Quantile optimizer in action

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In a nutshell

- investors with quantile preferences maximize τ-quantile of future portfolio returns
  - a level of investor's risk aversion is captured by τ (a smaller τs correspond to greater risk aversion)

This paper:

- study out-of-sample performance of portfolios formed by such investors
  - no closed-form solution and computationally hard
- document great heterogeneity across quantiles in terms of performance as well as portfolio compositions
- compare with mainstream portfolio selection methods

Common theme in the "confusion matrix"

	q005	q01	q02	q03	q04	q05	q06	q07	q08	q09	q095	q099
5%	-5.575	-5.744	-6.02	-6.851	-8.721	-10.07	-10.91	-13.31	-14.01	-15.31	-15.64	-14.52
10%	-3.713	-3.823	-3.941	-4.224	-5.615	-6.964	-7.871	-9.247	-10.66	-10.55	-11.75	-9.738
20%	-1.877	-2.018	-2.155	-2.451	-2.687	-3.131	-4.219	-5.181	-6.16	-6.076	-6.35	-5.666
30%	-0.591	-0.695	-0.601	-1.059	-1.196	-1.344	-2.115	-2.464	-3.053	-3.472	-3.517	-2.685
40%	0.342	0.273	0.664	0.015	0.147	0.068	-0.529	-0.45	-1.122	-1.204	-1.26	-0.882
50%	1.158	1.212	1.365	1.226	1.381	1.491	1.438	1.347	1.008	0.67	0.825	1.01
60%	2.225	2.186	2.274	2.393	2.652	3.028	3.327	3.115	3.198	2.814	2.7	3.258
70%	3.096	3.282	3.274	3.598	3.901	5.079	5.0	5.59	5.92	5.679	5.425	5.135
80%	4.158	4.337	4.428	5.014	5.362	7.038	7.34	8.679	9.432	8.786	8.746	8.496
90%	5.766	5.959	6.0	7.402	7.955	9.997	11.47	12.46	13.85	14.26	14.17	13.25
95%	7.268	7.035	8.194	9.172	10.26	13.52	14.36	15.19	17.84	19.16	19.17	19.13
99%	10.23	10.54	10.3	13.58	14.5	18.88	20.01	22.95	26.26	31.18	29.64	30.68

Contents

τ-quantile preferences

- alternative preferences to Expected Utility representation
- given some fixed $\tau \in (0,1)$ a τ-quantile preference $\succeq$ is defined in eq. (1):

$$X \succeq Y \iff Q_\tau[u(X)] \geq Q_\tau[u(Y)] \tag{1}$$

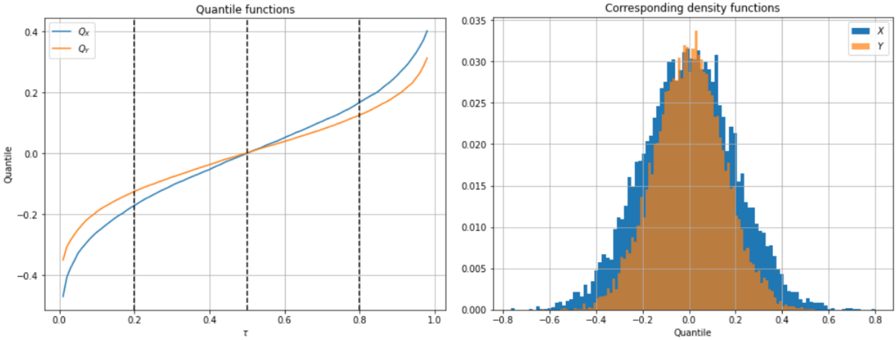
- u is some general utility function and X and Y are random variables

Redundancy of utility function

$$Q_\tau[u(X)] \geq Q_\tau[u(Y)] \iff Q_\tau[X] \geq Q_\tau[Y] \qquad (2)$$

- a continuous and strictly increasing utility function u
- the invariance of quantiles w.r.t. monotone transformations
- the concavity of the utility function does not determine the risk attitude of an investor with quantile preferences!
  - τ does
  - the smaller the τ, the higher the risk aversion

