GRAVITATIONAL LENSING

16 - AXIALLY SYMMETRIC LENSES

Massimo Meneghetti

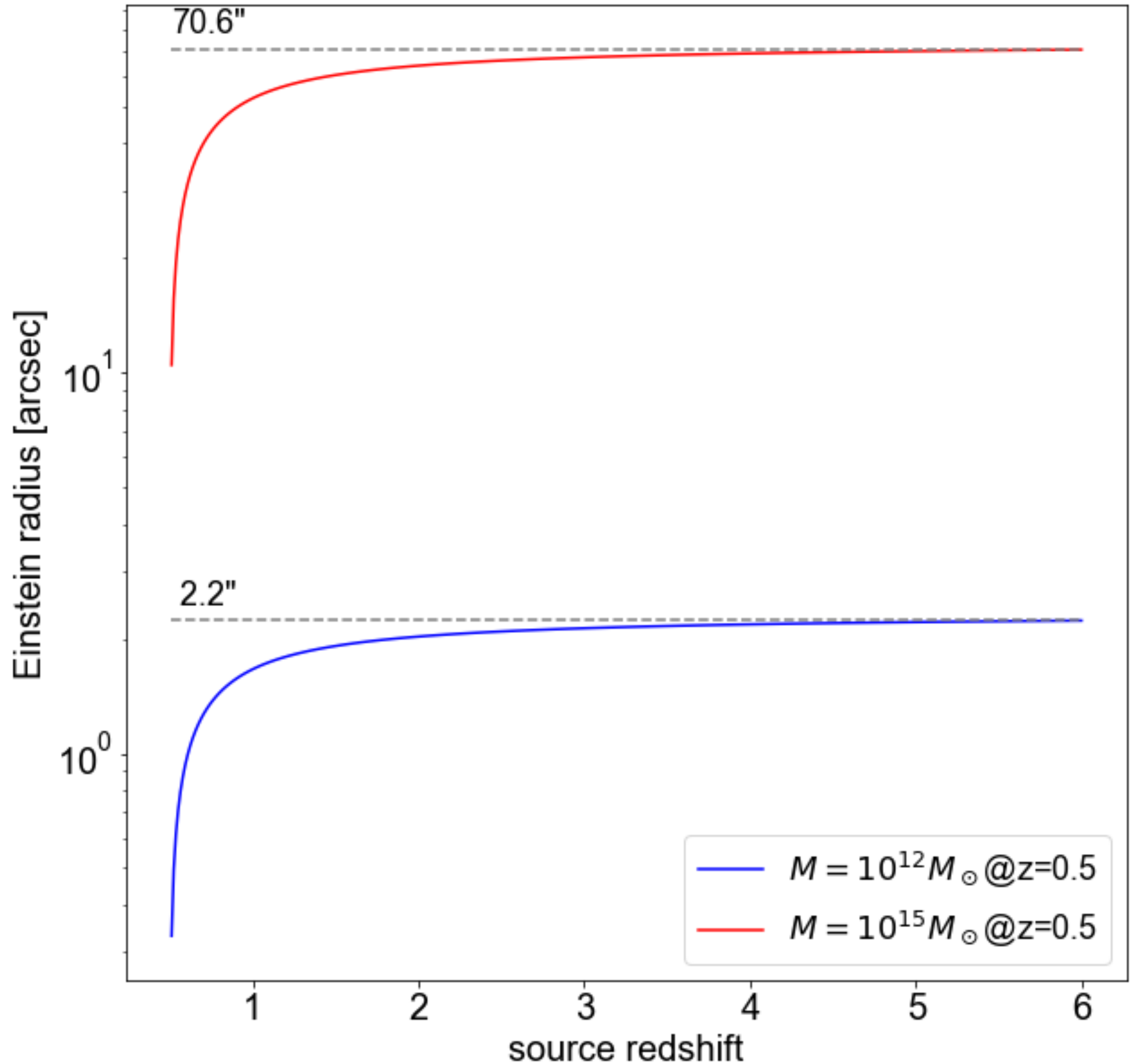
AA 2019-2020

LENSING BY GALAXIES AND GALAXY CLUSTERS

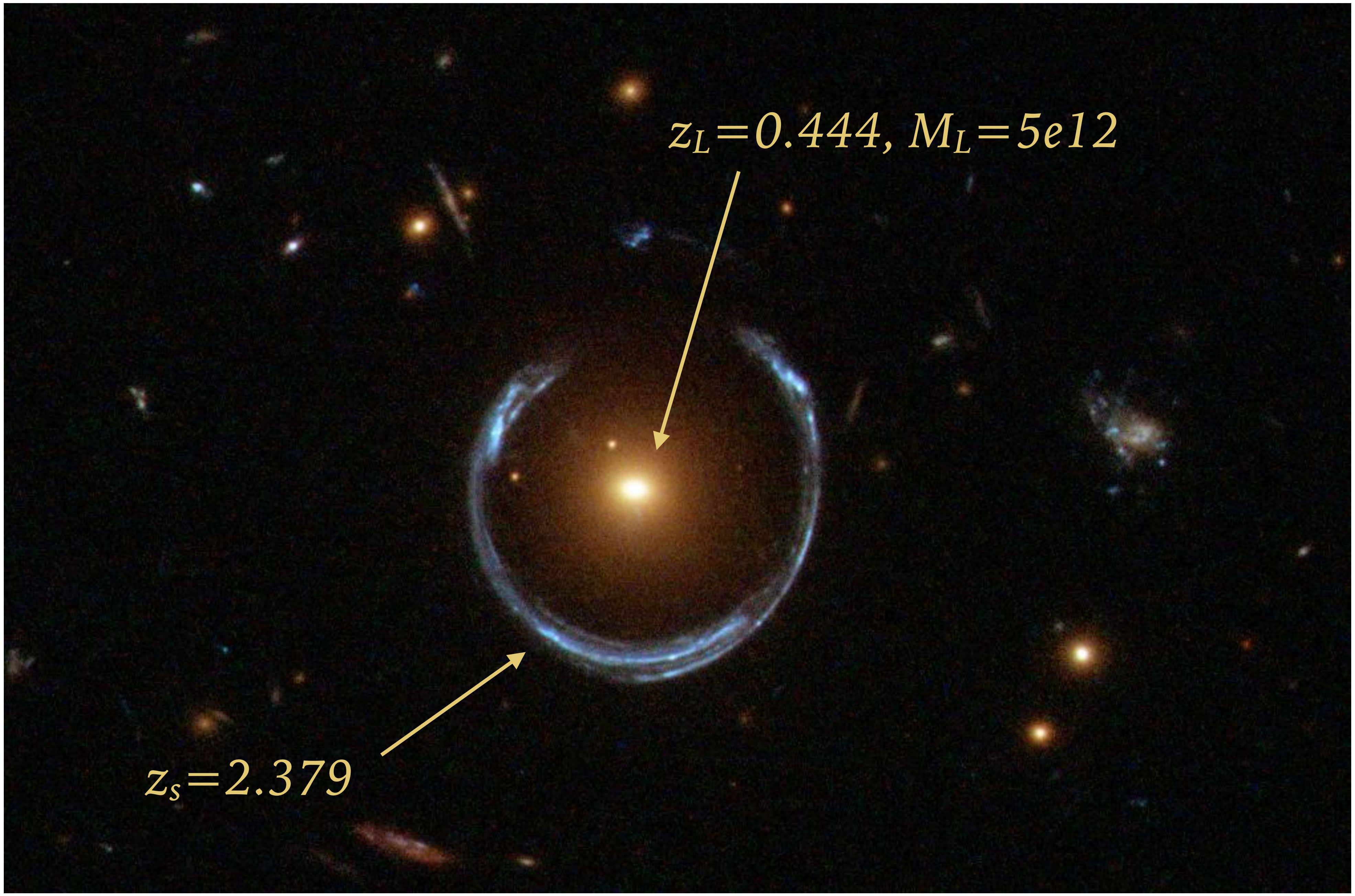
- Much more massive than stars!
- At cosmological distances
- Extended!
- Complex (more parameters)

