GRAVITATIONAL LENSING

16 - AXIALLY SYMMETRIC LENSES

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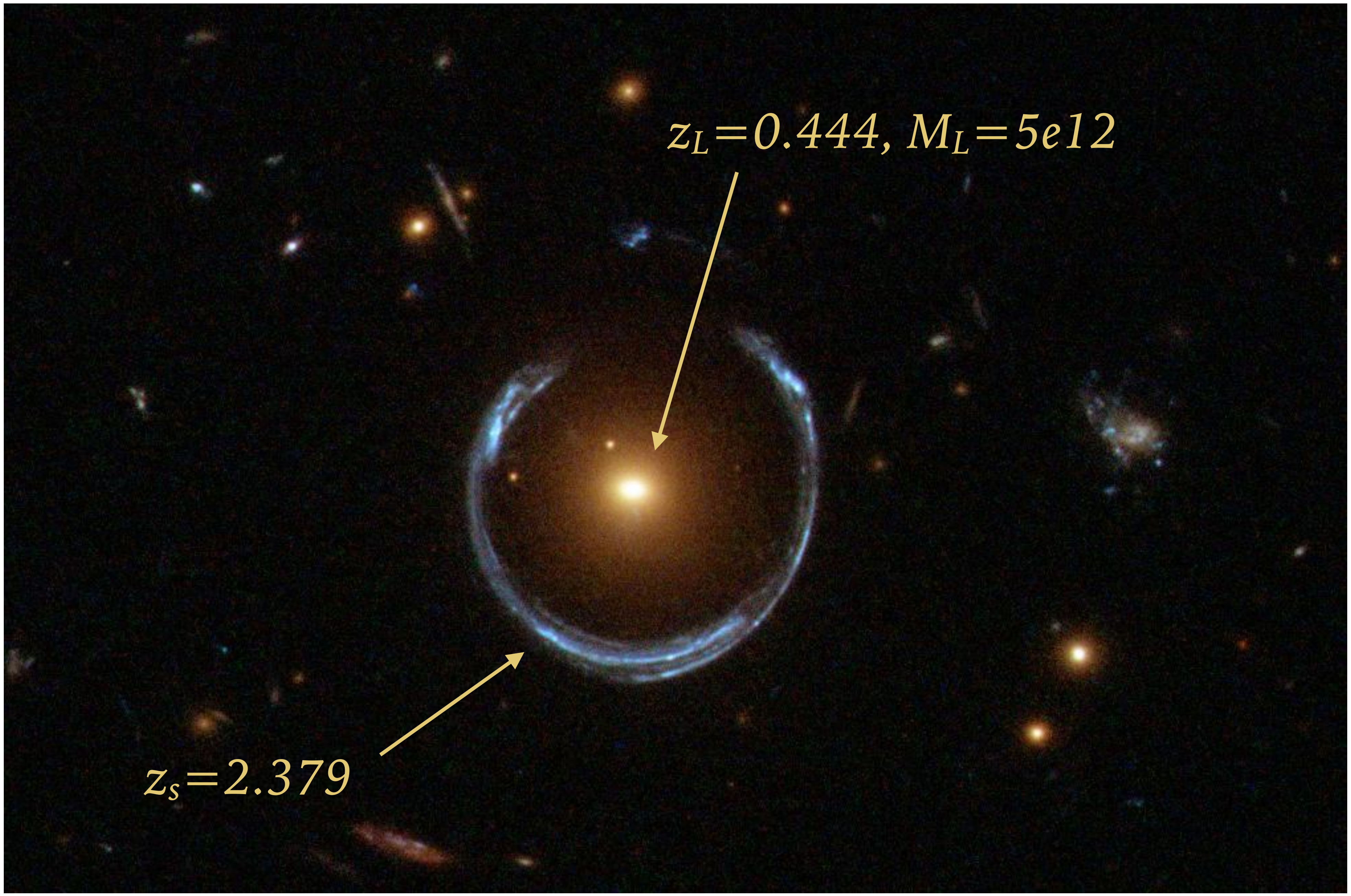
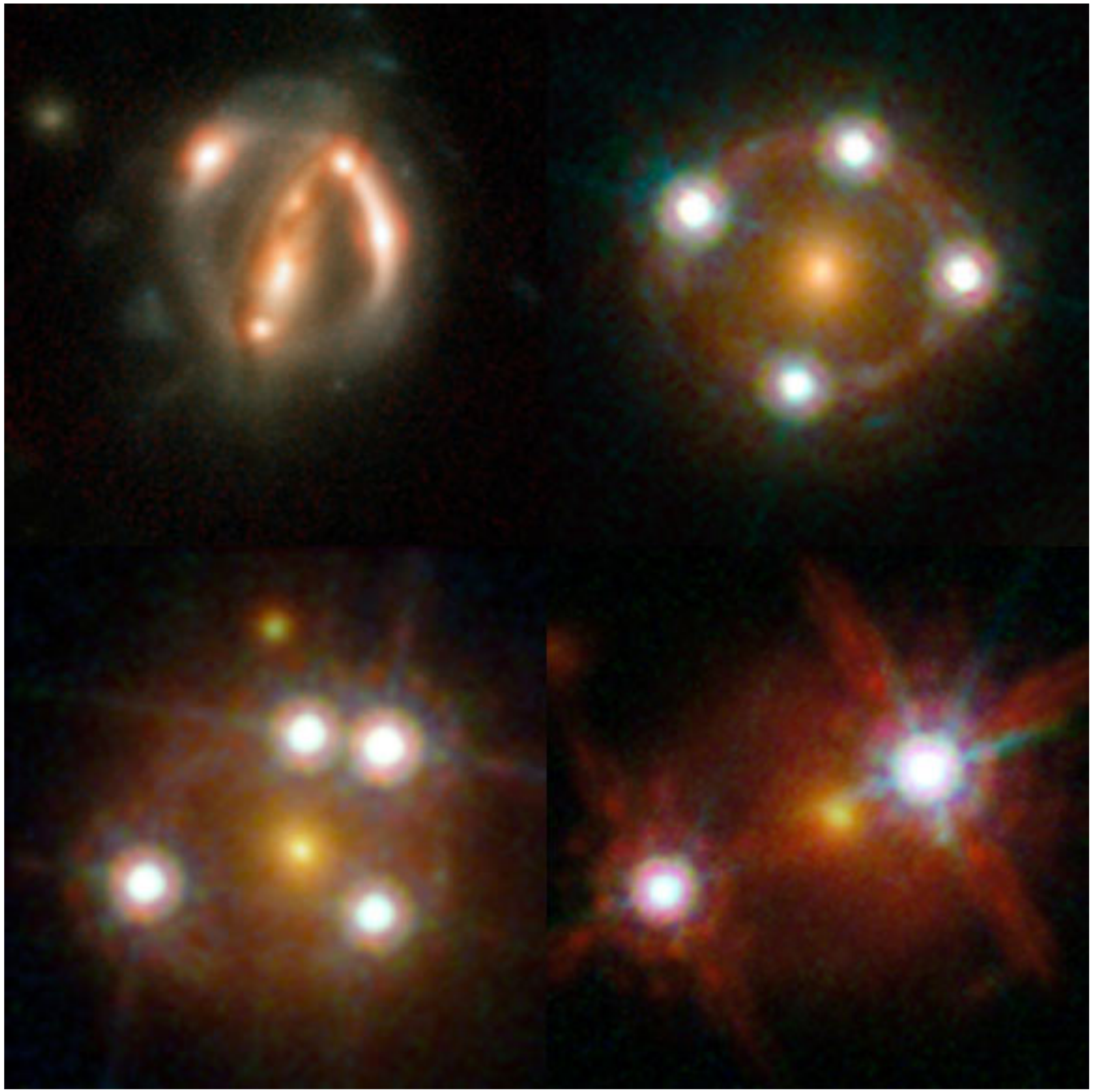
AA 2017-2018

EXTENDED LENSES

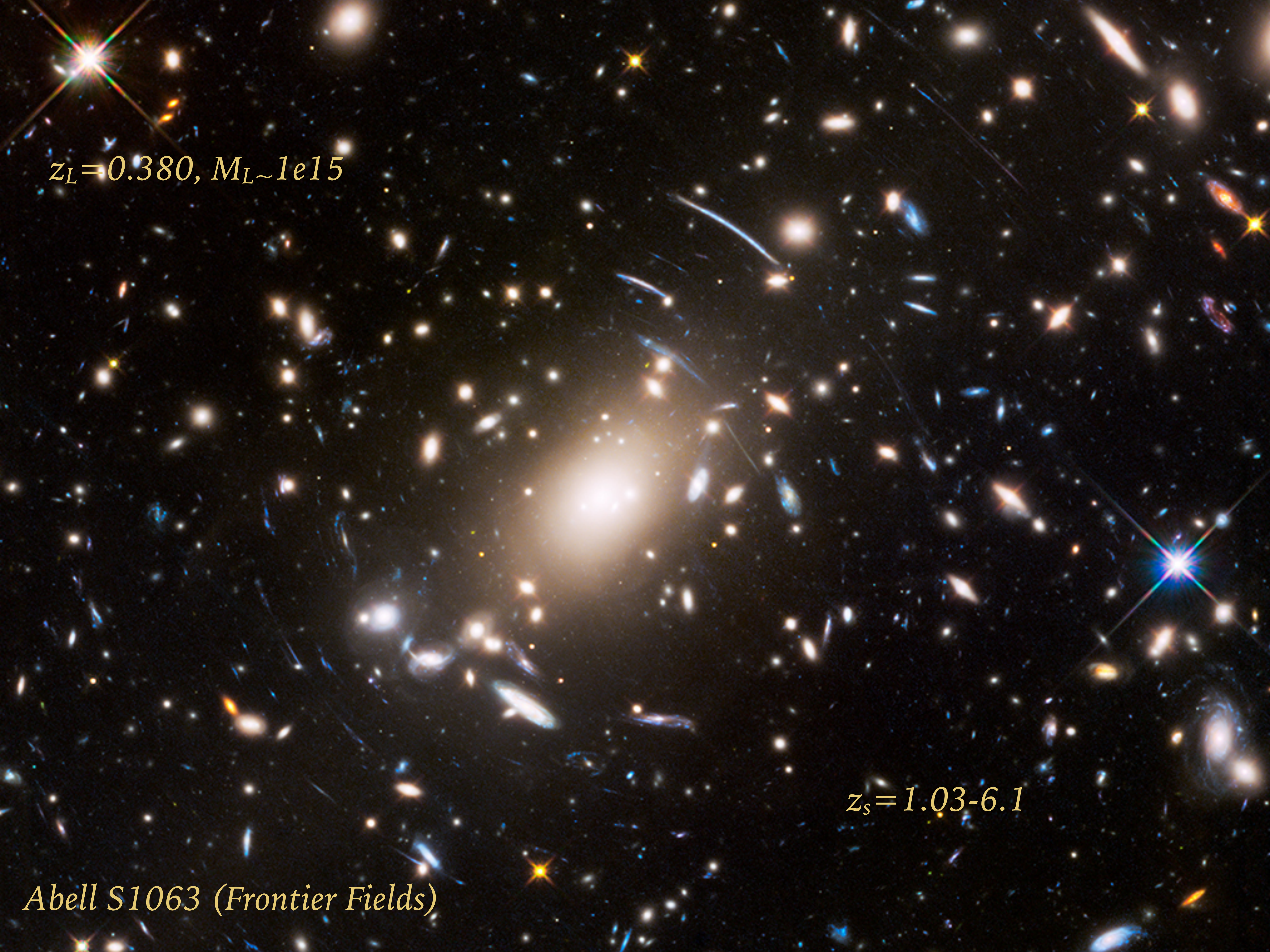


Cosmic horseshoe (Belokurov et al. 2007)

EXTENDED LENSES



Suyu et al. (H0LiCOW team)


$z_L = 0.380$, $M_L \sim 1e15$
$z_s = 1.03\text{-}6.1$
Abell S1063 (Frontier Fields)

EXTENDED LENSES

- Cosmic structures like galaxies and galaxy clusters are characterized by bound mass distributions, which cannot be approximated by point lenses
- Indeed these are *extended lenses*, and their lensing properties are determined by e.g. their surface mass density:

$$\Sigma(\vec{\xi}) = \int \rho(\vec{\xi}, z)\ \mathrm{d}z$$

$$\vec{\alpha}(\vec{\xi}) = \frac{4G}{c^2} \int \frac{(\vec{\xi} - \vec{\xi}')\Sigma(\vec{\xi}')}{|\vec{\xi} - \vec{\xi}'|^2}\ \mathrm{d}^2\xi'$$

EXTENDED LENSES

- Recall that the surface density is related to the lensing potential by

$$\triangle_\theta \Psi(\vec{\theta}) = 2\kappa(\vec{\theta})$$

$$\kappa(\vec{\theta}) \equiv \frac{\Sigma(\vec{\theta})}{\Sigma_{\rm cr}} \quad \text{with} \quad \Sigma_{\rm cr} = \frac{c^2}{4\pi G}\frac{D_{\rm S}}{D_{\rm L} D_{\rm LS}}$$

WHAT ARE THE RELEVANT PROPERTIES OF THE LENSES?

- The surface density distribution of a lens (and its potential) can be characterized by means of
  - the profile
  - the shape of the iso-density (iso-potential) contours
  - the smoothness
  - the environment where the lens resides
- In this and in the following lessons, we will study how these features determine the ability of a mass distribution to produce lensing effects.
- We will do that by building analytical models with increasing level of complexity.

AXIALLY SYMMETRIC, CIRCULAR LENSES

- Axially symmetric, circular models are the simplest lens models for describing extended mass distributions
- For these lenses $\hat{\Psi}(\vec{\theta}) = \hat{\Psi}(\theta)$
- Several quantities relevant for lensing can be derived in a simple manner by using the symmetry properties of the lens.
- One example is the deflection angle…

DEFLECTION ANGLE OF AN AXIALLY SYMMETRIC LENS

$$\vec{\nabla}_\theta \equiv D_{\rm L}\left(\frac{\partial}{\partial\xi}\vec{e}_\xi + \frac{1}{\xi}\frac{\partial}{\partial\phi}\vec{e}_\phi\right) = \left(\frac{\partial}{\partial\theta}\vec{e}_\theta + \frac{1}{\theta}\frac{\partial}{\partial\phi}\vec{e}_\phi\right)$$

$$\nabla_\theta\hat{\Psi}(\vec{\theta}) = \hat{\Psi}'(\theta)\vec{e}_\theta = \vec{\alpha}(\vec{\theta}) = \alpha(\theta)\vec{e}_\theta$$

For an axially symmetric lens, the deflection is "radial": it depends only on the distance from the lens center.

DEFLECTION ANGLE OF AN AXIALLY SYMMETRIC LENS

$$\frac{1}{\theta}\frac{\partial}{\partial\theta}\left(\theta\frac{\partial}{\partial\theta}\right)\hat{\Psi}(\theta) = 2\kappa(\theta)$$

From this equation, we obtain

$$\begin{aligned}
\alpha(\theta) &= \frac{2\int_0^\theta \kappa(\theta')\theta' d\theta'}{\theta} \\
&= \frac{2\int_0^\theta \Sigma(\theta')\theta' d\theta'}{\theta\Sigma_{\mathrm{cr}}} \\
&= \frac{D_{\mathrm{LS}}}{D_{\mathrm{S}}}\frac{4GM(\theta)}{c^2 D_{\mathrm{L}}\theta} \\
&= \frac{D_{\mathrm{LS}}}{D_{\mathrm{S}}}\hat{\alpha}(\theta) \,.
\end{aligned}$$

DEFLECTION ANGLE OF AN AXIALLY SYMMETRIC LENS

$$\frac{1}{\theta}\frac{\partial}{\partial\theta}\left(\theta\frac{\partial}{\partial\theta}\right)\hat{\Psi}(\theta) = 2\kappa(\theta)$$

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&= \frac{2\int_0^\theta \Sigma(\theta')\theta' d\theta'}{\theta\Sigma_{\rm cr}} \\
&= \frac{D_{\rm LS}}{D_{\rm S}}\frac{4GM(\theta)}{c^2 D_{\rm L}\theta} \\
&= \frac{D_{\rm LS}}{D_{\rm S}}\hat{\alpha}(\theta)\,.
\end{aligned}$$

Identical to point-mass lens!

