Quantile optimizer in action

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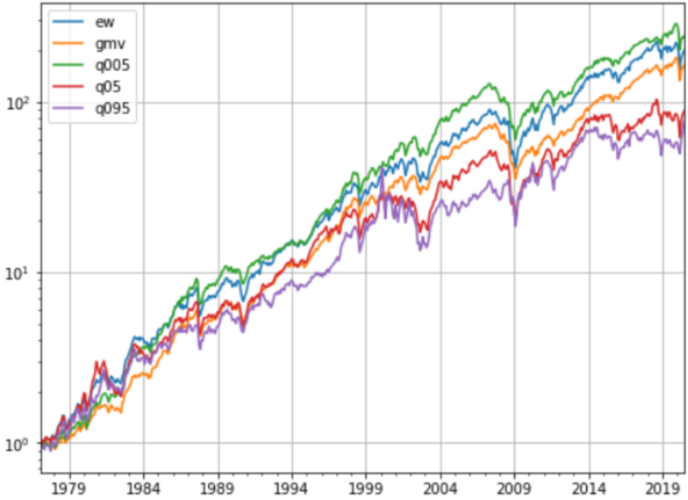
In a nutshell

- investors with quantile preferences maximize τ-quantile of future portfolio returns
  - a level of investor's risk aversion is captured by τ (a smaller τs correspond to greater risk aversion)

This paper:

- study out-of-sample performance of portfolios formed by such investors
  - no closed-form solution and computationally hard
- document great heterogeneity across quantiles in terms of performance as well as portfolio compositions
- compare with mainstream portfolio selection methods

Common theme in the OOS performance



Common theme in the "confusion matrix"

	q005	q01	q02	q03	q04	q05	q06	q07	q08	q09	q095	q099
5%	-5.575	-5.744	-6.02	-6.851	-8.721	-10.07	-10.91	-13.31	-14.01	-15.31	-15.64	-14.52
10%	-3.713	-3.823	-3.941	-4.224	-5.615	-6.964	-7.871	-9.247	-10.66	-10.55	-11.75	-9.738
20%	-1.877	-2.018	-2.155	-2.451	-2.687	-3.131	-4.219	-5.181	-6.16	-6.076	-6.35	-5.666
30%	-0.591	-0.695	-0.601	-1.059	-1.196	-1.344	-2.115	-2.464	-3.053	-3.472	-3.517	-2.685
40%	0.342	0.273	0.664	0.015	0.147	0.068	-0.529	-0.45	-1.122	-1.204	-1.26	-0.882
50%	1.158	1.212	1.365	1.226	1.381	1.491	1.438	1.347	1.008	0.67	0.825	1.01
60%	2.225	2.186	2.274	2.393	2.652	3.028	3.327	3.115	3.198	2.814	2.7	3.258
70%	3.096	3.282	3.274	3.598	3.901	5.079	5.0	5.59	5.92	5.679	5.425	5.135
80%	4.158	4.337	4.428	5.014	5.362	7.038	7.34	8.679	9.432	8.786	8.746	8.496
90%	5.766	5.959	6.0	7.402	7.955	9.997	11.47	12.46	13.85	14.26	14.17	13.25
95%	7.268	7.035	8.194	9.172	10.26	13.52	14.36	15.19	17.84	19.16	19.17	19.13
99%	10.23	10.54	10.3	13.58	14.5	18.88	20.01	22.95	26.26	31.18	29.64	30.68

Contents

τ-quantile preferences

- alternative preferences to Expected Utility representation
- given some fixed $\tau \in (0,1)$ a τ-quantile preference $\succeq$ is defined in eq. (1):

$$X \succeq Y \iff Q_\tau[u(X)] \geq Q_\tau[u(Y)] \tag{1}$$

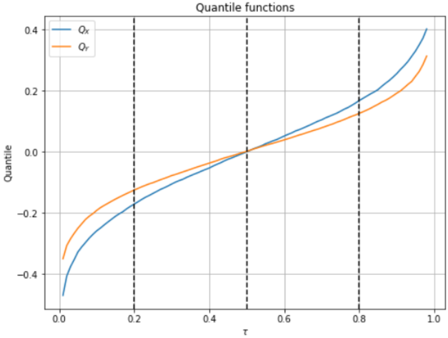
- u is some general utility function and X and Y are random variables

