GRAVITATIONAL LENSING

1 – DEFLECTION OF LIGHT

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AA 2019-2020

CONTACTS

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THE COURSE

- Basics of Gravitational Lensing Theory
- Applications of Gravitational Lensing:
  - microlensing in the MW
  - lensing by galaxies and galaxy clusters
- Python

LEARNING RESOURCES

- https://github.com/maxmen/LensingLectures
- available materials:
  - lecture notes (partially)
  - lecture slides
  - python notebooks
- Suggested books:
  - Principles of Gravitational Lensing - Congdon & Keeton
  - Gravitational Lensing - Dodelson
  - Gravitational Lensing: strong, weak and micro - Schneider, Kochaneck & Wambsganss

FINAL EXAM

- three questions: the first at your choice
- all the topics discussed during the course
- you are encouraged to complement the material distributed during the course with other papers, books, etc.
- for what regards the python examples: you are strongly encouraged to study and understand the codes to fully understand the algorithms
- programming will not be part of the exam, but the knowledge of the algorithms will be required

CONTENTS OF TODAY'S LESSON

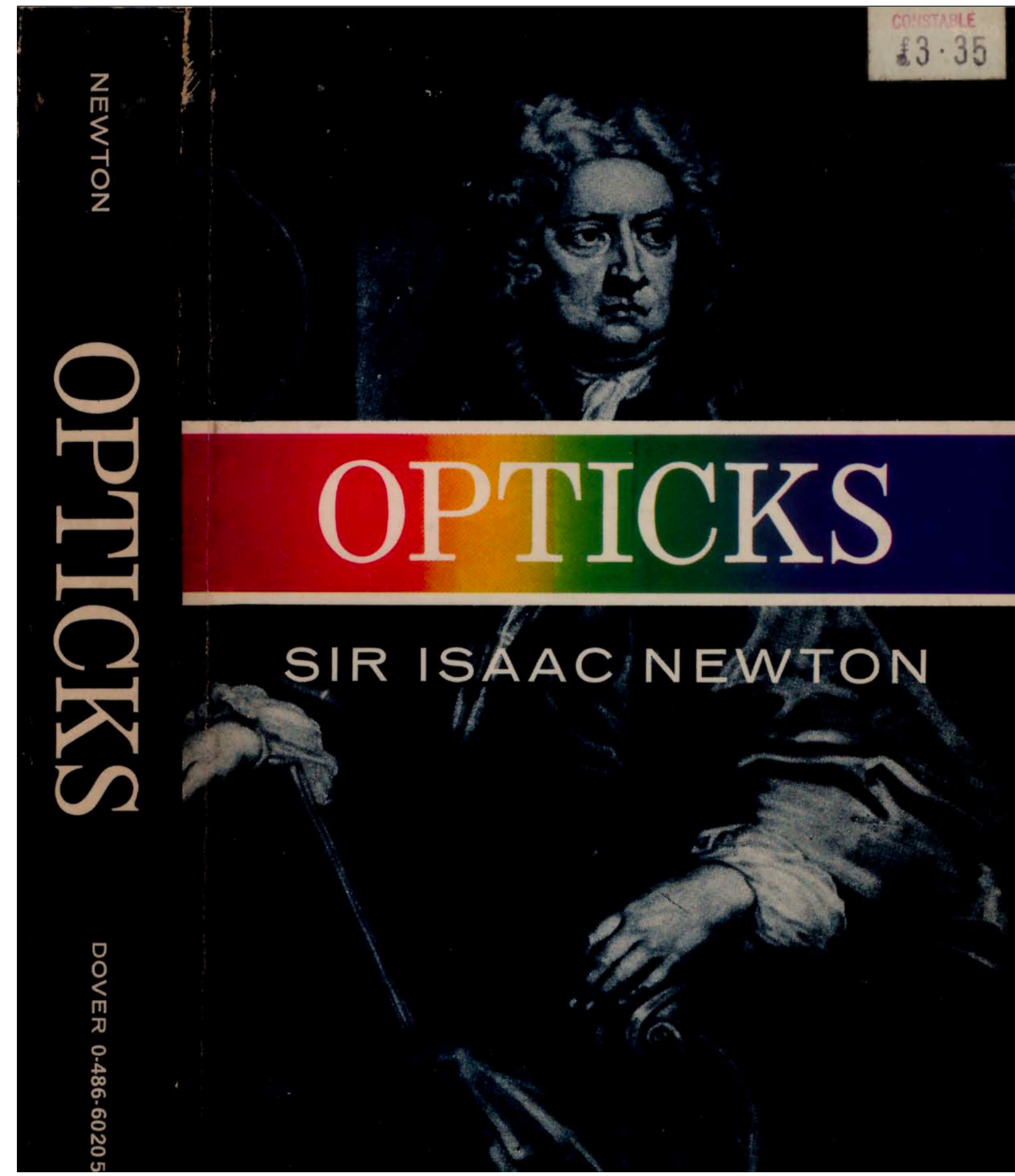
- Few historical remarks
- Deflection of light in the Newtonian limit
- Gravitational lensing in the context of general relativity
- The deflection angle

CORPUSCULAR THEORY OF LIGHT

- I. Newton, Opticks (1704-1730)
- Third volume ends with 31 queries:

Query 1. Do not Bodies act upon Light at a distance, and by their action bend its Rays; and is not this action (*cæteris paribus*) strongest at the least distance?

Qu. 29. Are not the Rays of Light very small Bodies emitted from shining Substances? For such Bodies will pass through uniform Mediums in right Lines without bending into the Shadow, which is the Nature of the Rays of Light. They will also be capable



- 1678: wave theory, Huygens

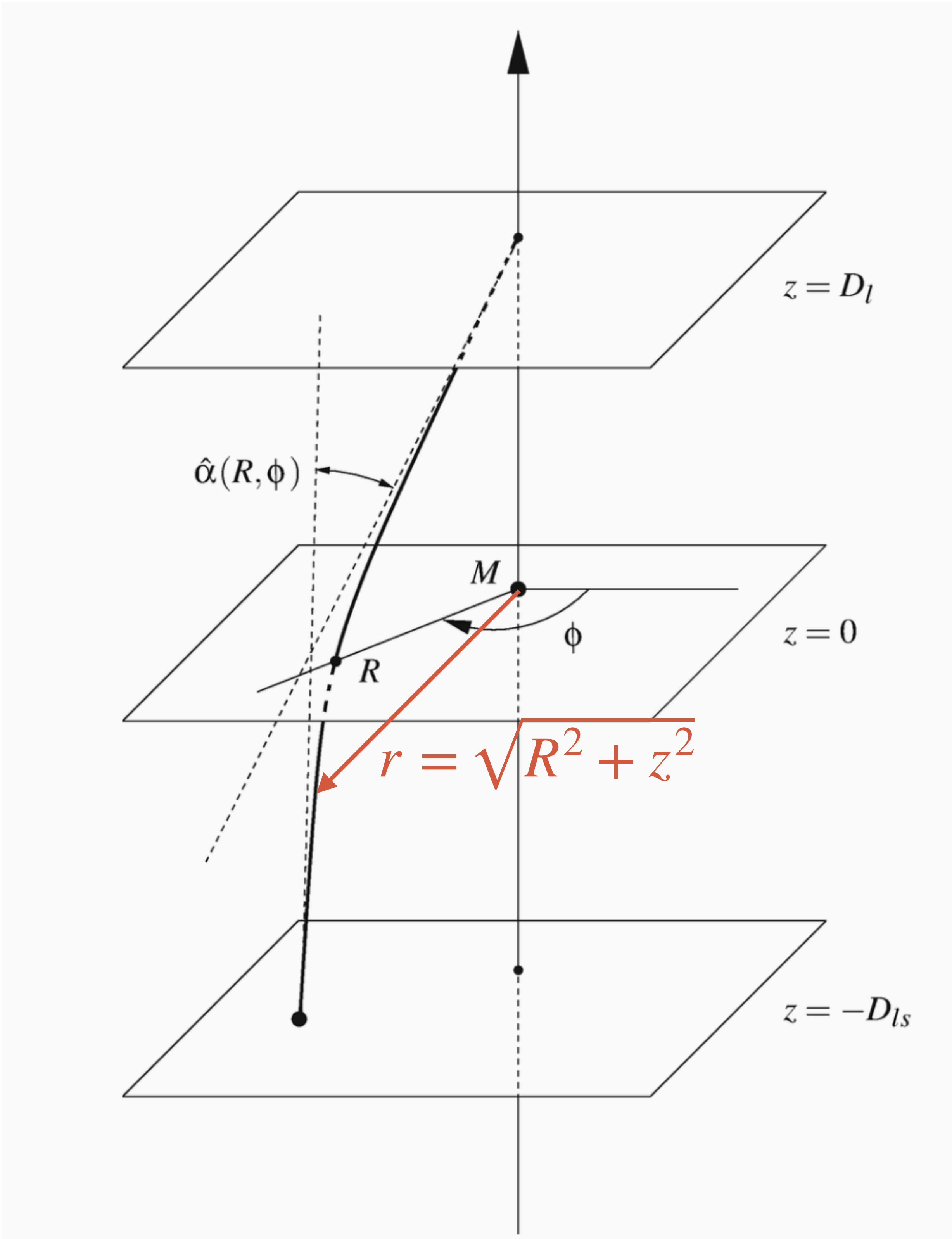
TIMELINE

- 1783: John Michell writes to Henry Cavendish. A light corpuscle might not be capable of escaping a massive star if

$$E \equiv \frac{1}{2}mv^2 - \frac{GmM}{R} \leq 0 \qquad v = c \qquad R < R_s \equiv \frac{2GM}{c^2} \; .$$

- 1784: Henry Cavendish calculates the deflection of a light corpuscle by a mass M. Unpublished until beginning of '900.
- 1801: Johan Soldner independently repeats the same calculation and publish it

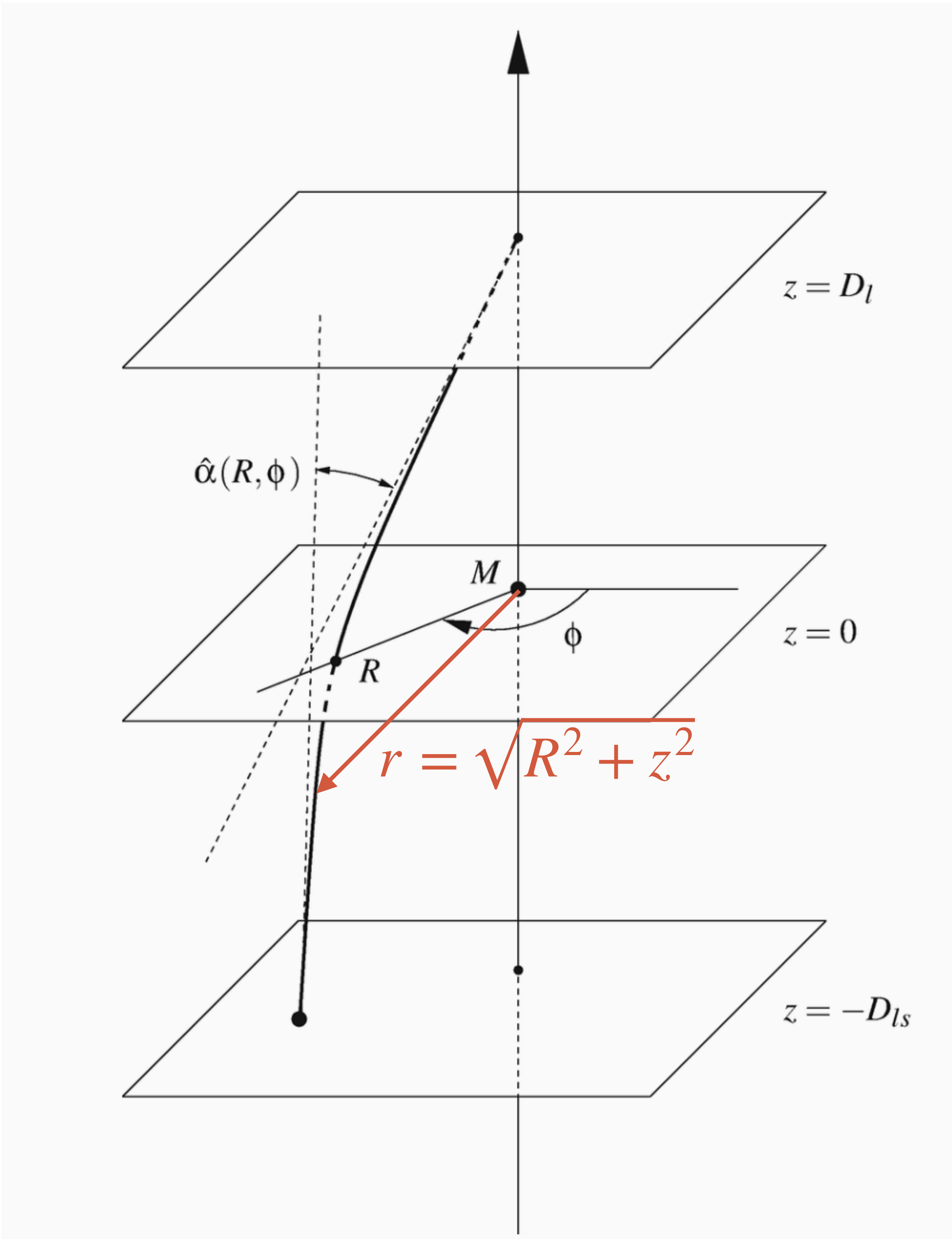
DEFLECTION OF A LIGHT CORPUSCLE



- Assumptions:
  - photons have mass and feel gravity
  - Newton's law of gravitation
  - Newton's 2nd law of motion
  - speed of light is finite

