GRAVITATIONAL LENSING

LECTURE 24

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AA 2016-2017

THE MODEL

$$\mathbf{p} = [\mathbf{q}, \mathbf{m}, \mathbf{s}, \mathbf{x_c}]$$

OBSERVABLES

- multiple images (astrometric constraints)
- image distortions
- flux ratios
- time delays
- spectra

- in addition: complementary mass measurements (stellar kinematics, X-ray emission via assumption of hydrostatic equilibrium)

ASTROMETRIC CONSTRAINTS AND IMAGE DISTORTIONS

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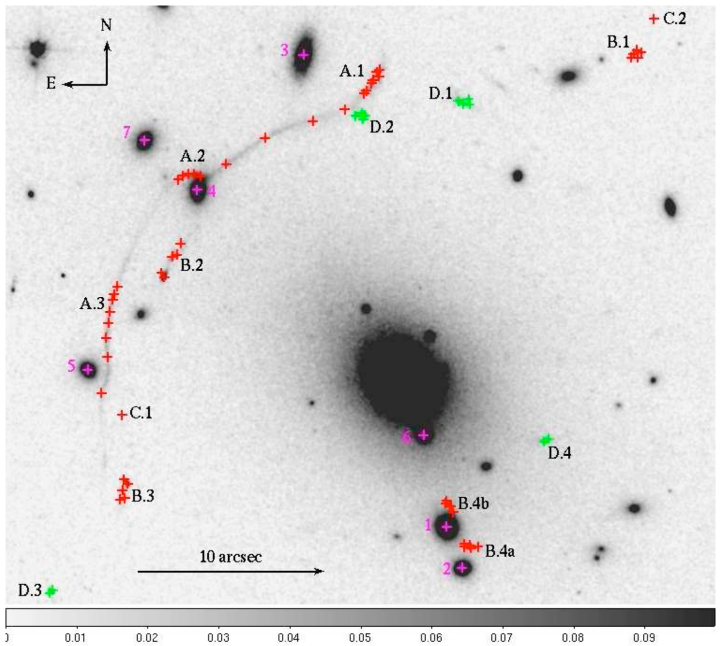

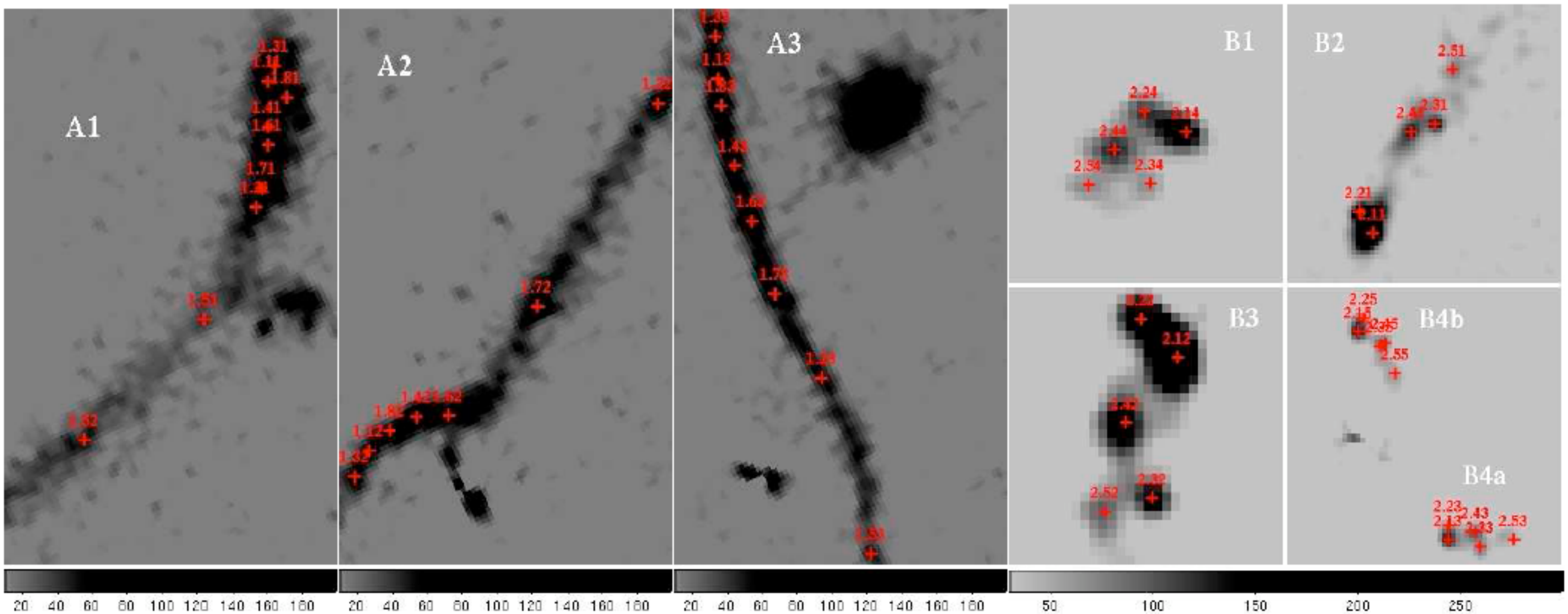

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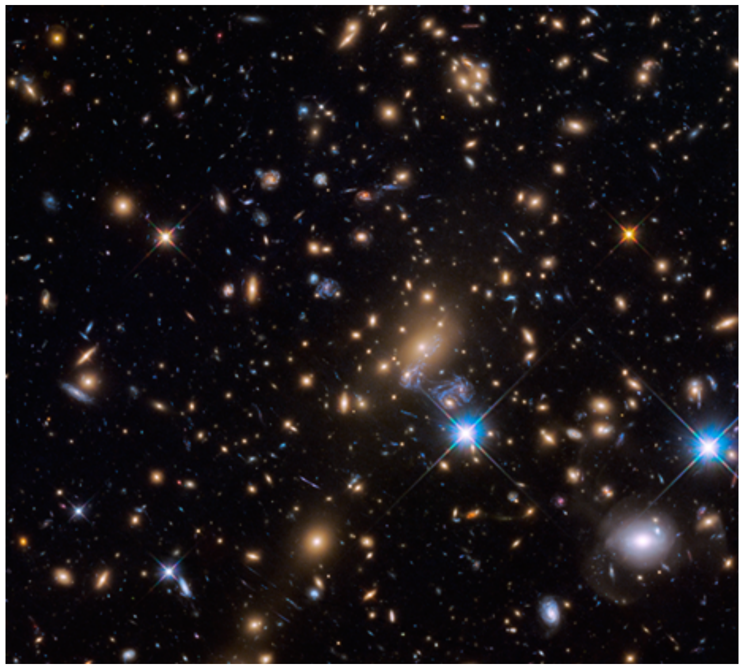

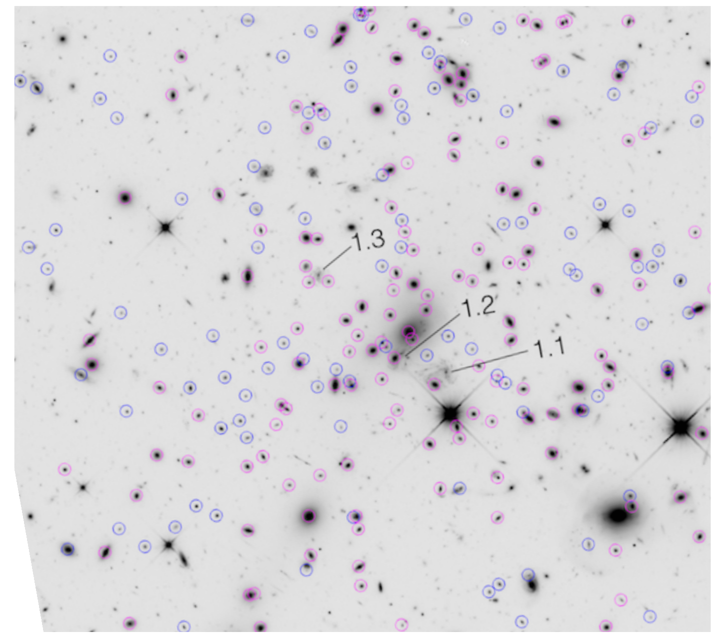

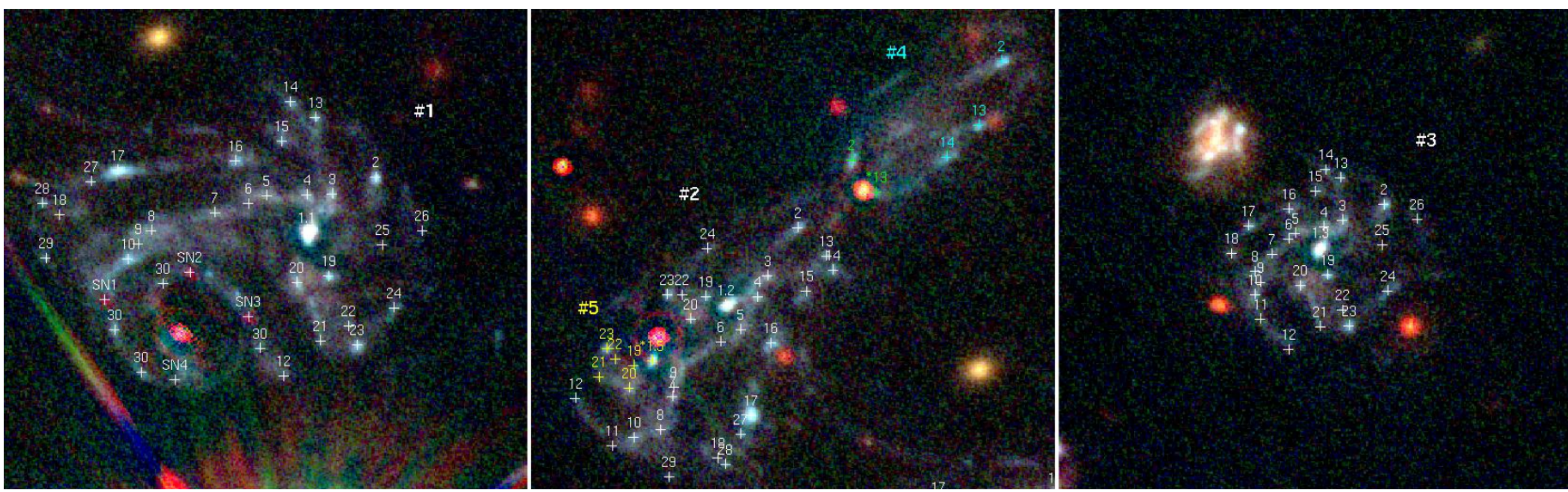

LENS OPTIMIZATION

- lensing likelihood:
- minimization of χ^2 to find the best **p** fitting the data
- for example: using astrometric constraints
- iterate between image and source plane
- or optimization in the source plane

Number of systems

$$\mathcal{L} = \Pr(D|\mathbf{p}) = \prod_{i=1}^{N} \frac{1}{\prod_{j=1}^{n_i} \sigma_{ij}\sqrt{2\pi}} \exp -\frac{\chi_i^2}{2}$$

Number of images

Contribution from single system

$$\chi_i^2 = \sum_{j=1}^{n_i} \frac{[\vec{\theta}^j_{obs} - \vec{\theta}^j_{\mathbf{p}}]^2}{\sigma_{ij}^2}$$

$$\vec{\beta}^j_{\mathbf{p}} = \vec{\theta}^j_{obs} - \vec{\alpha}(\vec{\theta}^j_{obs}, \mathbf{p})$$

