14.385 Nonlinear Econometric Analysis

(Causal) matrix completion

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Outline

- Setup: Filling in missing entries in a matrix $\boldsymbol{Y}$.
    1. Recommender systems.
    2. Counterfactual outcomes in panel data.
- Recap: Singular value decompositions and principal components.
- Empirical risk minimization methods.
- Nearest neighbor methods.
- Missingness assumptions.
- Reweighting.
- The synthetic nearest neighbor algorithm.

Takeaways for this part of class

- Typical assumption:
  Y is a sum of a low rank matrix A and idiosyncratic noise E.

- Any matrix has a singular value decomposition.
  Principal components correspond to the largest singular values.

- Popular methods for matrix completion:
  1. *Empirical risk minimization*.
  2. *Nearest neighbors*.

- Standard methods suffer from bias with non-random missingness.

- For (conditionally) missing at random data,
  reweighting can provide a solution.

- An algorithm for more general missingness is *Synthetic nearest neighbors*.

Setup

Standard algorithms

Missingness assumptions

Synthetic nearest neigbhors

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Setup

- Random matrices with m rows, n columns:
  - Latent matrix A.
  - Error matrix E, where $E[E|A] = \mathbf{0}$.
  - Outcome matrix $Y = A + E$, thus $E[Y|A] = A$.
  - Observability matrix D.
  - Probability of observability: $P = E[D]$.
- Goal: Estimate the entries of the latent matrix A based on observations $(Y_{ij} \cdot D_{ij}, D_{ij})$.
- Typical loss function:

$$\frac{1}{m \cdot n} \sum_{i,j} \left(\hat{A}_{ij} - A_{ij} \right)^2 .$$

Interpretations

- Recommender systems:
  - Rows i index individuals, columns j index movies.
  - Y_{ij} are movie ratings by individual i for movie j.
  - D_{ij} are indicators for whether a movie was rated by an individual.
  - Goal: Recommend movies that would receive high ratings.
- Panel data causal inference (cf. synthetic controls):
  - Rows i index cross-sectional units, columns j index time-periods.
  - Y_{ij} are potential outcomes absent treatment.
  - D_{ij} are indicators for untreated units.
  - Goal: Recover missing Y_{ij} to recover causal effects.

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Reminder: Singular value decomposition

- Any real valued matrix A with m rows, n columns, rank $k = \text{rank}(A)$ can be decomposed as

$$A = U \cdot \Sigma \cdot V = \sum_{l=1}^{k} \sigma_l \cdot u_l \cdot v_l'.$$

  - U is an $m \times k$ matrix with orthonormal columns u_l.
  - V is an $n \times k$ matrix with orthonormal columns v_l.
  - Σ is a $k \times k$ diagonal matrix with entries σ_l of decreasing magnitude.

- Special case: Diagonalization of square matrices A.
- Consider the largest singular values σ_l:
  - Principal components: $\sigma_l \cdot u_l$.
  - Low-rank approximation: $A \approx \sum_{l=1}^{\kappa} \sigma_l \cdot u_l \cdot v_l'$, where $\kappa < k$.

Empirical risk minimization (ERM) methods

- Minimize average prediction error for observed outcomes:

$$\hat{A} = \operatorname*{argmin}_{a} \sum_{i,j} D_{ij} \cdot \left(a_{ij} - Y_{ij}\right)^2 + \lambda \cdot \boldsymbol{Reg}(a).$$

- Here ***Reg*** is one of several possible regualarization penalties, λ is a tuning parameter.
- Popular choice: Nuclear norm (or trace norm).

$$\boldsymbol{Reg}(a) = \operatorname{tr}(\sqrt{a' \cdot a}) = \sum_{l} \sigma_l(a).$$

The $\sigma_l(a)$ are the singular values of a.

$\Rightarrow$ Lasso penalty for the singular values of $\hat{A}$.

$\Rightarrow$ *SoftImpute* algorithm

- Variant: Rather than penalizing $\hat{A}$, constrain $\hat{A}$ to be low rank.

Nearest neighbor methods

- Consider a specific i, j with $D_{i,j} = 0$.
- Find a set $\mathcal{I}$ of k rows i', such that
  1. $D_{i',j} \neq 0$.
  2. Row i' is "similar" to row i.
- "Similar" often means a small distance of the vector of observed values,

$$\sum_{j'} D_{ij'} \cdot D_{i'j'} \cdot \left(Y_{ij'} - Y_{i'j'}\right)^2 .$$

- Impute an estimate for Y_{ij} as

$$\hat{Y}_{ij} = \frac{1}{k} \sum_{i' \in \mathcal{I}} Y_{i'j}.$$

Missingness assumptions

1. **Missing completely at random** (MCAR):

$$D_{ij}|Y \sim^{iid} Ber(p).$$

2. **Missing at random** (MAR):

$$D_{ij}|Y,X \sim Ber(P_{ij}(X)),$$

   independently across i,j,
   where X are observable controls, and $P_{ij}(X) > 0$.

3. **Missing not at random** (MNAR):
   - D and Y are not independent,
   - D_{ij} and $D_{i'j'}$ are not independent,
   - $P_{ij} = 0$ is allowed.

Practice problem

- Suppose MCAR holds.
  Consider any empirical risk minimization (ERM) algorithm.
  What is the expectation of the objective function for such an algorithm?

