Shallow water model

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1 Equations

1.1 Basic Equations

In Cartesian coordinates, the system is described by

$$u_t - fv = -g\eta_x + F_u \tag{1}$$
$$v_t + fu = -g\eta_y + F_v \tag{2}$$
$$\eta_t + (Hu)_x + (Hv)_y = F_\eta \tag{3}$$

Spherical coordinates with a being the radius of the earth, λ the longitude, and ϑ the latitude:

$$u_t - fv = -g\frac{1}{a\cos\vartheta}\frac{\partial\eta}{\partial\lambda} + F_u \tag{4}$$
$$v_t + fu = -g\frac{1}{a}\frac{\partial\eta}{\partial\vartheta} + F_v \tag{5}$$
$$\eta_t + \frac{1}{a\cos\vartheta}\frac{\partial Hu}{\partial\lambda} + \frac{1}{a\cos\vartheta}\frac{\partial\cos\vartheta Hv}{\partial\vartheta} = F_\eta \tag{6}$$

1.2 Forcing and Damping

The forcing and damping terms F_u in (1) and F_v in (2) represent one or the sum of some of the following terms.

1.2.1 Bottom Friction Term

The model can use both linear and quadratic bottom friction. The linear friction term is described by

$$\frac{\partial(u,v)}{\partial t} = X - \frac{r}{H}(u,v) \tag{7}$$

The discretized form following an implicit scheme is

$$u|_u^{i,j,l+1} = \frac{u|_u^{i,j,l} + X_u}{1 + \frac{\Delta t r}{H|_u^{i,j}}} \tag{8}$$

$$v|_v^{i,j,l+1} = \frac{v|_v^{i,j,l} + X_v}{1 + \frac{\Delta t r}{H|_v^{i,j}}} \tag{9}$$

where r is the linear bottom friction coefficient and X_u and X_v represents the residual terms of the momentum equation. We use an implicit scheme to gain unconditional numerical stability of the linear friction term.

The quadratic friction term is described by

$$F = -\frac{k}{H}\sqrt{u^2+v^2}\,(u,v) \tag{10}$$

Discretization using an explicit scheme leads to

$$F_u|_u^{i,j} = -\frac{k}{H|_u^{ij}}\sqrt{(u|_u^{ijl})^2 + \left(\frac{v|_v^{i,j+1,l} + v|_v^{ijl} + v|_v^{i-1,j+1,l} + v|_v^{i-1,j,l}}{4}\right)^2}\,u|_u^{ijl} \tag{11}$$

$$F_v|_v^{i,j} = -\frac{k}{H|_v^{ij}}\sqrt{\left(\frac{u|_u^{i+1,j,l+1} + u|_u^{i+1,j-1,l+1} + u|_u^{i,j,l+1} + u|_u^{i,j-1,l+1}}{4}\right)^2 + (v|_v^{ijl})^2}\,v|_v^{ijl} \tag{12}$$

where k is the empirical quadratic bottom friction coefficient.

1.2.2 Lateral Mixing of Momentum

The lateral mixing of momentum is described by

$$F_u = A_h f^\lambda \tag{13}$$

$$F_v = A_h f^\vartheta \tag{14}$$

where A_h is the horizontal eddy viscosity. Following Bryan (1969) the lateral mixing of momentum is parameterized in spherical coordinates with consideration of conservation of momentum by

$$\begin{aligned} f^\lambda =& \frac{1}{a^2\cos^2\vartheta H}\frac{\partial^2 uH}{\partial\lambda^2} + \frac{1}{a^2\cos\vartheta H}\frac{\partial}{\partial\vartheta}\left(\cos\vartheta\frac{\partial uH}{\partial\vartheta}\right) \\ &+ \frac{1}{a^2 H}\left((1-\tan^2\vartheta)uH - \frac{2\tan\vartheta}{\cos\vartheta}\frac{\partial vH}{\partial\lambda}\right) \end{aligned} \tag{15}$$

$$\begin{aligned} f^\vartheta =& \frac{1}{a^2\cos^2\vartheta H}\frac{\partial^2 vH}{\partial\lambda^2} + \frac{1}{a^2\cos\vartheta H}\frac{\partial}{\partial\vartheta}\left(\cos\vartheta\frac{\partial vH}{\partial\vartheta}\right) \\ &+ \frac{1}{a^2 H}\left((1-\tan^2\vartheta)vH - \frac{2\tan\vartheta}{\cos\vartheta}\frac{\partial uH}{\partial\lambda}\right) \end{aligned} \tag{16}$$

