Quantile maximizing portfolio selection

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$$\max_{w} E[U(W)]$$

$$? \; U(W)$$

In a nutshell

- investors with quantile preferences maximize τ-quantile of future portfolio returns
  - a level of investor's risk aversion is captured by τ (a smaller τs correspond to greater risk aversion)

This paper:

- study out-of-sample performance of portfolios formed by such investors
  - no closed-form solution and computationally hard
- document great heterogeneity across quantiles in terms of performance as well as portfolio compositions
- compare with mainstream portfolio selection methods

Common theme in the "confusion matrix"

	q005	q01	q02	q03	q04	q05	q06	q07	q08	q09	q095	q099
5%	-5.575	-5.744	-6.02	-6.851	-8.721	-10.07	-10.91	-13.31	-14.01	-15.31	-15.64	-14.52
10%	-3.713	-3.823	-3.941	-4.224	-5.615	-6.964	-7.871	-9.247	-10.66	-10.55	-11.75	-9.738
20%	-1.877	-2.018	-2.155	-2.451	-2.687	-3.131	-4.219	-5.181	-6.16	-6.076	-6.35	-5.666
30%	-0.591	-0.695	-0.601	-1.059	-1.196	-1.344	-2.115	-2.464	-3.053	-3.472	-3.517	-2.685
40%	0.342	0.273	0.664	0.015	0.147	0.068	-0.529	-0.45	-1.122	-1.204	-1.26	-0.882
50%	1.158	1.212	1.365	1.226	1.381	1.491	1.438	1.347	1.008	0.67	0.825	1.01
60%	2.225	2.186	2.274	2.393	2.652	3.028	3.327	3.115	3.198	2.814	2.7	3.258
70%	3.096	3.282	3.274	3.598	3.901	5.079	5.0	5.59	5.92	5.679	5.425	5.135
80%	4.158	4.337	4.428	5.014	5.362	7.038	7.34	8.679	9.432	8.786	8.746	8.496
90%	5.766	5.959	6.0	7.402	7.955	9.997	11.47	12.46	13.85	14.26	14.17	13.25
95%	7.268	7.035	8.194	9.172	10.26	13.52	14.36	15.19	17.84	19.16	19.17	19.13
99%	10.23	10.54	10.3	13.58	14.5	18.88	20.01	22.95	26.26	31.18	29.64	30.68

Contents

τ-quantile preferences

- alternative preferences to Expected Utility representation
  - robust - not requiring existence of moments of any orders
- given some fixed $\tau \in (0,1)$ a τ-quantile preference $\succeq$ is defined in eq. (1):

$$X \succeq Y \iff Q_\tau[u(X)] \geq Q_\tau[u(Y)] \tag{1}$$

- u is some general utility function and X and Y are random variables

Redundancy of utility function

$$Q_\tau[u(X)] \geq Q_\tau[u(Y)] \iff Q_\tau[X] \geq Q_\tau[Y] \tag{2}$$

- a continuous and strictly increasing utility function u
- the invariance of quantiles w.r.t. monotone transformations
- the concavity of the utility function does not determine the risk attitude of an investor with quantile preferences!
  - τ does
  - the smaller the τ, the higher the risk aversion

Literature

- introduced by Manski (1988)
- axiomatized by Chambers (2009), Rostek (2010), de Castro and Galvao (2019)
- Giovannetti (2013) models a two-period economy with agents with quantile preferences
- de Castro et al. (2020) run behavioural experiment
- He and Zhou (2011) study a portfolio choice model in continuous time
- Benati and Rizzi (2007) propose mean-VaR model
  - show it is a NP-hard problem (because of VaR)
  - introduce Mixed integer linear programming reformulation
- Benati (2015) use medians instead of means in number of portfolio selection models

