LDPC Codes, Construction and Applications

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What is an LDPC Code

A linear error correcting code

All codewords should satisfy

$$\mathbf{Hx} = \mathbf{0} \tag{1}$$

Any vector that satisfy above equation is a codeword

A block code

If **H** is *M* x *N* and has rank *R* then the code has dimension

$$K = N - R$$

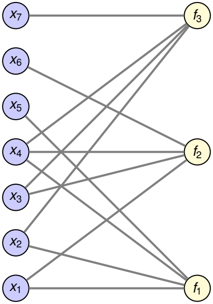
Block length of the code is *N*.

Sparse **H**

Here we are dealing with binary LDPC codes, so H has few 1s rest all zeros

Tanner Graph

$$\mathbf{H} = \begin{pmatrix} 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 \end{pmatrix} \quad (2)$$



(a)

Degree Distribution

Let λ and ρ be vectors such that their i^{th} component λ_i and ρ_i represent the fraction of edges connecting to a variable node of degree i and check node of degree i respectively. Thus

$$\lambda(x) = \sum_i \lambda_i x^{i-1} \quad \text{and} \quad \rho(x) = \sum_i \rho_i x^{i-1}$$

are called variable and check degree distribution polynomials respectively.

- Regular LDPC Codes
- Irregular LDPC Codes
- Ensemble LDPC(n, λ, ρ)

Marginalization by Message Passing

For

$$f(x_1, x_2, x_3, x_4, x_5, x_6) = f_1(x_1, x_2, x_3) f_2(x_1, x_4, x_6) f_3(x_4) f_4(x_4, x_5)$$

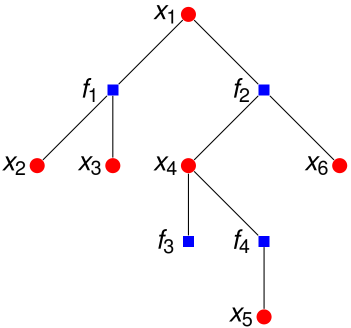
$$f(x_1) = \sum_{x_2, x_3, x_4, x_5, x_6} f(x_1, x_2, x_3, x_4, x_5, x_6)$$

$$= \left[\sum_{x_2, x_3} f_1(x_1, x_2, x_3)\right] \left[\sum_{x_4} f_3(x_4) \left(\sum_{x_6} f_2(x_1, x_4, x_6)\right) \left(\sum_{x_5} f_4(x_4, x_5)\right)\right]$$

Factor Graph

For

$$\left[\sum_{x_2,x_3} f_1(x_1, x_2, x_3)\right] \left[\sum_{x_4} f_3(x_4) \left(\sum_{x_6} f_2(x_1, x_4, x_6)\right) \left(\sum_{x_5} f_4(x_4, x_5)\right)\right]$$