Discretization and consideration of the free-slip boundary condition (see 3.3) leads to

$$
\begin{aligned}
f^{\lambda}|_u^{i,j,l} = & \frac{1}{\Delta\lambda^2 a^2 \cos^2\vartheta|_u^{j} \; H|_u^{i,j}} \left(u|_u^{i+1,j,l} \; H|_u^{i+1,j} - 2\, u|_u^{i,j,l} \; H|_u^{i,j} + u|_u^{i-1,j,l} \; H|_u^{i-1,j} \right) \\
& - \frac{\tan\vartheta|_u^{j}}{2\Delta\vartheta a^2 \, H|_u^{i,j}} \left(u|_u^{i,j+1,l} \; H|_u^{i,j+1} \, \mathfrak{o}^{i,j+1} - u|_u^{i,j-1,l} \; H|_u^{i,j-1} \, \mathfrak{o}^{i,j} - (\mathfrak{o}^{i,j+1} - \mathfrak{o}^{i,j})\, u|_u^{i,j,l} \; H|_u^{i,j} \right) \\
& + \frac{1}{\Delta\vartheta^2 a^2 \, H|_u^{i,j}} \left(u|_u^{i,j+1,l} \; H|_u^{i,j+1} \, \mathfrak{o}^{i,j+1} - (\mathfrak{o}^{i,j+1} + \mathfrak{o}^{i,j})\, u|_u^{i,j,l} \; H|_u^{i,j} + u|_u^{i,j-1,l} \; H|_u^{i,j-1} \, \mathfrak{o}^{i,j} \right) \\
& + \frac{1 - \tan^2\vartheta|_u^{j}}{a^2 \, H|_u^{i,j}} \, u|_u^{i,j,l} \; H|_u^{i,j} \\
& - \frac{\tan\vartheta|_u^{j}}{a^2 \cos\vartheta|_u^{j} \, \Delta\lambda \, H|_u^{i,j}} \left(v|_v^{i,j+1,l} \; H|_v^{i,j+1} \, \mathfrak{o}^{i,j+1} + v|_v^{i,j,l} \; H|_v^{i,j} \, \mathfrak{o}^{i,j} \right) \\
& + \frac{\tan\vartheta|_u^{j}}{a^2 \cos\vartheta|_u^{j} \, \Delta\lambda \, H|_u^{i,j}} \left(v|_v^{i-1,j+1,l} \; H|_v^{i-1,j+1} \, \mathfrak{o}^{i,j+1} + v|_v^{i-1,j,l} \; H|_v^{i-1,j} \, \mathfrak{o}^{i,j} \right)
\end{aligned}
\tag{17}
$$

$$
\begin{aligned}
f^{\vartheta}|_v^{i,j,l} = & \frac{1}{\Delta\lambda^2 a^2 \cos^2\vartheta|_v^{j} \; H|_v^{i,j}} \left(v|_v^{i+1,j,l} \; H|_v^{i+1,j} \, \mathfrak{o}^{i+1,j} - (\mathfrak{o}^{i+1,j} + \mathfrak{o}^{i,j})\, v|_v^{i,j,l} \; H|_v^{i,j} + v|_v^{i-1,j,l} \; H|_v^{i-1,j} \, \mathfrak{o}^{i,j} \right) \\
& - \frac{\tan\vartheta|_v^{j}}{2\Delta\vartheta a^2 \, H|_v^{i,j}} \left(v|_v^{i,j+1,l} \; H|_v^{i,j+1} - v|_v^{i,j-1,l} \; H|_v^{i,j-1} \right) \\
& + \frac{1}{\Delta\vartheta^2 a^2 \, H|_v^{i,j}} \left(v|_v^{i,j+1,l} \; H|_v^{i,j+1} - 2\, v|_v^{i,j,l} \; H|_v^{i,j} + v|_v^{i,j-1,l} \; H|_v^{i,j-1} \right) \\
& + \frac{1 - \tan^2\vartheta|_v^{j}}{a^2 \, H|_v^{i,j}} \, v|_v^{i,j,l} \; H|_v^{i,j} \\
& - \frac{\tan\vartheta|_v^{j}}{a^2 \cos\vartheta|_v^{j} \, \Delta\lambda \, H|_v^{i,j}} \left(u|_u^{i+1,j,l} \; H|_u^{i+1,j} + u|_u^{i+1,j-1,l} \; H|_u^{i+1,j-1} \right) \\
& + \frac{\tan\vartheta|_v^{j}}{a^2 \cos\vartheta|_v^{j} \, \Delta\lambda \, H|_v^{i,j}} \left(u|_u^{i,j,l} \; H|_u^{i,j} + u|_u^{i,j-1,l} \; H|_u^{i,j-1} \right)
\end{aligned}
\tag{18}
$$

where $\mathfrak{o}$ is the ocean mask defined on the H-grid.