Note: computed using

$$\theta_E = \sqrt{\frac{4GM}{c^2}\frac{D_{LS}}{D_L D_S}}$$

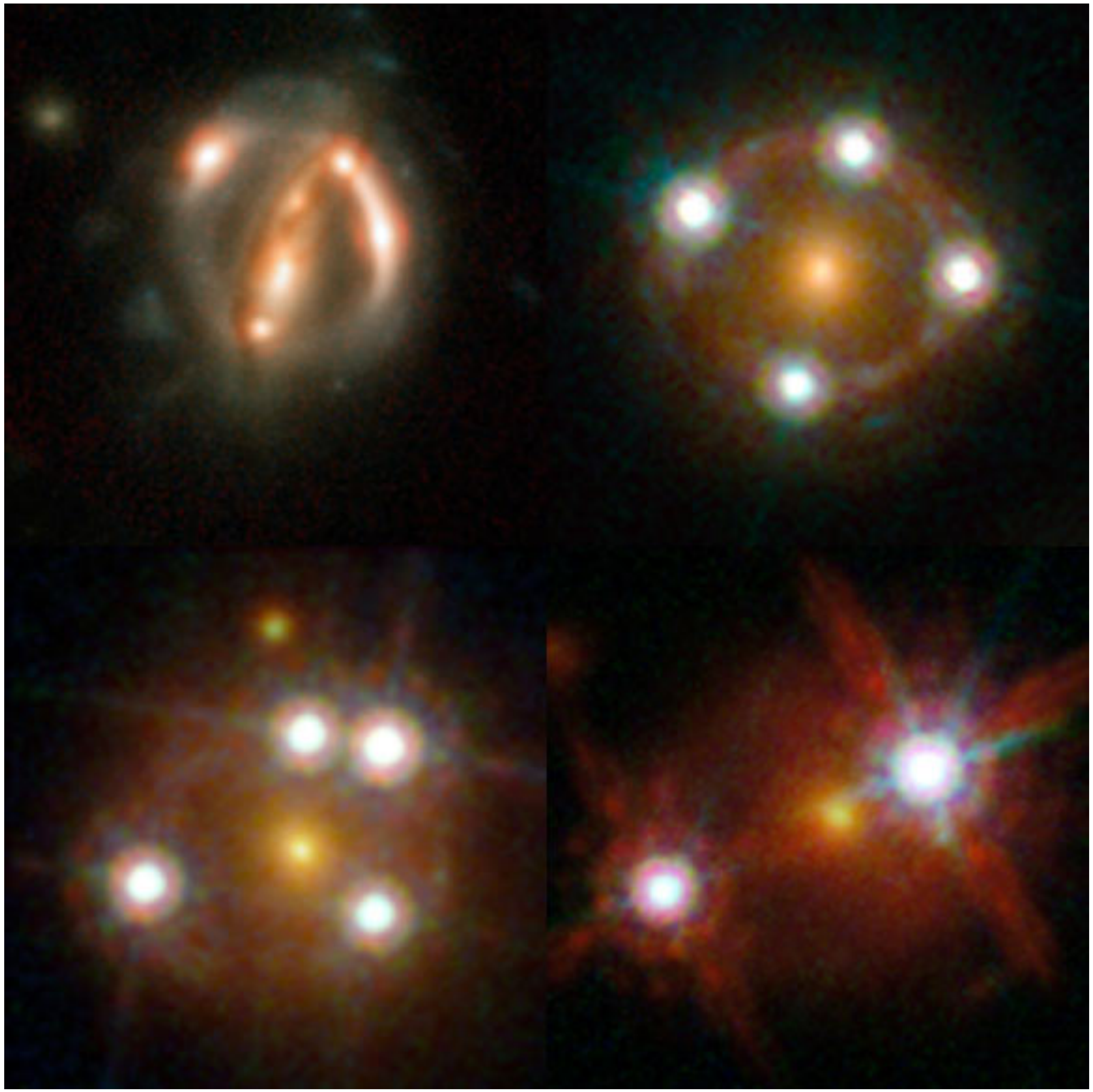


EXTENDED LENSES

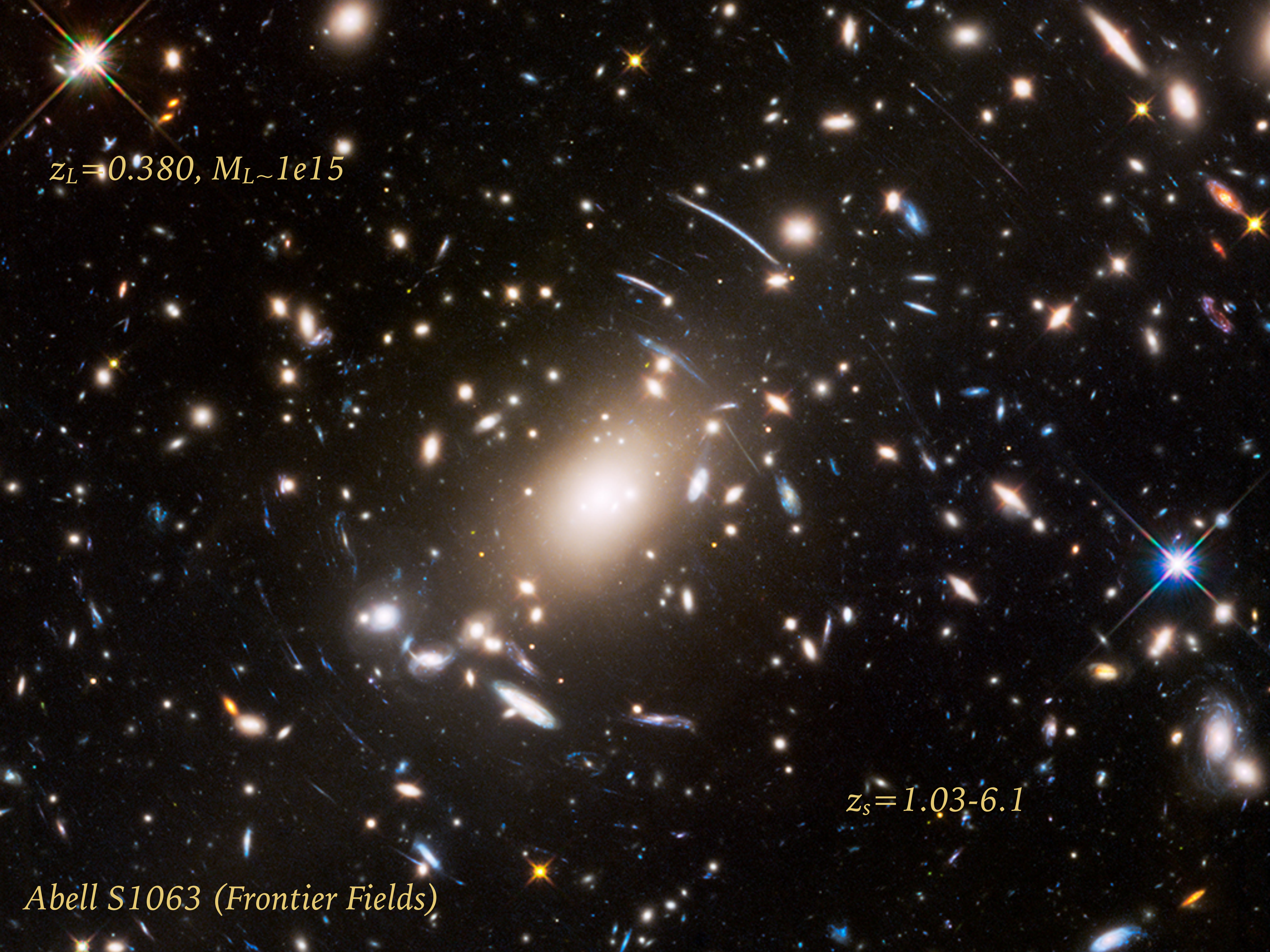


Cosmic horseshoe (Belokurov et al. 2007)