DEFLECTION ANGLE OF AN AXIALLY SYMMETRIC LENS

Dimensionless form:

$$
\begin{aligned}
\alpha(x) &= \frac{D_{\mathrm{L}} D_{\mathrm{LS}}}{\xi_0 D_{\mathrm{S}}} \hat{\alpha}(\xi_0 x) \\
&= \frac{D_{\mathrm{L}} D_{\mathrm{LS}}}{\xi_0 D_{\mathrm{S}}} \frac{4GM(\xi_0 x)}{c^2 \xi} \frac{\pi \xi_0}{\pi \xi_0} \\
&= \frac{M(\xi_0 x)}{\pi \xi_0^2 \Sigma_{\mathrm{cr}}} \frac{1}{x} \equiv \frac{m(x)}{x} \, ,
\end{aligned}
$$

Dimensionless mass

$$
\alpha(x) = \frac{2}{x} \int_0^x x' \kappa(x') \mathrm{d}x' \Rightarrow m(x) = 2 \int_0^x x' \kappa(x') \mathrm{d}x'
$$

LENS EQUATION

$$\vec{y} = \vec{x} - \vec{\alpha}(\vec{x}) \qquad \vec{\alpha}(\vec{x}) = \frac{m(\vec{x})}{x^2}\vec{x}$$

Given that the deflection angle and x are parallel, so will be y!

$$y = x - \frac{m(x)}{x}$$

CONVERGENCE

$$\frac{1}{\theta}\frac{\partial}{\partial\theta}\left(\theta\frac{\partial}{\partial\theta}\right)\hat{\Psi}(\theta) = 2\kappa(\theta)$$

$$\downarrow$$

$$\kappa(\theta) = \frac{1}{2}\left(\hat{\Psi}''(\theta) + \frac{\hat{\Psi}'(\theta)}{\theta}\right)$$

$$\downarrow \quad \hat{\Psi}'(\theta) = \alpha(\theta)$$

$$\kappa(\theta) = \frac{1}{2}\left(\alpha'(\theta) + \frac{\alpha(\theta)}{\theta}\right)$$

$$\downarrow \quad \alpha'(x) = \frac{m'(x)}{x} - \frac{m(x)}{x^2}$$

$$\kappa(x) = \frac{1}{2}\frac{m'(x)}{x}$$

SHEAR

The shear components are derived from the second derivatives of the potential or from the first derivatives of the deflection angle components:

$$\begin{aligned}
\frac{\partial}{\partial \theta_1} &= \cos\phi \frac{\partial}{\partial \theta} - \frac{\sin\phi}{\theta} \frac{\partial}{\partial \phi} \\
\frac{\partial}{\partial \theta_2} &= \sin\phi \frac{\partial}{\partial \theta} + \frac{\cos\phi}{\theta} \frac{\partial}{\partial \phi}
\end{aligned}$$

$$\begin{aligned}
\alpha_1(\theta) &= \alpha(\theta)\cos\phi \\
\alpha_2(\theta) &= \alpha(\theta)\sin\phi
\end{aligned}$$

SHEAR

$$
\begin{aligned}
\frac{\partial}{\partial \theta_1} &= \cos\phi \frac{\partial}{\partial \theta} - \frac{\sin\phi}{\theta}\frac{\partial}{\partial \phi} \\
\frac{\partial}{\partial \theta_2} &= \sin\phi \frac{\partial}{\partial \theta} + \frac{\cos\phi}{\theta}\frac{\partial}{\partial \phi}
\end{aligned}
$$

$$
\begin{aligned}
\alpha_1(\theta) &= \alpha(\theta)\cos\phi \\
\alpha_2(\theta) &= \alpha(\theta)\sin\phi
\end{aligned}
$$

$$
\begin{aligned}
\gamma_1(\theta) &= \frac{1}{2}\left(\frac{\partial}{\partial \theta_1}\alpha_1(\theta) - \frac{\partial}{\partial \theta_2}\alpha_2(\theta)\right) \\
&= \frac{1}{2}\left[(\cos^2\phi - \sin^2\phi)\alpha'(\theta) - (\cos^2\phi - \sin^2\phi)\frac{\alpha(\theta)}{\theta}\right] \\
&= \frac{\cos 2\phi}{2}\left(\alpha'(\theta) - \frac{\alpha(\theta)}{\theta}\right),
\end{aligned}
$$

SHEAR

$$\begin{aligned}\frac{\partial}{\partial\theta_1} &= \cos\phi\frac{\partial}{\partial\theta} - \frac{\sin\phi}{\theta}\frac{\partial}{\partial\phi}\\ \frac{\partial}{\partial\theta_2} &= \sin\phi\frac{\partial}{\partial\theta} + \frac{\cos\phi}{\theta}\frac{\partial}{\partial\phi}\end{aligned}$$

$$\begin{aligned}\alpha_1(\theta) &= \alpha(\theta)\cos\phi\\ \alpha_2(\theta) &= \alpha(\theta)\sin\phi\end{aligned}$$

$$\begin{aligned}\gamma_2(\theta) &= \frac{\partial}{\partial\theta_2}\alpha_1(\theta)\\ &= \left(\sin\phi\cos\phi\,\alpha'(\theta) + \sin\phi\cos\phi\frac{\alpha(\theta)}{\theta}\right)\\ &= \frac{\sin 2\phi}{2}\left(\alpha'(\theta) - \frac{\alpha(\theta)}{\theta}\right).\end{aligned}$$

SHEAR

$$\begin{aligned}\gamma_1(\theta) &= \frac{1}{2}\left(\frac{\partial}{\partial\theta_1}\alpha_1(\theta)-\frac{\partial}{\partial\theta_2}\alpha_2(\theta)\right)\\ &= \frac{1}{2}\left[(\cos^2\phi-\sin^2\phi)\alpha'(\theta)-(\cos^2\phi-\sin^2\phi)\frac{\alpha(\theta)}{\theta}\right]\\ &= \frac{\cos 2\phi}{2}\left(\alpha'(\theta)-\frac{\alpha(\theta)}{\theta}\right),\end{aligned}$$

$$\begin{aligned}\gamma_2(\theta) &= \frac{\partial}{\partial\theta_2}\alpha_1(\theta)\\ &= \left(\sin\phi\cos\phi\,\alpha'(\theta)+\sin\phi\cos\phi\,\frac{\alpha(\theta)}{\theta}\right)\\ &= \frac{\sin 2\phi}{2}\left(\alpha'(\theta)-\frac{\alpha(\theta)}{\theta}\right).\end{aligned}$$

$$\begin{aligned}\alpha(x) &= \frac{D_{\rm L}D_{\rm LS}}{\xi_0 D_{\rm S}}\hat{\alpha}(\xi_0 x)\\ &= \frac{D_{\rm L}D_{\rm LS}}{\xi_0 D_{\rm S}}\frac{4GM(\xi_0 x)}{c^2\xi}\frac{\pi\xi_0}{\pi\xi_0}\\ &= \frac{M(\xi_0 x)}{\pi\xi_0^2\Sigma_{\rm cr}}\frac{1}{x}\equiv\frac{m(x)}{x},\end{aligned}$$

$$\alpha'(x)=\frac{m'(x)}{x}-\frac{m(x)}{x^2}$$

$$\begin{aligned}\gamma(x) &= \frac{1}{2}\left|\frac{m'(x)}{x}-\frac{2m(x)}{x^2}\right|\\ &= \left|\kappa(x)-\overline{\kappa}(x)\right|,\end{aligned}$$

$$\overline{\kappa}(x)=\frac{m(x)}{x^2}=2\pi\frac{\int_0^x x'\kappa(x')\mathrm{d}x'}{\pi x^2}$$

LENSING JACOBIAN

$$A = \left[1 - \frac{m'(x)}{2x}\right] I - \frac{1}{2}\left[\frac{m'(x)}{x} - \frac{2m(x)}{x^2}\right] \begin{pmatrix} \cos 2\phi & \sin 2\phi \\ \sin 2\phi & -\cos 2\phi \end{pmatrix}$$

$$\Downarrow$$

$$A = I + \frac{m}{x^2} C(\phi) - \frac{m'(x)}{2x}\left[I + C(\phi)\right]$$

$$\begin{aligned} C(\phi) &= \begin{pmatrix} \cos 2\phi & \sin 2\phi \\ \sin 2\phi & -\cos 2\phi \end{pmatrix} \\ &= \begin{pmatrix} \cos^2\phi - \sin^2\phi & 2\sin\phi\cos\phi \\ 2\sin\phi\cos\phi & \sin^2\phi - \cos^2\phi \end{pmatrix} \end{aligned}$$

$$\begin{aligned} I + C(\phi) &= \begin{pmatrix} 1 + \cos 2\phi & \sin 2\phi \\ \sin 2\phi & 1 - \cos 2\phi \end{pmatrix} \\ &= 2\begin{pmatrix} \cos^2\phi & \sin\phi\cos\phi \\ \sin\phi\cos\phi & \sin^2\phi \end{pmatrix} \end{aligned}$$

LENSING JACOBIAN

$$A = I + \frac{m}{x^2} C(\phi) - \frac{m'(x)}{2x}[I + C(\phi)]$$

$$\begin{aligned} C(\phi) &= \begin{pmatrix} \cos 2\phi & \sin 2\phi \\ \sin 2\phi & -\cos 2\phi \end{pmatrix} \\ &= \begin{pmatrix} \cos^2\phi - \sin^2\phi & 2\sin\phi\cos\phi \\ 2\sin\phi\cos\phi & \sin^2\phi - \cos^2\phi \end{pmatrix} \end{aligned}$$

$$\begin{aligned} I + C(\phi) &= \begin{pmatrix} 1+\cos 2\phi & \sin 2\phi \\ \sin 2\phi & 1-\cos 2\phi \end{pmatrix} \\ &= 2\begin{pmatrix} \cos^2\phi & \sin\phi\cos\phi \\ \sin\phi\cos\phi & \sin^2\phi \end{pmatrix} \end{aligned}$$

$$\begin{aligned} A &= I + \frac{m(x)}{x^2}\begin{pmatrix} \cos^2\phi - \sin^2\phi & 2\sin\phi\cos\phi \\ 2\sin\phi\cos\phi & \sin^2\phi - \cos^2\phi \end{pmatrix} - \\ &\quad \frac{m'(x)}{x}\begin{pmatrix} \cos^2\phi & \sin\phi\cos\phi \\ \sin\phi\cos\phi & \sin^2\phi \end{pmatrix}. \end{aligned}$$

LENSING JACOBIAN (CARTESIAN COORDINATES)

$$A = I + \frac{m(x)}{x^2}\begin{pmatrix} \cos^2\phi - \sin^2\phi & 2\sin\phi\cos\phi \\ 2\sin\phi\cos\phi & \sin^2\phi - \cos^2\phi \end{pmatrix} - \frac{m'(x)}{x}\begin{pmatrix} \cos^2\phi & \sin\phi\cos\phi \\ \sin\phi\cos\phi & \sin^2\phi \end{pmatrix}.$$

$$(x_1, x_2) = (x\cos\phi, x\sin\phi),$$

$$A = I + \frac{m(x)}{x^4}\begin{pmatrix} x_1^2 - x_2^2 & 2x_1x_2 \\ 2x_1x_2 & x_2^2 - x_1^2 \end{pmatrix} - \frac{m'(x)}{x^3}\begin{pmatrix} x_1^2 & x_1x_2 \\ x_1x_2 & x_2^2 \end{pmatrix}.$$

DETERMINANT OF THE LENSING JACOBIN

$$y = x - \frac{m(x)}{x}$$

$$\begin{aligned}
\det A &= \frac{y}{x}\frac{\mathrm{d}y}{\mathrm{d}x} = \left[1 - \frac{\alpha(x)}{x}\right]\left[1 - \alpha'(x)\right] \\
&= \left[1 - \frac{m(x)}{x^2}\right]\left[1 + \frac{m(x)}{x^2} - \frac{m'(x)}{x}\right] \\
&= \left[1 - \overline{\kappa}(x)\right]\left[1 + \overline{\kappa}(x) - 2\kappa(x)\right] .
\end{aligned}$$

CRITICAL LINES

$$
\begin{aligned}
\det A &= \frac{y}{x}\frac{\mathrm{d}y}{\mathrm{d}x} = \left[1 - \frac{\alpha(x)}{x}\right]\left[1 - \alpha'(x)\right] \\
&= \left[1 - \frac{m(x)}{x^2}\right]\left[1 + \frac{m(x)}{x^2} - \frac{m'(x)}{x}\right] \\
&= \left[1 - \overline{\kappa}(x)\right]\left[1 + \overline{\kappa}(x) - 2\kappa(x)\right] .
\end{aligned}
$$

First critical line: $\alpha(x)/x = m(x)/x^2 = \overline{\kappa}(x) = 1$

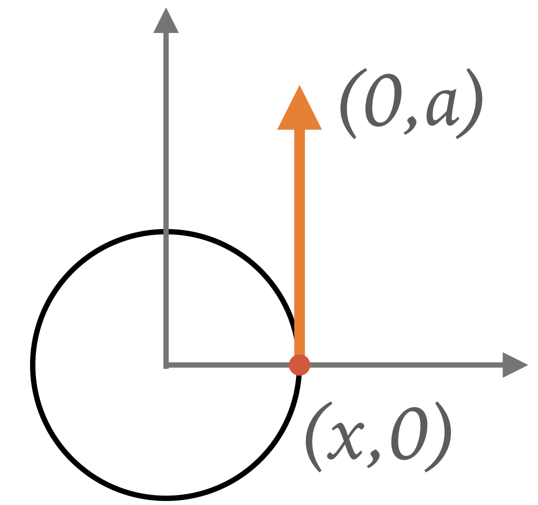
Second critical line: $\alpha'(x) = m'(x)/x - m/x^2 = 2\kappa(x) - \overline{\kappa}(x) = 1$

CRITICAL LINES

First critical line:

$$\alpha(x)/x = m(x)/x^2 = \overline{\kappa}(x) = 1$$

Given the symmetry of the lens, the critical lines are circles.
We consider the point (x,0) on the critical line:



$$\begin{aligned} A &= I + \frac{m(x)}{x^4}\begin{pmatrix} x_1^2 - x_2^2 & 2x_1x_2 \\ 2x_1x_2 & x_2^2 - x_1^2 \end{pmatrix} \\ &\quad - \frac{m'(x)}{x^3}\begin{pmatrix} x_1^2 & x_1x_2 \\ x_1x_2 & x_2^2 \end{pmatrix} . \end{aligned}$$

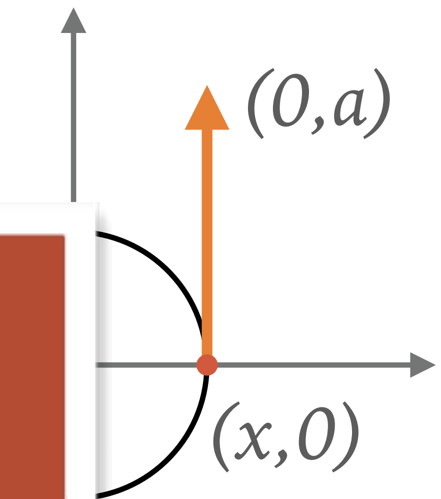
$$A(x,0) = I + \frac{m(x)}{x^2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \frac{m'(x)}{x}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = A(x,0)\begin{pmatrix} 0 \\ a \end{pmatrix} \qquad \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \left[1 - \frac{m(x)}{x^2}\right]\begin{pmatrix} 0 \\ a \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

CRITICAL LINES

First critical line: $$\alpha(x)/x = m(x)/x^2 = \overline{\kappa}(x) = 1$$

Given the symmetry of the lens, the critical lines are circles.
We consider the point (x,0) on the critical line:



$A = I -$

A vector tangent to the critical line is an eigenvector of A with zero eigenvalue: this is the tangential critical line!

