Gaussian Impulse Response and EE50

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July 2020

1 Gaussian PSF

Given the 50% Encircled Energy (EE50) of some Gaussian Point Spread Function (PSF), let us find the standard deviation σ of the Gaussian PSF corresponding to the EE50. Let the Gaussian PSF be:

$$f(x, y) = \exp\left[-\left(\frac{x^2}{2\sigma_x^2} + \frac{y^2}{2\sigma_y^2}\right)\right]$$

Let $r^2 = x^2 + y^2$ and $\sigma_x = \sigma_y = \sigma$. We can then write the Gaussian PSF in polar coordinates as:

$$f(r) = \exp\left[-\left(\frac{r^2}{2\sigma^2}\right)\right]$$

We first find the total energy E_{Total} of the PSF over all space. That is:

$$\begin{aligned}
E_{Total} &= \iint_{\mathbb{R}} f(x, y)\, dx\, dy \\
&= \int_0^{2\pi} \int_0^{\infty} f(r) r\, dr\, d\theta \\
&= \int_0^{2\pi} \int_0^{\infty} \exp\left[-\left(\frac{r^2}{2\sigma^2}\right)\right] r\, dr\, d\theta \\
&= 2\pi\sigma^2
\end{aligned}$$

Let the total energy of the PSF from the centroid (peak) to some r be $E(r)$. $E(r)$ is:

$$\begin{aligned}
E(r) &= \int_0^{2\pi} \int_0^{r} f(\tau)\tau\, d\tau\, d\theta \\
&= \int_0^{2\pi} \int_0^{r} \exp\left[-\left(\frac{\tau^2}{2\sigma^2}\right)\right] \tau\, d\tau\, d\theta \\
&= 2\pi\sigma^2 - 2\pi\sigma^2 \exp\left[-\frac{r^2}{2\sigma^2}\right]
\end{aligned}$$

Let the encircled energy at some r be $EE(r)$. By definition:

$$\begin{aligned}
EE(r) &= \frac{E(r)}{E_{Total}} \\
&= \frac{2\pi\sigma^2 - 2\pi\sigma^2 \exp\left[-\frac{r^2}{2\sigma^2}\right]}{2\pi\sigma^2} \\
&= 1 - \exp\left[-\frac{r^2}{2\sigma^2}\right]
\end{aligned}$$

EE50 is defined to be the r such that at $r = \text{EE50}$, $EE(\text{EE50}) = 0.50$. To find σ given some EE50, we take the above expression for $E(r)$ and rearrange to solve for σ:

$$\begin{aligned} EE(\text{EE50}) &= 0.5 \\ 1 - \exp\left[-\frac{(\text{EE50})^2}{2\sigma^2}\right] &= 0.5 \\ \exp\left[-\frac{(\text{EE50})^2}{2\sigma^2}\right] &= 0.5 \\ \frac{(\text{EE50})^2}{2\sigma^2} &= -\ln 0.5 \\ \sigma &= \sqrt{\frac{-(\text{EE50})^2}{2\ln 0.5}} \end{aligned}$$

2 Gaussian LSF

Let us now find the Gaussian Line Spread Function (LSF) $f_X(x)$ in the x-direction, corresponding to the Gaussian PSF $f(x, y)$. The LSF can be found by:

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f(x, y)\, dy \\ &= \int_{-\infty}^{\infty} \exp\left[-\left(\frac{x^2}{2\sigma_x^2} + \frac{y^2}{2\sigma_y^2}\right)\right] dy \\ &= \int_{-\infty}^{\infty} \exp\left[-\left(\frac{x^2 + y^2}{2\sigma^2}\right)\right] dy \\ &= \sqrt{2\pi}\sigma \exp\left[-\left(\frac{x^2}{2\sigma^2}\right)\right] \end{aligned}$$