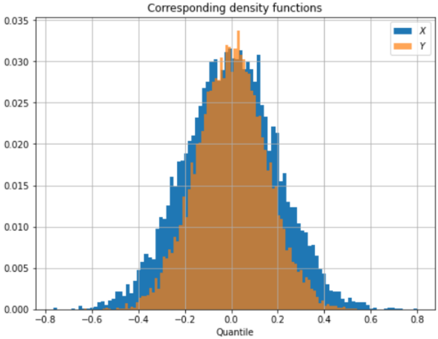
Redundancy of utility function

$$Q_\tau[u(X)] \geq Q_\tau[u(Y)] \iff Q_\tau[X] \geq Q_\tau[Y] \tag{2}$$

- a continuous and strictly increasing utility function u
- the invariance of quantiles w.r.t. monotone transformations
- the concavity of the utility function does not determine the risk attitude of an investor with quantile preferences!
  - τ does
  - the smaller the τ, the higher the risk aversion

Using quantile functions





Literature

- introduced by Manski (1988)
- axiomatized by Chambers (2009), Rostek (2010), de Castro and Galvao (2019)
- Giovannetti (2013) models a two-period economy with agents with quantile preferences
- de Castro et al. (2020) run behavioural experiment
- He and Zhou (2011) study a portfolio choice model in continuous time
- Benati and Rizzi (2007) propose mean-VaR model
  - show it is a NP-hard problem (because of VaR)
  - introduce Mixed integer linear probramming reformulation
- Benati (2015) use medians instead of means in number of portfolio selection models

Portfolio selection setting

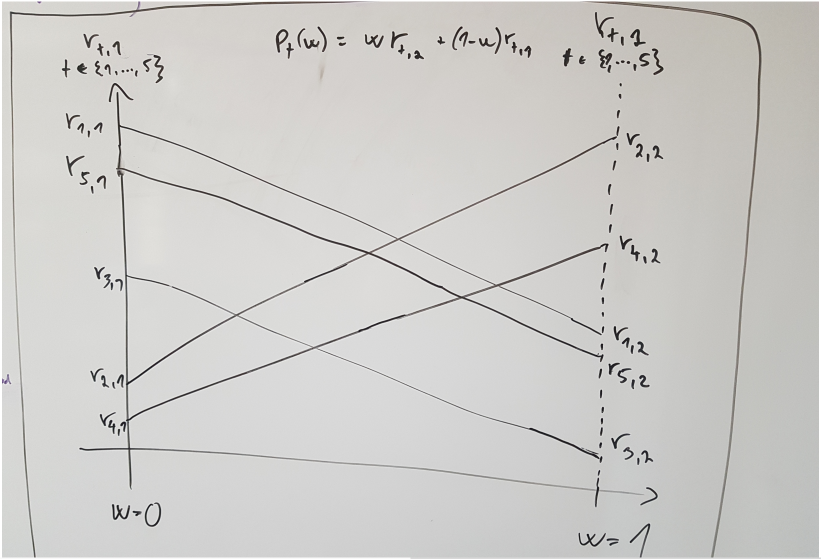
- N assets[1]
- T periods[2]
- weights: $w = (w_1, \ldots, w_N)$
  - long-only: $w_i > 0; \quad \forall i \in \{1, \ldots, N\}$
  - full-investment: $\sum_{i=1}^{N} w_i = 1$
- return of an asset i in period t : $r_{t,i}$
- portfolio return in period t: $p_t = \sum_{i=1}^{N} w_i r_{t,i}$
- portfolio returns over T periods: $P = (p_1, \ldots, p_T)$
- $\tau \in 0, 1$
- τ-quantile of portfolio returns: $Q_\tau(P)$

[1] Can change over time.
[2] Length of the estimation sample.

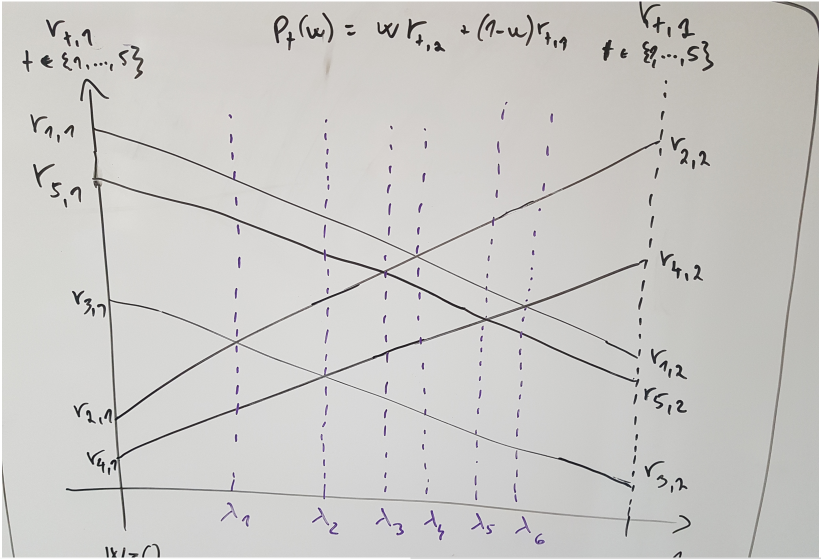
Portfolio Optimization via Quantile Maximization