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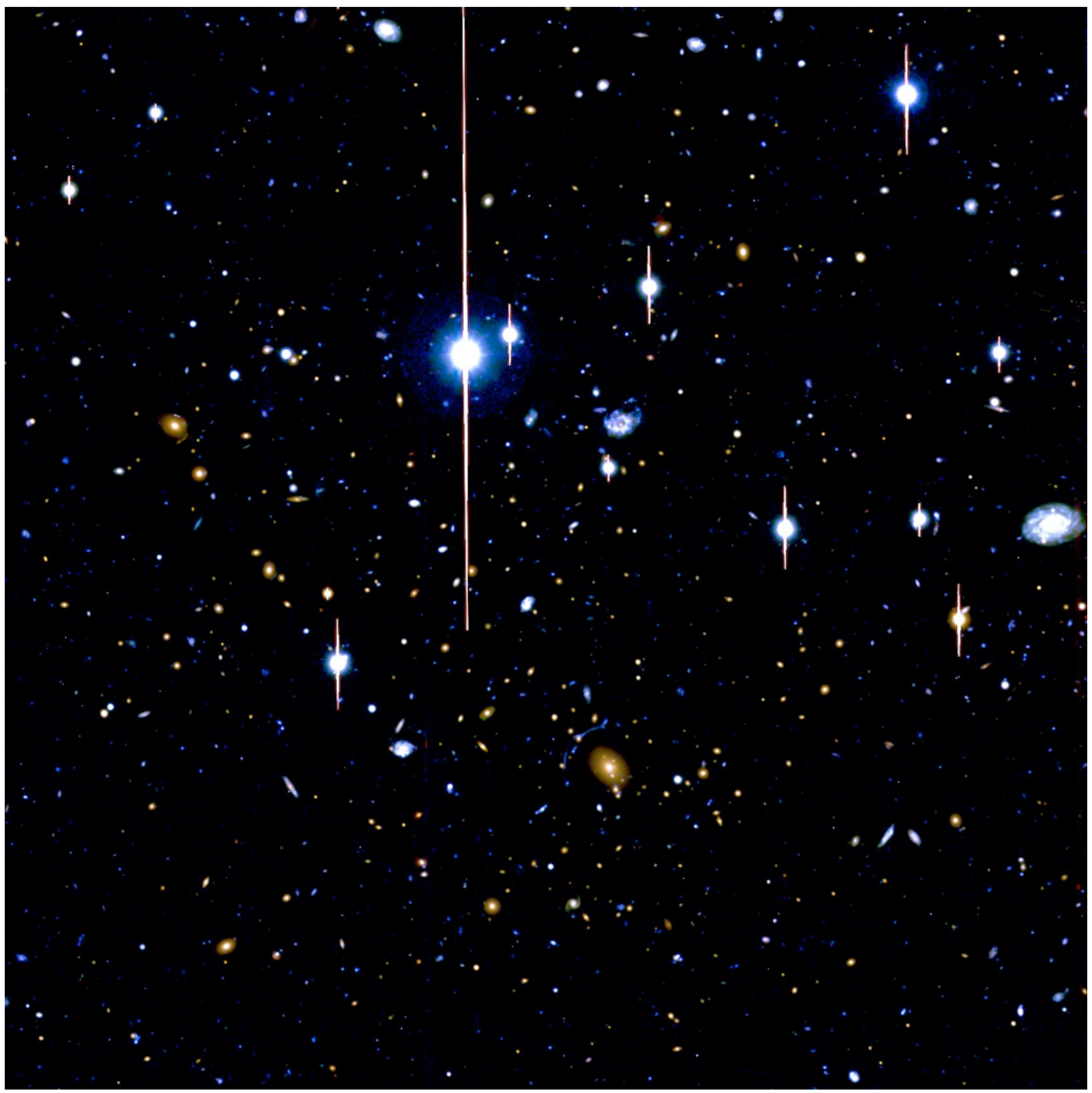
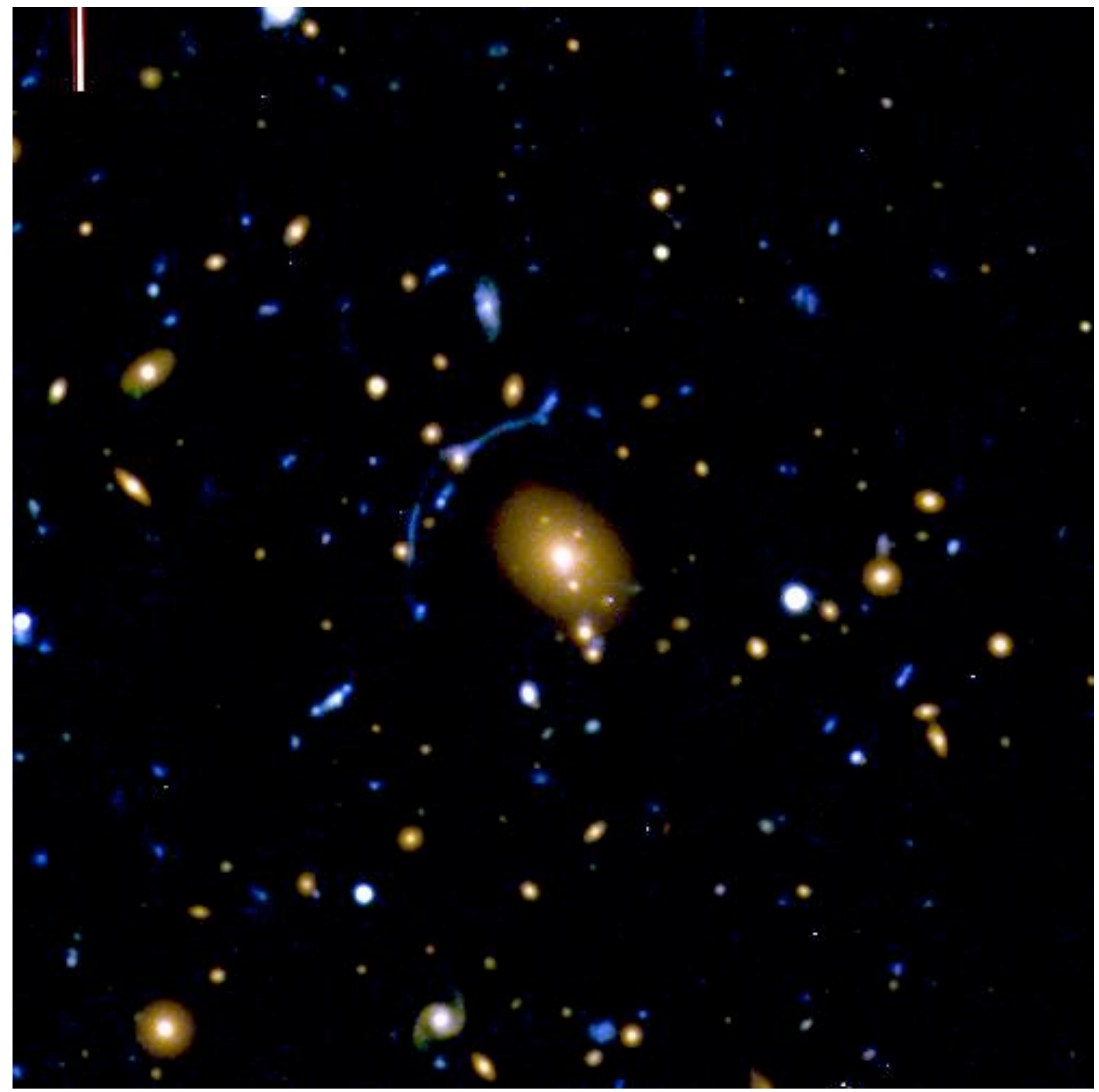
$$\chi_{S,i}^2 = \sum_{j=1}^{n_i} \frac{[\beta^j_{\mathbf{p}} - \langle\beta^j_{\mathbf{p}}\rangle]}{\mu_j^{-2}\sigma_{ij}^2}$$

