CONSTRUCTION OF THE CIRCLE IN UNIMATH

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Overview

From the perspective of homotopy type theory, the standard CW-complex presentation of the circle $\mathbb{S}^1$—built by attaching a loop to a point—describes its universal property as a *higher inductive type*. The corresponding induction principle says that to define a function $\mathbb{S}^1 \to X$ valued in another type X one must specify a point $x : X$ together with a loop $p : x =_X x$.

There is another classical homotopy theoretic construction of the circle, as the homotopy type of the classifying space $B\mathbb{Z}$ of the group $\mathbb{Z}$ of integers. In homotopy type theory, $B\mathbb{Z}$ has an appealing definition as the sum over all pairs comprised of a type X and an endomorphism $f : X \to X$ so that (X, f) is merely equivalent to $(\mathbb{Z}, \mathrm{succ})$. The objective of this paper is to show that the type $B\mathbb{Z}$ has the universal property of $\mathbb{S}^1$ described above (where the full induction principle allows the type X to depend on $B\mathbb{Z}$.

Recommendation

The paper is well-written and carefully argued: indeed, the main result and its supporting lemmas have been formalized in the *UniMath* library, the code for which is publicly available on `github`. The mathematical result is undoubtably of interest to homotopy type theorists: given the conservative conventions adopted by the *UniMath* library, which prohibits the use of higher inductive types, the author's work gives the first construction of the circle in that library.

The question is whether this result is of broader interest outside of the homotopy type theory community, such as would warrant its publication in a more general-audience journal. I believe a case for this can be made, though in the current version the authors largely neglect to do so.

My recommendation to the authors is two-fold:

(1) Expand the discussion in the introduction about the controversy surrounding higher inductive types (and their semantics) that led to the choice of *UniMath* to exclude them. This discussion should be addressed to a non-specialist reader, so it might start, for instance, by describing how semantics relates to "the burden of proving the consistency of a formal system."

One reason that a non-specialist might be interested in homotopy type theory is due to Shulman's result that homotopy type theory can be interpreted in any ∞-topos. Crucially, from the point of view of this paper, Shulman's result encompasses standard homotopy type theory, with univalent universes, but not higher inductive types (at least at present). Now it's not so hard to give a direct construction of $\mathbb{S}^1$ in an ∞-topos — for instance, it can be understood as the homotopy

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coequalizer of two copies of the identity map between the terminal object and itself — but the author's main theorem still gives a new result: that this object gives rise to a classifier for $\mathbb{Z}$-torsors. In my view, this is a story worth sharing with the reader. Thus:

(2) Add a section (perhaps between §5 and §6) that sketches the construction of $B\mathbb{Z}$ in any ∞-topos, explain the interpretation of its dependent universal property (which can be applied, for instance, to construct an equivalence with the circle $\mathbb{S}^1$ defined above, and conclude that $\mathbb{S}^1$ classifies $\mathbb{Z}$-torsors.

I'd like to see a revised version of the paper that makes a case for the broader interest of this result along these lines. At that point, I'd advocate for its publication in JPAA with reservations.

Several minor expository suggestions follow.

Minor comments

(p2) In the sentence "The traditional algebra notion" clarify that the $\mathbb{Z}$-torsors are the *objects* of the connected groupoid you are describing. Also "trivial $\mathbb{Z}$-torsor" confuses me. Perhaps "canonical $\mathbb{Z}$-torsor — namely $\mathbb{Z}$ itself — whose automorphism group is $\mathbb{Z}$."

(p2) The ∞-topos motivation for this work could be discussed in a paragraph right above "Now we give a few more details."

(p3) On page 3 you switch from $B\mathbb{Z}$ to $T\mathbb{Z}$ without explanation: say $B\mathbb{Z} = (T\mathbb{Z}, \mathrm{pt})$.

(p3) It might be kind to the reader to slow down a little around the paragraph "Considering pairs" to remind them exactly what a $\mathbb{Z}$-torsor is and thus why $T\mathbb{Z}$ defines the correct type. Eg you could observe here that the automorphism group of $(\mathbb{Z}, s)$ is $\mathbb{Z}$ as mentioned above, while $\mathrm{Aut}(\mathbb{Z})$ would be too big.

(p3) My instinct would be to footnote the parenthetical about Grayson's formalization, or leave it unparenthesized.

(p5) Display the formula for the propositional truncation.

(2.1) Maybe "positive integer" in place of "positive number."

(p5) You introduce S for the successor on $\mathbb{N}$ without comment.

(p6) When you "recall the equivalence ev_0" I'd just explain it again.

(p7) Unmatched left parenthesis in $\prod_{z:\mathbb{Z}}(P(z)$ on the very first line.

(p8) It would be great if you could somehow fit the product decomposition of $\varphi^{-1}(p)$ into two lines so the matched parentheses are easier to see.

(§4) This seems to interrupt the flow from §3 to §5. Perhaps it could appear as §2 or be relegated to an appendix?

(p8) "their types" could be "the types of the terms b_1 and b_2".

(4.3) Does the proof construct an inverse equivalence? Or does it reduce the statement that blah is an equivalence to the case where blah is the identity function? Same questions for the proof of 4.9.

(§5) At the start of the section perhaps write "In this section, pairs (X, f) will be ..." Also perhaps remind the reader where to look for the various formalizations.

(5.2) This definition references the unit map $|-|: A \to \|A\|$ without introducing the notation.

(p11) I wasn't aware that the induction principle for propositional truncation but not the computation rule holds in UniMath. Could you say more about why this is? Or is there a reference to a discussion the reader can look at?

(p11) Around "The aim of this section" it would be helpful to state the types of both the recursion and induction principles for the reader to look at and compare.

(p11) It wasn't clear to me why $Q_p(X, f)$ is a proposition. Is this using an inhabitant of X to define a composite path $a' = a = a''$ between two terms of $Q_p(X, f)$?

(p12) $\vec{c}_p(-)$ is meant to be a function from what to what?

(§5.2) Add a reference to §4 (particularly if it moves to the appendix) at the start of §5.2. Also add a reference the first time that rrfl is used to remind the reader that this is not a typo!

(§6) In the conclusion "we have proved" remind the reader that A is any type family over $T\mathbb{Z}$.

(§6) In the "future research," what exactly are you hoping to prove about the type TG?