$$
\begin{aligned}
&\max_{w} Q_\tau(P) \\
&\text{s.t.} \sum_{i=1}^{n} w_i = 1 \\
&p_t = \sum_{i=1}^{N} w_i r_{t,i} \quad \forall t \in \{1, \ldots, T\} \\
&w_i \geq 0 \quad \forall i \in \{1, \ldots, N\}
\end{aligned} \tag{3}
$$

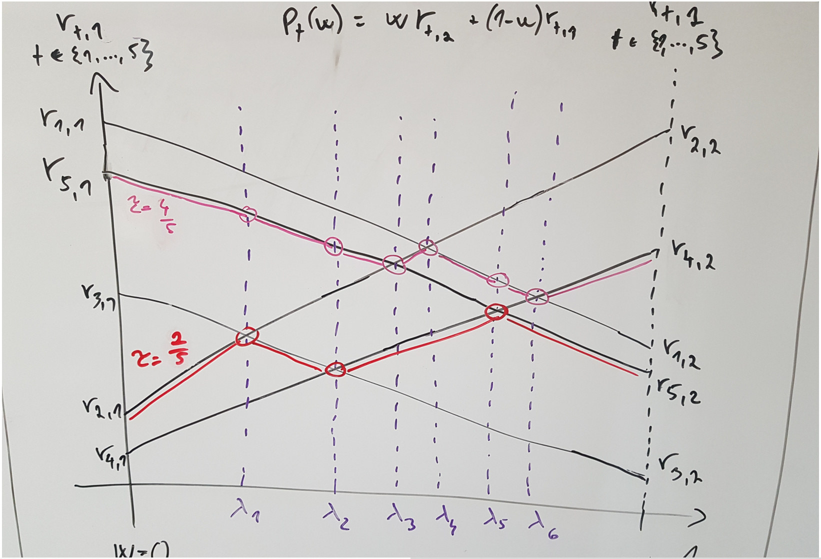
Finding quantile of portfolio returns (I)



Finding quantile of portfolio returns (II)



Finding quantile of portfolio returns (III)



NP-hardness

- the quantile function is non-differentiable piece-wise linear function
- Benati and Rizzi (2007) show eq. (3) is Approximable hard
  - a class of NP-hard[3] problem for which a Polynomial Time Approximation scheme does not exist, unless $P = NP$
  - it rules out the existence of polynomial and pseudopolynomial time algorithms (no hope (unless $P = NP$) to obtain an approximation algorithm
- eq. (3) can be reformulated as an Mixed Integer Linear Programming model shown in eq. (4)

[3] Non-deterministic polynomial-time hardness.

Mixed Integer Programming (MIP) formulation

$$
\begin{aligned}
&\max_{w,y,Q_\tau} \; Q_\tau(P) \\
\text{s.t.} \quad & Q_\tau \leq \sum_{i=1}^{N} w_i r_{t,i} + L(1 - y_t) \;\; \forall t \in \{1, \ldots, T\} \\
& \sum_{t=1}^{T} y_i \leq \tau T \\
& \sum_{i=1}^{n} w_i = 1 \\
& y_i \in \{0, 1\} \;\; \forall t \in \{1, \ldots, T\} \\
& x_i \geq 0 \;\; \forall i \in \{1, \ldots, N\}
\end{aligned}
\tag{4}
$$

Solvable special cases

- eq. (4) is NP-hard because T and N can grow arbitrarily large
- Benati and Rizzi (2007) show 2 polynomial time solvable special cases
  - finite and small T: $O(2^T)$
  - finite and small N: $O(\lambda^N)$
    - λ corresponds to the number of candidate solutions which is dependent on τ
- Benati (2015) apply this to maximizing median of portfolio returns ($N = 60$ and $T \in \{21, 31, 41\}$)

Branch and bound algorithm

- a systematic enumeration of candidate solutions by means of state space search
  - grid-search (brute-force) approach unfeasible: e.g. grid of 1000 values (0.1% step) in case of 10 assets 1000^{10} options
- the set of candidate solutions as a rooted tree with the full set at the root
- branches of the tree as subsets of the solution set (branching alone = brute-force enumeration)
- saves bounds on the minimum in the branch and "prunes" the search space [4]

[4] Eliminating candidate solutions which cannot be optimum.