GALAXY CLUSTERS AS WEAK GRAVITATIONAL LENSES





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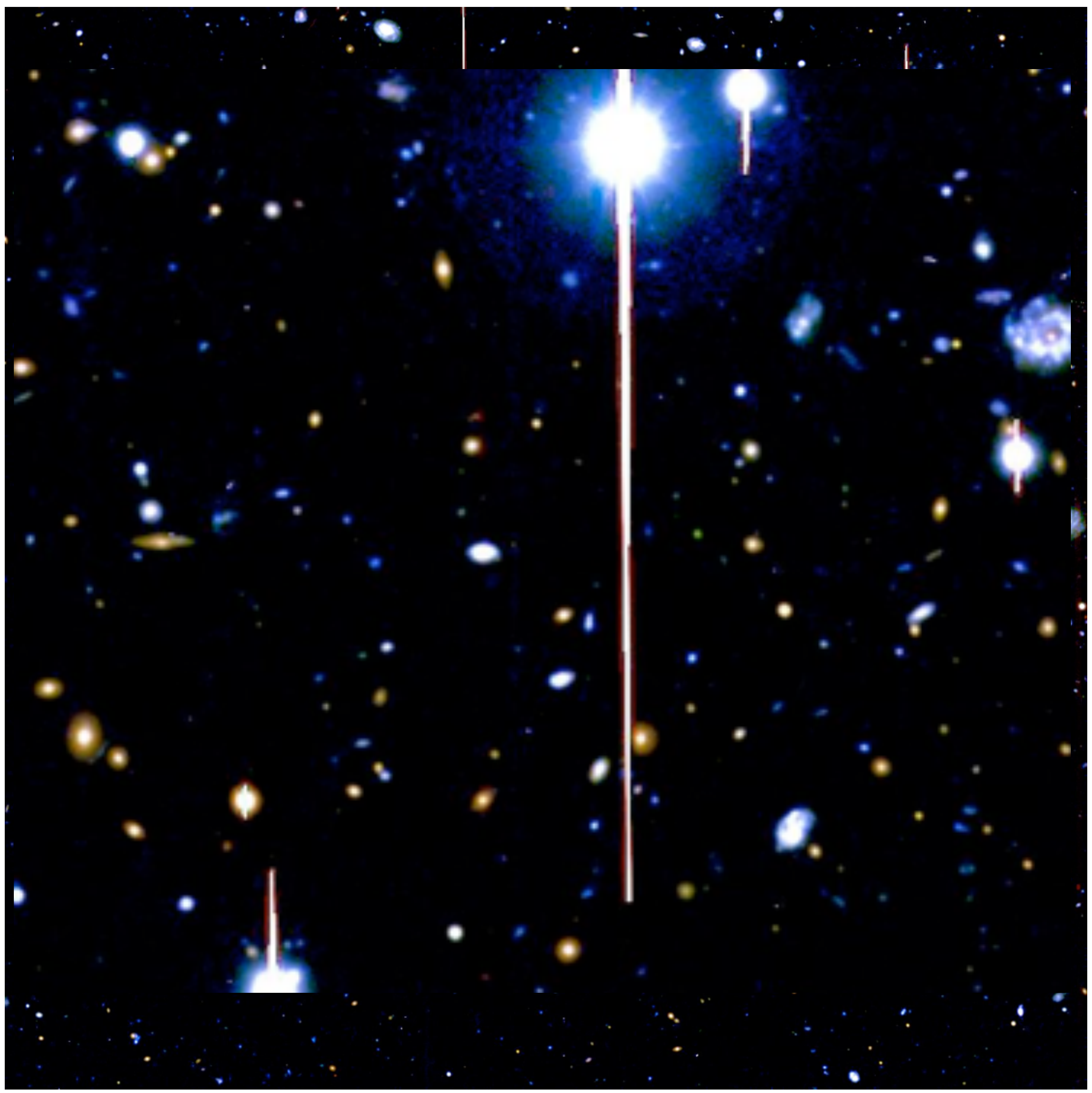
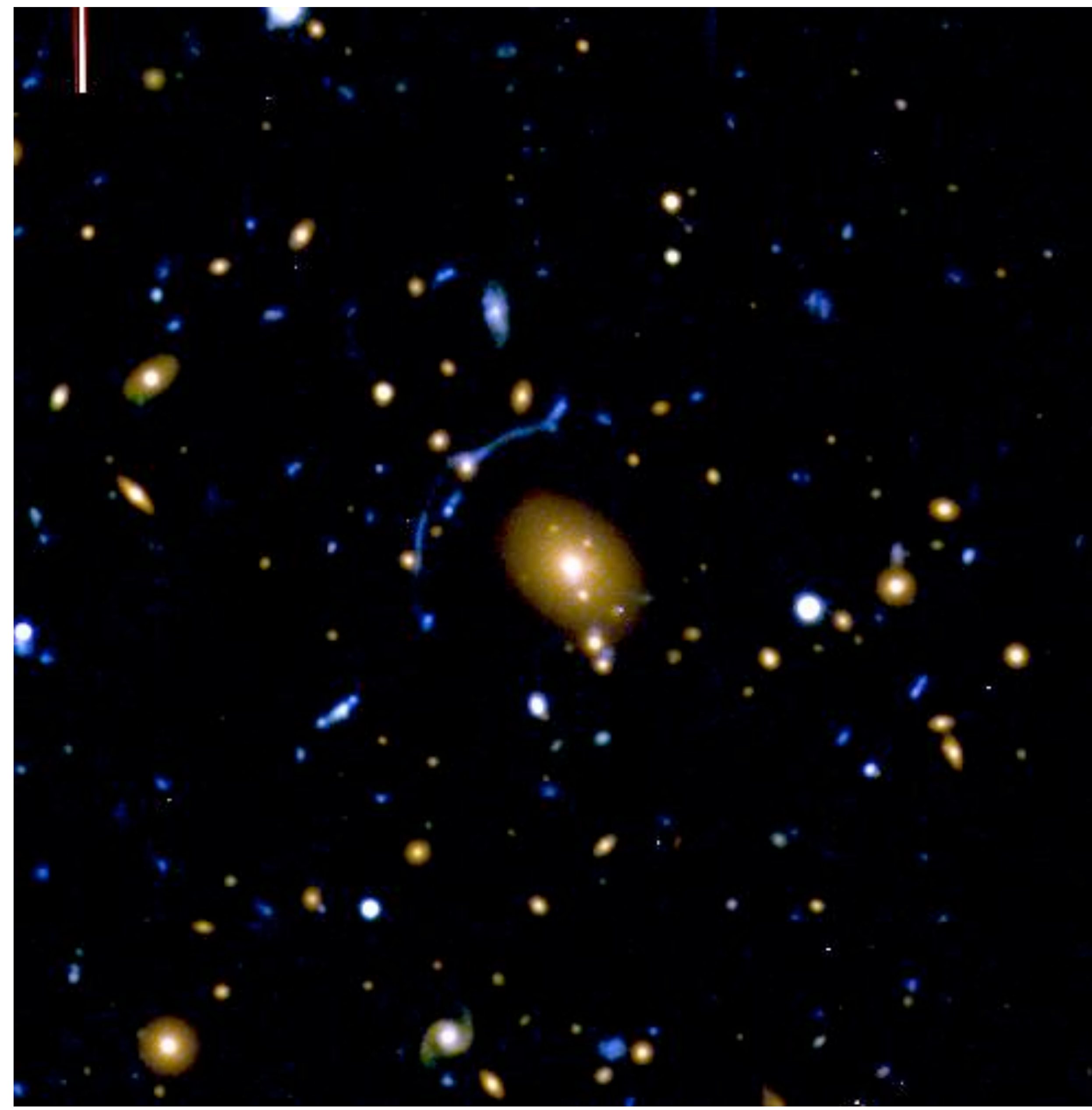
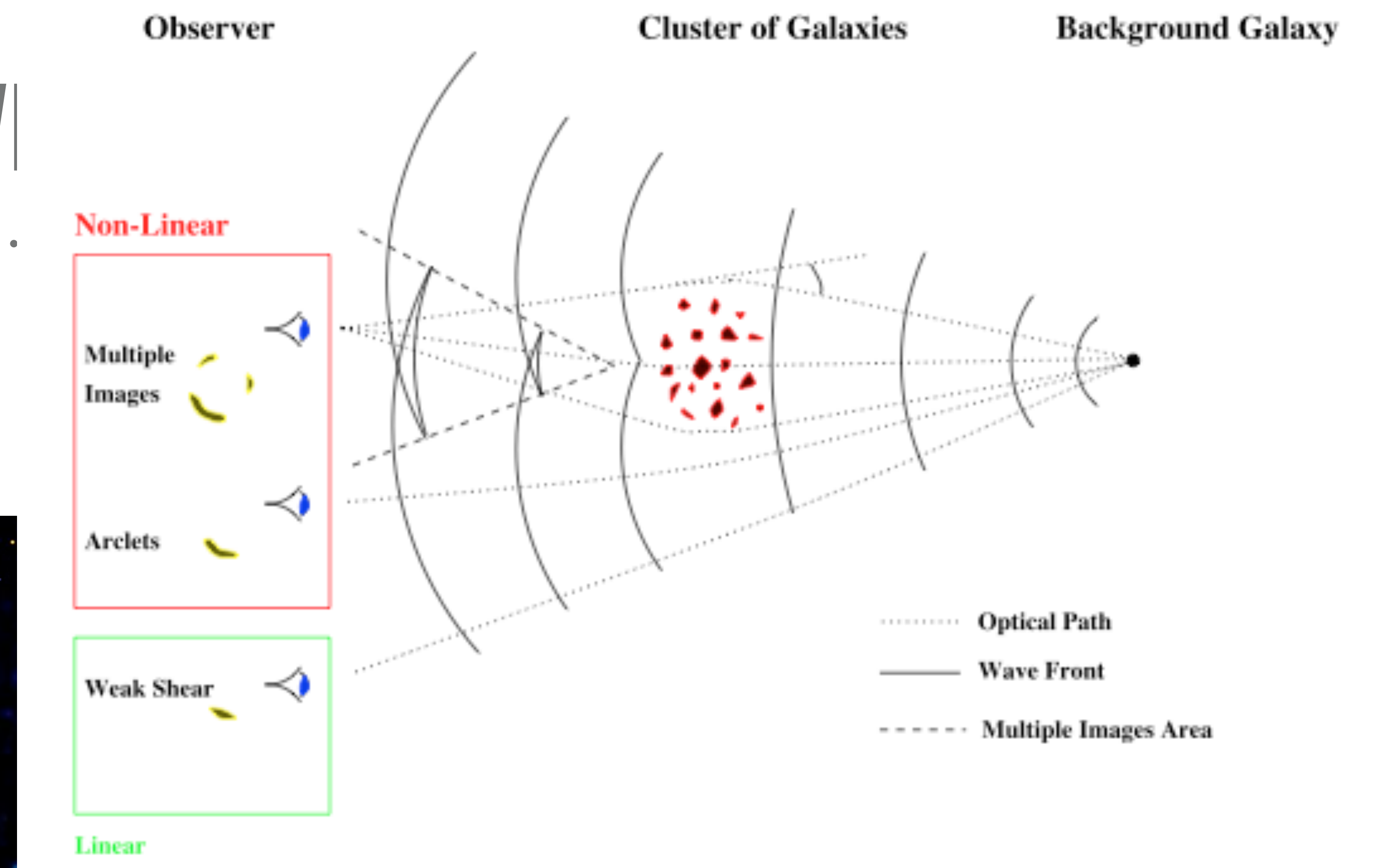
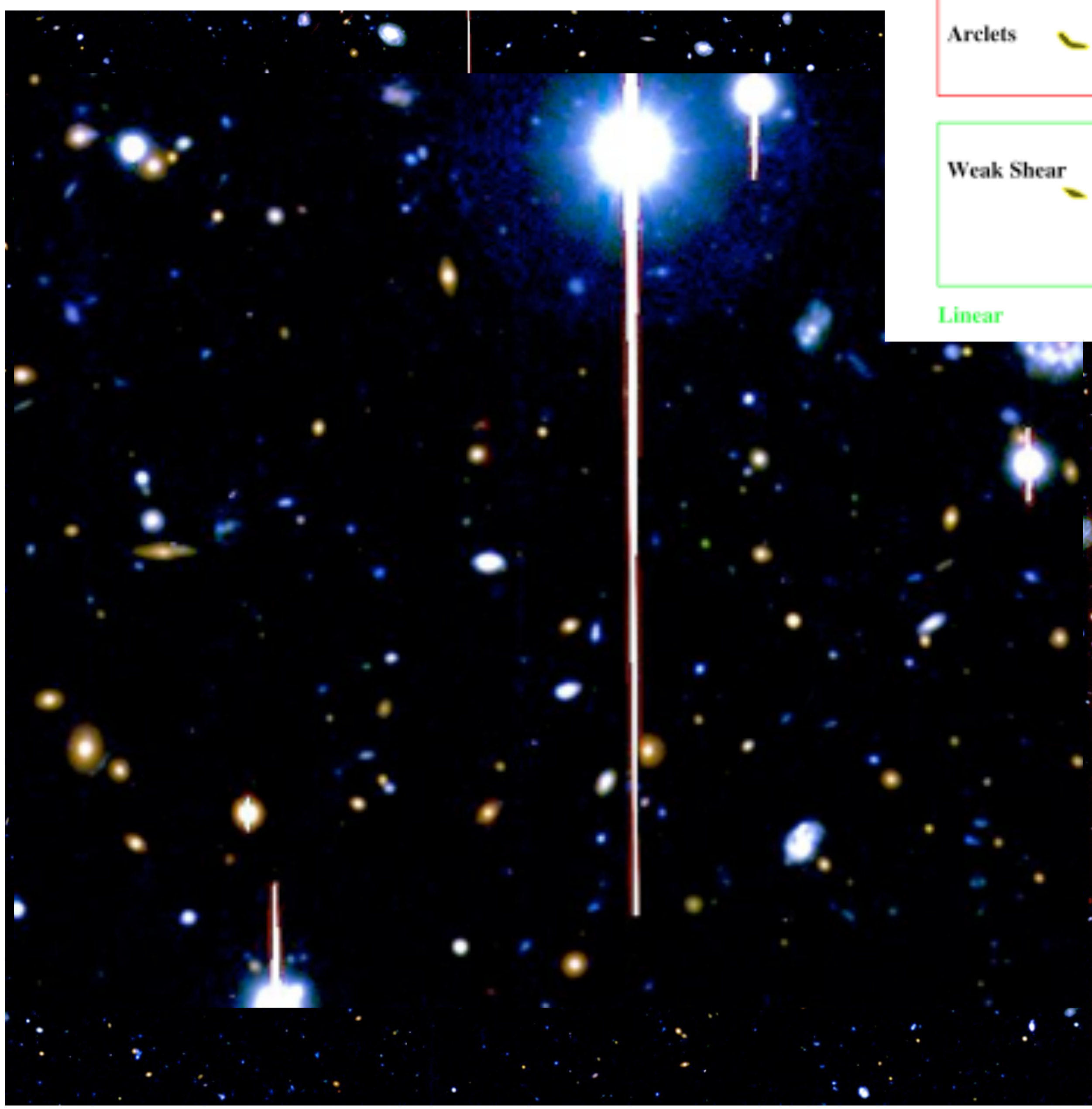
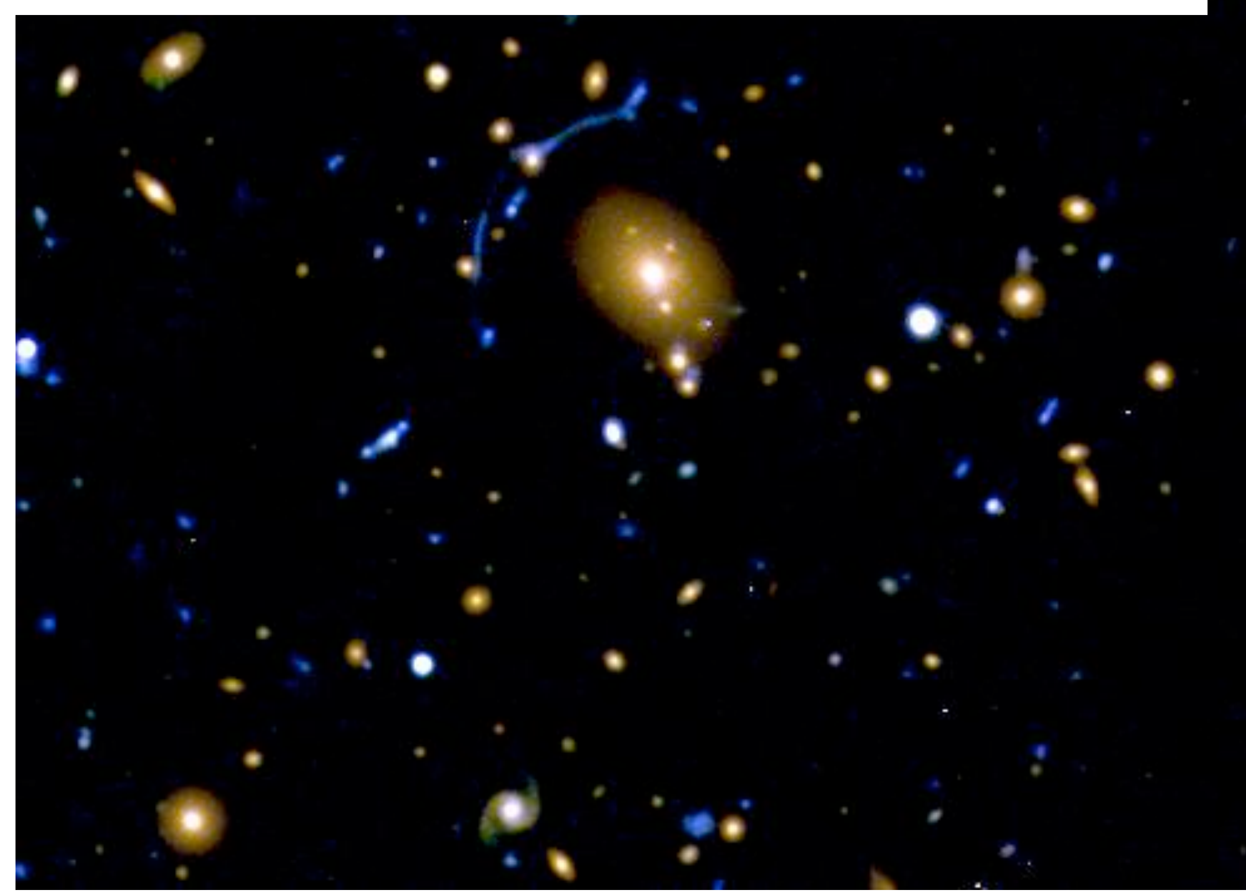