EXTENDED LENSES

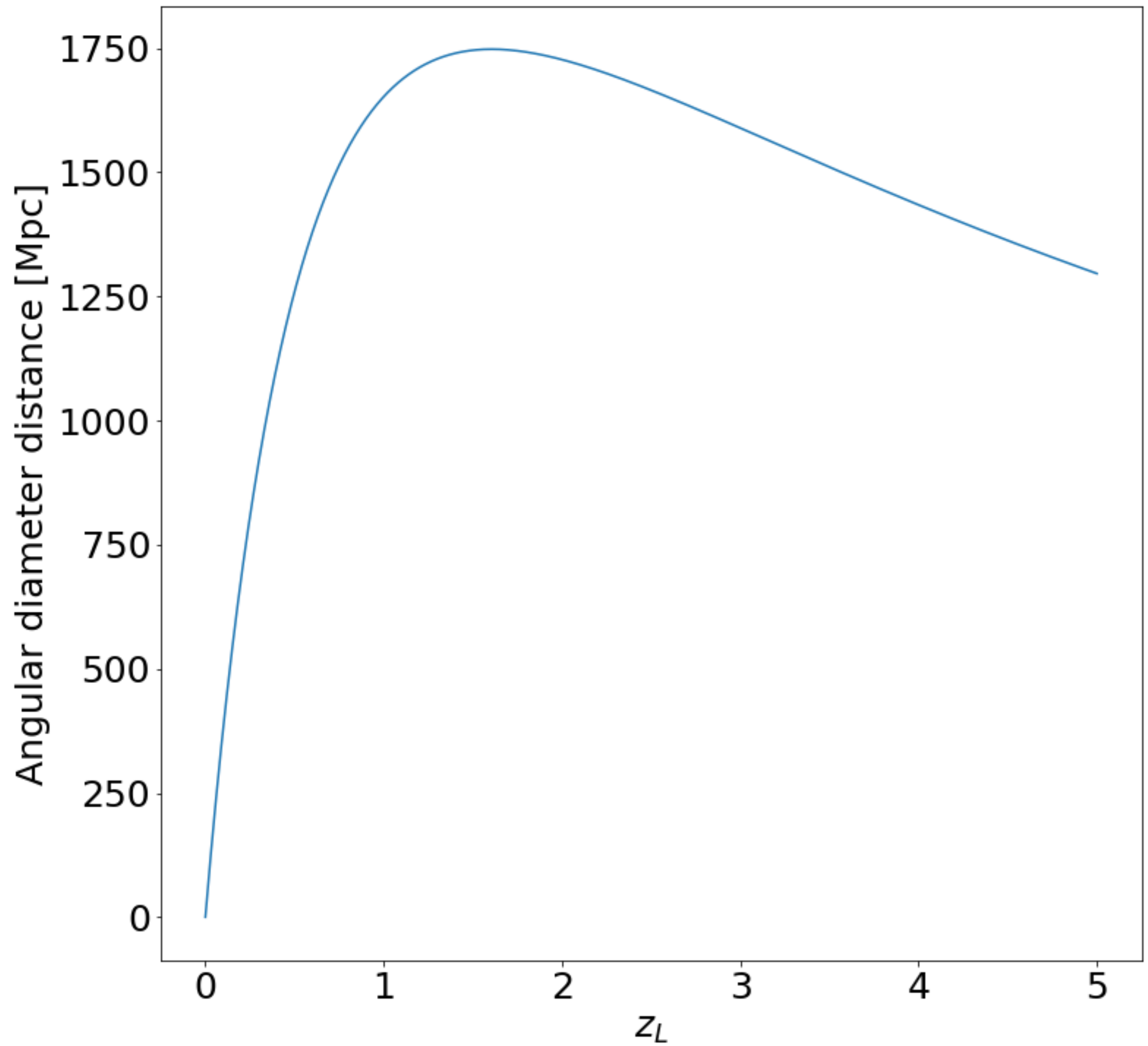


Suyu et al. (H0LiCOW team)


z_L=0.380, M_L~1e15
z_s=1.03-6.1
Abell S1063 (Frontier Fields)

LENSING BY GALAXIES AND GALAXY CLUSTERS

- Much more massive than stars!
- At cosmological distances
- Extended!
- Complex (more parameters)