Portfolio selection setting

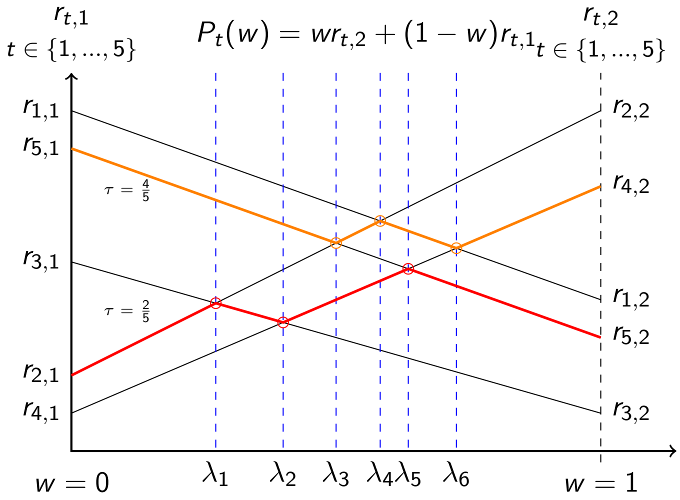
- N assets[1]
- T periods[2]
- weights: $w = (w_1, \dots, w_N)$
  - long-only: $w_i > 0; \quad \forall i \in \{1, \dots, N\}$
  - full-investment: $\sum_{i=1}^{N} w_i = 1$
- return of an asset i in period t : $r_{t,i}$
- portfolio return in period t: $p_t = \sum_{i=1}^{N} w_i r_{t,i}$
- portfolio returns over T periods: $P = (p_1, \dots, p_T)$
- $\tau \in 0, 1$
- τ-quantile of portfolio returns: $Q_\tau(P)$

[1] Can change over time.
[2] Length of the estimation sample.

Portfolio Optimization via Quantile Maximization

$$
\begin{aligned}
&\max_{w} Q_\tau(P) \\
&\text{s.t.} \sum_{i=1}^{n} w_i = 1 \\
&p_t = \sum_{i=1}^{N} w_i r_{t,i} \quad \forall t \in \{1, \ldots, T\} \\
&w_i \geq 0 \;\; \forall i \in \{1, \ldots, N\}
\end{aligned}
\tag{3}
$$

Finding quantile of portfolio returns



NP-hardness

- the quantile function is non-differentiable piece-wise linear function
- Benati and Rizzi (2007) show eq. (3) is Approximable hard
  - a class of NP-hard[3] problem for which a Polynomial Time Approximation scheme does not exist, unless $P = NP$
  - it rules out the existence of polynomial and pseudopolynomial time algorithms (no hope (unless $P = NP$) to obtain an approximation algorithm
- eq. (3) can be reformulated as an Mixed Integer Linear Programming model shown in eq. (4)

[3] Non-deterministic polynomial-time hardness.

Mixed Integer Programming (MIP) formulation

$$
\begin{aligned}
& \max_{w,y,Q_\tau} \; Q_\tau \\
\text{s.t.} \quad & Q_\tau \leq \sum_{i=1}^{N} w_i r_{t,i} + L(1 - y_t) \;\; \forall t \in \{1, \ldots, T\} \\
& \sum_{t=1}^{T} y_i \leq \tau T \\
& \sum_{i=1}^{n} w_i = 1 \\
& y_i \in \{0, 1\} \;\; \forall t \in \{1, \ldots, T\} \\
& x_i \geq 0 \;\; \forall i \in \{1, \ldots, N\}
\end{aligned}
\tag{4}
$$

Solvable special cases

- eq. (4) is NP-hard because T and N can grow arbitrarily large
- Benati and Rizzi (2007) show 2 polynomial time solvable special cases
  - finite and small T: $O(2^T)$
  - finite and small N: $O(\lambda^N)$
    - λ corresponds to the number of candidate solutions which is dependent on τ
- Benati (2015) apply this to maximizing median of portfolio returns ($N = 60$ and $T \in \{21, 31, 41\}$)