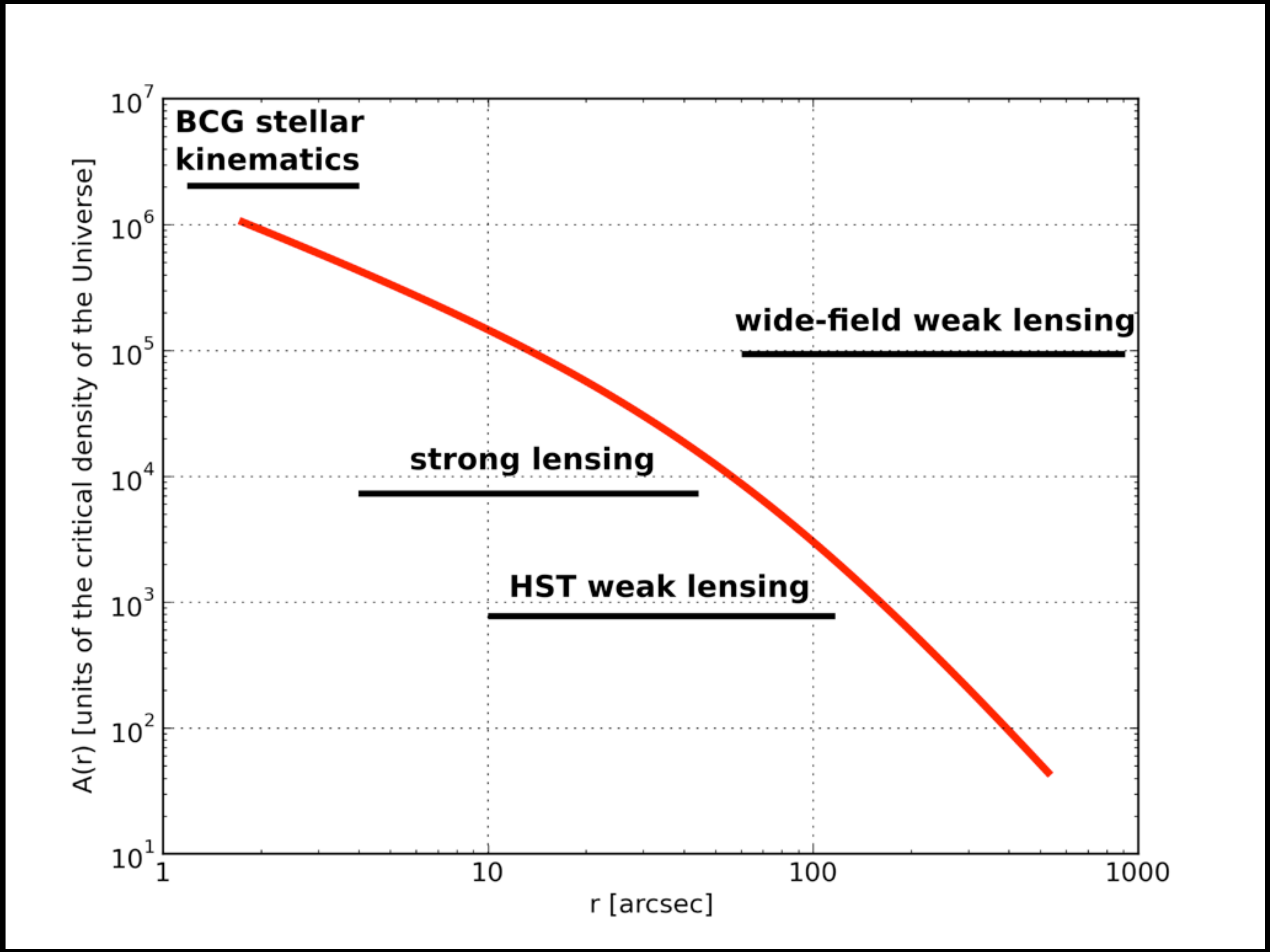
GALAXY CLUSTERS AS W



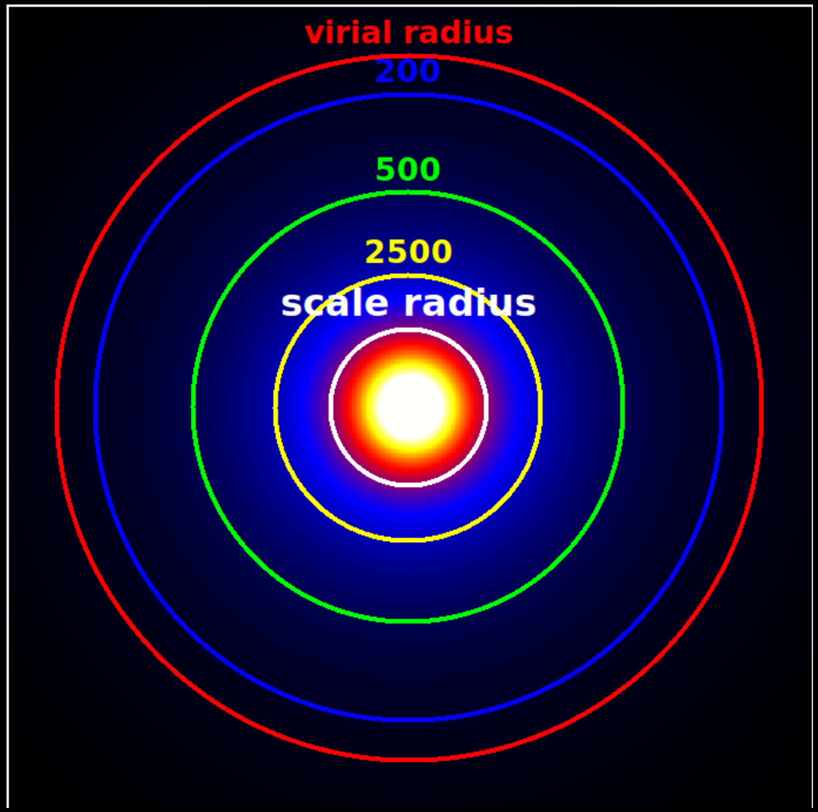




The dark matter density profile



$$A(r) = \frac{A_s}{\frac{r}{r_s}\left(1+\frac{r}{r_s}\right)^2}$$



typical characterizations:

r_2500 (SL, SZ)
r_500 (X-ray)
r_200 (WL)
r_vir ("Theory")

$$c_\Delta = \frac{r_\Delta}{r_s} \qquad x = \frac{r}{r_s}$$

$$M(r) = 4\pi A_s r_s^3 \left(\ln(1+x) - \frac{x}{1+x} \right)$$

THE WEAK LENSING REGIME

- Let remind how lensing works in the limit of small deflections

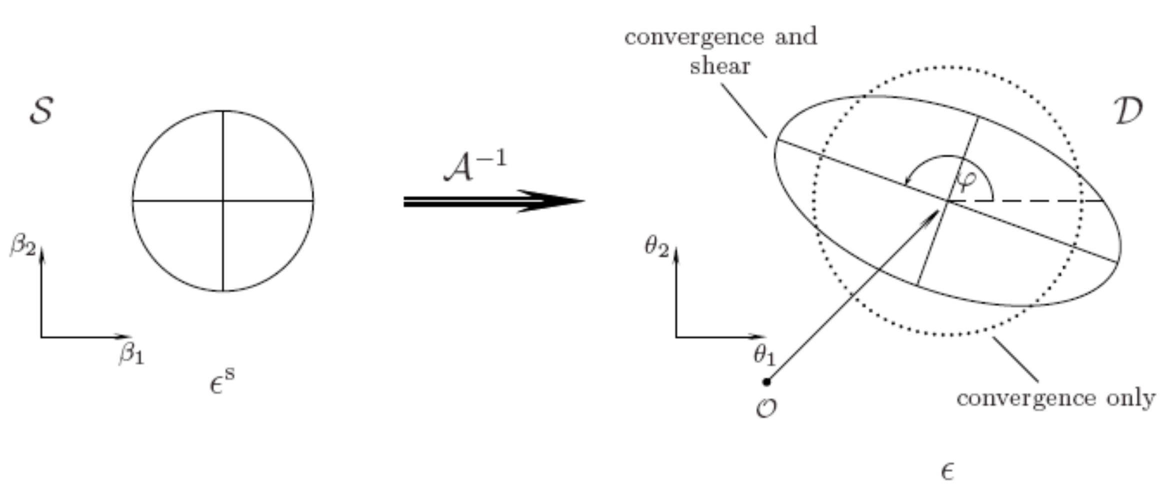
- As we have seen, in this regime, the lens equation can be linearized and the lens mapping is described by the Jacobian matrix

- Circular sources are mapped on elliptical images

$$\boldsymbol{\beta} - \boldsymbol{\beta}_0 = \mathcal{A}(\boldsymbol{\theta}_0) \cdot (\boldsymbol{\theta} - \boldsymbol{\theta}_0)$$

$$\mathcal{A}(\boldsymbol{\theta}) = (1-\kappa) \begin{pmatrix} 1 - g_1 & -g_2 \\ -g_2 & 1 + g_1 \end{pmatrix}$$

$$g(\boldsymbol{\theta}) = \frac{\gamma(\boldsymbol{\theta})}{[1 - \kappa(\boldsymbol{\theta})]}$$

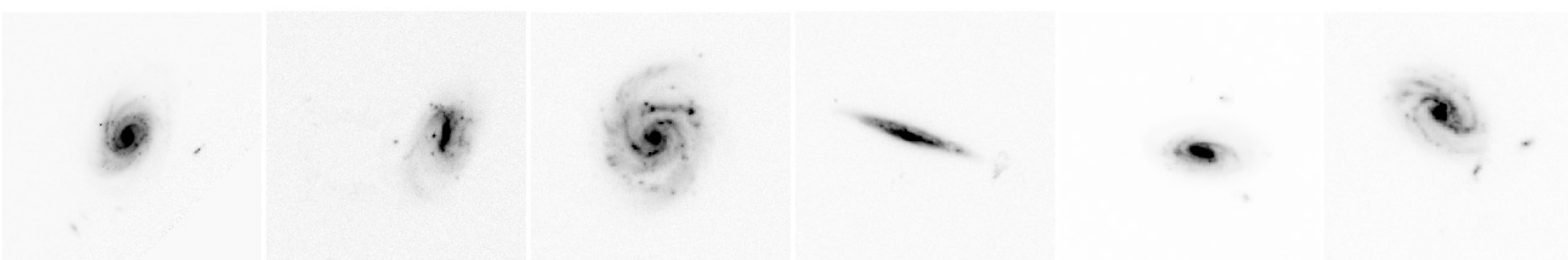


$$a = \frac{r}{1-\kappa-\gamma} \quad , \quad b = \frac{r}{1-\kappa+\gamma}$$

$$\epsilon = \frac{a-b}{a+b} = \frac{2\gamma}{2(1-\kappa)} = \frac{\gamma}{1-\kappa} \approx \gamma$$

$$|g| = \frac{1-b/a}{1+b/a} \quad \Leftrightarrow \quad \frac{b}{a} = \frac{1-|g|}{1+|g|}$$

WHAT IS THE IMAGE ELLIPTICITY?



MEASUREMENTS OF GALAXY SHAPES

Observable: brightness distribution

First moment: image centroid $$\bar{\boldsymbol{\theta}} \equiv \frac{\int \mathrm{d}^2\theta \, I(\boldsymbol{\theta}) \, q_I[I(\boldsymbol{\theta})] \, \boldsymbol{\theta}}{\int \mathrm{d}^2\theta \, I(\boldsymbol{\theta}) \, q_I[I(\boldsymbol{\theta})]}$$

$$q_I(I) = \mathrm{H}(I - I_{\mathrm{th}})$$

Define a tensor of second order brightness moments:

$$Q_{ij} = \frac{\int \mathrm{d}^2\theta \, I(\boldsymbol{\theta}) \, q_I[I(\boldsymbol{\theta})] \, (\theta_i - \bar{\theta}_i) \, (\theta_j - \bar{\theta}_j)}{\int \mathrm{d}^2\theta \, I(\boldsymbol{\theta}) \, q_I[I(\boldsymbol{\theta})]} \, , \quad i, j \in \{1, 2\}$$

Diagonalizing the tensor, we find the principal axes of the ellipse. The eigenvalues are:

$$\lambda_{\pm} = \frac{1}{2}\left[Q_{11} + Q_{22} \pm \sqrt{(Q_{11} - Q_{22})^2 - 4Q_{12}}\right]$$

giving the squares of the ellipse semi-axes.