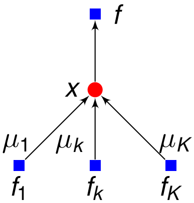
Message Passing Rules

$\mu(x) = f(x)$

initialization at leaf nodes

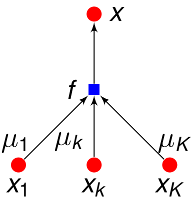
$\mu(x) = 1$

$\mu(x) = \prod_{k=1}^{K} \mu_k(x)$



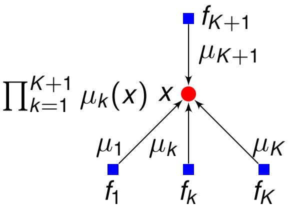
variable/check node processing

$\mu(x) = \sum_{\sim x} f(x, x_1, \cdots, x_K) \prod_{k=1}^{K} \mu_k(x)$

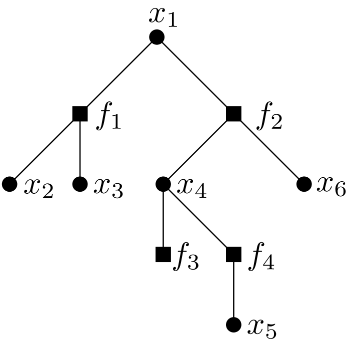


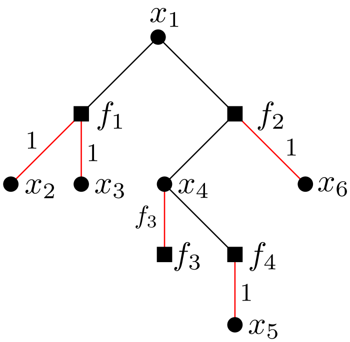
Message Passing Rules

marginalization

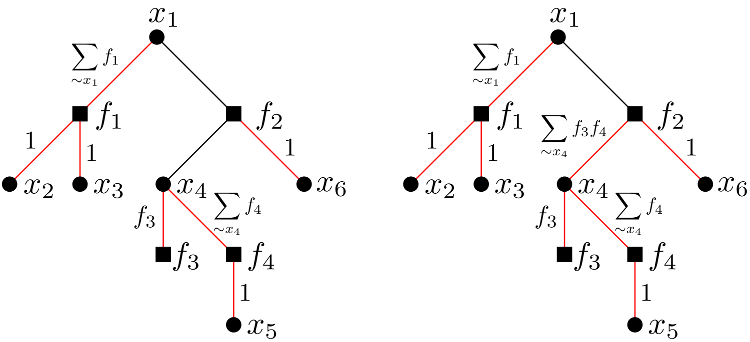


Message Passing Example

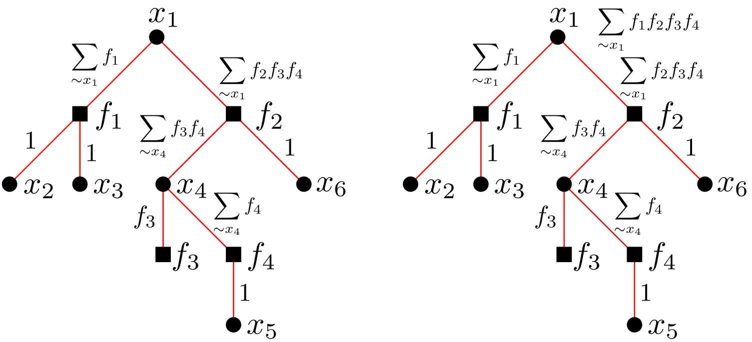




Message Passing Example



Message Passing Example



Belief Propagation Decoding

Bitwise maximum a posteriori (MAP) decoding for memoryless channel without feedback.

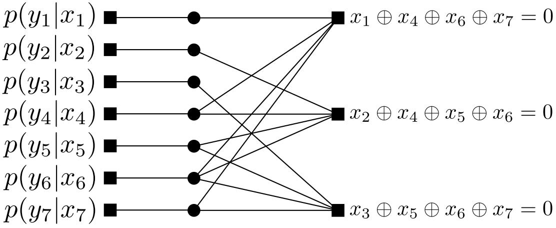
$$\begin{aligned}
\hat{x}_i(y) &= \arg\max_{x_i} p_{X_i|Y}(x_i|y) && (3)\\
&= \arg\max_{x_i} \sum_{\sim x_i} p_{X|Y}(x|y) && (4)\\
&= \arg\max_{x_i} \sum_{\sim x_i} p_{Y|X}(y|x) p_X(x) && (5)\\
&= \arg\max_{x_i} \sum_{\sim x_i} \prod_j p_{Y_j|X_j}(y_j|x_j) p_X(x) && (6)\\
&= \arg\max_{x_i} \sum_{\sim x_i} \prod_j p_{Y_j|X_j}(y_j|x_j) \mathbf{1}_{\{x \in C\}} && (7)
\end{aligned}$$

An Example

Given the parity check matrix

$$\mathbf{H} = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{pmatrix} \tag{8}$$

$$\hat{x}_i(y) = \arg\max_{x_i} \sum_{\sim x_i} \prod_j p_{Y_j|X_j}(y_j|x_j) \mathbf{1}_{\{x_1+x_4+x_6+x_7=0\}} \mathbf{1}_{\{x_2+x_4+x_5+x_6=0\}} \mathbf{1}_{\{x_3+x_5+x_6+x_7=0\}} \tag{9}$$



Message Passing Rules - Simplifications

- In the binary case the message is $(\mu(1), \mu(-1))$.
- $(\mu(1), \mu(-1)) \leftarrow \left(p_{y_i|x_i}(y_i|1), p_{y_i|x_i}(y_i|-1)\right)$
- At a variable node of degree $K+1$,

$$\mu(1) = \prod_{k=1}^{K} \mu_k(1), \quad \mu(-1) = \prod_{k=1}^{K} \mu_k(-1)$$

- $r_k \leftarrow \mu_k(1)/\mu_k(-1)$

$$r = \frac{\mu(1)}{\mu(-1)} = \frac{\prod_{k=1}^{K} \mu_k(1)}{\prod_{k=1}^{K} \mu_k(-1)} = \prod_k r_k$$