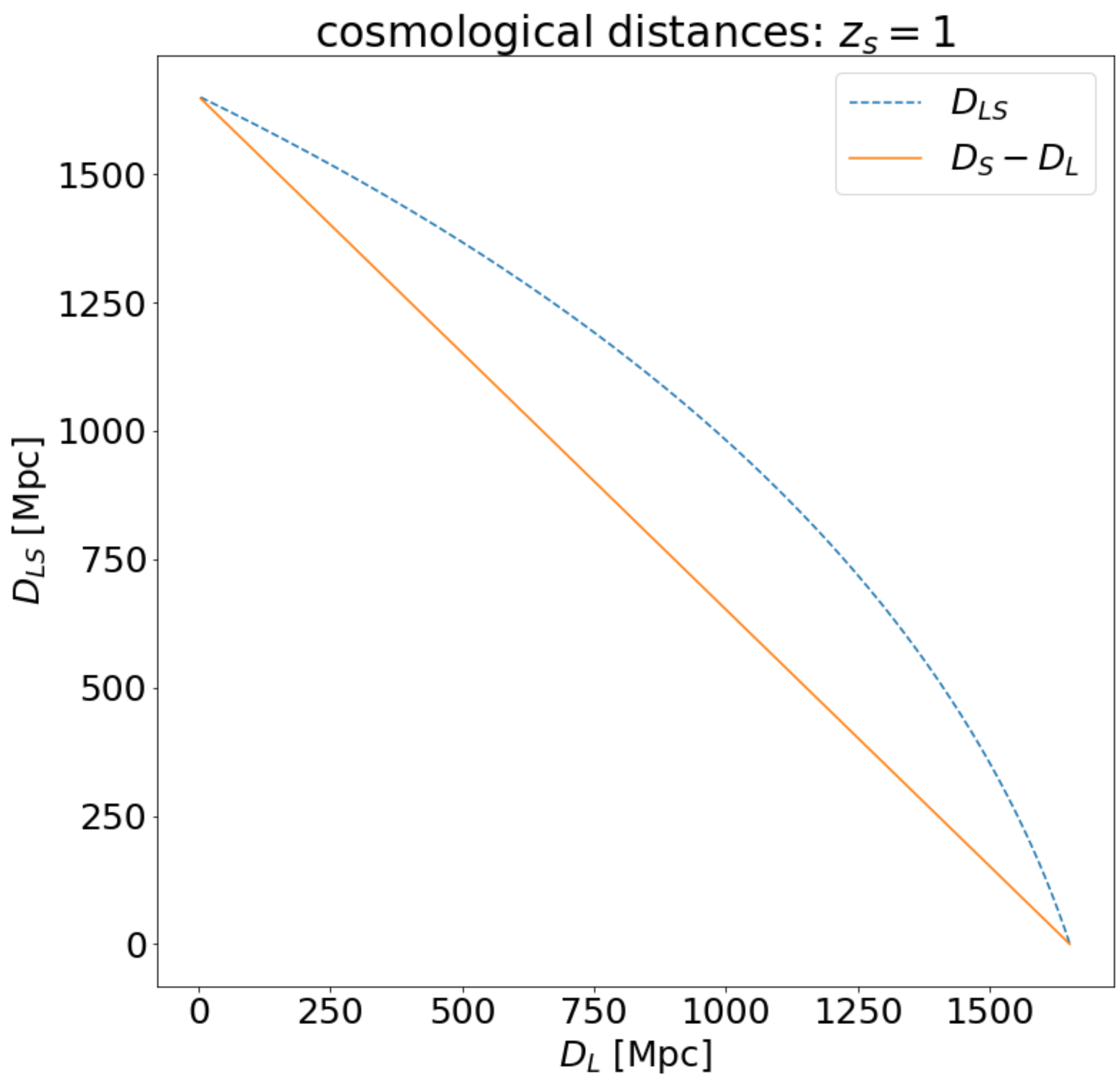


$$D_A(z) = \frac{c}{1+z}\int_0^z \frac{dz}{H(z)}$$

$$H^2(z) = H_0^2[\Omega_m(z)(1+z)^3 + \Omega_{DE}(1+z)^{3(1+w)}]$$

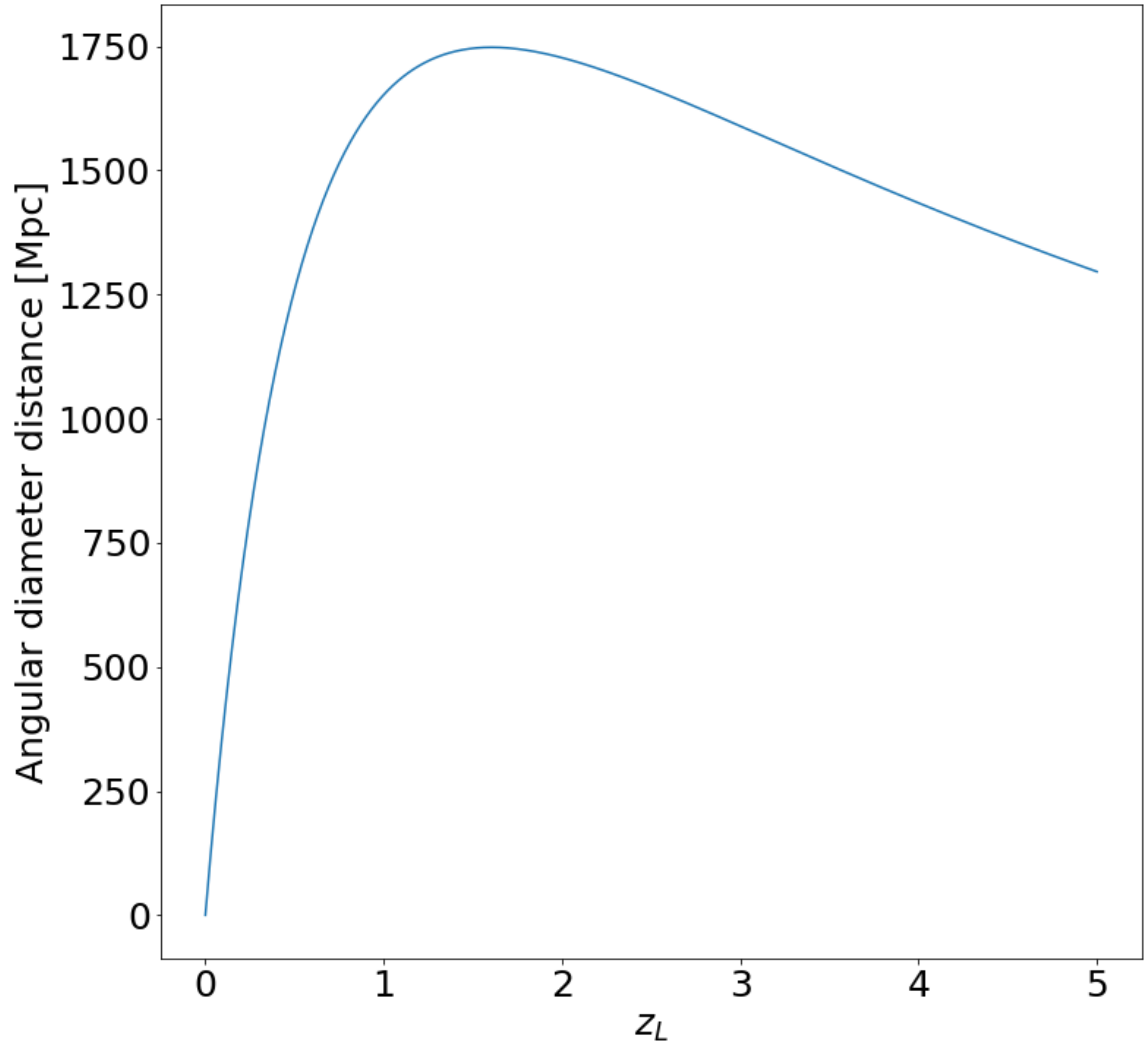
AGAIN ON ANGULAR DIAMETER DISTANCES…

On cosmological scales $D_{LS} \neq D_S - D_L$



LENSING BY GALAXIES AND GALAXY CLUSTERS

- Much more massive than stars!
- At cosmological distances
- Extended!
- Complex (more parameters)



$$D_A(z) = \frac{c}{1+z}\int_0^z \frac{dz}{H(z)}$$

$$H^2(z) = H_0^2[\Omega_m(z)(1+z)^3 + \Omega_{DE}(1+z)^{3(1+w)}]$$

EXTENDED LENSES

- Cosmic structures like galaxies and galaxy clusters are characterized by bound mass distributions, which cannot be approximated by point lenses
- Indeed these are *extended lenses*, and their lensing properties are determined by e.g. their surface mass density:

$$\Sigma(\vec{\xi}) = \int \rho(\vec{\xi}, z)\ \mathrm{d}z$$

$$\vec{\alpha}(\vec{\xi}) = \frac{4G}{c^2} \int \frac{(\vec{\xi} - \vec{\xi'})\Sigma(\vec{\xi'})}{|\vec{\xi} - \vec{\xi'}|^2}\ \mathrm{d}^2\xi'$$

EXTENDED LENSES

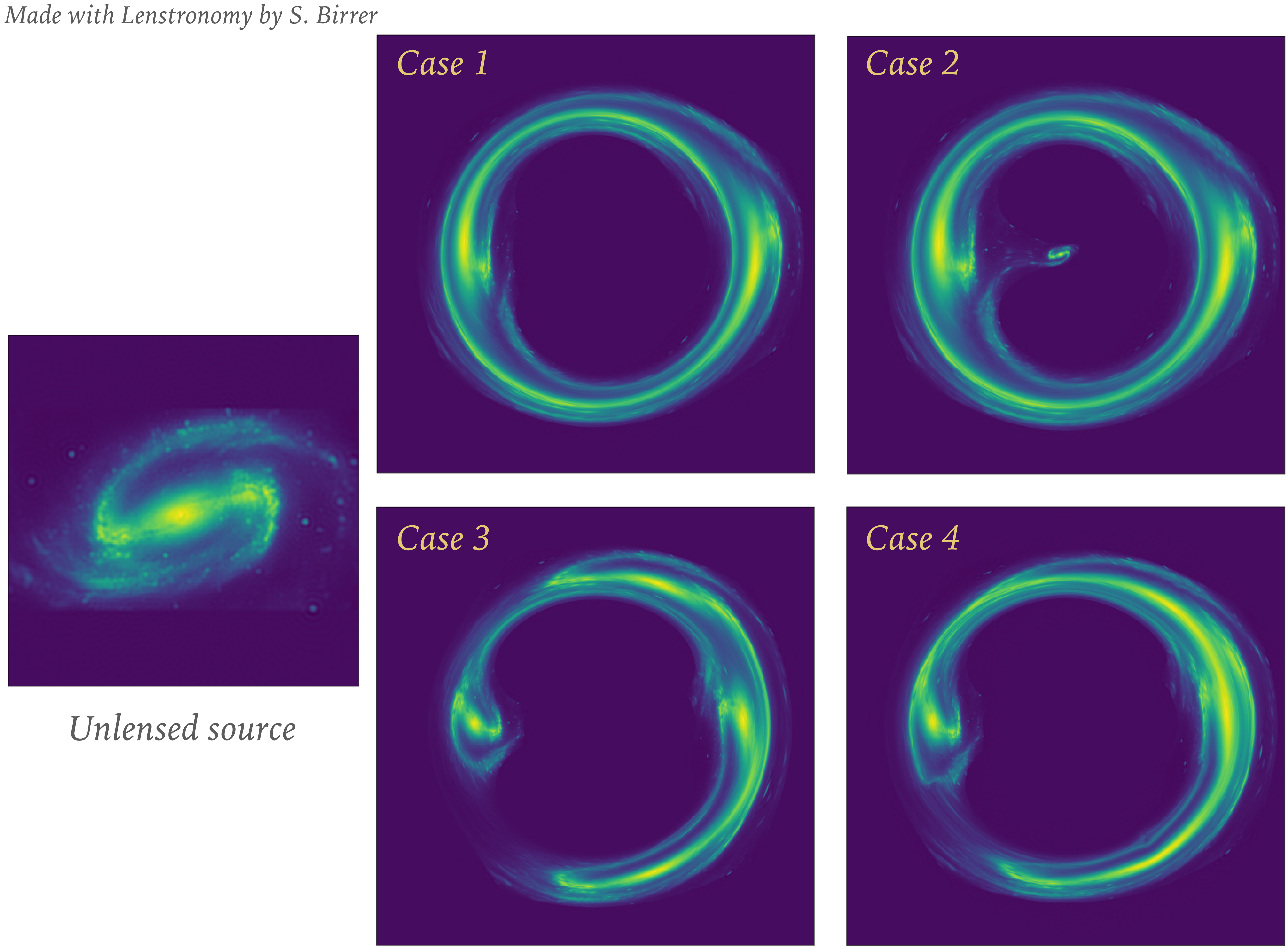
- Recall that the surface density is related to the lensing potential by

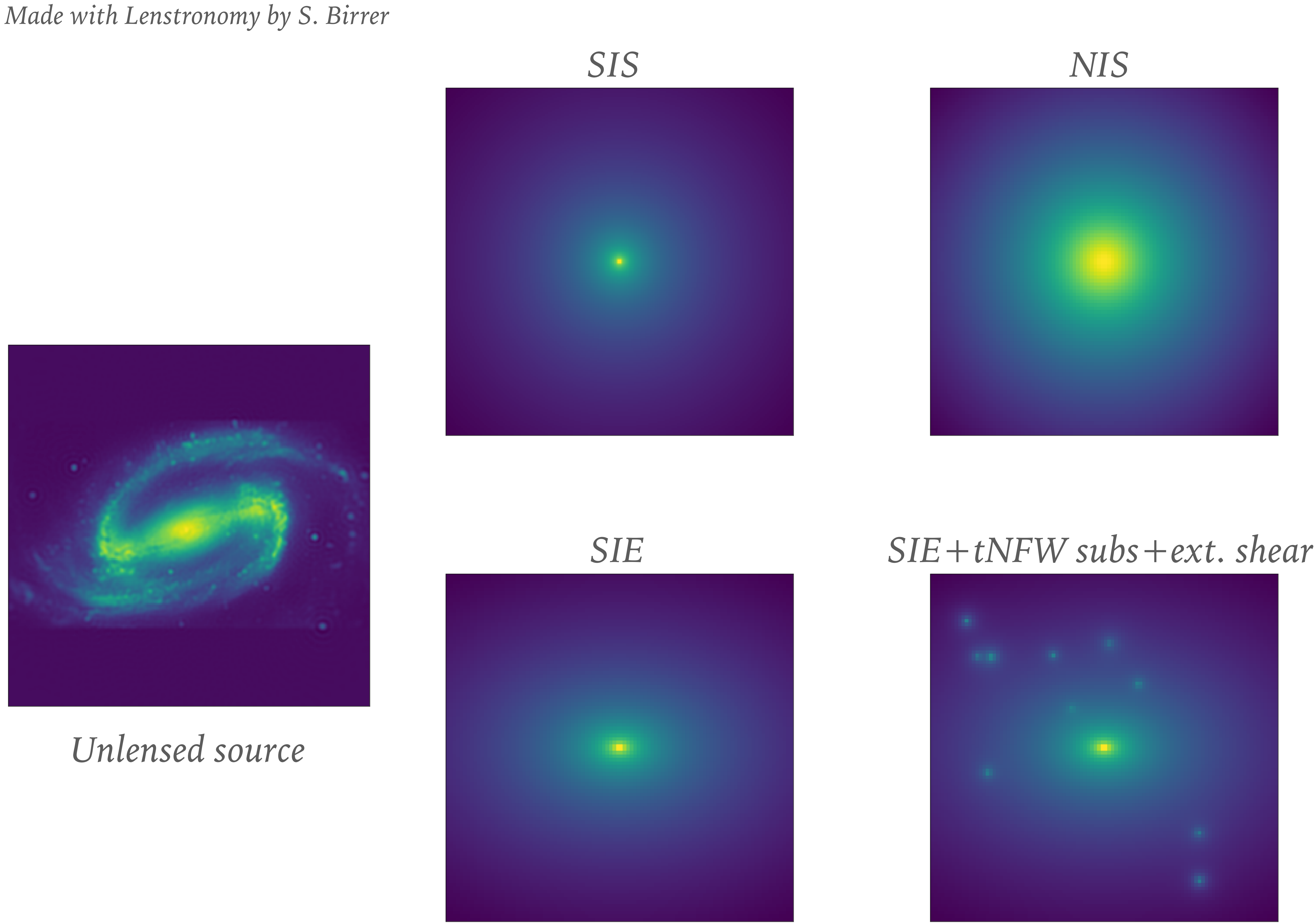
$$\triangle_\theta \Psi(\vec{\theta}) = 2\kappa(\vec{\theta})$$