$A(x,0)$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = A(x,0) \begin{pmatrix} 0 \\ a \end{pmatrix} \qquad \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \left[1 - \frac{m(x)}{x^2}\right] \begin{pmatrix} 0 \\ a \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

CRITICAL LINES

The tangential critical line occurs where $\alpha(x)/x = m(x)/x^2 = \overline{\kappa}(x) = 1$

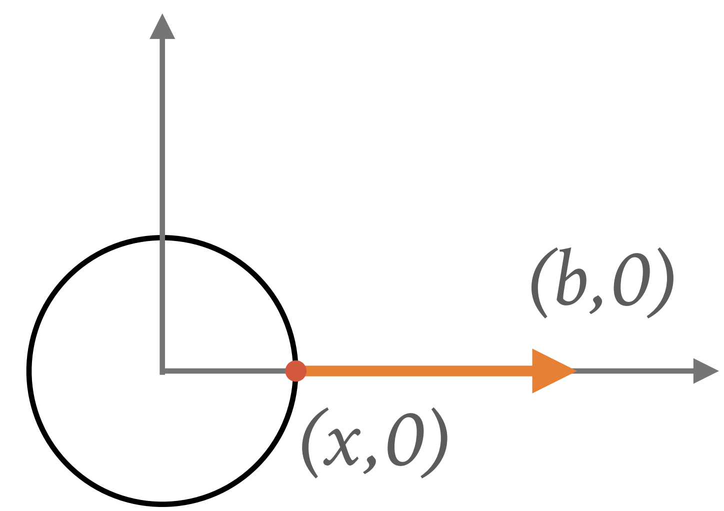
$$M(\theta_E) = \pi \Sigma_{\mathrm{cr}} \theta_E^2 D_{\mathrm{L}}^2$$

$$\theta_E = \sqrt{\frac{4GM(\theta_E)}{c^2} \frac{D_{\mathrm{LS}}}{D_{\mathrm{L}} D_{\mathrm{S}}}}$$

CRITICAL LINES

Second critical line: $\alpha'(x) = m'(x)/x - m/x^2 = 2\kappa(x) - \overline{\kappa}(x) = 1$

We consider another point (x,0) on this critical line and a vector (b,0) radially oriented.



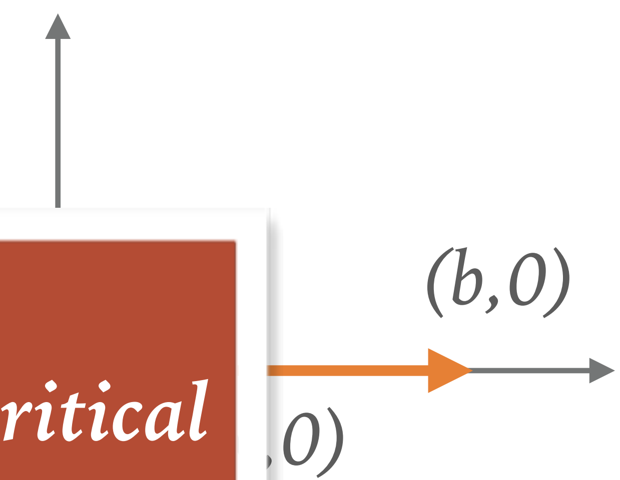
$$A(x,0) = I + \frac{m(x)}{x^2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \frac{m'(x)}{x}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = A(x,0)\begin{pmatrix} b \\ 0 \end{pmatrix} = \left[1 + \frac{m(x)}{x^2} - \frac{m'(x)}{x}\right]\begin{pmatrix} b \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

CRITICAL LINES

Second critical line: $\alpha'(x) = m'(x)/x - m/x^2 = 2\kappa(x) - \overline{\kappa}(x) = 1$

We consider another point (x,0) on this critical line and a vector (b,0) radially oriented.

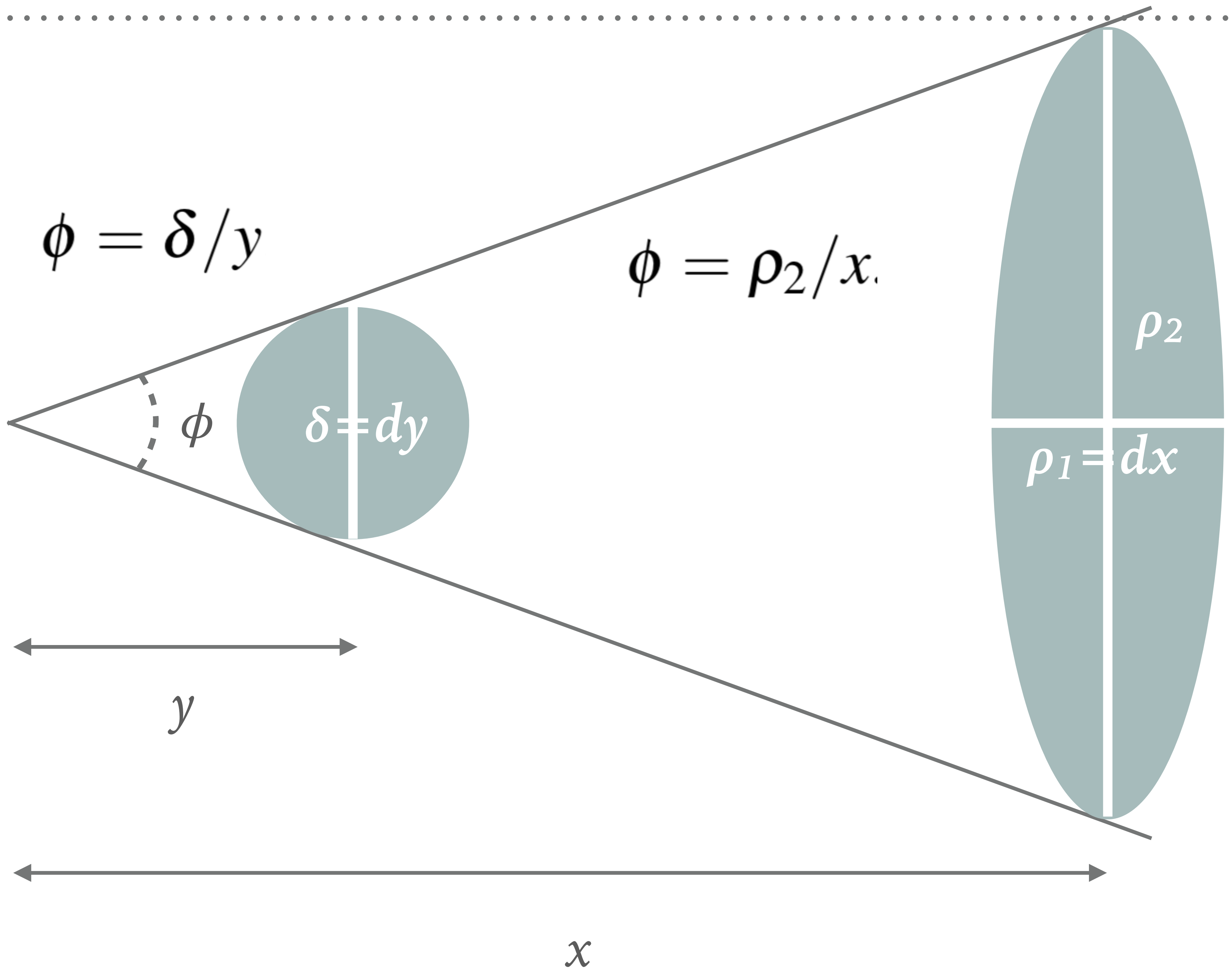


$A(x,0) \quad = \quad I + \ldots$

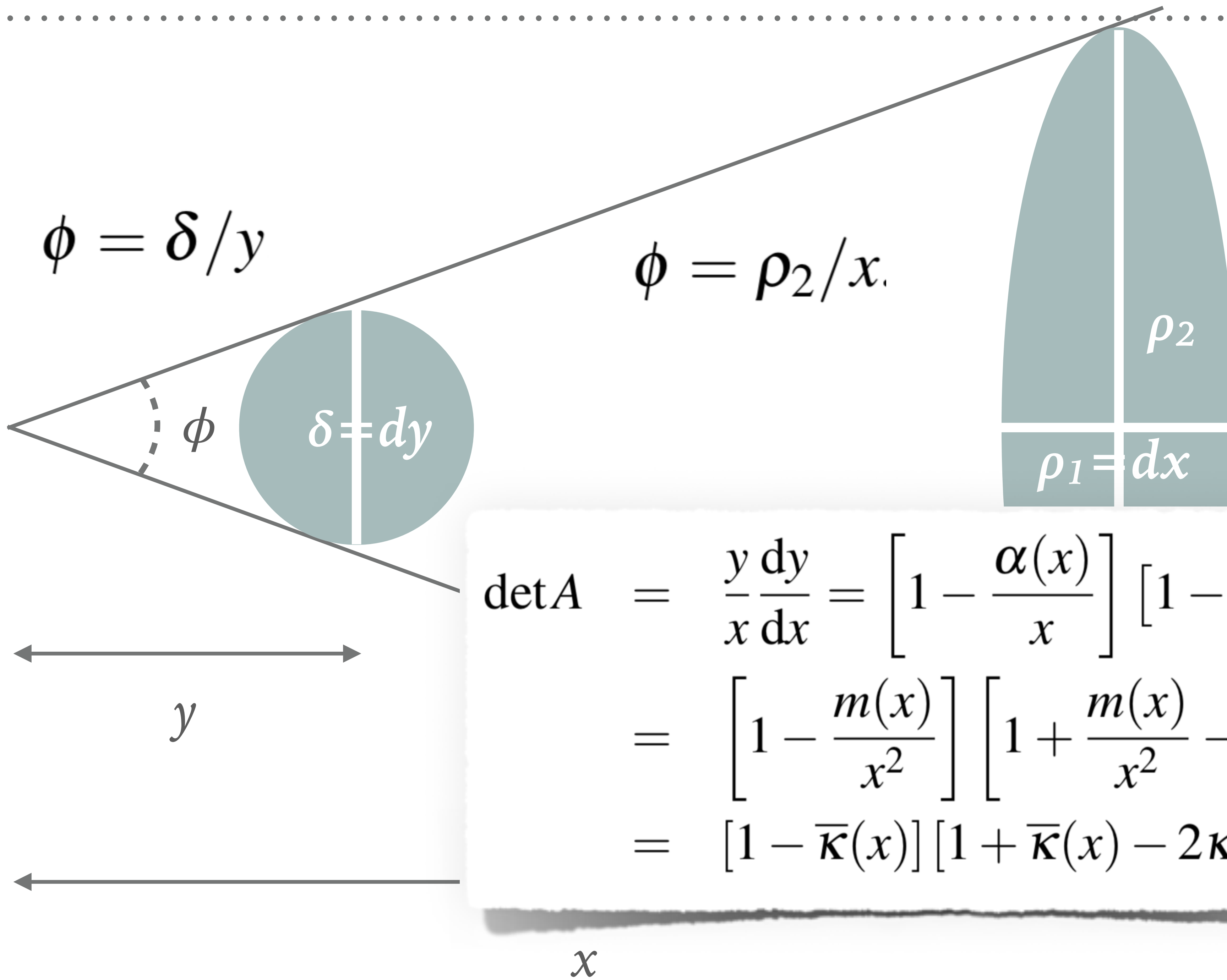
A (radial) vector, perpendicular to the critical line, is an eigenvector of A with zero eigenvalue: this is the radial critical line!

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = A \ldots = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

RADIAL AND TANGENTIAL MAGNIFICATION

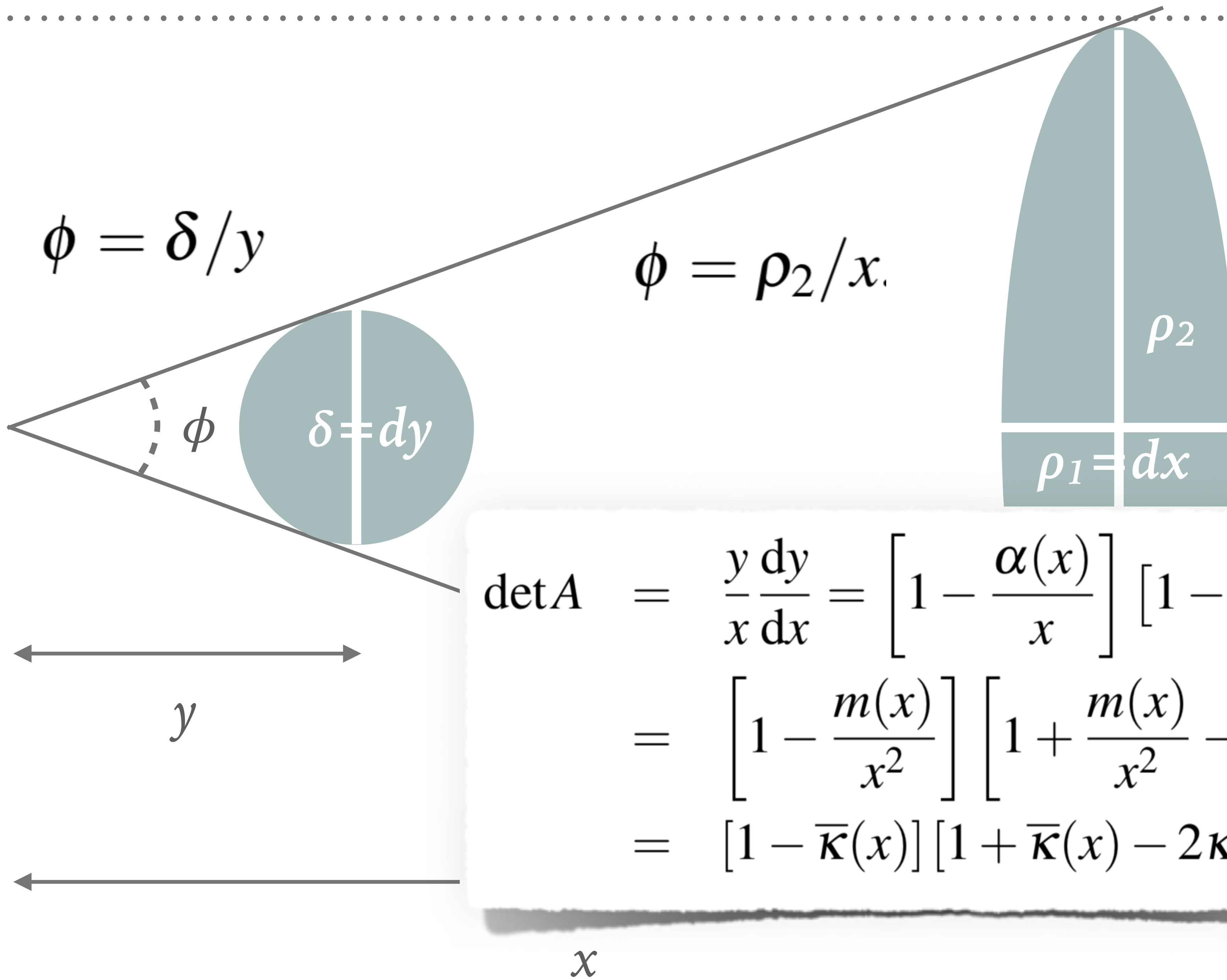


RADIAL AND TANGENTIAL MAGNIFICATION



$$
\begin{aligned}
\det A &= \frac{y}{x}\frac{\mathrm{d}y}{\mathrm{d}x} = \left[1 - \frac{\alpha(x)}{x}\right]\left[1 - \alpha'(x)\right] \\
&= \left[1 - \frac{m(x)}{x^2}\right]\left[1 + \frac{m(x)}{x^2} - \frac{m'(x)}{x}\right] \\
&= \left[1 - \overline{\kappa}(x)\right]\left[1 + \overline{\kappa}(x) - 2\kappa(x)\right] .
\end{aligned}
$$

