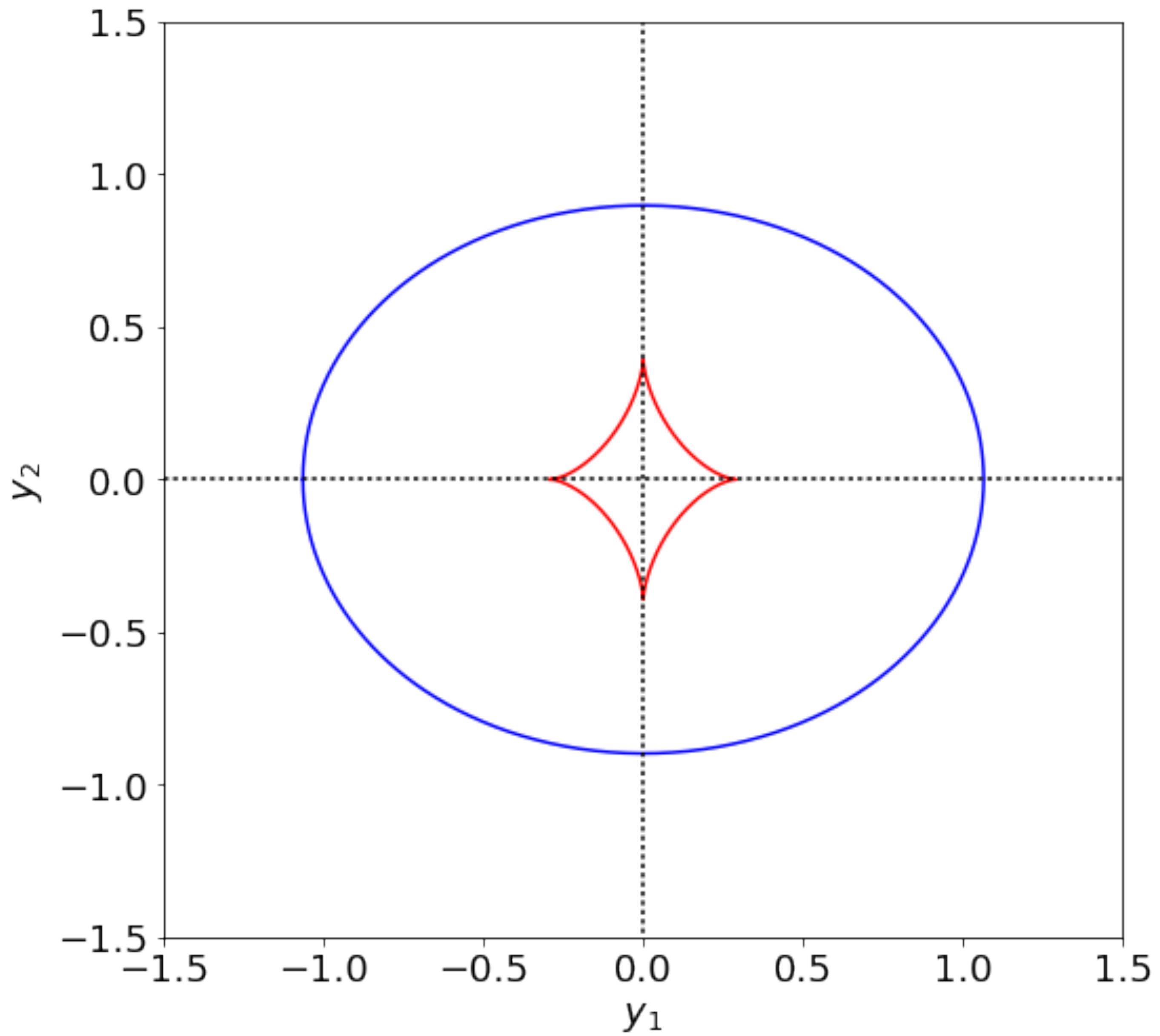
GRAVITATIONAL LENSING

21 - ELLIPTICAL MODELS, EXTERNAL PERTURBATIONS , SUBSTRUCTURES

Massimo Meneghetti

AA 2018-2019

CRITICAL LINE, CUT, CAUSTIC



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\begin{aligned}
y_{t,1} &= \frac{\sqrt{f}}{\Delta(\varphi)}\cos\varphi - \frac{\sqrt{f}}{f'}\operatorname{arcsinh}\left(\frac{f'}{f}\cos\varphi\right) \\
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$$f' = \sqrt{1-f^2}$$

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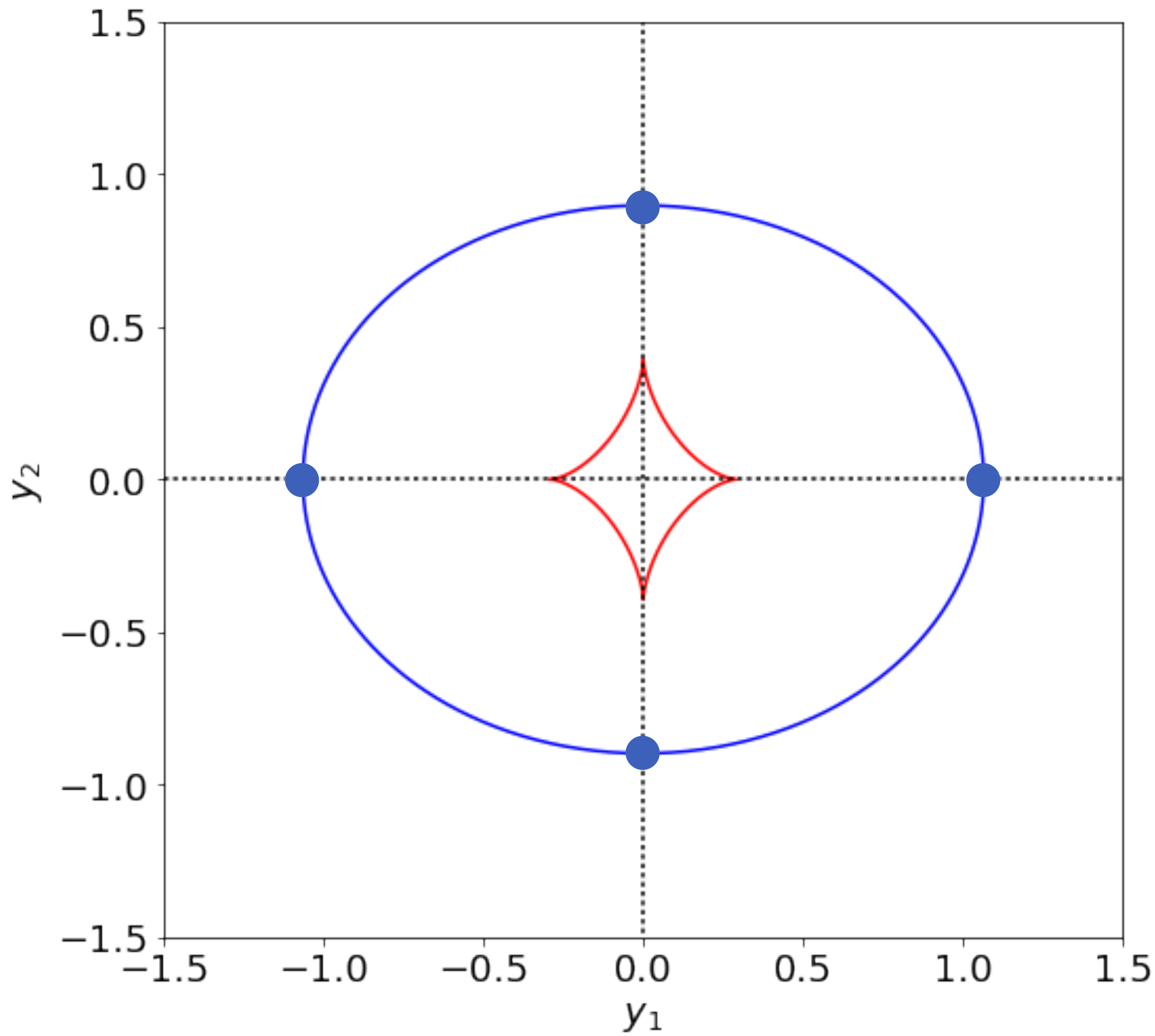