Hardware and software

- Hardware
  - CPU Intel Xeon Gold 6126 2.60GHz (48 cores)
  - RAM 428 GB
- Optimization software
  - IBM ILOG CPLEX (commercial with free academic license)
    - $\approx$ 20x faster than open-sourced GNU Linear Programming Kit (GLPK)
    - alternative (commercial) solvers: GAMS, Gurobi (free academic license)
    - contains modelling layer (interfaces) to Python/MATLAB

Test datasets

- number of datasets typically used in the portfolio selection literature
  - equal-weighted portfolios from French data library
  - individual stocks from Center for Research in Security Prices (CRSP)
- daily and monthly returns

Dataset	N	T	Start-End	Freq.
Industry Portfolios	5,17,48	250,120,60	1972-2020	d,m
Portfolios - B/M x Size	6,25,100	250,120,60	1972-2020	d,m
CRSP large	30,50	250	1971-2018	d
CRSP random	30,50	250	1971-2018	d

Out-of-sample performance evaluation

- rolling estimation window of 250 days or 60 months[5]
- portfolio weights are obtained as a solution to the optimization problem using estimation window only
- rebalancing monthly
  - daily data: rebalancing each 21 days ("month")

[5] Alternatively 120 months.

OOS quantile "confusion matrix"

	q005	q01	q02	q03	q04	q05	q06	q07	q08	q09	q095	q099
5%	-5.575	-5.744	-6.02	-6.851	-8.721	-10.07	-10.91	-13.31	-14.01	-15.31	-15.64	-14.52
10%	-3.713	-3.823	-3.941	-4.224	-5.615	-6.964	-7.871	-9.247	-10.66	-10.55	-11.75	-9.738
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30%	-0.591	-0.695	-0.601	-1.059	-1.196	-1.344	-2.115	-2.464	-3.053	-3.472	-3.517	-2.685
40%	0.342	0.273	0.664	0.015	0.147	0.068	-0.529	-0.45	-1.122	-1.204	-1.26	-0.882
50%	1.158	1.212	1.365	1.226	1.381	1.491	1.438	1.347	1.008	0.67	0.825	1.01
60%	2.225	2.186	2.274	2.393	2.652	3.028	3.327	3.115	3.198	2.814	2.7	3.258
70%	3.096	3.282	3.274	3.598	3.901	5.079	5.0	5.59	5.92	5.679	5.425	5.135
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90%	5.766	5.959	6.0	7.402	7.955	9.997	11.47	12.46	13.85	14.26	14.17	13.25
95%	7.268	7.035	8.194	9.172	10.26	13.52	14.36	15.19	17.84	19.16	19.17	19.13
99%	10.23	10.54	10.3	13.58	14.5	18.88	20.01	22.95	26.26	31.18	29.64	30.68

Figure 1: OOS quantiles of portfolio returns. Portfolios formed by maximizing IS τ-quantile, with $\tau \in \{5\%, \dots, 99\%\}$ Universe: 100 FF portfolios formed by double-sorting on Book-to-market and Size.

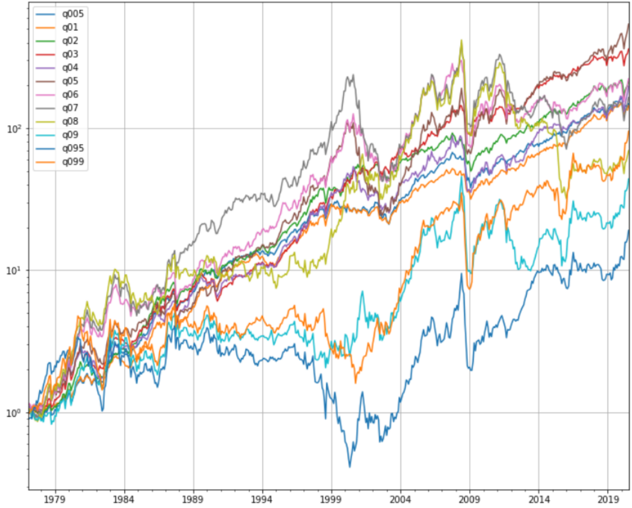

Figure 2: Cumulative log-returns of portfolios formed by maximizing various τ-quantiles.

