MATH3401 Complex Analysis

Condensed notes for open-book exam

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1 General results and definitions

Definition 1.1 (Modulus). $|z| = \sqrt{x^2 + y^2}$.

Definition 1.2 (Triangle Inequality). $|z_1 + z_2| \le |z_1| + |z_2|$.

Definition 1.3 (Complex conjugates).

$$\bar{z} = x - iy, \quad \overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}, \quad \overline{z_1 z_2} = \bar{z}_1 \bar{z}_2, \quad \frac{\bar{z}_1}{\bar{z}_2} = \left(\overline{\frac{z_1}{z_2}}\right).$$

Definition 1.4 (Equivalent forms). *Since* $x = r\cos\theta$, $y = r\sin\theta$,

$$z = x + iy \Leftrightarrow z = r(\cos\theta + i\sin\theta).$$

Definition 1.5 (Exponential form). $z = re^{i\theta}$. *Properties:*

$$z_1 = r_1 e^{i\theta_1},\ z_2 = r_2 e^{i\theta_2} \Rightarrow z_1 z_2 = (r_1 + r_2)e^{i(\theta_1+\theta_2)},$$
$$\Rightarrow \frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)}.$$

Definition 1.6 (Arg Properties).

$$\arg(z_1 z_2) = \arg(z_1) + \arg(z_2),\ \arg(z_1^{-1}) = -\arg(z_1),$$
$$\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2).$$

Definition 1.7 (Exponential function). $e^z = e^x e^{iy}$, $e^{iy} = \cos\,y + i\,\sin\,y$.
Properties: $|e^z| = e^x$, $\arg(e^z) = y + 2n\pi$, $e^z \neq 0$, $e^{z+2\pi i} = e^z$, e^z *can be negative.*

Definition 1.8 (Logarithm function). $\log z = \ln|z| + i\arg z$. *Same properties as in* $\mathbb{R}$.

Definition 1.9 (Power function). $z^c = e^{c\log z}$. *Same properties as in* $\mathbb{R}$. *The principal value of* z^c: *P.V* $z^c = e^{c\,\mathrm{Log}\,z}$ *which is also the principle branch of the function* z^c *on* $|z| > 0$, $-\pi < \mathrm{Arg}\ z < \pi$.

Definition 1.10 (de Moivre's formula). $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$.
Properties:

$$\cos 2\theta = \cos^2\theta - \sin^2\theta,$$
$$\sin 2\theta = 2\sin\theta\cos\theta.$$

Definition 1.11 (Trigonometric functions (p.103)). *Trigonometric functions are entire on* $\mathbb{C}$. *Derivatives defined same as in* $\mathbb{R}$.

$$\sin\,z = \frac{e^{iz} - e^{-iz}}{2i}, \quad \cos z = \frac{e^{iz} + e^{-iz}}{2}$$

Properties:

$$\begin{aligned}
&\sin(-z) = -\sin z (\textit{odd function}), && \cos(-z) = \cos z\ (\textit{even function})\\
&\sin(z_1 + z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2, && \cos(z_1 + z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2\\
&\sin(2z) = 2\sin z\ \ \cos z, && \cos(2z) = \cos^2 z - \sin^2 z,\\
&\sin z = \sin x\ \ \cosh y + i\cos x\ \ \sinh y, && \cos z = \cos x\ \ \cosh y - i\sin x\ \ \sinh y,\\
&\sin z = 0\ \textit{iff}\ z = n\pi, && \cos z = 0\ \textit{iff}\ z = \pi/2 + n\pi\\
&\sin^2 z + \cos^2 z = 1
\end{aligned}$$

Definition 1.12 (Hyperbolic Trig Functions (p.109)). *Derivatives defined same as in* $\mathbb{R}$

$$\sinh z = \frac{e^z - e^{-z}}{2}, \quad \cosh z = \frac{e^z + e^{-z}}{2} \tag{1}$$

Hyperbolic sine and cosine functions are related to the trigonometric functions by,

$$-i\sinh(iz) = \sin z, \quad \cosh(iz) = \cos z,$$
$$-i\sin(iz) = \sinh z, \quad \cos(iz) = \cosh z$$

Frequently used identities,

$$\sinh(-z) = -\sinh z, \quad \cosh(-z) = \cosh z \tag{2}$$

2 General Theorems

Theorems on Limits

Theorem 2.1 (Uniqueness of Limits). *When a limit of a function $f(z)$ exists at a point z_0, it is unique.*

Theorem 2.2. *Suppose that for $z = x+iy$, $f(z) = u(x,y)+iv(x,y)$ and $z_0 = x_0+iy_0$, $w_0 = u_0+iv_0$. Then, if*

$$\lim_{(x,y)\to(x_0,y_0)} u(x,y) = u_0 \text{ and } \lim_{(x,y)\to(x_0,y_0)} v(x,y) = v_0,$$

then $\lim_{z\to z_0} f(z) = w_0$.