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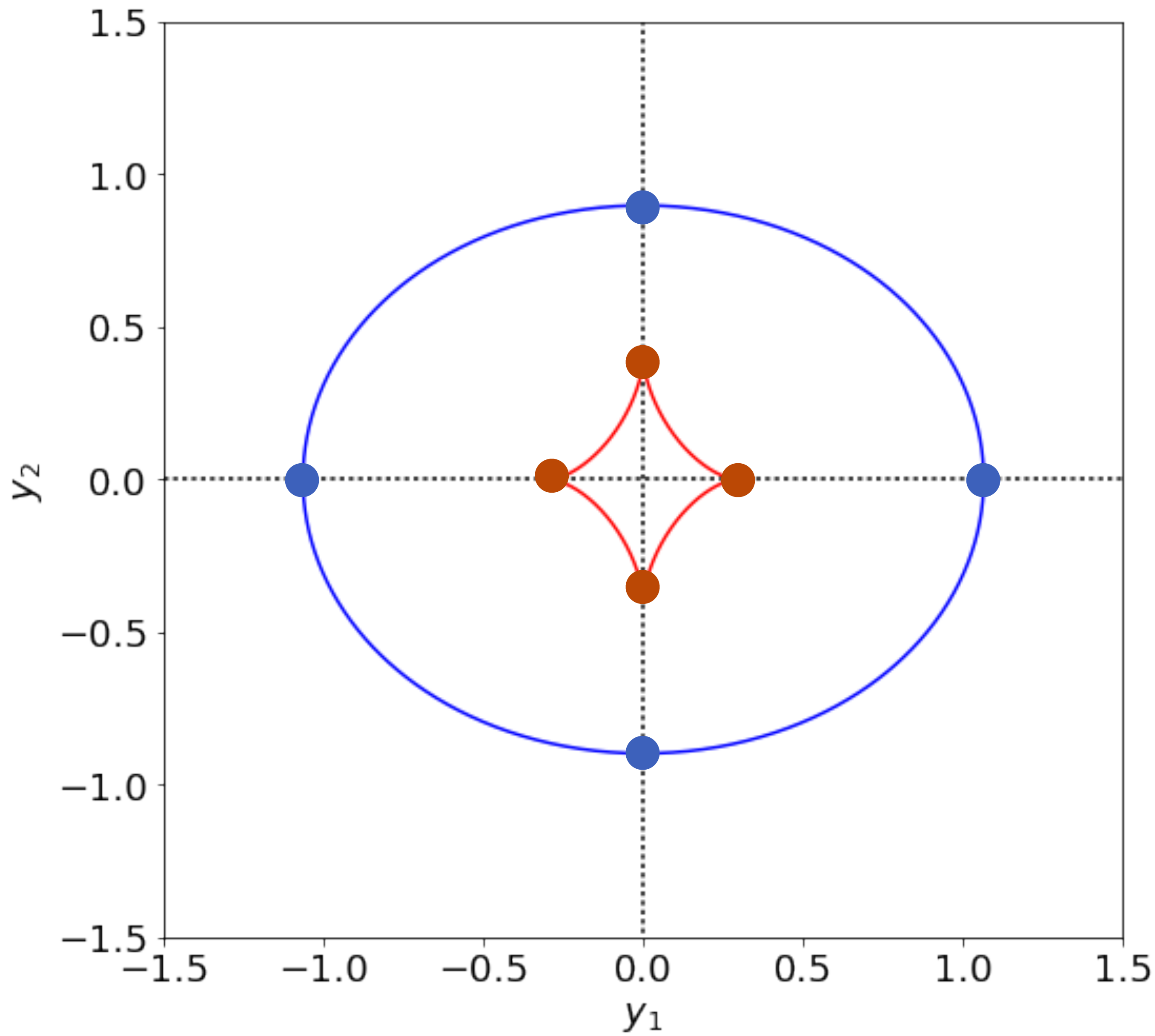

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CRITICAL LINE, CUT, CAUSTIC

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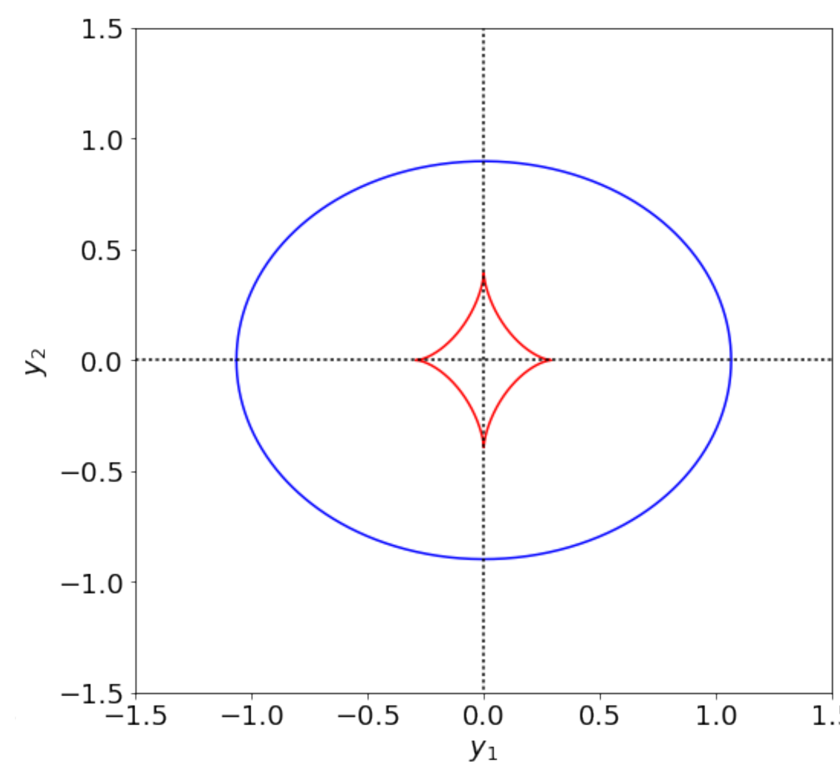
We can easily see that

$$s_{1,c} > s_{1,t}$$

independent on f.



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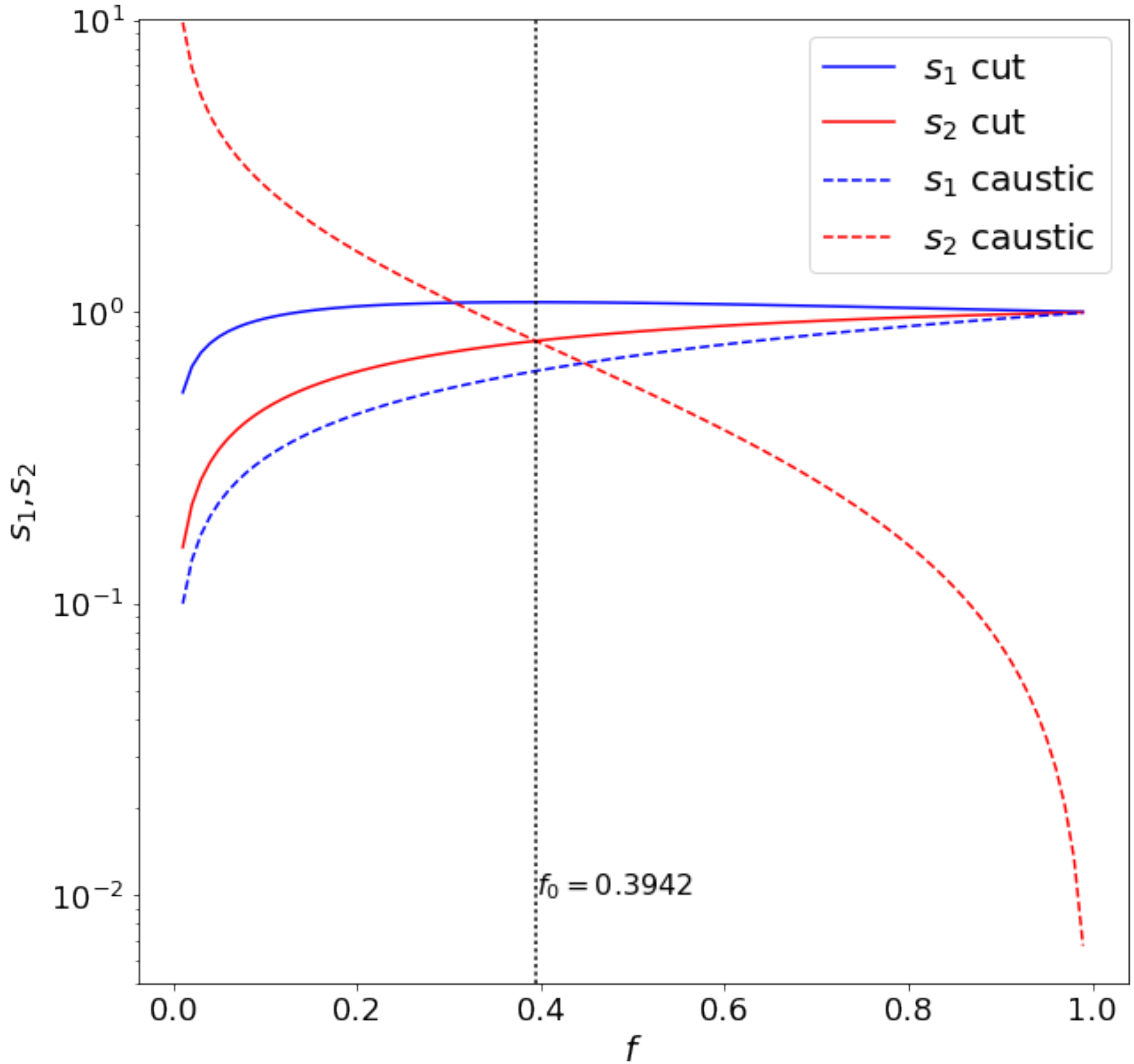