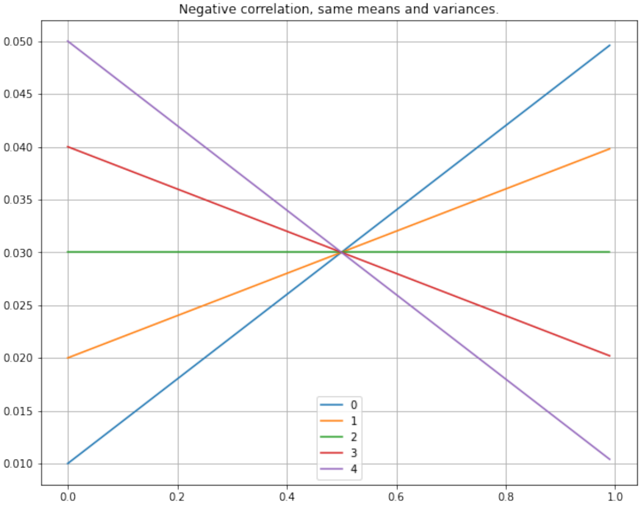
Hardware and software

- Hardware
  - CPU Intel Xeon Gold 6126 2.60GHz (48 cores)
  - RAM 428 GB
- Optimization software
  - IBM ILOG CPLEX (commercial with free academic license)
    - $\approx$ 20x faster than open-sourced GNU Linear Programming Kit (GLPK)
    - alternative (commercial) solvers: GAMS, Gurobi (free academic license)
    - contains modelling layer (interfaces) to Python/MATLAB

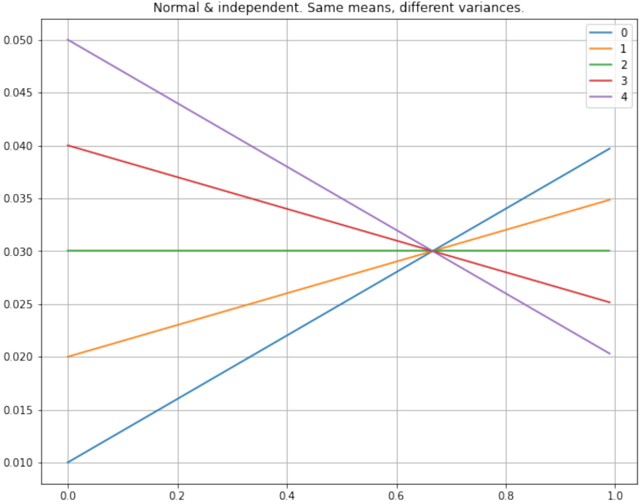
Simulations - naive case with closed-form solution

- if $r_1, \ldots, r_N$ are mutually independent normal random variables with means $\mu_1, \ldots, \mu_N$ and variances $\sigma_1, \ldots, \sigma_N$ $(P = \sum_{i=1}^{N} w_i r_i)$
- $Var(P) = \sum_{i=1}^{N} w_i^2 \sigma_i$
- $F^{-1}(p) = \mu + \sigma\sqrt{2}erf^{-1}(2p-1), \quad p \in (0,1)$
- a quantile of portfolio returns is just its rescaled variance plus its mean

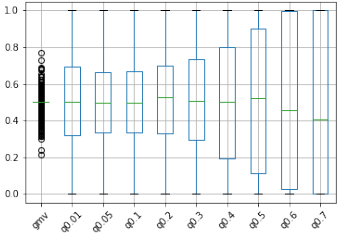
Toy example 1



Toy example 2



Normal and independent



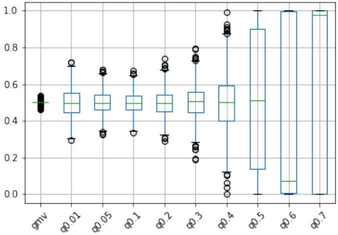


Figure 1: Independently and normally distributed returns with the same variances. $T = 21$ on the left $T = 1000$ on the right.