Theorem 2.3. *If z_0 and w_0 are points in the z and w planes respectively, then*

1. $\lim_{z\to z_0} f(z) = \infty$ *if* $\lim_{z\to z_0} \frac{1}{f(z)} = 0$,
2. $\lim_{z\to\infty} f(z) = w_0$ *if* $\lim_{z\to 0} f\left(\frac{1}{z}\right) = w_0$,
3. $\lim_{z\to\infty} f(z) = \infty$ *if* $\lim_{z\to 0} \frac{1}{f(\frac{1}{z})} = 0$.

Theorems on Continuity

Theorem 2.4. *If two functions are continuous at a point, then their sum and product are continuous at that point. A composition of continuous functions is itself continuous. If a function $f(z)$ is continuous and non-zero at z_0, then there exists some neighbourhood of z_0 where $f(z) \neq 0$.*

Theorem 2.5. *A function f is continuous at a point z_0 if all of the following conditions are satisfied*

1. $\lim_{z\to z_0} f(z)$ *exists,*
2. $f(z_0)$ *exists,*
3. $\lim_{z\to z_0} f(z) = z_0$.

Theorems on Derivatives

Definition 2.1 (Derivatives in $\mathbb{C}$). *Let f be a function whose domain of definition contains a neighbourhood $|z - z_0| < \varepsilon$ of a point z_0. The derivate of f at z_0 is the limit,*

$$f'(z_0) = \lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0}.$$

By change of variable $\Delta z = z - z_0$, we have

$$f(z_0) = \lim_{\Delta z\to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}.$$

Theorem 2.6 (Rules for differentiation). *Derivatives defined same as in $\mathbb{R}$. The quotient rule for complex functions is given by*

$$\frac{d}{dz}\left[\frac{f(z)}{g(z)}\right] = \frac{g(z)f'(z) - f(z)g'(z)}{g(z)^2}.$$

Theorem 2.7 (Cauchy-Riemann (CR) Equations). *Suppose that $f(z) = u(x,y) + iv(x,y)$ and that $f'(z)$ exists at a point $z_0 = x_0 + iy_0$. Then u_x, u_y, v_x, v_y must exist at $(x_0, y_0$ and they must satisfy the CR equations*

$$u_x = v_y, \quad u_y = -v_x.$$

Since CR are necessary conditions, they can be used to test where f is not differentiable.

Theorem 2.8 (Sufficient conditions for differentiability). *Let $f(z) = u(x, y) + iv(x, y)$ be defined on some ε-neighbourhood of a point $z_0 = x_0 + iy_0$ and suppose*

1. *1st order partials exist everywhere in the neighbourhood.*
2. *Those partials are continuous at (x_0, y_0) and satisfy CR.*

Then $f'(z_0) = u_x + iv_x$, where partials are evaluated at (x_0, y_0).

Theorem 2.9 (CR in Polar coordinates). *$ru_r = v_\theta$, $u_\theta = -rv_r$. Then $f'(z_0)$ exists, and $f'(z_0) = e^{i\theta}(u_r + iv_r)$ evaluated at (r_0, θ_0).*

Definition 2.2 (Analytic functions). *A function f is analytic in an open set S if it has a derivative everywhere in that set. It is analytic at a point if it is analytic in some neighbourhood. An **entire** function is entire if it is analytic at each point in the plane.*

Definition 2.3 (Harmonic functions). *A function H of two real variables x and y is said to be harmonic in a given domain if throughout that domain it has continuous partials of the first and second order and satisfies Laplace's equation*

$$H_{xx}(x, y) + H_{yy}(x, y) = 0.$$

Theorem 2.10. *If a function $f(z) = u(x, y) + iv(x, y)$ is analytic in a domain D, then its component functions u and v are harmonic in D.*

Definition 2.4 (Harmonic conjugates). *If a function $f(z) = u(x, y) + iv(x, y)$ is analytic in a domain D, then the real valued functions u and v are harmonic in that domain. (That is they have continuous 1st and 2nd order partials and satisfy Laplace's eq). If the first-order partials satisfy CR throughout D, then v is said to be the harmonic conjugate of u.*