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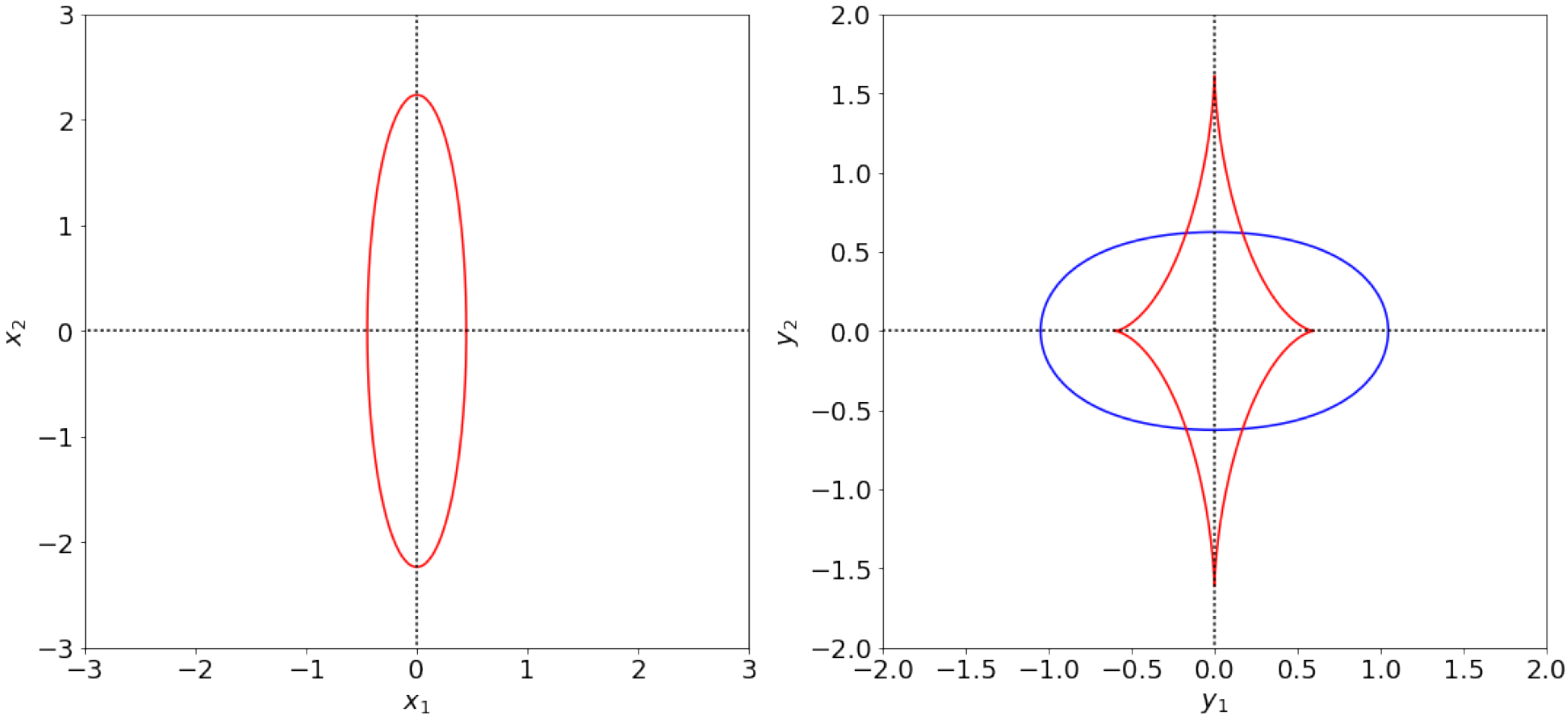

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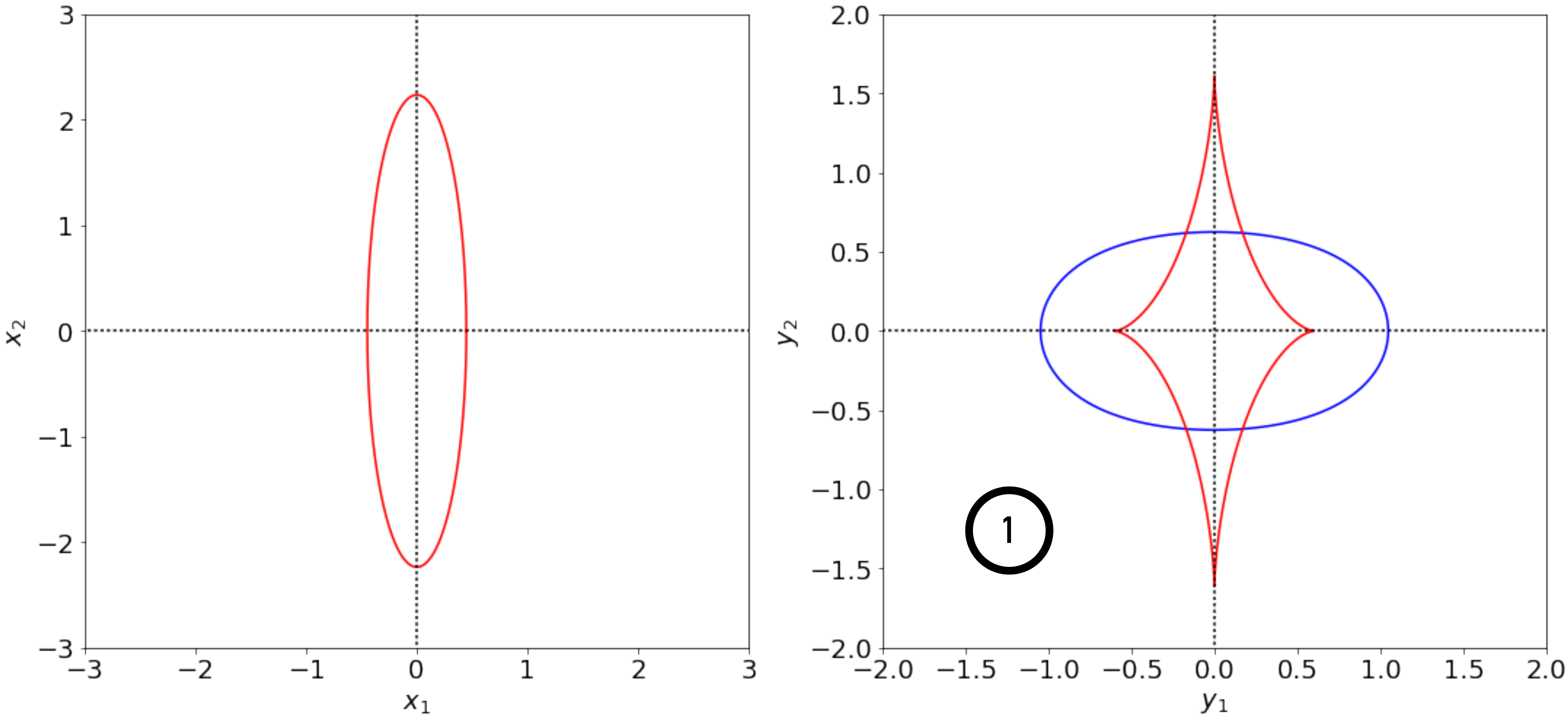