	q005	q01	q02	q03	q04	q05	q06	q07	q08	q09	q095	q099
std	4.18	4.13	4.36	5.03	5.86	7.46	8.02	9.03	10.04	10.62	11.01	10.26
min	-21.18	-21.89	-21.52	-22.54	-28.36	-31.58	-35.59	-34.08	-37.47	-37.94	-37.94	-37.94
max	14.80	13.60	13.74	16.90	19.97	26.88	27.15	30.30	34.85	40.52	78.68	59.04
vol p.a.	14.50	14.30	15.09	17.42	20.29	25.86	27.78	31.28	34.77	36.80	38.12	35.53
skew	-0.69	-0.62	-0.64	-0.30	-0.48	-0.62	-0.44	-0.28	-0.11	0.12	0.79	0.33
kurt	2.56	2.38	2.12	1.53	2.19	2.78	2.10	1.26	0.96	1.40	5.39	3.09
MDD	-43.46	-39.06	-46.07	-38.69	-60.42	-77.82	-75.56	-82.25	-92.38	-79.22	-89.90	-79.08
CVaR 99%	-9.19	-8.74	-9.50	-10.57	-12.81	-16.67	-17.44	-19.63	-20.80	-21.93	-21.24	-21.48

Figure 3: Risk-based portfolio performance metrics.

	q005	q01	q02	q03	q04	q05	q06	q07	q08	q09	q095	q099
mean	1.052	1.042	1.13	1.261	1.191	1.5	1.37	1.397	1.325	1.291	1.152	1.397
CAGR	12.19	12.1	13.14	14.5	12.92	15.56	13.23	12.45	10.24	9.085	7.013	11.04
Sharpe	0.572	0.572	0.612	0.619	0.491	0.528	0.437	0.398	0.333	0.303	0.249	0.349

Figure 4: Reward-based portfolio performance metrics.

OOS quantile comparison with EW and GMV portfolios

	ew	gmv	q005	q01	q05	q09
5%	-7.515	-5.818	-5.94	-5.831	-8.718	-11.43
10%	-4.822	-3.606	-3.816	-4.154	-6.262	-7.738
20%	-2.562	-1.715	-1.814	-1.957	-3.506	-3.948
30%	-0.795	-0.348	-0.342	-0.521	-1.445	-1.521
40%	0.6	0.756	0.649	0.58	0.149	0.105
50%	1.672	1.429	1.539	1.474	1.571	1.271
60%	2.628	2.286	2.464	2.344	2.899	2.732
70%	3.917	2.976	3.337	3.47	4.005	4.542
80%	4.834	3.921	4.467	4.667	5.697	6.769
90%	7.01	5.569	5.972	6.349	7.991	9.411
95%	8.076	6.902	7.395	7.894	9.649	11.94
99%	11.41	9.693	9.463	10.43	15.47	21.48

Figure 5: OOS quantiles of portfolio returns. Portfolios formed by maximizing IS τ-quantile, with $\tau \in \{5\%, 10\%, 50\%, 90\%\}$. Compared with minimum-variance and equal-weighted portfolios.

Risk-based comparison with EW and GMV portfolios

	ew	gmv	q005	q01	q05	q09
std	5.109	4.108	4.355	4.608	6.075	7.654
min	-26.390	-23.944	-23.117	-23.953	-28.329	-30.389
max	16.117	14.097	13.921	15.882	22.569	37.854
vol p.a.	17.697	14.230	15.086	15.963	21.044	26.515
skew	-0.948	-1.047	-1.153	-0.918	-0.562	-0.030
kurt	3.165	4.400	3.971	3.446	2.269	2.577
MDD	-54.651	-52.579	-53.107	-55.788	-60.542	-66.261
CVaR 99%	-11.891	-9.388	-10.211	-10.527	-13.676	-16.628

Figure 6: Risk-based portfolio performance metrics with equal-weighted portfolios and minimum-variance portfolios.