- Suppose MAR holds.
  How could you modify empirical risk minimization, to avoid biases?

Reweighting under MAR

- Suppose that P takes the form

$$P_{ij} = g\left(X_i \cdot \beta_x + W_j \cdot \beta_w + \delta_i + \gamma_j\right),$$

where $g(\cdot)$ is a link function; e.g. the logistic $g(x) = \frac{\exp(x)}{1+\exp(x)}$.

- We can estimate P by logistic regression of D_{ij} on X_i, W_j, and row and column fixed effects.

- Reweighted ERM:

$$\hat{A} = \underset{a}{\operatorname{argmin}} \sum_{i,j} \frac{D_{ij}}{\hat{P}_{ij}} \cdot \left(a_{ij} - Y_{ij}\right)^2 + \lambda \cdot Reg(a).$$

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A restricted form of MNAR

Assummptions:

1. **Low rank factor model**:
   $\text{rank}(A) = k < \min(m,n)$, so that

$$A = \sum_{l=1}^{k} \sigma_l \cdot u_l \cdot v_l'.$$

2. **Selection on latent factors**:

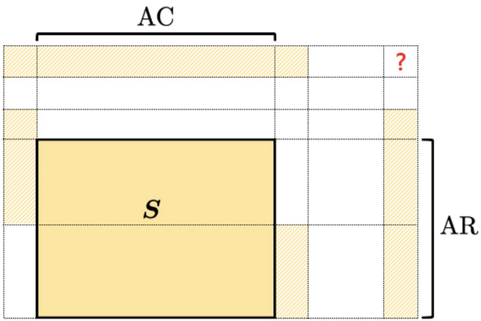
$$E[E|U,V,D] = 0.$$

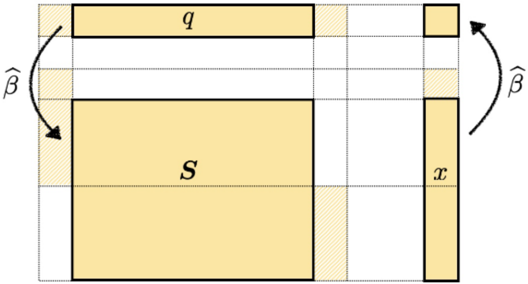
3. **Linear span inclusion**:
   Any set of k rows of U has full rank.

Identification

- Assumption 3 could be weakened, but holds generically.
- Immediate implication of these assumptions:
  - Fix a pair (i,j).
  - Let $\mathscr{I}$ be such that $D_{i'j} = 1$ for all $i' \in \mathscr{I}$ and $|\mathscr{I}| \geq k$.
  - Then there is a β such that
$$u_{i,\cdot} = \sum_{i' \in \mathscr{I}} \beta_{i'} \cdot u_{i',\cdot}.$$
  - Furthermore,
$$A_{ij} = \sum_{i' \in \mathscr{I}} \beta_{i'} \cdot E[Y_{i'j}|U,V,D].$$
- Thus:
  - Suppose we could estimate β.
  - Then we could impute
$$\hat{A}_{ij} = \sum_{i' \in \mathscr{I}} \beta_{i'} \cdot Y_{i'j}.$$

Synthetic nearest neighbors





Synthetic nearest neighbors (1)

Algorithm proposed by Agarwal et al. (2021);

1. Fix tuning parameter $\kappa \in \mathbb{N}$ (rank of approximations).
2. Consider some (i,j) for which we want to estimate A_{ij}.
3. Find a set of rows and columns AR and AC such that

$$D_{i'j'} = D_{i'j} = D_{ij'} = 1$$

   for all $i' \in AR$ and $j' \in AC$.
   Let S be the submatrix of Y corresponding to rows AR, columns AC.

4. Find the singular value decomposition

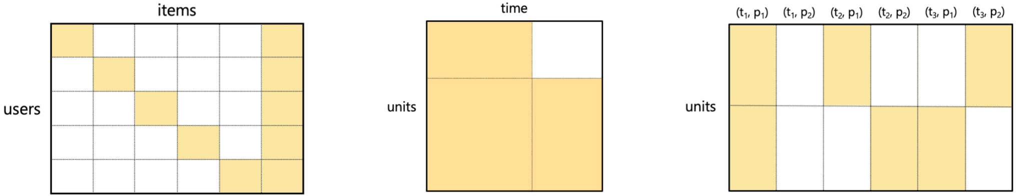
$$S = \sum_{l=1}^{k} \sigma_i \cdot \hat{u}_i \cdot \hat{v}_i'$$

Synthetic nearest neighbors (2)

5. Estimate
$$\hat{\beta} = \left(\sum_{l=1}^{\kappa} \frac{1}{\sigma_i} \cdot \hat{u}_i \cdot \hat{v}_i' \right) \cdot A_{i,AC}.$$
(Note we are truncating the sum at κ.)
6. Impute
$$\hat{A}_{ij} = \sum_{i' \in \mathscr{I}} \beta_{i'} \cdot Y_{i'j}.$$
7. Repeat for different rows AC, columns AR, and average.

Role of columns and rows could be switched, without affecting the estimate.

Typical patterns of missingness



(a) Recommender systems. **(b)** Panel data. **(c)** Sequential decision-making.

References

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- *Agarwal, A., Dahleh, M., Shah, D., and Shen, D. (2021). Causal matrix completion.* arXiv preprint arXiv:2109.15154