Congdon & Keeton, 2018

DEFLECTION OF A LIGHT CORPUSCLE



$$\vec{v}_0 = c\,\vec{e}_z$$

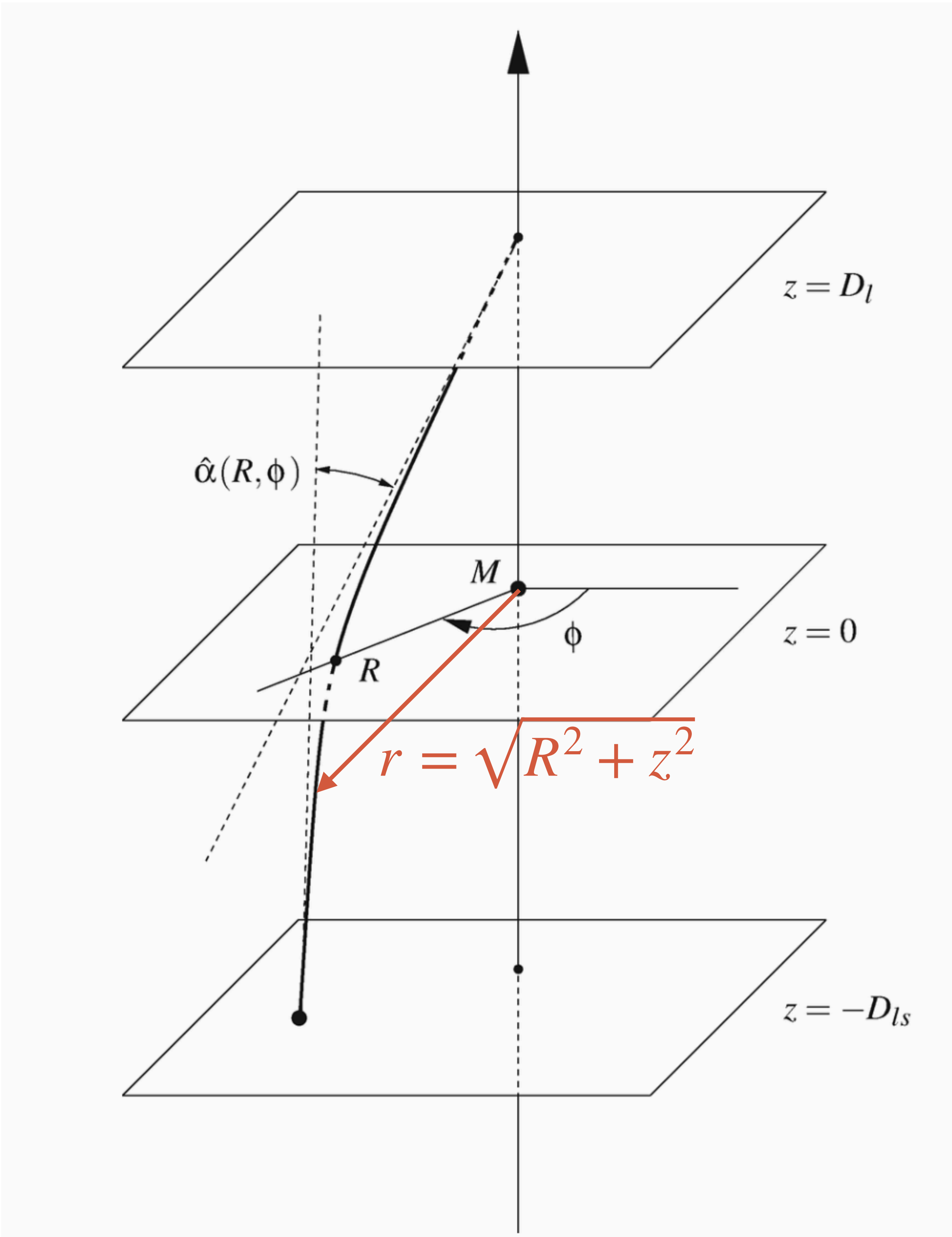
$$r = \sqrt{R^2 + z^2} \qquad dt = \frac{dz}{c}$$

$$\vec{\Delta} v = \int_{t_s}^{t_o} \vec{a}\, dt = \frac{1}{c} \int_{z_s}^{z_o} \vec{a}\, dz$$

$$\vec{a} = -\vec{\nabla}\Phi$$

$$\vec{\Delta} v = -\frac{1}{c} \int_{z_s}^{z_o} \vec{\nabla}\Phi dz$$

DEFLECTION OF A LIGHT CORPUSCLE

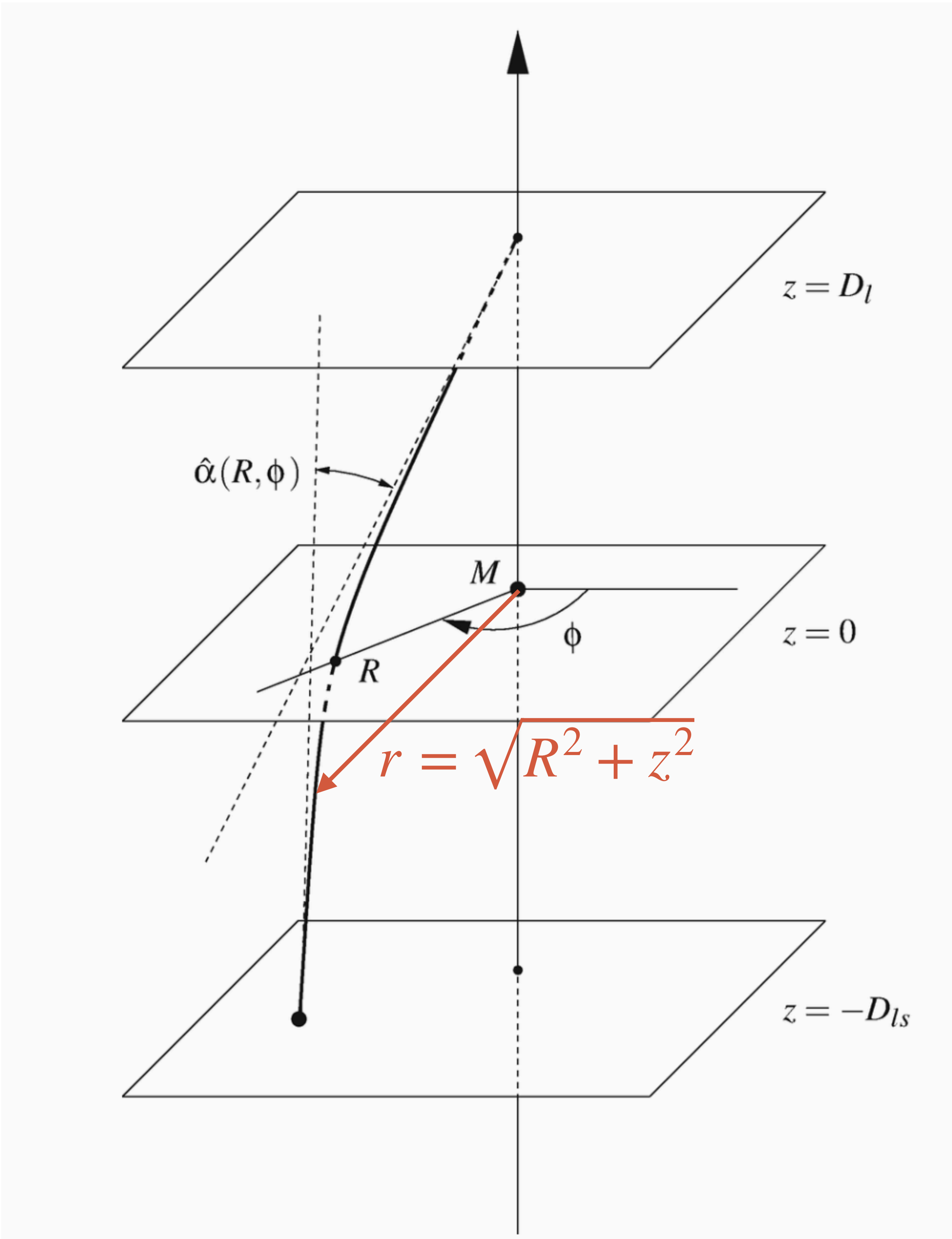


$$\overrightarrow{\Delta} v = -\frac{1}{c}\int_{z_s}^{z_o} \overrightarrow{\nabla}\Phi dz$$

$$\overrightarrow{\Delta} v = \Delta v_{||}\overrightarrow{e}_{||} + \Delta v_{\perp}\overrightarrow{e}_R$$

$$\overrightarrow{\nabla}\Phi = \frac{\partial \Phi}{\partial z}\overrightarrow{e}_z + \frac{\partial \Phi}{\partial R}\overrightarrow{e}_R$$

DEFLECTION OF A LIGHT CORPUSCLE



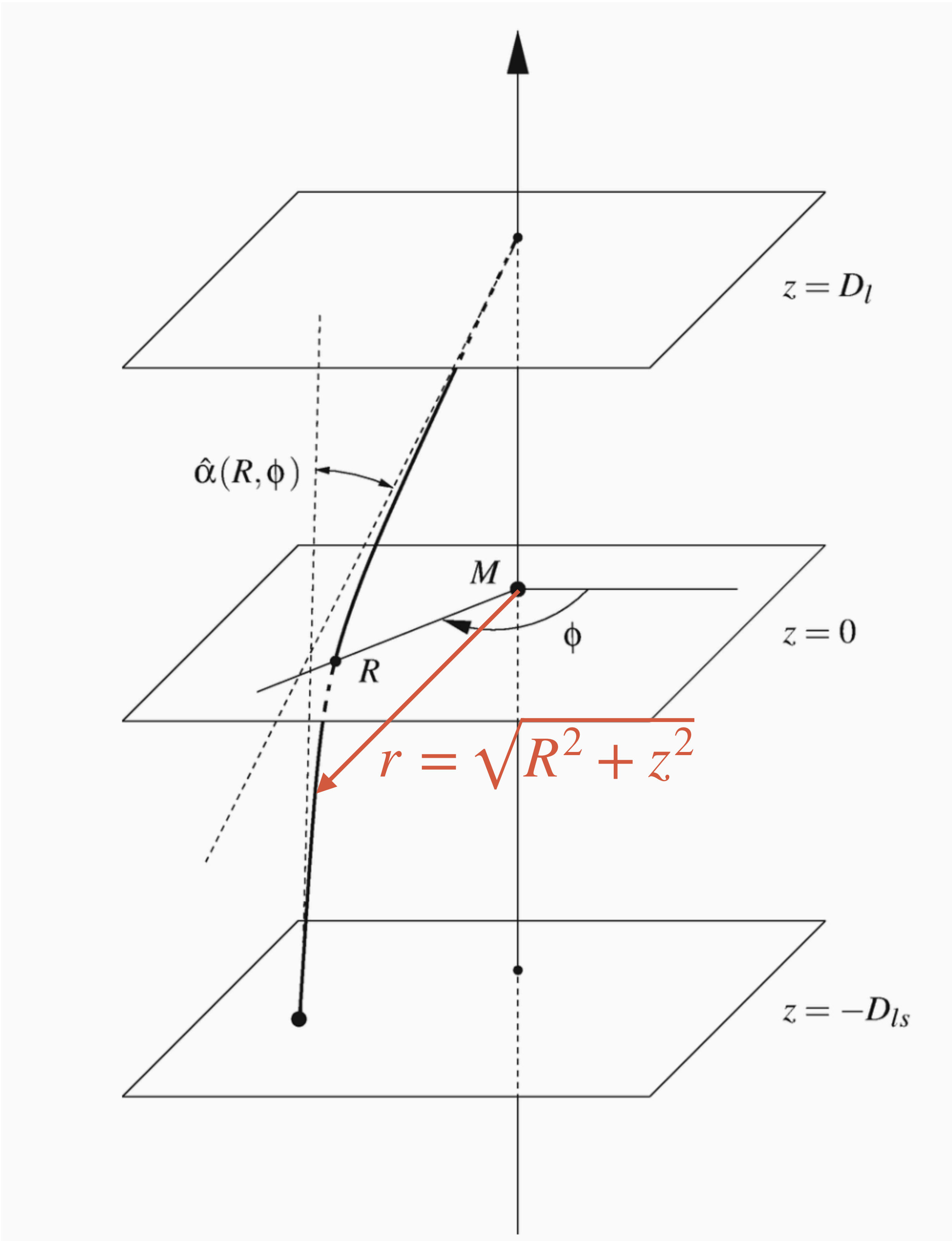
$$\overrightarrow{\Delta} v = -\frac{1}{c}\int_{z_s}^{z_o} \overrightarrow{\nabla}\Phi dz = \Delta v_{||}\overrightarrow{e}_{||} + \Delta v_{\perp}\overrightarrow{e}_{R}$$

$$\overrightarrow{\nabla}\Phi = \frac{\partial \Phi}{\partial z}\overrightarrow{e}_z + \frac{\partial \Phi}{\partial R}\overrightarrow{e}_R$$

$$\Delta v_{||} = -\frac{1}{c}\int_{z_s}^{z_o} \frac{\partial \Phi}{\partial z} dz$$

$$= -\frac{1}{c}[\Phi(\overrightarrow{R}, z_o) - \Phi(\overrightarrow{R}, z_s)]$$

DEFLECTION OF A LIGHT CORPUSCLE



$$\Delta v_{||} = -\frac{1}{c}\int_{z_s}^{z_o} \frac{\partial \Phi}{\partial z} dz$$

$$= -\frac{1}{c}[\Phi(\overrightarrow{R}, z_o) - \Phi(\overrightarrow{R}, z_s)]$$

Example: point mass

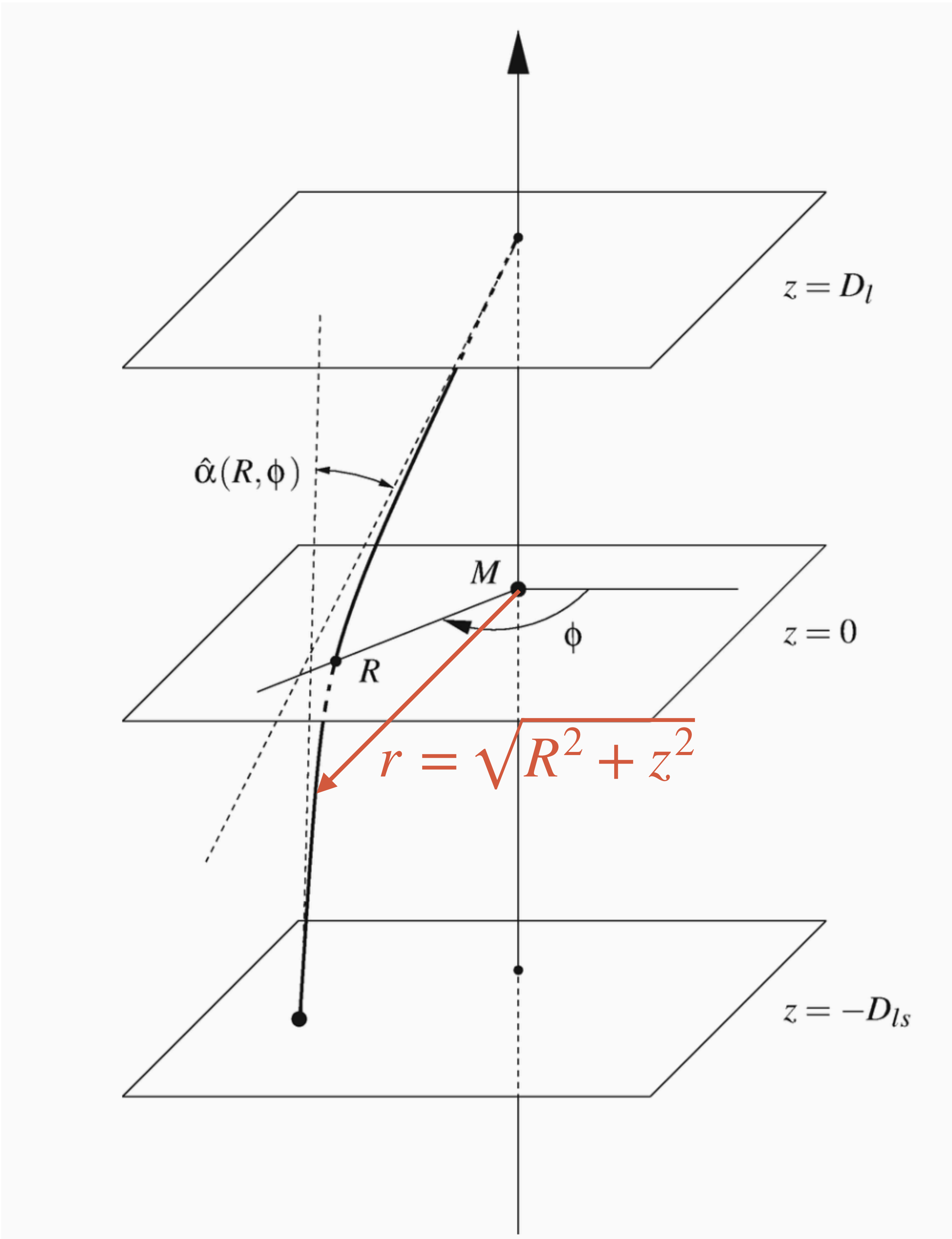
$$\Phi(r) = -\frac{GM}{r} = -\frac{GM}{\sqrt{R^2 + z^2}}$$

$$\lim_{|z| \to \infty} \Phi(r) = 0$$

If $|z_s|$ and $|z_o|$ are large:

$$\Delta v_{||} = 0$$

DEFLECTION OF A LIGHT CORPUSCLE



$$\Delta v_\perp = -\frac{1}{c}\int_{z_s}^{z_o} \frac{\partial \Phi}{\partial R} dz$$

