Numerically Solving the Tolman-Oppenheimer-Volkoff Equations to Model the Internal Structure of Neutron Stars

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Abstract

The modelling of a the internal structure of a spherically symmetric body in gravitational equilibrium can be used to investigate neutron stars. In this investigation, the Tolman-Oppenheimer-Volkhoff equations are solved with a numerical, python based solver to constrain parameters of neutron stars. In a representative model, mass of a neutron star is constrained to $1.65 M_\odot$ at a radius of 14.29 km. Furthermore, a Mass-Radius relation is determined to set constraints on the maximum mass and radius of a neutron star. Further parameters, physical and computational, are investigated to determine in which regime the model is physically relevant.

1 Introduction

The Tolman-Oppenheimer-Volkoff (TOV) equations describe the internal structure of a spherically symmetric body that is in gravitational equilibrium. Derived in a general relativistic framework, it can be used to model the interior of neutron stars.

This investigation looks at the use of a numerical, python based TOV equation solver to model the interior of a spherical body and constrain neutron star parameters. Neutron stars are the result of a collapsed supergiant star, and classified as small, dense, and static objects. Neutron star masses are in the range of $1.0M_{\odot}$ to $2.5M_{\odot}$ and a radii of the order of 10 km. The TOV equations, solved numerically, explain the step by step profile of the object's pressure and mass. This profile is heavily dependent on the central density of the star, a quantity that sets the internal pressure at the core.

Along with this paper, a code guide to accompany the TOV-solver is submitted. The code for this project is publicly available in a Github-repository and can adapted for further investigations:
https://github.com/mauritzwicker/tov-solver

2 Background

The TOV equations are derived from Einstein's equations and are solved in a General-Relativistic framework. They are therefore also called the general-relativisitic-hydrostatic-equilibrium equations. In this section, the theoretical framework is set to be able to solve the TOV equations. Then, the computational methods needed to solve the equations numerically are explained.

2.1 Theoretical Framework

The TOV equations are not exclusive to a neutron star. It is a generalized set of equations that, within the framework of general relativity, describes an isotropic and spherically symmetric body in gravitational equilibrium.

The TOV equations are based on a set of assumptions:

- Matter constitutes a perfect fluid
- The object is spherically symmetric

Therefore the metric is given as:

$$ds^2 = e^{2\nu}dt^2 - e^{2\lambda}dr^2 - r^2d\theta^2 - r^2d\Omega^2 \tag{1}$$

where:

$$d\Omega^2 = r^2d\theta^2 + r^2sin^2\theta d\phi^2 \tag{2}$$

The Einstein Field Equations are:

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{3}$$

where $G_{\mu\nu}$ is the Einstein tensor and $T_{\mu\nu}$ is the stress energy tensor. It can also be given as:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \tag{4}$$

where $R_{\mu\nu}$ is the Ricci-tensor and R is the Ricci-scalar.
By solving for Ricci scalars in the Field Equations we can arrive at the TOV Equations that describes the radial distribution of Pressure in our system:

$$\frac{dP(r)}{dr} = \frac{-G}{r^2}(\rho(1 + \frac{\epsilon}{c^2}) + \frac{p}{c^2})\frac{m(r) + 4\pi r^3 \frac{p}{c^2}}{1 - \frac{r_s}{r}} \tag{5}$$

where the Schwarzschild radius is given as:

$$r_s = \frac{2Gm(r)}{c^2} \tag{6}$$

The mass distribution in a gravitationally bound, spherically symmetric object is given as:

$$\frac{dm}{dr} = 4\pi r^2 \rho(1 + \frac{\epsilon}{c^2}) \tag{7}$$

The metric potential is given as:

$$\frac{d\phi}{dr} = \frac{m + 4\pi r^3 \frac{p}{c^2}}{r^2(1 - \frac{r_s}{r})} \tag{8}$$

In order to solve the TOV equations for a fluid, we need to associate pressure and density, which is done by assuming an equation of state. We assume a polytropic equation of state, a non-perfect assumption that allows for a good approximation of a neutron star. The polytropic equation of state is defined as follows:

$$p_{poly} = K \cdot \rho_{poly}^{\gamma} \tag{9}$$

$$\rho_{poly} = \left(\frac{P_{poly}}{K}\right)^{1/\gamma} \tag{10}$$

where K is the adiabatic coefficient, and γ is the adiabatic exponent.

2.2 Numerical Framework

The TOV equation needs to be analytically solved, which means we need to build on a set of assumptions and solve the equations step-wise. To do this we use the iterative methods called the Runge-Kutta methods. The family of Runge-Kutta methods is characterized by increasing complexity and precision.