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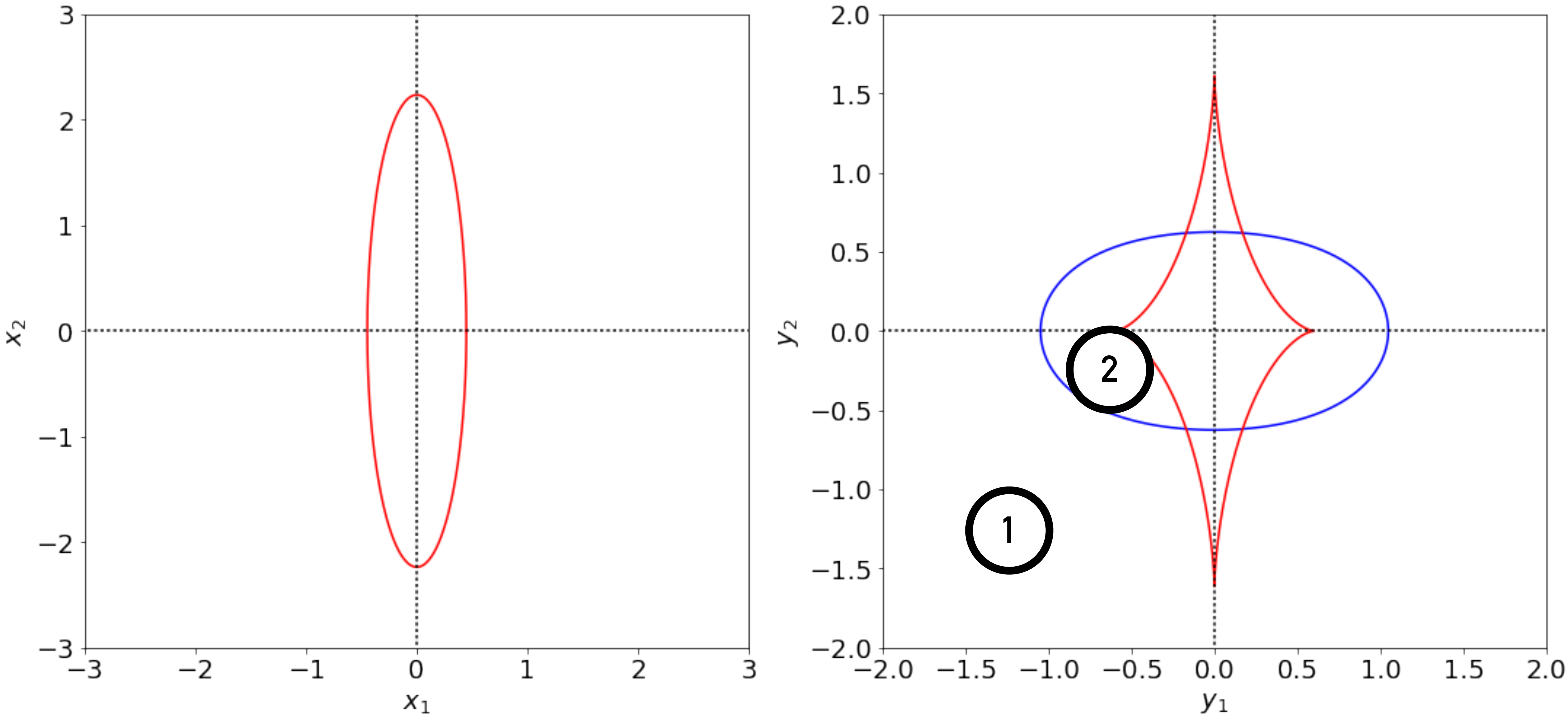

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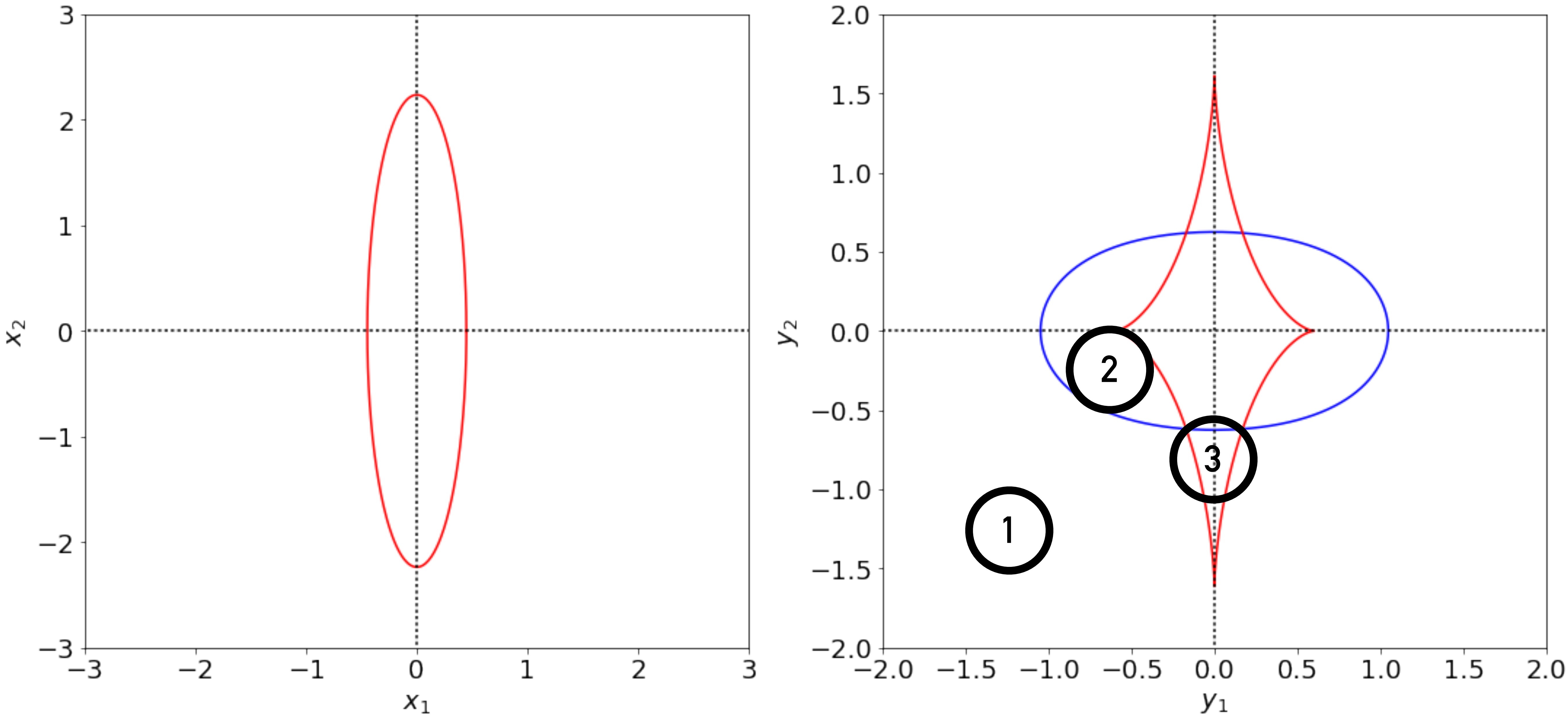