Example: point mass

$$\Phi(r) = -\frac{GM}{r} = -\frac{GM}{\sqrt{R^2 + z^2}}$$

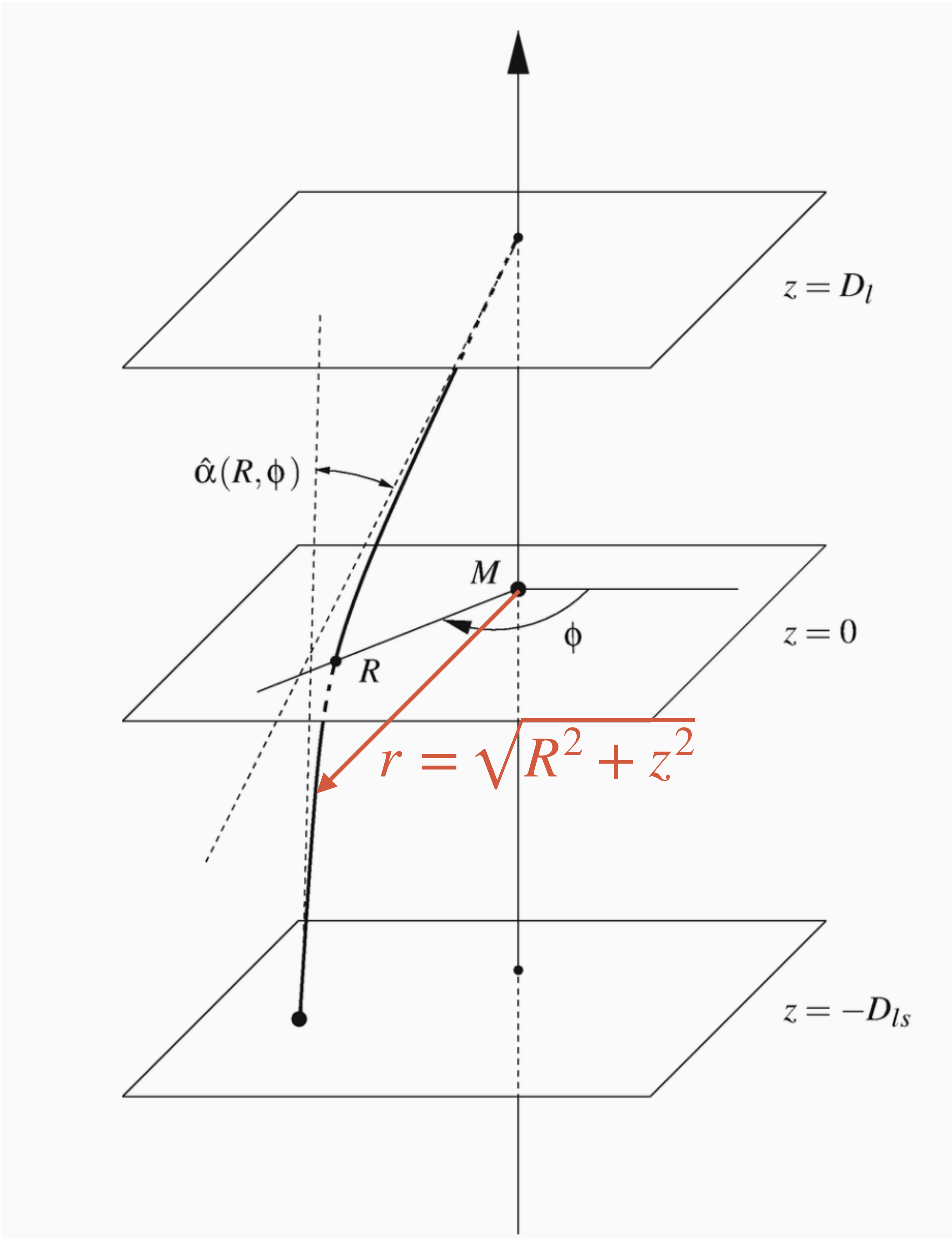
$$\Delta v_\perp = -\frac{GMR}{c}\int_{z_s}^{z_o} (R^2 + z^2)^{-3/2} dz$$

$$\tan(u) = \frac{z}{R} \qquad \frac{dz}{R} = \frac{du}{\cos^2(u)}$$

As before, if $|z_s|$ *and* $|z_o|$ *are large:*

$$\Delta v_\perp = -\frac{GM}{cR}\int_{-\pi/2}^{\pi/2} \cos(u) du = -\frac{2GM}{cR}$$

DEFLECTION OF A LIGHT CORPUSCLE



$$\Delta v_\perp = -\frac{GM}{cR}\int_{-\pi/2}^{\pi/2} \cos(u)du = -\frac{2GM}{cR}$$

Before the lens:

$$\vec{v}_0 = c\,\vec{e}_z = c\,\vec{e}_{in}$$

After the lens:

$$\vec{v} = v\,\vec{e}_{out} = c\,\vec{e}_{in} - \frac{2GM}{cR}\vec{e}_R$$

$$|v| = \sqrt{c^2 + \frac{4G^2M^2}{c^2R^2}} \simeq c$$

$$\vec{\hat{\alpha}}(R,\phi) = \vec{e}_{in} - \vec{e}_{out} = \frac{2GM}{c^2R}\vec{e}_R$$

DEFLECTION OF A PHOTON GRAZING THE SURFACE OF THE SUN

$$G = 6.67 \times 10^{-11} N\, m^{-2}\, kg^{-2}$$

$$c = 299792 \text{ km } s^{-1}$$

$$R_\odot = 695700 \text{ km}$$

$$M_\odot = 1.989 \times 10^{30}\, kg$$

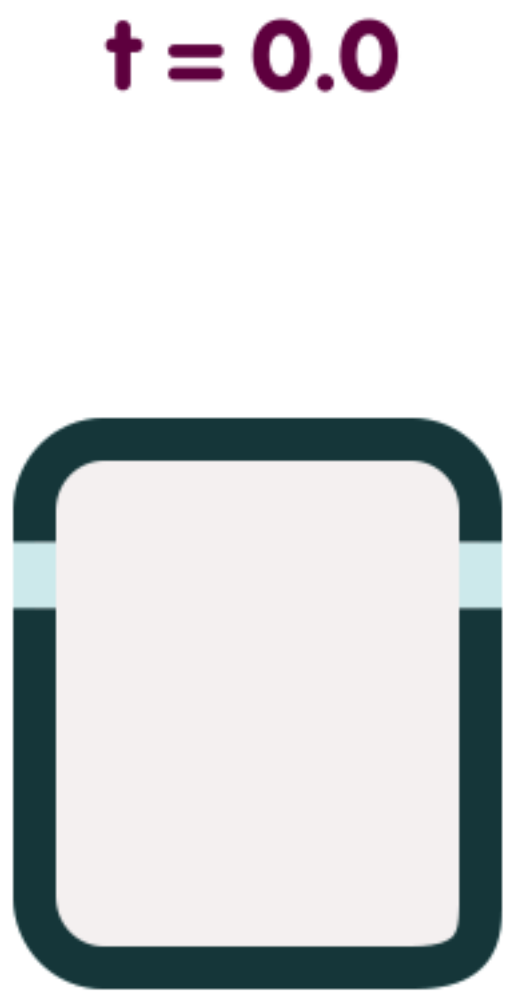
$$\boxed{\hat{\alpha}(R_\odot) = \frac{2GM_\odot}{c^2 R_\odot} = 0.875''}$$

TIMELINE

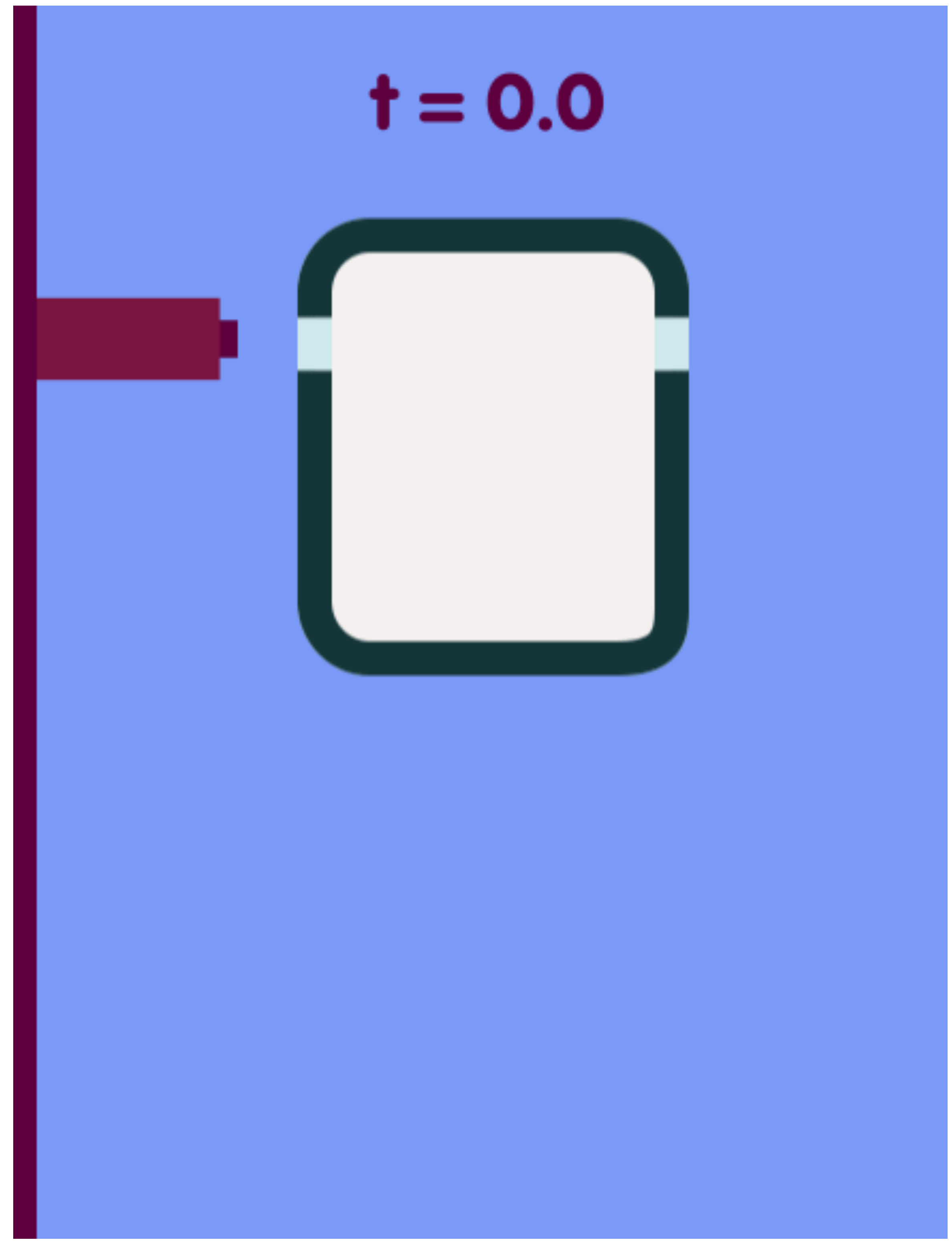
- 1801: Thomas Young demonstrates the wave nature of light using diffraction
- 1907-1911: Einstein resumes the idea of light deflection using special relativity and equivalence principle: *"In an arbitrary gravitational field, at any given spacetime point, we can choose a locally inertial reference frame such that, in a sufficiently small region surrounding that point, all physical laws take the same form they would take in absence of gravity, namely the form prescribed by Special Relativity"*