$$
\mathfrak{o}^{i,j} = \begin{cases} 0 & \text{if} \quad H|_H^{i,j} = 0 \\ 1 & \text{else} \end{cases}
\tag{19}
$$

Note that $u|_u^{l}$ is used in the v-equation instead of $u|_u^{l+1}$.

1.2.3 Surface Stress Forcing

Wind stress τ_s can be used to force the model. The forcing term in the u- and v-equation can be written as

$$
F_u = \frac{\tau_s^{\lambda}}{\rho_0 H}
\tag{20}
$$

$$
F_v = \frac{\tau_s^{\vartheta}}{\rho_0 H}
\tag{21}
$$

where τ_s^λ is the zonal and τ_s^ϑ the meridional component of the wind stress. Since both components are provided on different grids, the zonal on the u-grid, the meridional on the v-grid, discretization yields[1]

$$F_u|_u^{i,j} = \frac{\tau_s^\lambda|_u^{i,j}}{\rho_0 \, H|_u^{i,j}} \tag{22}$$

$$F_v|_v^{i,j} = \frac{\tau_s^\vartheta|_v^{i,j}}{\rho_0 \, H|_v^{i,j}} \tag{23}$$

1.2.4 Reynolds Stress Forcing

Reynolds stresses derived from observations can be used to force the model. These stresses are known at the surface and are supposed to decrease linearly with depth to zero. So the forcing terms in the u- and v-equation can be written as

$$\begin{aligned} F_u &= -\frac{1}{H}\int\limits_{-H}^{0} \frac{\partial}{\partial x}\left[\overline{u'u'}\left(\frac{z+H}{H}\right)\right]\mathrm{d}z - \frac{1}{H}\int\limits_{-H}^{0} \frac{\partial}{\partial y}\left[\overline{u'v'}\left(\frac{z+H}{H}\right)\right]\mathrm{d}z \\ &= -\frac{1}{2Ha\cos\vartheta}\frac{\partial}{\partial\lambda}\left(\overline{u'u'}\,H\right) - \frac{1}{2Ha\cos\vartheta}\frac{\partial}{\partial\vartheta}\left(\cos\vartheta\,\overline{u'v'}\,H\right) \end{aligned} \tag{24}$$

And analog

$$F_v = -\frac{1}{2Ha\cos\vartheta}\frac{\partial}{\partial\lambda}\left(\overline{u'v'}\,H\right) - \frac{1}{2Ha\cos\vartheta}\frac{\partial}{\partial\vartheta}\left(\cos\vartheta\,\overline{v'v'}\,H\right) \tag{25}$$

Discretization[2] leads to

$$\begin{aligned} F_u|_u^{i,j} = -&\frac{1}{2\,H|_u^{i,j}\,a\cos\vartheta|_u^j}\left(\frac{\overline{u'u'}\,|_\eta^{i,j}\;H|_\eta^{i,j} \;-\; \overline{u'u'}\,|_\eta^{i-1,j}\;H|_\eta^{i-1,j}}{\Delta\lambda}\right. \\ &\left.-\frac{\cos\vartheta|_H^{j+1}\;\overline{u'v'}\,|_H^{i,j+1}\;H|_H^{i,j+1} \;-\; \cos\vartheta|_H^{j}\;\overline{u'v'}\,|_H^{i,j}\;H|_H^{i,j}\,)}{\Delta\vartheta}\right) \end{aligned} \tag{26}$$

$$\begin{aligned} F_v|_v^{i,j} = -&\frac{1}{2\,H|_v^{i,j}\,a\cos\vartheta|_v^j}\left(\frac{\cos\vartheta|_\eta^{j}\;\overline{v'v'}\,|_\eta^{i,j}\;H|_\eta^{i,j} \;-\; \cos\vartheta|_\eta^{j-1}\;\overline{v'v'}\,|_\eta^{i,j-1}\;H|_\eta^{i,j-1}}{\Delta\vartheta}\right. \\ &\left.-\frac{\overline{u'v'}\,|_H^{i+1,j}\;H|_H^{i+1,j} \;-\; \overline{u'v'}\,|_H^{i,j}\;H|_H^{i,j}\,)}{\Delta\lambda}\right) \end{aligned} \tag{27}$$