In first order, Runge-Kutta 1 (RK1) or Euler-method, the next iteration of a variable is given by the sum of the previous one and the rate of change. Therefore, given the TOV equation and assuming initial parameters for the zeroth index (nearest point to the center of the object/star), we can determine, for increasing distance r, every next value.

$$p_{n+1} = p_n + \Delta r \frac{dp}{dr}(r_n) \tag{11}$$

$$m_{n+1} = m_n + \Delta r \frac{dm}{dr}(r_n) \tag{12}$$

In a generalized form, we can thus define the Runge-Kutta methods as follows. For an ordinary differential equation (like the TOV equations):

$$\frac{dy}{dx} = f(x, y) \tag{13}$$

we can define the factors,

$$k_1 = f^{'}(x_n, f_n) \tag{14}$$

$$k_2 = f^{'}(x_n + \frac{\Delta x}{2}, f_n + \frac{k_1}{2}) \tag{15}$$

$$k_3 = f^{'}(x_n + \frac{\Delta x}{2}, f_n + \frac{k_2}{2}) \tag{16}$$

$$k_4 = f^{'}(x_n + \Delta x, f_n + k_3) \tag{17}$$

which will define our Runge-Kutta orders. Our Runge-Kutta orders are given as:

$$f_{n+1}^{(RK1)} = f_n + \Delta x \cdot k_1 \tag{18}$$

$$f_{n+1}^{(RK2)} = f_n + \Delta x \cdot k_2 \tag{19}$$

$$f_{n+1}^{(RK3)} = f_n + \Delta x \left(\frac{k1 + 4k_2 + k_3}{6} \right) \tag{20}$$

$$f_{n+1}^{(RK4)} = f_n + \Delta x \left(\frac{k1 + 2k_2 + 2k_3 + k_4}{6} \right) \tag{21}$$

3 Results

Here I investigate the results for the TOV-equations for different physical scenarios, such different central density definitions or equation of state constants, as well as for different numerical scenarios, such as Runge-Kutta orders and the simulation stepsize.

3.1 A Representative Model

To use the TOV-equations to model a neutron star that accurately represents observations, we need to define our initial variables precisely. As a baseline for our model, and later for the parameter analysis, we define the following parameters.

Equation of State	Polytropic
Adiabatic Coefficient (K)	$1.982 \cdot 10^{-6}$
Adiabatic Exponent (γ)	2.75
Central Density	$2 \cdot \rho_{nuc}$
Gridsize	$1 \cdot 10^{6}$
Stepsize	100 cm
Runge-Kutta Order	4th

Table 1: Initial Parameters for TOV-Model

For these values, we run the Runge-Kutta function until the pressure of the system has dropped to zero. At this point, the star's radius and mass have been reached.

Radius	14.29 km
Mass	$1.65 M_{\odot}$
Simulation Steps	142955

Table 2: Results for TOV-model

The result of $1.65 M_{\odot}$ is equivalent to $3.289 \cdot 10^{33}$ g. To see how the mass, pressure, and density of the object evolve in the model, we look at their profile for increasing distance from the center.