THE ELEVATOR THOUGHT EXPERIMENT

Inside a free falling elevator



Outside a free falling elevator

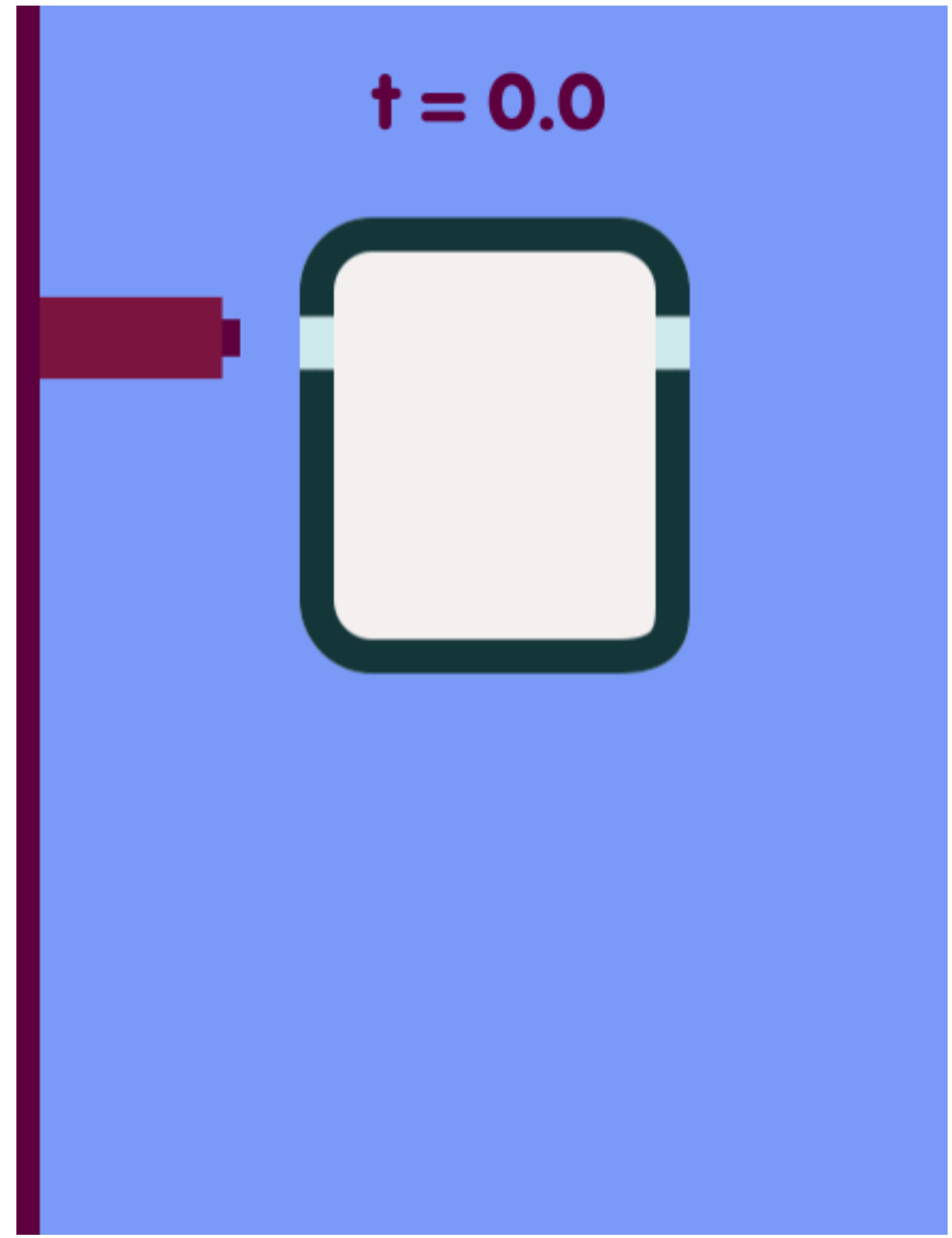


https://www.einstein-online.info/en/spotlight/equivalence_light

THE ELEVATOR THOUGHT EXPERIMENT

Inside a free falling elevator

Outside a free falling elevator



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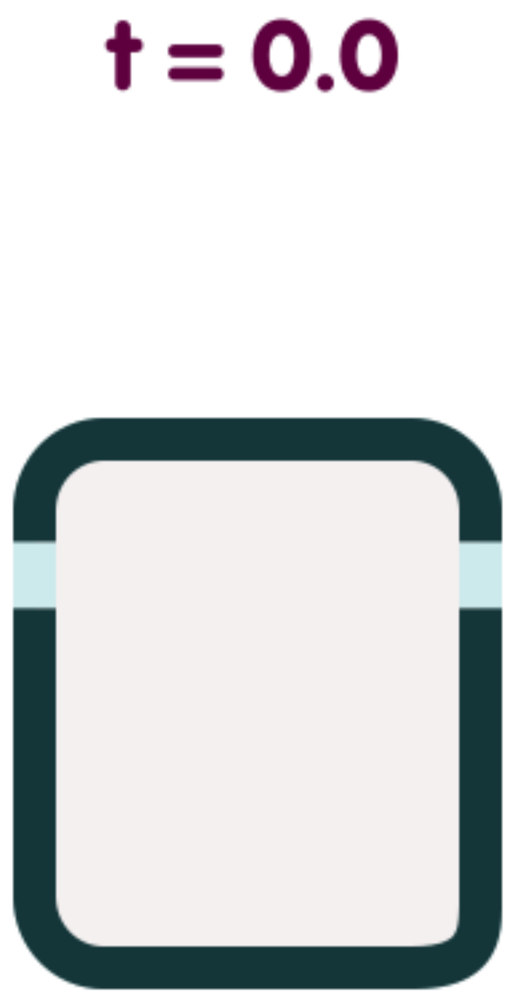
THE ELEVATOR THOUGHT EXPERIMENT

Inside a free falling elevator



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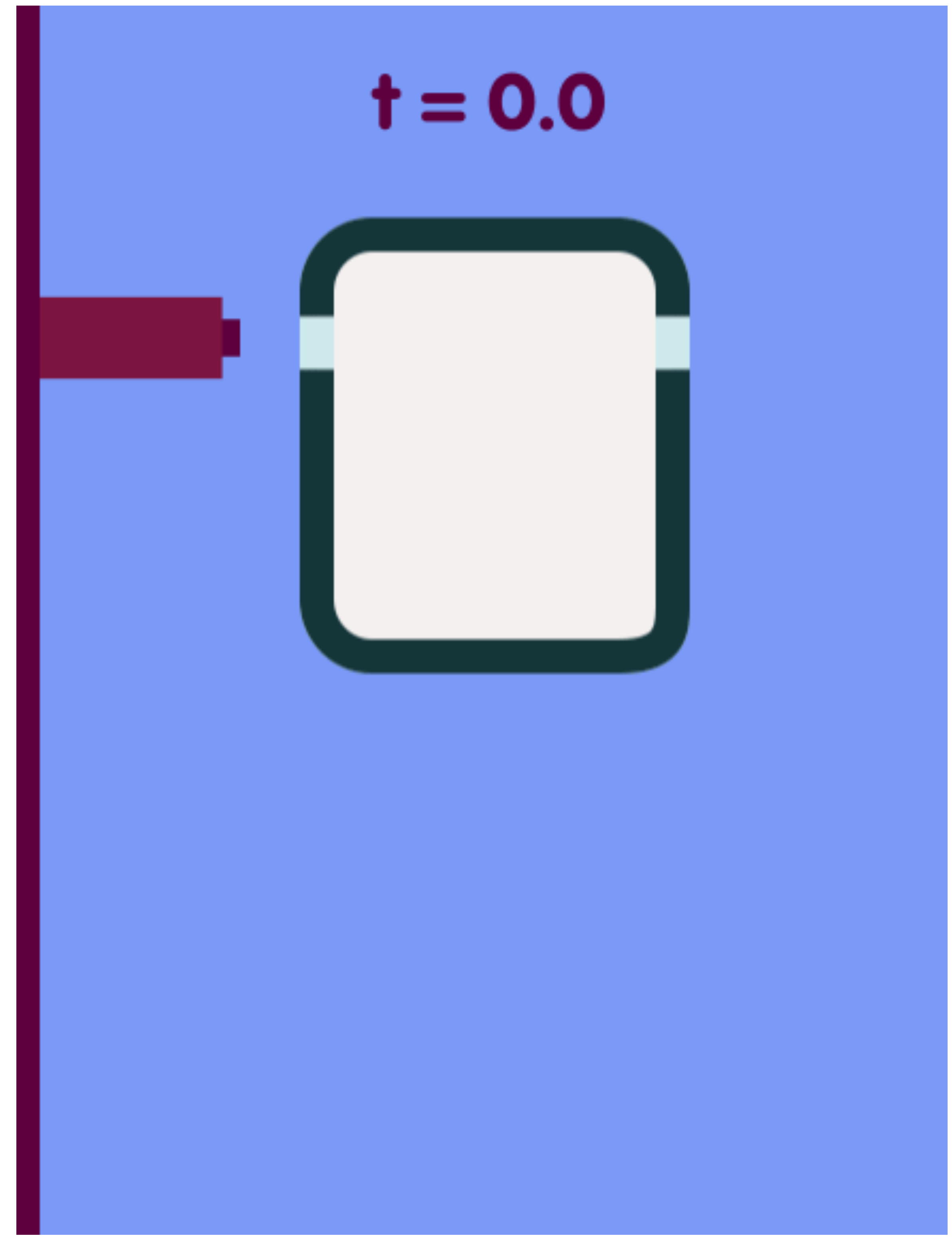

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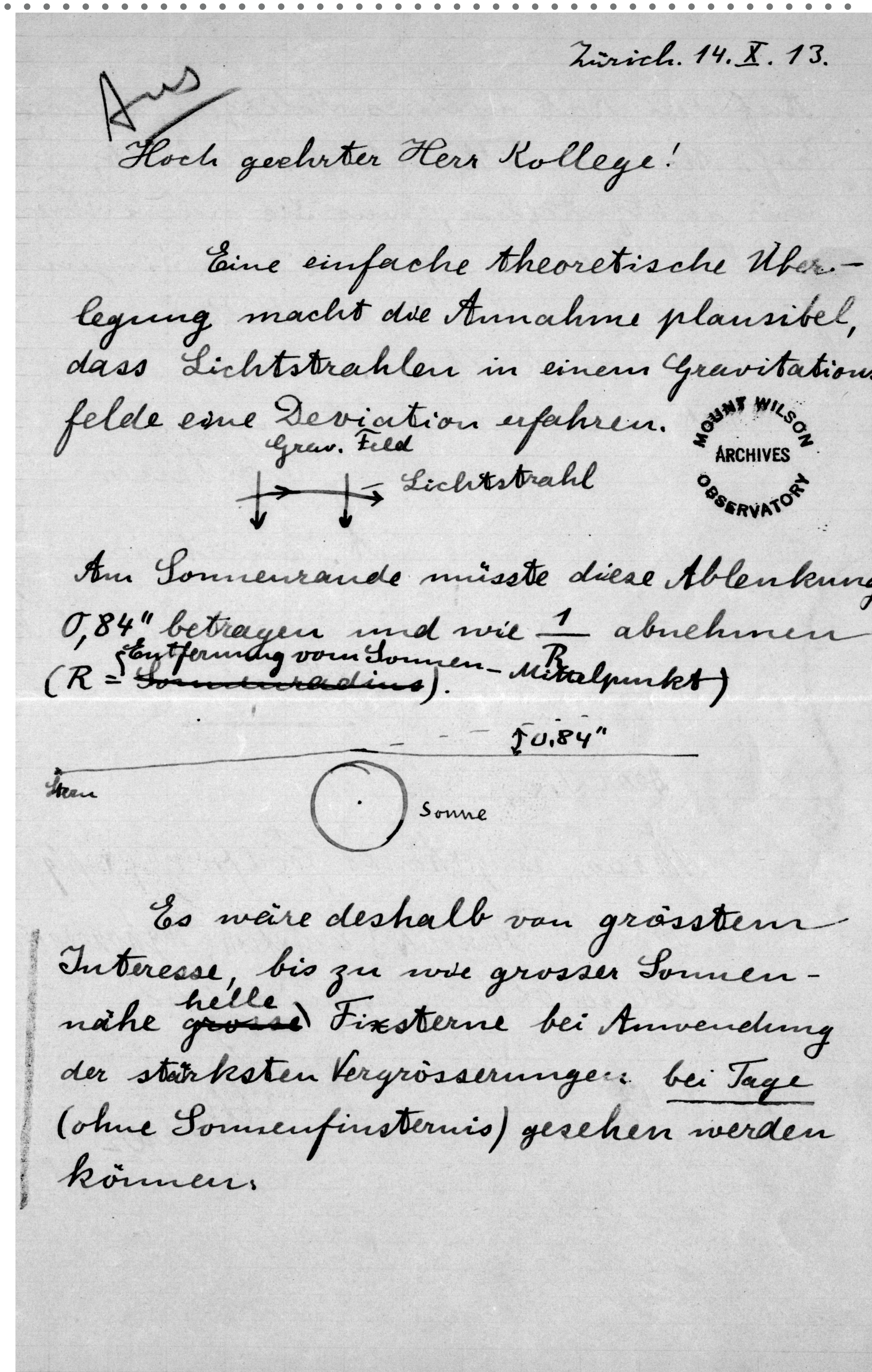
TIMELINE

- result of calculation of deflection is identical to that from Newtonian gravity
- 1913: A. Einstein writes to George Ellery Hale (Director of Mount Wilson Observatory), asking for advice on how to observe the position of stars in sun-light…



Zürich. 14. X. 13.

Aus

Hoch geehrter Herr Kollege!

Eine einfache theoretische Überlegung macht die Annahme plausibel, dass Lichtstrahlen in einem Gravitationsfelde eine Deviation erfahren.

Grav. Feld

Lichtstrahl

MOUNT WILSON ARCHIVES OBSERVATORY

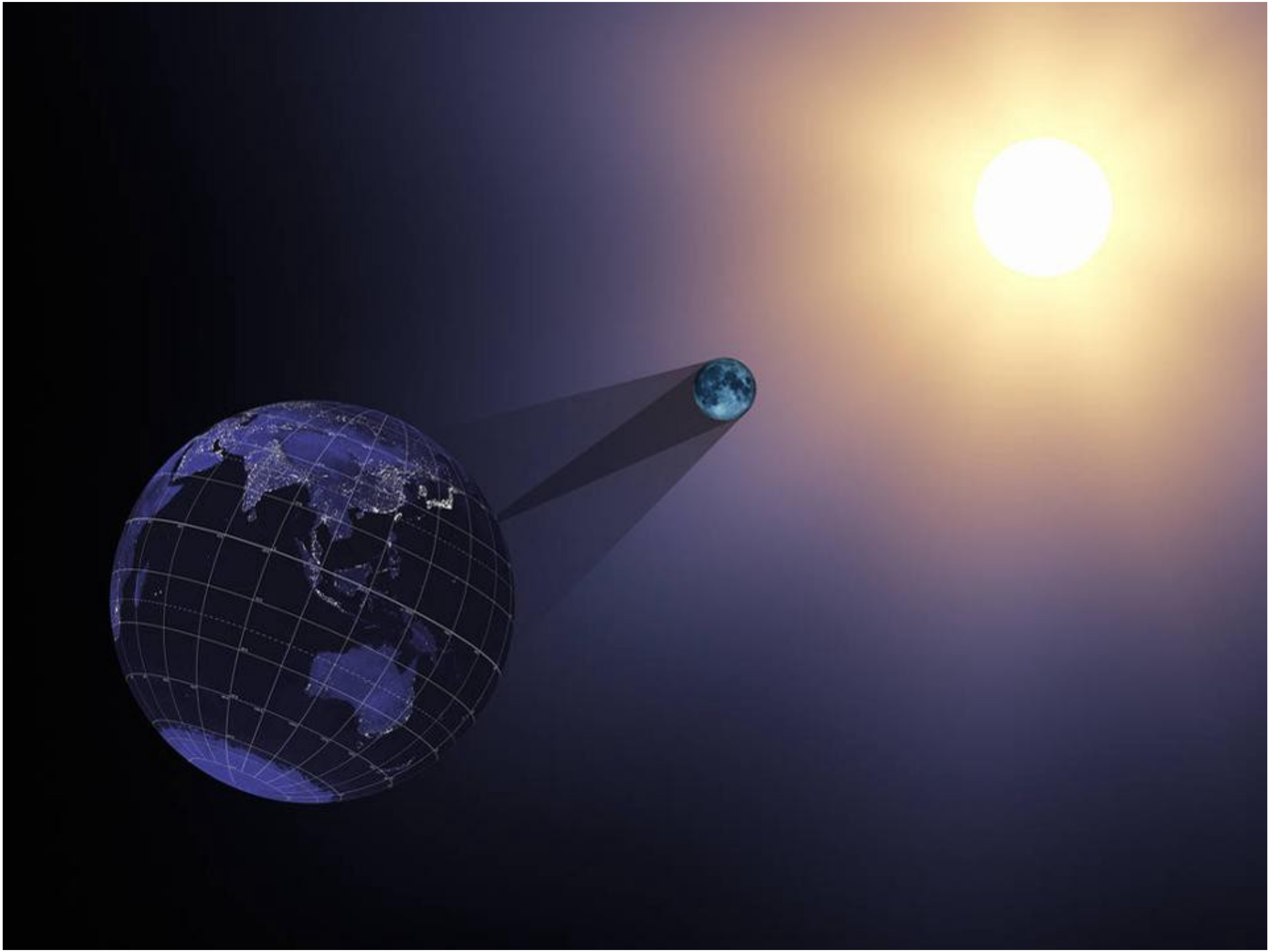
Am Sonnenrande müsste diese Ablenkung 0,84" betragen und wie $\frac{1}{R}$ abnehmen (R = ~~Sonnenradius~~ Entfernung vom Sonnen-Mittelpunkt).