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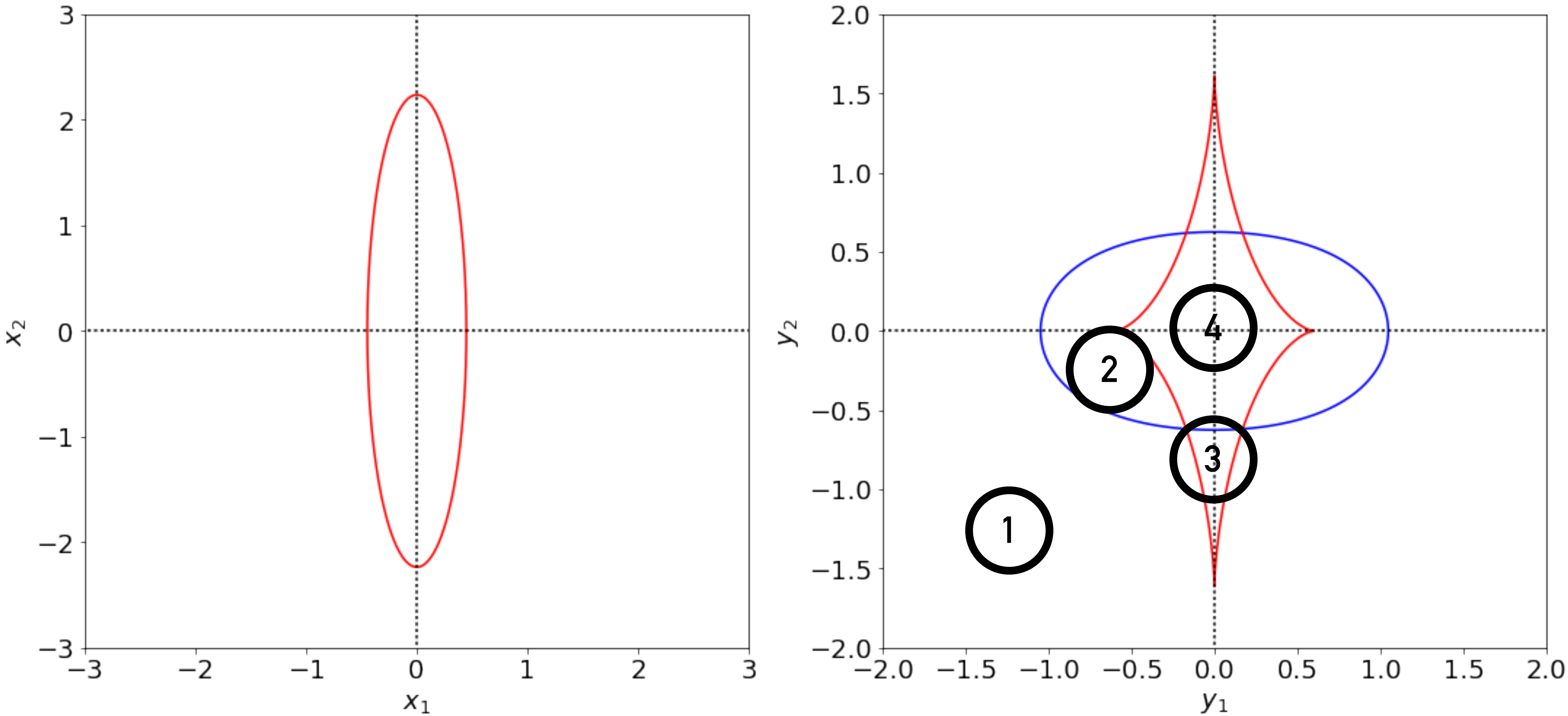


NON-SINGULAR-ISOTHERMAL-ELLIPSOID

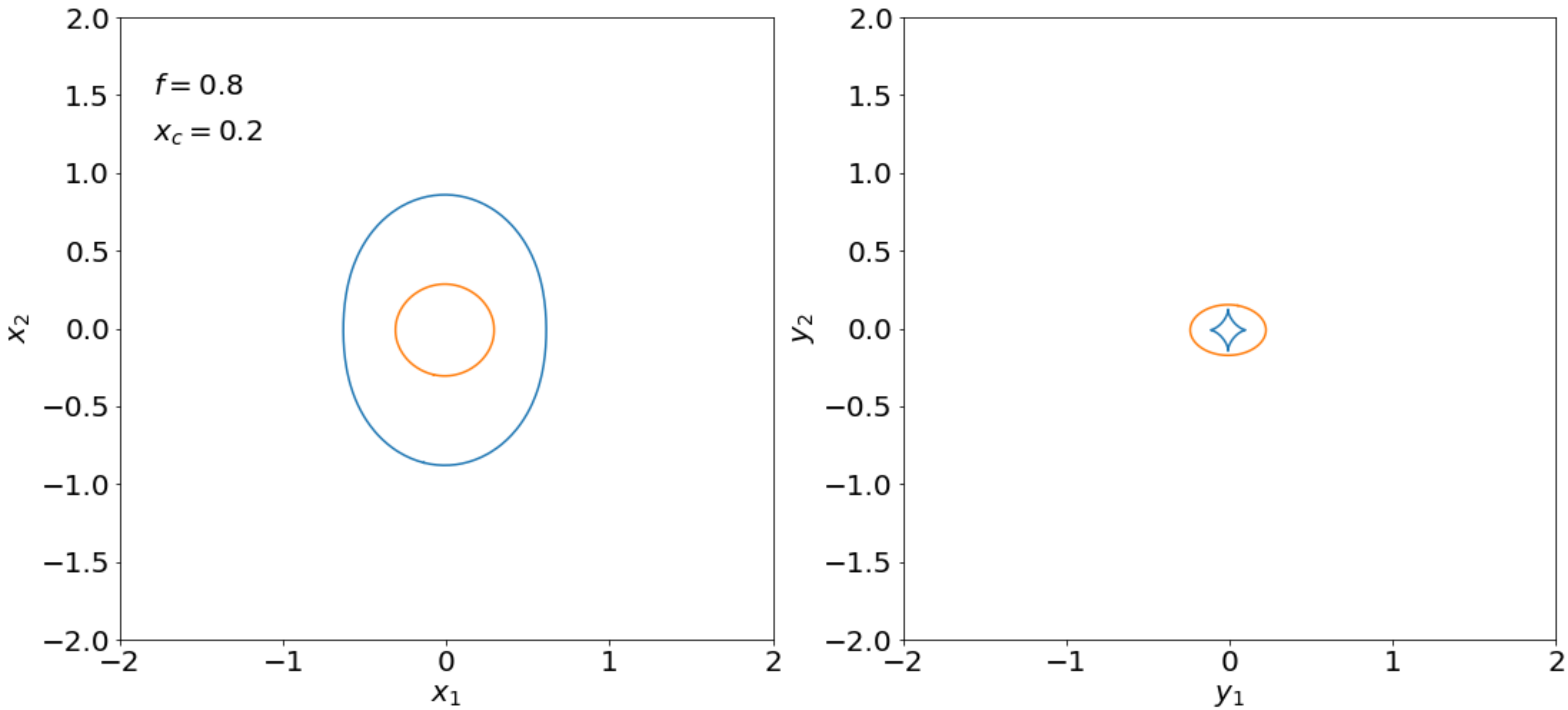
The SIE can be turned into a non-singular model by adding a core:

$$\Sigma(\vec{\xi}) = \frac{\sigma^2}{2G} \frac{\sqrt{f}}{\sqrt{\xi_1^2 + f^2 \xi_2^2 + \xi_c^2}}$$