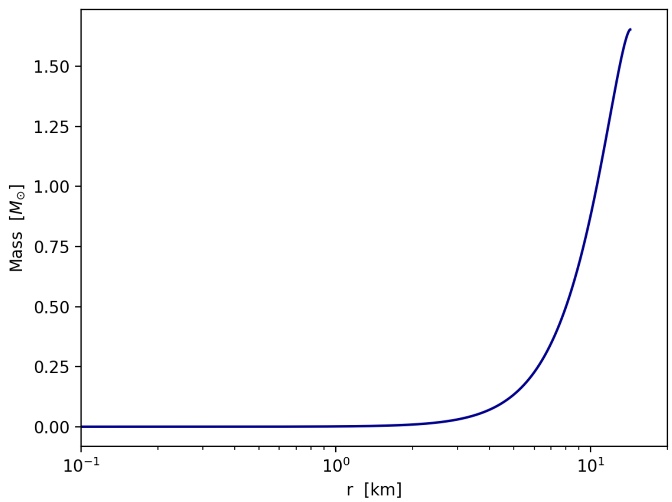

Figure 1: Mass profile for TOV-model

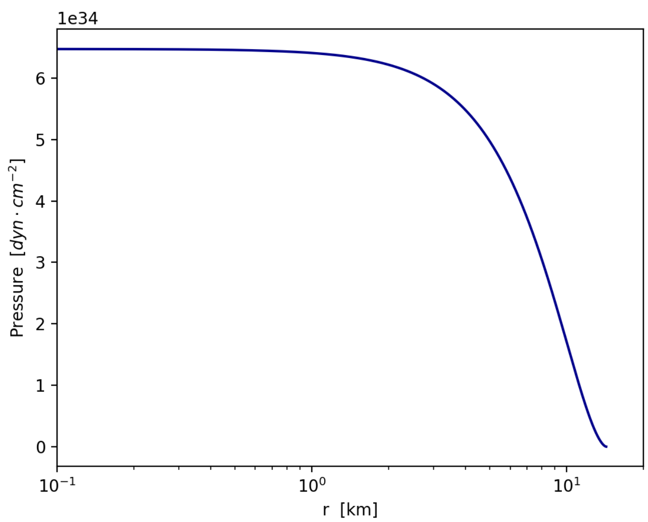

Figure 2: Pressure profile for TOV-model

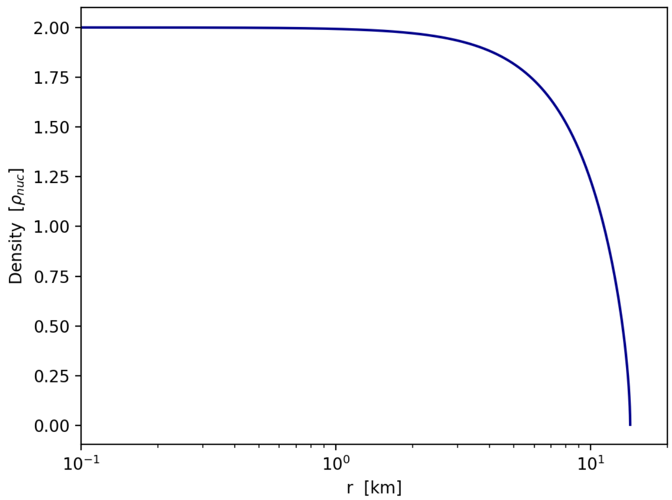

Figure 3: Density profile for TOV-model

This simulation for a neutron star shows that the TOV-equations can be used to simulate a neutron star to values representative of observed values. The mass, pressure, and density profiles are as expected for a spherically symmetric body that is in gravitational equilibrium.To further understand the limits and implications of the model, we will now look at the effect of parameters on our results.

3.2 Mass-Radius Relation

An essential initial parameter of the TOV-equations is the central density of the star. For the same equation of state, changing the central density alters the star's mass and radius at which the pressure becomes zero. By running the simulation for many different central densities, we can see how these values are altered.

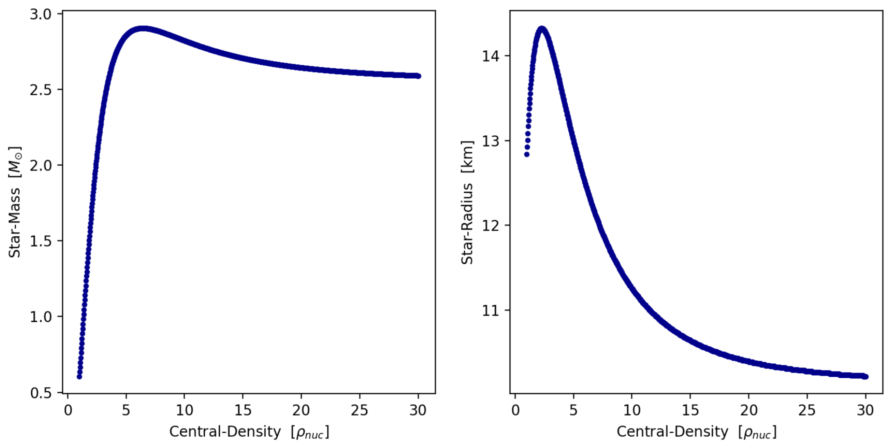

Figure 4: Neutron Star Mass (left) and Radius (right) for different central densities

This also allow us to directly plot a relation between the star mass and star radius.

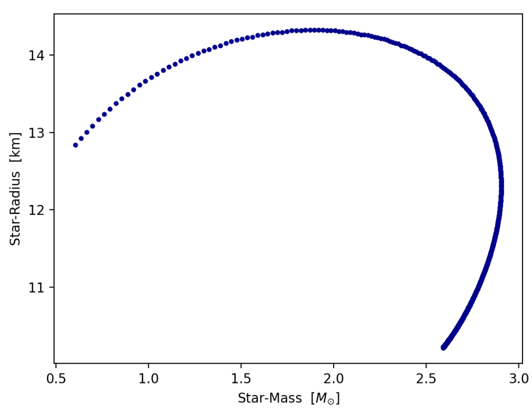

Figure 5: Neutron Star Radius-Mass relation

We can see that there are central density values, at which the mass and radius are largest. After this point the star radius decreases very quickly, while the mass does not drop as quickly. To find out the maximum mass and radius our model can obtain, we want to look at the central density values in this region more closely.

We can see that the evolution of mass, pressure, and density for different initial central densities is very similar, only showing significant mass differences when reaching the edge of the sphere.