0,84"

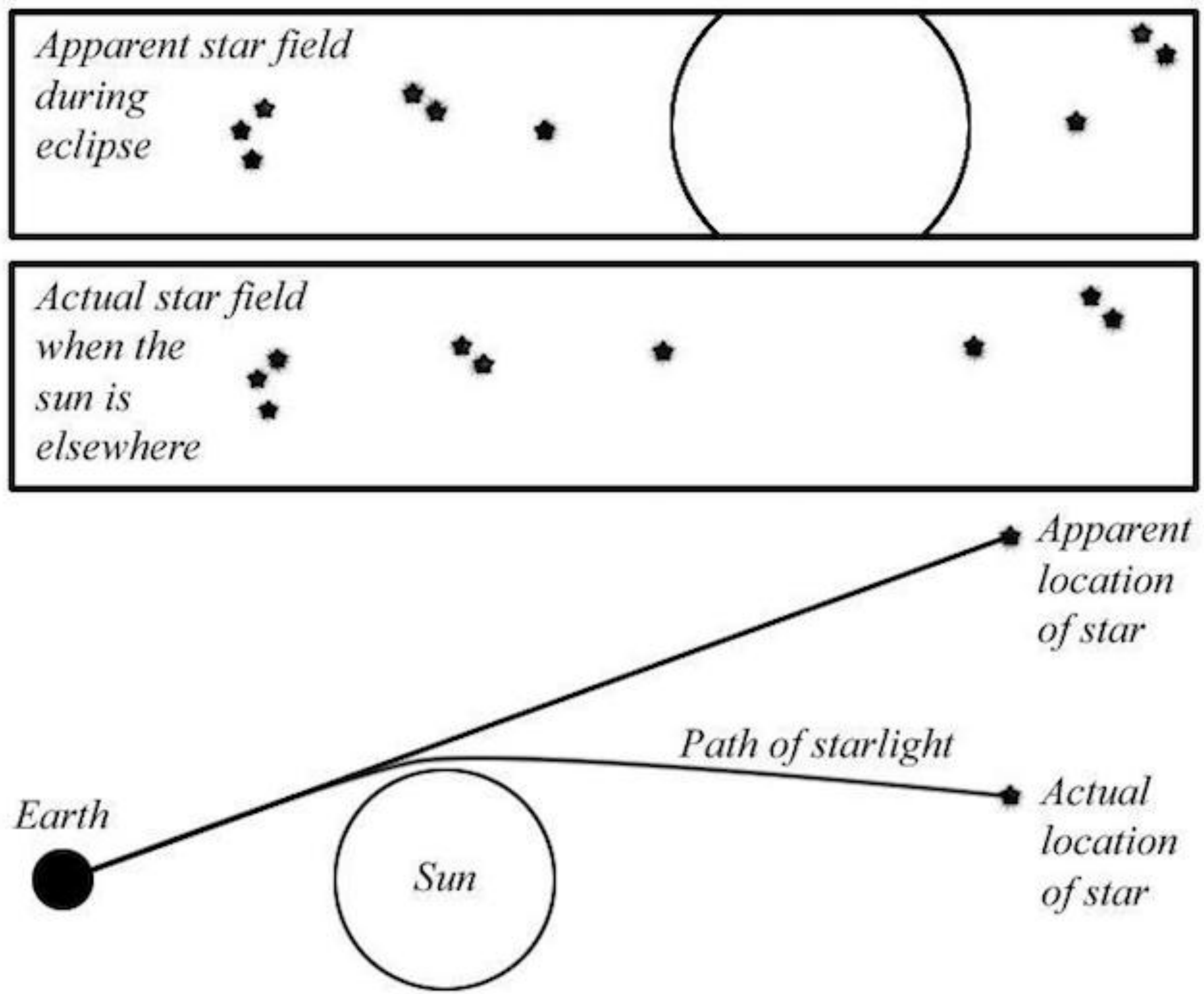
Stern

Sonne

Es wäre deshalb von grösstem Interesse, bis zu wie grosser Sonnennähe ~~grosse~~ helle Fixsterne bei Anwendung der stärksten Vergrösserungen bei Tage (ohne Sonnenfinsternis) gesehen werden können.



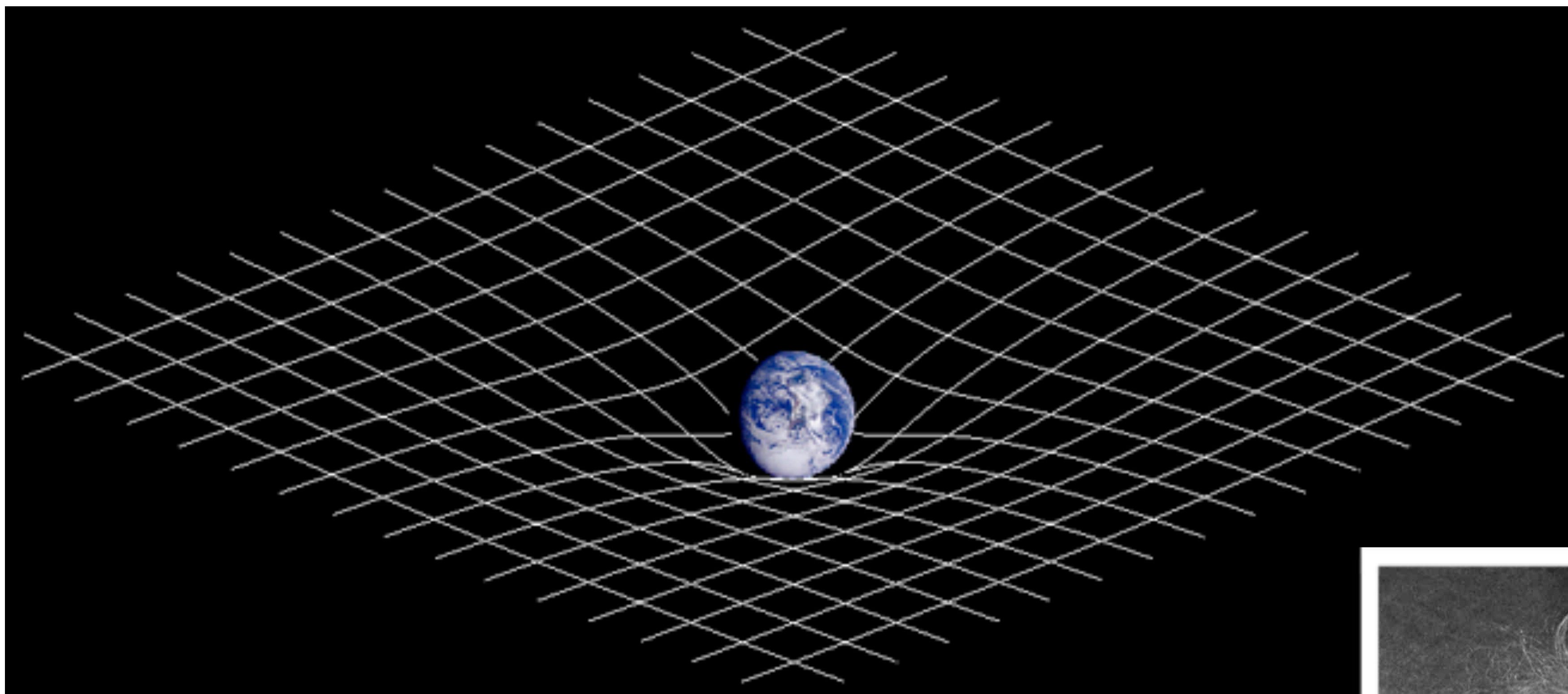
TESTING THE DEFLECTION OF LIGHT DURING A SOLAR ECLIPSE

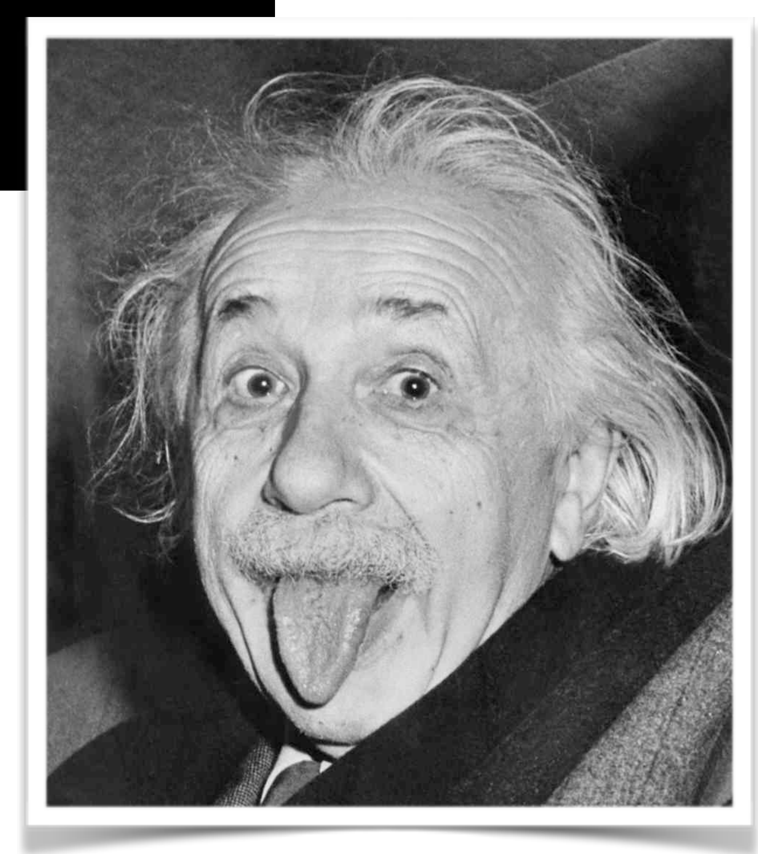


TIMELINE

- 1914 (August): Total solar eclipse in Crimea:
  - Attempts by Erwin Finlay-Freundlich and William Wallace Campbell
  - Unfortunately, WWI began and Russia entered into the war on Aug. 1st
  - Erwin Finlay-Freundlich (German citizen) arrested
  - William Wallace Campbell had his instrumentation confiscated
- 1915: Einstein publishes the Theory of General Relativity…

DEFLECTION OF LIGHT IN GENERAL RELATIVITY





DEFLECTION OF LIGHT IN GENERAL RELATIVITY

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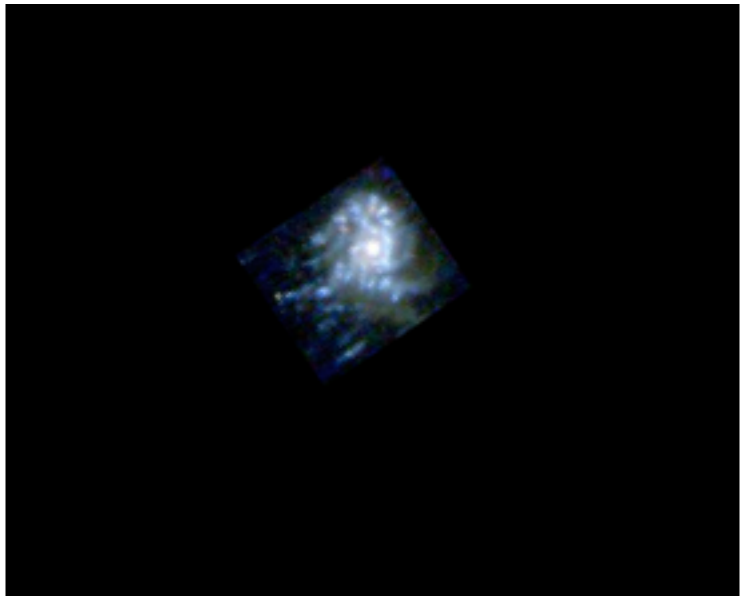
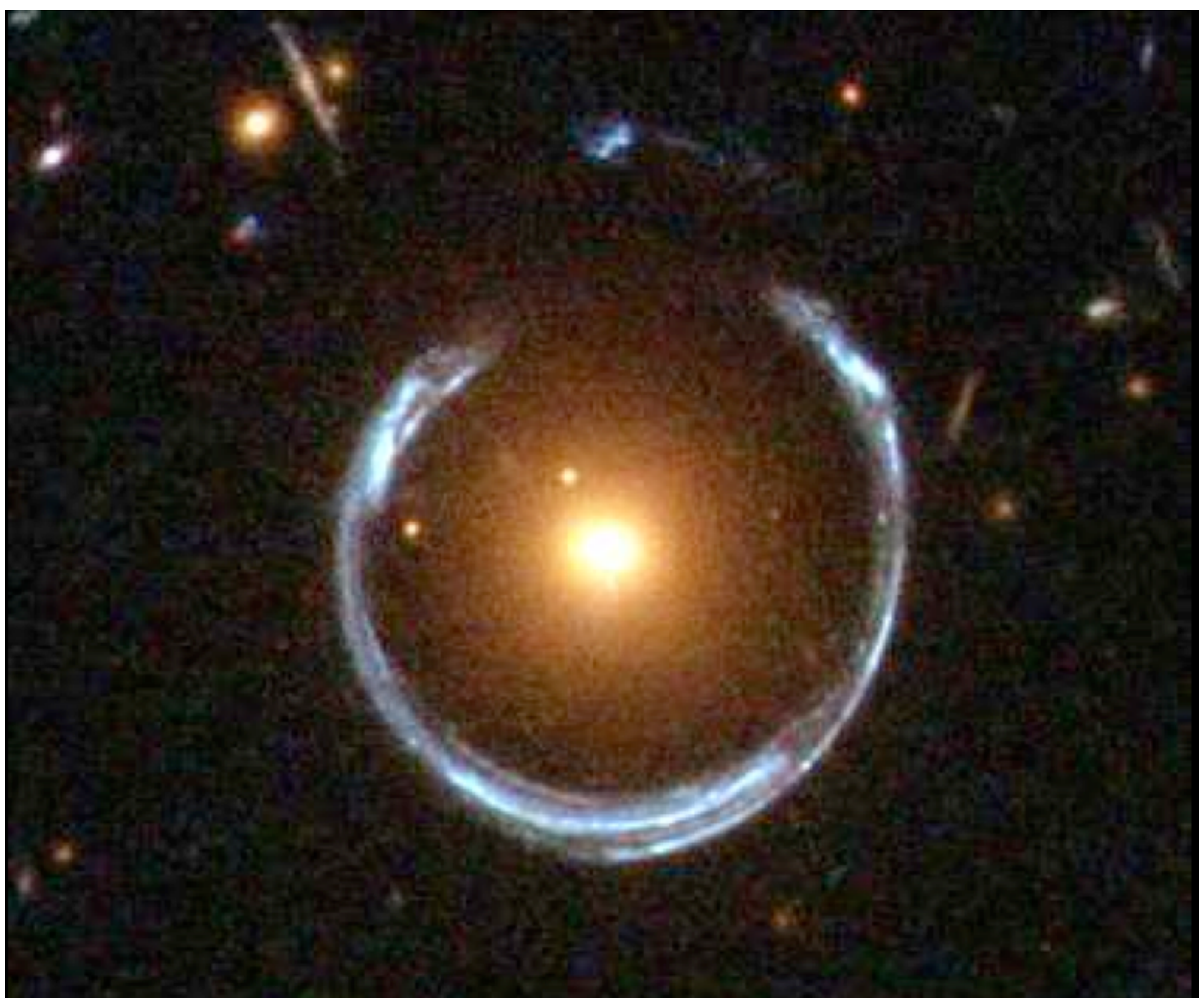
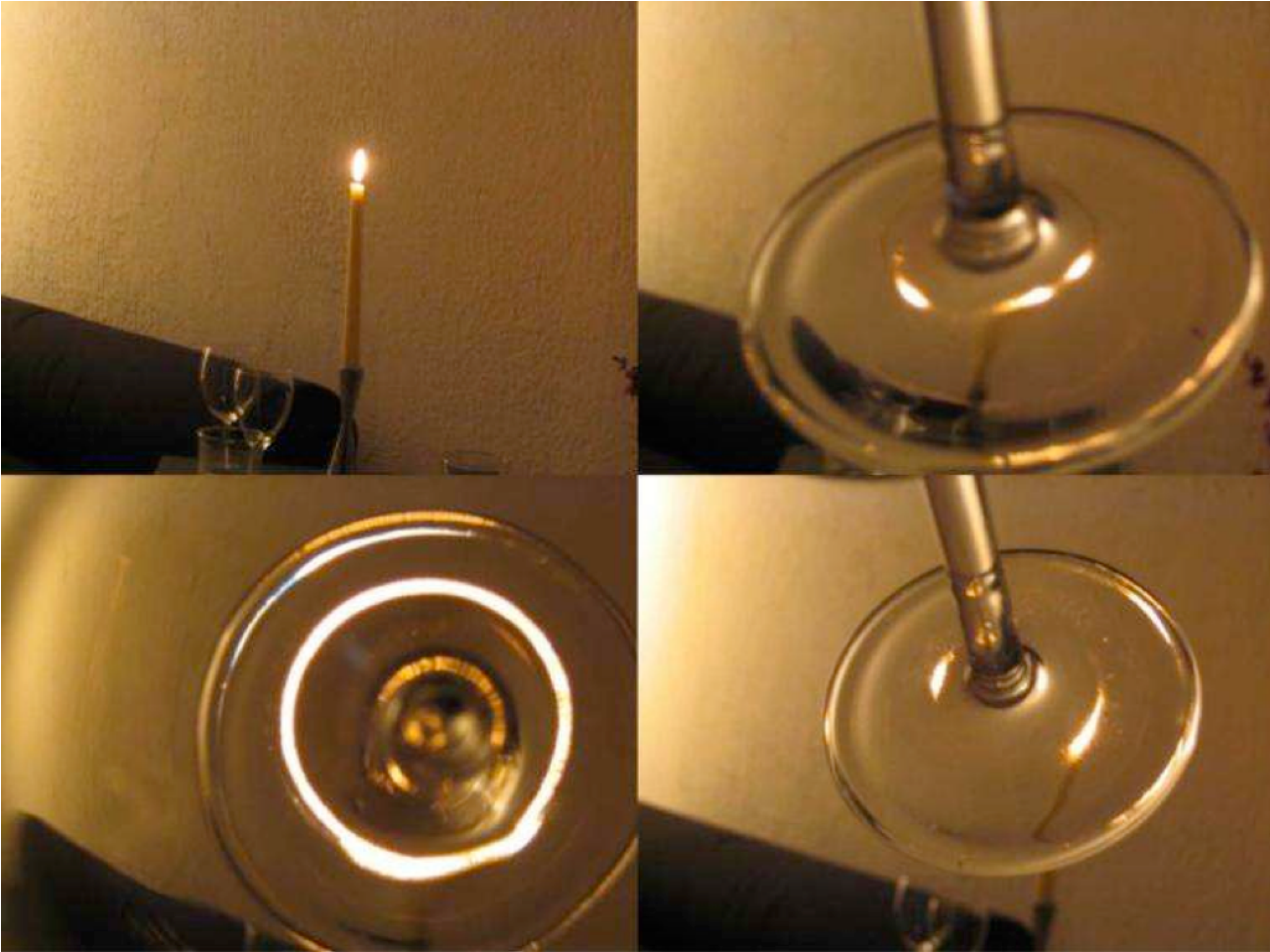


