# GRAVITATIONAL LENSING

## 25 – WEAK LENSING BY GALAXY CLUSTERS

*Massimo Meneghetti*

*AA 2018-2019*

# WEAK LENSES OR LARGE DISTANCES FROM STRONG LENSES

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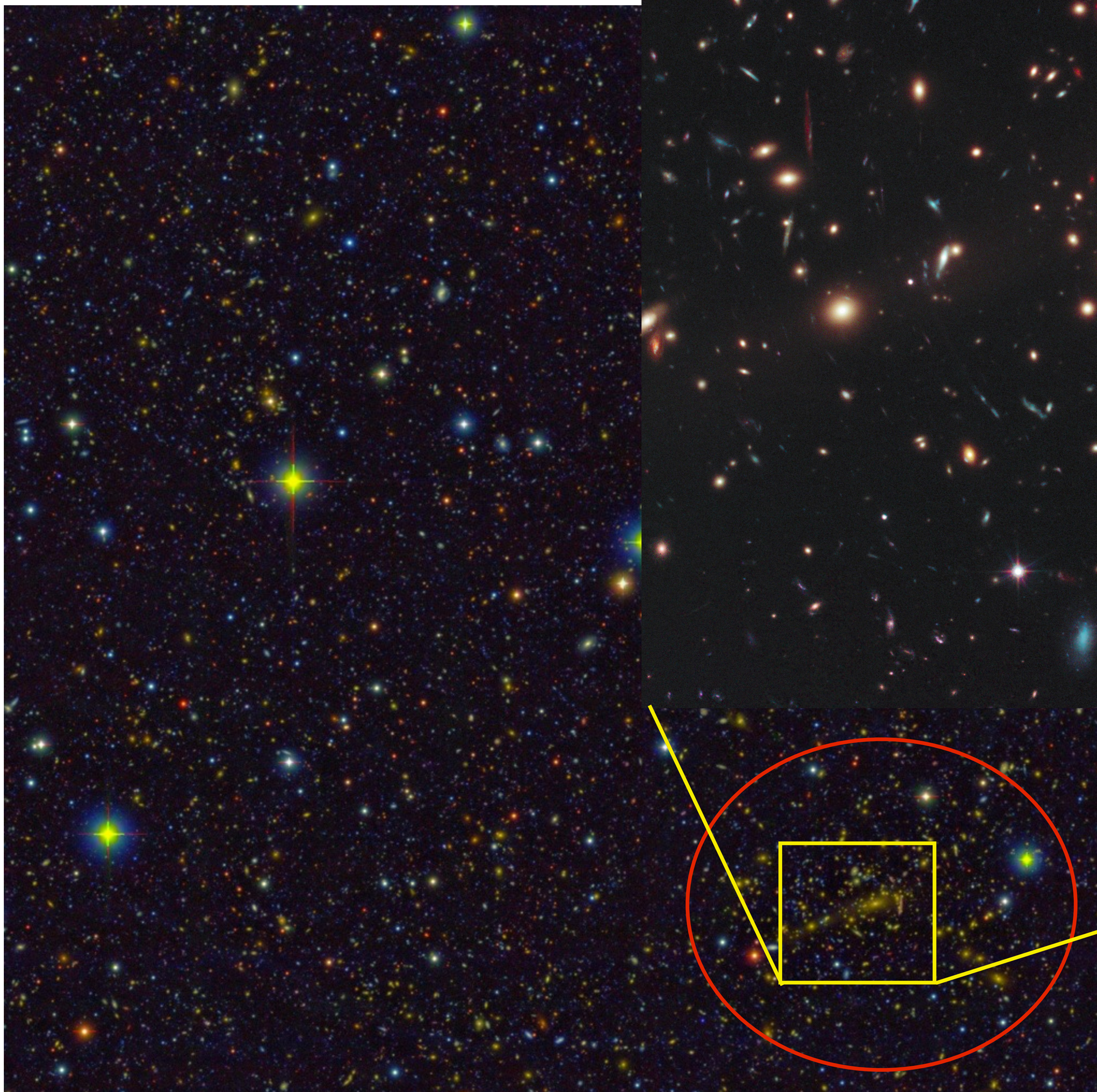
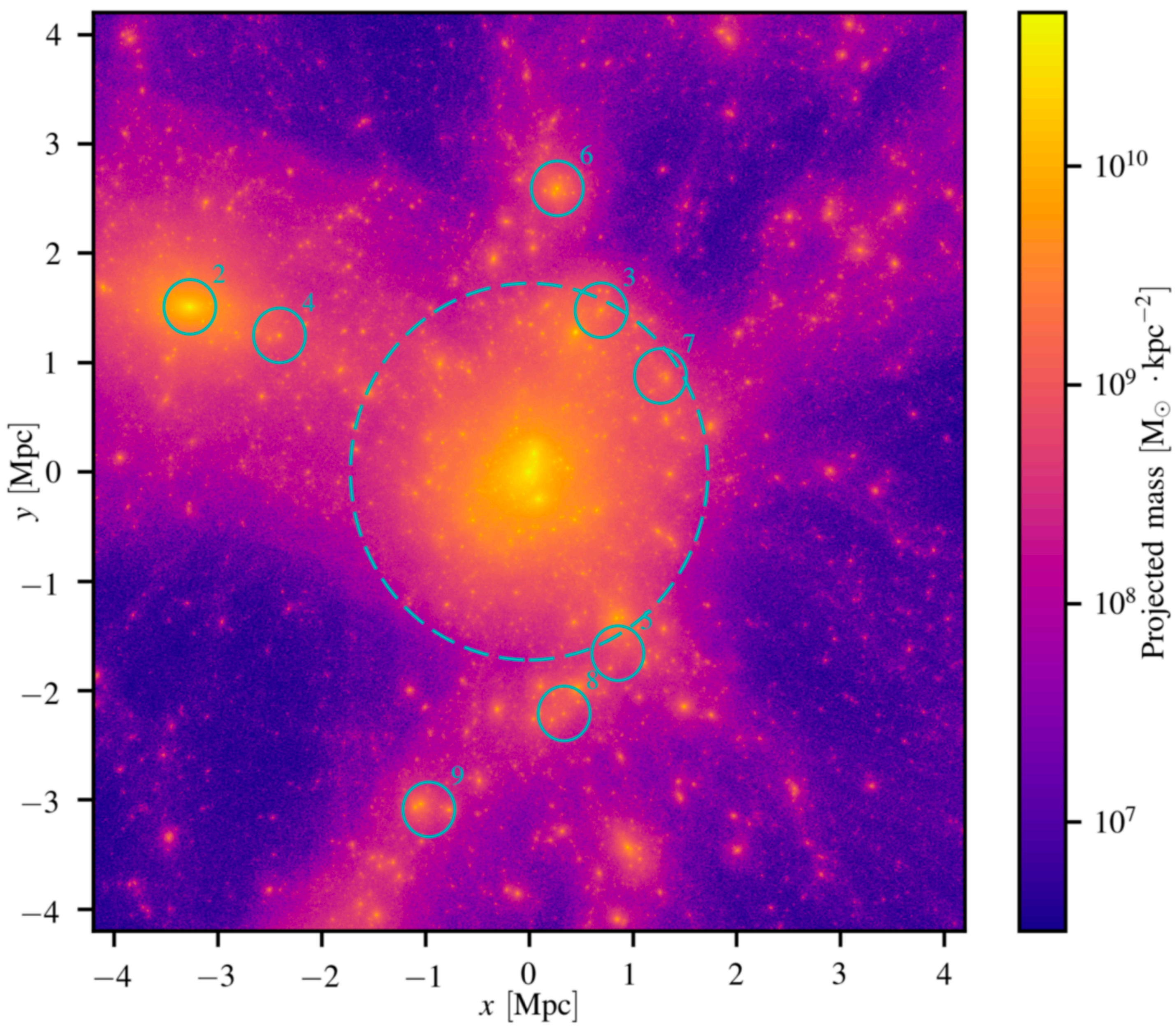
# WEAK LENSES OR LARG



# WHY MEASURING WEAK LENSING AROUND CLUSTERS?

- Trace the assembly of cosmic structures to large distances from the cluster center
- Test predictions of CDM and nature of DM: NFW profile, shapes, collisional nature
- cosmology: measure cluster masses out to the virial radius, build the cluster mass function, whose shape and evolution is extremely sensitive to cosmological parameters such as $\sigma_8$, $\Omega_M$, $w$