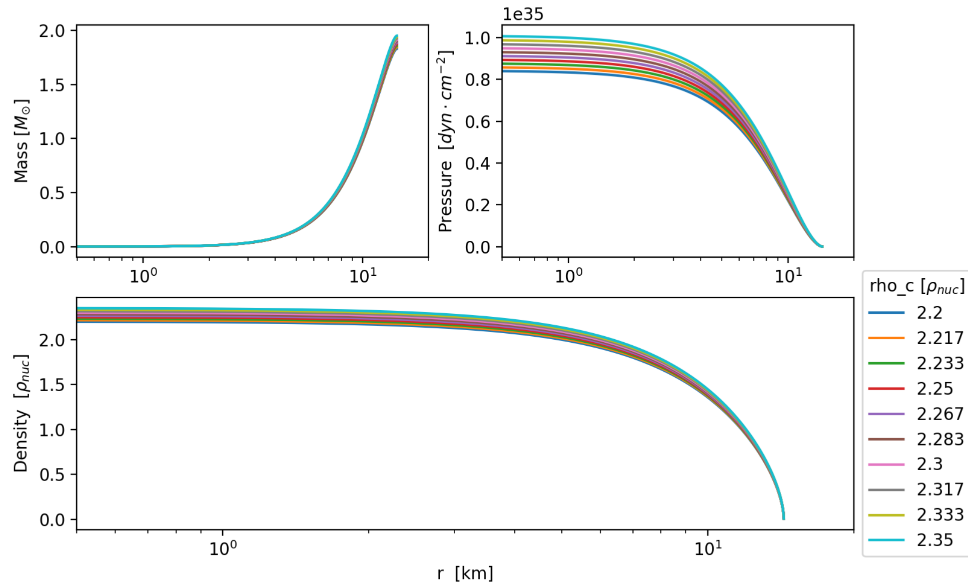

Figure 6: Mass, Pressure, and Density profiles for different central densities

We see that the maximum radius is achieved for a central density in the range of 2.25 to 2.32 nuclear densities. With a larger grid and smaller stepsize, we are able to see the results. Running over 1 million steps per star, with a stepsize of 1cm, the following is obtained.

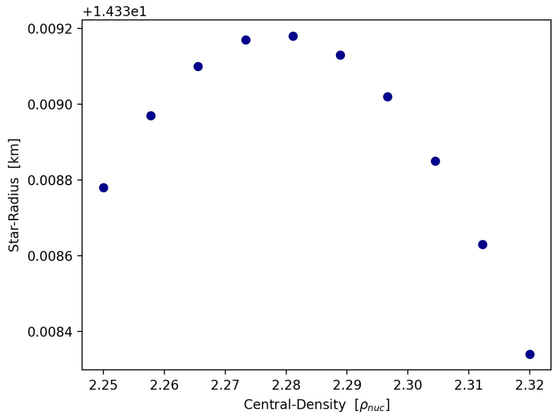

Figure 7: Star-Radius values for different central-densities

The maximum radius for our model with a polytropic equation of state is achieved at a central density of 2.28889 nuclear densities. The results are summarized below.

Central Density	$2.28889\rho_{nuc}$
Star Radius	14.339 km
Star Mass	$1.903M_\odot$

Table 3: Results for Radius-Maximizing TOV-model

Similarly, we can find the point at which the star's mass is maximized. For the same simulation parameters, the results are below:

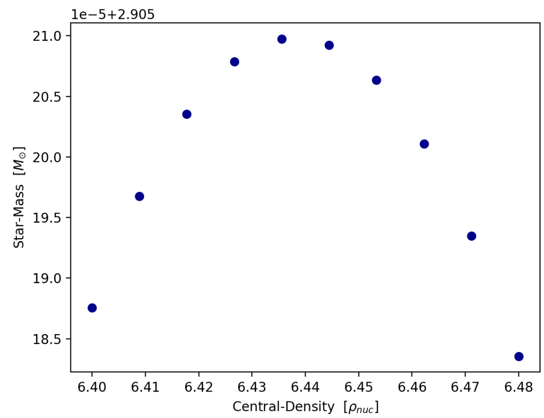

Figure 8: Star-Mass values for different central-densities

Central Density	$6.43556\rho_{nuc}$
Star Radius	12.323 km
Star Mass	$2.9051M_\odot$

Table 4: Results for Mass-Maximizing TOV-model

3.3 Numerical Parameters: Step-size

After having looked at the significance of some physical parameters, we will now look at the numerical parameters that influence our model. As the TOV-equations are solved numerically, we have to define a grid and solve the equations step by step. The step-size therefore is critical, as we need to consider the weigh off between accuracy and computational difficulty.
For step-sizes from 1 cm to 10^6 cm we can see how the star's mass and radius change. Only for very large step-sizes do we notice a significant deviation from the most accurate small step-size model.