RADIAL AND TANGENTIAL MAGNIFICATION



$$
\begin{aligned}
\det A &= \frac{y}{x}\frac{\mathrm{d}y}{\mathrm{d}x} = \left[1 - \frac{\alpha(x)}{x}\right]\left[1 - \alpha'(x)\right] \\
&= \left[1 - \frac{m(x)}{x^2}\right]\left[1 + \frac{m(x)}{x^2} - \frac{m'(x)}{x}\right] \\
&= \left[1 - \overline{\kappa}(x)\right]\left[1 + \overline{\kappa}(x) - 2\kappa(x)\right] .
\end{aligned}
$$

$$\frac{\delta}{\rho_2} = 1 - \frac{m(x)}{x^2}$$

$$\frac{\delta}{\rho_1} = 1 + \frac{m(x)}{x^2} - 2\kappa(x)$$

POWER-LAW LENS

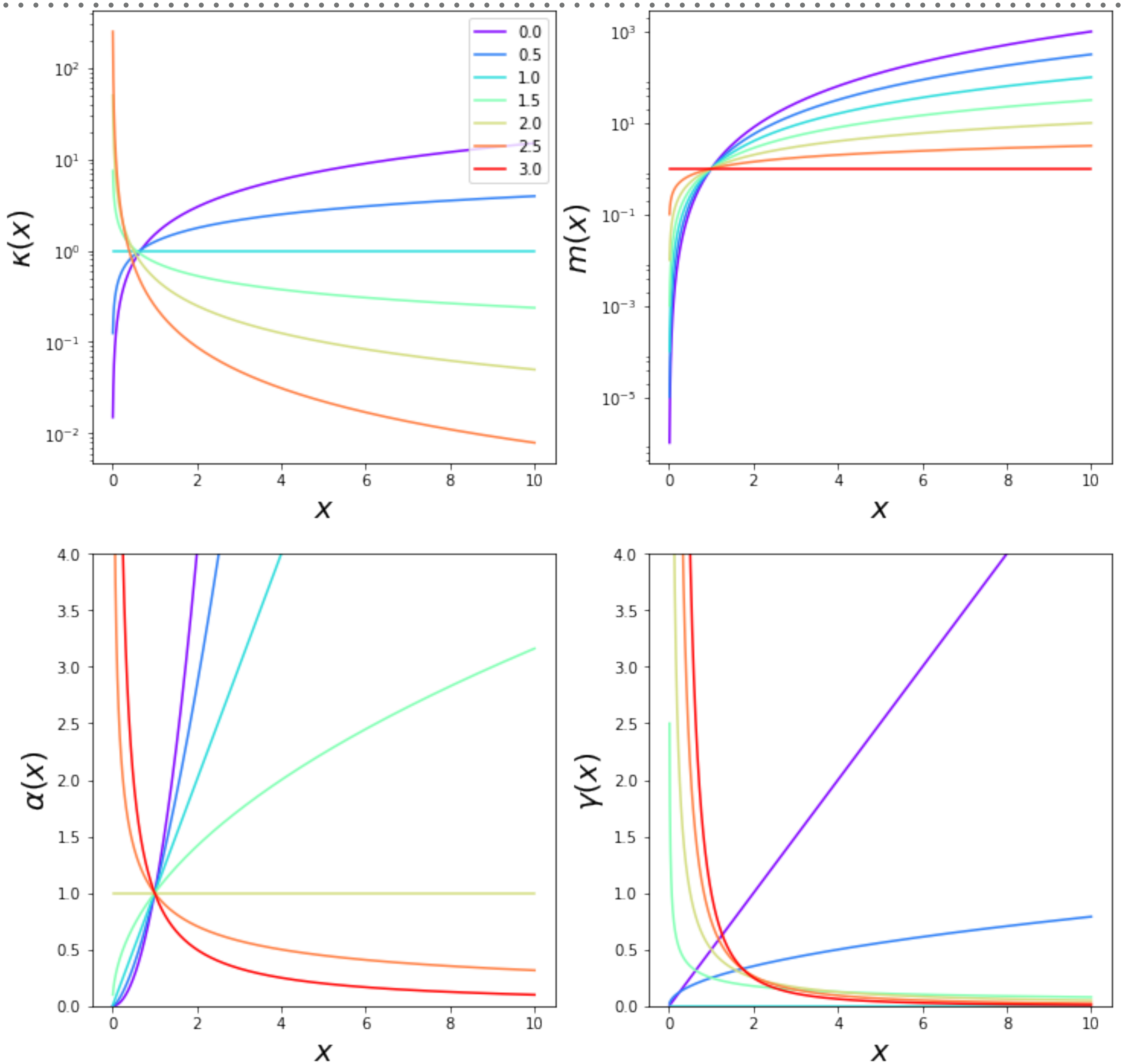
$$\kappa(x) = \frac{3-n}{2} x^{1-n}$$

$$m(x) = x^{3-n}$$

$$\alpha(x) = \frac{m(x)}{x} = x^{2-n}$$

$$\gamma(x) = \frac{m(x)}{x^2} - \kappa(x) = \frac{n-1}{2} x^{1-n}$$

POWER-LAW LENS



POWER-LAW LENS: CRITICAL LINES AND CAUSTICS

The tangential critical line has equation $x=1$ for any value of the slope parameter n.

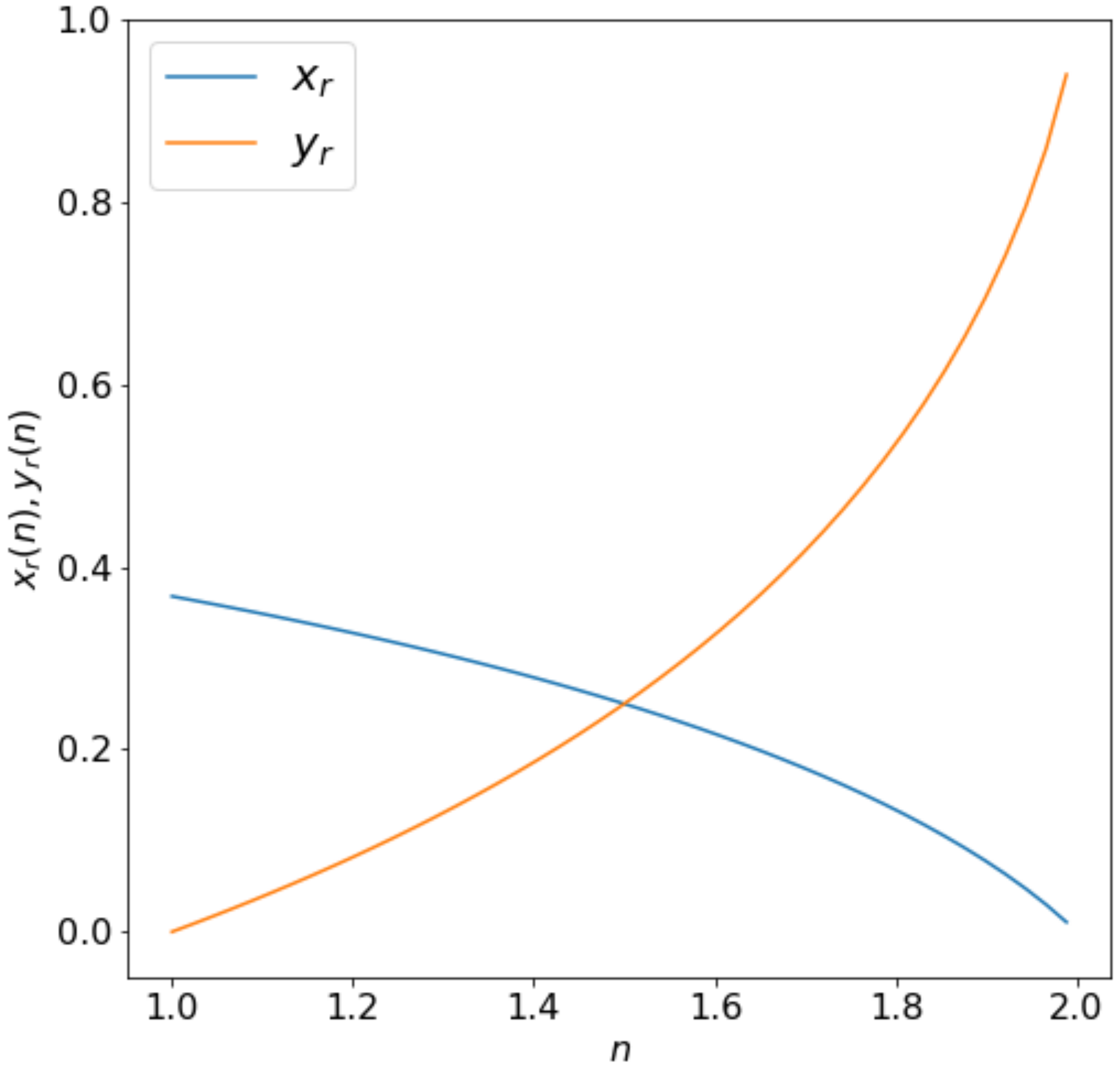
The caustic is the point $y=0$

Instead, the size of the radial critical line depends on n:

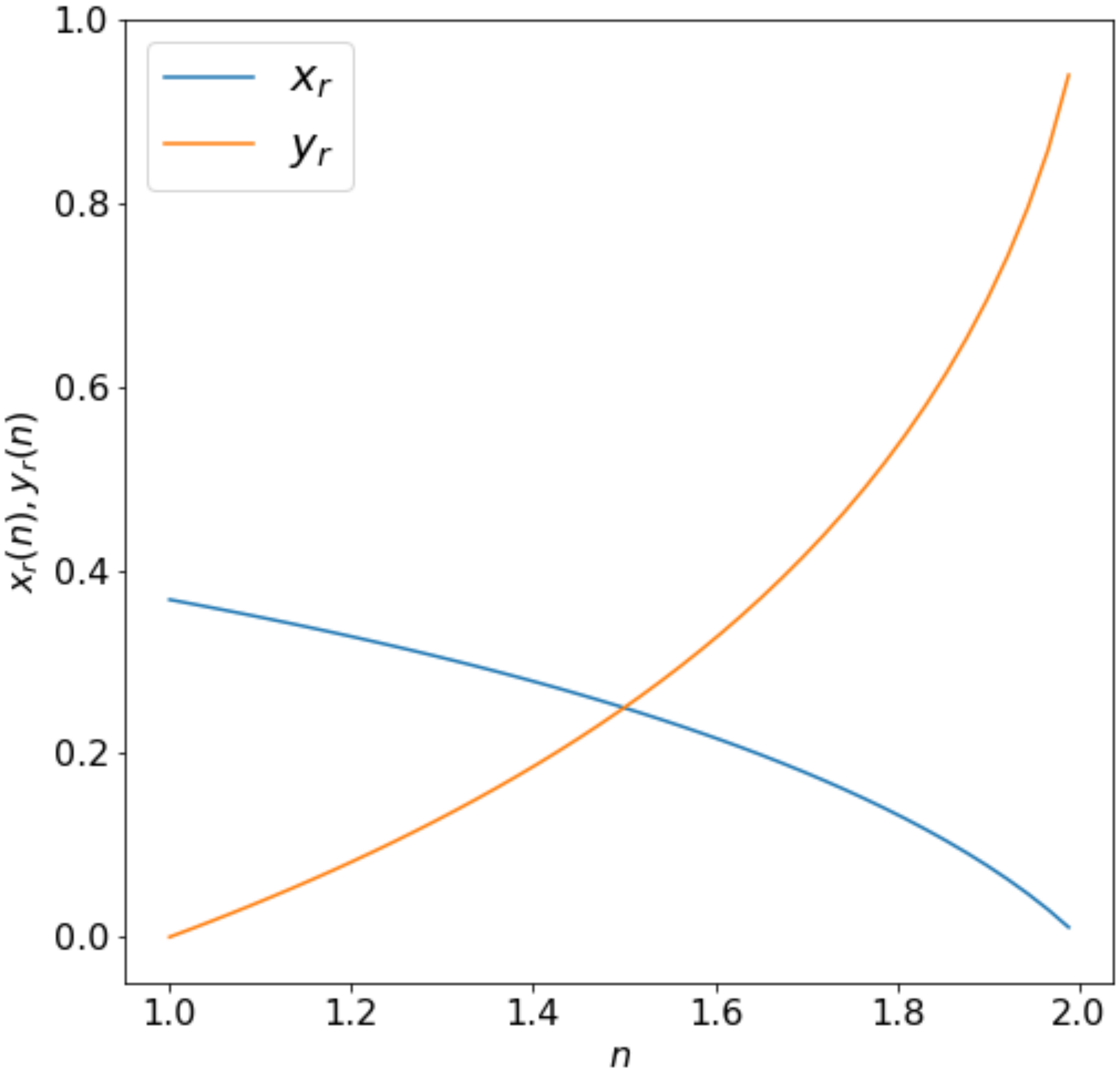
$$(2-n)x_r^{1-n} = 1$$

$$x_r = (2-n)^{1/(n-1)}$$

RADIAL CRITICAL LINE

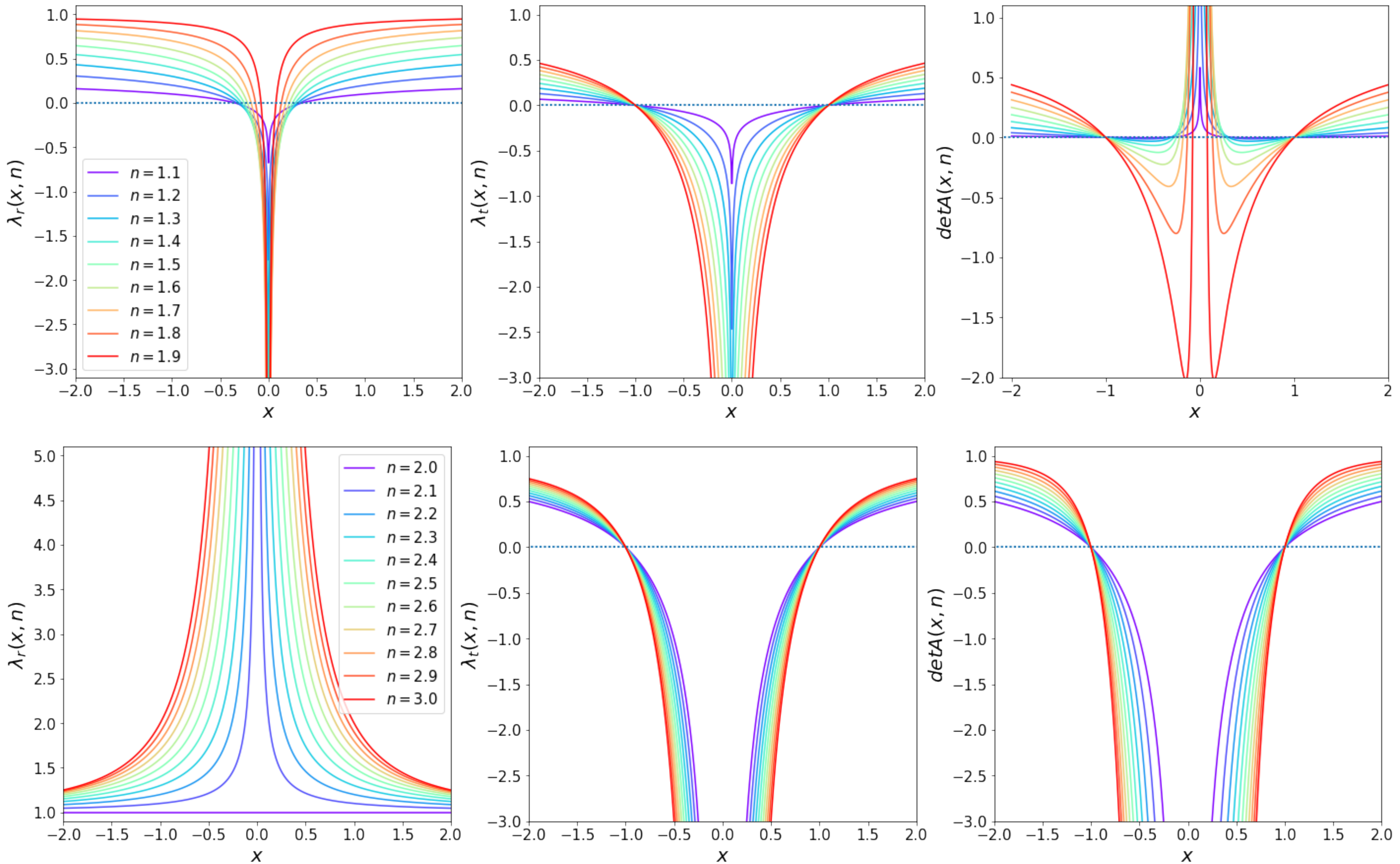


RADIAL CRITICAL LINE



Large n, small radial critical line

$$\lambda_t(x) = 1 - x^{1-n}$$
$$\lambda_r(x) = 1 - (2-n)x^{1-n}$$



no radial critical line if n>=2!

$$\lambda_t(x) = 1 - x^{1-n}$$
$$\lambda_r(x) = 1 - (2-n)x^{1-n}$$

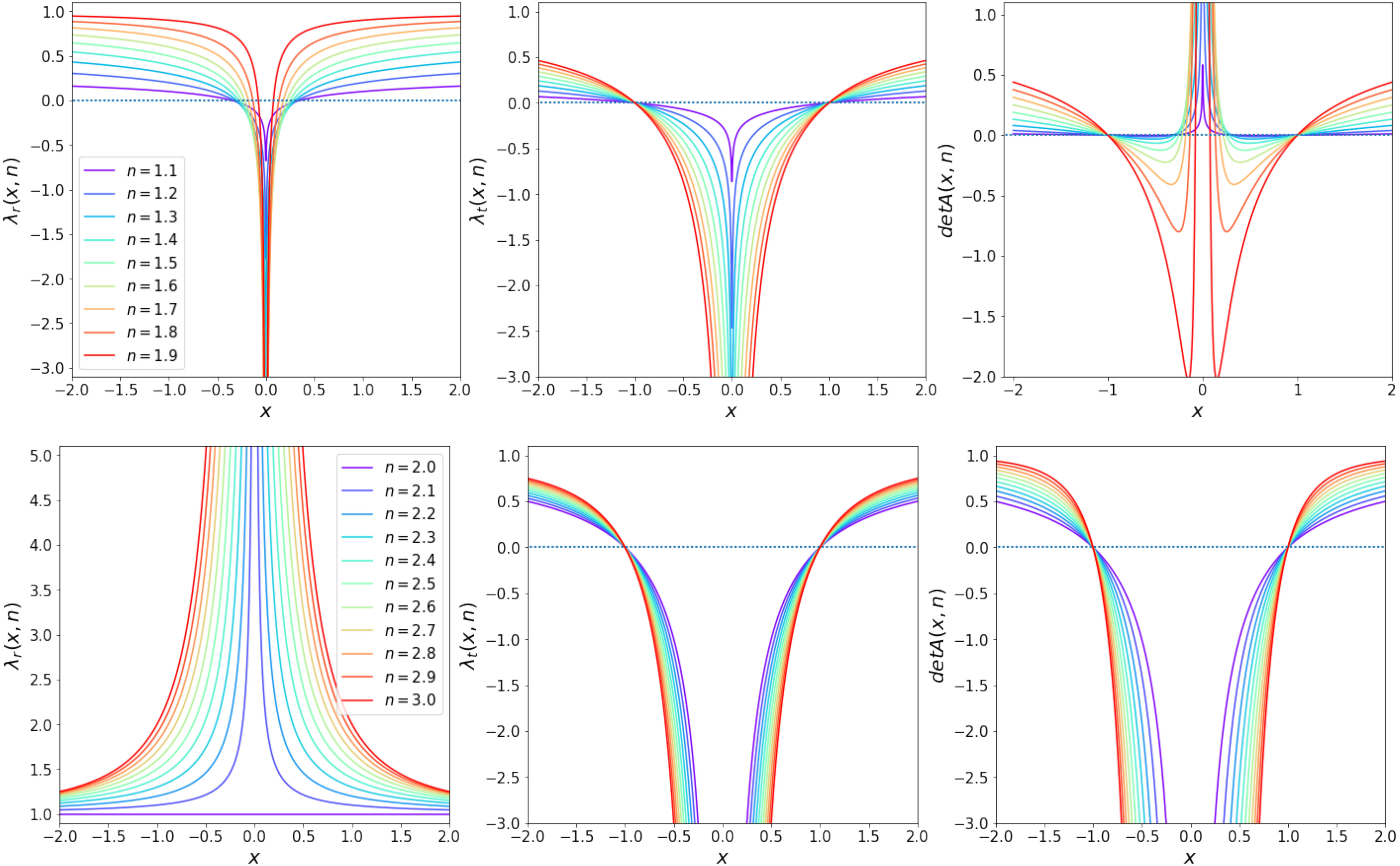


IMAGE DIAGRAM (EXAMPLE: N=1.7)

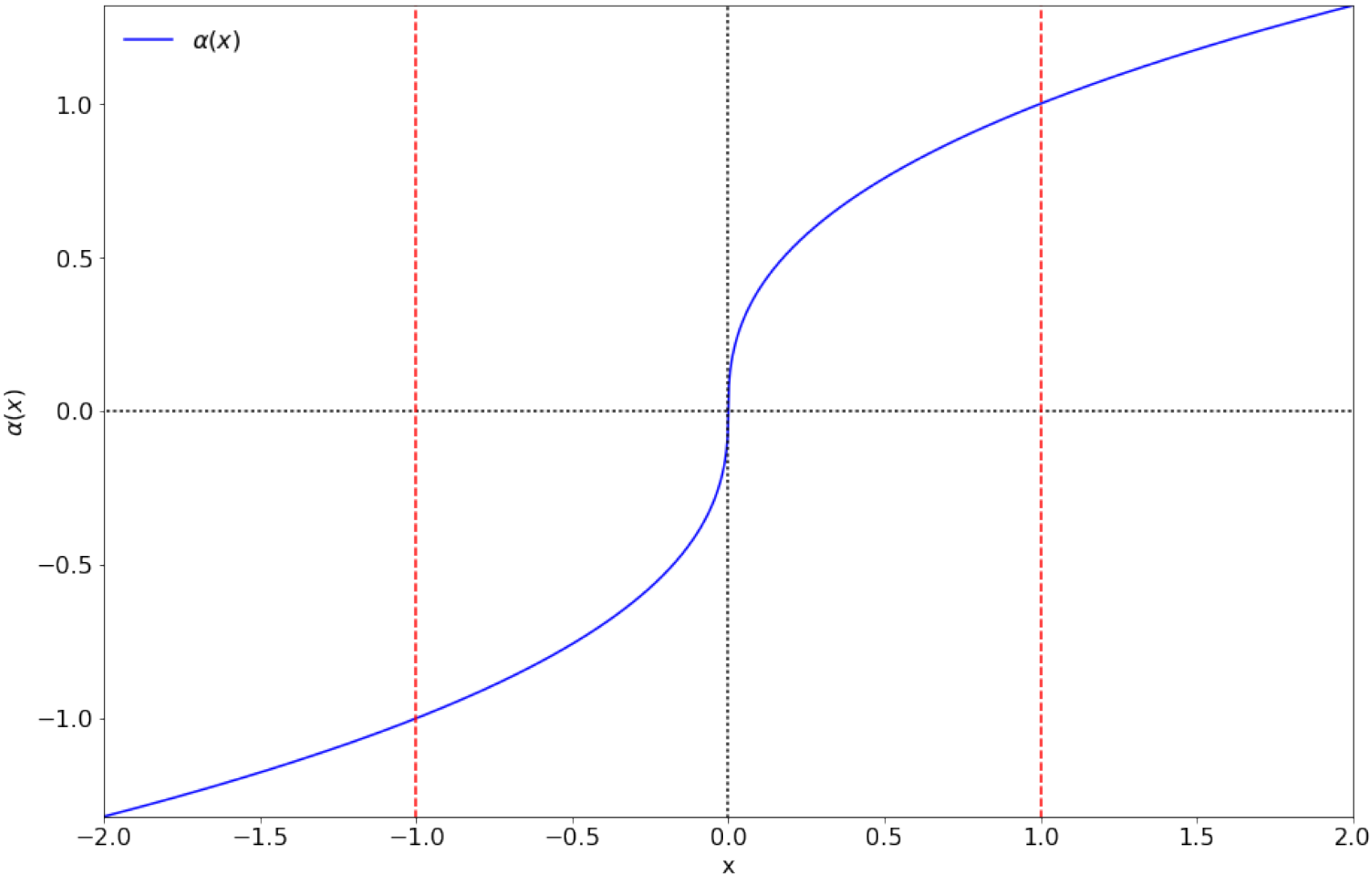


IMAGE DIAGRAM (EXAMPLE: N=1.7)

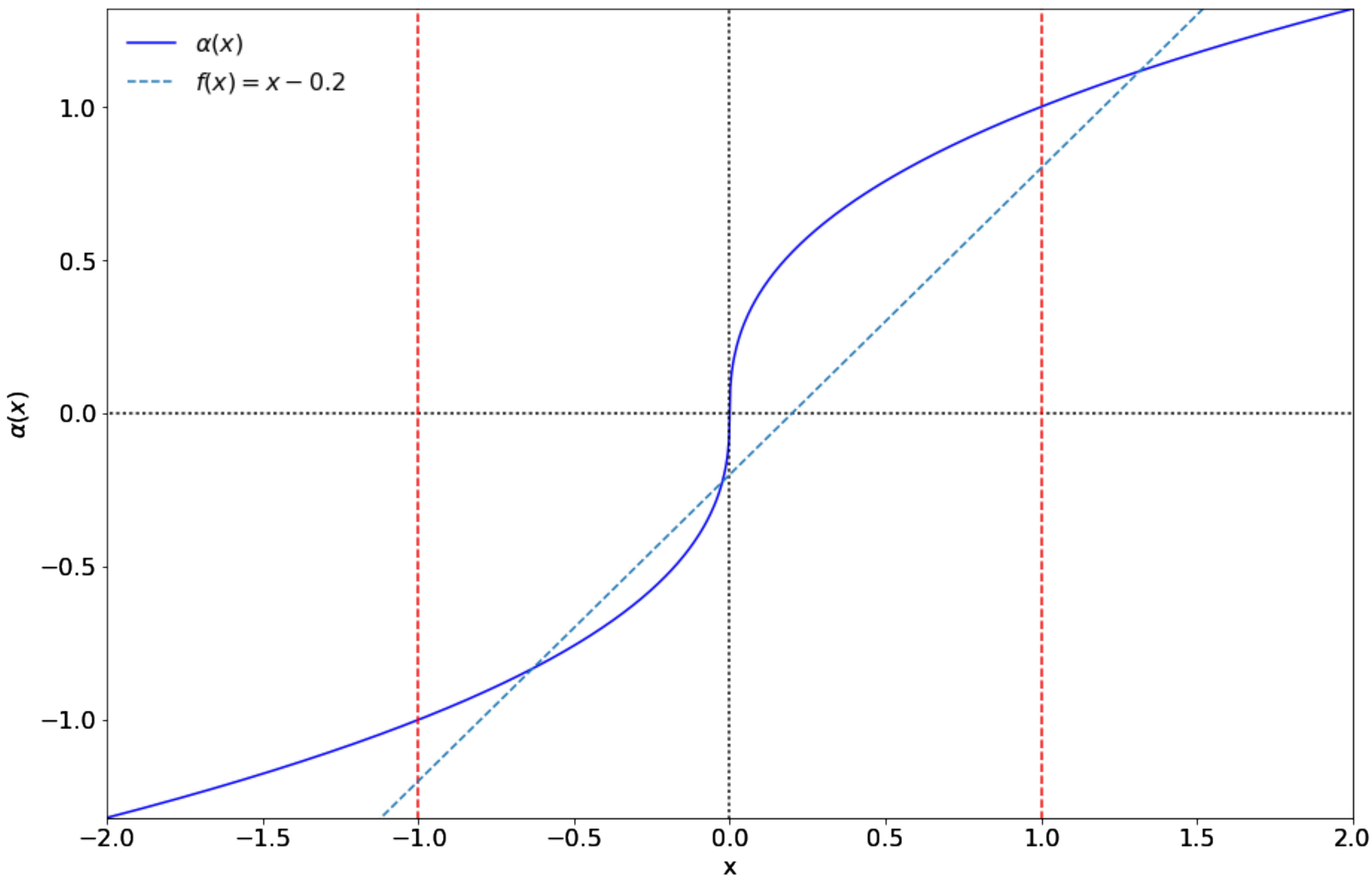


IMAGE DIAGRAM (EXAMPLE: N=1.7)