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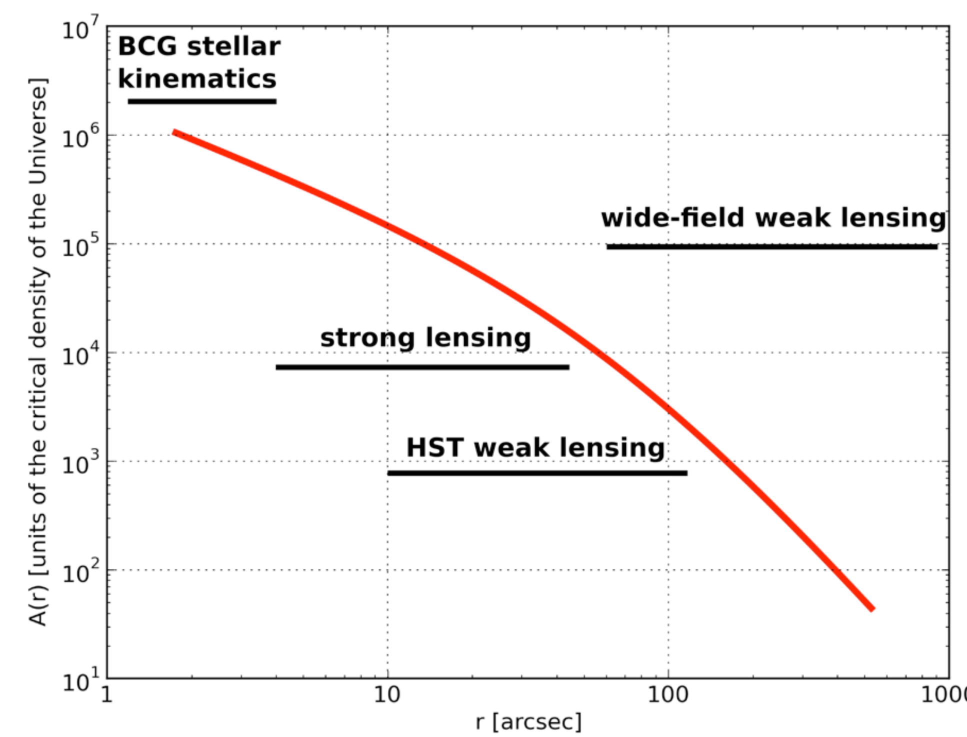
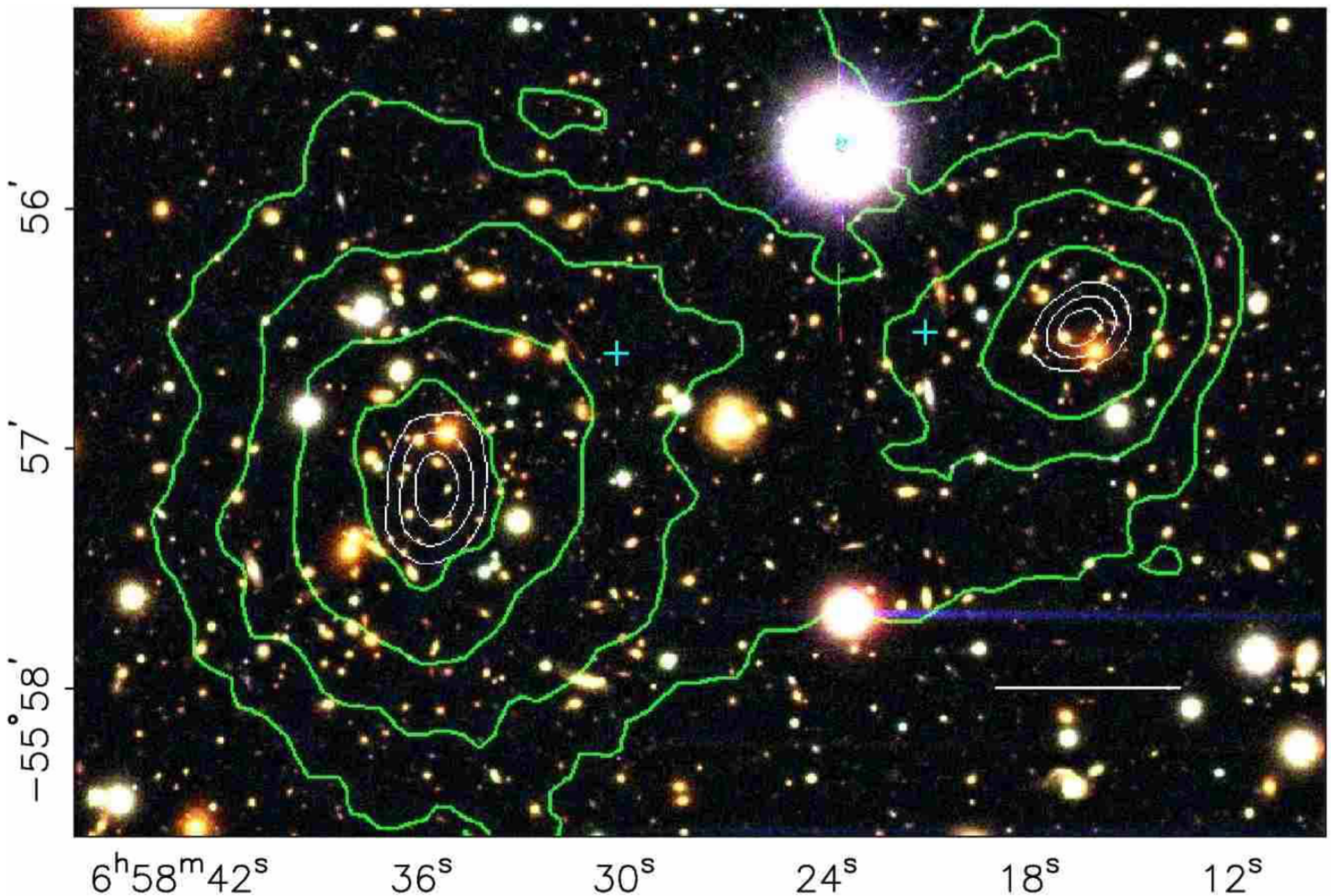
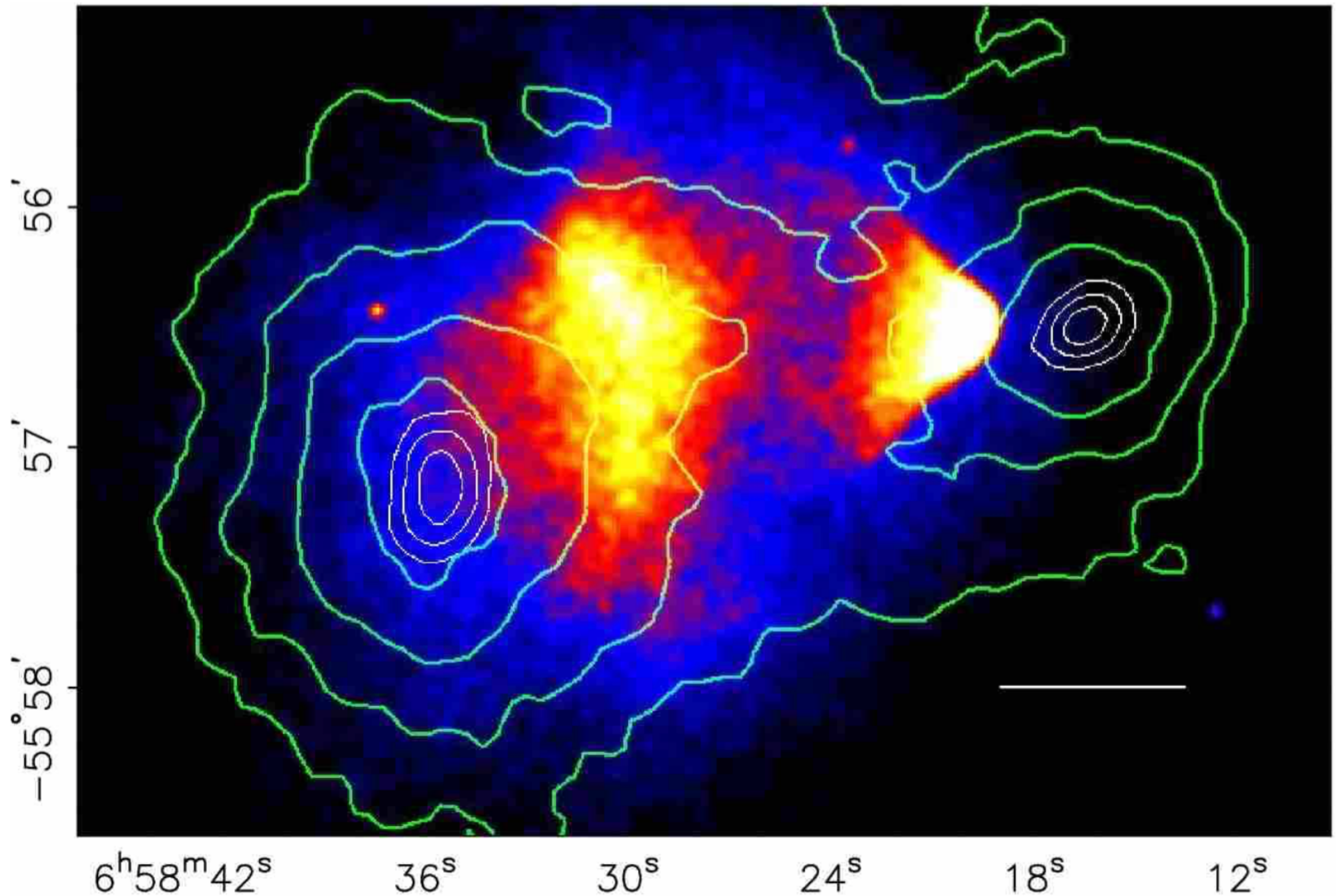


# THE BULLET CLUSTER: A PAIR OF COLLIDING CLUSTERS



# THE NATURE OF DM FROM WEAK LENSING

X-ray emission from the bullet cluster



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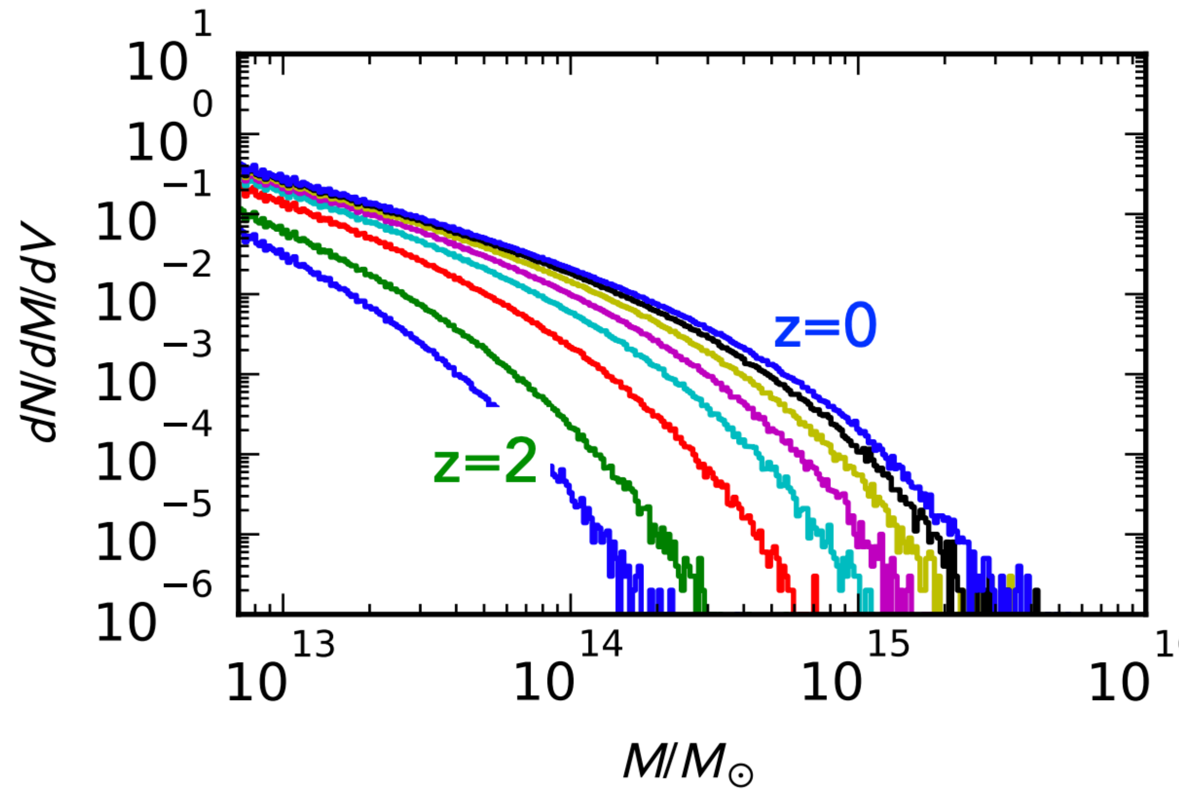