Using quantile functions



Literature

- introduced by Manski (1988)
- axiomatized by Chambers (2009), Rostek (2010), de Castro and Galvao (2019)
- Giovannetti (2013) models a two-period economy with agents with quantile preferences
- de Castro et al. (2020) run behavioural experiment
- He and Zhou (2011) study a portfolio choice model in continuous time
- Benati and Rizzi (2007) propose mean-VaR model
  - show it is a NP-hard problem (because of VaR)
  - introduce Mixed integer linear probramming reformulation
- Benati (2015) use medians instead of means in number of portfolio selection models

Portfolio selection setting

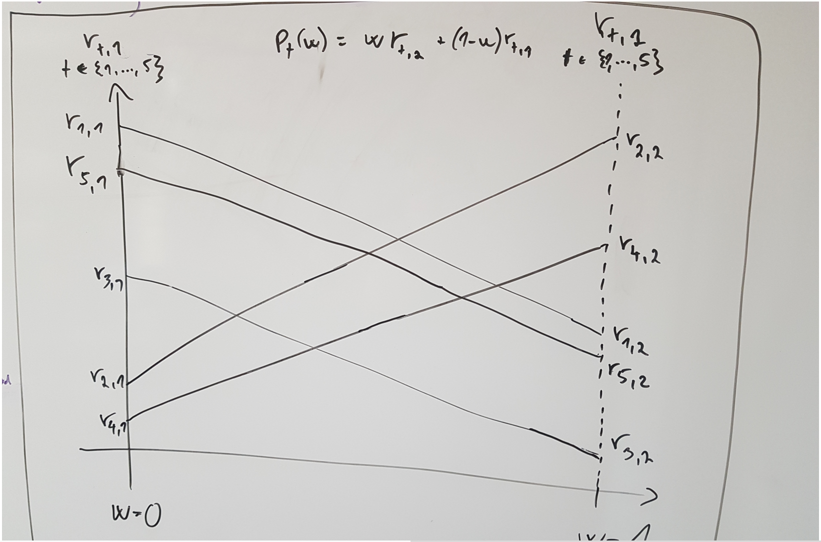
- N assets[1]
- T periods[2]
- weights: $w = (w_1, \ldots, w_N)$
  - long-only: $w_i > 0; \quad \forall i \in \{1, \ldots, N\}$
  - full-investment: $\sum_{i=1}^{N} w_i = 1$
- return of an asset i in period t : $r_{t,i}$
- portfolio return in period t: $p_t = \sum_{i=1}^{N} w_i r_{t,i}$
- portfolio returns over T periods: $P = (p_1, \ldots, p_T)$
- $\tau \in 0, 1$
- τ-quantile of portfolio returns: $Q_\tau(P)$

[1] Can change over time.
[2] Length of the estimation sample.

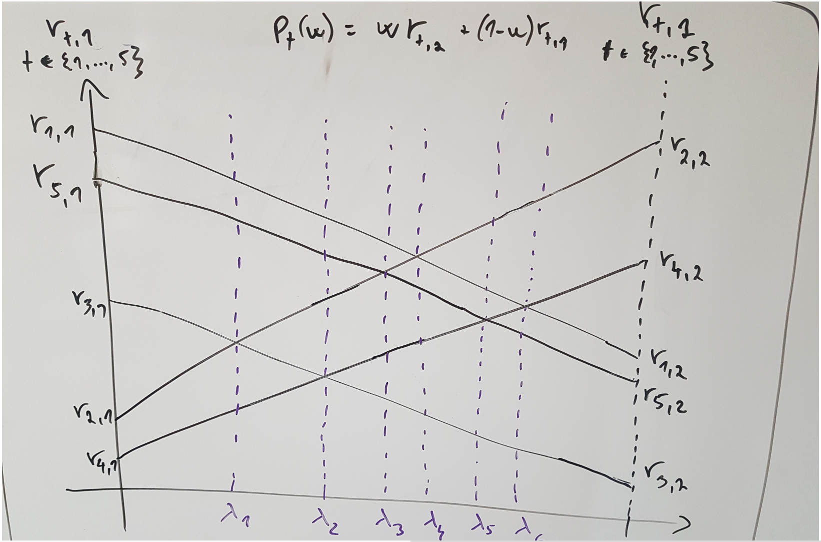
Portfolio Optimization via Quantile Maximization

$$\begin{aligned}
&\max_{w} Q_\tau(P) \\
&\text{s.t.} \sum_{i=1}^{n} w_i = 1 \\
&p_t = \sum_{i=1}^{N} w_i r_{t,i} \quad \forall t \in \{1, \dots, T\} \\
&w_i \geq 0 \;\; \forall i \in \{1, \dots, N\}
\end{aligned} \tag{3}$$