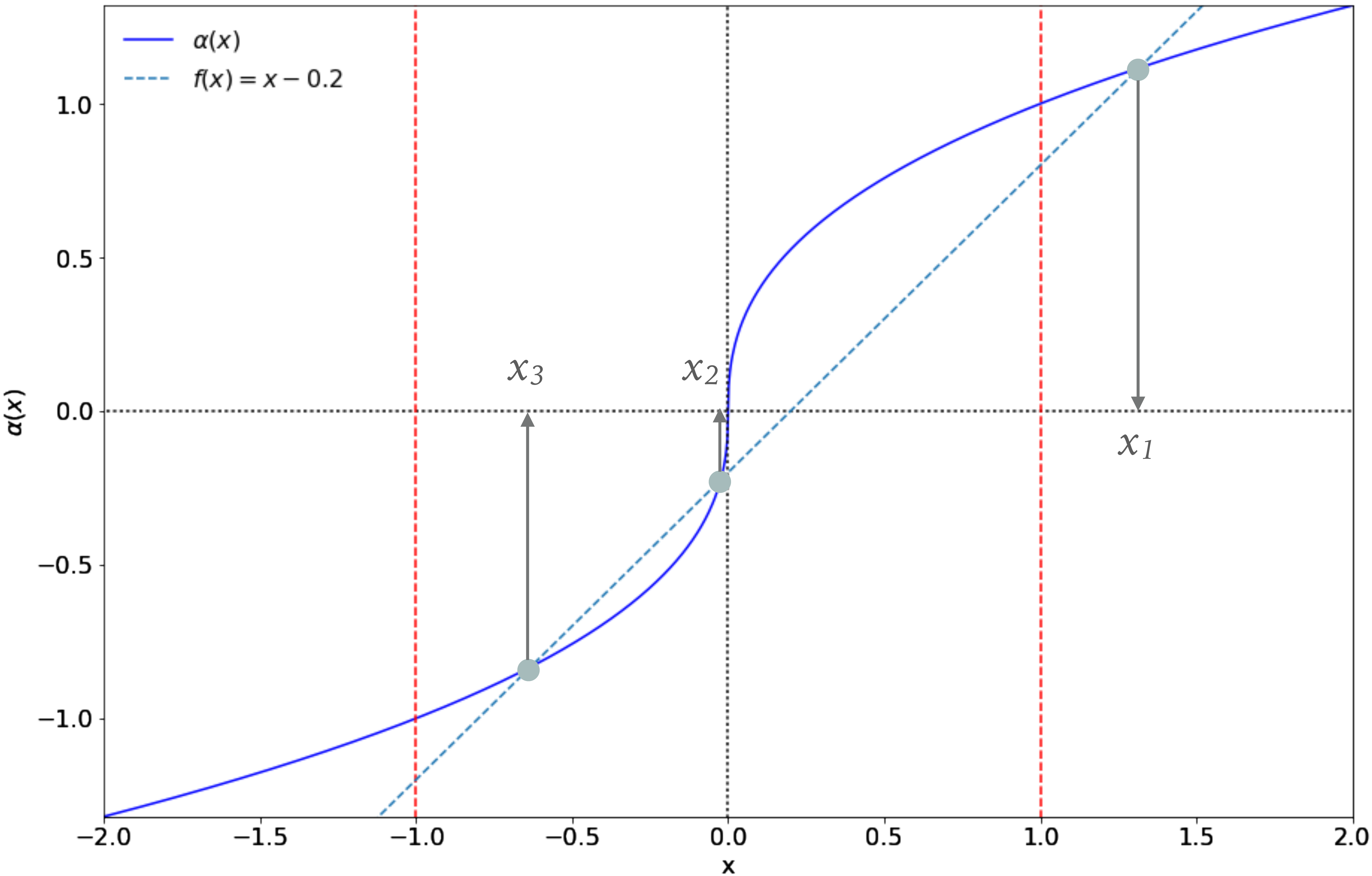


IMAGE DIAGRAM (EXAMPLE: N=1.7)

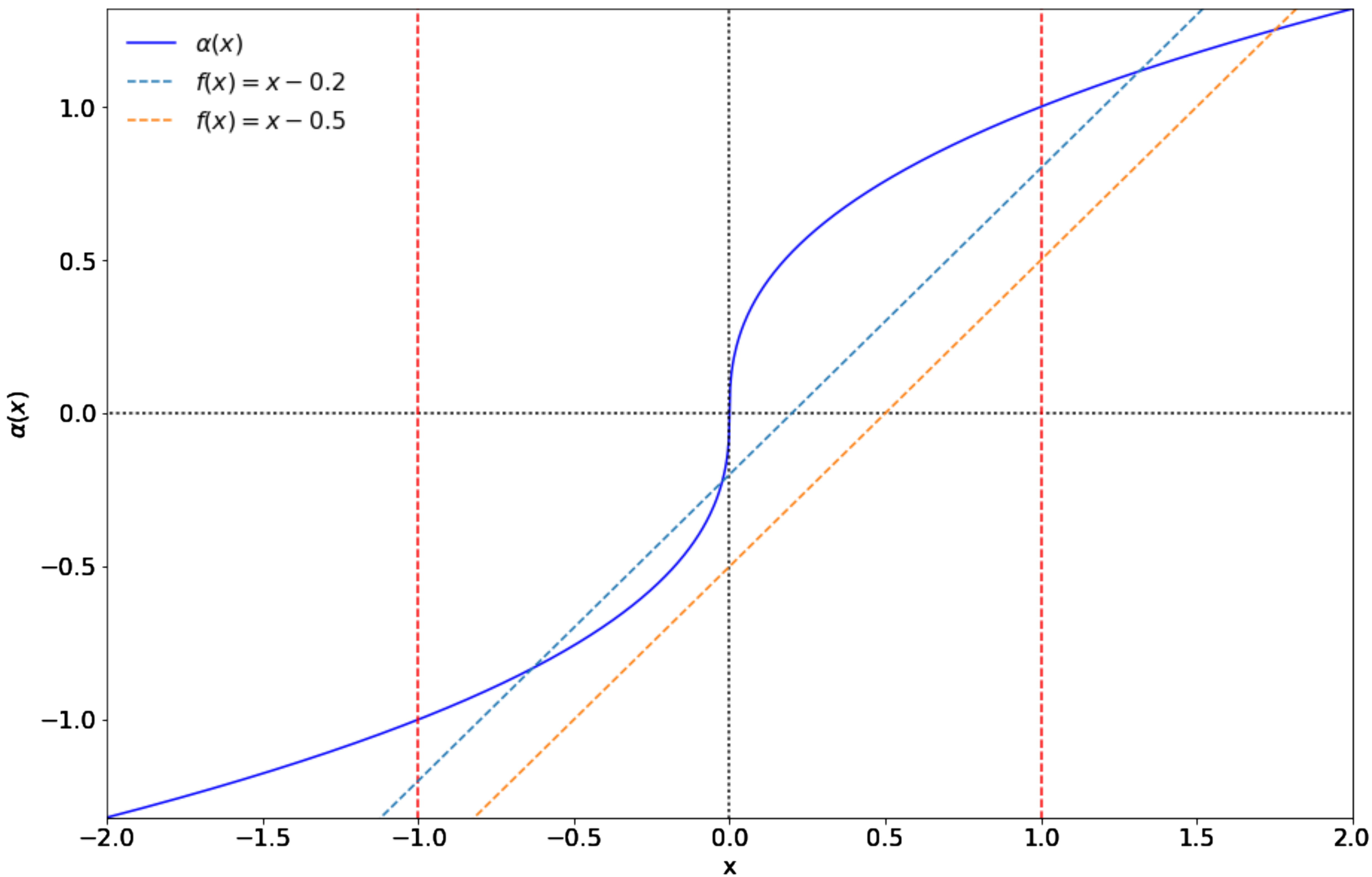


IMAGE DIAGRAM (EXAMPLE: N=1.7)

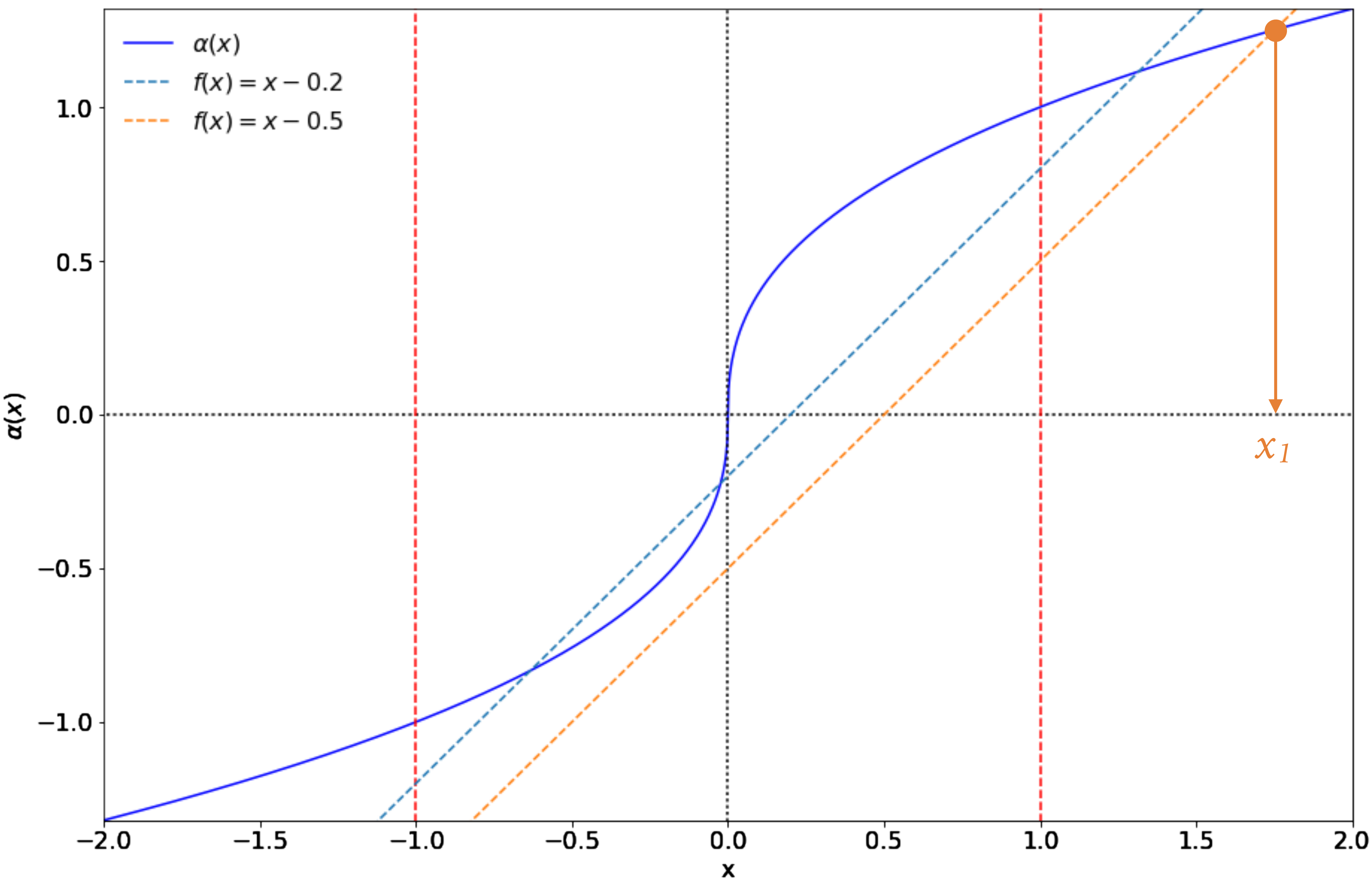


IMAGE DIAGRAM (EXAMPLE: N=1.7)

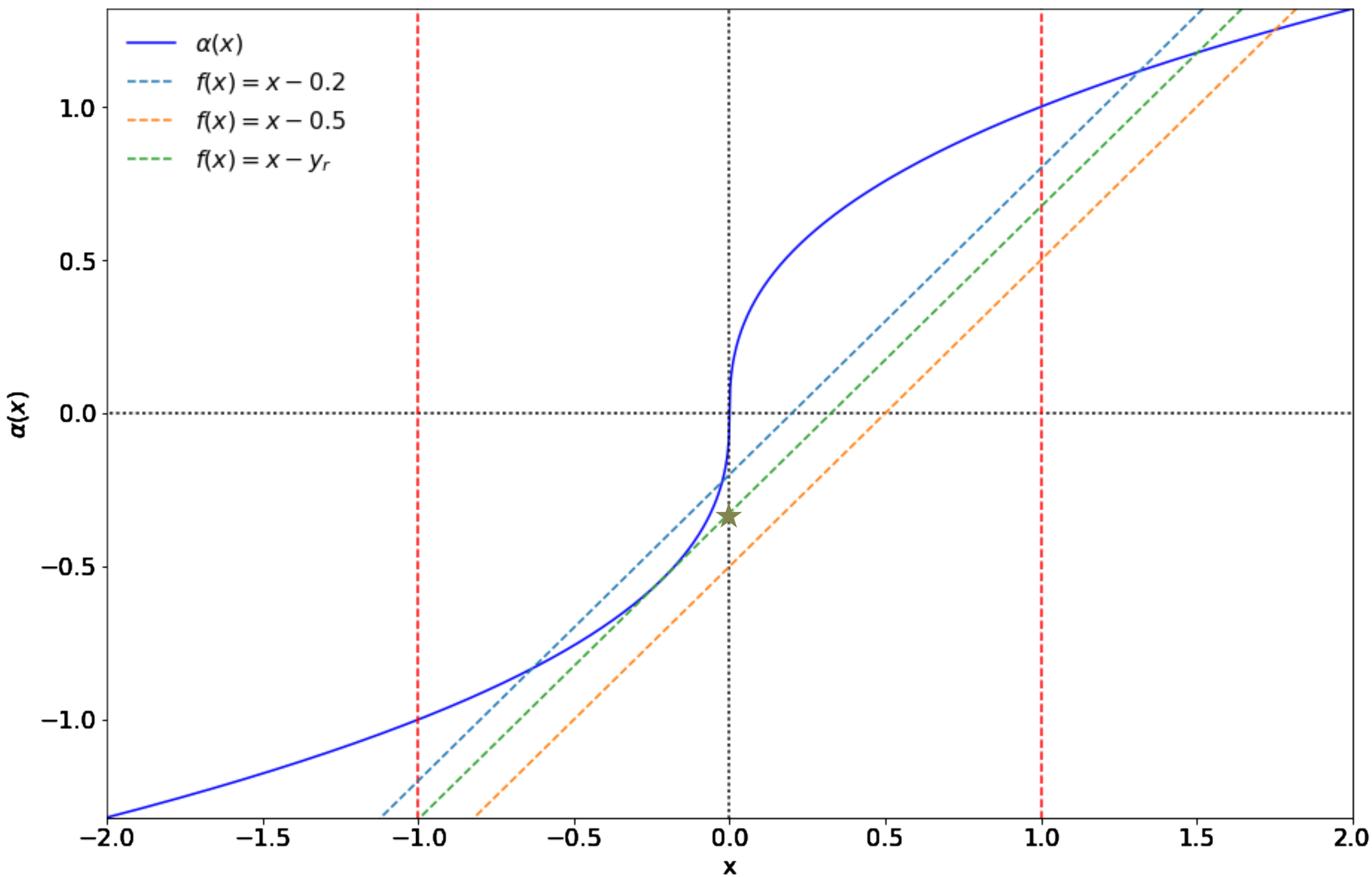


IMAGE DIAGRAM (EXAMPLE: N=1.7)