$$\kappa(\vec{\theta}) \equiv \frac{\Sigma(\vec{\theta})}{\Sigma_{\rm cr}} \quad \text{with} \quad \Sigma_{\rm cr} = \frac{c^2}{4\pi G} \frac{D_{\rm S}}{D_{\rm L} D_{\rm LS}}$$

LENSING BY GALAXIES AND GALAXY CLUSTERS

- Much more massive than stars!
- At cosmological distances
- Extended!
- Complex (more parameters)


Made with Lenstronomy by S. Birrer
Case 1
Case 2
Unlensed source
Case 3
Case 4

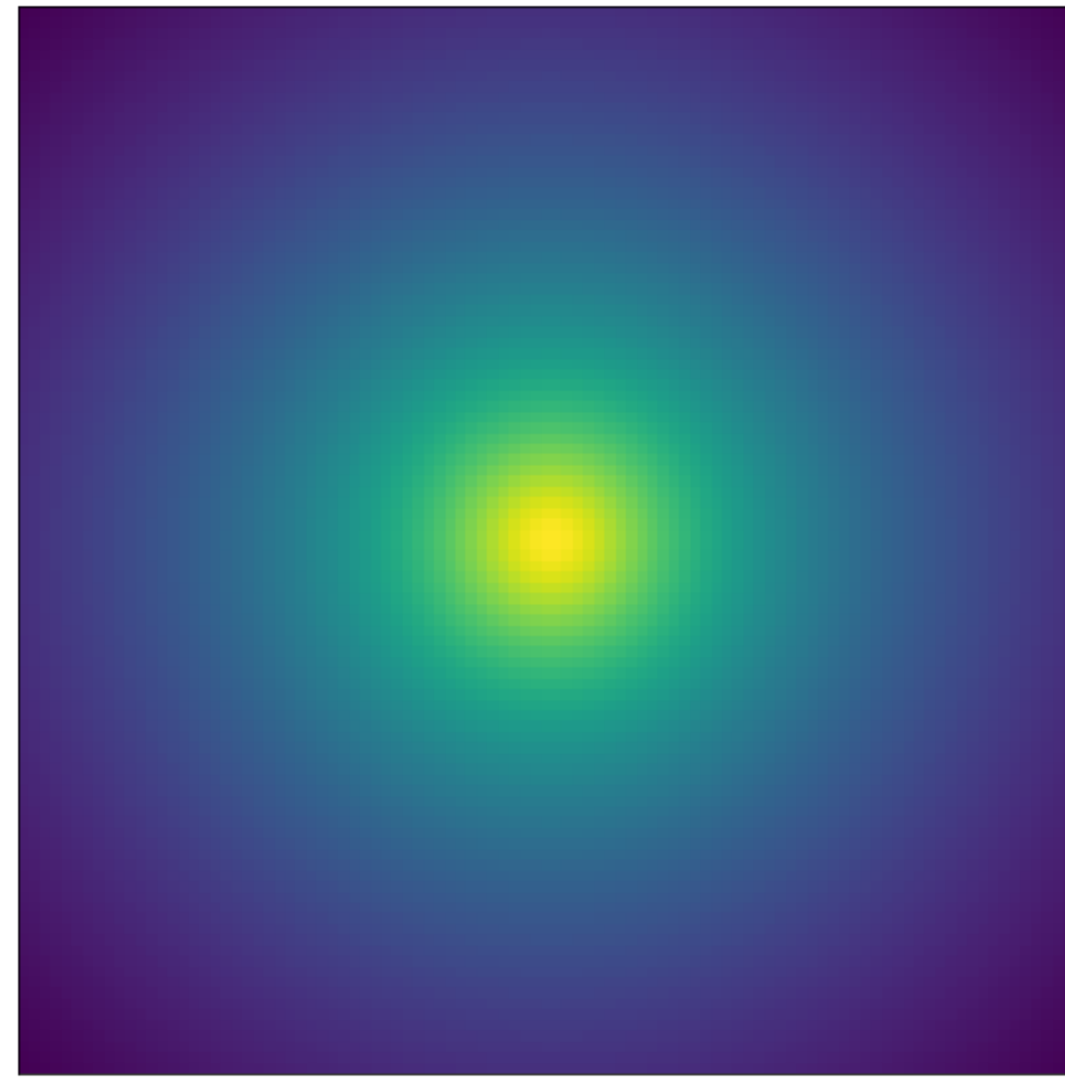

Made with Lenstronomy by S. Birrer
SIS
NIS
Unlensed source
SIE
SIE+tNFW subs+ext. shear

WHAT ARE THE RELEVANT PROPERTIES OF THE LENSES?

- The surface density distribution of a lens (and its potential) can be characterized by means of
  - the profile
  - the shape of the iso-density (iso-potential) contours
  - the smoothness
  - the environment where the lens resides
- In this and in the following lessons, we will study how these features determine the ability of a mass distribution to produce lensing effects.
- We will do that by building analytical models with increasing level of complexity.

AXIALLY SYMMETRIC, CIRCULAR LENSES

- Axially symmetric, circular models are the simplest lens models for describing extended mass distributions
- For these lenses $\hat{\Psi}(\overrightarrow{\theta}) = \hat{\Psi}(\theta)$
- Several quantities relevant for lensing can be derived in a simple manner by using the symmetry properties of the lens.
- One example is the deflection angle…



POLAR COORDINATES IN THE LENS OR IN THE SOURCE PLANES

$$\bar{\xi} = \xi_1 \cdot \bar{e}_{\xi_1} + \xi_2 \cdot \bar{e}_{\xi_2}$$

$$\xi_1 = \xi \cdot \cos\phi \qquad \xi_2 = \xi \cdot \sin\phi$$

$$\bar{e}_{\xi_1} = \cos\phi \cdot \bar{e}_{\xi} - \sin\phi \cdot \bar{e}_{\phi}$$

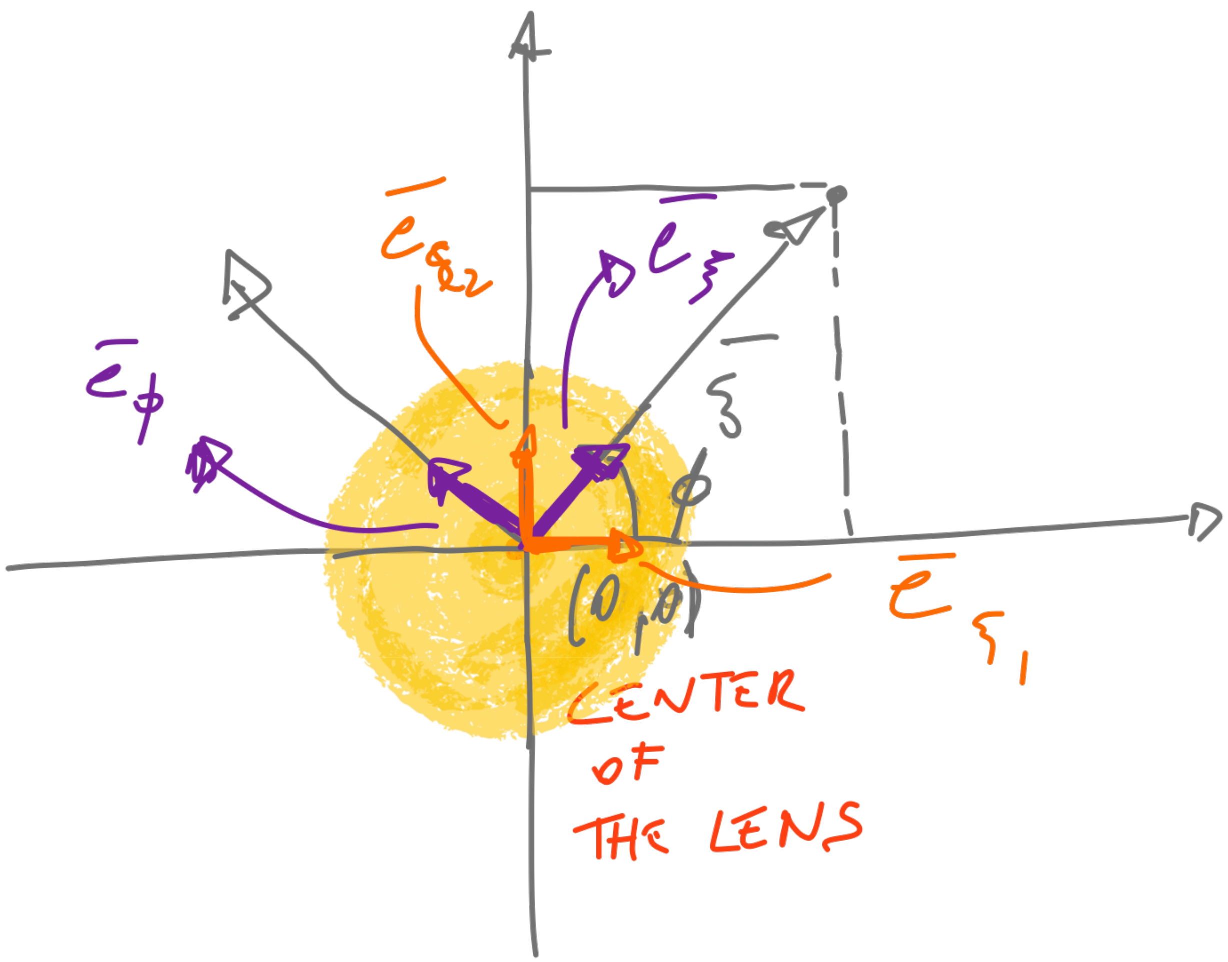
$$\bar{e}_{\xi_2} = \sin\phi \cdot \bar{e}_{\xi} + \cos\phi \cdot \bar{e}_{\phi}$$