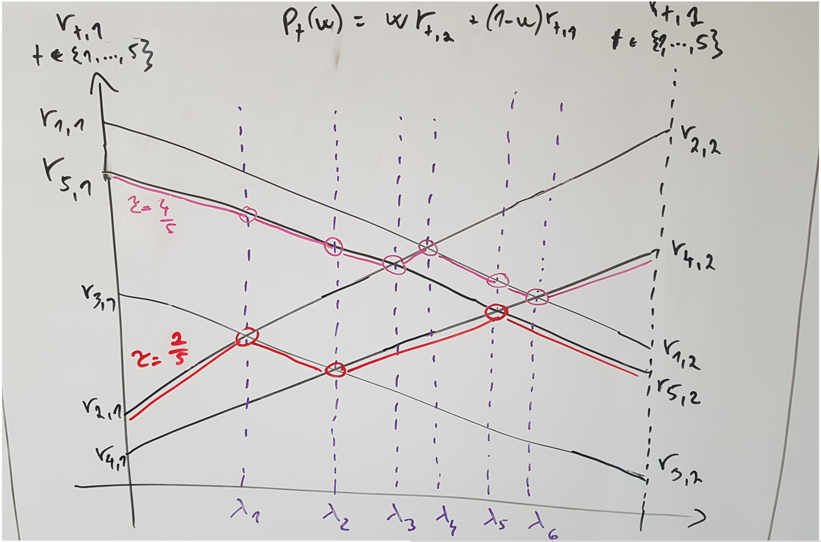
Finding quantile of portfolio returns (I)



Finding quantile of portfolio returns (II)



Finding quantile of portfolio returns (III)



NP-hardness

- the quantile function is non-differentiable piece-wise linear function
- Benati and Rizzi (2007) show eq. (3) is Approximable hard
  - a class of NP-hard[3] problem for which a Polynomial Time Approximation scheme does not exist, unless $P = NP$
  - it rules out the existence of polynomial and pseudopolynomial time algorithms (no hope (unless $P = NP$) to obtain an approximation algorithm
- eq. (3) can be reformulated as an Mixed Integer Linear Programming model shown in eq. (4)

[3]Non-deterministic polynomial-time hardness.

Mixed Integer Programming (MIP) formulation

$$
\begin{aligned}
& \max_{w,y,Q_\tau} \; Q_\tau \\
\text{s.t.} \quad & Q_\tau \leq \sum_{i=1}^{N} w_i r_{t,i} + L(1 - y_t) \;\; \forall t \in \{1, \ldots, T\} \\
& \sum_{t=1}^{T} y_i \leq \tau T \\
& \sum_{i=1}^{n} w_i = 1 \\
& y_i \in \{0, 1\} \;\; \forall t \in \{1, \ldots, T\} \\
& x_i \geq 0 \;\; \forall i \in \{1, \ldots, N\}
\end{aligned} \tag{4}
$$

Solvable special cases

- eq. (4) is NP-hard because T and N can grow arbitrarily large
- Benati and Rizzi (2007) show 2 polynomial time solvable special cases
  - finite and small T: $O(2^T)$
  - finite and small N: $O(\lambda^N)$
    - λ corresponds to the number of candidate solutions which is dependent on τ
- Benati (2015) apply this to maximizing median of portfolio returns ($N = 60$ and $T \in \{21, 31, 41\}$)

Branch and bound algorithm

- a systematic enumeration of candidate solutions by means of state space search
  - grid-search (brute-force) approach unfeasible: e.g. grid of 1000 values (0.1% step) in case of 10 assets 1000^{10} options
- the set of candidate solutions as a rooted tree with the full set at the root
- branches of the tree as subsets of the solution set (branching alone = brute-force enumeration)
- saves bounds on the minimum in the branch and "prunes" the search space [4]

[4] Eliminating candidate solutions which cannot be optimum.