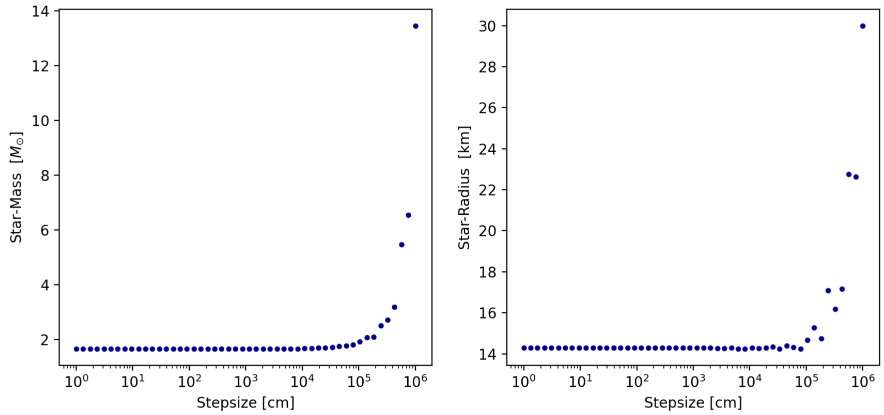

Figure 9: Neutron Star Mass (left) and Radius (right) values for different simulation step-sizes

The mass, pressure, and density profiles no longer represent an accurate model for the largest step-sizes, as expected.

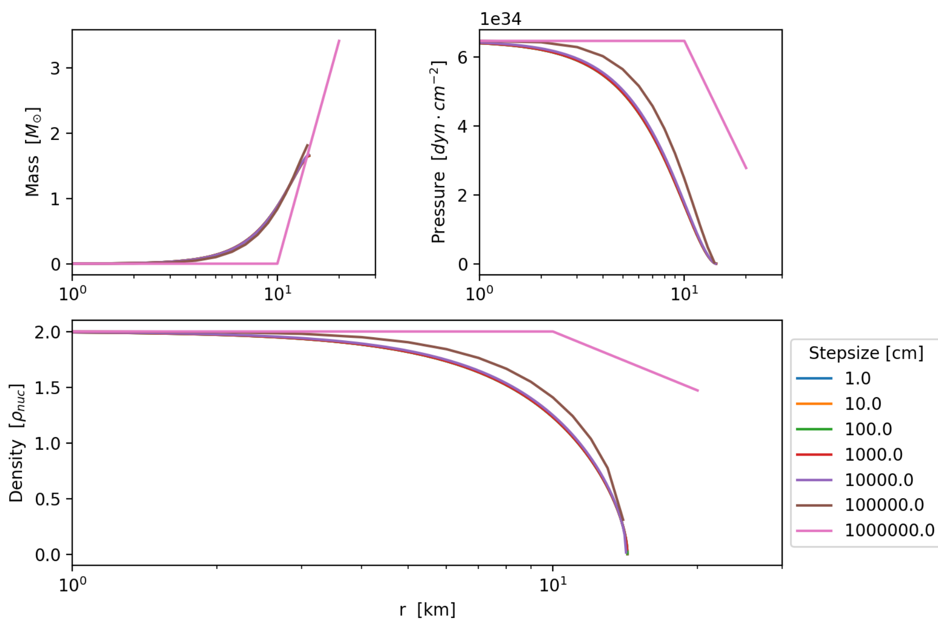

Figure 10: Mass, Pressure, and Density profiles for different simulation step-sizes

3.4 Numerical Parameters: Runge-Kutta

In our numiercal simulation, we have to use an integrator as defined in Section 2. We can see how different Runge-Kutta methods alter the accuracy of our results by seeing how they deviate from each other. As we can see below, the deviation in results is very minuscule and order's one, two, and three do not deviate much from the most accurate fourth order.
For our representative model used above we obtain the following star mass and radius values for each Runge-Kutta order.

RK-Order	Radius [km]	Mass [$M_\odot$]
1	14.29135	1.65393948
2	14.29155	1.65392319
3	14.29145	1.65392584
4	14.29155	1.65392320

Table 5: Star Radius and Mass values for different Runge-Kutta Orders

We can see how the mass profile changes relative to the fourth order method, noting that although very similar, the second order method is not exactly the same as the fourth order.

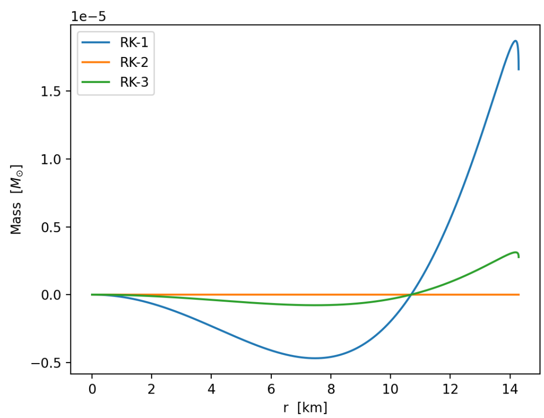

Figure 11: Deviation of Mass-Profile for RK method 1,2,3 from RK-4 method

As the complexity of the model increases with each order, the method becomes more computationally expensive. This means the runtime of each simulation is longer. In order to investigate this and to get a relevant time statistic, we run 1000 simulations for each Runge-Kutta method and compare the times relative to the fourth order method.