Theorem 2.11. *If a harmonic function $u(x, y)$ is defined on a simply connected domain D, it always has a harmonic conjugate $v(x, y)$ in D.*

Theorems on Integrals

Theorem 2.12 (Contour integrals). *Suppose we have a function $z = z(t)$ $(a \le t \le b)$ that represents a contour extending from $z_1 = z(a)$ to $z_2 = z(b)$. Assuming $f[z(t)]$ is piecewise continuous, then*

$$\int_C f(z)\, dz = \int_a^b f[z(t)]z'(t)\, dt. \tag{3}$$

The value of the contour is invariant under a change in representation.

Theorem 2.13. *If $w(t)$ is a piecewise continuous function defined on an interval $a \le t \le b$, then*

$$\left|\int_a^b w(t)\, dt\right| \le \int_a^b |w(t)|\, dt. \tag{4}$$

Theorem 2.14 (ML Inequality). *Let C denote a contour of length L, and suppose that a function $f(z)$ is piecewise continuous on C. If M is a non-negative constant such that $|f(z)| \le M$ for all points z on C at which $f(z)$ is defined, then*

$$\left|\int_C f(z)dz\right| \le ML. \tag{5}$$

Theorem 2.15 (Antiderivatives). *For a function $f(z)$, continuous in a domain D, if any of the following are true, they all are:*

1. *$f(z)$ has an antiderivative $F(z)$ on D;*
2. $\displaystyle\int_{z_1}^{z_2} f(z)dz = F(z)|_{z_1}^{z_2} = F(z_2) - F(z_1)$;
3. *The integrals of $f(z)$ around closed contours lying in D all have value 0.*

Theorem 2.16 (Cauchy-Goursat). *If a function f is analytic at all points interior to and on a simple closed contour C, then*

$$\int_C f(z)\ dz = 0. \tag{6}$$

E.g., if C is any simple closed contour in either direction $\int_C \sin(z^2)\ dz = 0$ because $\sin(z^2)$ is analytic everywhere and $2z\cos(2z)$ is continuous everywhere.

Theorem 2.17 (Cauchy-Integral formula). *Let f be analytic everywhere inside and on a simple closed contour C, taken in the positive sense. If z_0 is any point interior to C, then*

$$f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)\ dz}{z - z_0}. \tag{7}$$

Theorem 2.18 (CI Extension). *Let f be analytic inside and on a simple closed contour C, taken in the positive sense. If z_0 is any point interior to C, then*

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_C \frac{f(z)\ dz}{(z - z_0)^{n+1}}. \tag{8}$$

E.g., if C is the positively oriented unit circle $|z| = 1$ and $f(z) = \exp(2z)$, then

$$\int_C \frac{\exp(2z)\ dz}{z^4} = \int_C \frac{f(z)\ dz}{(z-0)^{3+1}} = \frac{2\pi i}{3!}f'''(0) = \frac{8\pi i}{3}.$$

Theorem 2.19 (Cauchy's Inequality). *Suppose that a function f is analytic inside and on a positively oriented circle C_R, centered at z_0 and with radius R. If M_R denotes the maximum value of $|f(z)|$ on C_R, then*

$$|f^{(n)}(z_0) \leq \frac{n!M_R}{R^n}. \tag{9}$$

Theorem 2.20 (Liouville's Theorem). *If a function f is entire and bounded in the complex plane, then $f(z)$ is constant throughout the plane.*

Theorem 2.21. *If a function f is analytic and not constant in a given domain D, then $f(z)$ has no maximum value in D. There is no point z_0 in the domain such that $f(z) \leq f(z_0)\ \forall z \in D$.*

Definition 2.5 (Convergence of sequences). *An infinite sequence $z_1, z_2, \ldots, z_n, \ldots$ of complex numbers has a limit z, if for each positive number ε, $\exists$ a positive integer n_0 such that*

$$|z_n - z| < \varepsilon \text{ whenever } n > n_0. \tag{10}$$

When the limit z exists, the sequence is said to converge to z: $\lim_{n\to\infty} z_n = z$.