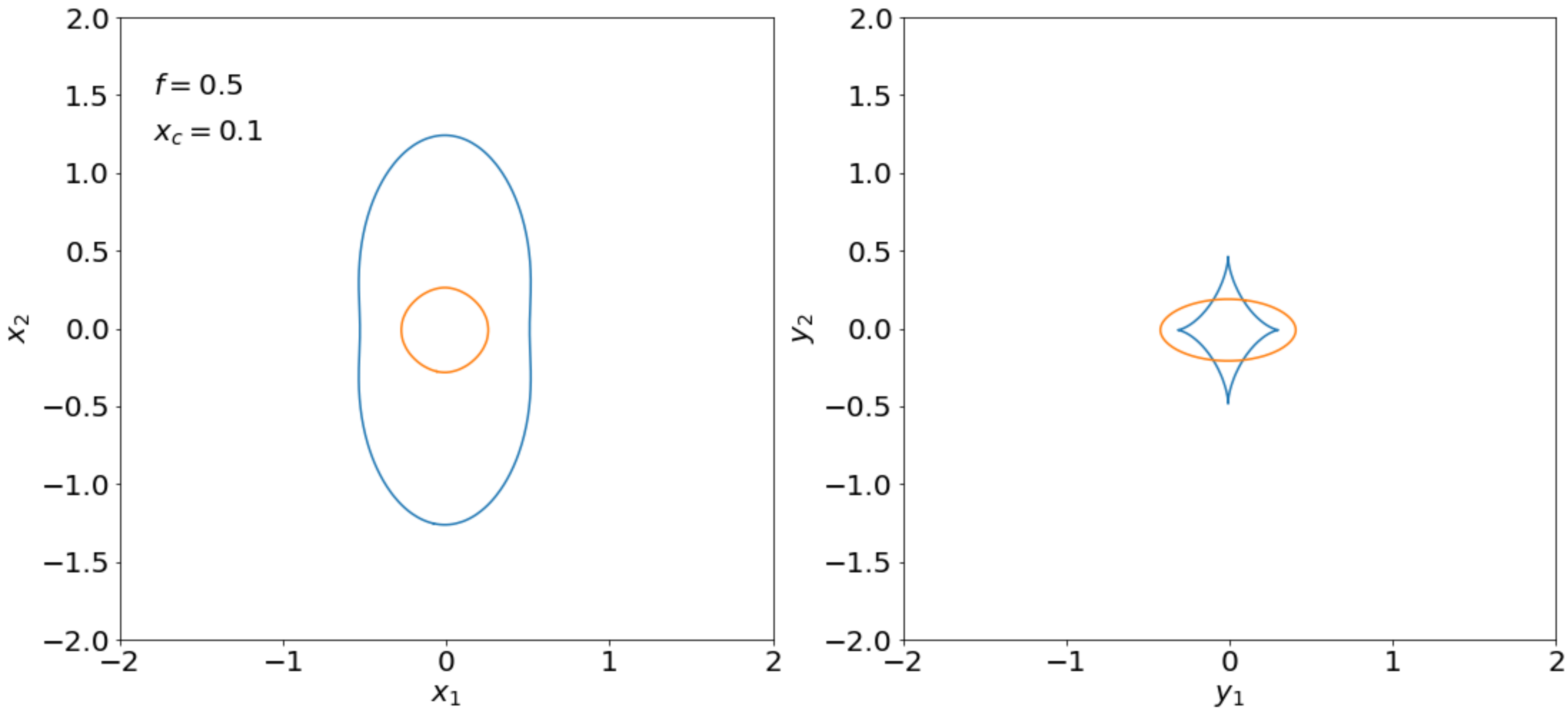
$$\kappa(\vec{x}) = \frac{\sqrt{f}}{2\sqrt{x_1^2 + f^2 x_2^2 + x_c^2}} .$$

In this case, the analytical treatment of the lens is much more complicated. We limit the discussion to the topology of the critical lines and caustics and infer information about the image multiplicities…