1.2.5 Newtonian Cooling

If the setup is a $1\frac{1}{2}$-Layer baroclinic model, diapycnal mixing at the interface is parameterized by a Newtonian cooling term which is described by

$$\frac{\partial\eta}{\partial t} = X - \gamma\eta \tag{28}$$

[1] There is an implicit two-point averaging of H to the u- and v-grid

[2] There is an implicit four point averaging from the H-grid to the η-grid since all stress terms and topography are provided on the H-grid

Discretization using a implicit scheme leads to

$$\eta|_{\eta}^{i,j,l+1} = \frac{X_\eta + \eta|_{\eta}^{i,j,l}}{1 + \Delta t \gamma} \tag{29}$$

where X represents the residual terms of the continuity equation (i.e. the horizontal mass flux divergence)

2 Grid

We want the grid to be as natural as possible (i.e., no averaging in the η-gradients and in the continuity equation): The fields will all carry integer indices which

H	v	H	v	H	v
u	eta	u	eta	u	eta
H	v	H	v	H	v

Figure 1: Grid arrangement. We use an Arakawa C-grid with the topography on the free grid points.

are to be interpreted according to

$$H|_H^{ij} = H(\Delta x \cdot i, \Delta y \cdot j) \tag{30}$$

$$\eta|_\eta^{ijl} = \eta(\Delta x \cdot (i + 1/2), \Delta y \cdot (j + 1/2), \Delta t \cdot l) \tag{31}$$

$$u|_u^{ijl} = u(\Delta x \cdot i, \Delta y \cdot (j + 1/2), \Delta t \cdot l) \tag{32}$$

$$v|_v^{ijl} = v(\Delta x \cdot (i + 1/2), \Delta y \cdot j, \Delta t \cdot l) \tag{33}$$

An expression like $H|_u^{ij}$ means that the depth is interpolated to the u-grid by simple linear interpolation which is two-point averaging here. In the Coriolis term, we'll need four point averaging from the v-grid to the u-grid and from the u-grid to the v-grid. The x- and y-derivatives in the continuity equation will all need two point averaging from the H-grid to the u- and v-grid.

3 Boundary conditions

3.1 Coastlines / Land mask

We'll define land to be where H is zero. This implies that u- and v-grid points are over land if both the closest H-grid points are over land. η-grid points are over land, if all four surrounding H-grid points are over land. It will probably be computationally effective to define masks for all three dynamic variables.

3.2 Periodic boundary conditions

For a global model, we'll need to implement periodic boundary conditions in the zonal direction. In the meridional direction we might get away without as we won't be able to extend the model domain to the poles anyway (CFL will become a problem there).

3.3 Free-slip Boundary condition

The model fulfill the free-slip boundary conditions which are $u = 0$ and $v = 0$ if the corresponding u- and v-grid points are over land, and $(uH)_y = 0$ and $(vH)_x = 0$ if the derivative is measured across a boundary, i.e. the H grid point in between the two u- or v-grid points is land. Please note that second boundary condition only influences the lateral mixing terms.

4 Time stepping scheme

Use Heaps (1972) which basically amounts to[3]

$$\eta^{l+1} = \eta^l - \Delta t(Hu^l)_x - \Delta t(Hv^l)_y + \Delta t F_\eta \tag{34}$$
$$u^{l+1} = u^l + \Delta t f v^l - \Delta t g \eta_x^{l+1} + \Delta t F_u \tag{35}$$
$$v^{l+1} = v^l - \Delta t f u^{l+1} - \Delta t g \eta_y^{l+1} + \Delta t F_v \tag{36}$$