Hardware and software

- Hardware
  - CPU Intel Xeon Gold 6126 2.60GHz (48 cores)
  - RAM 428 GB
- Optimization software
  - IBM ILOG CPLEX (commercial with free academic license)
    - $\approx$ 20x faster than open-sourced GNU Linear Programming Kit (GLPK)
    - alternative (commercial) solvers: GAMS, Gurobi (free academic license)
    - contains modelling layer (interfaces) to Python/MATLAB

Normal and independent

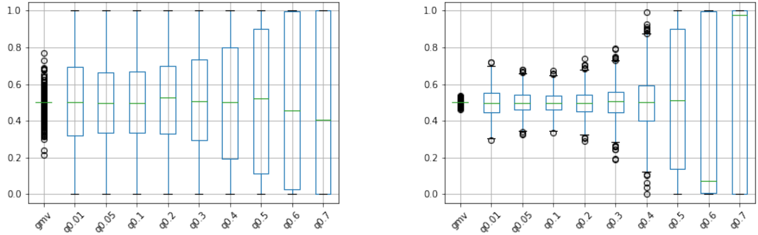
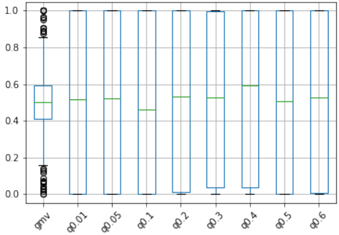


Figure 1: Independently and normally distributed returns with the same variances. $T = 21$ on the left $T = 1000$ on the right.

Normal and (highly) dependent



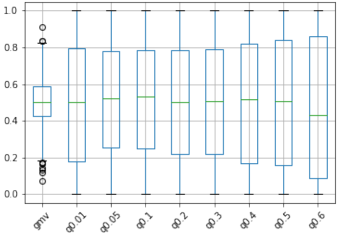


Figure 2: Dependent normally distributed with the same variances. $T = 21$ on the left $T = 1000$ on the right.

Non-normal and independent

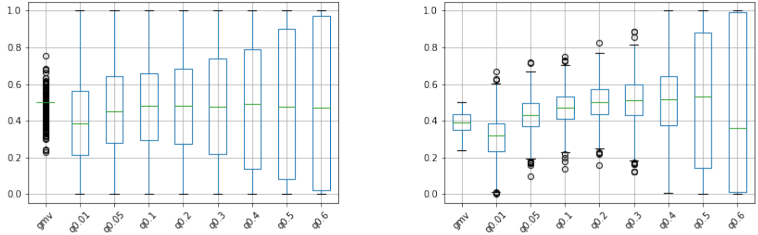


Figure 3: Independent α-stable distribution with same variances and different αs. $T = 21$ on the left $T = 1000$ on the right.

Test datasets

- number of datasets typically used in the portfolio selection literature
  - equal-weighted portfolios from French data library
  - individual stocks from Center for Research in Security Prices (CRSP)
- daily and monthly returns

Dataset	N
Industry Portfolios	5,17,48
Portfolios - B/M x Size	6,25
Portfolios - INV x Size	6,25
Portfolios - Prior 12-2 x Size	6,25
Portfolios - Prior 1-0 x Size	6,25
Europe Portfolios - B/M x Size	6
Japan Portfolios - B/M x Size	6
CRSP large	10,20,30
CRSP random	10,20,30

Out-of-sample performance evaluation

- monthly rolling estimation windows of 36,60,120 months
- daily rolling estimation windows of 21, 63 and 125 days
- portfolio weights are obtained as a solution to the optimization problem using estimation window only
- rebalancing monthly
    - daily data: rebalancing each 21 days ("month")
- evaluated using:
    - τ-quantile
    - variance
    - Sharpe ratio
    - weights concentration

OOS quantile "confusion matrix"

	q005	q01	q02	q03	q04	q05	q06	q07	q08	q09	q095	q099
5%	-5.575	-5.744	-6.02	-6.851	-8.721	-10.07	-10.91	-13.31	-14.01	-15.31	-15.64	-14.52
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99%	10.23	10.54	10.3	13.58	14.5	18.88	20.01	22.95	26.26	31.18	29.64	30.68

Figure 4: OOS quantiles of portfolio returns. Portfolios formed by maximizing IS τ-quantile, with $\tau \in \{5\%, \dots, 99\%\}$ Universe: 100 FF portfolios formed by double sorting on Book to market and Size

Detailed results

In the paper.

Conclusion

- max (IS) τ-quantile maximizer experiences the highest (OOS) τ-quantile among other τ-quantile maximizers
- however, GMV portfolio achieves higher τ-quantile for low τs
- high weights concentration for high and low τ-quantile maximizer
- the lowest concentration for τ around 0.5

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