., b_\rho) \in \mathcal{C}_b^{(i)}$. If we replace

each b_i $(1 \leq i \leq \rho)$ by its location vector $z(b_i)$, we get a $z(b) = (z(b_1), z(b_2), ..., z(b_\rho))$, which is a ρq-tuple of weight ρ over $GF(2)$.

2. Arrange all q ρq-tuple of $\mathcal{C}_b^{(i)}$ as rows of a matrix and call this matrix as $\mathbf{A}_i$. The weight of each column of $\mathbf{A}_i$ is 1.

3. Choose a positive integer γ, such that $1 \leq \gamma \leq q$. Then the parity check matrix $\mathbf{H}$ of size $\gamma q \times \rho q$ is defined as:

$$\mathbf{H} \stackrel{\triangle}{=} \begin{bmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 \\ \vdots \\ \mathbf{A}_\gamma \end{bmatrix} \tag{23}$$

Since each column of $\mathbf{A}_i$ has weight 1, weight of an each column of $\mathbf{H}$ is γ. So, $\mathbf{H}$ is a (γ, ρ)-regular matrix. Each row in $\mathbf{A}_i$ is a coset member, so each row in $\mathbf{A}_i$ is different,

1. Two rows in $\mathbf{A}_i$ do not have single common element,

2. Two codewords in $\mathcal{C}_b$ agree at at most one symbol location ($d_{min} = \rho - 1$).

From Property 1 and 2 above, no two rows from $\mathbf{A}_i,\ \mathbf{A}_j,\ i \neq j$ agree at more than a single element. This will imply that the bipartite graph corresponding to $\mathbf{H}$ is free of length 4 cycles. To see this, notice that our construction guarantees a $\mathbf{H}$ which contains no rectangle having all the corner points non-zero. This explains the construction of a (γ, ρ) regular RS-LDPC.

5.2 Construction of Parity Check Matrix using Progressive Edge Growth

In this section, we discuss the progressive edge growth (PEG) method for the construction of regular and irregular LDPC code. A bipartite graph can be described using variable nodes, check nodes and set of edges E. In this method, edges between variable nodes and check nodes are established in a progressive manner. For given variable node, an edge connects the one of the check node such that girth is maximum. Thus, PEG algorithm has large girth when compared to codes constructed using random methods. Hence, code constructed using PEG algorithm has low error floor in comparison with code constructed using random methods.
Let us denote the variable nodes by $x_i, i \in \{1, 2, \cdots, n\}$ and denote the check nodes by $f_i, i \in \{1, 2, \cdots, m\}$ For a given variable node x_i denote the check nodes reachable by breadth first search of up-to l layers as $\mathcal{N}_{x_i}^l$. Let us denote it's complement as $\bar{\mathcal{N}}_{x_i}^l$. Let us describe variable node degree sequence as follow:

$$D_x = \{d_{x1}, d_{x2}, \cdots, d_{xn}\},\ d_{x1} \leq d_{x2} \leq \cdots \leq d_{xn}$$

Where d_{xi} presents the degree of i^{th} variable node x_i.

Algorithm 1 Progressive Edge Growth Algorithm[4]

Input: sequence D_x
Output: parity check matrix $\mathbf{H}$
initialize all check nodes with degree 0
for $i = 1$ to n **do**
    for $j = 1$ to d_{xi} **do**
        if $j = 1$ **then**
            Put an edge between f_k and x_i where f_k the check node with smallest degree.
        else
            Expand the graph from x_i up-to l layers such that either the cardinality of $\mathcal{N}_{x_i}^l$ stops growing, or $\bar{\mathcal{N}}_{x_i}^l \neq \Phi$ and $\bar{\mathcal{N}}_{x_i}^{l+1} = \Phi$. Then add an edge between f_k and x_i where f_k is the node with minimum degree in $\bar{\mathcal{N}}_{x_i}^l$
        end if
    end for
end for

Check node degree distribution obtained using Algorithm 1 is almost uniform. Whenever multiple choices are available to pick check node from set $\bar{\mathcal{N}}_{x_i}^l$, we can pick any check node from set $\bar{\mathcal{N}}_{x_i}^l$.