The position angle is: $\tan(2\phi) = \dfrac{2Q_{12}}{Q_{11} - Q_{22}}$

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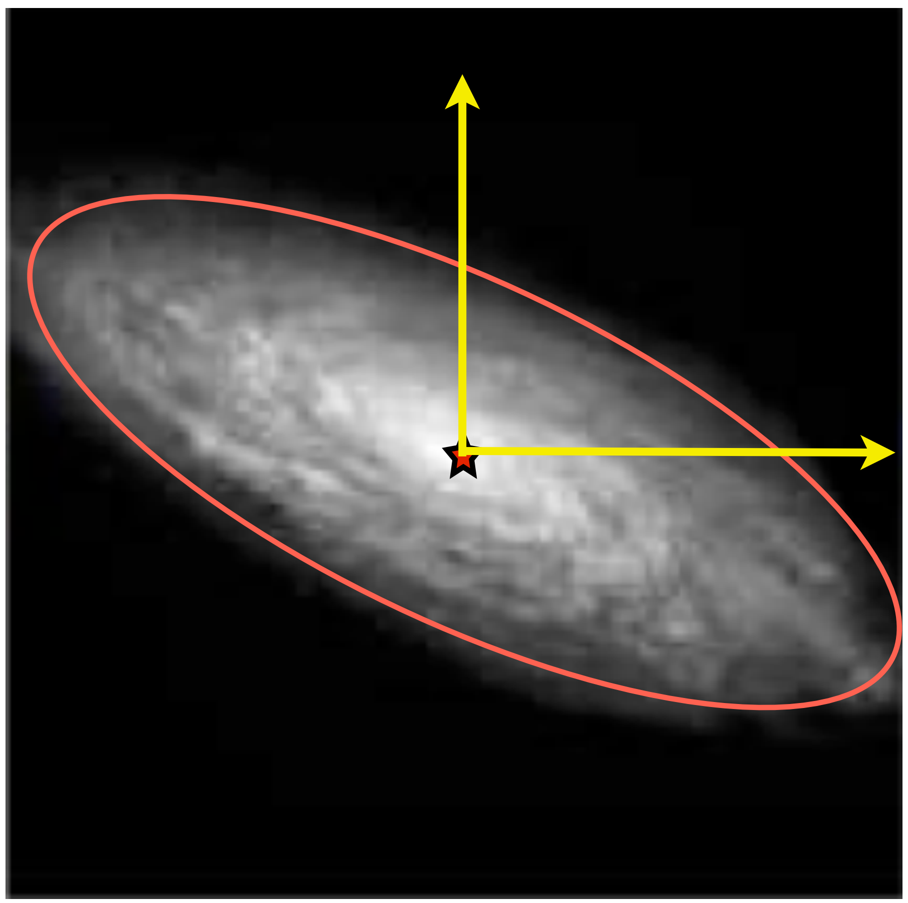

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COMPLEX ELLIPTICITY AND SHEAR

For the shear, we defined two components: $\gamma = (\gamma_1, \gamma_2)$

$$\gamma_1 = \gamma \cos(2\phi)$$
$$\gamma_2 = \gamma \sin(2\phi)$$

It is very common to use a complex notation to write the shear as:

$$\gamma = \gamma_1 + i\gamma_2 = |\gamma| e^{2i\phi}$$

Similarly, we can define the complex reduced shear and ellipticity:

$$g = \frac{\gamma}{1-\kappa} = g_1 + ig_2 = |g| e^{2i\phi}$$

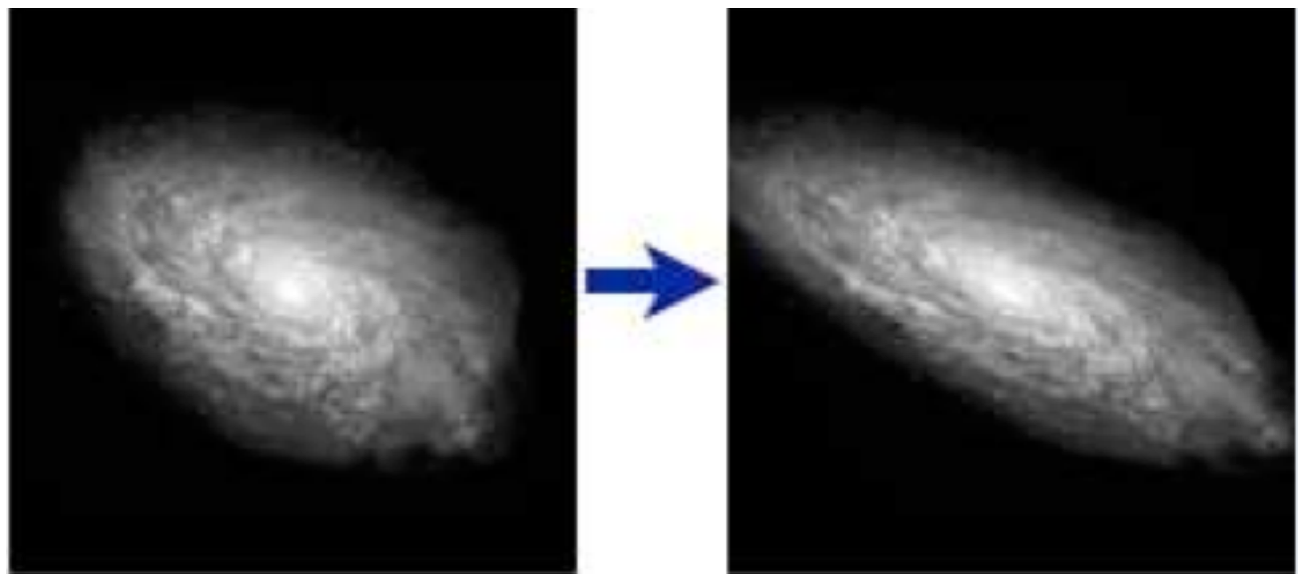
$$\epsilon = \epsilon_1 + i\epsilon_2 = |\epsilon| e^{2i\phi}$$

Using the previous formulae:

$$\epsilon \equiv \frac{Q_{11} - Q_{22} + 2\mathrm{i}Q_{12}}{Q_{11} + Q_{22} + 2(Q_{11}Q_{22} - Q_{12}^2)^{1/2}}$$

FROM SOURCE TO IMAGE ELLIPTICITY

$I_s(\vec{\beta})$



$I(\vec{\theta})$

How are the two brightness distributions linked?

FROM SOURCE TO IMAGE ELLIPTICITY

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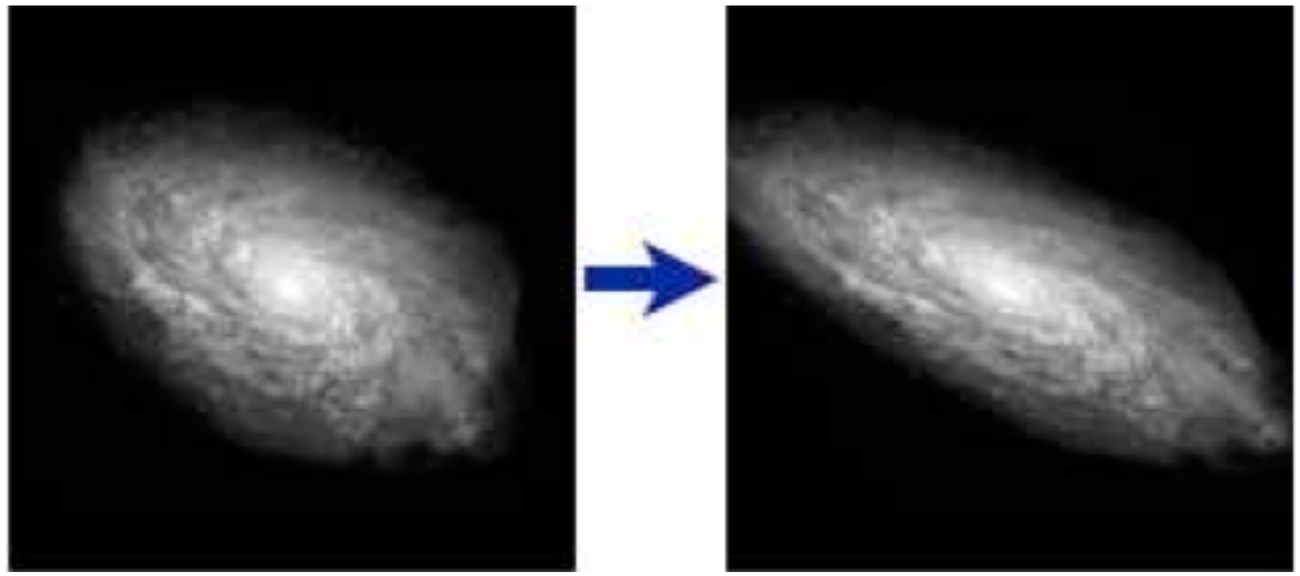
$$I(\boldsymbol{\theta}) = I^{(\mathrm{s})}[\boldsymbol{\beta}_0 + \mathcal{A}(\boldsymbol{\theta}_0) \cdot (\boldsymbol{\theta} - \boldsymbol{\theta}_0)]$$

FROM SOURCE TO IMAGE ELLIPTICITY



$$Q^{(\mathrm{s})}_{ij} = \frac{\int \mathrm{d}^2\beta\, I^{(\mathrm{s})}(\boldsymbol{\theta})\, q_I[I^{(\mathrm{s})}(\boldsymbol{\beta})]\, (\beta_i - \bar{\beta}_i)\, (\beta_j - \bar{\beta}_j)}{\int \mathrm{d}^2\beta\, I^{(\mathrm{s})}(\boldsymbol{\theta})\, q_I[I^{(\mathrm{s})}(\boldsymbol{\beta})]} \qquad Q_{ij} = \frac{\int \mathrm{d}^2\theta\, I(\boldsymbol{\theta})\, q_I[I(\boldsymbol{\theta})]\, (\theta_i - \bar{\theta}_i)\, (\theta_j - \bar{\theta}_j)}{\int \mathrm{d}^2\theta\, I(\boldsymbol{\theta})\, q_I[I(\boldsymbol{\theta})]}\,, \quad i,j \in \{1,2\}$$

Given that $\beta - \bar{\beta} = \mathcal{A}\left(\boldsymbol{\theta} - \bar{\boldsymbol{\theta}}\right)$ $\quad \mathrm{d}^2\beta = \det \mathcal{A}\, \mathrm{d}^2\theta,$ $\quad I(\boldsymbol{\theta}) = I^{(\mathrm{s})}[\boldsymbol{\beta}_0 + \mathcal{A}(\boldsymbol{\theta}_0)\cdot(\boldsymbol{\theta} - \boldsymbol{\theta}_0)]$

We find that $Q^{(\mathrm{s})} = \mathcal{A}\, Q\, \mathcal{A}^T = \mathcal{A}\, Q\, \mathcal{A}$

which gives:

$$\epsilon^{(\mathrm{s})} = \begin{cases} \dfrac{\epsilon - g}{1 - g^*\epsilon} & \text{if } |g| \le 1\ ; \\[2ex] \dfrac{1 - g\epsilon^*}{\epsilon^* - g^*} & \text{if } |g| > 1\ . \end{cases}$$

The inverse transformations are found by changing the source and the image ellipticities and g with -g

EXTRACTING THE SIGNAL

Now we have what we need to separate the lensing induced ellipticity from the intrinsic ellipticity…

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Remember that ellipticities are complex numbers characterized by a phase.

Suppose that sources have intrinsically random phases.

In this case, averaging over a number of sources, the expectation value of the ellipticity is...

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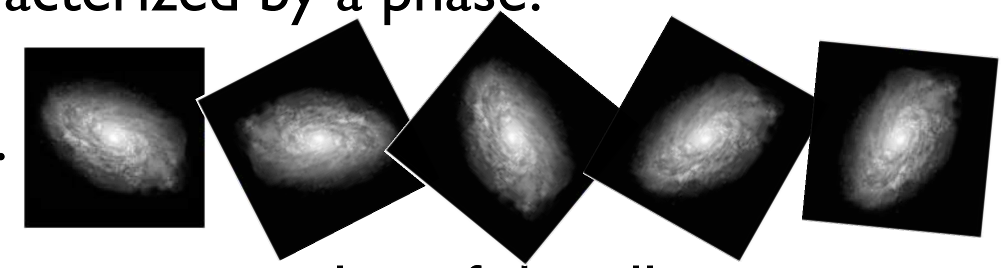

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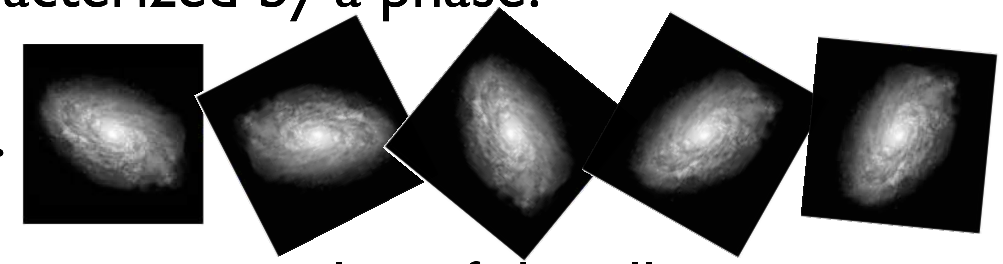

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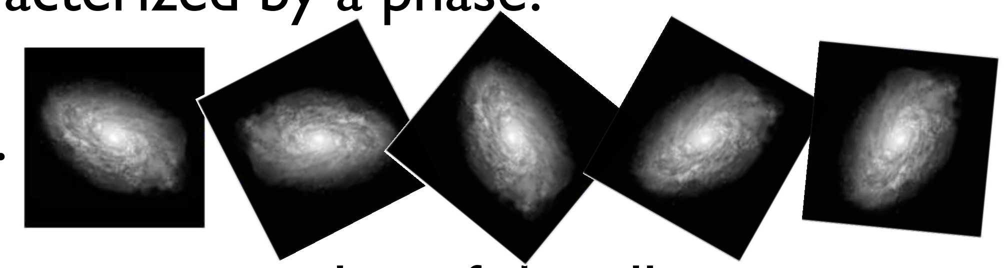