- If $l_k = \log(r_k)$, processing rule becomes $l = \sum_k l_k$

Message Passing Rules - Simplifications

- At a check node of degree $J+1$ the computation is

$$\mu(x) = \sum_{x_1, x_2, \cdots, x_J} \mathbf{1}_{\{\prod_{j=1}^{J} x_j = x\}} \prod_{j=1}^{J} \mu_j(x_j) \tag{10}$$

$$r = \frac{\mu(1)}{\mu(-1)} = \frac{\sum_{\{x_1, x_2, \cdots, x_J : \prod_{j=1}^{J} x_j = 1\}} \prod_{j=1}^{J} \mu_j(x_j)}{\sum_{\{x_1, x_2, \cdots, x_J : \prod_{j=1}^{J} x_j = -1\}} \prod_{j=1}^{J} \mu_j(x_j)} \tag{11}$$

$$= \frac{\sum_{\{x_1, x_2, \cdots, x_J : \prod_{j=1}^{J} x_j = 1\}} \prod_{j=1}^{J} \frac{\mu_j(x_j)}{\mu_j(-1)}}{\sum_{\{x_1, x_2, \cdots, x_J : \prod_{j=1}^{J} x_j = -1\}} \prod_{j=1}^{J} \frac{\mu_j(x_j)}{\mu_j(-1)}} \tag{12}$$

Message Passing Rules - Simplifications

$$
\begin{aligned}
r &= \frac{\sum_{\{x_1, x_2, \cdots, x_J : \prod_{j=1}^{J} x_j = 1\}} \prod_{j=1}^{J} r_j^{(1+x_j)/2}}{\sum_{\{x_1, x_2, \cdots, x_J : \prod_{j=1}^{J} x_j = -1\}} \prod_{j=1}^{J} r_j^{(1+x_j)/2}} && (13) \\
&= \frac{\prod_{j=1}^{J}(r_j + 1) + \prod_{j=1}^{J}(r_j - 1)}{\prod_{j=1}^{J}(r_j + 1) - \prod_{j=1}^{J}(r_j - 1)} && (14) \\
&= \frac{1 + \dfrac{\prod_{j=1}^{J}(r_j - 1)}{\prod_{j=1}^{J}(r_j + 1)}}{1 - \dfrac{\prod_{j=1}^{J}(r_j - 1)}{\prod_{j=1}^{J}(r_j + 1)}} && (15) \\
\implies \frac{r - 1}{r + 1} &= \prod_{j=1}^{J} \frac{r_j - 1}{r_j + 1} && (16)
\end{aligned}
$$

Message Passing Rules - Simplifications

$$
\begin{aligned}
r &= \exp(l) && (17)\\
\implies \tanh(l/2) &= \frac{r-1}{r+1} && (18)\\
&= \prod_{j=1}^{J} \frac{r_j - 1}{r_j + 1} && (19)\\
&= \prod_{j=1}^{J} \tanh(l_j/2) && (20)\\
\implies l &= 2\tanh^{-1}\left(\prod_{j=1}^{J} \tanh(l_j/2)\right) && (21)
\end{aligned}
$$

Construction of Parity Check Matrices

- Ideally we need tree.
- In practice we try to achieve a minimum girth in the tanner graph.
- Construction of Parity Check matrix from Reed Solomon codes.
  - Minimum girth of 4 is guaranteed.
  - Constructs a regular LDPC code.
- Progressive Edge Growth construction
  - Starts with a graph with no edges.
  - Progressively introduce edges such that it has minimum effect on the girth
  - Outputs both regular and irregular LDPC codes.