6 Encoding

If the parity check matrix is given as $[\mathbf{I}\,\mathbf{P}]$, where $\mathbf{I}$ is $N-K$ x $N-K$ identity matrix, $\mathbf{P}$ is some $N-K$ x K matrix, a codeword $\mathbf{x} = \left[\mathbf{x_p^T}\,\mathbf{x_d^T}\right]^\mathsf{T}$ can be formed by assigning

$$\mathbf{x_p} = \mathbf{P} \cdot \mathbf{x_d}$$

We obtain $\mathbf{G} = [\mathbf{I}\,\mathbf{P}]$ from $\mathbf{H}$ by Column permutation, Row additions and Row permutations. This algorithm is explained in Algorithm 2. Suppose f represents the permutation in obtaining $\mathbf{G}$ from $\mathbf{H}$, and $\mathbf{G} \cdot \mathbf{x} = \mathbf{0}$, then $\mathbf{H} \cdot f(\mathbf{x}) = \mathbf{0}$ This is explained in Figure 7.

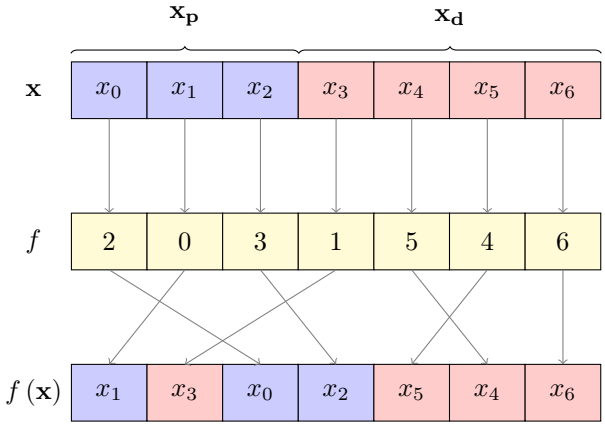


Figure 7: Encoding Procedure

Algorithm 2 Construction of $\mathbf{G} = [\mathbf{I}\ \mathbf{P}]$

Input: Parity Check matrix $\mathbf{H}$
Output: $\mathbf{G} = [\mathbf{I}\ \mathbf{P}]$
$nchk \leftarrow$ Number of rows of $\mathbf{H}$
$nvar \leftarrow$ Number of columns of $\mathbf{H}$
$limit \leftarrow$ Number of rows of $\mathbf{H}$
$rank \leftarrow 0$
$i \leftarrow 0$
$f \leftarrow (1, 2, 3, \ldots, nvar)$
while $i < limit$ **do** ▷ Pivoting at (i, i)
  $found \leftarrow$ False ▷ Flag indicating if the row contains non-zero entry
  for j from i to $nvar$ **do**
    if $\mathbf{H}[i, j] = 1$ **then** ▷ Encountering a non-zero entry at (i, j)
      $found \leftarrow$ True
      $rank \leftarrow rank + 1$ ▷ Increment rank
      swap columns i and j ▷ Now the entry at (i, i) is 1
      swap $f(i)$ and $f(j)$ ▷ Keeping track of the column permutation
      break
    end if
  end for
  if $found =$ true **then**
    for k from i to $nchk$ **do**
      if $\mathbf{H}[k, i] = 1$ **then** ▷ Checking for non-zero entries below (i, i)
        Add row i^{th} row to k^{th} row
      end if
    end for
  end if
  if $found =$ false **then** ▷ We encounter a row with all zeros
    for $rows$ in i to $limit$ **do**
      swap row $rows$ with $rows + 1$ ▷ Pushing the all-zero row to the bottom
    end for
    $limit \leftarrow limit - 1$
  end if
end while

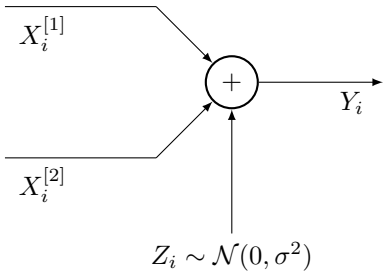


Figure 8: AWGN Multiple Access Channel

7 Belief Propagation Decoding for 2 User Gaussign MAC

The mathematical model of 2 user Gaussign Multiple Access channel is given in Figure 8. The capacity region is given by

$$\begin{aligned} R^{[1]} &\leq I(X_1; Y|X_2) \\ R^{[2]} &\leq I(X_2; Y|X_1) \\ R^{[1]} + R^{[2]} &\leq I(X_1, X_2; Y) \end{aligned}$$