Weight differences

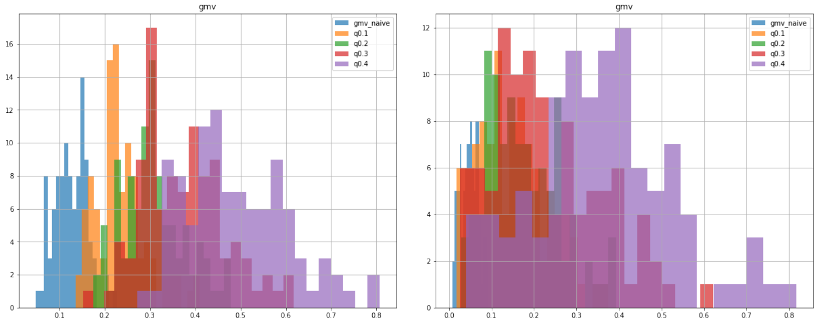
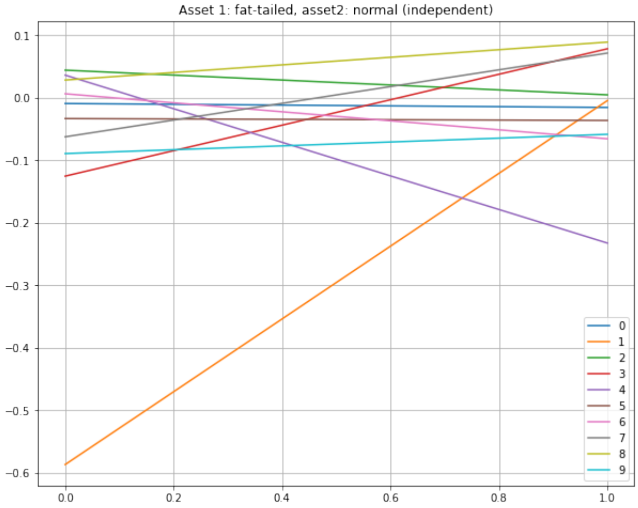


Figure 2: Independently and normally distributed returns with the same variances. $T = 36$ on the left $T = 200$ on the right.

Toy example 3



Non-normal and independent

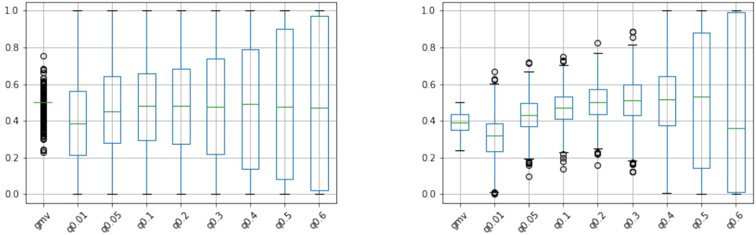


Figure 4: Independent α-stable distribution with same variances and different αs. $T = 21$ on the left $T = 1000$ on the right.

Test datasets

- number of datasets typically used in the portfolio selection literature
  - equal-weighted portfolios from French data library
  - individual stocks from Center for Research in Security Prices (CRSP)
- daily and monthly returns

Dataset	N
Industry Portfolios	5,17,48
Portfolios - B/M x Size	6,25
Portfolios - INV x Size	6,25
Portfolios - Prior 12-2 x Size	6,25
Portfolios - Prior 1-0 x Size	6,25
Europe Portfolios - B/M x Size	6
Japan Portfolios - B/M x Size	6
CRSP large	10,20,30
CRSP random	10,20,30

Out-of-sample performance evaluation

- monthly rolling estimation windows of 36,60,120 months
- daily rolling estimation windows of 21, 63 and 125 days
- portfolio weights are obtained as a solution to the optimization problem using estimation window only
- rebalancing monthly
    - daily data: rebalancing each 21 days ("month")
- evaluated using:
    - τ-quantile
    - variance
    - Sharpe ratio
    - weights concentration

OOS quantile "confusion matrix"

	q005	q01	q02	q03	q04	q05	q06	q07	q08	q09	q095	q099
5%	-5.575	-5.744	-6.02	-6.851	-8.721	-10.07	-10.91	-13.31	-14.01	-15.31	-15.64	-14.52
10%	-3.713	-3.823	-3.941	-4.224	-5.615	-6.964	-7.871	-9.247	-10.66	-10.55	-11.75	-9.738
20%	-1.877	-2.018	-2.155	-2.451	-2.687	-3.131	-4.219	-5.181	-6.16	-6.076	-6.35	-5.666
30%	-0.591	-0.695	-0.601	-1.059	-1.196	-1.344	-2.115	-2.464	-3.053	-3.472	-3.517	-2.685
40%	0.342	0.273	0.664	0.015	0.147	0.068	-0.529	-0.45	-1.122	-1.204	-1.26	-0.882
50%	1.158	1.212	1.365	1.226	1.381	1.491	1.438	1.347	1.008	0.67	0.825	1.01
60%	2.225	2.186	2.274	2.393	2.652	3.028	3.327	3.115	3.198	2.814	2.7	3.258
70%	3.096	3.282	3.274	3.598	3.901	5.079	5.0	5.59	5.92	5.679	5.425	5.135
80%	4.158	4.337	4.428	5.014	5.362	7.038	7.34	8.679	9.432	8.786	8.746	8.496
90%	5.766	5.959	6.0	7.402	7.955	9.997	11.47	12.46	13.85	14.26	14.17	13.25
95%	7.268	7.035	8.194	9.172	10.26	13.52	14.36	15.19	17.84	19.16	19.17	19.13
99%	10.23	10.54	10.3	13.58	14.5	18.88	20.01	22.95	26.26	31.18	29.64	30.68