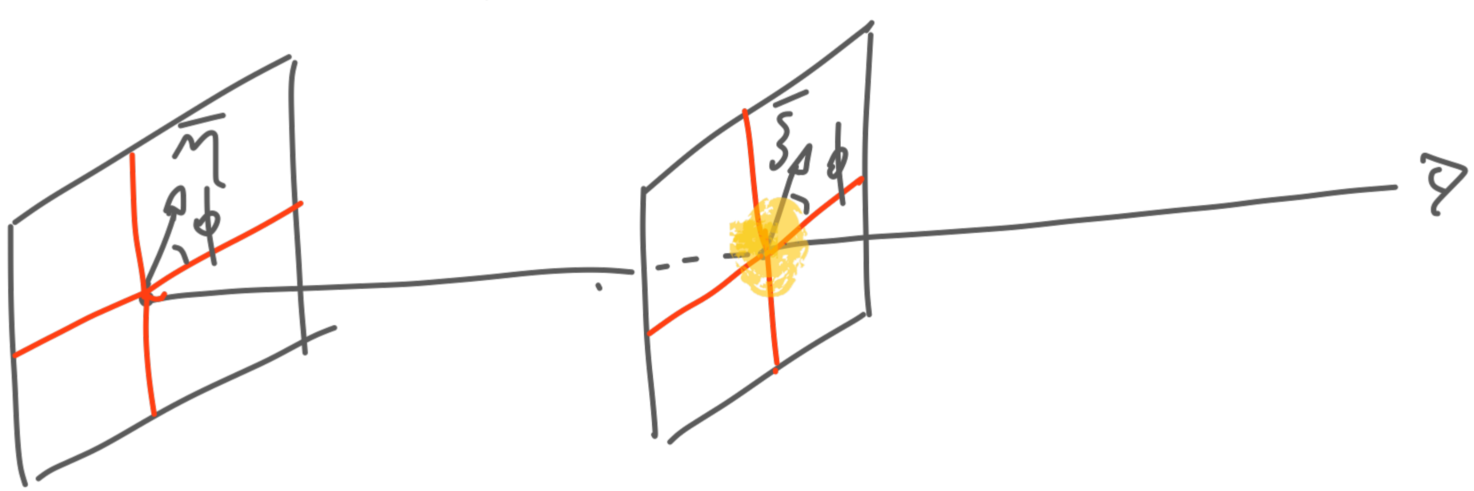
Analogously:

$$\eta_1 = \eta \cdot \cos\phi$$

$$\eta_2 = \eta \cdot \sin\phi$$

...





DEFLECTION ANGLE OF AN AXIALLY SYMMETRIC LENS

$$\vec{\nabla}_{\xi} = \frac{\partial}{\partial \xi_1}\vec{e}_{\xi_1} + \frac{\partial}{\partial \xi_2}\vec{e}_{\xi_2} = \frac{\partial}{\partial \xi}\vec{e}_{\xi} + \frac{1}{\xi}\frac{\partial}{\partial \phi}\vec{e}_{\phi}$$

$$\vec{\nabla}_{\theta} = D_L\left(\frac{\partial}{\partial \xi}\vec{e}_{\xi} + \frac{1}{\xi}\frac{\partial}{\partial \phi}\vec{e}_{\phi}\right) = \left(\frac{\partial}{\partial \theta}\vec{e}_{\theta} + \frac{1}{\theta}\frac{\partial}{\partial \phi}\vec{e}_{\phi}\right)$$

$$\vec{\nabla}_{\theta}\hat{\Psi}(\theta) = \frac{\partial \hat{\Psi}(\theta)}{\partial \theta}\vec{e}_{\theta} = \Psi'(\theta)\vec{e}_{\theta} = \alpha(\theta)\vec{e}_{\theta} = \vec{\alpha}(\vec{\theta})$$

For an axially symmetric lens, the deflection is "radial": it depends only on the distance from the lens center.

DEFLECTION ANGLE OF AN AXIALLY SYMMETRIC LENS

$$\nabla_\theta^2 = \frac{1}{\theta}\frac{\partial}{\partial\theta}\left(\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\theta^2}\frac{\partial^2}{\partial\phi^2}$$

$$\frac{1}{\theta}\frac{\partial}{\partial\theta}\left(\theta\frac{\partial}{\partial\theta}\right)\hat{\Psi}(\theta) = 2\kappa(\theta)$$

From this equation, we obtain

$$\begin{aligned}\alpha(\theta) &= \frac{2\int_0^\theta \kappa(\theta')\theta' d\theta'}{\theta} \\ &= \frac{2\int_0^\theta \Sigma(\theta')\theta' d\theta'}{\theta\Sigma_{\rm cr}} \\ &= \frac{D_{\rm LS}}{D_{\rm S}}\frac{4GM(\theta)}{c^2 D_{\rm L}\theta} \\ &= \frac{D_{\rm LS}}{D_{\rm S}}\hat{\alpha}(\theta)\,.\end{aligned}$$

$$\kappa(\theta) = \frac{\Sigma(\theta)}{\Sigma_{cr}}$$

$$\Sigma_{cr} = \frac{c^2}{4\pi G}\frac{D_S}{D_L D_{LS}}$$

DEFLECTION ANGLE OF AN AXIALLY SYMMETRIC LENS

$$\nabla_\theta^2 = \frac{1}{\theta}\frac{\partial}{\partial\theta}\left(\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\theta^2}\frac{\partial^2}{\partial\phi^2}$$

$$\frac{1}{\theta}\frac{\partial}{\partial\theta}\left(\theta\frac{\partial}{\partial\theta}\right)\hat{\Psi}(\theta) = 2\kappa(\theta)$$

From this equation, we obtain

$$\begin{aligned} \alpha(\theta) &= \frac{2\int_0^\theta \kappa(\theta')\theta' d\theta'}{\theta} \\ &= \frac{2\int_0^\theta \Sigma(\theta')\theta' d\theta'}{\theta\Sigma_{\rm cr}} \\ &= \frac{D_{\rm LS}}{D_{\rm S}}\frac{4GM(\theta)}{c^2 D_{\rm L}\theta} \\ &= \frac{D_{\rm LS}}{D_{\rm S}}\hat{\alpha}(\theta)\,. \end{aligned}$$

$$\kappa(\theta) = \frac{\Sigma(\theta)}{\Sigma_{cr}}$$

$$\Sigma_{cr} = \frac{c^2}{4\pi G}\frac{D_S}{D_L D_{LS}}$$

Identical to point-mass lens!