The capcity region looks like the curve shown in Figure 9. We are intereseted in the rate pairs on the dominant face $\mathcal{D}$ of the capacity region for the points on $\mathcal{D}$ gives the maximum sum rate. The corner points are achievable by successive decoding. The other pointes are achieved through rate splitting or time sharing. Here we present joing decoding of the two users using belief propogation to achieve a general point on $\mathcal{D}$ without rate splitting or time sharing. To decode $x_i^{[i]}$, the i^{th} bit of user 1, we have the following MAP decoding rule.

$$\begin{aligned} \hat{x}_i^{[1]} &\stackrel{\Delta}{=} \arg\max_{x_i} p_{X_i^{[1]}|Y}(x_i^{[1]}|y) && (24) \\ &= \arg\max_{x_i} \sum_{\sim x_i^{[1]}} p_{X^{[1]},X^{[2]}|Y}(x^{[1]}, x^{[2]}|y) && (25) \\ &= \arg\max_{x_i} \sum_{\sim x_i^{[1]}} p_{Y|X^{[1]},X^{[2]}}(y|x^{[1]}, x^{[2]}) p_{X^{[1]},X^{[2]}}(x^{[1]}, x^{[2]}) && (26) \\ &= \arg\max_{x_i} \sum_{\sim x_i^{[1]}} p_{Y|X^{[1]},X^{[2]}}(y|x^{[1]}, x^{[2]}) p_{X^{[1]}}(x^{[1]}) p_{X^{[2]}}(x^{[2]}) && (27) \\ &= \arg\max_{x_i} \sum_{\sim x_i^{[1]}} \prod_j p_{Y_j|X_j^{[1]},X_j^{[2]}}(y_j|x_j^{[1]}, x_j^{[2]}) \mathbf{1}_{\{x^{[1]} \in \mathcal{C}^{[1]}\}} \mathbf{1}_{\{x^{[2]} \in \mathcal{C}^{[2]}\}} && (28) \end{aligned}$$

To obtain Equation 25 from Equation 24, we use the marginal probability rule. We get Equation 26 fron Equation 25 we use the Bayes' rule. By applying the fact that the data from one user is independent of the data from the other user, we get Equation 27. Finally we use that fact that the channel is memoryless, and each codeword is equiprobable to obtain Equation 28. Figure 10 represents the factor graph for the marginalization shown in Equation 28.

The message passing rules at the variable nodes and the check nodes remain the same as given for AWGN channel. [8] suggests the following rule for the message passing at the function nodes.

$$\mu_{s_i \to x_i^{[2]}} = \log\left(\frac{\exp(\mu_{x_i^{[1]} \to s_i}) p(y|x_i^{[1]} = -1, x_i^{[2]} = -1) + p(y|x_i^{[1]} = 1, x_i^{[2]} = -1)}{\exp(\mu_{x_i^{[1]} \to s_i}) p(y|x_i^{[1]} = -1, x_i^{[2]} = 1) + p(y|x_i^{[1]} = 1, x_i^{[2]} = 1)}\right) \quad (29)$$

$$\mu_{s_i \to x_i^{[1]}} = \log\left(\frac{\exp(\mu_{x_i^{[2]} \to s_i}) p(y|x_i^{[1]} = -1, x_i^{[2]} = -1) + p(y|x_i^{[1]} = -1, x_i^{[2]} = 1)}{\exp(\mu_{x_i^{[2]} \to s_i}) p(y|x_i^{[1]} = 1, x_i^{[2]} = -1) + p(y|x_i^{[1]} = 1, x_i^{[2]} = 1)}\right) \quad (30)$$

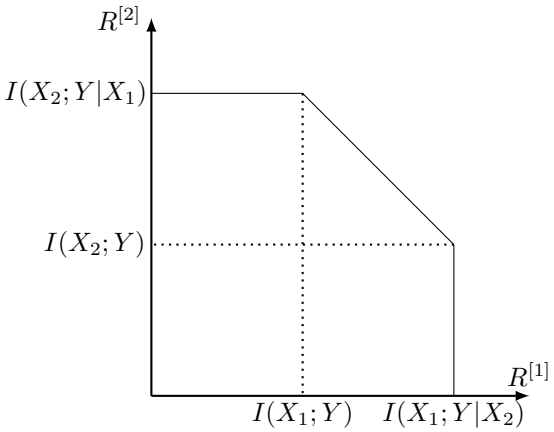


Figure 9: Capacity of AWGN Multiple Access Channel

The above rules have been verified in simulation. It is not clear at first hand why this rule works. Here we present a possible explanation.