HH-index - portfolios

	17_IP	25	25_INV	25_Pr12	25_Pr1	48_IP
ew	0.059	0.040	0.040	0.040	0.040	0.021
gmv	0.258	0.303	0.312	0.292	0.304	0.165
q001	0.541	0.705	0.707	0.705	0.694	0.402
q005	0.405	0.560	0.527	0.517	0.584	0.281
q01	0.361	0.494	0.485	0.430	0.468	0.250
q02	0.356	0.469	0.457	0.446	0.441	0.264
q03	0.347	0.398	0.417	0.436	0.443	0.243
q04	0.397	0.382	0.433	0.445	0.439	0.303
q05	0.484	0.422	0.431	0.452	0.423	0.410
q06	0.566	0.460	0.473	0.488	0.476	0.494
q07	0.631	0.494	0.486	0.582	0.556	0.630
q08	0.699	0.576	0.585	0.583	0.609	0.712
q09	0.819	0.715	0.731	0.751	0.781	0.845
q095	0.890	0.917	0.846	0.859	0.822	0.921
q099	1.000	1.000	1.000	1.000	1.000	1.000

HH-index - stocks

	L10	R10	L20	R20	L30	R30
ew	0.100	0.100	0.050	0.050	0.033	0.033
gmv	0.207	0.141	0.134	0.081	0.109	0.059
q001	0.428	0.399	0.319	0.312	0.277	0.268
q005	0.351	0.343	0.268	0.260	0.221	0.218
q01	0.312	0.327	0.231	0.227	0.204	0.192
q02	0.303	0.317	0.237	0.220	0.223	0.195
q03	0.325	0.326	0.271	0.242	0.253	0.208
q04	0.403	0.378	0.370	0.301	0.367	0.274
q05	0.488	0.460	0.475	0.402	0.465	0.359
q06	0.562	0.544	0.554	0.479	0.549	0.426
q07	0.694	0.669	0.653	0.602	0.601	0.558
q08	0.815	0.752	0.758	0.705	0.719	0.671
q09	0.842	0.877	0.804	0.809	0.821	0.797
q095	0.941	0.932	0.873	0.912	0.882	0.899
q099	1.000	1.000	1.000	1.000	1.000	1.000

5% OOS quantile - portfolios

	25_INV	25_Pr12	25_Pr1	25_BM	17_IP	48_IP
ew	-7.323	-7.443	-7.868	-7.001	-6.303	-6.497
gmv	-5.964	-6.088	-6.138	-5.637	-4.854	-4.626
q001	-6.017	-6.956	-6.78	-6.466	-5.333	-5.397
q005	-6.386	-6.357	-6.485	-6.422	-5.588	-5.351
q01	-6.414	-7.206	-6.116	-6.145	-6.075	-6.135
q02	-6.424	-6.875	-7.199	-6.575	-5.84	-6.24
q03	-6.568	-7.018	-7.156	-6.13	-5.886	**-7.202**
q04	-7.476	-7.407	-7.393	-7.143	**-7.81**	**-8.79**
q05	-8.008	**-9.374**	-8.15	-7.639	**-8.529**	**-10.416**
q06	-8.417	**-9.571**	**-9.111**	-8.025	**-8.928**	**-9.948**
q07	**-9.123**	**-9.412**	**-9.822**	**-9.055**	**-9.703**	**-11.627**
q08	**-9.245**	**-9.177**	**-9.383**	**-9.966**	**-10.053**	**-13.574**
q09	**-9.922**	**-9.625**	**-9.883**	**-11.037**	**-9.622**	**-14.162**
q095	**-9.763**	**-9.28**	**-9.953**	**-9.437**	**-10.452**	**-15.325**
q099	**-10.108**	**-10.607**	**-9.99**	**-9.82**	**-11.643**	**-15.438**