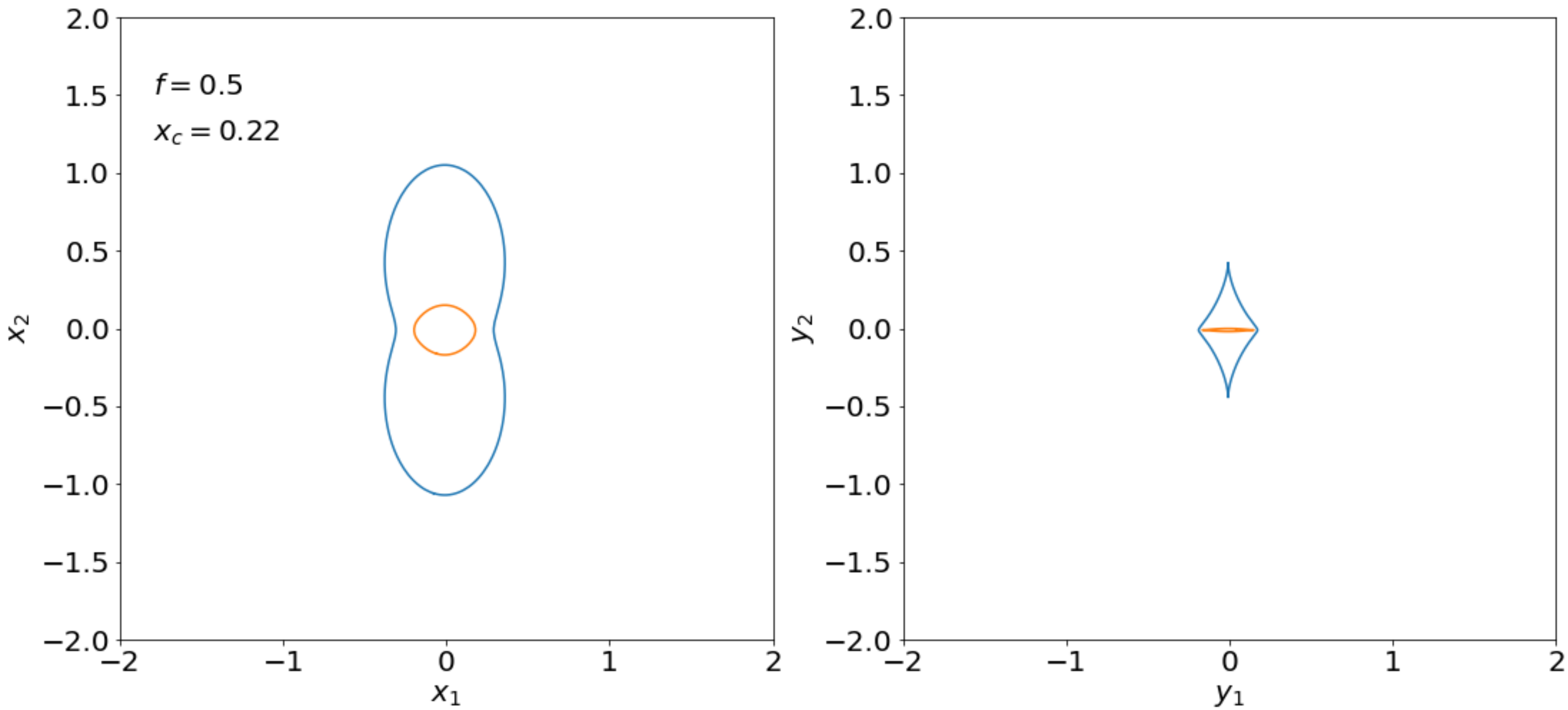
SMALL CORE RADIUS AND ELLIPTICITY



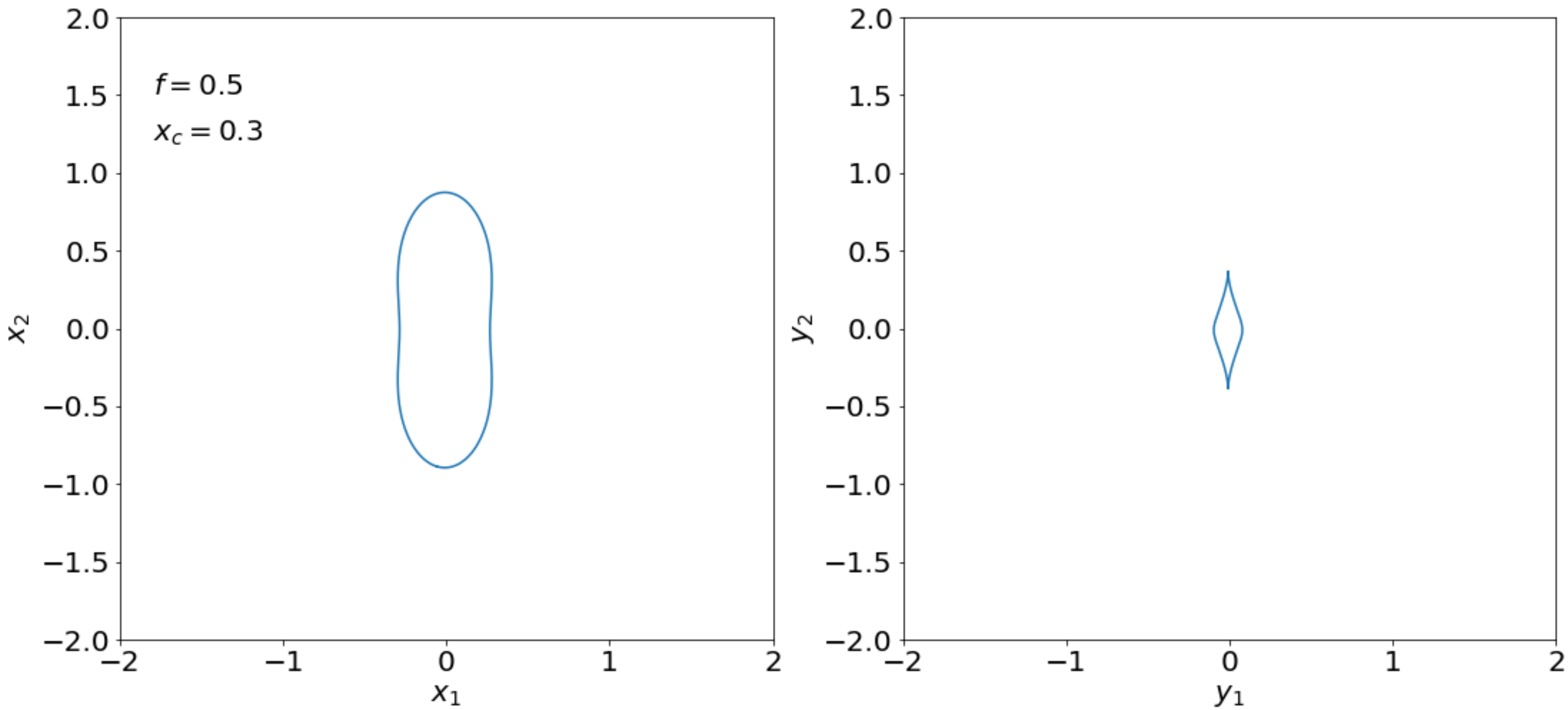
NAKED CUSP



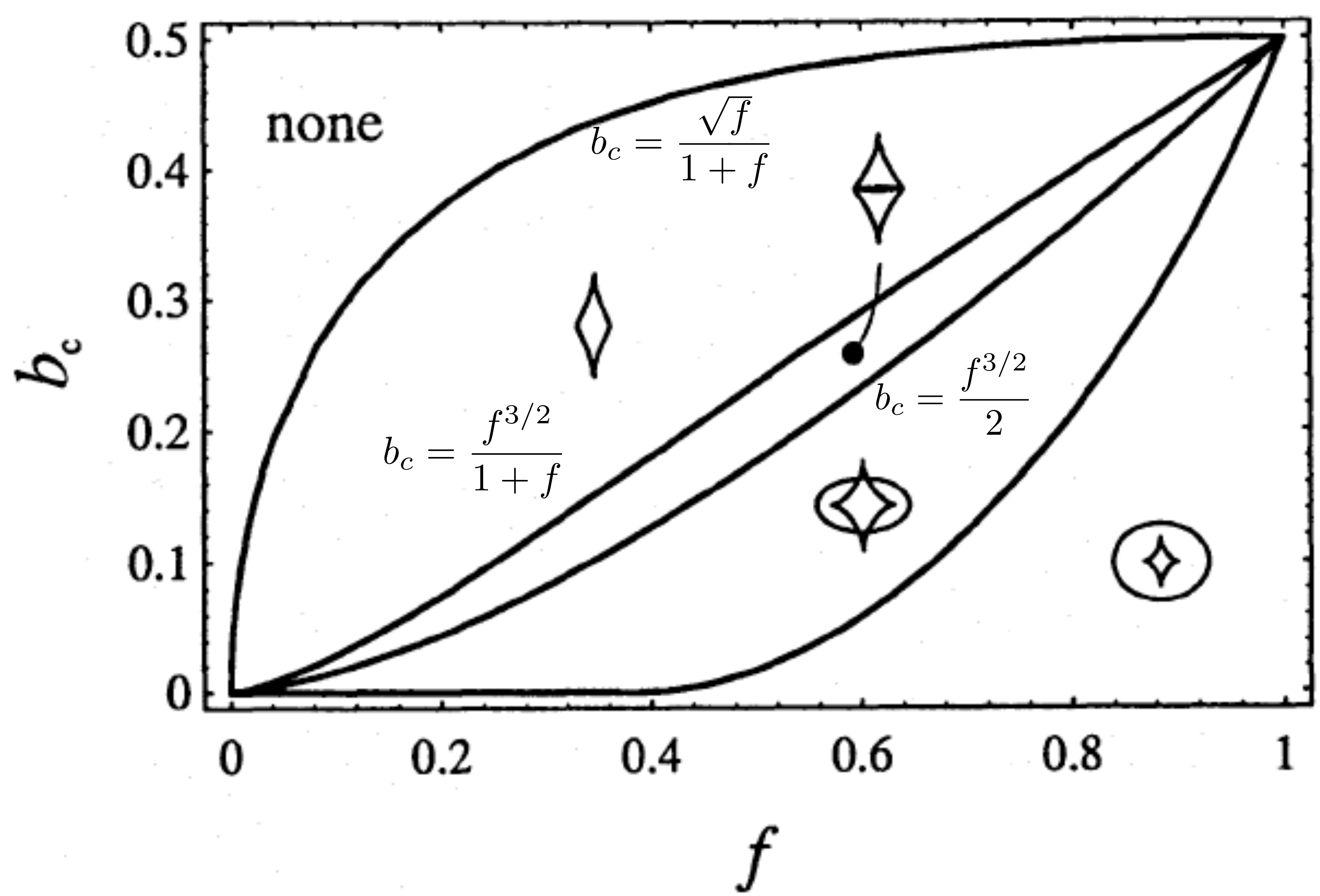
INCREASING THE CORE SIZE…



NO RADIAL CRITICAL LINE AND CAUSTIC



CAUSTIC TOPOLOGIES (SEE KORMANN, BARTELMANN & SCHNEIDER, 1994)



Q0957+561

EXTERNAL PERTURBATIONS

Lenses are often not isolated. Therefore, it is sometimes necessary to embed the lens into an external mass distribution mimicking the presence of nearby structures. How can such perturbation be modeled?

One can think to use a potential, defined such that

$$\begin{aligned}
\gamma_1 &= \frac{1}{2}(\Psi_{11} - \Psi_{22}) = \text{const.} \\
\gamma_2 &= \Psi_{12} = \text{const.} \\
\kappa &= \frac{1}{2}(\Psi_{11} + \Psi_{22}) = \text{const.}
\end{aligned}$$

EXTERNAL PERTURBATIONS

If both the sum ad the difference of the 2nd derivatives must be constant, the two derivatives must be constant separately:

$$\Psi = Cx_1^2 + C'x_2^2 + Dx_1x_2 + E$$

$$\begin{aligned}
\frac{1}{2}(\Psi_{11} - \Psi_{22}) &= C - C' = \gamma_1 \\
\Psi_{12} &= D = \gamma_2 \\
\frac{1}{2}(\Psi_{11} + \Psi_{22}) &= C + C' = \kappa
\end{aligned}$$

EXTERNAL PERTURBATIONS: EXAMPLE

In the case of a constant sheet of matter, there is no shear involved:

$$\begin{aligned}\frac{1}{2}(\Psi_{11}-\Psi_{22}) &= C-C' = \gamma_1\\ \Psi_{12} &= D = \gamma_2\\ \frac{1}{2}(\Psi_{11}+\Psi_{22}) &= C+C' = \kappa\end{aligned}$$

$$\Psi = Cx_1^2 + C'x_2^2 + Dx_1x_2 + E$$

$$\vec{\alpha} = \vec{\nabla}\Psi_\kappa = \kappa\vec{x}$$

$$\vec{y} = \vec{x} - \vec{\alpha} = \vec{x}(1-\kappa)$$

A sheet of constant density will change the focussing properties of the lens. For example, if κ=1…

EXTERNAL PERTURBATIONS: EXAMPLE

In the case of a constant sheet of matter, there is no shear involved:

$$\begin{aligned}\frac{1}{2}(\Psi_{11}-\Psi_{22}) &= C-C' = \gamma_1 = 0\\ \Psi_{12} &= D = \gamma_2\\ \frac{1}{2}(\Psi_{11}+\Psi_{22}) &= C+C' = \kappa\end{aligned}$$

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$$C = \kappa/2$$

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EXTERNAL PERTURBATIONS

Instead, if the perturber does not contribute to the convergence:

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$$C = -C' \Rightarrow C = \frac{\gamma_1}{2}$$

$$\Psi_\gamma = \frac{\gamma_1}{2}(x_1^2 - x_2^2) + \gamma_2 x_1 x_2$$

$$\Psi_\gamma = \frac{\gamma}{2} x^2 \cos 2(\phi - \phi_\gamma)$$

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TIME DELAYS

As seen earlier, lensing introduces a time delay:

$$t(x) = \frac{(1+z_L)}{c}\frac{D_L D_S}{D_{LS}}\frac{\xi_0^2}{D_L^2}\left[\frac{1}{2}(x-y)^2 - \Psi(x)\right] = \frac{(1+z_L)}{c}\frac{D_L D_S}{D_{LS}}\tau(x)$$