RK-1 : RK-4	0.257 ± 0.132
RK-2 : RK-4	0.566 ± 0.200
RK-3 : RK-4	0.757 ± 0.304

Table 6: Star Radius and Mass values for different Runge-Kutta Orders

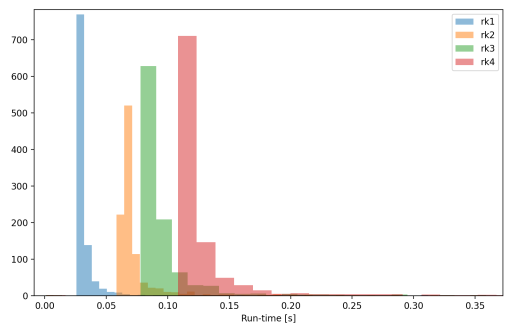

Figure 12: Histogram of Runtimes for Different RK-methods

As we can see in the plot, there are a few outliers. That is why the median was used to determine this statistics. The deviation in the statistics is quite large, which can also be due to the fact that the core used for the simulations is not very powerful or consistent. For more exact results, more runs should be done.

3.5 Polytropic Equation of State Parameters

To solve the TOV-equations, an equation of state must be assumed. For an accurate model, the polytropic equation of state is used as defined in Section 2. This equation rests on two constants, the adiabatic exponent γ and the adiabatic coefficient K. We will now look at how these two parameters influence our simulation results.

3.5.1 Adiabatic Exponent: Gamma

We again assume the standard input defined above. We see that for a γ less than 2.6 the model does not reach a representative mass by the time the pressure has dropped to zero. Similarly for a γ larger than 3.1 we see that pressure is initially so high, that the star's mass and radius achieved are no longer representative.

