GRAVITATIONAL LENSING

4 - LENS MAPPING

Massimo Meneghetti

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SUMMARY OF THE PREVIOUS LESSON

$$\hat{\vec{\alpha}}(\vec{\xi}) = \frac{4G}{c^2}\int \frac{\vec{\xi} - \vec{\xi}'}{|\vec{\xi} - \vec{\xi}'|^2}\Sigma(\vec{\xi}')d\xi'^2$$

Deflection angle:

$$\hat{\vec{\alpha}}(\vec{\theta}) = \frac{4G}{c^2 D_L}\int \frac{\vec{\theta} - \vec{\theta}'}{|\vec{\theta} - \vec{\theta}'|^2}\Sigma(\vec{\theta}')d\theta'^2$$

Lens equation: $\vec{\theta} = \frac{\vec{\xi}}{D_L}$ $\qquad \vec{\beta} = \frac{\vec{\eta}}{D_S}$

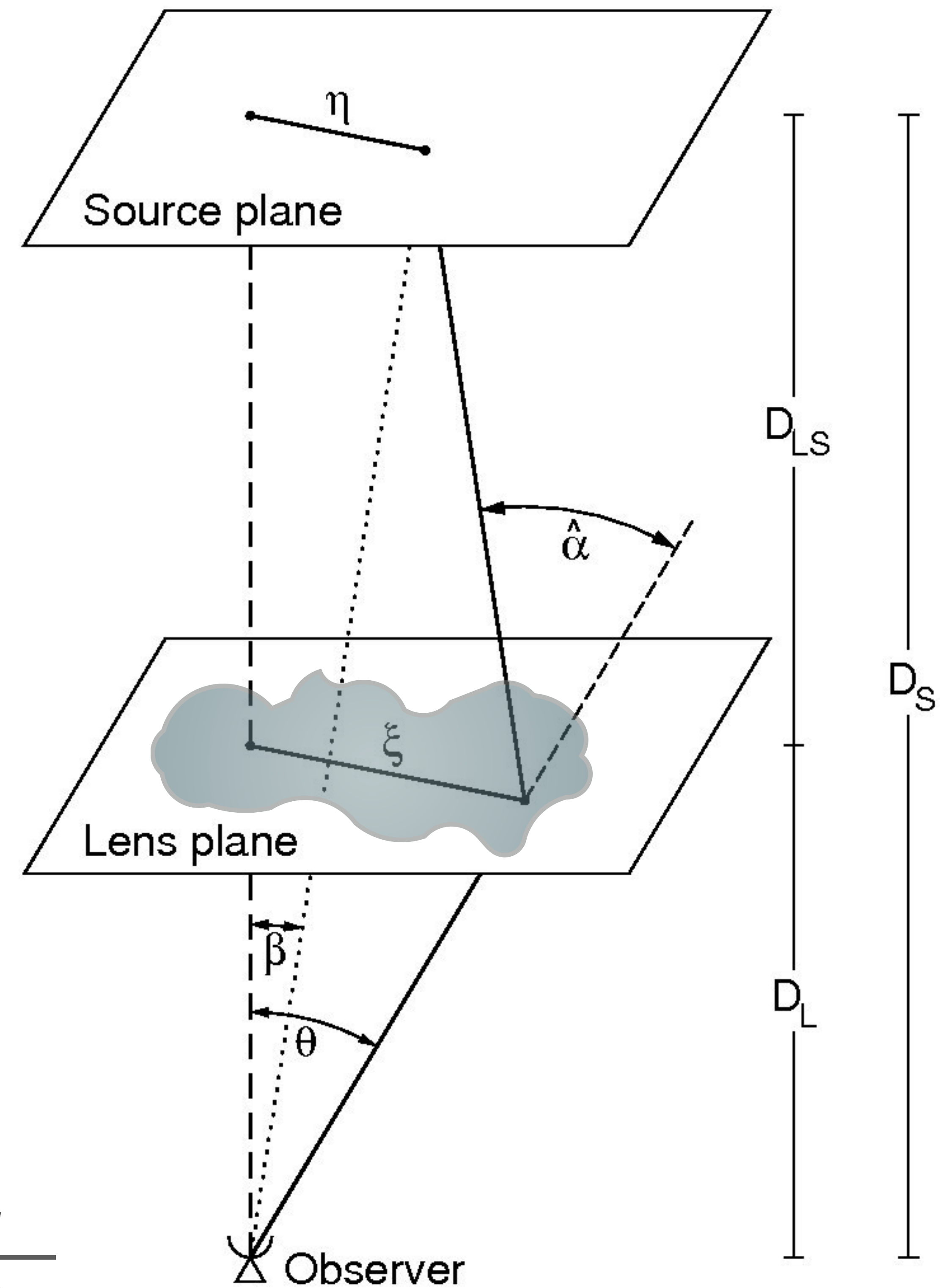
$$\vec{\beta} = \vec{\theta} - \frac{D_{LS}}{D_S}\hat{\vec{\alpha}}(\vec{\theta}) = \vec{\theta} - \vec{\alpha}(\vec{\theta})$$