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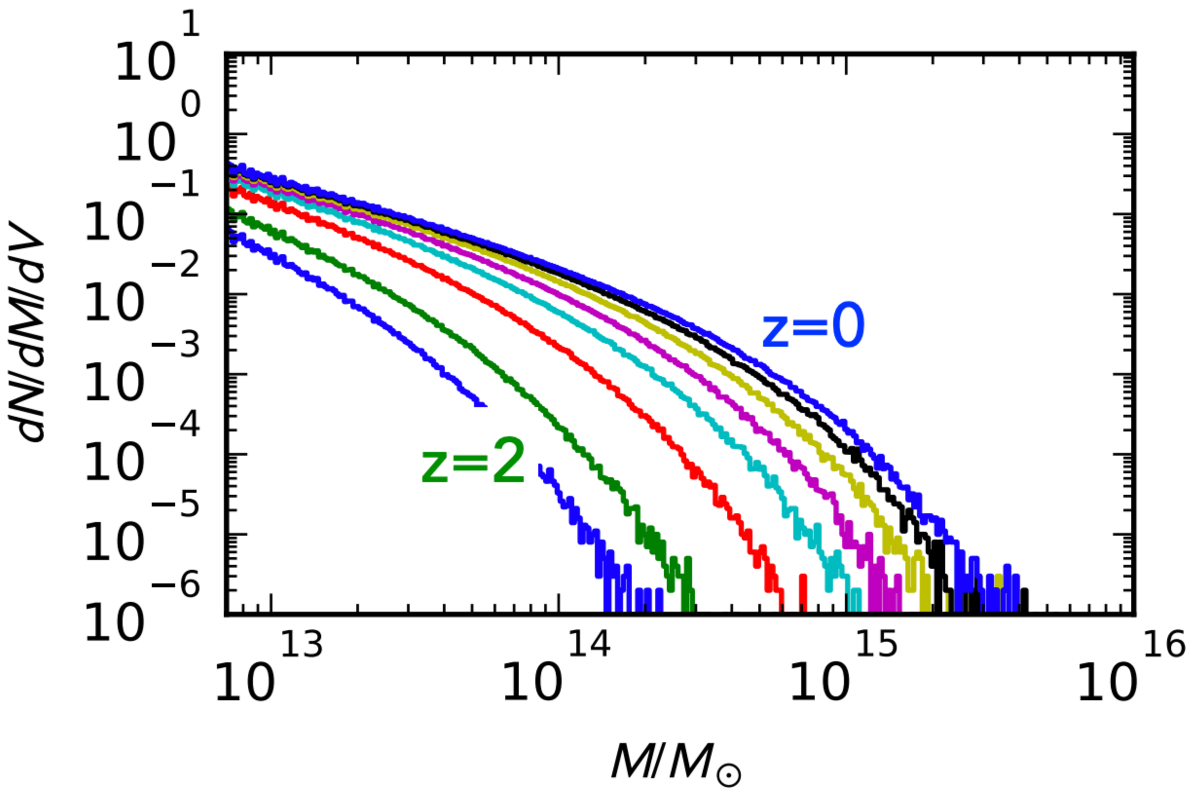
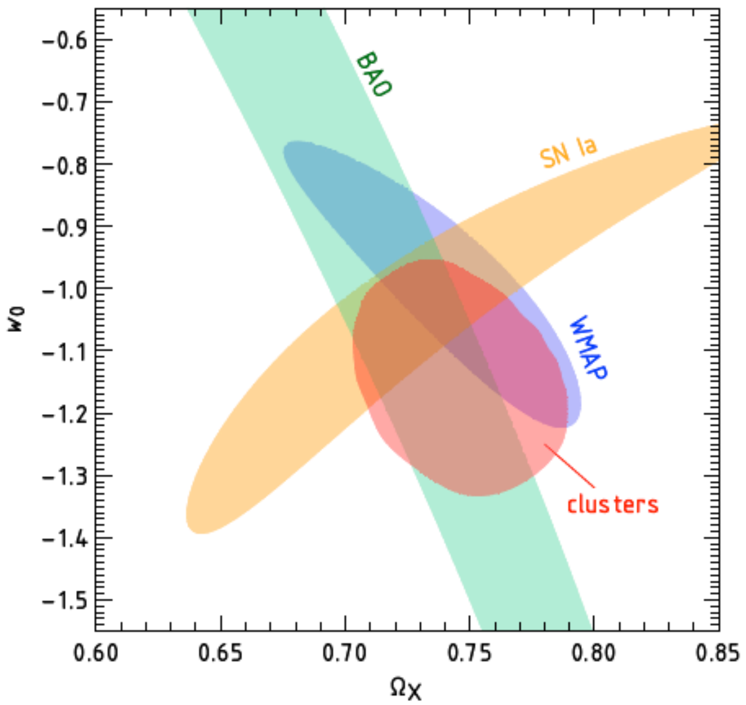


# THE WEAK LENSING REGIME: THE BASIC IDEA

- *$\kappa$ and $\gamma$ are small and nearly constant over the scale of a galaxy*

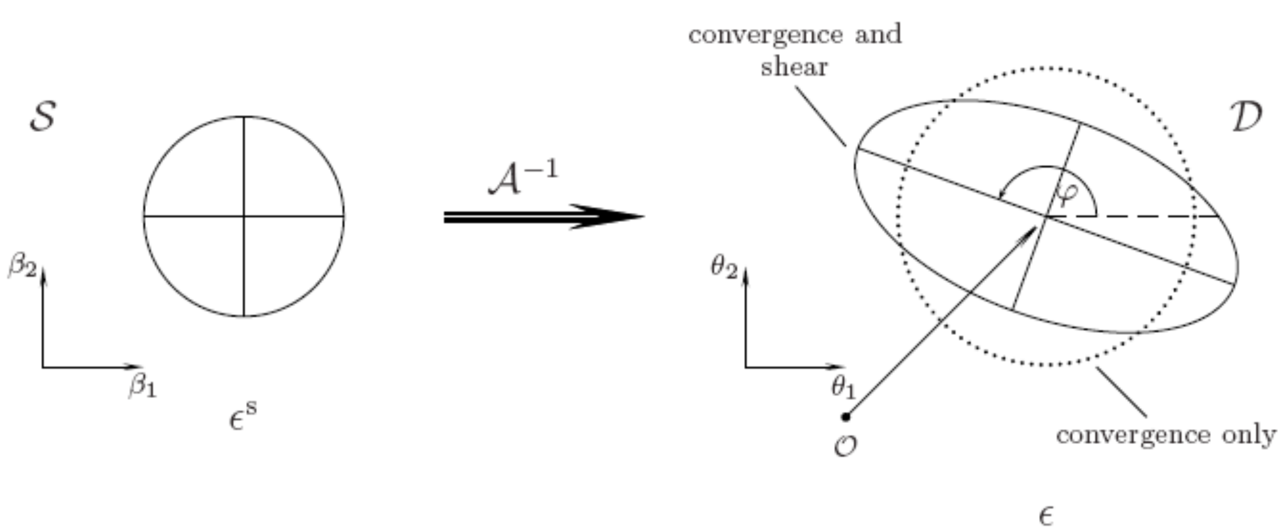
- *Circular sources are mapped into elliptical images*

- *If galaxies were circles, we would be able to measure the reduced shear from the image axis ratio and the orientation of the major axis*

$$\boldsymbol{\beta} - \boldsymbol{\beta}_0 = \mathcal{A}(\boldsymbol{\theta}_0) \cdot (\boldsymbol{\theta} - \boldsymbol{\theta}_0)$$

$$\mathcal{A}(\boldsymbol{\theta}) = (1-\kappa) \begin{pmatrix} 1-g_1 & -g_2 \\ -g_2 & 1+g_1 \end{pmatrix}$$

$$g(\boldsymbol{\theta}) = \frac{\gamma(\boldsymbol{\theta})}{[1-\kappa(\boldsymbol{\theta})]}$$

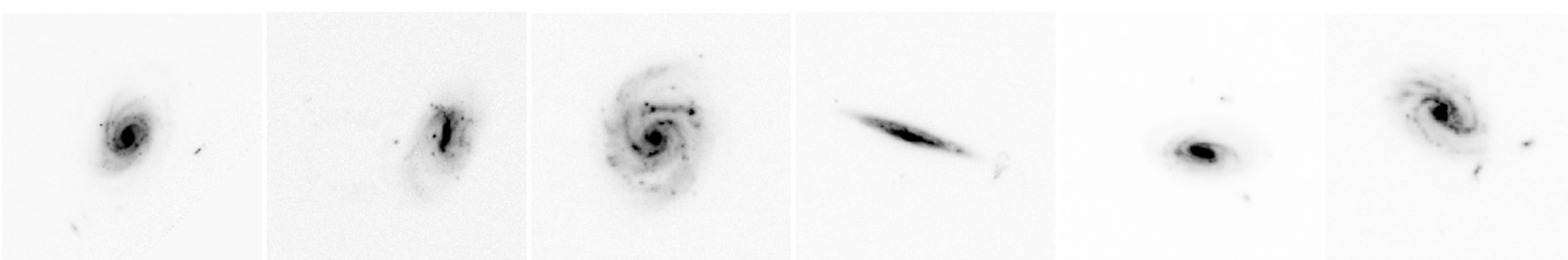



$$a = \frac{r}{1-\kappa-\gamma} \quad , \quad b = \frac{r}{1-\kappa+\gamma}$$

$$\epsilon = \frac{a-b}{a+b} = \frac{2\gamma}{2(1-\kappa)} = \frac{\gamma}{1-\kappa} \approx \gamma$$

$$|g| = \frac{1-b/a}{1+b/a} \quad \Leftrightarrow \quad \frac{b}{a} = \frac{1-|g|}{1+|g|}$$