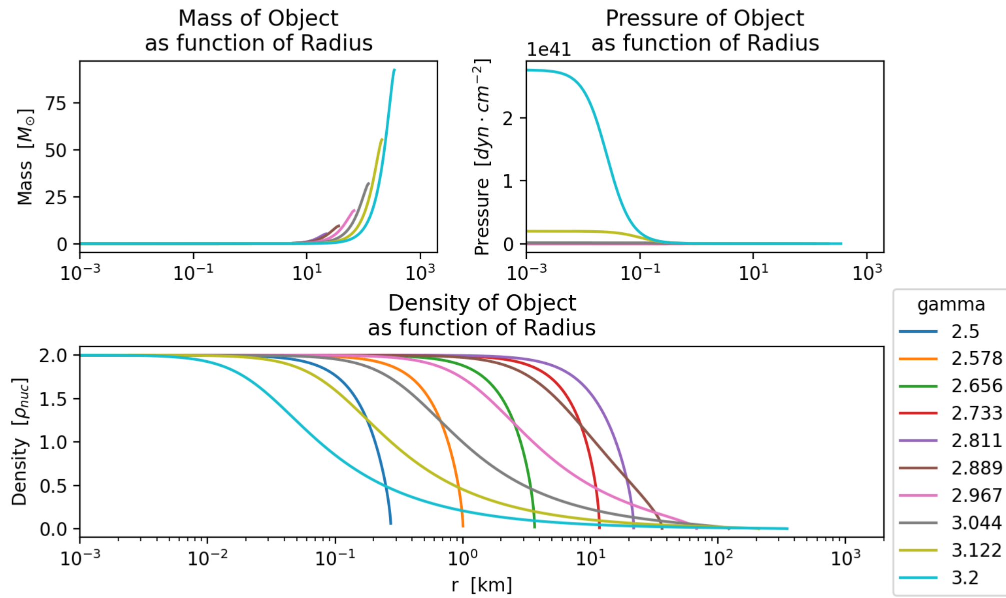

Figure 13: Mass, Density, and Pressure Profile for different Gamma-values

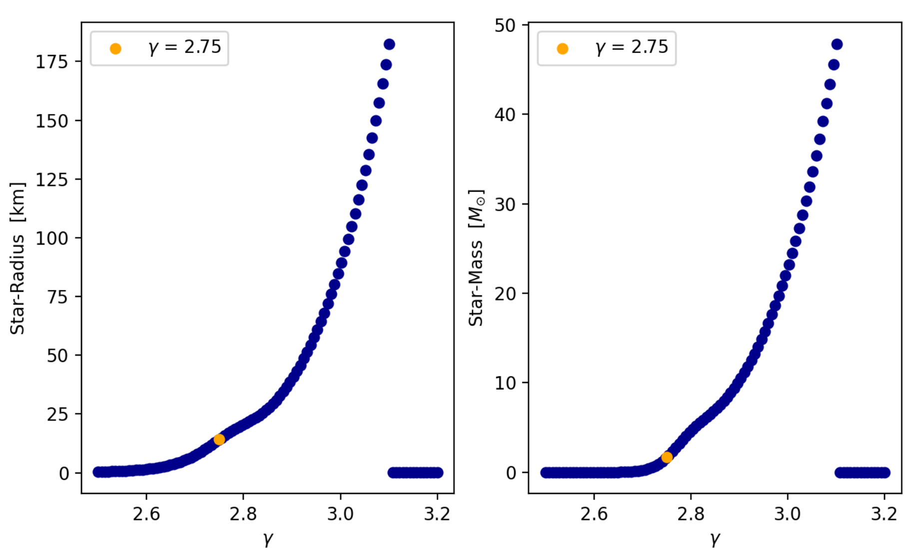

Figure 14: Star Mass and Radius for different Gamma-values

3.5.2 Adiabatic Constant: K

Similarly to γ, we see that there is a range of K values for which the results are representative. For values smaller than $1 \cdot 10^{-6}$ the simulation results in a very low star radius and mass. For values larger than 0.5, the model also does not achieve representative values because the initial pressure is very high.

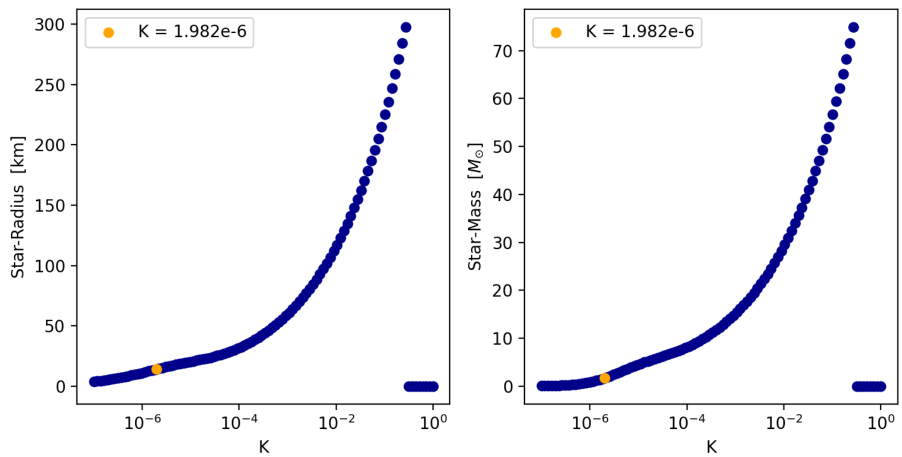

Figure 15: Star Mass and Radius for different K-values

3.6 Implementation of Specific Internal Energy

For a polytropic fluid, we can determine the specific internal energy ϵ from the first law of thermodynamics. In the TOV-equations, this value is represented in the pressure and mass differential equations. If we do not ignore this, instead determining ϵ for every step, we obstain slightly altered results.

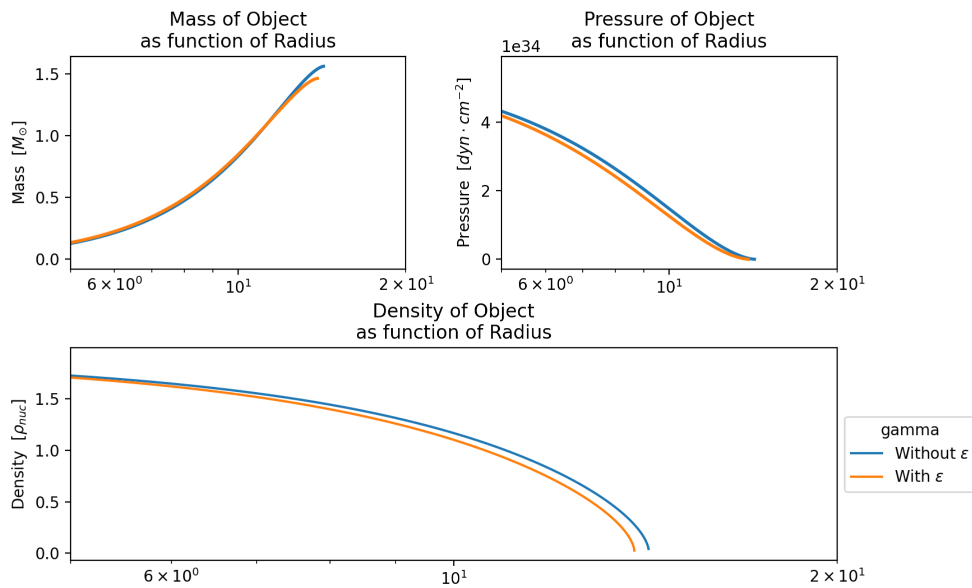

Figure 16: Mass, Density, Pressure Profile with and without specific internal energy consideration