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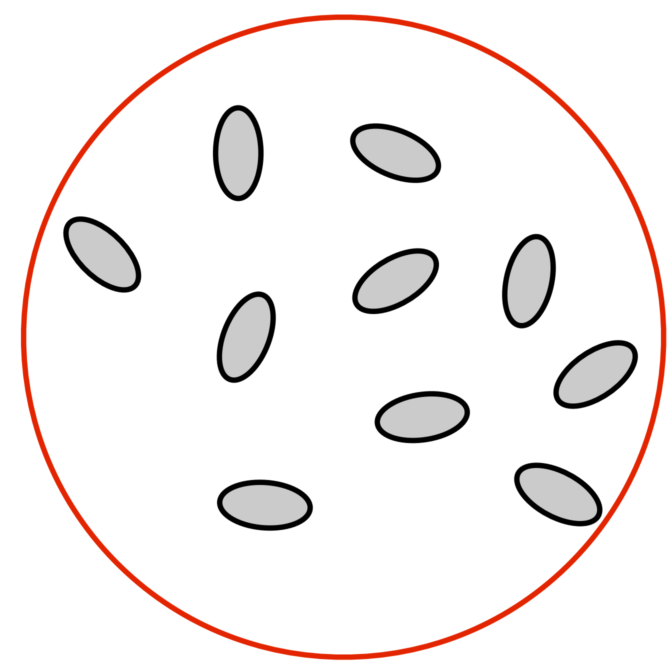
Thus, from the above equations:

$$\mathrm{E}(\epsilon) = \begin{cases} g & \text{if } |g| \le 1 \\ 1/g^* & \text{if } |g| > 1 \end{cases}$$

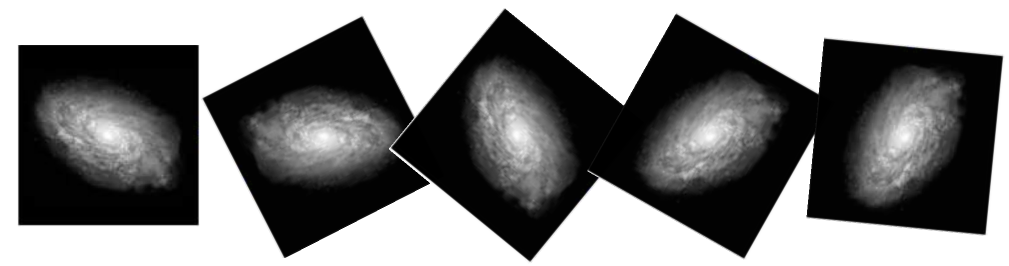


Which means that we can estimate the reduced shear by averaging over a number of sources:

$$g \approx \langle \epsilon \rangle$$

NOISE

The noise is given by the dispersion in the intrinsic ellipticity distribution



$$\sigma_\epsilon = \sqrt{\left\langle \epsilon^{(\mathrm{s})} \epsilon^{(\mathrm{s})*} \right\rangle}$$

Averaging over N galaxies, the 1-σ deviation from the mean ellipticity is

$$\sigma_\epsilon / \sqrt{N}$$

Thus, we can beat the noise by averaging over many galaxies!

- select a number of galaxies in a region and assume that the shear is constant within the region
- if the region is too large, the shear is smoothed
- increase the number density of galaxies

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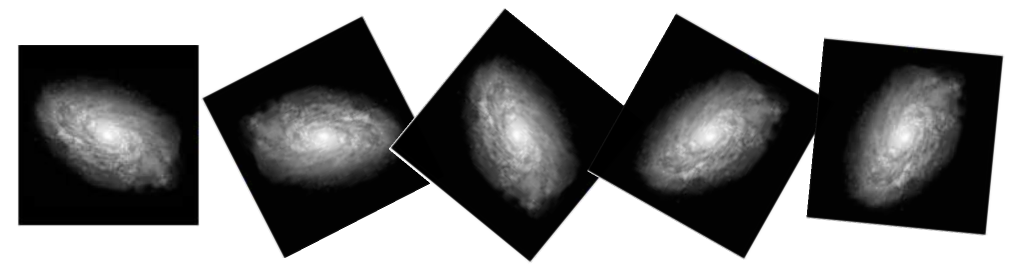

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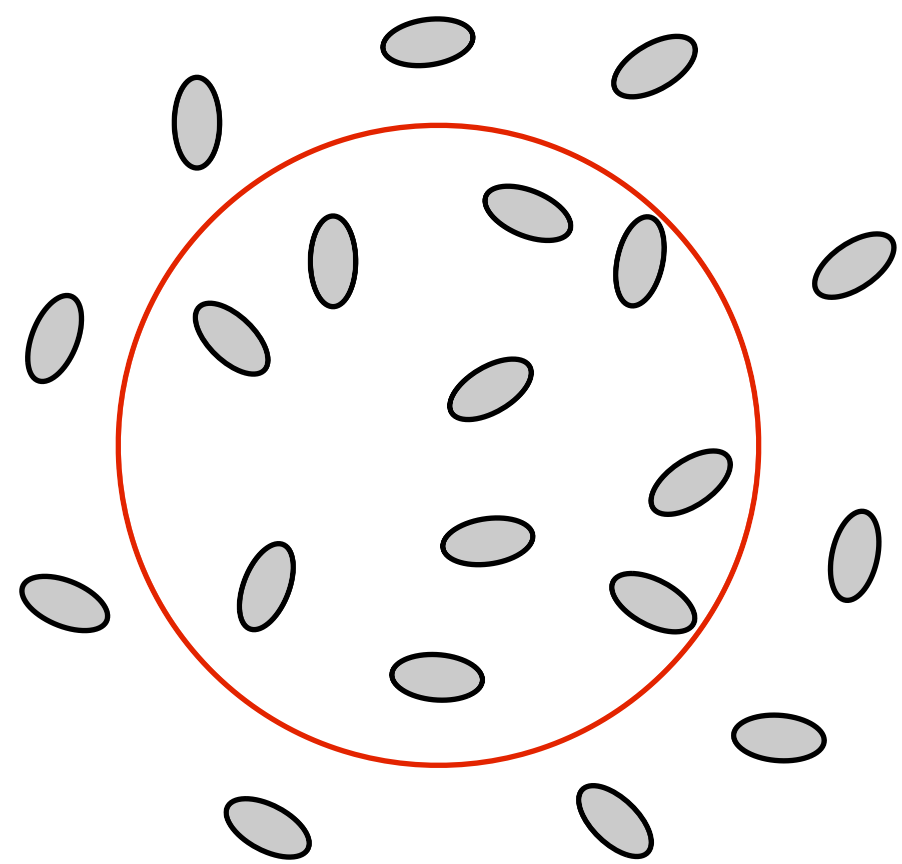

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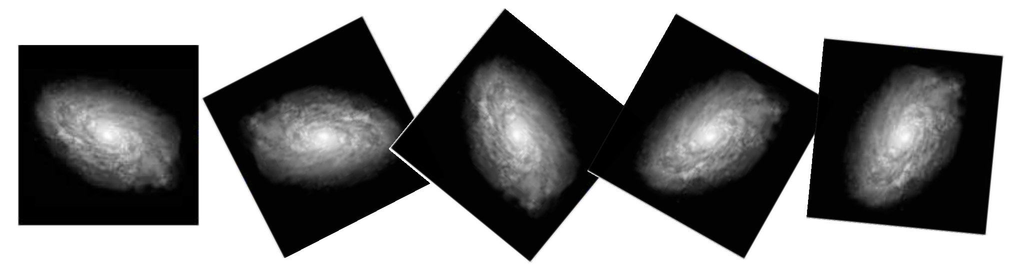

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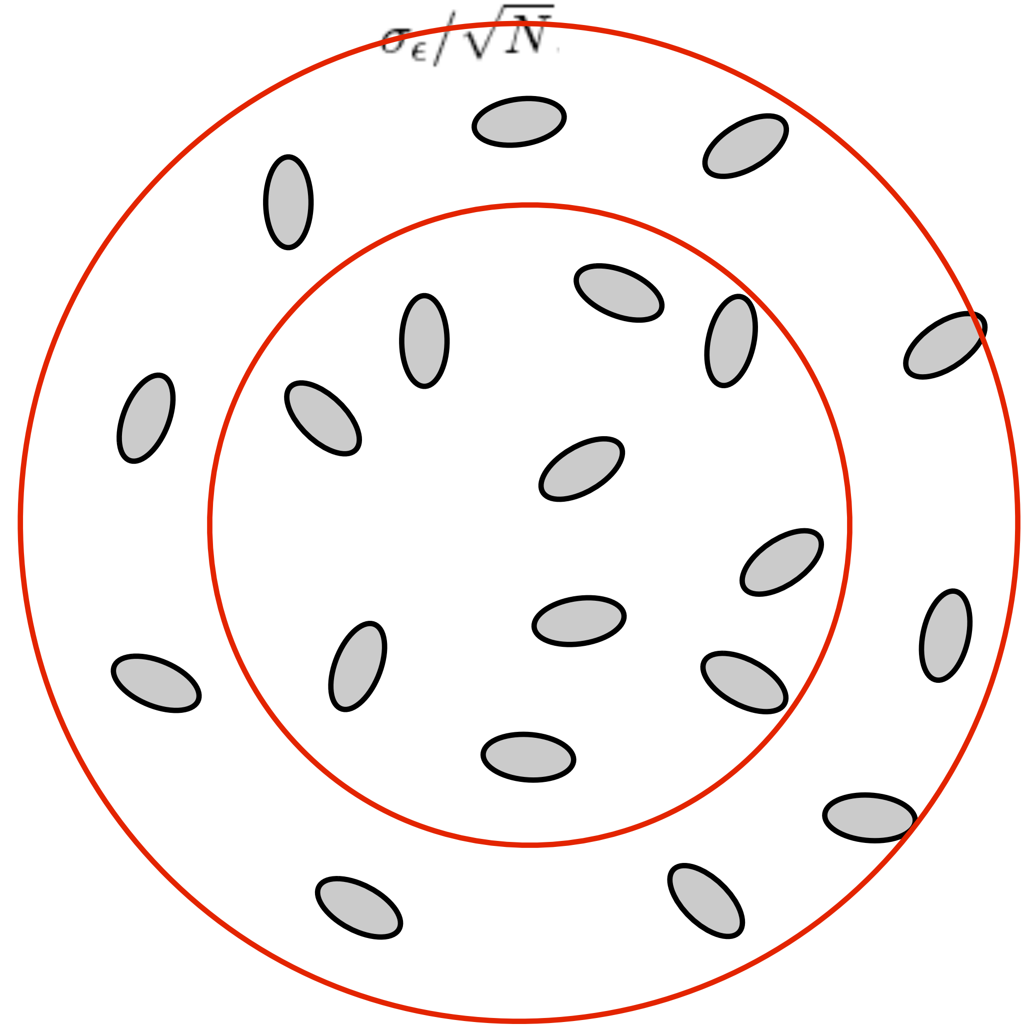

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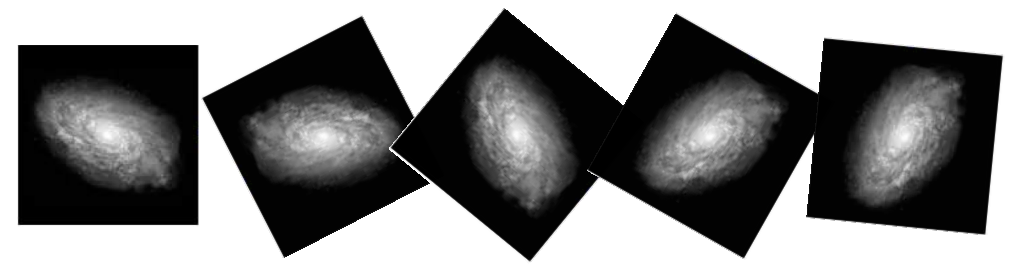

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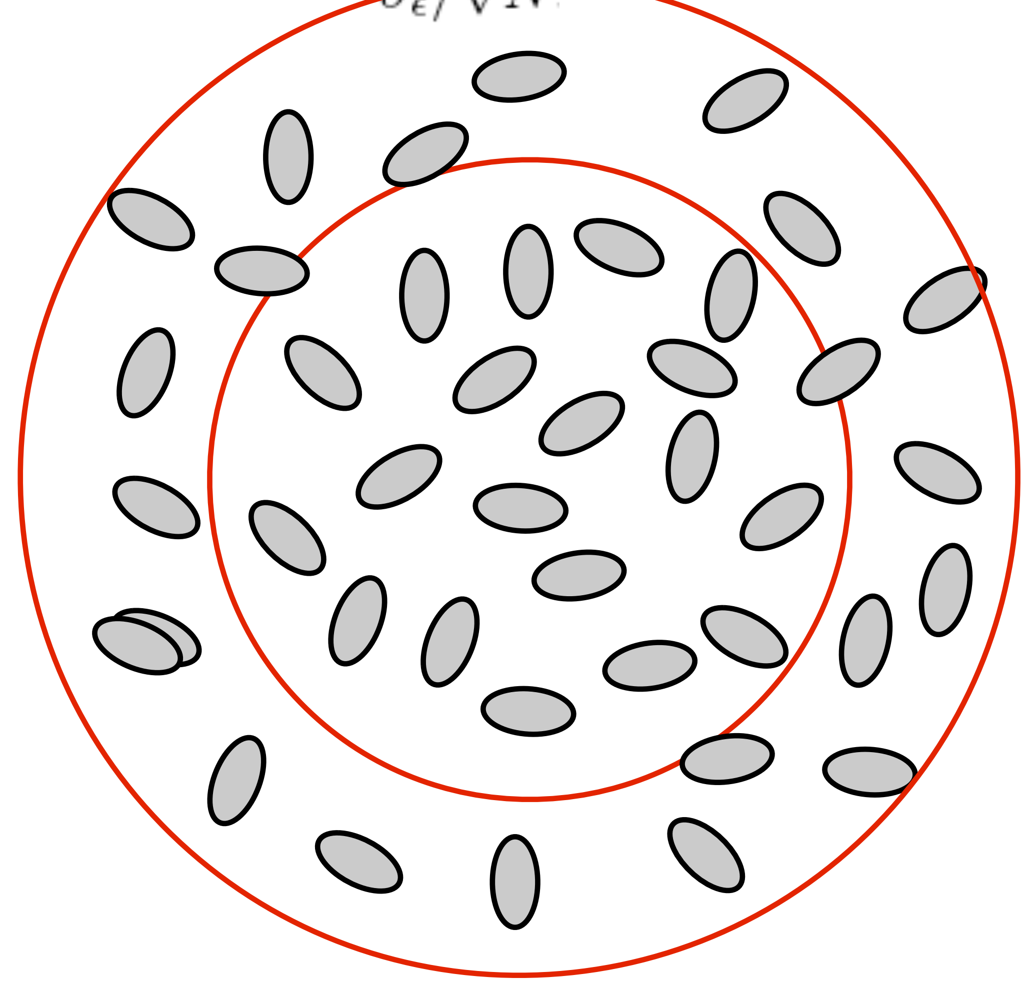

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ONCE MEASURED THE SHEAR, WHAT DO WE DO?