Spherical coordinates with full[4] glory:

$$\begin{aligned}\eta|_\eta^{i,j,l+1} =& \eta|_\eta^{ijl} + \Delta t\, F_\eta|_\eta^{i,j} \\ &- \frac{\Delta t}{a\cos\vartheta|_\eta^j \Delta\lambda}\left(H|_u^{i+1,j} u|_u^{i+1,j,l} - H|_u^{ij} u|_u^{ijl} \right) \\ &- \frac{\Delta t}{a\cos\vartheta|_\eta^j \Delta\vartheta}\left(H|_v^{i,j+1} \cos\vartheta|_v^{j+1} v|_v^{i,j+1,l} - H|_v^{ij} \cos\vartheta|_v^j v|_v^{ijl} \right)\end{aligned} \tag{37}$$

$$u|_u^{i,j,l+1} = u|_u^{ijl} + \Delta t f v|_u^{ijl} - \frac{\Delta t g}{a\cos\vartheta|_u^j \Delta\lambda}\left(\eta|_\eta^{i,j,l+1} - \eta|_\eta^{i-1,j,l+1} \right) + \Delta t\, F_u|_u^{i,j} \tag{38}$$

$$v|_v^{i,j,l+1} = v|_v^{ijl} - \Delta t f u|_v^{i,j,l+1} - \frac{\Delta t g}{a\Delta\vartheta}\left(\eta|_\eta^{i,j,l+1} - \eta|_\eta^{i,j-1,l+1} \right) + \Delta t\, F_v|_v^{i,j} \tag{39}$$

In the model, we'll calculate $H|_u^{ij}$ and $H|_v^{ij}$ once in the beginning and then use it throughout the simulation. However, for defining which grid points are over land, it is still helpful to use the original H-grid. Note that the Rayleigh friction (linear bottom friction) and the Newtonian cooling is discretized with an implicit scheme.

5 Stability

Note that we will not consider optimization of roundoff errors at this point. We'll stick to SI-units and readable code as long as possible.

There are two criteria which may be of importance: $|\Delta t \cdot f| < 1$ which comes from the Coriolis term, and the CFL criterion $c\frac{\Delta t}{\Delta x} = \sqrt{gH}\frac{\Delta t}{\Delta x} < 1$. The CFL criterion is by far the most dangerous. From the first criterion, Δt varies from

[3]Note that the superscript of v in the second equation is mean to be an l while all the others are meant to be $l+1$. In each line, you use the newest information that is available ...

[4]There still is an implicit two point averaging from the H-grid to the u- and v-grids as well as an implicit four point averaging in the Coriolis term.

infinity at the equator to half a day at the poles, while from CFL with high resolution, high latitudes and large H time steps of much less than 10 seconds can easily arise.

$$\Delta t < \frac{\Delta\lambda/deg \cdot 111km \cdot \cos\vartheta^{max}}{\sqrt{gH}} \tag{40}$$

For $H = O(10km)$, $\Delta\lambda = 1/10^o$ and a model domain which extends up to $89N$, this results in as short a time steps as 0.6 seconds.

Since western boundary currents must be resolved by the model, there are two other criteria. When using linear bottom friction the western boundary current width (WBCW) is defined by the width of the stommel layer which gives us a criteria for the friction parameter

$$r > \Delta x H \beta$$

When using later mixing WBWC is defined by the width of the munk layer

$$A_h > \Delta x^3 \beta$$

There is also an issue with the lateral mixing term. This term exhibits instability for large values of A_h. Instability was observed in a Equatorial $1\frac{1}{2}$-Layer setup with $\Delta t = 3000s; \Delta\lambda = 0.1°; A_h = 10^4 m^2 s^{-1}$. The centers of growth where located at the edges of the rectangular basin, which can be a hint for further investigation.

6 Estimation of computational effort

In the end, we might want to simulate an almost global model (say $\lambda = 0 \ldots 360°$, $\vartheta = -89 \ldots 89°$) with a resolution of up to $1/10°$ for more than one year of model time. There are $360 * 178 * 100 \approx 6.5 \cdot 10^6$ grid points and for each grid point we need $O(20)$ multiplications and additions. The time step is $\approx 0.6s$ and one year of model time has $\approx 3.15 \cdot 10^7 s$. Altogether, we need $N \approx 6.7 \cdot 10^{15}$ operations which on a machine with $8GFLOPS$ can be done in $8.5 \cdot 10^5 s \approx 10d$. For a smaller domain, which only extends to 85° in latitude, only about $2d$ are necessary.

7 First test: Rectangular domain

Domain:

$$\begin{aligned} \lambda &= -20° \ldots 20° \\ \vartheta &= -20° \ldots 20° \\ \Delta\lambda &= \Delta\vartheta = 0.4° \\ H &= 1000m = const. \\ \Rightarrow \Delta t &= 422s \end{aligned} \tag{41}$$

Initial conditions: Adjustment should be done after $O(20 \cdot 2 \cdot 111km/\sqrt{gH}) = O(4.5 \cdot 10^4 s) = O(100)\Delta t$.