Lensing Potential:

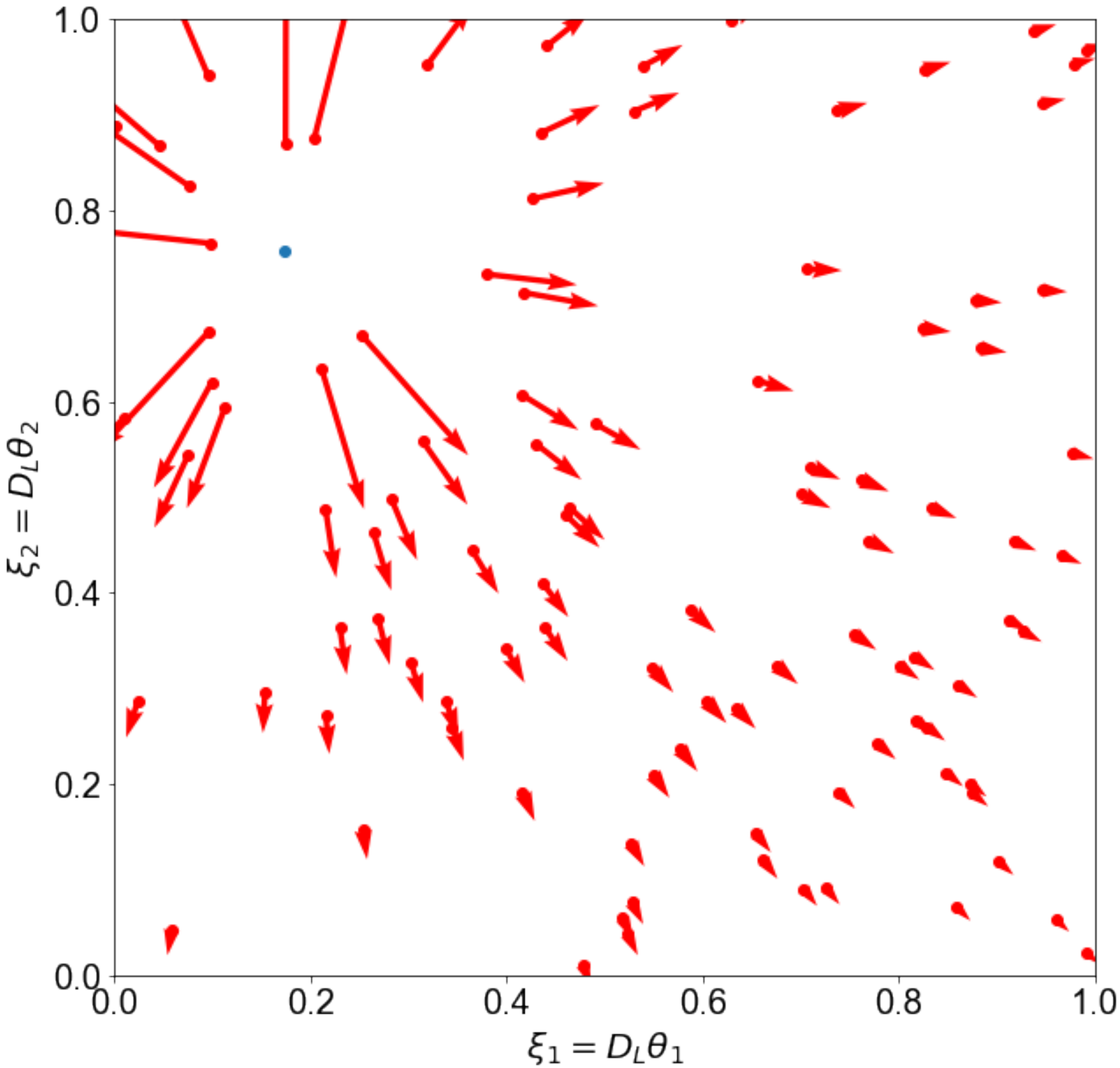
$$\hat{\Psi}(\vec{\theta}) = \frac{D_{LS}}{D_L D_S}\frac{2}{c^2}\int_0^\infty \Phi(D_L\vec{\theta})dz$$