Several ways to convert the shear measurement into a mass estimate:

- some methods are parametric
- other methods are free-form

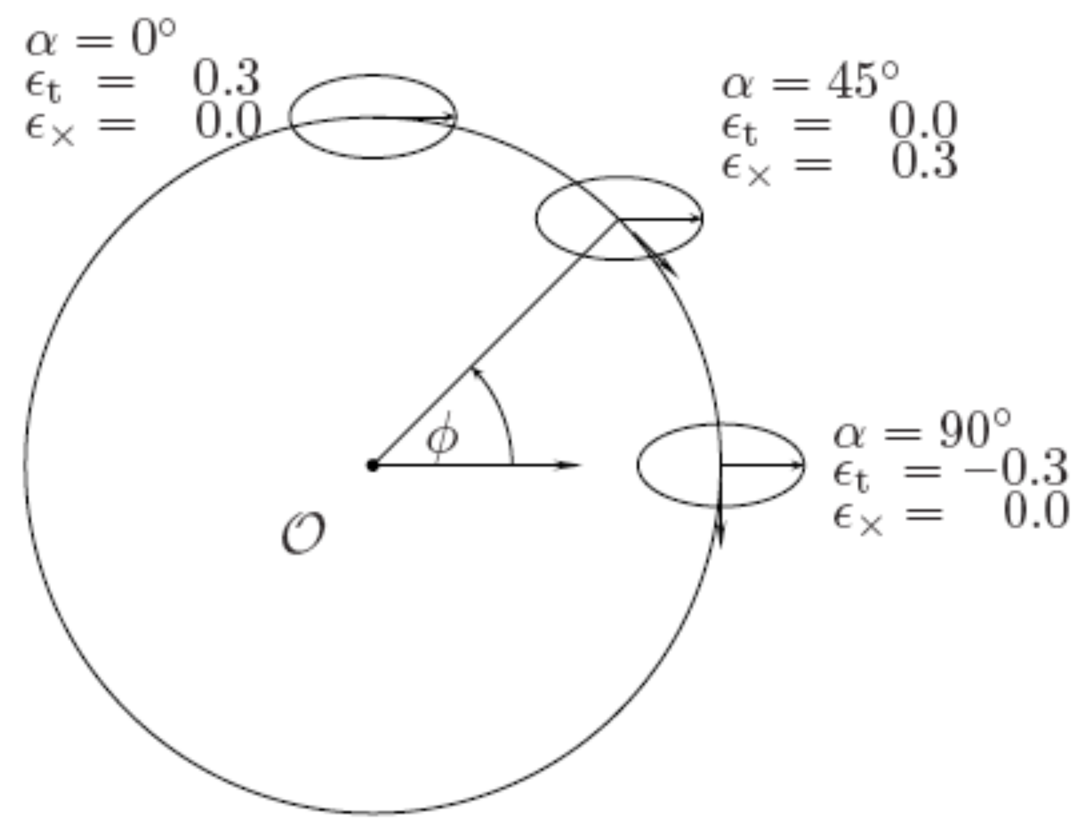
TANGENTIAL AND CROSS COMPONENT OF THE SHEAR

Given a direction ϕ we can define a tangential and a cross component of the ellipticity/shear relative to this direction.

$$\gamma_{\rm t} = -\mathcal{R}{\rm e}\left[\gamma\, {\rm e}^{-2{\rm i}\phi}\right] \quad , \quad \gamma_\times = -\mathcal{I}{\rm m}\left[\gamma\, {\rm e}^{-2{\rm i}\phi}\right]$$

Note that, under this convention, "tangential" means both tangentially and radially oriented ellipticities

With this we want to emphasize that lensing, being caused by a scalar potential is curl-free



The signs are chosen such that the tangential component is positive for tangentially distorted images, and it is negative for radially distorted images.

FIT OF THE TANGENTIAL SHEAR PROFILE

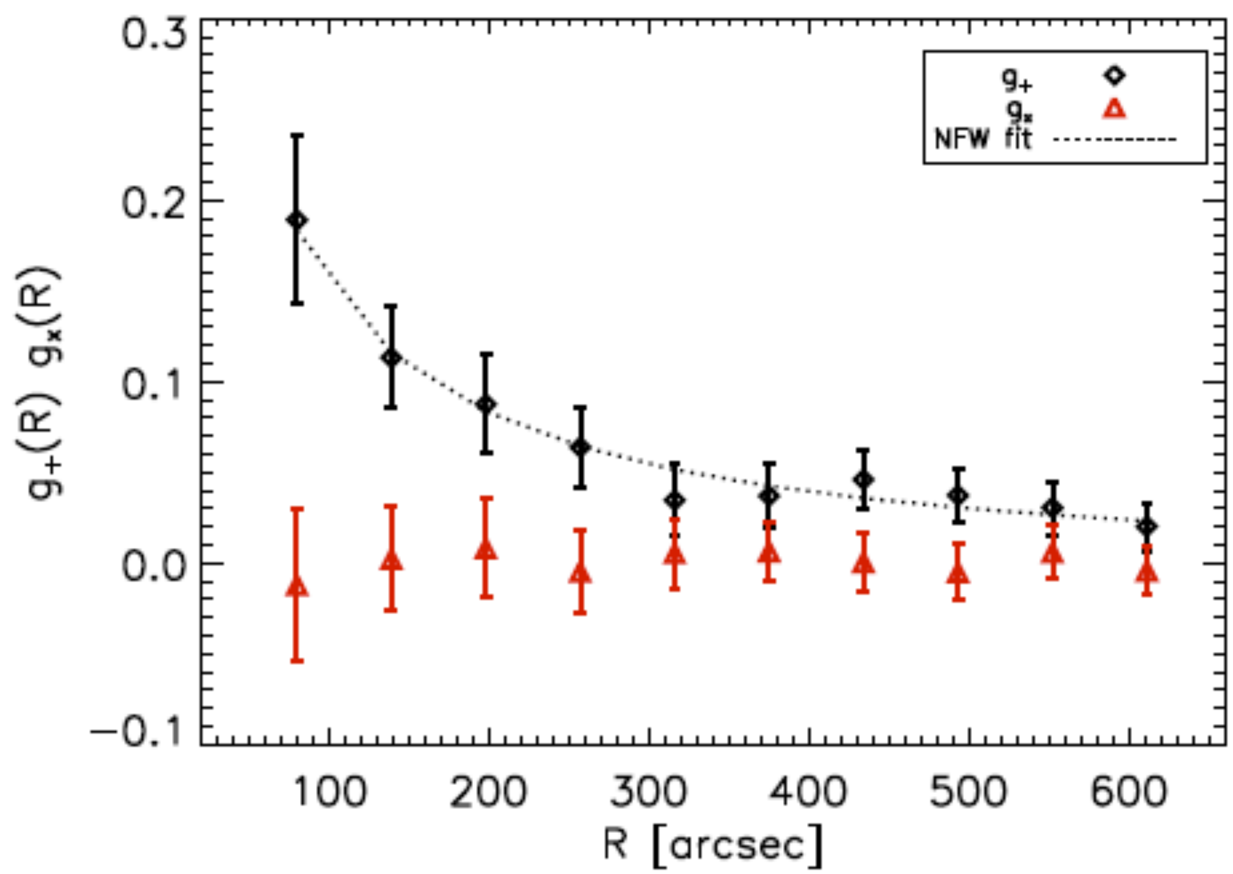
Having measured the tangential shear profile, we can fit it with some parametric model

SIS $\gamma(x) = (\gamma_1^2 + \gamma_2^2)^{1/2} = \frac{1}{2x} = \kappa(x)$

NFW $\kappa(x) = \frac{\Sigma(\xi_0 x)}{\Sigma_{cr}} = 2\kappa_s \frac{f(x)}{x^2 - 1}$

$$f(x) = \begin{cases} 1 - \frac{2}{\sqrt{x^2-1}} \text{ arctan}\sqrt{\frac{x-1}{x+1}} & (x > 1) \\ 1 - \frac{2}{\sqrt{1-x^2}} \text{ arctanh}\sqrt{\frac{1-x}{1+x}} & (x < 1) \\ 0 & (x = 1) \end{cases}$$

$$\gamma(x) = \overline{\kappa}(x) - \kappa(x)$$



$$l_\gamma = \sum_{i=1}^{N_\gamma} \left[\frac{|\epsilon_i - g(\theta_i)|^2}{\sigma^2[g(\theta_i)]} + 2 \ln \sigma[g(\theta_i)] \right]$$

THE KAISER & SQUIRES INVERSION ALGORITHM

Fourier transform: $\hat{f}(k) = \int_{-\infty}^{+\infty} f(x) e^{-ikx} dx \qquad f(x) = \int_{-\infty}^{+\infty} \hat{f}(k) e^{ikx} dk$

$$\hat{f}(\vec{k}) = \int_{-\infty}^{+\infty} f(\vec{x}) e^{-2i\vec{k}\vec{x}} d^2k \quad f(\vec{x}) = \int_{-\infty}^{+\infty} \hat{f}(\vec{k}) e^{2i\vec{k}\vec{x}} d^2x$$

Shear and convergence:

$$\kappa = \frac{1}{2}(\psi_{11} + \psi_{22}) \quad \Rightarrow \quad \hat{\kappa} = -\frac{1}{2}(k_1^2 + k_2^2)\hat{\psi}$$

$$\gamma_1 = \frac{1}{2}(\psi_{11} - \psi_{22}) \quad \Rightarrow \quad \hat{\gamma}_1 = -\frac{1}{2}(k_1^2 - k_2^2)\hat{\psi}$$

$$\gamma_2 = \psi_{12} \quad \Rightarrow \quad \hat{\gamma}_2 = -k_1 k_2 \hat{\psi}\ ,$$

Real space — Fourier space

THE KAISER & SQUIRES INVERSION ALGORITHM

From:

$$\begin{aligned}
\kappa = \frac{1}{2}(\psi_{11} + \psi_{22}) \quad &\Rightarrow \quad \hat{\kappa} = -\frac{1}{2}(k_1^2 + k_2^2)\hat{\psi} \\
\gamma_1 = \frac{1}{2}(\psi_{11} - \psi_{22}) \quad &\Rightarrow \quad \hat{\gamma}_1 = -\frac{1}{2}(k_1^2 - k_2^2)\hat{\psi} \\
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\end{aligned}$$

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$$\begin{pmatrix} \hat{\gamma}_1 \\ \hat{\gamma}_2 \end{pmatrix} = k^{-2} \begin{pmatrix} k_1^2 - k_2^2 \\ 2k_1 k_2 \end{pmatrix} \hat{\kappa}$$

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using:

$$\left[k^{-2} \begin{pmatrix} k_1^2 - k_2^2 \\ 2k_1 k_2 \end{pmatrix} \right] \left[k^{-2} \begin{pmatrix} k_1^2 - k_2^2 & 2k_1 k_2 \end{pmatrix} \right] = 1$$

$$\hat{\kappa} = k^{-2} \begin{pmatrix} k_1^2 - k_2^2 & 2k_1 k_2 \end{pmatrix} \begin{pmatrix} \hat{\gamma}_1 \\ \hat{\gamma}_2 \end{pmatrix} = k^{-2}[(k_1^2 - k_2^2)\hat{\gamma}_1 + 2k_1 k_2 \hat{\gamma}_2]$$

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using:

$$\left[k^{-2} \begin{pmatrix} k_1^2 - k_2^2 \\ 2k_1k_2 \end{pmatrix}\right] [k^{-2}(\; k_1^2 - k_2^2 \quad 2k_1k_2 \;)] = 1$$

$$(f \hat{*} g) = \hat{f}\hat{g}$$

$$\hat{\kappa} = k^{-2}(\; k_1^2 - k_2^2 \quad 2k_1k_2 \;) \begin{pmatrix} \hat{\gamma}_1 \\ \hat{\gamma}_2 \end{pmatrix} = k^{-2}[(k_1^2 - k_2^2)\hat{\gamma}_1 + 2k_1k_2\hat{\gamma}_2]$$