From the single user case, the the message incoming into the variable nodes from the channel is of the type, $\log\left(p(y_i|x_i^{[k]}=-1)/p(y_i|x_i^{[k]}=1)\right)$. At the channel output we have the posterior probabilities $p(y_i|x_i^{[1]},x_i^{[2]})$. Given below is the computation of likelihood for user 2 that is passed to the variable node.

$$\frac{p(y_i|x_i^{[2]}=-1)}{p(y_i|x_i^{[2]}=1)}=\frac{p(y_i|x_i^{[1]}=-1,x_i^{[2]}=-1)p(x_i^{[1]}=-1|x_i^{[2]})+p(y_i|x_i^{[1]}=1,x_i^{[2]}=-1)p(x_i^{[1]}=1|x_i^{[2]})}{p(y_i|x_i^{[1]}=-1,x_i^{[2]}=1)p(x_i^{[1]}=-1|x_i^{[2]})+p(y_i|x_i^{[1]}=1,x_i^{[2]}=1)p(x_i^{[1]}=1|x_i^{[2]})}$$

This should be read as a ratio of marginal probabilities where $p(x_i^{[1]}|x_i^{[2]})$ acts as the weights of the conditional probabilities. To start with $x_i^{[1]}$ and $x_i^{[2]}$ are independent. So we can use $p(x_i^{[1]})$ instead of $p(x_i^{[1]}|x_i^{[2]})$. After running an iteration of message passing on one user, the posterior probability $p(x_i^{[1]}|y)$ changes and this should be used as the weights in the above equation. So we should update the above eqation as

$$\begin{aligned}\frac{p(y_i|x_i^{[2]}=-1)}{p(y_i|x_i^{[2]}=1)} &= \frac{p(y_i|x_i^{[1]}=-1,x_i^{[2]}=-1)p(x_i^{[1]}=-1|y)+p(y_i|x_i^{[1]}=1,x_i^{[2]}=-1)p(x_i^{[1]}=1|y)}{p(y_i|x_i^{[1]}=-1,x_i^{[2]}=1)p(x_i^{[1]}=-1|y_i)+p(y_i|x_i^{[1]}=1,x_i^{[2]}=1)p(x_i^{[1]}=1|y_i)}\\ &= \frac{p(y_i|x_i^{[1]}=-1,x_i^{[2]}=-1)\dfrac{p(x_i^{[1]}=-1|y)}{p(x_i^{[1]}=1|y)}+p(y_i|x_i^{[1]}=1,x_i^{[2]}=-1)}{p(y_i|x_i^{[1]}=-1,x_i^{[2]}=1)\dfrac{p(x_i^{[1]}=-1|y_i)}{p(x_i^{[1]}=1|y_i)}+p(y_i|x_i^{[1]}=1,x_i^{[2]}=1)}\end{aligned}$$

By the message passing rules, $\mu_{x_i^{[1]}\to s_i}$ is the sum of the messages incident on $x_i^{[1]}$ from all the check nodes connected to $x_i^{[1]}$. Recall that we are operating in log-likelihood domain. So what we have is,

$$\mu_{x_i^{[1]}\to s_i} = \log\left(\frac{p(x_i^{[1]}=-1|y)}{p(x_i^{[1]}=1|y)}\right) \tag{31}$$

$$\implies \frac{p(x_i^{[1]}=-1|y)}{p(x_i^{[1]}=1|y)} = \exp(\mu_{x_i^{[1]}\to s_i}) \tag{32}$$

8 Outcomes and Results

GNU Radio blocks for LDPC encoder and decoder were designed as part of this project. These can be installed as an Out of Tree module in GNU Radio. The source can be downloaded from [11]