DEFLECTION ANGLE OF AN AXIALLY SYMMETRIC LENS

Dimensionless form:

$$
\begin{aligned}
\alpha(x) &= \frac{D_{\rm L}D_{\rm LS}}{\xi_0 D_{\rm S}}\hat{\alpha}(\xi_0 x) \\
&= \frac{D_{\rm L}D_{\rm LS}}{\xi_0 D_{\rm S}}\frac{4GM(\xi_0 x)}{c^2\xi}\frac{\pi\xi_0}{\pi\xi_0} \\
&= \frac{M(\xi_0 x)}{\pi\xi_0^2\Sigma_{\rm cr}}\frac{1}{x} \equiv \frac{m(x)}{x}\,,
\end{aligned}
$$

Dimensionless mass

$$
\alpha(x) = \frac{2}{x}\int_0^x x'\kappa(x')\mathrm{d}x' \Rightarrow m(x) = 2\int_0^x x'\kappa(x')\mathrm{d}x'
$$

LENS EQUATION

$$\vec{y} = \vec{x} - \vec{\alpha}(\vec{x}) \qquad \vec{\alpha}(\vec{x}) = \frac{m(\vec{x})}{x}\vec{e}_x$$

Given that the deflection angle and x are parallel, so will be y!

$$y = x - \alpha(x) = x - \frac{m(x)}{x}$$

CONVERGENCE

$$\frac{1}{\theta}\frac{\partial}{\partial\theta}\left(\theta\frac{\partial}{\partial\theta}\right)\hat{\Psi}(\theta) = 2\kappa(\theta) \rightarrow \frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial}{\partial x}\right)\Psi(x) = 2\kappa(x)$$

$$\kappa(\theta) = \frac{1}{2}\left(\hat{\Psi}''(\theta) + \frac{\hat{\Psi}'(\theta)}{\theta}\right) \rightarrow \kappa(x) = \frac{1}{2}\left(\Psi''(x) + \frac{\Psi'(x)}{x}\right)$$

$$\hat{\Psi}'(\theta) = \alpha(\theta)$$

$$\kappa(\theta) = \frac{1}{2}\left(\alpha'(\theta) + \frac{\alpha(\theta)}{\theta}\right) \rightarrow \kappa(x) = \frac{1}{2}\left(\alpha'(x) + \frac{\alpha(x)}{x}\right)$$

$$\alpha'(x) = \frac{m'(x)}{x} - \frac{m(x)}{x^2}$$

$$\kappa(x) = \frac{1}{2}\frac{m'(x)}{x}$$

SHEAR

The shear components are derived from the second derivatives of the potential or from the first derivatives of the deflection angle components:

$$\frac{\partial}{\partial\theta_1} = \cos\phi\frac{\partial}{\partial\theta} - \frac{\sin\phi}{\theta}\frac{\partial}{\partial\phi}$$

$$\frac{\partial}{\partial\theta_2} = \sin\phi\frac{\partial}{\partial\theta} + \frac{\cos\phi}{\theta}\frac{\partial}{\partial\phi}$$

$$\alpha_1 = \alpha\cos\phi$$

$$\alpha_2 = \alpha\sin\phi$$

SHEAR

$$\frac{\partial}{\partial\theta_1} = \cos\phi\frac{\partial}{\partial\theta} - \frac{\sin\phi}{\theta}\frac{\partial}{\partial\phi}$$

$$\frac{\partial}{\partial\theta_2} = \sin\phi\frac{\partial}{\partial\theta} + \frac{\cos\phi}{\theta}\frac{\partial}{\partial\phi}$$

$$\alpha_1 = \alpha\cos\phi$$

$$\alpha_2 = \alpha\sin\phi$$

$$\begin{aligned}
\gamma_1(\theta) &= \frac{1}{2}\left[\frac{\partial}{\partial\theta_1}\alpha_1(\theta) - \frac{\partial}{\partial\theta_2}\alpha_2(\theta)\right] \\
&= \frac{1}{2}\left[(\cos^2\phi - \sin^2\phi)\alpha'(\theta) - (\cos^2\phi - \sin^2\phi)\frac{\alpha(\theta)}{\theta}\right] \\
&= \frac{\cos 2\phi}{2}\left[\alpha'(\theta) - \frac{\alpha(\theta)}{\theta}\right] ,
\end{aligned}$$

SHEAR

$$\frac{\partial}{\partial\theta_1} = \cos\phi\frac{\partial}{\partial\theta} - \frac{\sin\phi}{\theta}\frac{\partial}{\partial\phi}$$

$$\frac{\partial}{\partial\theta_2} = \sin\phi\frac{\partial}{\partial\theta} + \frac{\cos\phi}{\theta}\frac{\partial}{\partial\phi}$$

$$\alpha_1 = \alpha\cos\phi$$

$$\alpha_2 = \alpha\sin\phi$$

$$\begin{aligned}
\gamma_2(\theta) &= \frac{\partial}{\partial\theta_2}\alpha_1(\theta) \\
&= \left[\sin\phi\cos\phi\,\alpha'(\theta) - \sin\phi\cos\phi\,\frac{\alpha(\theta)}{\theta}\right] \\
&= \frac{\sin 2\phi}{2}\left[\alpha'(\theta) - \frac{\alpha(\theta)}{\theta}\right] .
\end{aligned}$$

SHEAR

$$\begin{aligned}\gamma_1(\theta) &= \frac{1}{2}\left[\frac{\partial}{\partial\theta_1}\alpha_1(\theta) - \frac{\partial}{\partial\theta_2}\alpha_2(\theta)\right] \\ &= \frac{1}{2}\left[(\cos^2\phi - \sin^2\phi)\alpha'(\theta) - (\cos^2\phi - \sin^2\phi)\frac{\alpha(\theta)}{\theta}\right] \\ &= \frac{\cos 2\phi}{2}\left[\alpha'(\theta) - \frac{\alpha(\theta)}{\theta}\right],\end{aligned}$$

$$\begin{aligned}\gamma_2(\theta) &= \frac{\partial}{\partial\theta_2}\alpha_1(\theta) \\ &= \left[\sin\phi\cos\phi\,\alpha'(\theta) - \sin\phi\cos\phi\frac{\alpha(\theta)}{\theta}\right] \\ &= \frac{\sin 2\phi}{2}\left[\alpha'(\theta) - \frac{\alpha(\theta)}{\theta}\right].\end{aligned}$$

$$\alpha(x) = \frac{m(x)}{x} \qquad \alpha'(x) = \frac{m'(x)}{x} - \frac{m(x)}{x^2}$$

$$\begin{aligned}\gamma(x) &= \frac{1}{2}\left|\frac{m'(x)}{x} - \frac{2m(x)}{x^2}\right| \\ &= \left|\kappa(x) - \overline{\kappa}(x)\right|,\end{aligned}$$

$$\kappa(x) = \frac{1}{2}\frac{m'(x)}{x}$$

$$\overline{\kappa}(x) = \frac{m(x)}{x^2} = 2\pi\frac{\int_0^x x'\kappa(x')\mathrm{d}x'}{\pi x^2}$$

LENSING JACOBIAN

$$A = \left[1 - \frac{m'(x)}{2x}\right] I - \frac{1}{2}\left[\frac{m'(x)}{x} - \frac{2m(x)}{x^2}\right]\begin{pmatrix} \cos 2\phi & \sin 2\phi \\ \sin 2\phi & -\cos 2\phi \end{pmatrix}$$

$$A = I + \frac{m}{x^2} C(\phi) - \frac{m'(x)}{2x}[I + C(\phi)]$$

$$\begin{aligned} C(\phi) &= \begin{pmatrix} \cos 2\phi & \sin 2\phi \\ \sin 2\phi & -\cos 2\phi \end{pmatrix} \\ &= \begin{pmatrix} \cos^2\phi - \sin^2\phi & 2\sin\phi\cos\phi \\ 2\sin\phi\cos\phi & \sin^2\phi - \cos^2\phi \end{pmatrix} \end{aligned}$$

$$\begin{aligned} I + C(\phi) &= \begin{pmatrix} 1 + \cos 2\phi & \sin 2\phi \\ \sin 2\phi & 1 - \cos 2\phi \end{pmatrix} \\ &= 2\begin{pmatrix} \cos^2\phi & \sin\phi\cos\phi \\ \sin\phi\cos\phi & \sin^2\phi \end{pmatrix} \end{aligned}$$

DETERMINANT OF THE LENSING JACOBIN

$$y = x - \alpha(x) = 1 - \frac{m(x)}{x}$$

$$\begin{aligned}
\det A &= \frac{y}{x}\frac{\mathrm{d}y}{\mathrm{d}x} = \left[1 - \frac{\alpha(x)}{x}\right]\left[1 - \alpha'(x)\right] \\
&= \left[1 - \frac{m(x)}{x^2}\right]\left[1 + \frac{m(x)}{x^2} - \frac{m'(x)}{x}\right] \\
&= \left[1 - \overline{\kappa}(x)\right]\left[1 + \overline{\kappa}(x) - 2\kappa(x)\right] \ .
\end{aligned}$$