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$$\hat{\kappa} = k^{-2} (\; k_1^2 - k_2^2 \quad 2k_1k_2 \;) \begin{pmatrix} \hat{\gamma}_1 \\ \hat{\gamma}_2 \end{pmatrix} = k^{-2}[(k_1^2 - k_2^2)\hat{\gamma}_1 + 2k_1k_2\hat{\gamma}_2]$$

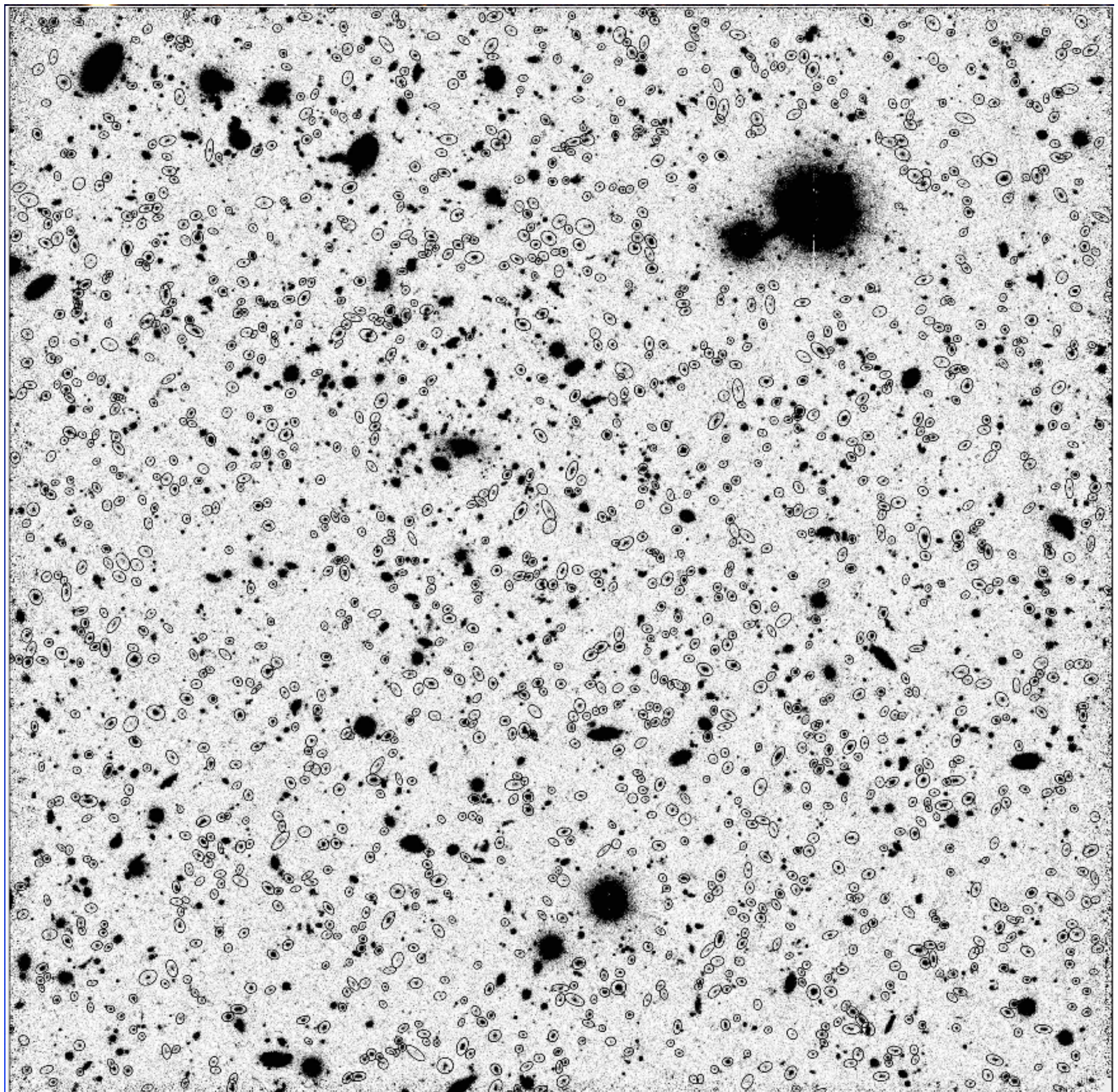
$$(f \hat{*} g) = \hat{f}\hat{g}$$

$$\boxed{\kappa(\vec{\theta}) = \frac{1}{\pi} \int \mathrm{d}^2\theta' [D_1(\vec{\theta} - \vec{\theta}')\gamma_1 + D_2(\vec{\theta} - \vec{\theta}')\gamma_2]}$$

THE KAISER & SQUIRES INVERSION ALGORITHM

CL1232-1250
(Clowe et al.)

THE KAISER & SQUIRES INVERSION ALGORITHM



CL1232-1250
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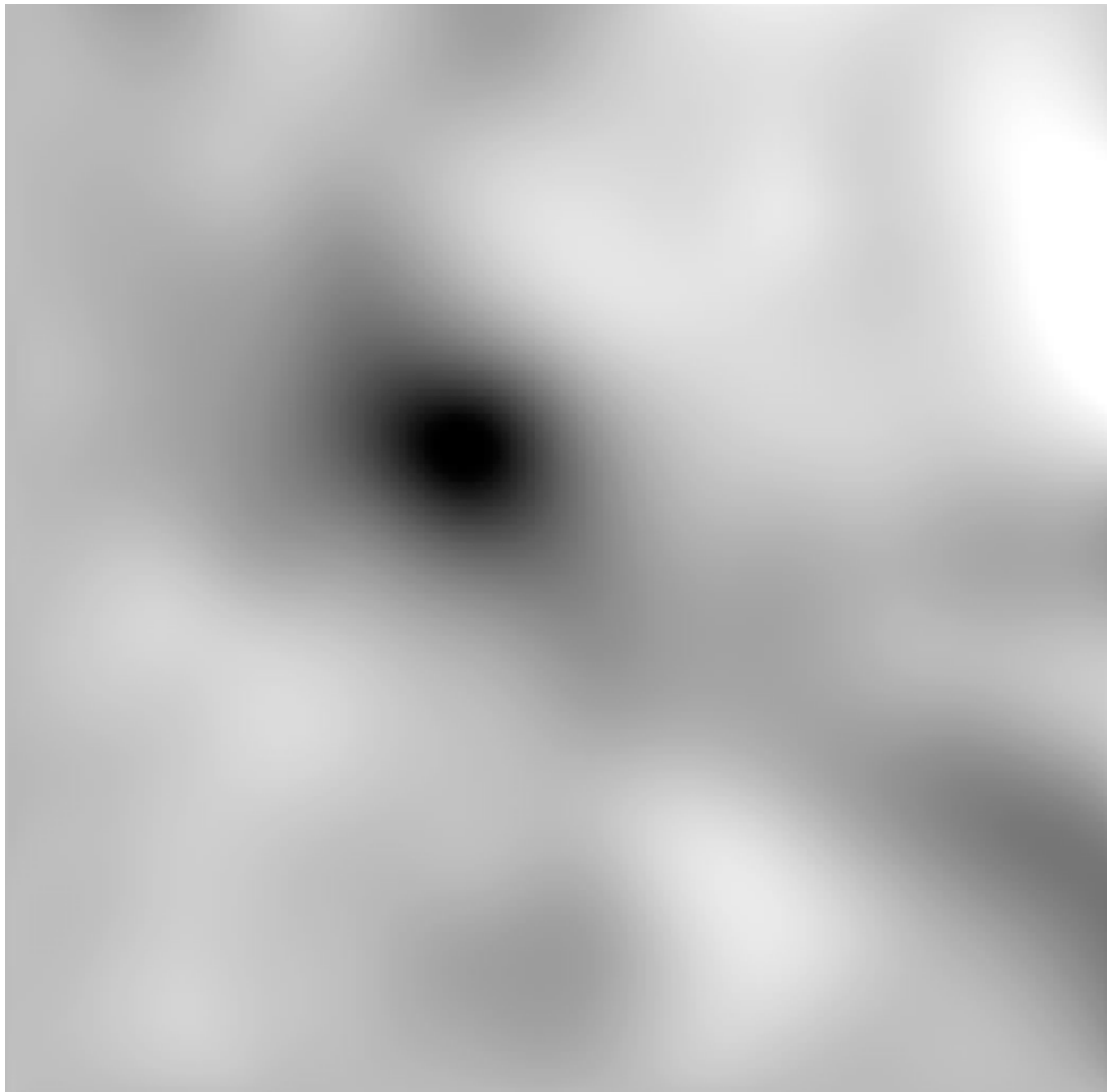
THE KAISER & SQUIRES INVERSION ALGORITHM



CL1232-1250
(Clowe et al.)

THE KAISER & SQUIRES INVERSION ALGORITHM

- infinite fields would be required: wide field + boundary conditions.
- ellipticity measures the reduced shear, not the shear:

$$\kappa(\vec{\theta}) = \frac{1}{\pi} \int \mathrm{d}^2\theta' [D_1(\vec{\theta} - \vec{\theta}')g_1(1-\kappa) + D_2(\vec{\theta} - \vec{\theta}')g_2(1-\kappa)]$$

This equation can be solved iteratively starting from $\kappa=0$

A FUNDAMENTAL LIMIT OF MASS MODELING: THE MASS-SHEET DEGENERACY

$$\psi(\vec{x}) \rightarrow \psi'(\vec{x}) = \frac{1-\lambda}{2}x^2 + \lambda\psi$$

$$\vec{\alpha}'(\vec{x}) = \vec{\nabla}\psi'(\vec{x}) = (1-\lambda)\vec{x} + \lambda\vec{\alpha}(\vec{x})$$

$$\vec{y}' = \vec{x} - \vec{\alpha}'(\vec{x}) = \lambda[\vec{x} - \vec{\alpha}(\vec{x})] = \lambda\vec{y}$$

Thus, under the transformation of the potential above, the image positions are unchanged, provided that we apply an isotropic scaling to the source plane.

A FUNDAMENTAL LIMIT OF MASS MODELING: THE MASS-SHEET DEGENERACY

$$\vec{\alpha}'(\vec{x}) = \vec{\nabla}\psi'(\vec{x}) = (1-\lambda)\vec{x} + \lambda\vec{\alpha}(\vec{x})$$

If we derive further, we obtain:

$$\begin{aligned} \kappa &\rightarrow \kappa' = (1-\lambda) + \lambda\kappa \\ \gamma &\rightarrow \gamma' = \lambda\gamma \,, \end{aligned}$$

This is called the "mass-sheet degeneracy" (Falco et al. 1985). Note that:

$$\begin{aligned} \lambda_t &\rightarrow \lambda_t' = \lambda\lambda_t \\ \lambda_r &\rightarrow \lambda_r' = \lambda\lambda_r \,. \end{aligned}$$

Thus, the transformation does not change the location of the critical lines

MASS SHEET DEGENERACY FOR GALAXY SHAPES

The mass sheet transformation on shear and convergence is:

$$\kappa \to \kappa' = (1-\lambda) + \lambda\kappa$$

$$\gamma \to \gamma' = \lambda\gamma$$

The ellipticity is then: $$\epsilon' = g' = \frac{\lambda\gamma}{1-(1-\lambda)-\lambda\kappa} = \frac{\lambda\gamma}{\lambda(1-\kappa)} = g = \epsilon$$

Thus, weak lensing is also variant under mass-sheet transformations!

AFFECTED QUANTITIES: TIME DELAYS AND MAGNIFICATIONS

$$
\begin{aligned}
\Delta t' &\propto \frac{1}{2}(\vec{x}-\vec{y}')^2 - \psi'(\vec{x}) \\
&= \frac{\vec{x}-\lambda\vec{y}}{2} - \lambda\psi(\vec{x}) - \frac{1-\lambda}{2}x^2 \\
&= \lambda\left[\frac{1}{2}(\vec{x}-\vec{y})^2 - \psi(\vec{x})\right] + \frac{\lambda(\lambda-1)}{2}y^2 \\
&= \lambda\Delta t + \mathrm{const}
\end{aligned}
$$

$$
\mu' = (\lambda_t'\lambda_r')^{-1} = (\det A')^{-1} = (\lambda^2 \det A)^{-1} = \frac{\mu}{\lambda^2}
$$

Time delays and magnifications are changed by mass-sheet transformations.

In principle, they can be used to break the degeneracy.

BREAKING THE DEGENERACY

- Complementary measurements of the mass profiles
  - Example: using stellar kinematics, in the case of an elliptical galaxy
- Adopting a shape for the mass profile
- Assuming that the convergence goes to zero at large distances from the center of the lens
- Using sources at different redshifts
- Measuring the magnification statistically, or via galaxy number counts

YET ANOTHER LIMIT: PERTURBATIONS ALONG THE LINE OF SIGHT