Encoding

If the parity check matrix is given as $[\mathbf{I}\ \mathbf{P}]$, where $\mathbf{I}$ is $N-K$ x $N-K$ identity matrix, $\mathbf{P}$ is some $N-K$ x K matrix, a codeword $\mathbf{x} = \left[\mathbf{x}_\mathbf{p}^T\ \mathbf{x}_\mathbf{d}^T\right]^T$ can be formed by assigning

$$\mathbf{x}_\mathbf{p} = \mathbf{P} \cdot \mathbf{x}_\mathbf{d}$$

We obtain $\mathbf{G} = [\mathbf{I}\ \mathbf{P}]$ from $\mathbf{H}$ by

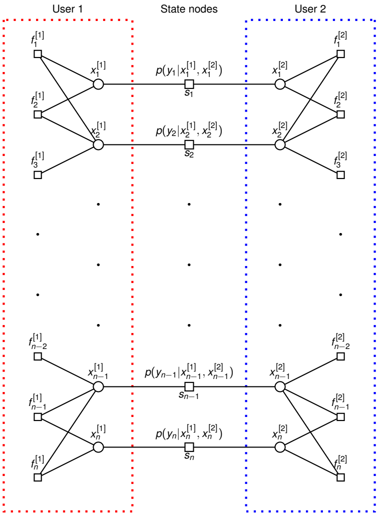
- Column permutation
- Row additions
- Row permutations

Suppose f represents the permutation in obtaining $\mathbf{G}$ from $\mathbf{H}$, and $\mathbf{G} \cdot \mathbf{x} = \mathbf{0}$, then $\mathbf{H} \cdot f(\mathbf{x}) = \mathbf{0}$

Belief Propagation Decoding for 2 User Gaussian MAC

$$\begin{aligned}
\hat{x}_i^{[1]} &\triangleq \arg\max_{x_i} p_{X_i^{[1]}|Y}(x_i^{[1]}|y) && (22)\\
&= \arg\max_{x_i} \sum_{\sim x_i^{[1]}} p_{X^{[1]},X^{[2]}|Y}(x^{[1]}, x^{[2]}|y) && (23)\\
&= \arg\max_{x_i} \sum_{\sim x_i^{[1]}} p_{Y|X^{[1]},X^{[2]}}(y|x^{[1]}, x^{[2]}) p_{X^{[1]},X^{[2]}}(x^{[1]}, x^{[2]}) && (24)\\
&= \arg\max_{x_i} \sum_{\sim x_i^{[1]}} p_{Y|X^{[1]},X^{[2]}}(y|x^{[1]}, x^{[2]}) p_{X^{[1]}}(x^{[1]}) p_{X^{[2]}}(x^{[2]}) && (25)\\
&= \arg\max_{x_i} \sum_{\sim x_i^{[1]}} \prod_j p_{Y_j|X_j^{[1]},X_j^{[2]}}(y_j|x_j^{[1]}, x_j^{[2]}) \mathbf{1}_{\{x^{[1]}\in\mathcal{C}^{[1]}\}} \mathbf{1}_{\{x^{[2]}\in\mathcal{C}^{[2]}\}} && (26)
\end{aligned}$$

Factor Graph for Joint Decoding



Message Passing Rules for Joint Decoding

- At variable nodes and at Check nodes the rules are the same.
- Rule for the function node

$$\mu_{s_i \to x_i^{[2]}} = \log \left(\frac{\exp(\mu_{x_i^{[1]} \to s_i}) p(y|x_i^{[1]} = -1, x_i^{[2]} = -1) + p(y|x_i^{[1]} = 1, x_i^{[2]} = -1)}{\exp(\mu_{x_i^{[1]} \to s_i}) p(y|x_i^{[1]} = -1, x_i^{[2]} = 1) + p(y|x_i^{[1]} = 1, x_i^{[2]} = 1)} \right) \quad (27)$$

$$\mu_{s_i \to x_i^{[1]}} = \log \left(\frac{\exp(\mu_{x_i^{[2]} \to s_i}) p(y|x_i^{[1]} = -1, x_i^{[2]} = -1) + p(y|x_i^{[1]} = -1, x_i^{[2]} = 1)}{\exp(\mu_{x_i^{[2]} \to s_i}) p(y|x_i^{[1]} = 1, x_i^{[2]} = -1) + p(y|x_i^{[1]} = 1, x_i^{[2]} = 1)} \right) \quad (28)$$

Results - Bit Error Rate Performance

(3, 6) Regular Code of block length $N = 96$, dimension $K = 50$, Number of parity checks $M = 48$, 10^7 Bytes Transferred