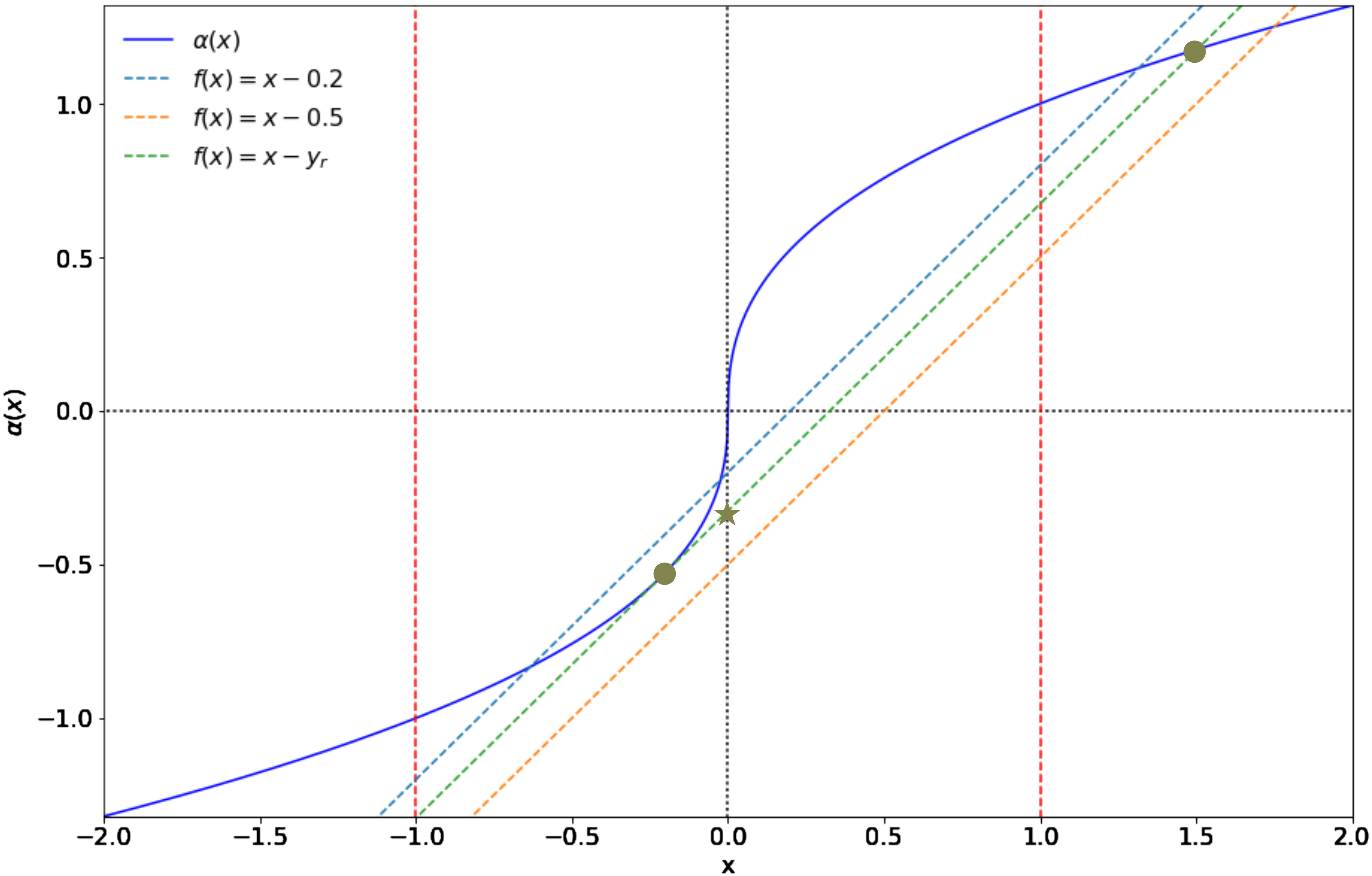


IMAGE DIAGRAM (EXAMPLE: N=1.7)

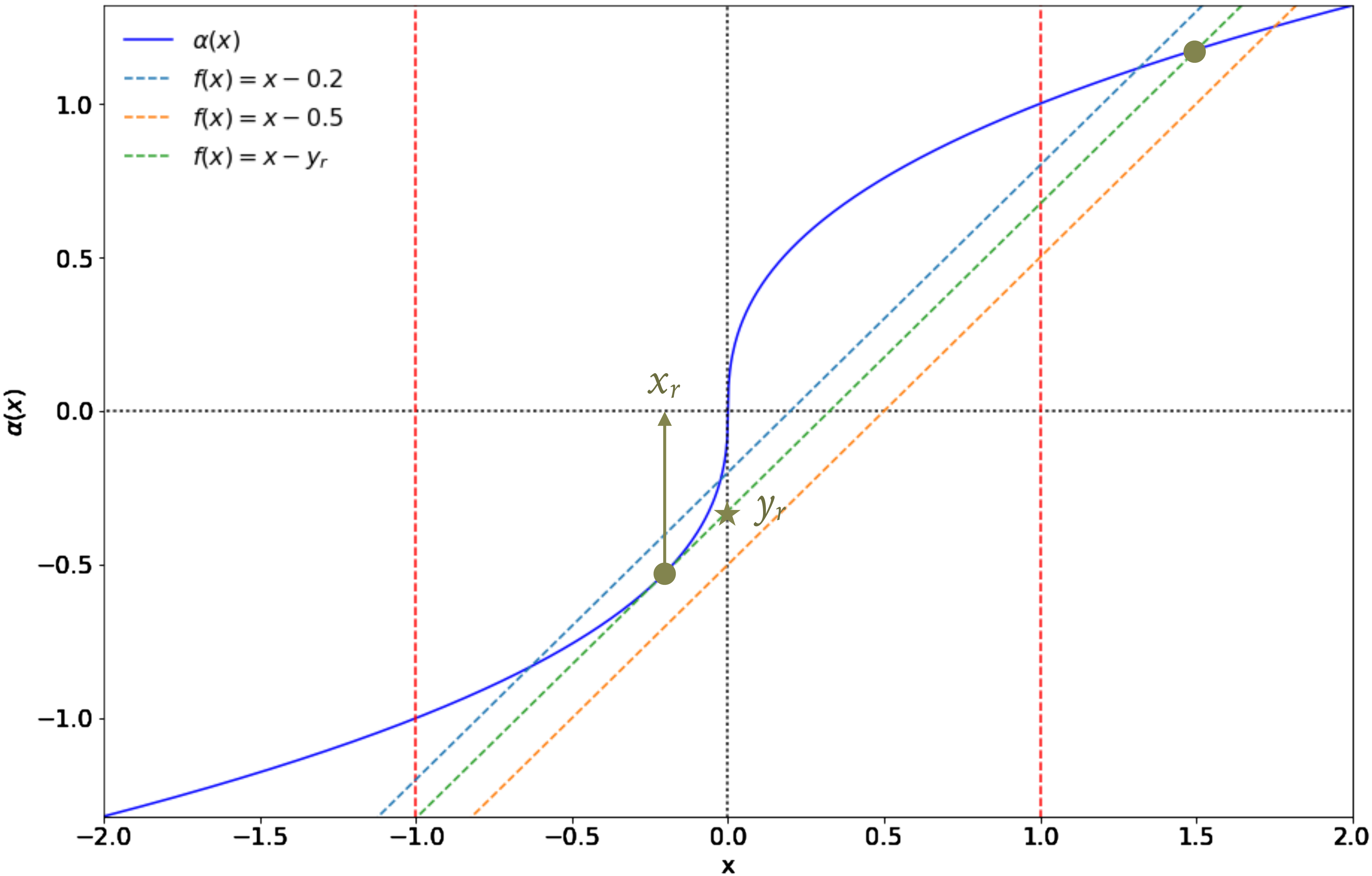
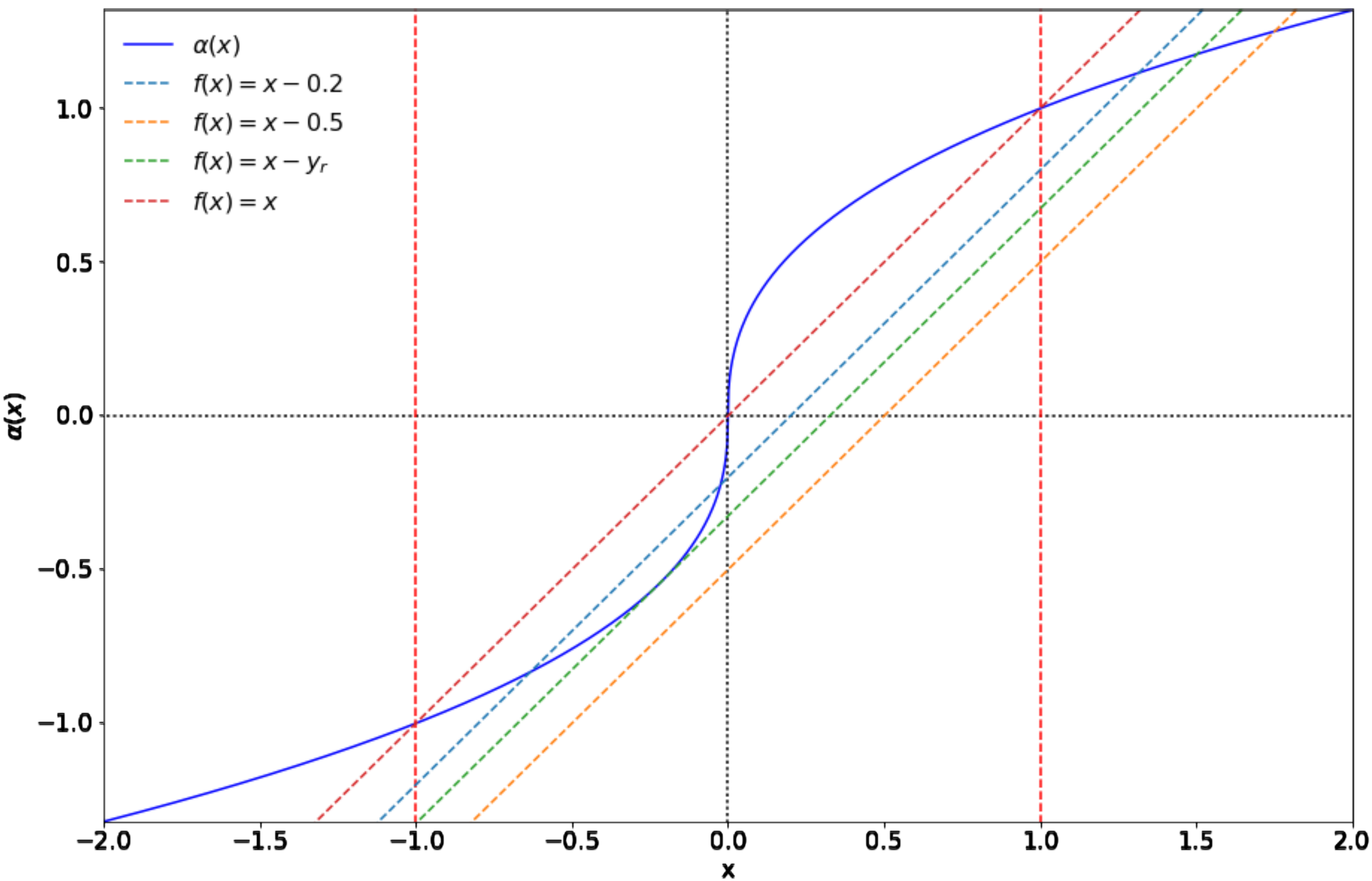


IMAGE DIAGRAM (EXAMPLE: N=1.7)



SOLUTIONS OF THE LENS EQUATION: IMAGE DIAGRAM

$$1 < n < 2$$

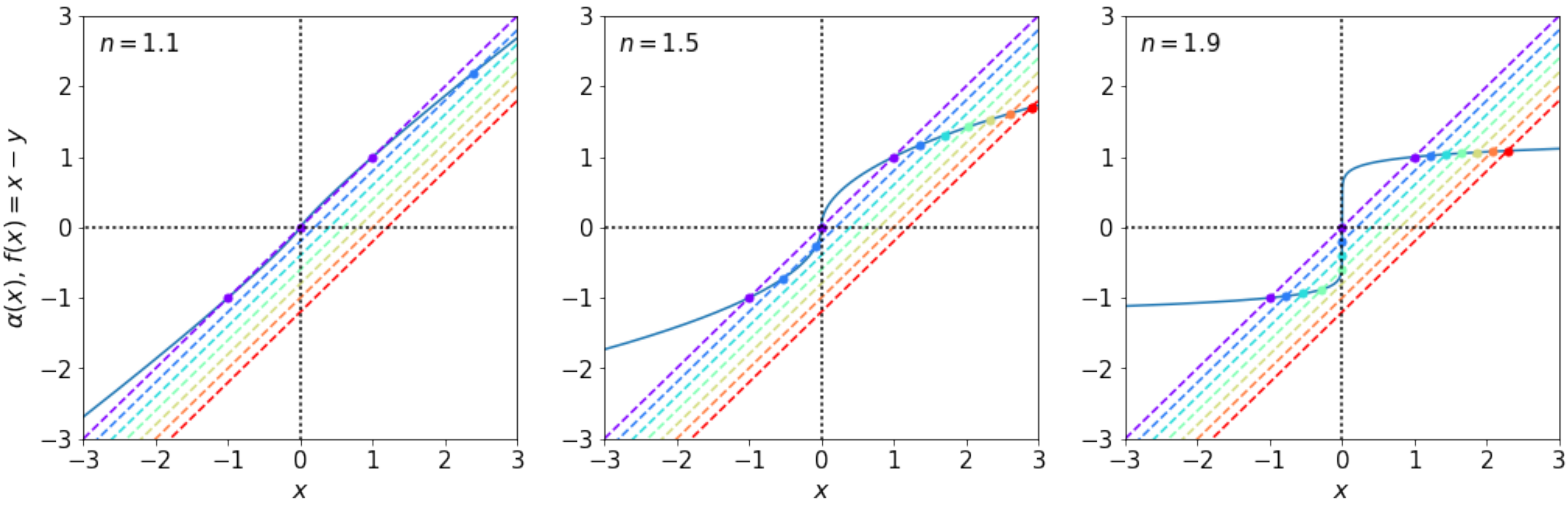
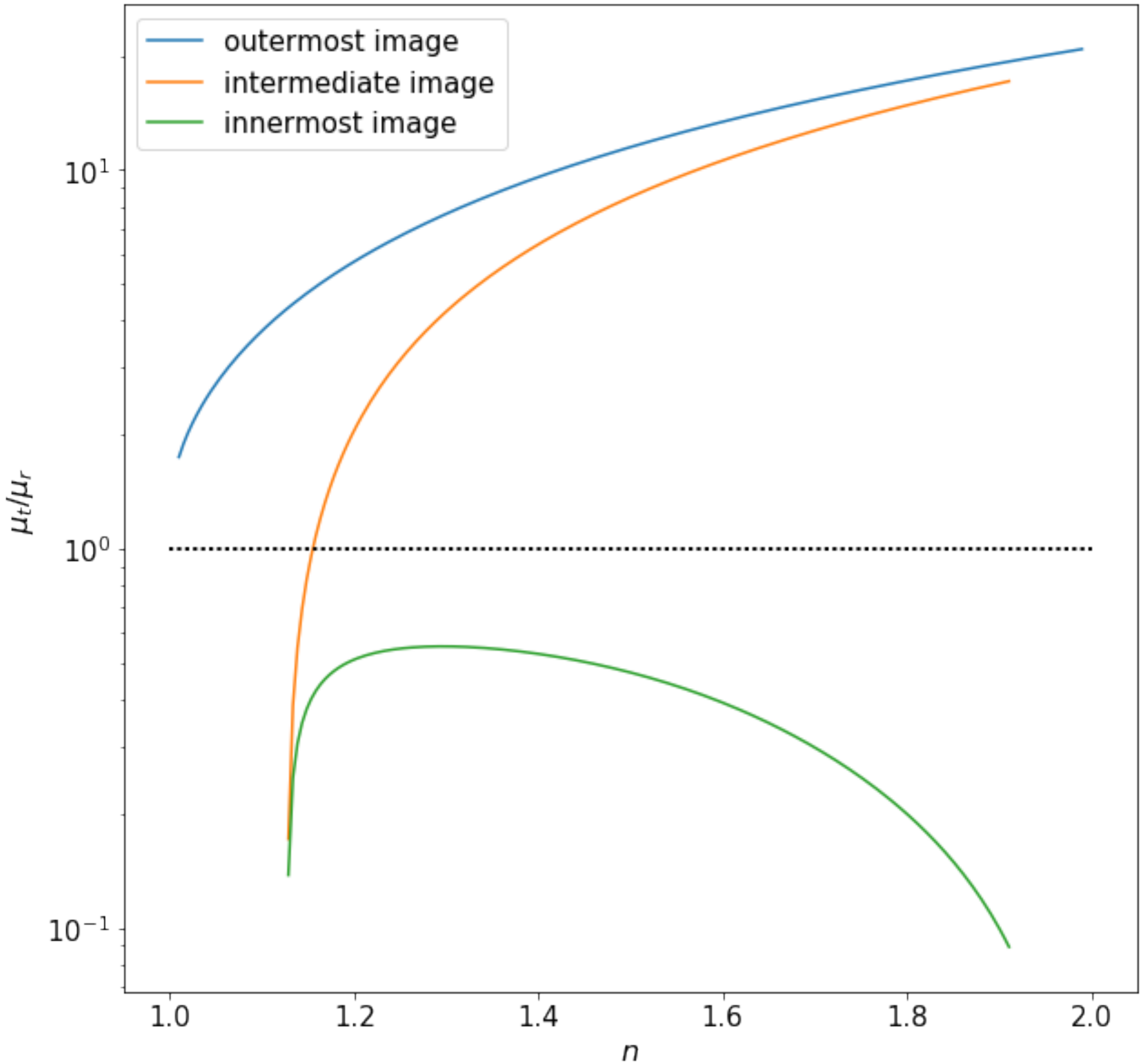