	Radius [km]	Mass [$M_\odot$]
Without ϵ	14.253	1.560
With ϵ	13.888	1.462

Table 7: Star Radius and Mass values with and without γ

We can see that when we account for the specific internal energy in the TOV-equations, the pressure drops more quickly and a lower star radius is achieved. The star only has about 93.7 percent the mass and 97.4 percent the radius when we include ϵ in the equations.

3.7 Newtonian Limit of TOV-Equations

As the TOV-equations are derived in a General-Relativistic framework, using the interior Schwarzschild metric, we can compare the results to a Newtonian approximation. The TOV-equations simplify to:

$$\frac{dP(r)}{dr} = \frac{-Gm\rho}{r^2} \tag{22}$$

$$\frac{dm}{dr} = 4\pi\rho r^2 \tag{23}$$

We see that the Newtonian pressure profile drops a lot less quickly, so the mass and radius of the object are significantly larger than for the GR model.

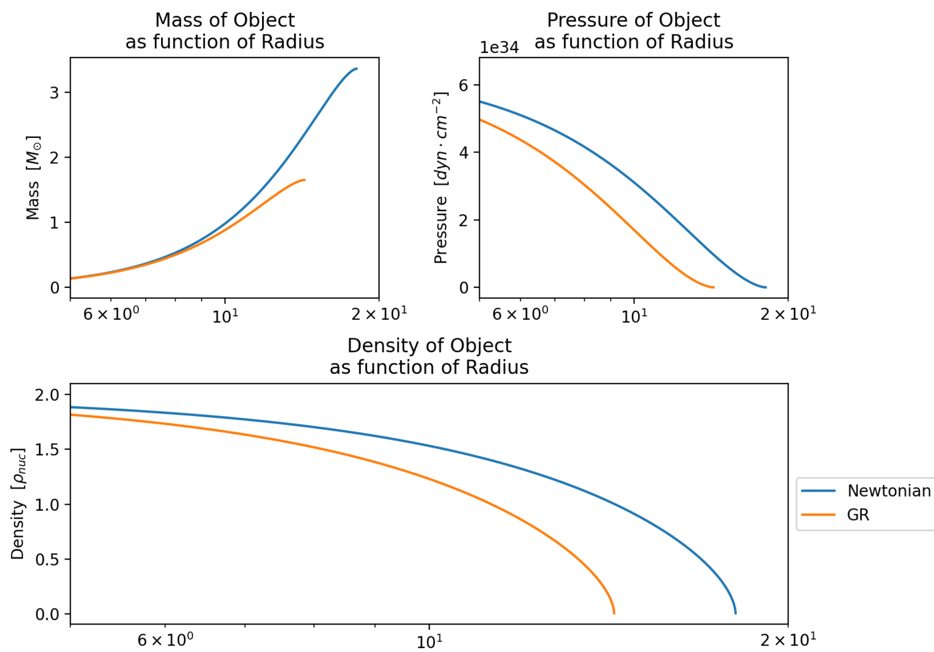

Figure 17: Mass, Density, Pressure Profile for GR and Newtonian Model

	Radius [km]	Mass $[M_\odot]$
GR	14.294	1.654
Newtonian	18.073	3.367

Table 8: Star Radius and Mass values with and without γ

The GR model achieves only 49.13 percent of the Mass and 79.09 percent of the Radius of the Newtonian model.

4 Summary and Evaluation

In this investigation I applied my numerical, python-based solver of the TOV-equations to model a spherically symmetric body that is in gravitational equilibrium. The goal of this was to create an accurate model to describe a neutron star. I was successful is creating a representative model, for a star with a mass of $1.65 M_{\odot}$ and a radius of 14.29 km. These values are well within the accepted range of neutron stars.

Furthermore, I was able to obtain a Mass-Radius relation for our bodies by altering the initial central density. By doing this I was able to constrain the maximum mass and radius achievable with this model. I find the maximum mass for a body in this model to be $2.91 M_{\odot}$. Similarly, the maximum radius was determined to be 14.34 km. We also considered numerical implications of the TOV-solver. The step-size and integration method are very important for the accuracy and computational cost of the model. Finally, we looked at parameters describing the equation of state, the internal energy, and the Newtonian limit and their influence on the TOV model.

Further investigations from this project could entail applying different equations of state to the model or including scalar fields in the TOV-equations.

Sources

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- C. D. Ott. Static Spherically-Symmetric Stellar Structure in General Relativity. 2013.
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