# WHAT IS THE IMAGE ELLIPTICITY?



# ELLIPTICITY FROM BRIGHTNESS MOMENTS

*Observable: brightness distribution*

*Image centroid* $\bar{\boldsymbol{\theta}} \equiv \frac{\int d^2\theta \, I(\boldsymbol{\theta}) \, q_I[I(\boldsymbol{\theta})] \, \boldsymbol{\theta}}{\int d^2\theta \, I(\boldsymbol{\theta}) \, q_I[I(\boldsymbol{\theta})]}$

*Weight function* $q_I(I) = \mathrm{H}(I - I_{\mathrm{th}})$

*Define a tensor of second order brightness moments:*

$$Q_{ij} = \frac{\int d^2\theta \, I(\boldsymbol{\theta}) \, q_I[I(\boldsymbol{\theta})] \, (\theta_i - \bar{\theta}_i) \, (\theta_j - \bar{\theta}_j)}{\int d^2\theta \, I(\boldsymbol{\theta}) \, q_I[I(\boldsymbol{\theta})]} \,, \quad i, j \in \{1, 2\}$$

*For an image with circular isophotes,* $Q_{11}=Q_{22}$ *and* $Q_{12}=Q_{21}=0$

*The trace of the Q describes the size of the image*

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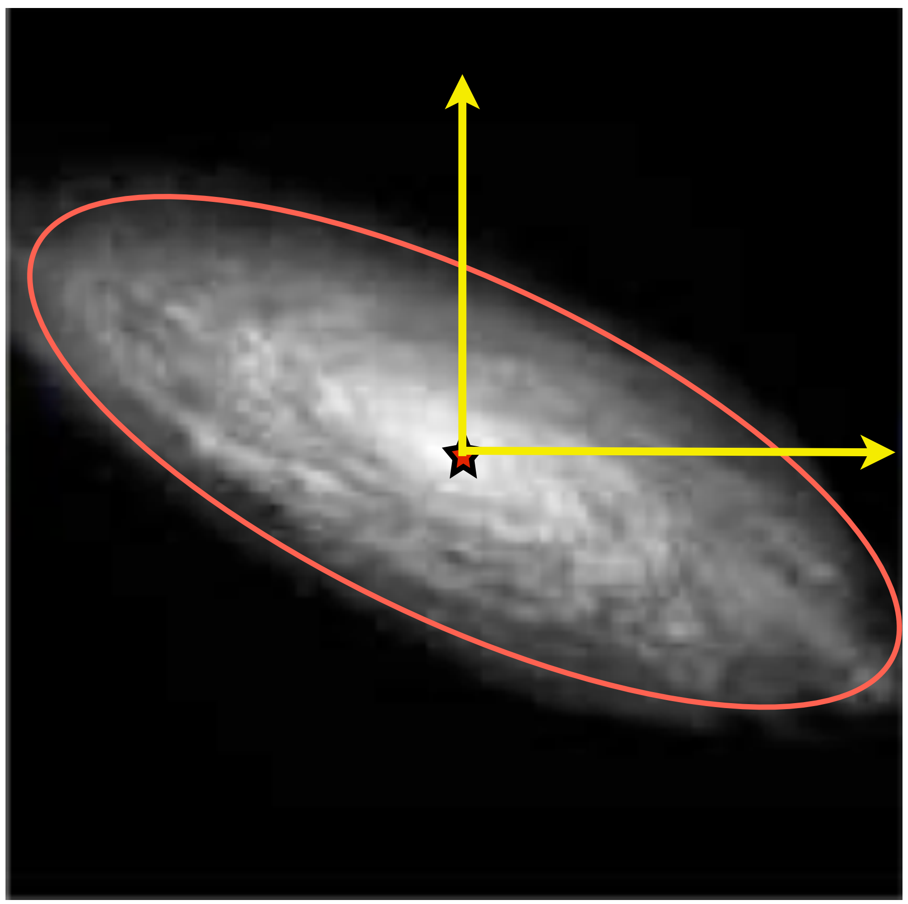

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*The trace of the Q describes the size of the image*

# COMPLEX ELLIPTICITY AND REDUCED SHEAR

*From $Q_{ij}$, one can define the complex ellipticity:*

1. *Diagonalize the matrix*

$$\det(Q_{ij} - \lambda\delta_{ij}) = 0$$

2. *Eigenvalues give the inverse square root of the semi-axes*

$$\lambda_+ = \frac{1}{2}\left(Q_{11} + Q_{22} + \sqrt{(Q_{11} - Q_{22})^2 + 4Q_{12}}\right) = \frac{1}{a^2}$$

$$\lambda_- = \frac{1}{2}\left(Q_{11} + Q_{22} - \sqrt{(Q_{11} - Q_{22})^2 + 4Q_{12}}\right) = \frac{1}{b^2}$$

3. *The modulus of the ellipticity is*

$$|\epsilon| = \frac{a-b}{a+b} = \frac{\sqrt{(Q_{11} - Q_{22})^2 + 4Q_{12}^2}}{Q_{11} + Q_{22} + 2(Q_{11}Q_{22} - Q_{12})^{1/2}}$$

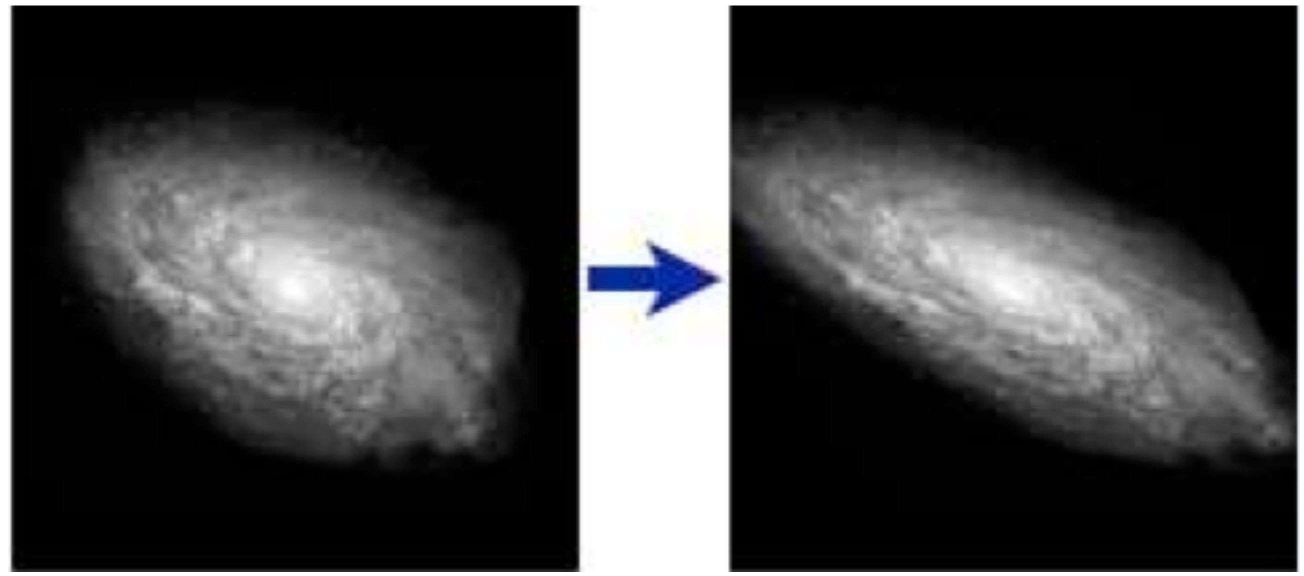
4. *In analogy with shear, we can define the complex ellipticity*

$$|\epsilon| = \sqrt{\epsilon\epsilon^*} = \sqrt{\epsilon_1^2 + \epsilon_2^2}$$

$$\epsilon = \epsilon_1 + i\epsilon_2 = \frac{Q_{11} - Q_{22} + 2iQ_{12}}{Q_{11} + Q_{22} + 2(Q_{11}Q_{22} - Q_{12})^{1/2}}$$

# UNFORTUNATELY GALAXIES ARE NOT CIRCULAR…

$I_s(\vec{\beta})$



$I(\vec{\theta})$

*We can use the lens mapping (at first order) to find the transformation between observed and intrinsic ellipticity*

$$Q_{ij}^{(s)} = \frac{\int d\beta^2 q_I[I^{(s)}(\beta)] I^{(s)}(\beta)(\beta_i - \overline{\beta}_i)(\beta_j - \overline{\beta}_j)}{\int d\beta^2 q_I[I^{(s)}(\beta)] I^{(s)}(\beta)} \quad i,j \in \{1,2\}$$

*With* $\boldsymbol{\beta} - \bar{\boldsymbol{\beta}} = \mathcal{A}\,(\boldsymbol{\theta} - \bar{\boldsymbol{\theta}})$ $\quad \mathrm{d}^2\beta = \det\mathcal{A}\,\mathrm{d}^2\theta,$ $\quad \mathcal{A} \equiv \mathcal{A}(\bar{\boldsymbol{\theta}})$

*We can show that* $Q^{(\mathrm{s})} = \mathcal{A}\,Q\,\mathcal{A}^T = \mathcal{A}\,Q\,\mathcal{A}$

# UNFORTUNATELY GALAXIES ARE NOT CIRCULAR…

*Using the fact that* $$\mathcal{A}(\boldsymbol{\theta}) = (1-\kappa)\begin{pmatrix} 1-g_1 & -g_2 \\ -g_2 & 1+g_1 \end{pmatrix}$$

*and that* $$\epsilon \equiv \frac{Q_{11} - Q_{22} + 2\mathrm{i}Q_{12}}{Q_{11} + Q_{22} + 2(Q_{11}Q_{22} - Q_{12}^2)^{1/2}}$$

*we can show that* $Q^{(\mathrm{s})} = \mathcal{A}\, Q\, \mathcal{A}^T = \mathcal{A}\, Q\, \mathcal{A}$ *implies*

$$\epsilon^{(\mathrm{s})} = \begin{cases} \dfrac{\epsilon - g}{1 - g^*\epsilon} & \text{if } |g| \le 1 \; ; \\[2ex] \dfrac{1 - g\epsilon^*}{\epsilon^* - g^*} & \text{if } |g| > 1 \; . \end{cases}$$