IMAGE MAGNIFICATION



$y_s = 0.05$

$$\lambda_t(x) = 1 - x^{1-n}$$
$$\lambda_r(x) = 1 - (2-n)x^{1-n}$$

IMAGE MAGNIFICATION

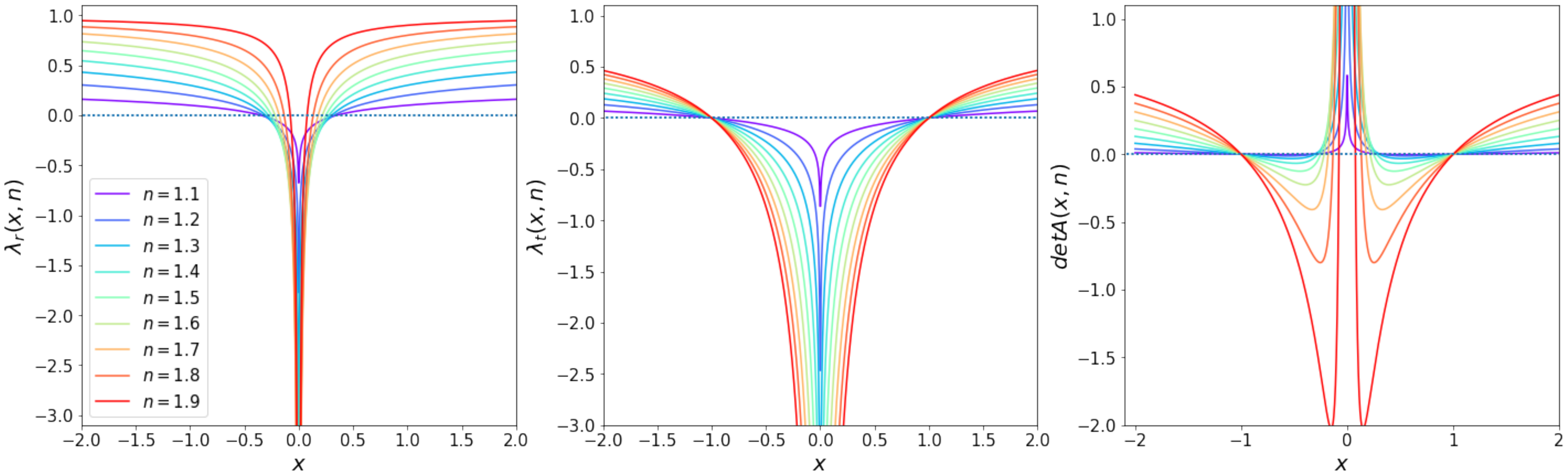


IMAGE MAGNIFICATION

higher radial magnification when n is small!

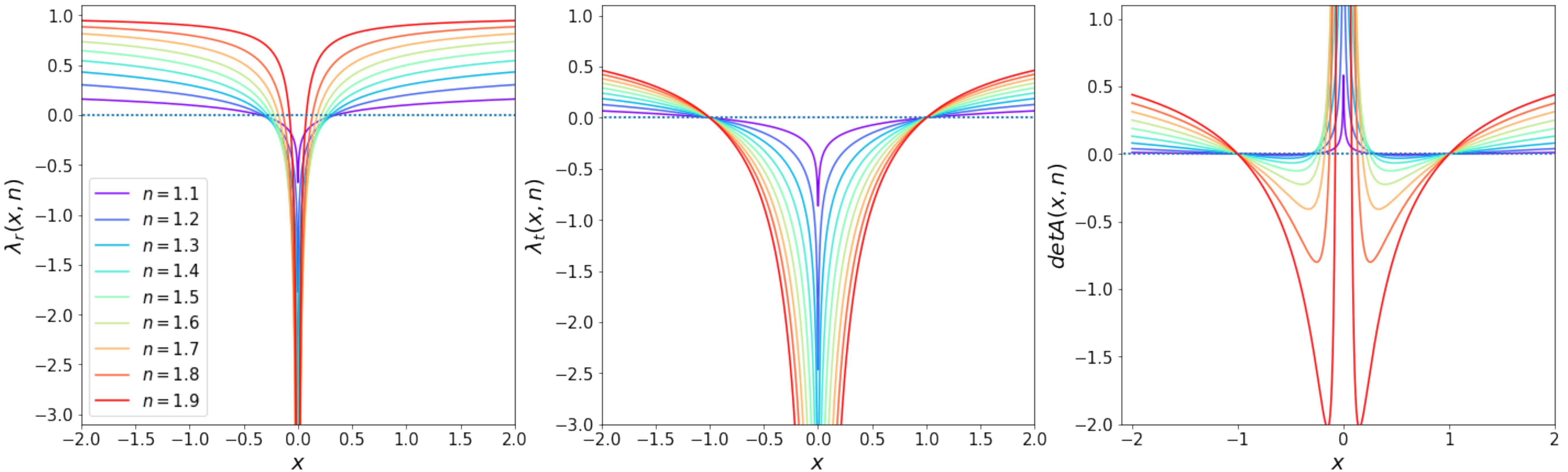
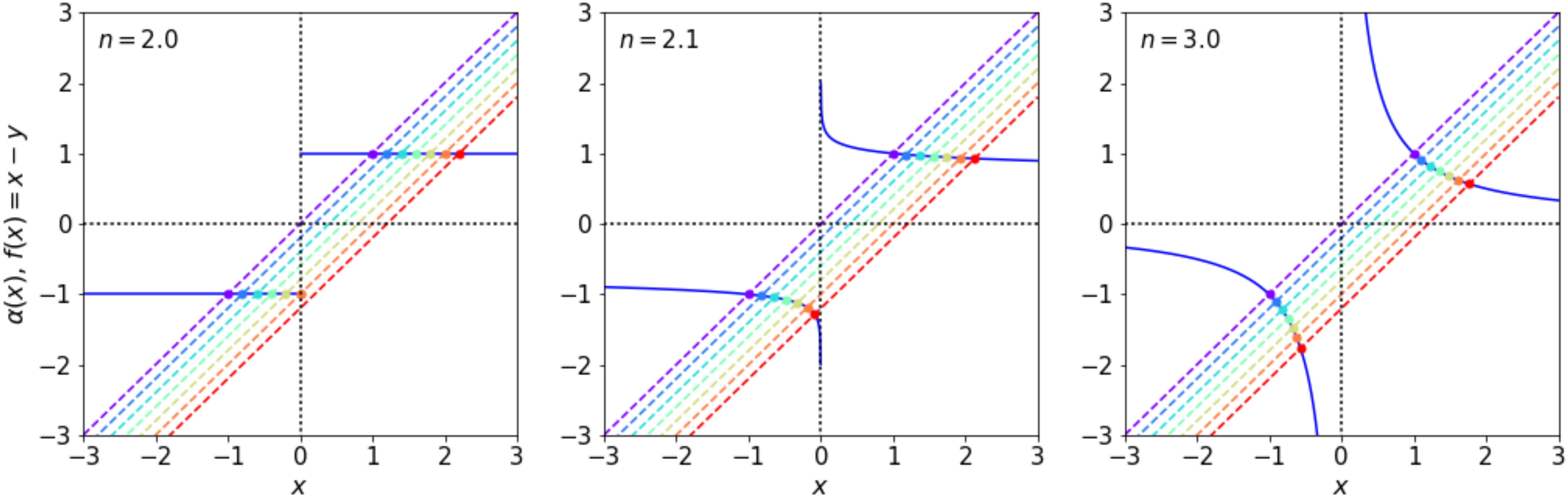


IMAGE DIAGRAM (N>2)



THE SINGULAR ISOTHERMAL SPHERE

The Singular Isothermal Sphere is a simple model to describe the distribution of matter in galaxies and clusters. It can be derived assuming that the matter content of the lens behaves like an ideal gas confined by a spherically symmetric gravitational potential. If the gas is in isothermal and hydrostatic equilibrium, its density profile is

$$\rho(r) = \frac{\sigma_v^2}{2\pi G r^2}$$

velocity dispersion of the gas particles

The profile is "unphysical"

- *singularity near the center*
- *mass is infinite*

THE SINGULAR ISOTHERMAL SPHERE

For lensing purposes, we are interested in the projection of this profile:

$$\rho(r) = \frac{\sigma_v^2}{2\pi G r^2}$$

$$\begin{aligned}
\Sigma(\xi) &= 2\frac{\sigma_v^2}{2\pi G}\int_0^\infty \frac{\mathrm{d}z}{\xi^2+z^2} \\
&= \frac{\sigma_v^2}{\pi G}\frac{1}{\xi}\left[\arctan\frac{z}{\xi}\right]_0^\infty \\
&= \frac{\sigma_v^2}{2G\xi} \ .
\end{aligned}$$

THE SINGULAR ISOTHERMAL SPHERE

As usual, we can switch to dimensionless units.

Let's take $$\xi_0 = 4\pi \left(\frac{\sigma_v}{c}\right)^2 \frac{D_L D_{LS}}{D_S}$$

Then: $$\Sigma(x) = \frac{\sigma_v^2}{2G\xi}\frac{\xi_0}{\xi_0} = \frac{1}{2x}\frac{c^2}{4\pi G}\frac{D_S}{D_L D_{LS}} = \frac{1}{2x}\Sigma_{cr} \,.$$

$$\kappa(x) = \frac{1}{2x}$$

Thus, the SIS lens is a power-law lens with n=2!

THE SINGULAR ISOTHERMAL SPHERE

The mass profile is readily computed:

$$m(x) = |x|$$

as well as the deflection angle:

$$\alpha(x) = \frac{x}{|x|}$$

The lens equation reads

$$y = x - \frac{x}{|x|}$$

How many solutions does this equation have?

THE SINGULAR ISOTHERMAL SPHERE

If 0<y<1, the solution are two:

$$x_- = y - 1$$

$$x_+ = y + 1$$

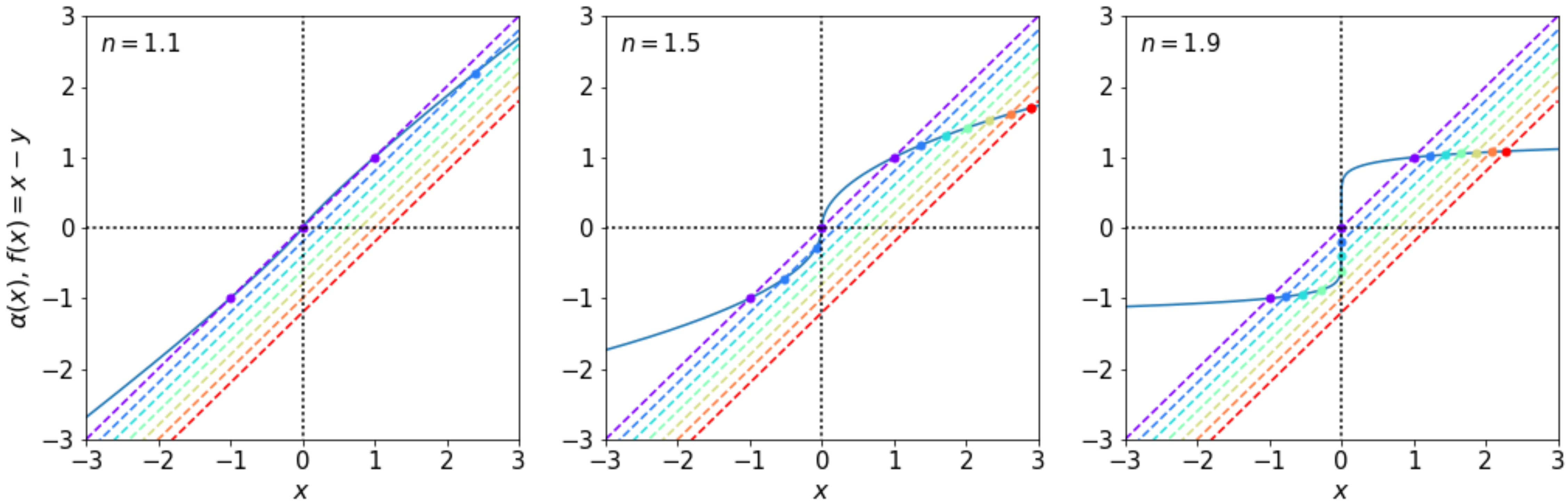
$$\theta_\pm = \beta \pm \theta_E$$

Otherwise, there is only one solution at

$$x_+ = y + 1$$

Thus, the circle of radius y=1 plays the same role of the radial caustic for the power-law lens with n<2, separating the source plane into regions with different image multiplicity.

IMAGE DIAGRAM



radial critical line: $$\left.\frac{d\alpha(x)}{dx}\right|_{x_r} = 1$$

THE SINGULAR ISOTHERMAL SPHERE

On the other hand, for the SIS: $d\bar{\alpha}/dx = 0$

This implies that the radial eigenvalue of the Jacobian matrix is always $\lambda_r = 1.$

Thus, the SIS lens does not magnify, neither de-magnifies the images in the radial direction.

THE SINGULAR ISOTHERMAL SPHERE

The shear can be computed easily:

$$\gamma(x) = \frac{m(x)}{x} - \kappa(x) = \frac{1}{2x}$$

$$\begin{aligned} \gamma_1 &= -\frac{1}{2}\frac{\cos 2\phi}{x} \\ \gamma_2 &= -\frac{1}{2}\frac{\sin 2\phi}{x} \end{aligned}$$

THE SINGULAR ISOTHERMAL SPHERE

as well as the magnification

$$\mu = \frac{|x|}{|x|-1}$$

$$\mu_+ = \frac{y+1}{y} = 1 + \frac{1}{y} \quad ; \quad \mu_- = \frac{|y-1|}{|y-1|-1} = \frac{-y+1}{-y} = 1 - \frac{1}{y}$$

THE SINGULAR ISOTHERMAL SPHERE