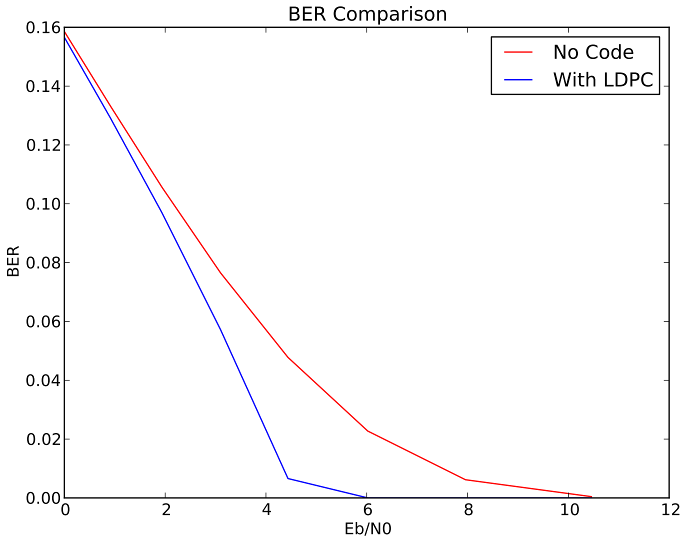
	No Code		With LDPC	
sigma	Errors	BER	Errors	BER
0.3	4338	0.0004338	0	0
0.4	61814	0.0061814	0	0
0.5	227228	0.0227228	31	3.1e-06
0.6	477744	0.0477744	65855	0.0065855
0.7	765326	0.0765326	572393	0.0572393
0.8	1055848	0.1055848	968856	0.0968856
0.9	1332039	0.1332039	1291208	0.1291208
1.0	1585653	0.1585653	1565803	0.1565803

Results - Block Error Rate Performance

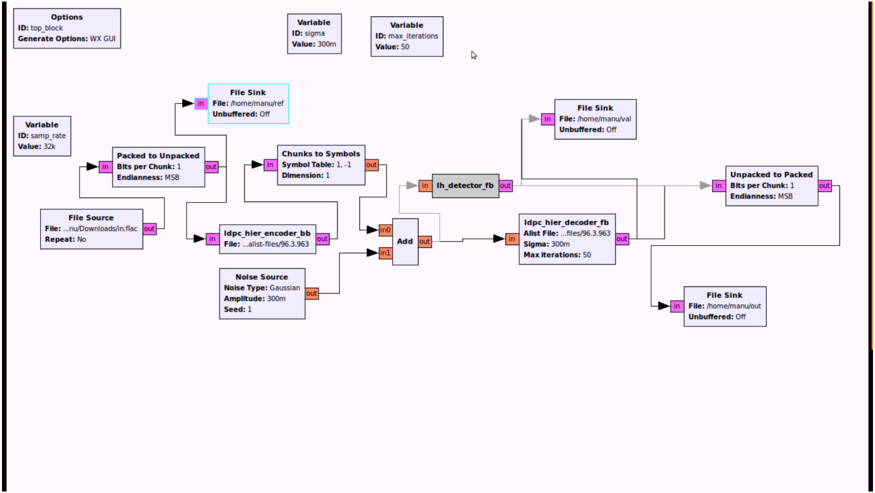
LDPC generated using PEG Algorithm,
Block length $N = 1008$, Dimension $K = 504$
Number of Parity Checks $M = 504$

	No Code		With LDPC	
sigma	Errors	BER	Errors	BER
0.3	3898	0.196461	0	0
0.4	18938	0.954488	0	0
0.5	19840	0.999994	0	0
0.6	19841	1.0	17	0.0008568
0.7	19841	1.0	19841	1.0

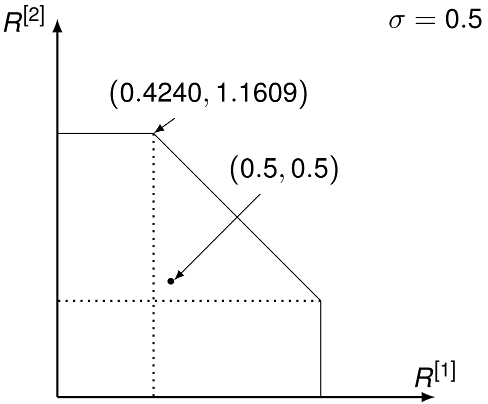
Results - Plots



gr-ldpc module in GNU Radio



Rate - Capacity comparison



For single user at $\sigma = 0.8$ (capacity = 0.678) all blocks were decoded correctly.

Future Work

- 2 users with sample offset of 0.5 symbol duration.
- Encoder using sparse LU decomposition
- Using VOLK