*The inverse relation can be obtained by changing g with -g.*

# ESTIMATING THE REDUCED SHEAR

*We assume that the intrinsic orientations of galaxies (phases of the complex ellipticity) are random. In this case,*

$$E(\epsilon^{(s)}) = 0$$

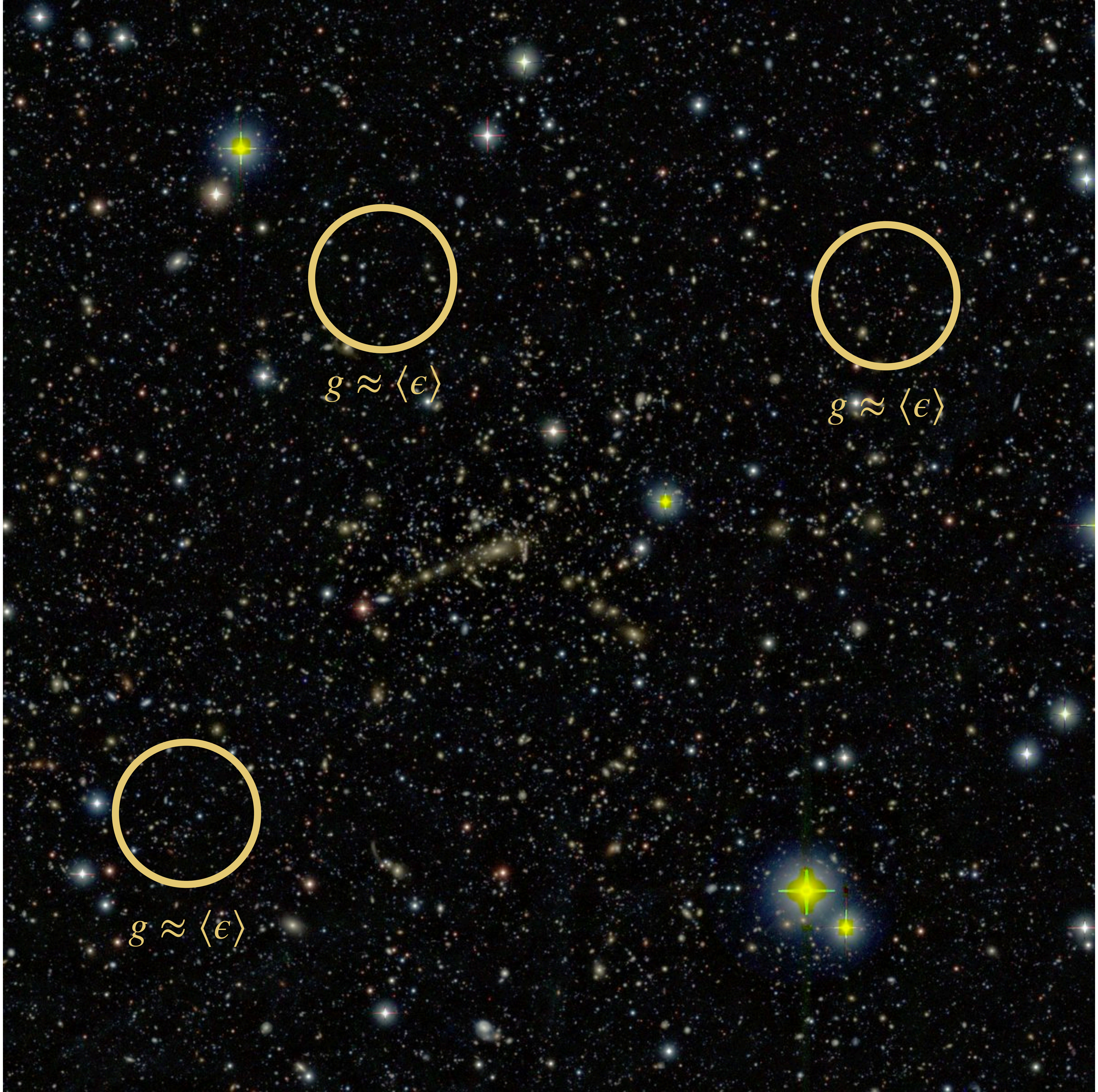
$$\epsilon^{(\mathrm{s})} = \begin{cases} \dfrac{\epsilon - g}{1 - g^*\epsilon} & \text{if } |g| \le 1 \;; \\[2ex] \dfrac{1 - g\epsilon^*}{\epsilon^* - g^*} & \text{if } |g| > 1 \;. \end{cases} \qquad \Rightarrow \qquad \mathrm{E}(\epsilon) = \begin{cases} g & \text{if } |g| \le 1 \\ 1/g^* & \text{if } |g| > 1 \end{cases}$$

*Each image ellipticity provides an un-biased estimate of the local shear. However this is very noisy. The noise is determined by the intrinsic ellipticity dispersion*

$$\sigma_\epsilon = \sqrt{\left\langle \epsilon^{(\mathrm{s})} \epsilon^{(\mathrm{s})*} \right\rangle}$$

*Noise can be beaten down by averaging over many galaxy images. The accuracy of the shear estimate depends on the local density of galaxies for which shape can be measured. Thus, deep imaging observations are required.*

$$\gamma \approx g \approx \langle \epsilon \rangle$$



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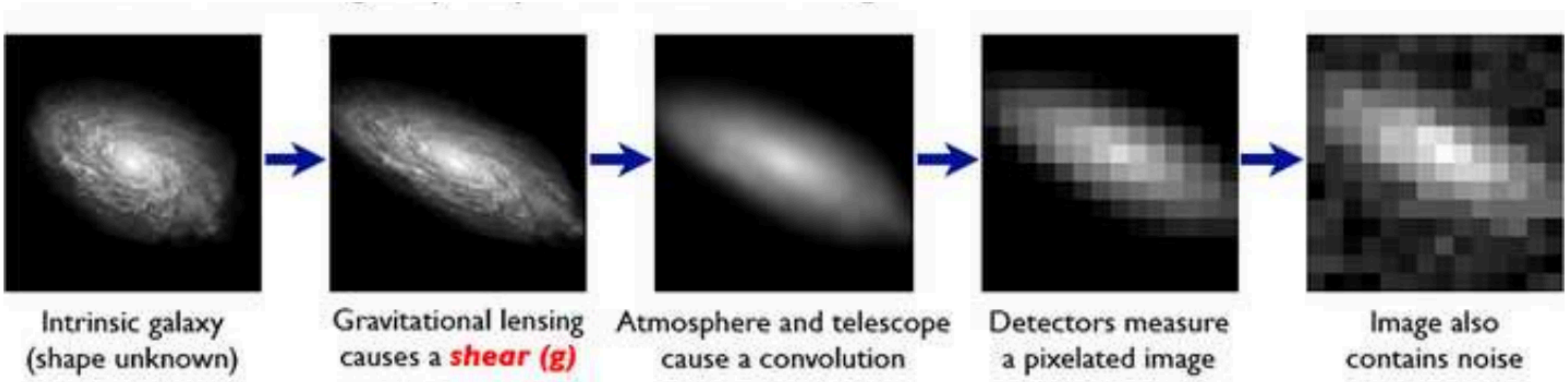
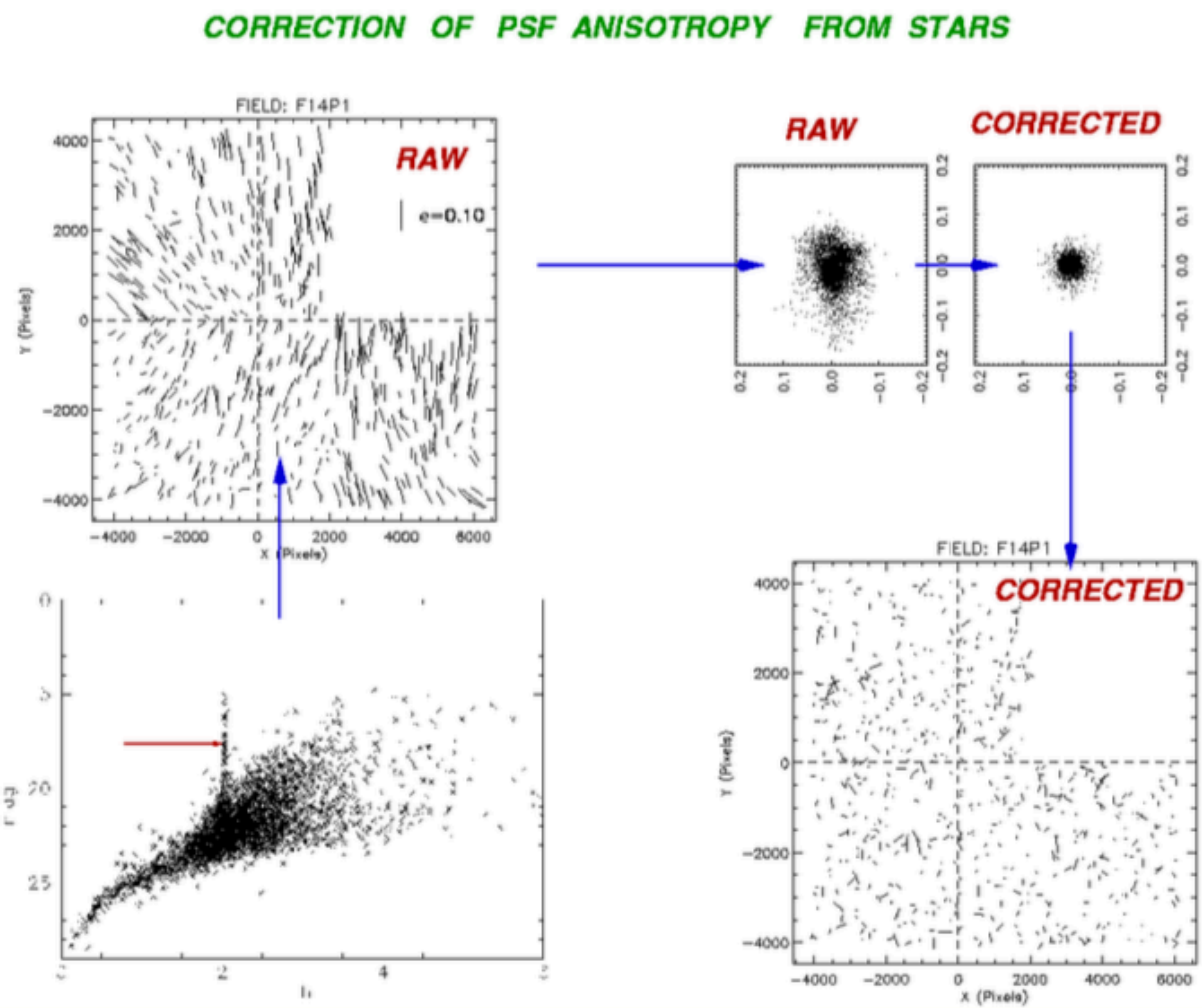
# NOT SO EASY… :-(