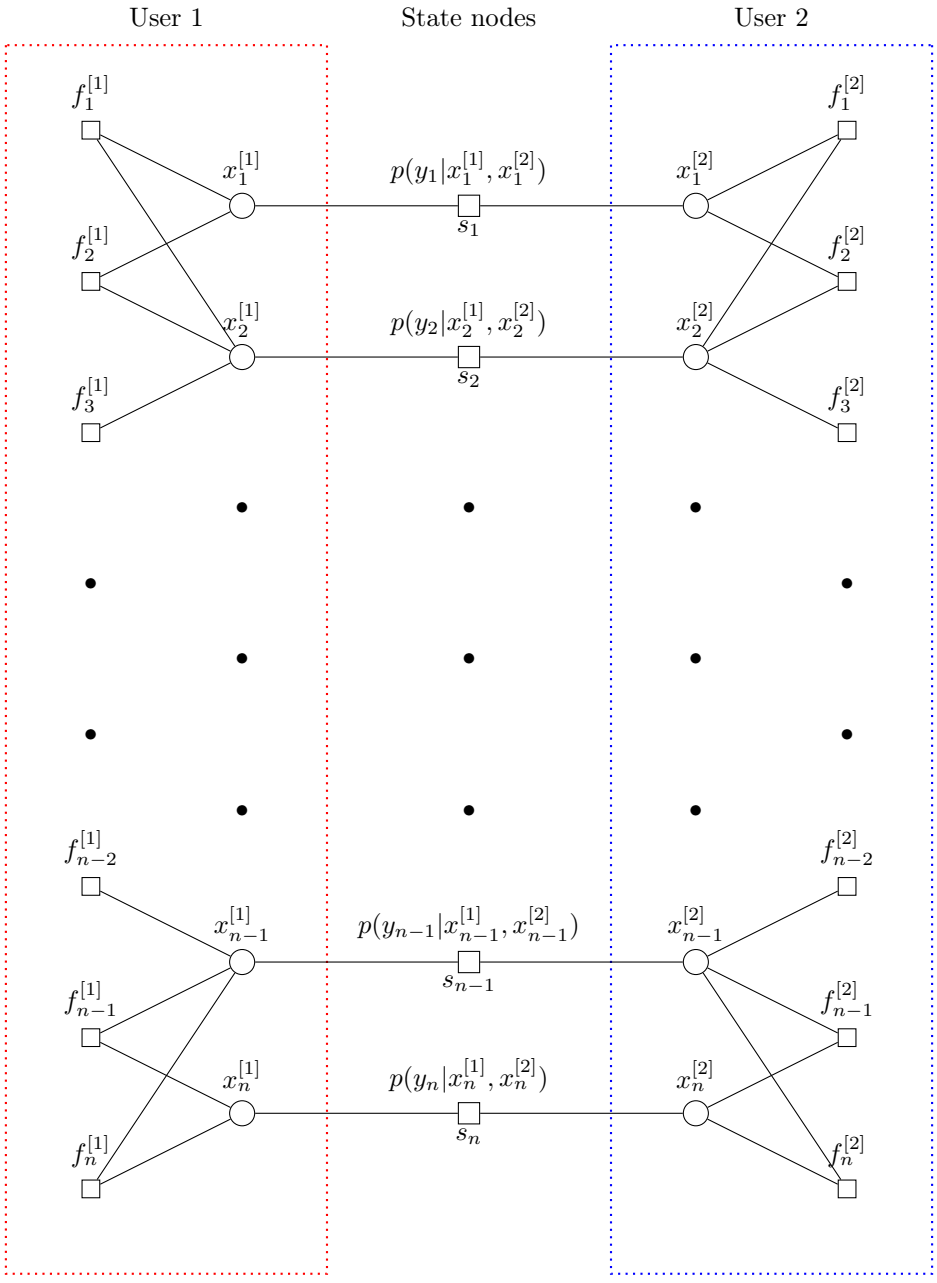


Figure 10: Factor graph for joint decoding of 2 users

Table 1: Bit Error Rate Comparison

sigma	Errors (No code)	BER (No Code)	Errors (With LDPC)	BER (With LDPC)
0.3	4338	0.0004338	0	0
0.4	61814	0.0061814	0	0
0.5	227228	0.0227228	31	3.1e-06
0.6	477744	0.0477744	65855	0.0065855
0.7	765326	0.0765326	572393	0.0572393
0.8	1055848	0.1055848	968856	0.0968856
0.9	1332039	0.1332039	1291208	0.1291208
1.0	1585653	0.1585653	1565803	0.1565803