Theorem 2.22. *Suppose that $z_n = x_n + iy_n$ and $z = x + iy$. Then,*

$$\lim_{n\to\infty} z_n = z \text{ iff } \lim_{n\to\infty} x_n = x, \text{ and } \lim_{n\to\infty} y_n = y. \tag{11}$$

Theorem 2.23 (Taylor's Theorem). *Suppose that a function f is analytic throughout a disk $|z - z_0| < R_0$, centered at z_0 with radius R_0. Then, $f(z)$ has the power series representation*

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n, \quad (|z - z_0| < R_0). \tag{12}$$

Any function which is analytic at a point z_0 must have a Taylor series about z_0.

Theorem 2.24 (Maclaurin Series). *When $z_0 = 0$ and f is assumed to be analytic throughout a disk $|z| < R_0$, the Taylor series becomes a Maclaurin series*

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n. \tag{13}$$

Theorem 2.25. *Useful Maclaurin series*

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n = 1 + z + z^2 + \dots \quad (|z| < 1) \tag{14}$$

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \dots \quad (|z| < \infty), \tag{15}$$

$$\sin(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots \quad (|z| < 1), \tag{16}$$

$$\cos(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \dots \quad (|z| < 1), \tag{17}$$

$$\sinh(z) = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!} = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \dots \quad (|z| < 1), \tag{18}$$

$$\cosh(z) = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!} = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \dots \quad (|z| < 1). \tag{19}$$

Theorem 2.26 (Laurent Series). *Suppose that a function f is analytic throughout an annular domain $R_1 < |z - z_0| < R_2$ centered at z_0, and let C denote any positively oriented simple closed contour around z_0 and lying in that domain. Then, at each point in the domain, $f(z)$ has the series representation*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=0}^{\infty} \frac{b_n}{(z - z_0)^n}, \quad (R_1 < |z - z_0| < R_2), \tag{20}$$

where

$$a_n = \frac{1}{2\pi i} \int_C \frac{f(z)\, dz}{(z - z_0)^{n+1}} \quad \text{and} \quad b_n = \frac{1}{2\pi i} \int_C \frac{f(z)\, dz}{(z - z_0)^{-n+1}}.$$

Theorem 2.27 (Absolute Convergence of Power Series). *If a power series $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ converges when $z = z_1$ ($z_1 \neq z_0$), then it is absolutely convergent at each point z in the open disk $|z - z_0| < R_1$ where $R_1 = |z_1 - z_0|$.*

Theorem 2.28 (Unique Series Representation). *Both Taylor and Laurent series representations of functions are unique.*

Theorem 2.29 (Cauchy's Residue Theorem). *Let C be a simple closed contour, oriented in the positive sense. If a function f is analytic inside and on C except for a finite number of singular points z_k ($k = 1, 2, \dots, n$) inside C, then*

$$\int_C f(z) dz = 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z = z_0} f(z). \tag{21}$$

Theorem 2.30 (Residues). *If a function f is analytic everywhere in the finite plane except for a finite number of singular points interior to a positively oriented simple closed contour C, then*

$$\int_C f(z)\,dz = 2\pi i \operatorname*{Res}_{z=0}\left[\frac{1}{z^2} f\left(\frac{1}{z}\right)\right]. \tag{22}$$

Definition 2.6 (Classifying isolated singular points). *If f has an isolated singular point at z_0, then $f(z)$ has a Laurent series representation.*

1. ***Removable singular points:*** *When every b_n is zero, so that*

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n, \tag{23}$$

z_0 is known as a removable singular point. If we define $f(z_0) = a_0$, then the singularity at z_0 is therefore removed.

2. ***Essential singular points:*** *If an infinite number of the coefficients b_n are nonzero, z_0 is said to be an essential singular point of f.*

3. ***Poles of order m:*** *If the principal part of f at z_0 contains at least one nonzero term but the number of terms is finite, there exists an integer m such that: $b_m \neq 0$ and $b_{m+1} \neq b_{m+2} = \cdots = 0$. Then, the isolated singular point z_0 is called a pole of order m. A pole of order $m = 1$ is a simple pole.*

Theorem 2.31 (Residues at Poles). *Let z_0 be an isolated singular point of a function f. The following two statements are equivalent*

1. *z_0 is a pole of order m of f.*
2. *$f(z)$ can be written in the form $f(z) = \dfrac{\phi(z)}{(z-z_0)^m}$, where $\phi(z)$ is analytic and nonzero at z_0.*

Moreover if both of these are true, then

$$\operatorname*{Res}_{z=z_0} f(z) = \phi(z_0) \text{ when } m = 1, \text{ and } \operatorname*{Res}_{z=z_0} f(z) = \frac{\phi^{(m-1)}(z_0)}{(m-1)!} \text{ when } m = 2, 3, \ldots \tag{24}$$