- images are blurred by instrument **PSF** and by the atmosphere. PSF tends to circularize shapes but can also introduce artificial elongations, i.e. spurious signal
- images are **pixelated**
- sometimes galaxies are **blended**
- several **instrumental effects** can mimic as shear signal (e.g. CTE, bad tracking, star saturations, ghosts, cosmic rays)

# CORRECTION OF PSF ANISOTROPY



# MORE CHALLENGES

- if there are **intrinsic alignments**, then the shear measurement will be biased
- lensing signal (shear) is **redshift** dependent. This needs to be taken into account, especially when dealing with deep observations
- only galaxies behind the lens (cluster) are lensed. Averaging over galaxies that are erroneously classified as background but in fact are in the cluster foreground or within the cluster causes **signal dilution**
- and more…

# THE KAISER & SQUIRES INVERSION ALGORITHM

Shear and convergence in Fourier space

$$\kappa = \frac{1}{2}(\psi_{11} + \psi_{22}) \quad \Rightarrow \quad \hat{\kappa} = -\frac{1}{2}(k_1^2 + k_2^2)\hat{\psi}$$
$$\gamma_1 = \frac{1}{2}(\psi_{11} - \psi_{22}) \quad \Rightarrow \quad \hat{\gamma}_1 = -\frac{1}{2}(k_1^2 - k_2^2)\hat{\psi}$$
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using:

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$$\hat{\kappa} = k^{-2} ( \ k_1^2 - k_2^2 \quad 2k_1 k_2 \ ) \begin{pmatrix} \hat{\gamma}_1 \\ \hat{\gamma}_2 \end{pmatrix} = k^{-2}[(k_1^2 - k_2^2)\hat{\gamma}_1 + 2k_1 k_2 \hat{\gamma}_2]$$

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$$\boxed{\kappa(\vec{\theta}) = \frac{1}{\pi}\int \mathrm{d}^2\theta' [D_1(\vec{\theta} - \vec{\theta}')\gamma_1 + D_2(\vec{\theta} - \vec{\theta}')\gamma_2]}$$

$$\begin{pmatrix} \hat{\gamma}_1 \\ \hat{\gamma}_2 \end{pmatrix} = k^{-2} \begin{pmatrix} k_1^2 - k_2^2 \\ 2k_1 k_2 \end{pmatrix} \hat{\kappa}$$

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# THE KAISER & SQUIRES INVERSION ALGORITHM

CL1232-1250
(Clowe et al.)

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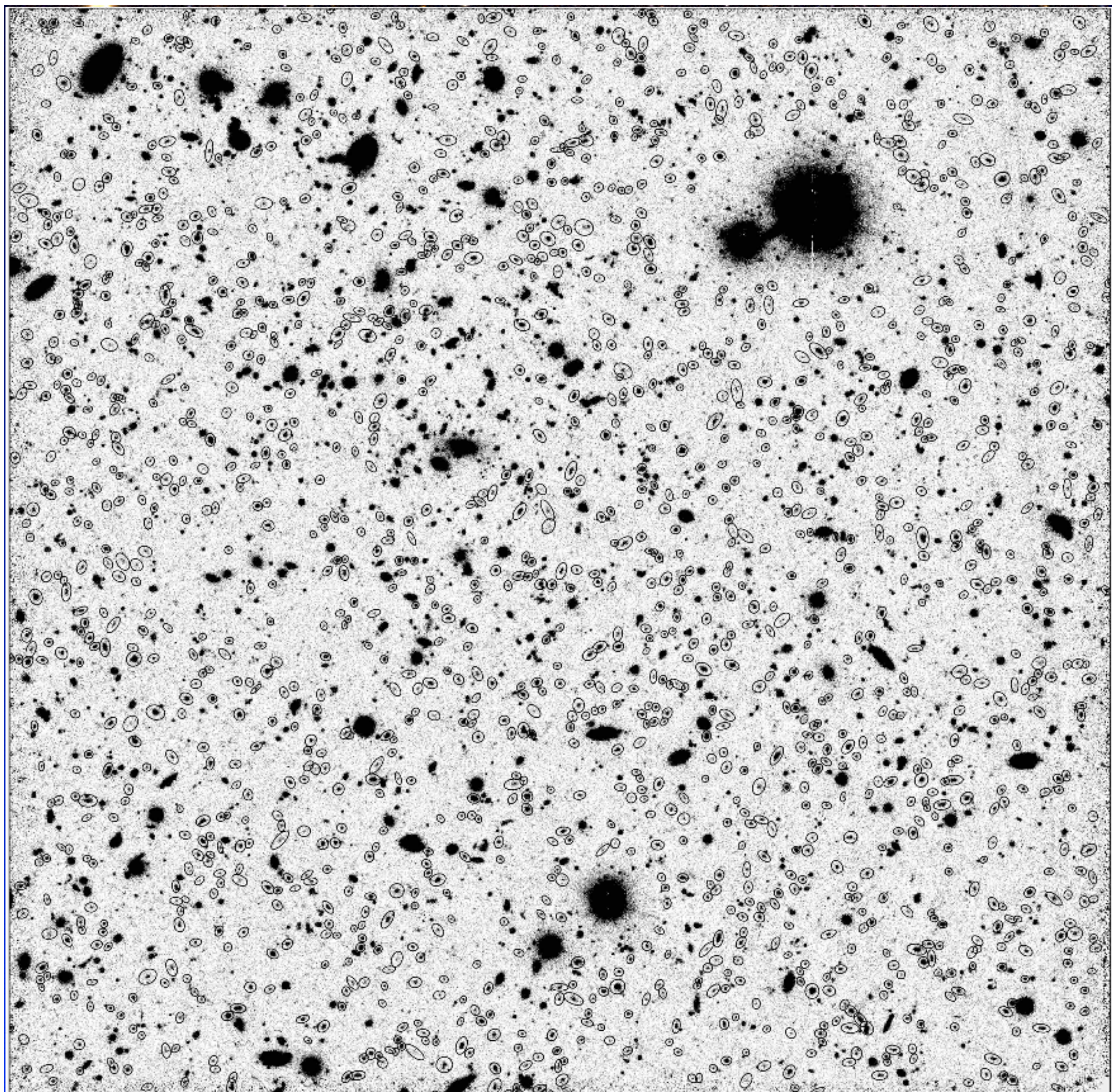

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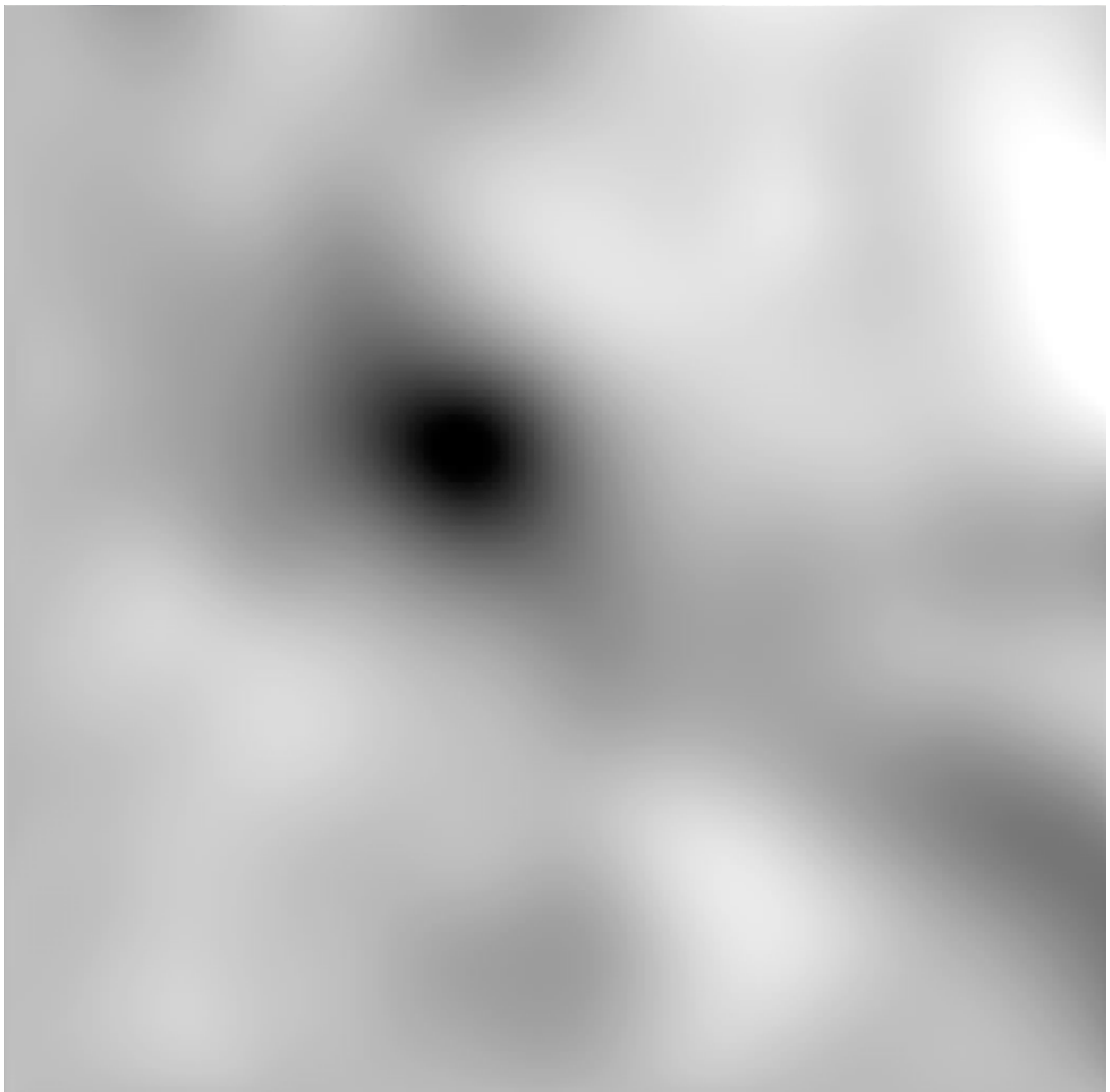