5% OOS quantile - stocks

	L10	R10	L20	R20	L30	R30
ew	-6.64	-9.029	-6.278	-8.76	-6.548	-7.448
gmv	-5.274	-7.659	-5.207	-6.195	**-4.939**	-5.781
q001	-6.386	-7.957	-6.144	-7.756	-5.811	-6.604
q005	-5.939	-7.279	-5.867	-7.411	-6.383	-6.136
q01	-5.831	-7.416	-6.59	-7.245	-6.464	-7.359
q02	-6.657	-7.989	**-7.956**	-9.76	-8.943	**-8.585**
q03	-6.867	**-10.458**	**-9.119**	**-9.976**	**-10.598**	**-11.148**
q04	**-7.565**	**-11.516**	**-9.323**	**-13.209**	**-10.317**	**-13.169**
q05	**-9.134**	**-15.945**	**-11.319**	**-17.178**	**-11.376**	**-17.447**
q06	**-9.353**	**-18.245**	**-11.453**	**-18.862**	**-13.616**	**-20.616**
q07	**-12.017**	**-23.183**	**-12.447**	**-23.481**	**-12.445**	**-23.633**
q08	**-12.007**	**-22.222**	**-12.096**	**-25.302**	**-13.225**	**-28.187**
q09	**-12.537**	**-26.715**	**-13.094**	**-30.851**	**-15.497**	**-27.91**
q095	**-12.74**	**-30.618**	**-13.53**	**-33.383**	**-15.47**	**-31.361**
q099	**-12.299**	**-32.091**	**-12.654**	**-34.589**	**-15.952**	**-33.554**

50% OOS quantile - portfolios

	25_INV	25_Pr12	25_Pr1	25_BM	17_IP	48_IP
ew	1.566	1.491	1.524	1.648	1.343	1.382
gmv	1.397	1.326	1.312	1.541	1.204	1.216
q001	1.185	1.354	1.590	1.563	1.153	1.092
q005	1.563	1.316	1.383	1.591	1.200	1.365
q01	1.651	1.355	1.227	1.681	1.212	1.383
q02	1.562	1.499	1.536	1.674	1.292	0.957
q03	1.618	1.819	1.518	1.643	1.318	1.355
q04	1.766	1.88	1.711	1.752	1.385	1.627
q05	1.771	1.946	1.820	1.446	1.249	1.225
q06	1.584	1.601	1.750	1.785	1.172	1.461
q07	1.638	1.756	1.811	1.805	0.968	1.210
q08	1.476	1.625	1.656	1.857	0.979	1.028
q09	1.697	1.317	1.704	1.391	0.670	0.674
q095	1.476	1.31	1.854	1.532	0.530	0.931
q099	1.526	**1.05**	1.600	1.155	0.510	0.950

50% OOS quantile - stocks

	L10	R10	L20	R20	L30	R30
ew	1.024	0.976	1.175	1.348	1.179	1.61
gmv	0.908	1.187	0.944	1.159	1.076	1.314
q001	1.045	1.029	0.962	1.181	1.186	1.023
q005	0.882	0.946	1.045	1.392	1.126	1.0
q01	0.853	0.999	0.742	1.481	1.037	1.244
q02	0.991	1.082	0.949	1.39	1.043	1.832
q03	0.821	1.131	1.279	0.945	1.385	1.044
q04	0.922	1.539	1.096	0.795	1.63	1.527
q05	0.791	1.204	1.133	0.232	1.444	0.939
q06	1.144	0.615	0.919	-0.49	1.332	0.998
q07	0.894	0.561	1.101	**-1.456**	1.111	-0.196
q08	1.033	0.0	1.124	**-1.988**	1.324	-0.0
q09	1.085	**-0.562**	0.993	**-1.429**	0.794	-0.633
q095	0.812	**-0.864**	0.439	**-1.695**	0.725	**-1.9**
q099	0.540	**-1.361**	0.345	**-1.818**	**0.359**	**-2.041**