The performance of a (3, 6) regular LDPC code with $N = 96, K = 50$, and $M = 48$

Table 2: Block Error Rate Performance

sigma	Errors (No Code)	BER (No Code)	Errors (With LDPC)	BER (With LDPC)
0.3	3898	0.196461	0	0
0.4	18938	0.954488	0	0
0.5	19840	0.999994	0	0
0.6	19841	1.0	17	0.0008568
0.7	19841	1.0	19841	1.0

The performance of an LDPC code generated using PEG with $N = 1008, K = 504$, and $M = 504$

Tables 1 and 2 compares the performance of these blocks with uncoded BPSK. The plot of Bit error performance is in Figure 11 The GNU Radio set up used in this experiment is shown in Figure 12

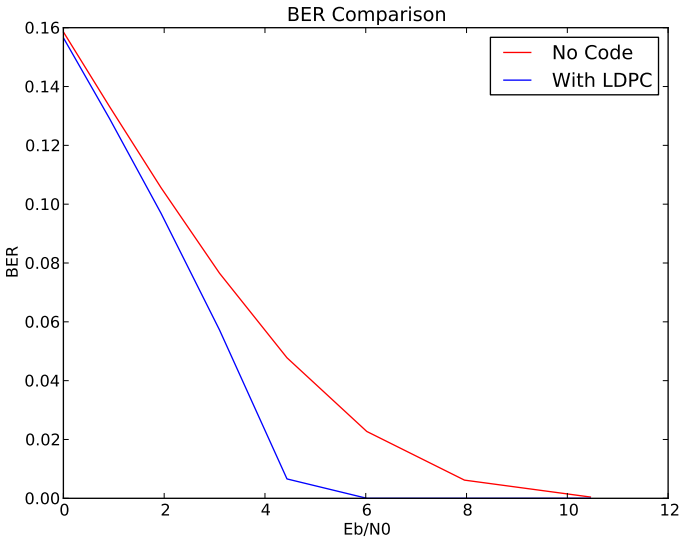


Figure 11: BER plot corresponding to Table 1

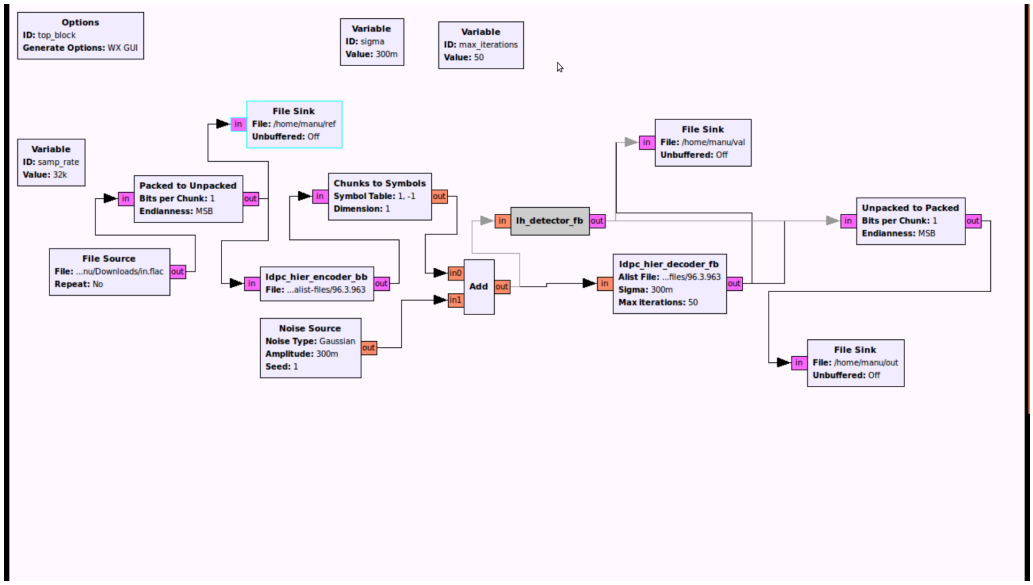


Figure 12: GNU Radio block diagram

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[11] `https://github.com/manuts/gr-ldpc`