CL1232-1250
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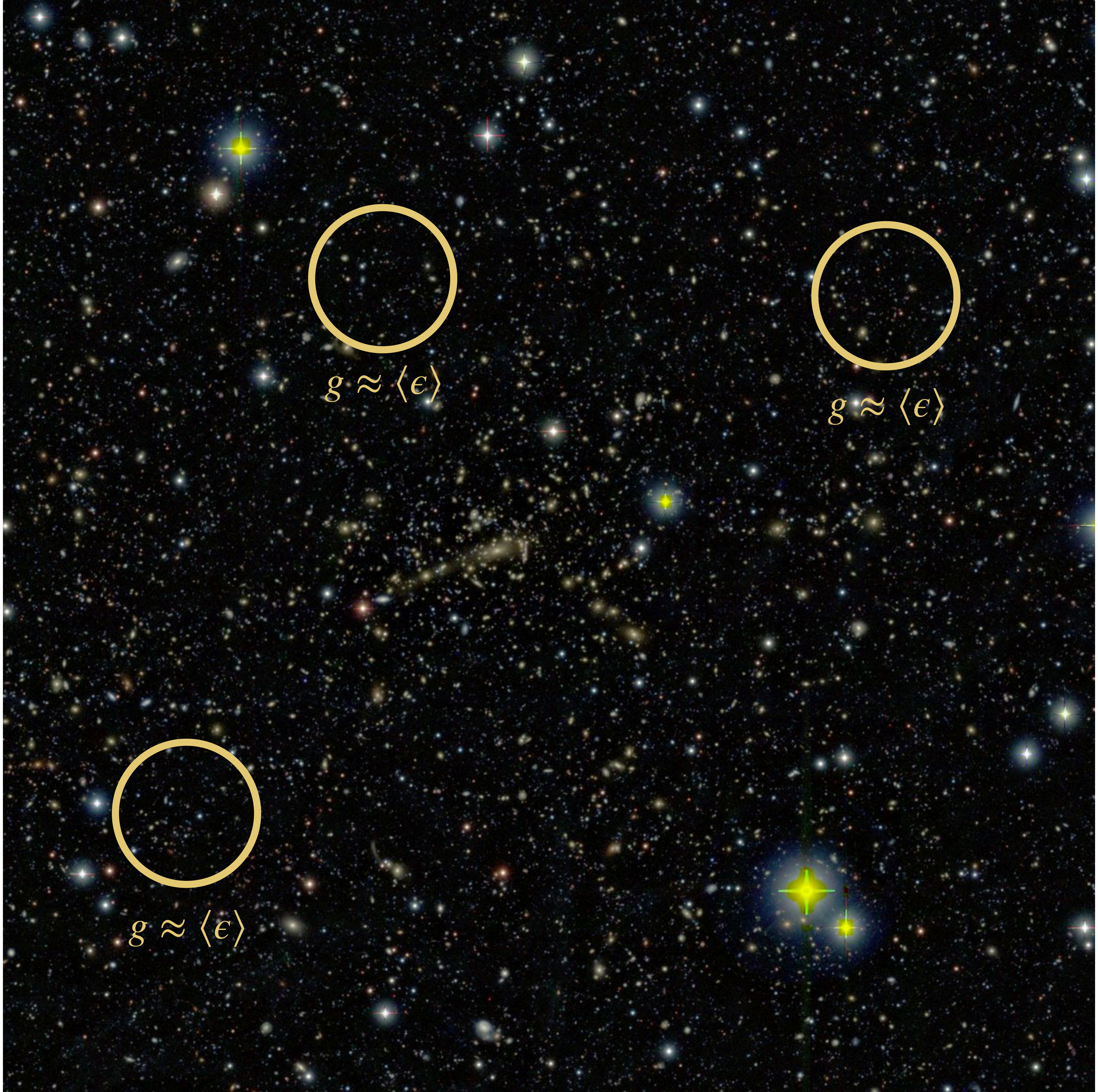
# THE KAISER & SQUIRES INVERSION ALGORITHM

- infinite fields would be required: wide field + boundary conditions.
- ellipticity measures the reduced shear, not the shear:

$$\kappa(\vec{\theta}) = \frac{1}{\pi} \int \mathrm{d}^2\theta' [D_1(\vec{\theta} - \vec{\theta}')g_1(1-\kappa) + D_2(\vec{\theta} - \vec{\theta}')g_2(1-\kappa)]$$

This equation can be solved iteratively starting from $\kappa=0$



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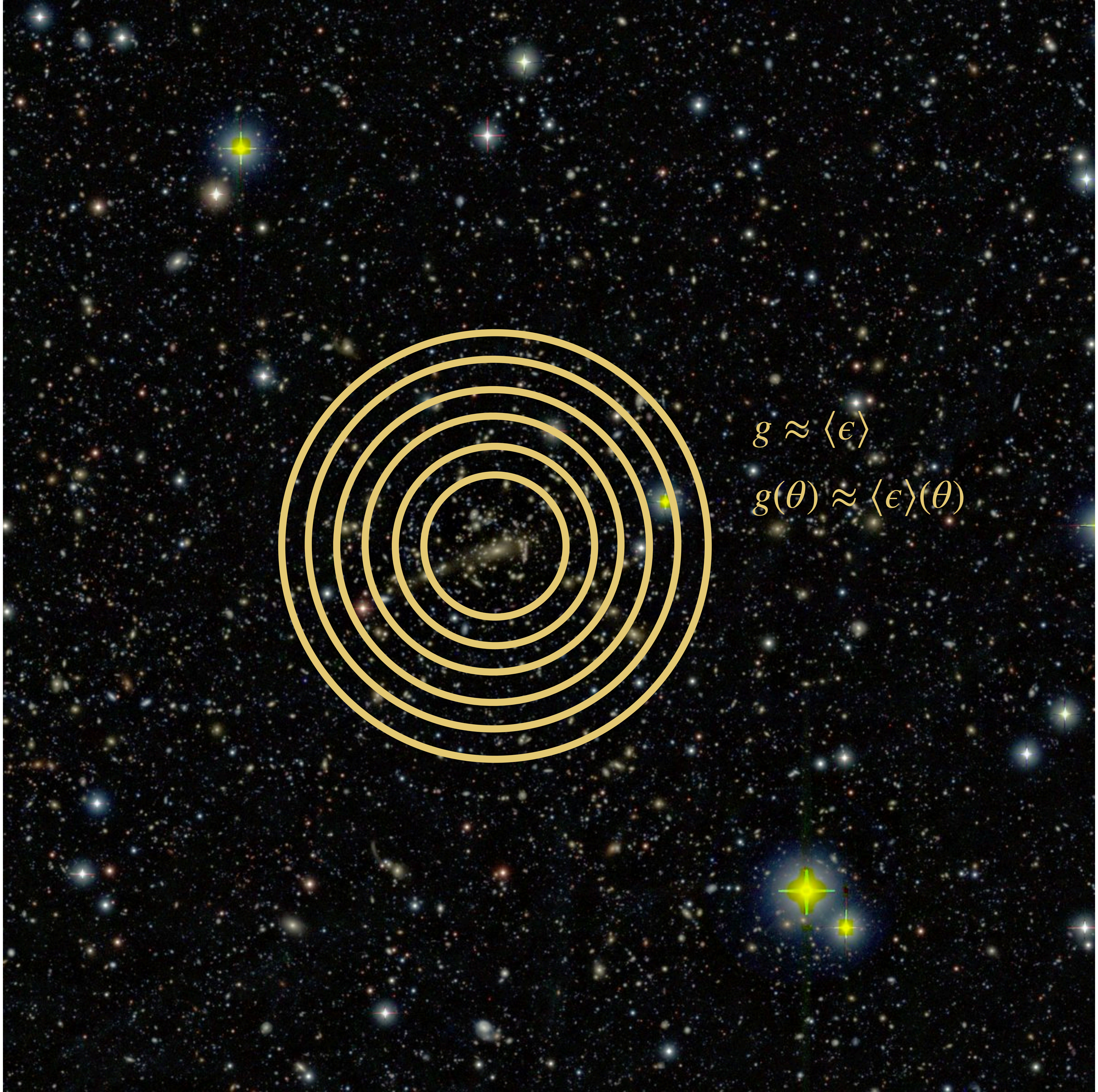

$g \approx \langle \epsilon \rangle$
$g(\theta) \approx \langle \epsilon \rangle(\theta)$

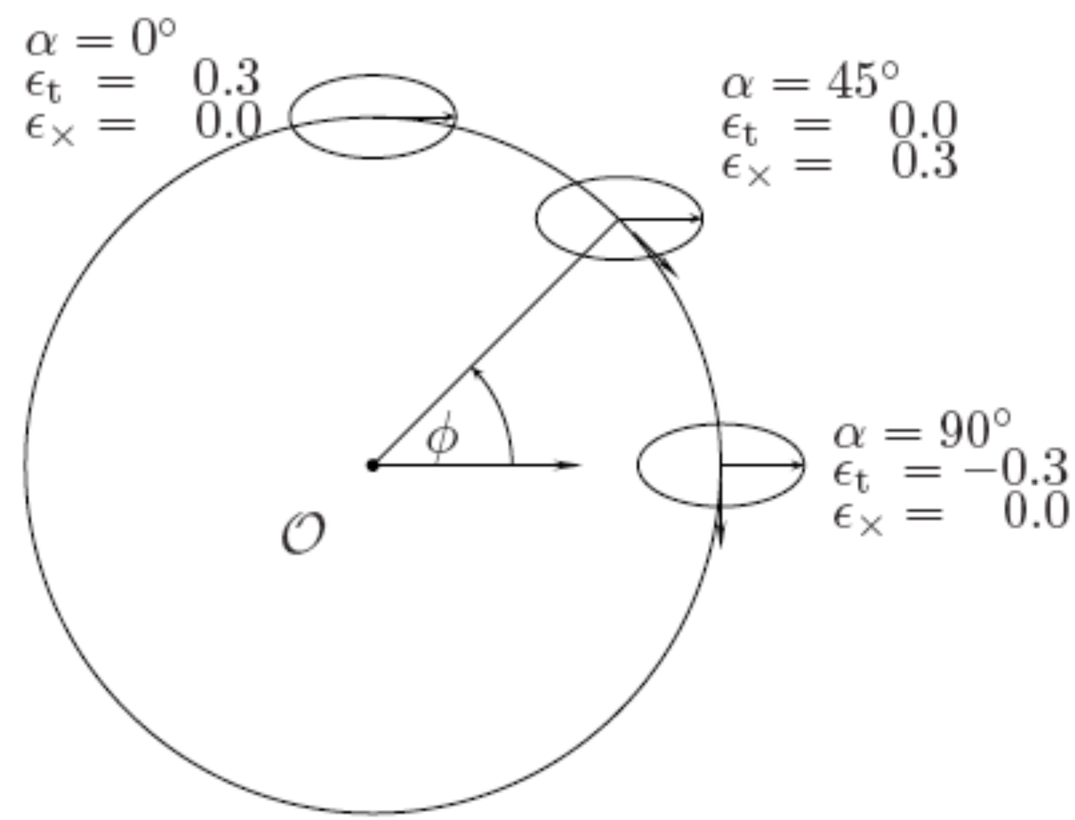
# TANGENTIAL AND CROSS COMPONENT OF THE SHEAR

Given a direction φ we can define a tangential and a cross component of the ellipticity/shear relative to this direction.

$$\gamma_{\rm t} = -\mathcal{R}{\rm e}\left[\gamma\,{\rm e}^{-2{\rm i}\phi}\right] \quad , \quad \gamma_\times = -\mathcal{I}{\rm m}\left[\gamma\,{\rm e}^{-2{\rm i}\phi}\right]$$

Note that, under this convention, “tangential” means both tangentially and radially oriented ellipticities

With this we want to emphasize that lensing, being caused by a scalar potential is curl-free




The signs are chosen such that the tangential component is positive for tangentially distorted images, and it is negative for radially distorted images.