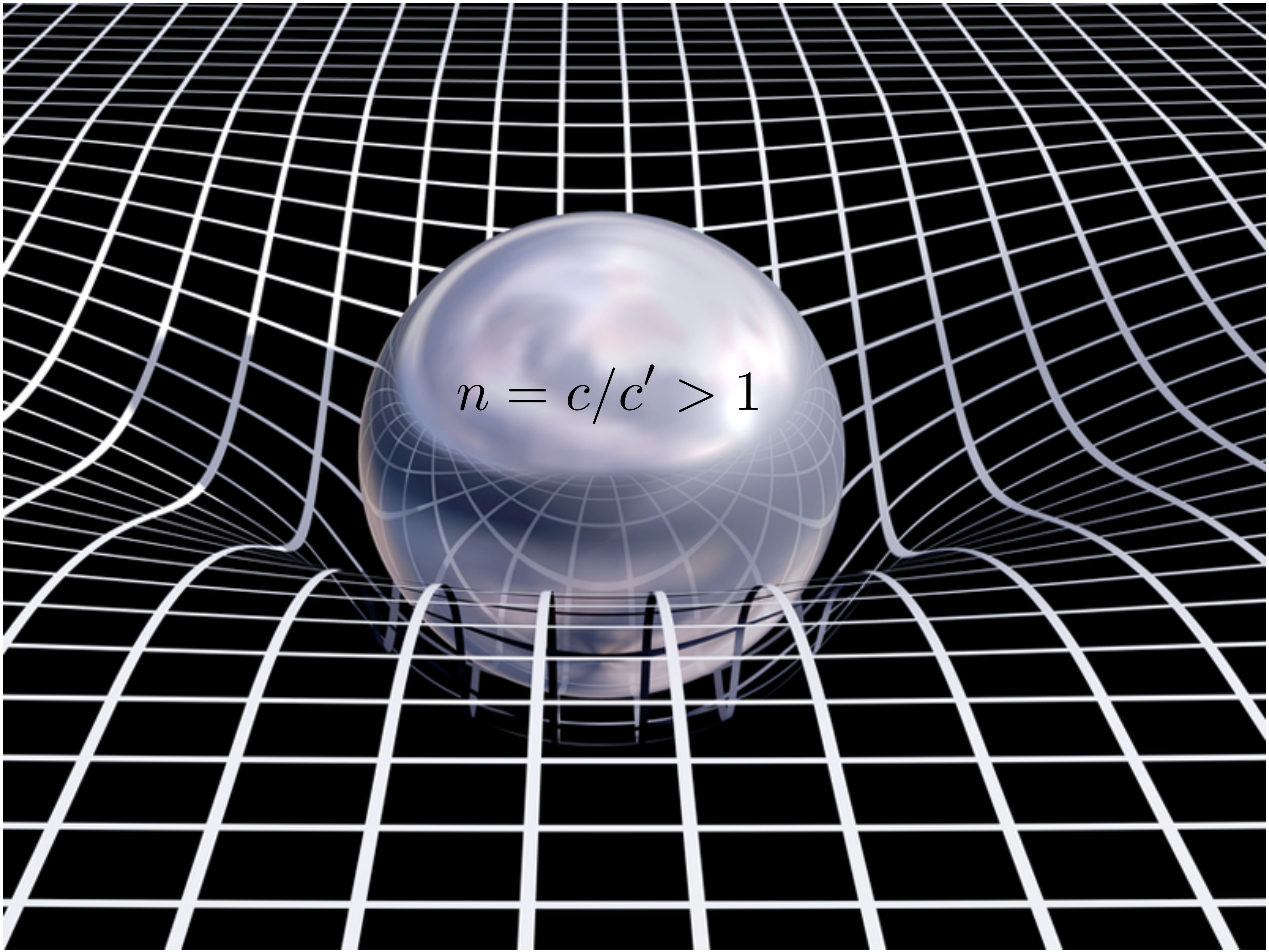


DEFLECTION OF LIGHT IN GENERAL RELATIVITY

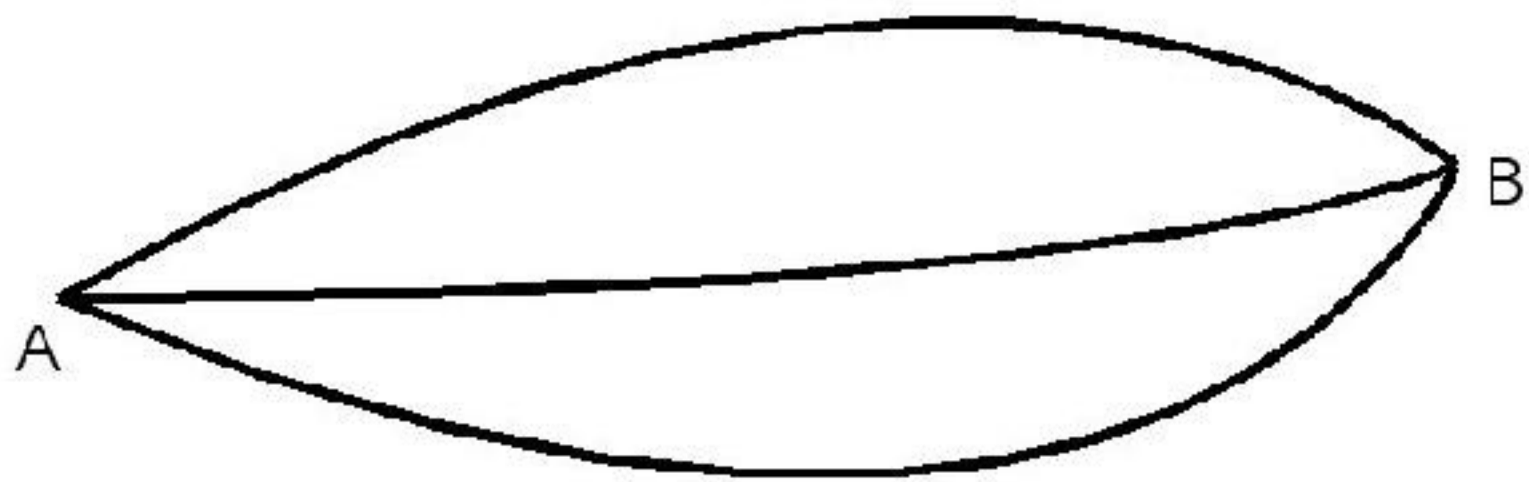
- We will now repeat the calculation of the deflection angle in the context of a locally curved space-time
- Assumptions:
  - the deflection occurs in small region of the universe and over time-scales where the expansion of the universe is not relevant
  - the weak-field limit can be safely applied: $|\Phi|/c^2 \ll 1$
  - perturbed region can be described in terms of an effective refractive index
  - Fermat principle

DEFLECTION OF LIGHT IN GENERAL RELATIVITY



DEFLECTION OF LIGHT IN GENERAL RELATIVITY

Travel time $= \int \frac{n}{c} \mathrm{d}l$



Fermat principle: $\delta \int_A^B n dl = 0$

DEFLECTION OF LIGHT IN GENERAL RELATIVITY

How to define the effective diffraction index?

absence of lens or in a locally inertial frame = locally flat space-time described by the Minkowski metric

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$\mathrm{d}s^2 = \eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = (\mathrm{d}x^0)^2 - (\mathrm{d}\vec{x})^2 = c^2\mathrm{d}t^2 - (\mathrm{d}\vec{x})^2$$

In a different frame (where the photon is accelerated because of a gravitational field), assuming that the field is weak

$$\eta_{\mu\nu} \to g_{\mu\nu} = \begin{pmatrix} 1+\frac{2\Phi}{c^2} & 0 & 0 & 0 \\ 0 & -(1-\frac{2\Phi}{c^2}) & 0 & 0 \\ 0 & 0 & -(1-\frac{2\Phi}{c^2}) & \\ 0 & 0 & 0 & -(1-\frac{2\Phi}{c^2}) \end{pmatrix}$$

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = \left(1+\frac{2\Phi}{c^2}\right)c^2\mathrm{d}t^2 - \left(1-\frac{2\Phi}{c^2}\right)(\mathrm{d}\vec{x})^2$$

DEFLECTION OF LIGHT IN GENERAL RELATIVITY

How to define the effective refractive index?

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = \left(1 + \frac{2\Phi}{c^2}\right) c^2 \mathrm{d}t^2 - \left(1 - \frac{2\Phi}{c^2}\right) (\mathrm{d}\vec{x})^2$$

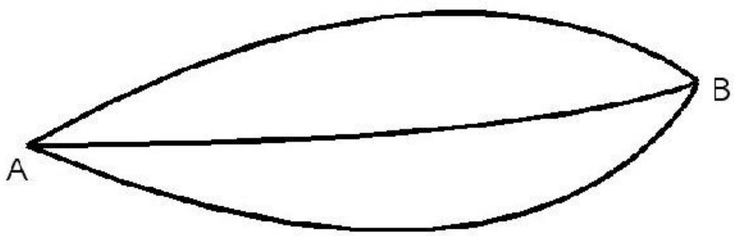
$$\left(1 + \frac{2\Phi}{c^2}\right) c^2 \mathrm{d}t^2 = \left(1 - \frac{2\Phi}{c^2}\right) (\mathrm{d}\vec{x})^2$$

$$c' = \frac{|\mathrm{d}\vec{x}|}{\mathrm{d}t} = c\sqrt{\frac{1 + \frac{2\Phi}{c^2}}{1 - \frac{2\Phi}{c^2}}} \approx c\left(1 + \frac{2\Phi}{c^2}\right)$$

$$n = \frac{c}{c'} \approx \frac{1}{1 + 2\Phi/c^2} \approx 1 - \frac{2\Phi}{c^2}$$

DEFLECTION OF LIGHT IN GENERAL RELATIVITY

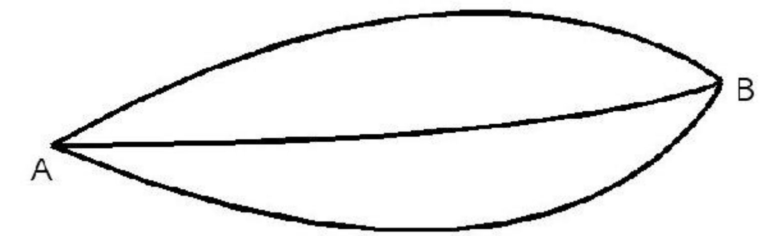
Let's use the Fermat principle



$$\delta \int_A^B n[\vec{x}(l)]\mathrm{d}l = 0$$

DEFLECTION OF LIGHT IN GENERAL RELATIVITY

Let's use the Fermat principle

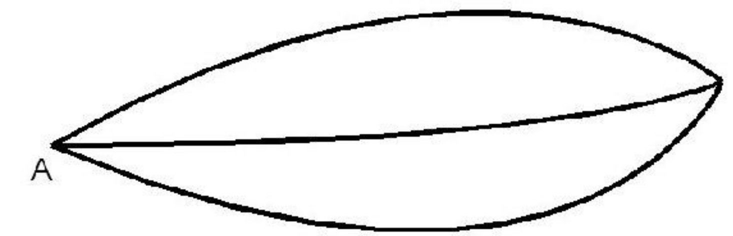


$$\delta \int_A^B n[\vec{x}(l)]\mathrm{d}l = 0$$

$$\mathrm{d}l = \left|\frac{\mathrm{d}\vec{x}}{\mathrm{d}\lambda}\right| \mathrm{d}\lambda$$