If there are multiple images, each of them is probing a different line of sight…

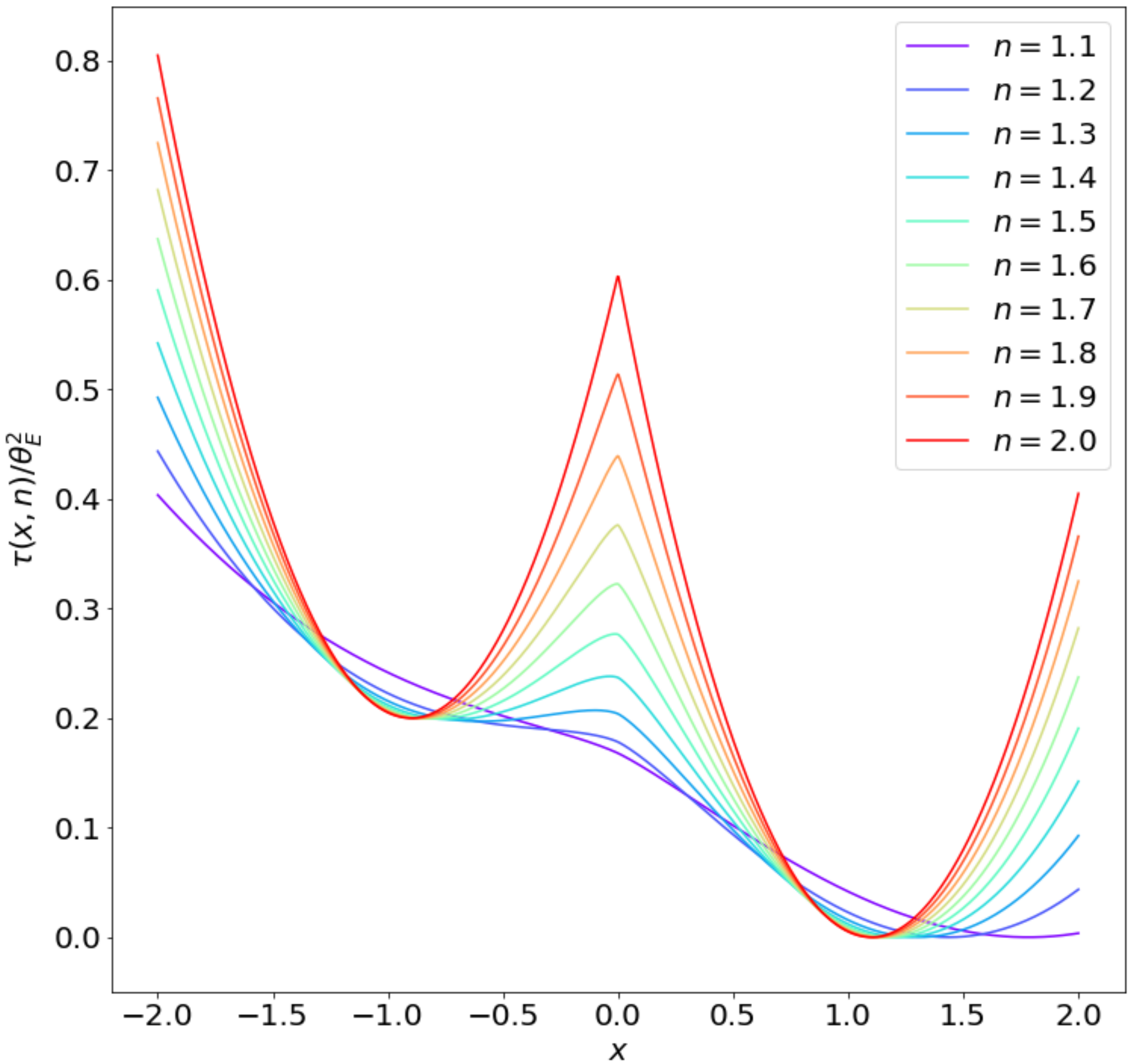
If the source is intrinsically variable, we may be able to measure a delay between the images.

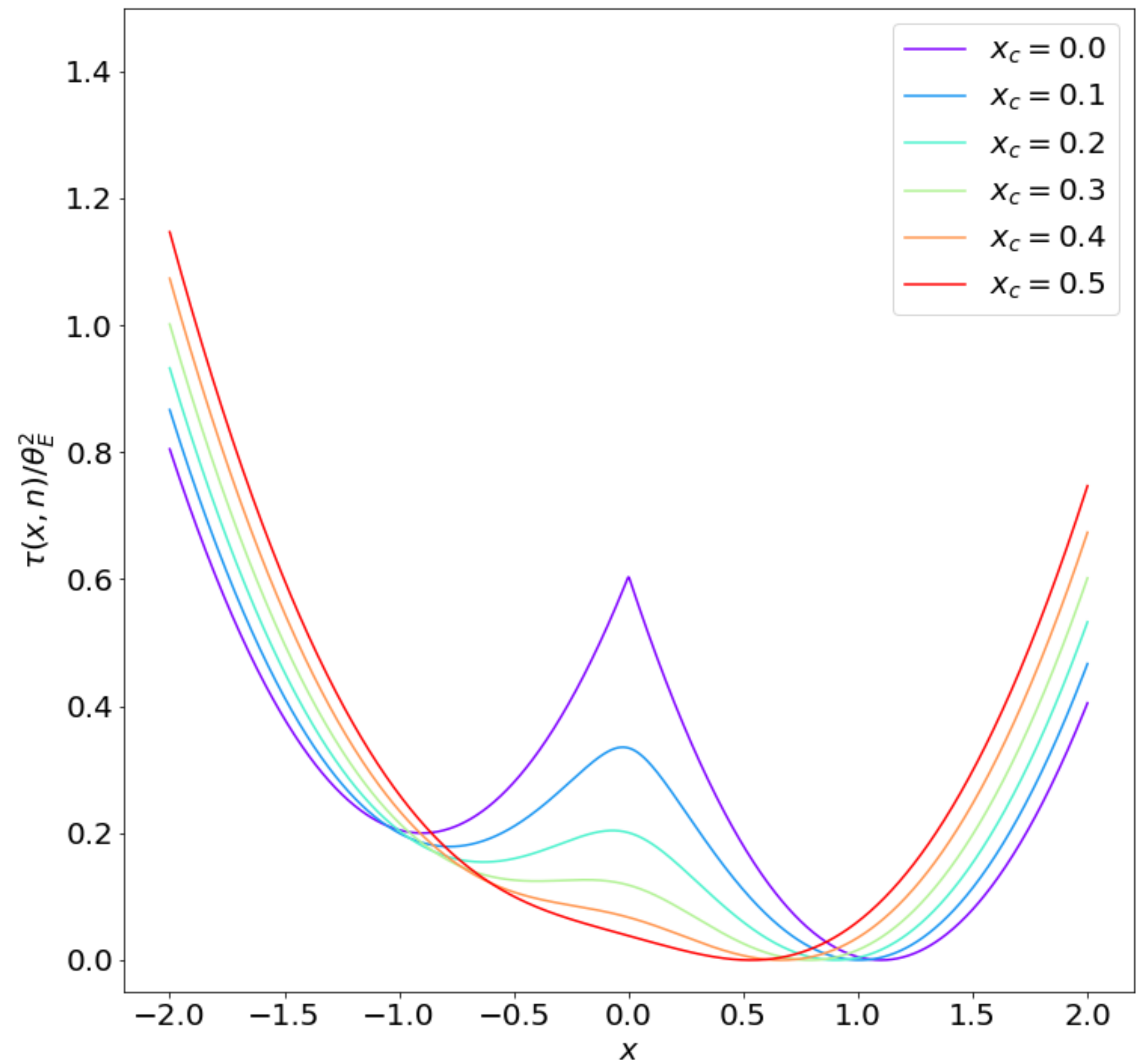
The models we have studied can be used to predict the time delay between the images. The fundamental ingredient is the lensing potential:

$$\Psi(x) = \frac{1}{3-n}x^{3-n}$$ *power-law*

$$\Psi(x, x_c) = \sqrt{x^2 + x_c^2} - x_c \ln\left(x_c + \sqrt{x^2 + x_c^2}\right)$$ *NIS*

TIME DELAYS


$n = 1.1$
$n = 1.2$
$n = 1.3$
$n = 1.4$
$n = 1.5$
$n = 1.6$
$n = 1.7$
$n = 1.8$
$n = 1.9$
$n = 2.0$
$\tau(x, n)/\theta_E^2$
x


$x_c = 0.0$
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$x_c = 0.4$
$x_c = 0.5$
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x