Theorem 2.32. *Let f denote a function that is analytic at a point z_0. The following two statements are equivalent*

1. *f has a zero of order m at z_0.*
2. *there is a function g, which is analytic and nonzero at z_0 such that $f(z) = (z-z_0)^m g(z)$.*

Theorem 2.33 (Zeros and Poles). *Suppose that*

1. *Two functions p and q are analytic at a point z_0.*
2. *$p(z_0) \neq 0$ and q has a zero of order m at z_0.*

Then the quotient $p(z)/q(z)$ has a pole of order m at z_0.

Theorem 2.34. *Let two functions p and q be analytic at a point z_0. If $p(z_0) \neq 0$, $q(z_0) = 0$ and $q'(z_0) \neq 0$, then z_0 is a simple pole of the quotient $p(z)/q(z)$, and*

$$\operatorname*{Res}_{z=z_0} \frac{p(z)}{q(z)} = \frac{p(z_0)}{q'(z_0)}. \tag{25}$$

Theorem 2.35 (Cauchy's principle value). *If a function $f(x)$ is continuous for all x,*

$$\int_{\infty}^{\infty} f(x)\ dx = \lim_{R_1\to\infty}\int_{-R_1}^{0} f(x)\ dx + \lim_{R_2\to\infty}\int_{0}^{R_2} f(x)\ dx. \tag{26}$$

Another value, Cauchy's principle value (PV) of the integral is

$$P.V. \int_{\infty}^{\infty} f(x)\ dx = \lim_{R\to\infty}\int_{-R}^{R} f(x)\ dx. \tag{27}$$

Theorem 2.36 (Method for integrating rational functions). *Let $f(x) = p(x)/q(x)$ where $p(x)$ and $q(x)$ are polynomials with real coefficients and no factors in common. $q(z)$ has no real zeros but at least 1 above the real axis.*

1. *Firstly, identify all distinct zeros of $p(z) = z_1, z_2, \ldots, z_n$ where n is less than or equal to the the degree of $q(z)$.*
2. *Integrate $p(x)/q(x)$ around the simple closed contour C_R where R is a number such that all zeros lie inside the closed path.*

The parametric representation $z = x(-R \le x \le R)$ of the segment of the real axis and Cauchy's Residue Theorem can be used to write

$$\int_{-R}^{R} f(x)\ dx + \int_{C_R} f(z)\ dz = 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f(z). \tag{28}$$

If $\lim_{R\to\infty}\int_{C_R} f(z)\ dz = 0$, then it follows that

$$P.V. \int_{-\infty}^{\infty} f(x)\ dx = 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f(z), \tag{29}$$

and if $f(x)$ is even

$$\int_{-\infty}^{\infty} f(x)\ dx = 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f(z), \tag{30}$$

and

$$\int_{0}^{\infty} f(x)\ dx = \pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f(z). \tag{31}$$

Theorem 2.37 (Jordan's Lemma). *Suppose that*

1. *a function $f(z)$ is analytic at all points in the upper half plane $y \ge 0$ that are exterior to a circle $|z| = R_0$.*
2. *C_R denotes a semicircle $z = Re^{i\theta}$ $(0 \le \theta \le \pi)$ where $R > R_0$*
3. *for all points z on C_R, there is a positive constant M_R such that $|f(z)| \le M_R$ and $\lim_{R\to\infty} M_R = 0$.*

Then for every positive constant a

$$\lim_{R\to\infty}\int_{C_R} f(z)e^{iaz}\ dz = 0. \tag{32}$$

Theorem 2.38 (Argument Principle)**.** *Let C denote a positively oriented simple closed contour, and suppose that*

1. *a function $f(z)$ is meromorphic in the domain interior to C.*
2. *$f(z)$ is analytic and nonzero on C.*
3. *counting multiplicities, Z is the # of zeros, P is the # of poles of $f(z)$.*

Then

$$\Delta_c \arg f(z) = 2\pi(z - p). \tag{33}$$

Theorem 2.39 (Rouche's Theorem)**.** *Let C denote a simple closed contour, and suppose that*

1. *two functions $f(z)$ and $g(z)$ are analytic inside and on C;*
2. *$|f(z) > g(z)|$ at each point on C.*

Then $f(z)$ and $f(z) + g(z)$ have the same number of zeros, counting multiplicities, inside C.