# FIT OF THE TANGENTIAL SHEAR PROFILE

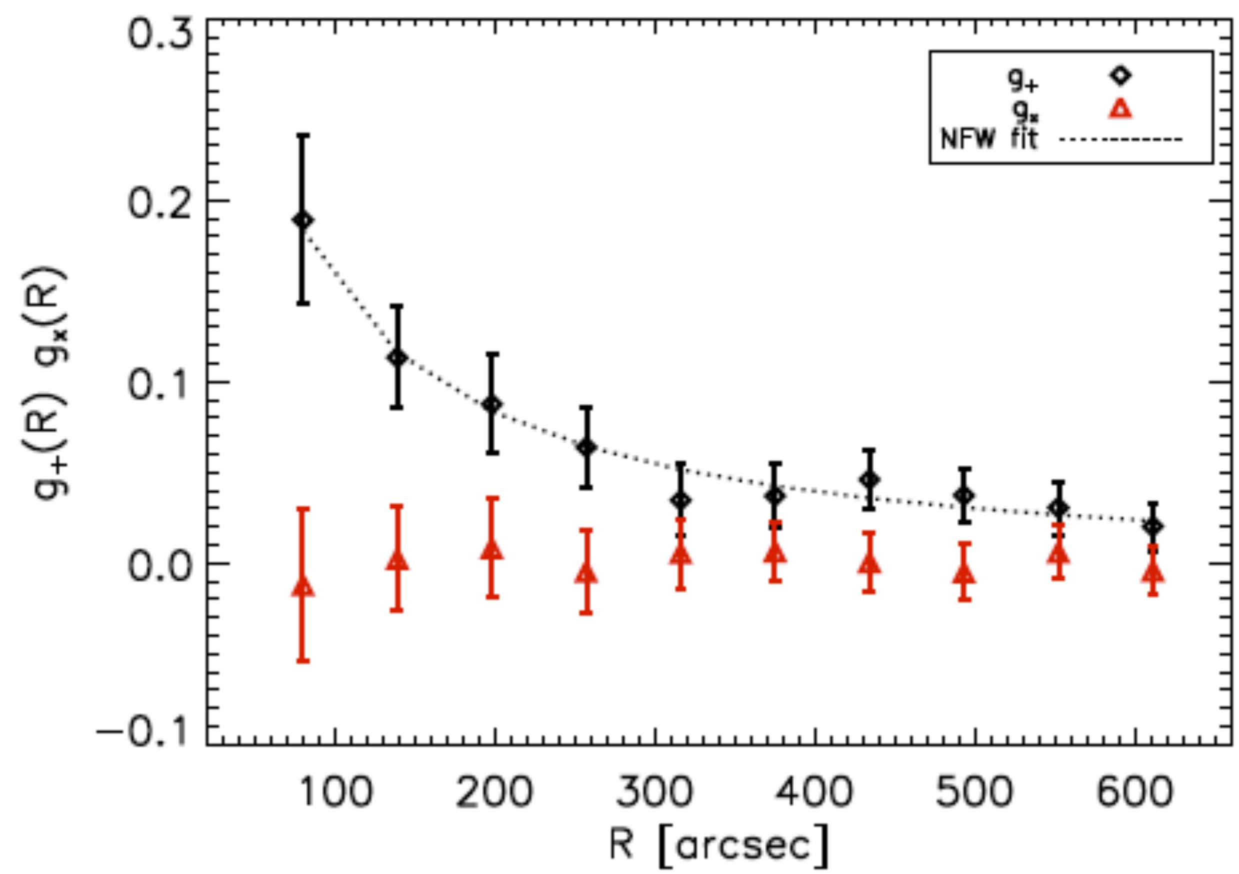
Having identified the galaxies in the background of the cluster, one can bin them in circular annuli and measure the average tangential and cross shear profiles.

The tangential shear profile, which contains the signal, can be fitted with some parametric model.
For example:

NFW $$\kappa(x) = \frac{\Sigma(\xi_0 x)}{\Sigma_{cr}} = 2\kappa_s \frac{f(x)}{x^2-1}$$

$$f(x) = \begin{cases} 1 - \frac{2}{\sqrt{x^2-1}} \operatorname{arctan}\sqrt{\frac{x-1}{x+1}} & (x>1) \\ 1 - \frac{2}{\sqrt{1-x^2}} \operatorname{arctanh}\sqrt{\frac{1-x}{1+x}} & (x<1) \\ 0 & (x=1) \end{cases}$$

$$\gamma(x) = \overline{\kappa}(x) - \kappa(x)$$




$$l_\gamma = \sum_{i=1}^{N_\gamma} \left[ \frac{|\epsilon_i - g(\theta_i)|^2}{\sigma^2[g(\theta_i)]} + 2\ln\sigma[g(\theta_i)] \right]$$

The profile of the cross component provides a check for systematics.

# A FUNDAMENTAL LIMIT OF MASS MODELING: THE MASS-SHEET DEGENERACY

$$\psi(\vec{x}) \rightarrow \psi'(\vec{x}) = \frac{1-\lambda}{2}x^2 + \lambda\psi$$

$$\vec{\alpha}'(\vec{x}) = \vec{\nabla}\psi'(\vec{x}) = (1-\lambda)\vec{x} + \lambda\vec{\alpha}(\vec{x})$$

$$\vec{y}' = \vec{x} - \vec{\alpha}'(\vec{x}) = \lambda[\vec{x} - \vec{\alpha}(\vec{x})] = \lambda\vec{y}$$

*Thus, under the transformation of the potential above, the image positions are unchanged, provided that we apply an isotropic scaling to the source plane.*

# A FUNDAMENTAL LIMIT OF MASS MODELING: THE MASS-SHEET DEGENERACY

$$\vec{\alpha}'(\vec{x}) = \vec{\nabla}\psi'(\vec{x}) = (1-\lambda)\vec{x} + \lambda\vec{\alpha}(\vec{x})$$

*If we derive further, we obtain:*

$$\begin{aligned} \kappa &\rightarrow \kappa' = (1-\lambda) + \lambda\kappa \\ \gamma &\rightarrow \gamma' = \lambda\gamma\,, \end{aligned}$$

*This is called the "mass-sheet degeneracy" (Falco et al. 1985). Note that:*

$$\begin{aligned} 1-\kappa-\gamma = \lambda_t &\rightarrow \lambda'_t = \lambda\lambda_t \\ 1-\kappa+\gamma = \lambda_r &\rightarrow \lambda'_r = \lambda\lambda_r\,. \end{aligned}$$

*Thus, the transformation does not change the location of the critical lines*

# MASS SHEET DEGENERACY FOR GALAXY SHAPES

The mass sheet transformation on shear and convergence is:

$$\kappa \to \kappa' = (1-\lambda) + \lambda\kappa$$
$$\gamma \to \gamma' = \lambda\gamma$$

The ellipticity is then: $$\epsilon' = g' = \frac{\lambda\gamma}{1-(1-\lambda)-\lambda\kappa} = \frac{\lambda\gamma}{\lambda(1-\kappa)} = g = \epsilon$$

Thus, weak lensing is also variant under mass-sheet transformations!

# AFFECTED QUANTITIES: TIME DELAYS AND MAGNIFICATIONS

$$\begin{aligned}\Delta t' &\propto \frac{1}{2}(\vec{x}-\vec{y}')^2 - \psi'(\vec{x}) \\ &= \frac{\vec{x}-\lambda\vec{y}}{2} - \lambda\psi(\vec{x}) - \frac{1-\lambda}{2}x^2 \\ &= \lambda\left[\frac{1}{2}(\vec{x}-\vec{y})^2 - \psi(\vec{x})\right] + \frac{\lambda(\lambda-1)}{2}y^2 \\ &= \lambda\Delta t + \mathrm{const}\end{aligned}$$

$$\mu' = (\lambda_t'\lambda_r')^{-1} = (\det A')^{-1} = (\lambda^2 \det A)^{-1} = \frac{\mu}{\lambda^2}$$

*Time delays and magnifications are changed by mass-sheet transformations.*

*Thus: to determine slope of mass profiles, absolute masses (away from the ER), Hubble constant, MSD must first be broken!*

# BREAKING THE DEGENERACY

- Using sources at different redshifts
- Complementary measurements of the mass profiles
  - Example: using stellar kinematics, in the case of an elliptical galaxy
- Adopting a shape for the mass profile
- Assuming that the convergence goes to zero at large distances from the center of the lens
- Measuring the magnification statistically, or via galaxy number counts

# USING GALAXY NUMBER COUNTS

$$n(> S) = n_0(S/\mu)/\mu$$

$$n_0(> S) \propto S^{-\alpha}$$

$$n(> S) \propto \frac{S^{-\alpha}}{\mu^{1-\alpha}}$$

$$n(> S)/n_0(> S) = \mu^{\alpha-1}$$

*Knowing the unlensed number density of galaxies, and the slope of the number counts, one can estimate the magnification and break the degeneracy*

# YET ANOTHER LIMIT: PERTURBATIONS ALONG THE LINE OF SIGHT