95% OOS quantile - portfolios

	25_INV	25_Pr12	25_Pr1	25_BM	17_IP	48_IP
ew	**8.06**	**8.09**	**8.049**	**8.149**	**7.631**	**7.655**
gmv	**7.123**	**7.003**	**7.019**	**7.162**	**6.186**	**6.102**
q001	**7.191**	**7.457**	**8.099**	**7.659**	**6.908**	**6.947**
q005	**7.673**	**8.12**	**8.023**	**8.001**	**6.745**	**7.466**
q01	**7.536**	**7.947**	**7.587**	**8.156**	**6.788**	**7.956**
q02	**8.108**	**8.577**	**7.915**	**8.113**	**7.584**	**8.634**
q03	**8.538**	**8.631**	**8.767**	**7.757**	**7.919**	**9.155**
q04	**8.524**	9.329	**9.94**	**8.33**	**8.713**	**9.685**
q05	**9.072**	9.808	**9.444**	**9.15**	**10.39**	**11.916**
q06	9.395	9.999	**10.261**	**9.528**	11.638	**13.402**
q07	9.708	11.722	11.313	9.729	11.811	**15.172**
q08	9.862	11.747	11.074	10.792	12.182	18.033
q09	10.737	11.885	11.778	11.242	12.388	18.024
q095	11.014	10.757	11.877	11.232	13.14	20.244
q099	11.579	12.392	11.642	11.42	13.14	19.112

95% OOS quantile - stocks

	L10	R10	L20	R20	L30	R30
ew	**8.025**	**11.409**	**7.929**	**11.072**	**7.382**	**10.18**
gmv	**7.11**	**9.374**	**6.908**	**8.464**	**6.845**	**8.709**
q001	**7.705**	**12.584**	**7.427**	**9.965**	**7.687**	**9.88**
q005	**8.347**	**10.197**	**8.07**	**9.093**	**7.778**	**9.755**
q01	**7.523**	**9.584**	**7.056**	**10.068**	**7.563**	**9.864**
q02	**8.242**	**11.736**	**8.552**	**11.823**	**8.619**	**12.085**
q03	**8.638**	**11.898**	**8.522**	**13.078**	**9.722**	**14.151**
q04	**10.367**	**15.859**	**11.054**	**16.562**	**11.949**	**17.799**
q05	**12.029**	**17.966**	13.474	**19.649**	14.324	**20.289**
q06	13.135	**20.28**	14.525	**21.304**	14.624	**21.74**
q07	15.295	25.199	14.806	**24.991**	14.514	**28.647**
q08	15.076	25.706	14.982	29.831	14.386	**30.341**
q09	15.383	29.36	16.148	39.628	16.677	37.446
q095	14.884	27.686	13.928	38.766	16.023	44.647
q099	14.977	27.802	13.634	44.66	15.812	38.04

Variance - portfolios

	25_INV	25_Pr12	25_Pr1	25_BM	17_IP	48_IP
ew	**0.251**	**0.268**	**0.272**	**0.253**	**0.212**	**0.223**
gmv	0.164	0.175	0.178	0.172	0.124	0.124
q001	0.179	0.199	0.213	0.208	**0.156**	**0.164**
q005	**0.212**	0.21	0.222	0.207	0.166	**0.172**
q01	**0.204**	**0.216**	0.211	**0.228**	**0.164**	**0.195**
q02	**0.214**	**0.241**	**0.243**	**0.235**	**0.175**	**0.238**
q03	**0.23**	**0.263**	**0.246**	**0.223**	**0.203**	**0.26**
q04	**0.265**	**0.311**	**0.303**	**0.255**	**0.285**	**0.371**
q05	**0.303**	**0.358**	**0.333**	**0.294**	**0.379**	**0.524**
q06	**0.327**	**0.387**	**0.372**	**0.326**	**0.449**	**0.597**
q07	**0.349**	**0.507**	**0.458**	**0.357**	**0.499**	**0.714**
q08	**0.374**	**0.487**	**0.474**	**0.417**	**0.527**	**0.923**
q09	**0.429**	**0.51**	**0.475**	**0.506**	**0.55**	**1.091**
q095	**0.468**	**0.478**	**0.506**	**0.504**	**0.56**	**1.242**
q099	**0.489**	**0.524**	**0.502**	**0.497**	**0.593**	**1.09**