DEFLECTION OF LIGHT IN GENERAL RELATIVITY

Let's use the Fermat principle



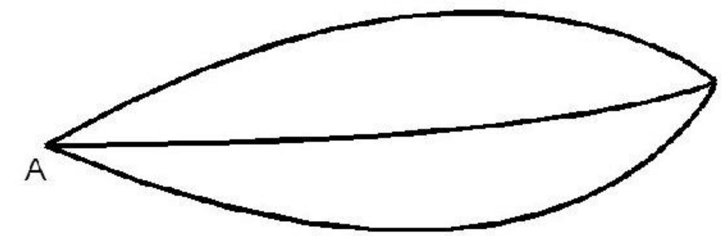
$$\delta \int_A^B n[\vec{x}(l)]\mathrm{d}l = 0$$

$$\mathrm{d}l = \left|\frac{\mathrm{d}\vec{x}}{\mathrm{d}\lambda}\right| \mathrm{d}\lambda$$

$$\delta \int_{\lambda_A}^{\lambda_B} \mathrm{d}\lambda\, n[\vec{x}(\lambda)] \left|\frac{\mathrm{d}\vec{x}}{\mathrm{d}\lambda}\right| = 0$$

DEFLECTION OF LIGHT IN GENERAL RELATIVITY

Let's use the Fermat principle



$$\delta \int_A^B n[\vec{x}(l)]\mathrm{d}l = 0$$

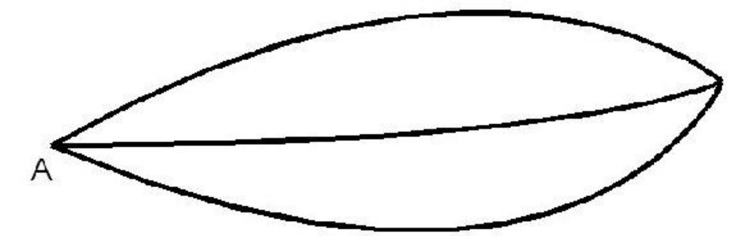
$$\mathrm{d}l = \left|\frac{\mathrm{d}\vec{x}}{\mathrm{d}\lambda}\right| \mathrm{d}\lambda$$

Time

$$\delta \int_{\lambda_A}^{\lambda_B} \mathrm{d}\lambda \, n[\vec{x}(\lambda)] \left|\frac{\mathrm{d}\vec{x}}{\mathrm{d}\lambda}\right| = 0$$

DEFLECTION OF LIGHT IN GENERAL RELATIVITY

Let's use the Fermat principle



$$\delta \int_A^B n[\vec{x}(l)]\mathrm{d}l = 0$$

$$\mathrm{d}l = \left|\frac{\mathrm{d}\vec{x}}{\mathrm{d}\lambda}\right| \mathrm{d}\lambda$$

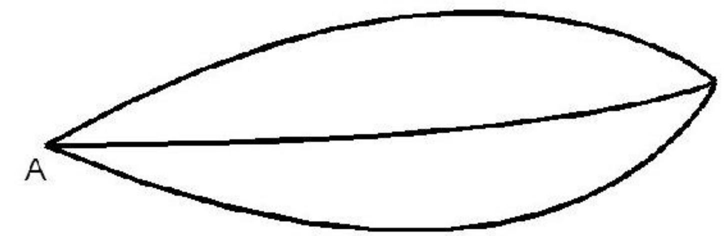
Time

$$\delta \int_{\lambda_A}^{\lambda_B} \mathrm{d}\lambda\, n[\vec{x}(\lambda)] \left|\frac{\mathrm{d}\vec{x}}{\mathrm{d}\lambda}\right| = 0$$

generalized coordinate

DEFLECTION OF LIGHT IN GENERAL RELATIVITY

Let's use the Fermat principle



$$\delta \int_A^B n[\vec{x}(l)]\mathrm{d}l = 0$$

$$\mathrm{d}l = \left|\frac{\mathrm{d}\vec{x}}{\mathrm{d}\lambda}\right| \mathrm{d}\lambda$$

Time

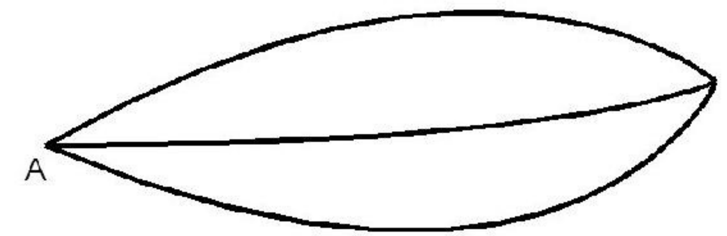
$$\delta \int_{\lambda_A}^{\lambda_B} \mathrm{d}\lambda \, n[\vec{x}(\lambda)] \left|\frac{\mathrm{d}\vec{x}}{\mathrm{d}\lambda}\right| = 0$$

generalized velocity $\dot{\vec{x}} \equiv \dfrac{\mathrm{d}\vec{x}}{\mathrm{d}\lambda}$

generalized coordinate

DEFLECTION OF LIGHT IN GENERAL RELATIVITY

Let's use the Fermat principle



$$\delta \int_A^B n[\vec{x}(l)]\mathrm{d}l = 0$$

$$\mathrm{d}l = \left|\frac{\mathrm{d}\vec{x}}{\mathrm{d}\lambda}\right| \mathrm{d}\lambda$$

Time

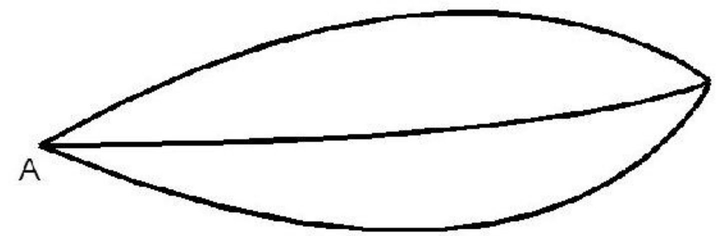
$$\delta \int_{\lambda_A}^{\lambda_B} \mathrm{d}\lambda\, n[\vec{x}(\lambda)] \left|\frac{\mathrm{d}\vec{x}}{\mathrm{d}\lambda}\right| = 0$$

generalized velocity $\dot{\vec{x}} \equiv \dfrac{\mathrm{d}\vec{x}}{\mathrm{d}\lambda}$

generalized coordinate

DEFLECTION OF LIGHT IN GENERAL RELATIVITY

Let's use the Fermat principle



$$\delta \int_A^B n[\vec{x}(l)]\mathrm{d}l = 0$$

$$\mathrm{d}l = \left|\frac{\mathrm{d}\vec{x}}{\mathrm{d}\lambda}\right| \mathrm{d}\lambda$$

Time

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generalized velocity $\dot{\vec{x}} \equiv \dfrac{\mathrm{d}\vec{x}}{\mathrm{d}\lambda}$

generalized coordinate

$$\boxed{n[\vec{x}(\lambda)] \left|\frac{\mathrm{d}\vec{x}}{\mathrm{d}\lambda}\right| \equiv L(\dot{\vec{x}}, \vec{x}, \lambda)}$$

Langrangian!

DEFLECTION OF LIGHT IN GENERAL RELATIVITY

Let's use the Fermat principle

$$\delta \int_{\lambda_A}^{\lambda_B} \mathrm{d}\lambda \, n[\vec{x}(\lambda)] \left| \frac{\mathrm{d}\vec{x}}{\mathrm{d}\lambda} \right| = 0$$

Euler-Langrange equation: $$\frac{\mathrm{d}}{\mathrm{d}\lambda} \frac{\partial L}{\partial \dot{\vec{x}}} - \frac{\partial L}{\partial \vec{x}} = 0$$

DEFLECTION OF LIGHT IN GENERAL RELATIVITY

Let's use the Fermat principle

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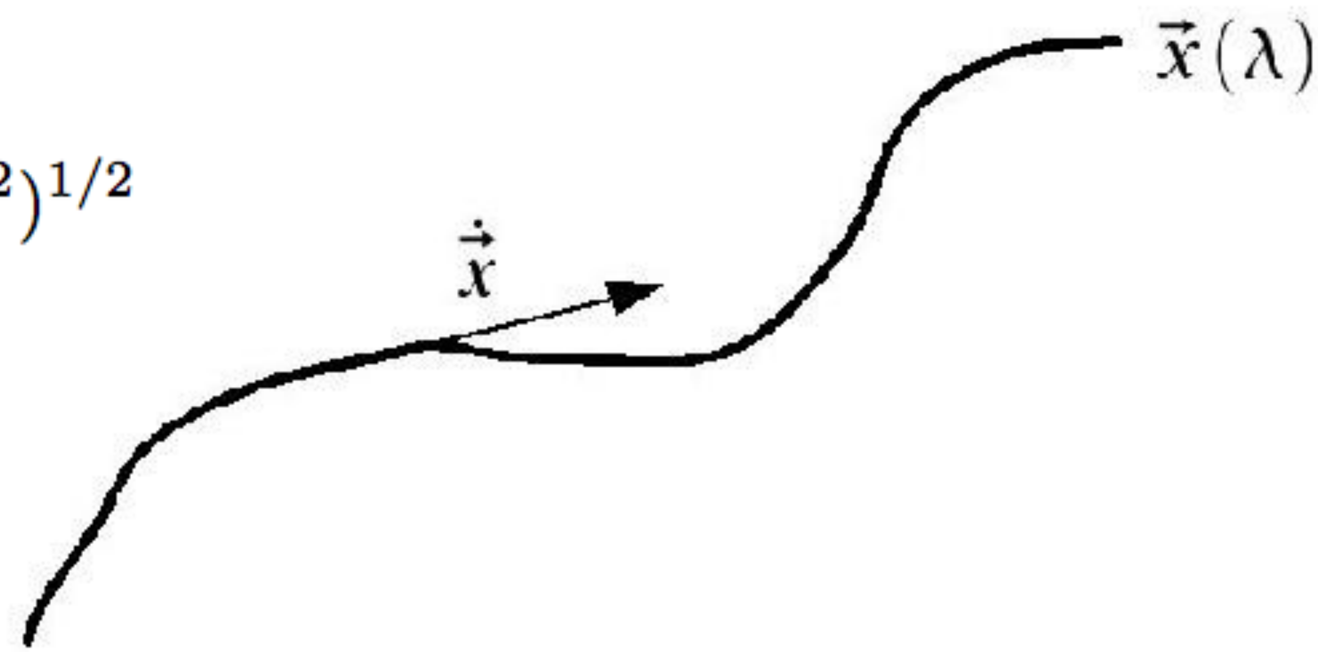

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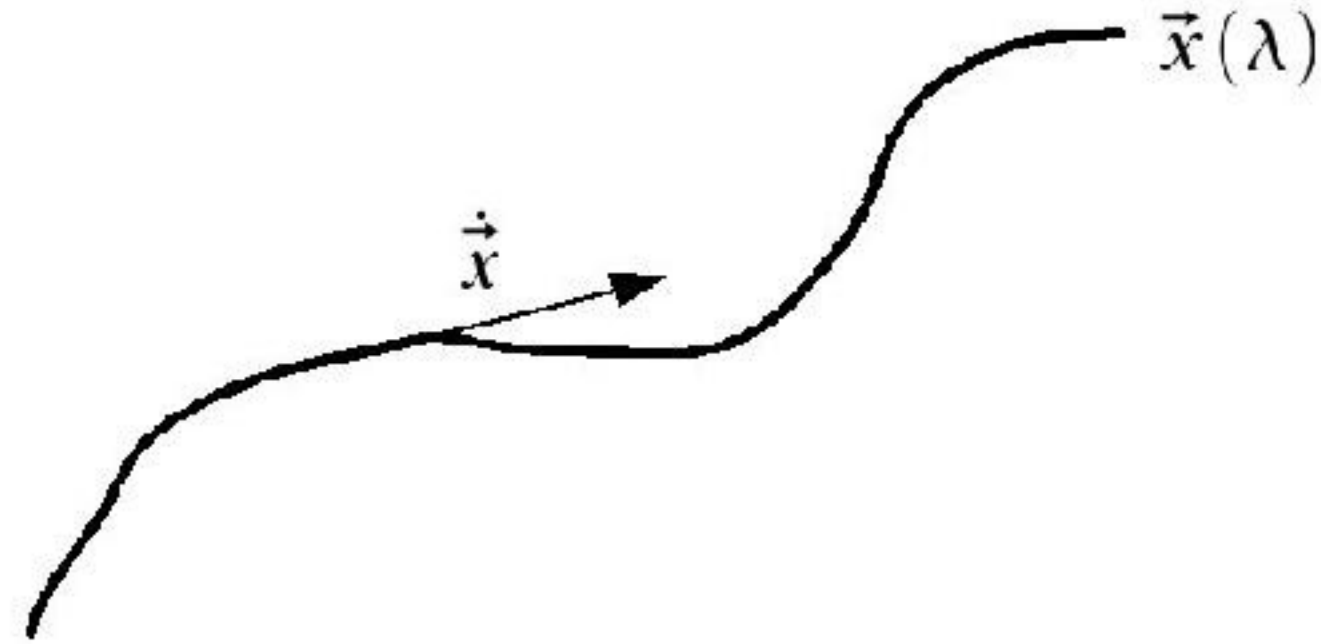

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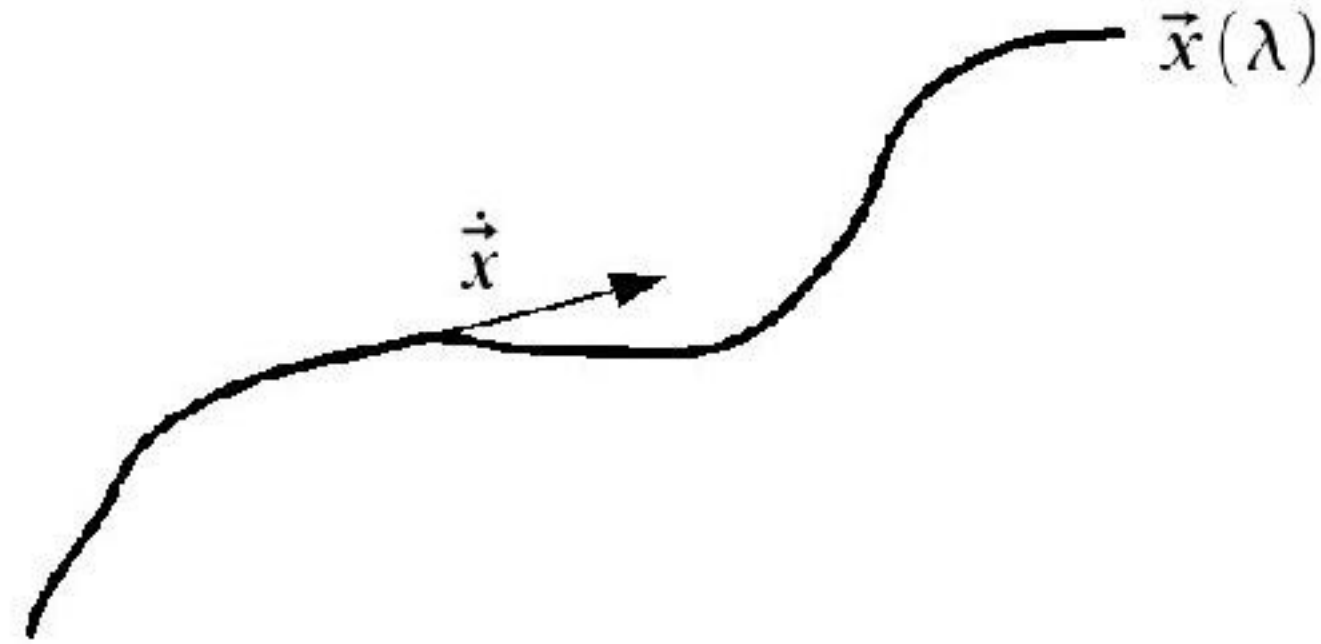