5% quantile - across models and datasets

	6	17	25	48	100
ew	-0.071	-0.064	-0.074	-0.069	-0.075
gmv	-0.061	-0.049	-0.06	-0.048	-0.058
q005	-0.063	-0.06	-0.069	-0.056	-0.059
q01	-0.064	-0.055	-0.062	-0.057	-0.058
q02	-0.061	-0.051	-0.063	-0.06	-0.069
q03	-0.065	-0.064	-0.067	-0.069	-0.071
q04	-0.066	-0.074	-0.064	-0.087	-0.076
q05	-0.077	-0.088	-0.079	-0.1	-0.087
q06	-0.075	-0.096	-0.085	-0.11	-0.083
q07	-0.087	-0.064	-0.088	-0.13	-0.099
q08	-0.094	-0.064	-0.1	-0.14	-0.12
q09	-0.097	-0.11	-0.1	-0.15	-0.11
q095	-0.095	-0.11	-0.11	-0.16	-0.095
q099	-0.097	-0.11	-0.1	-0.15	-0.11

- small τ optimizer achieves small OOS τ quantile (compared to EW portfolios as well as greater τs
- (slightly) lower OOS 5% quantile achieved by GMV compared to small τ optimizer

95% quantile - across models and datasets

	6	17	25	48	100
ew	0.078	0.076	0.08	0.077	0.081
gmv	0.071	0.061	0.071	0.061	0.069
q005	0.072	0.069	0.072	0.073	0.074
q01	0.077	0.068	0.081	0.07	0.079
q02	0.076	0.072	0.079	0.082	0.08
q03	0.077	0.076	0.081	0.092	0.084
q04	0.081	0.086	0.081	0.1	0.089
q05	0.08	0.11	0.087	0.14	0.096
q06	0.085	0.12	0.088	0.14	0.1
q07	0.096	0.076	0.1	0.15	0.11
q08	0.1	0.076	0.11	0.18	0.12
q09	0.1	0.12	0.11	0.19	0.12
q095	0.096	0.12	0.11	0.19	0.12
q099	0.1	0.13	0.11	0.19	0.11

- high τ optimizer achieves high OOS τ quantile (compared to EW portfolios as well as smaller τs

Variance - across models and datasets

	6	17	25	48	100
ew	0.0023	0.0021	0.0025	0.0022	0.0026
gmv	0.0018	0.0012	0.0017	0.0012	0.0017
q005	0.0019	0.0016	0.0021	0.0018	0.0019
q01	0.002	0.0016	0.0021	0.0017	0.0021
q02	0.002	0.0015	0.0022	0.0019	0.0023
q03	0.0022	0.0021	0.0024	0.0025	0.0026
q04	0.0023	0.0027	0.0025	0.0034	0.003
q05	0.0027	0.004	0.0029	0.0056	0.0037
q06	0.0029	0.0047	0.0033	0.0064	0.0042
q07	0.0035	0.0021	0.0039	0.0082	0.0046
q08	0.0039	0.0021	0.0043	0.01	0.0053
q09	0.0043	0.0056	0.0048	0.011	0.0059
q095	0.0041	0.0056	0.0053	0.012	0.0053
q099	0.0041	0.0053	0.0048	0.011	0.0064

- similar interpretation as in the case of 95% quantile
- smaller variance acieved by GMV compared to the small τ optimizer

Sharpe ratio - across models and datasets

	6	17	25	48	100
ew	0.53	0.51	0.53	0.52	0.54
gmv	0.59	0.62	0.59	0.6	0.6
q005	0.62	0.58	0.66	0.57	0.63
q01	0.71	0.62	0.75	0.57	0.62
q02	0.65	0.61	0.61	0.61	0.65
q03	0.65	0.51	0.6	0.62	0.64
q04	0.6	0.49	0.58	0.49	0.55
q05	0.44	0.38	0.44	0.53	0.39
q06	0.46	0.35	0.38	0.44	0.51
q07	0.37	0.51	0.4	0.4	0.38
q08	0.38	0.51	0.3	0.33	0.31
q09	0.34	0.16	0.3	0.3	0.39
q095	0.27	0.08	0.22	0.25	0.35
q099	0.31	0.12	0.34	0.35	0.27

- high τ optimizer not rewarded for taking more risk (both performance-wise as well as weights concentration-wise)
- GMV and small τ optimizer achieving similar Sharpe ratios despite different risk profiles $\rightarrow$ different mean returns profiles (in favour of small τ optimizer)

Dataset-specific (detailed) results

In Jupyter.

Discussion

- behaviour of forecasting errors? (compared to e.g. gmv)
- shrinkage via weight constraints?
- short-selling allowed? (with weight constraints)
- additional constraints?
  - risk-based
  - transaction costs-based
- characteristics of assets preferred by quantile optimizer? (covariance, higher co-moments)
- combining weights from different quantile opmizations?
- multi-quantile objective functions?

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