Variance - stocks

	L10	R10	L20	R20	L30	R30
ew	0.197	**0.483**	0.184	**0.437**	0.176	**0.372**
gmv	0.14	0.329	0.137	0.265	0.131	0.229
q001	0.187	**0.452**	0.184	**0.342**	0.186	**0.268**
q005	0.19	**0.413**	0.194	**0.342**	0.195	**0.305**
q01	0.184	**0.396**	0.19	**0.318**	0.196	**0.329**
q02	**0.232**	**0.433**	**0.248**	**0.459**	**0.28**	**0.49**
q03	**0.246**	**0.508**	**0.331**	**0.555**	**0.394**	**0.722**
q04	**0.354**	**0.745**	**0.462**	**0.994**	**0.515**	**1.037**
q05	**0.488**	**1.079**	**0.624**	**1.345**	**0.661**	**1.416**
q06	**0.526**	**1.774**	**0.705**	**2.101**	**0.766**	**2.047**
q07	**0.65**	**3.31**	**0.725**	**2.799**	**0.772**	**2.857**
q08	**0.69**	**3.824**	**0.772**	**4.557**	**0.787**	**4.188**
q09	**0.707**	**4.515**	**0.874**	**16.24**	**0.981**	**7.392**
q095	**0.701**	**3.756**	**0.738**	**7.103**	**0.944**	**8.189**
q099	**0.697**	**4.391**	**0.717**	**6.957**	**1.111**	**6.677**

Sharpe ratio - portfolios

	17_IP	25_BM	25_INV	25_Pr12	25_Pr1	48_IP
ew	0.532	0.576	0.590	0.542	0.538	0.549
gmv	0.600	0.612	0.642	0.562	0.576	0.592
q001	0.510	0.527	0.575	0.485	0.628	0.516
q005	0.532	0.649	0.685	0.613	0.585	0.612
q01	0.516	0.664	0.670	0.605	0.538	0.688
q02	0.539	0.658	0.655	0.709	0.647	0.483
q03	0.620	0.633	0.726	0.681	0.642	0.496
q04	0.485	0.636	0.638	0.665	0.618	0.373
q05	0.418	0.539	0.540	0.616	0.555	0.354
q06	0.402	0.478	0.539	0.490	0.537	0.525
q07	0.323	0.513	0.436	0.545	0.512	0.394
q08	0.326	0.422	0.442	0.594	0.523	0.415
q09	0.306	0.310	0.536	0.500	0.527	0.209
q095	0.179	0.376	0.463	0.462	0.521	0.282
q099	0.095	0.392	0.462	0.290	0.470	0.276

Sharpe ratio - stocks

	L10	R10	L20	R20	L30	R30
ew	0.445	0.320	0.467	0.631	0.490	0.615
gmv	0.498	0.507	0.556	0.687	0.631	0.711
q001	0.415	0.608	0.404	0.489	0.416	0.554
q005	0.435	0.440	0.439	0.577	0.471	0.552
q01	0.439	0.475	0.345	0.660	0.406	0.567
q02	0.492	0.459	0.351	0.538	0.277	0.657
q03	0.412	0.377	0.211	0.331	0.300	0.396
q04	0.349	0.423	0.198	0.294	0.264	0.477
q05	0.229	0.276	0.262	0.124	0.361	0.227
q06	0.407	0.272	0.190	-0.073	0.262	0.186
q07	0.338	0.146	0.281	-0.122	0.326	0.101
q08	0.289	0.167	0.257	-0.085	0.292	0.151
q09	0.290	0.044	0.216	0.152	0.151	0.271
q095	0.200	-0.232	-0.081	-0.058	0.059	0.181
q099	0.206	-0.336	-0.082	-0.074	-0.012	0.013

Conclusion

- high weights concentration for high and low τ-quantile maximizer
- the lowest concentration for τ around 0.5
- max (IS) τ-quantile maximizer experiences the highest (OOS) τ-quantile among other τ-quantile maximizers
- GMV portfolio achieves higher τ-quantile for low τs

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