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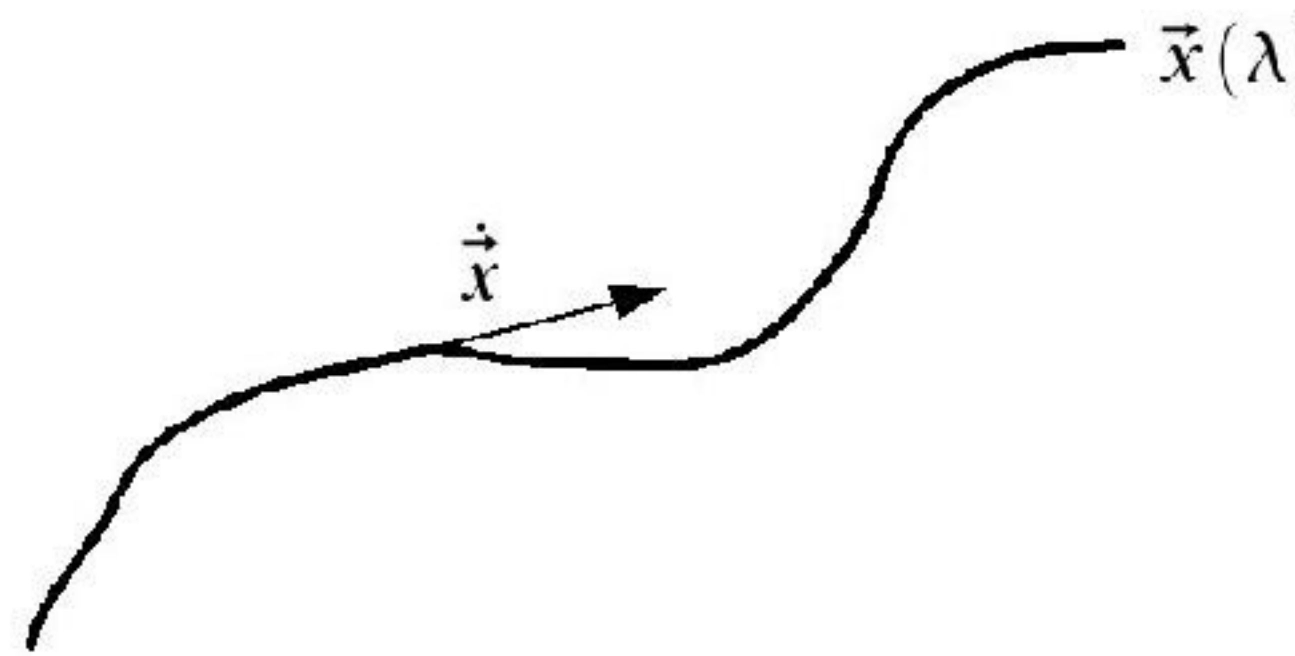

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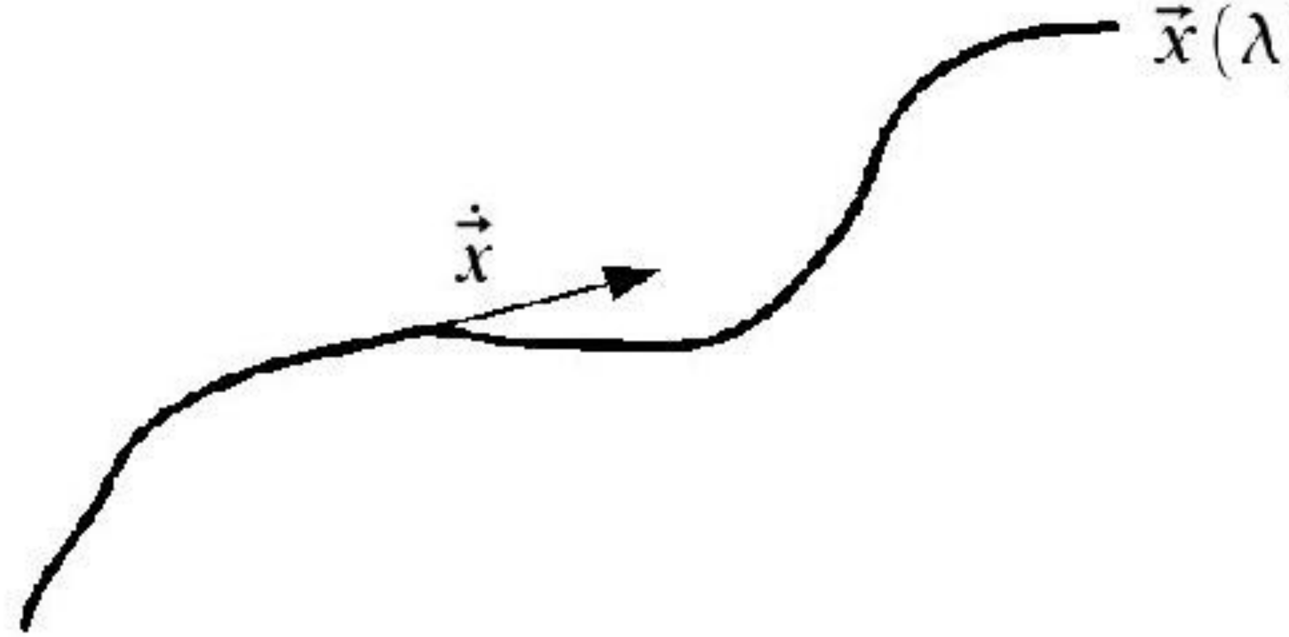

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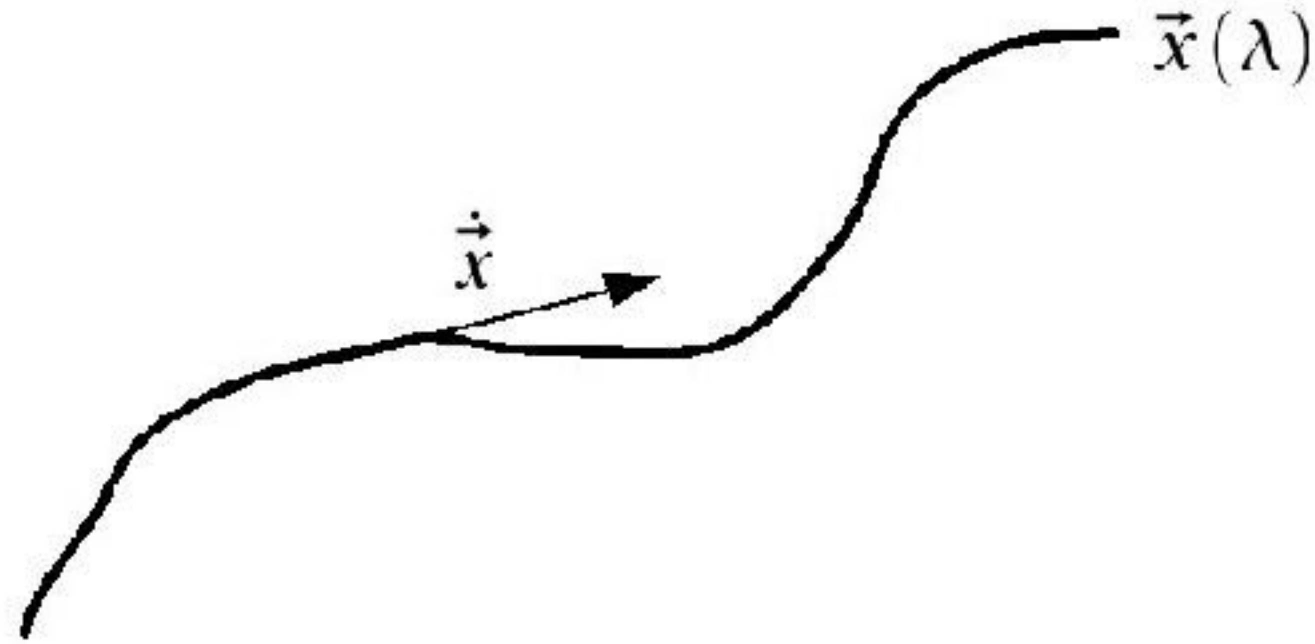

$$n = c/c' = 1 - \frac{2\phi}{c^2} \qquad \frac{\phi}{c^2} \ll 1$$

$$\ln n \approx -2\frac{\phi}{c^2}$$

$$\dot{\vec{e}} \approx -\frac{2}{c^2}\vec{\nabla}_\perp \Phi$$

$$\vec{e}_{out} = \vec{e}_{in} + \int_{\lambda_A}^{\lambda_B} \dot{e}\, d\lambda$$

Deflection angle

$$\hat{\vec{\alpha}} = \vec{e}_{in} - \vec{e}_{out} = \frac{2}{c^2}\int_{\lambda_A}^{\lambda_B} \vec{\nabla}_\perp \Phi\, d\lambda$$

HOW BIG ARE THESE DEFLECTIONS?

The potential has the dimension of a squared velocity, that is the characteristic velocity of a particle orbiting that potential. Therefore

- *galaxy: ~200 km/s*
- *galaxy cluster: ~1000 km/s*

This means that deflections from astrophysical masses are very small

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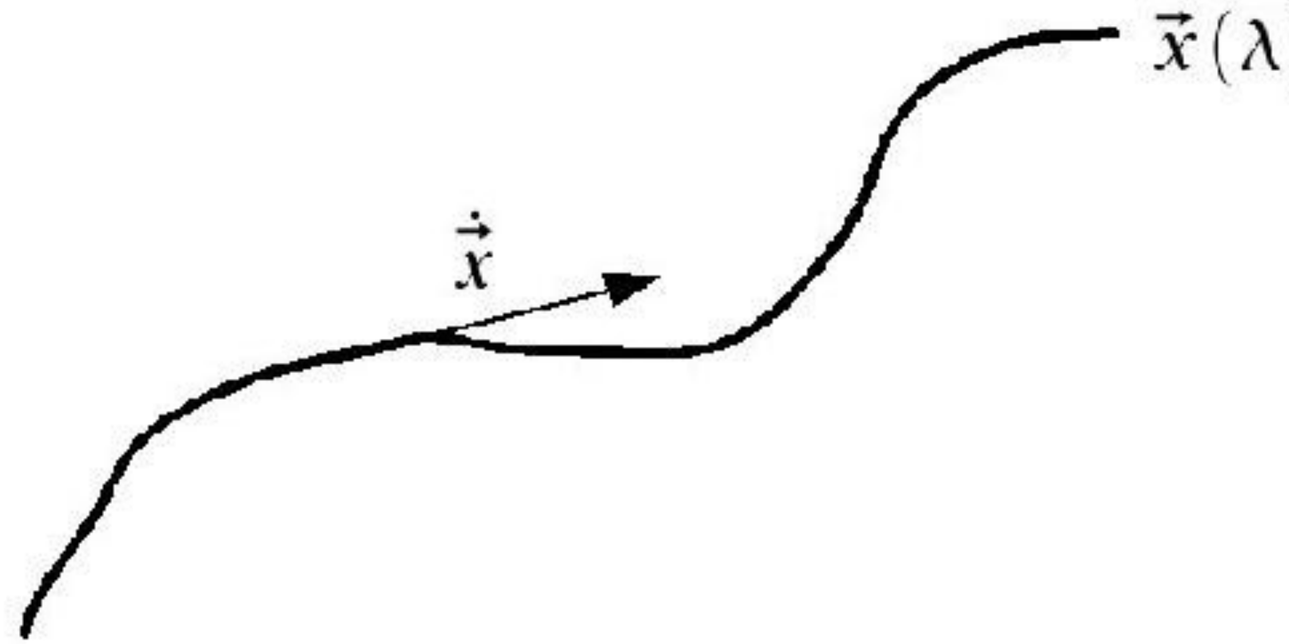

Good news!

The integral above should be carried out over the actual light path, but it can be approximated by integrating over the straight, undeflected light path (like in Born's approximation of scattering theory).

$$\hat{\vec{\alpha}} = \vec{e}_{in} - \vec{e}_{out} = \frac{2}{c^2}\int_{-\infty}^{+\infty} \vec{\nabla}_{\perp}\Phi dz$$

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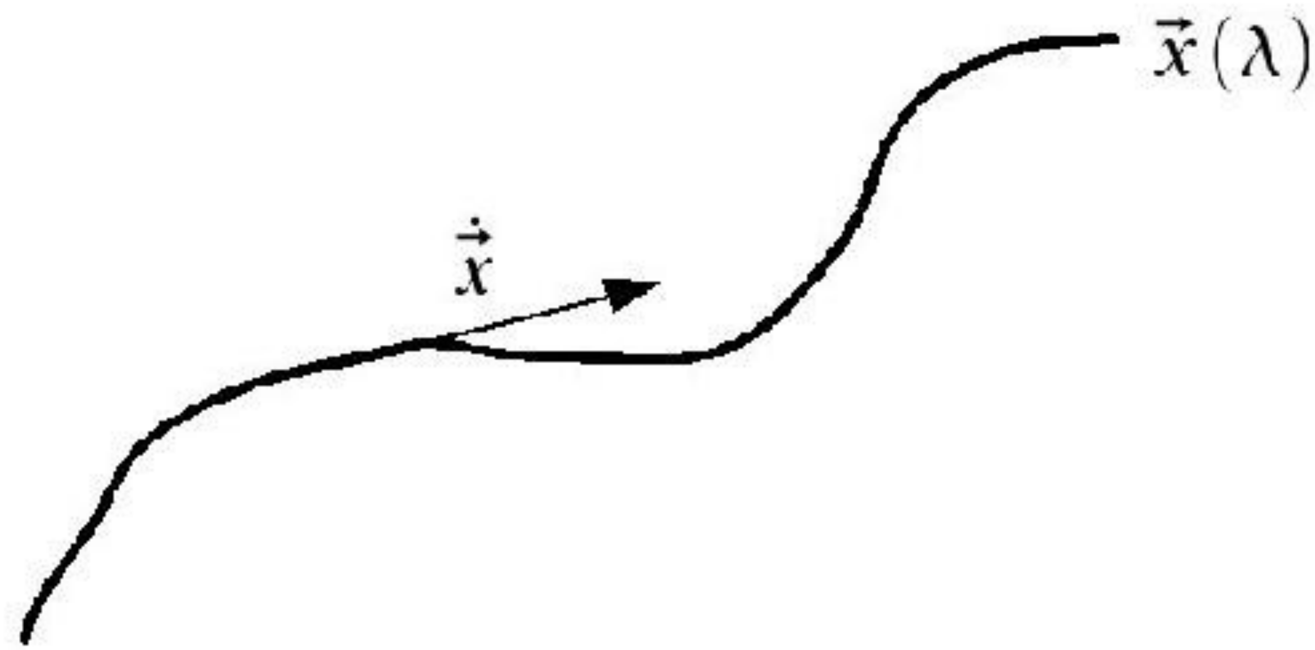

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A LIGHT RAY GRAZING THE SURFACE OF THE SUN

General relativity:

$$\hat{\alpha} = \frac{4GM_\odot}{c^2 R_\odot} = 1.75"$$

Newtonian gravity:

$$\hat{\alpha} = \frac{2GM_\odot}{c^2 R_\odot} = 0.875"$$

The reason for the factor of 2 difference is that ***both the space and time coordinates are bent*** *in the vicinity of massive objects — it is the four-dimensional space–time which is bent by the Sun.*