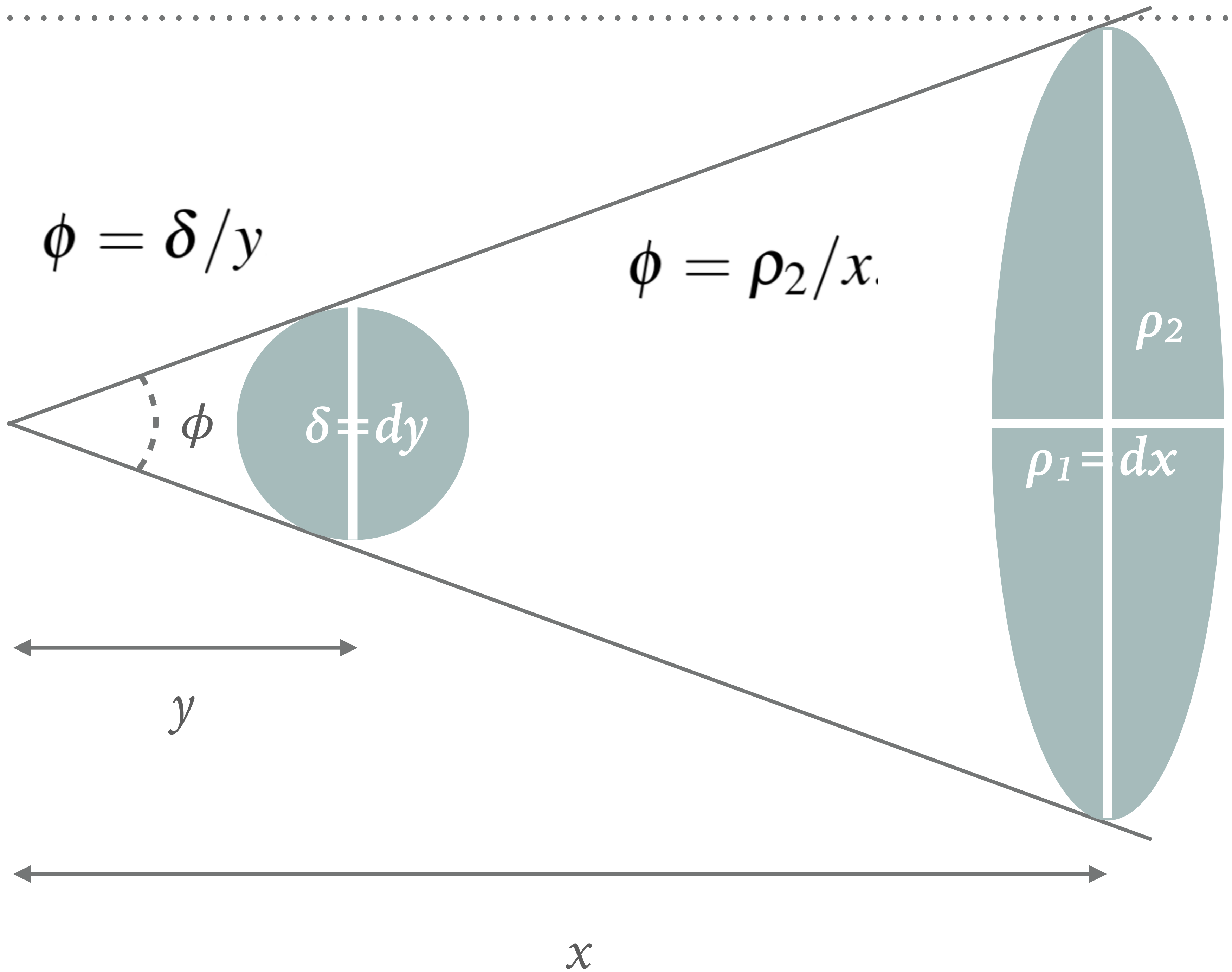
CRITICAL LINES

$$
\begin{aligned}
\det A &= \frac{y}{x}\frac{\mathrm{d}y}{\mathrm{d}x} = \left[1 - \frac{\alpha(x)}{x}\right]\left[1 - \alpha'(x)\right] \\
&= \left[1 - \frac{m(x)}{x^2}\right]\left[1 + \frac{m(x)}{x^2} - \frac{m'(x)}{x}\right] \\
&= \left[1 - \overline{\kappa}(x)\right]\left[1 + \overline{\kappa}(x) - 2\kappa(x)\right] .
\end{aligned}
$$

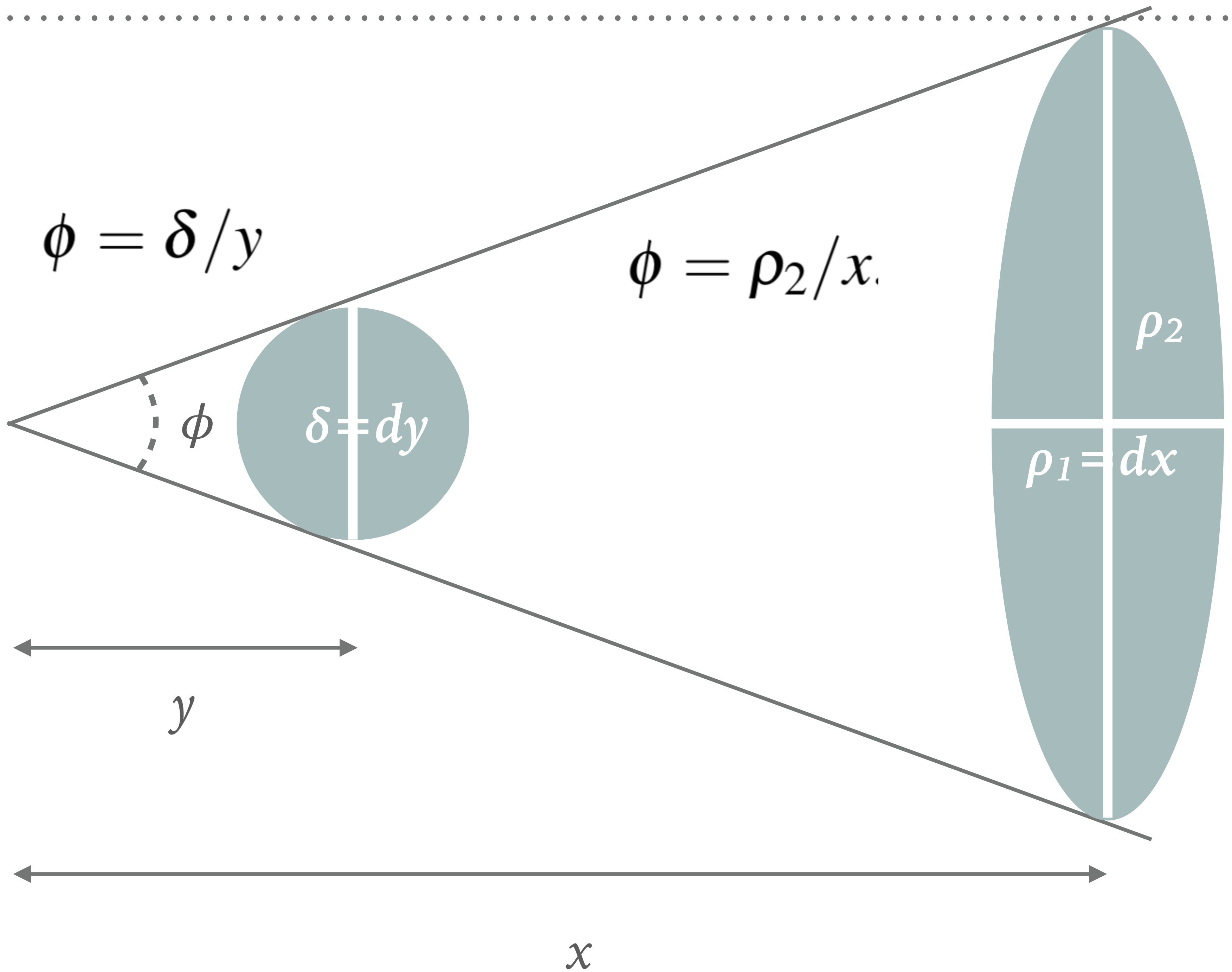
First critical line: $\alpha(x)/x = m(x)/x^2 = \overline{\kappa}(x) = 1$

Second critical line: $\alpha'(x) = m'(x)/x - m/x^2 = 2\kappa(x) - \overline{\kappa}(x) = 1$

RADIAL AND TANGENTIAL MAGNIFICATION



RADIAL AND TANGENTIAL MAGNIFICATION



$$\frac{\delta}{\rho_2} = 1 - \frac{m(x)}{x^2}$$

$$\frac{\delta}{\rho_1} = 1 + \frac{m(x)}{x^2} - 2\kappa(x)$$

CRITICAL LINES

The tangential critical line occurs where $\alpha(x)/x = m(x)/x^2 = \overline{\kappa}(x) = 1$

$$\kappa(\vec{\theta}) \equiv \frac{\Sigma(\vec{\theta})}{\Sigma_{\mathrm{cr}}} \quad \text{with} \quad \Sigma_{\mathrm{cr}} = \frac{c^2}{4\pi G}\frac{D_{\mathrm{S}}}{D_{\mathrm{L}}D_{\mathrm{LS}}}$$

$$M(\theta_E) = \pi\Sigma_{\mathrm{cr}}\theta_E^2 D_{\mathrm{L}}^2$$

$$\theta_E = \sqrt{\frac{4GM(\theta_E)}{c^2}\frac{D_{\mathrm{LS}}}{D_{\mathrm{L}}D_{\mathrm{S}}}}$$