Convergence:

$$\kappa(\vec{\theta}) = \frac{\Sigma(\vec{\theta})}{\Sigma_{cr}} = \frac{1}{2}\Delta_\theta\hat{\Psi}(\vec{\theta}) \qquad \Sigma_{cr} = \frac{c^2}{4\pi G}\frac{D_S}{D_L D_{LS}}$$



LENSING CAUSES A SHIFT OF THE APPARENT POSITION OF THE SOURCE

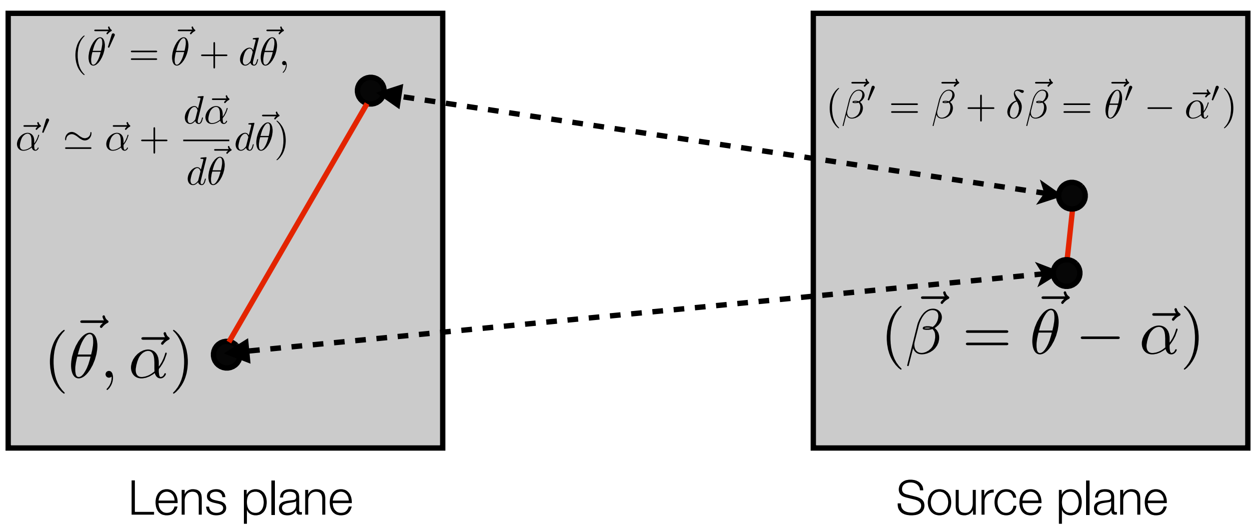


LENS MAPPING (FIRST ORDER)

- *Let's go back to the lens equation*

$$\vec{\beta} = \vec{\theta} - \vec{\alpha}(\vec{\theta}) \qquad \vec{\alpha}(\vec{\theta}) = \frac{4G}{c^2}\frac{D_{LS}}{D_L D_S}\int \frac{\vec{\theta} - \vec{\theta}\,'}{|\vec{\theta} - \vec{\theta}\,'|^2}\Sigma(\vec{\theta}\,')d\theta'^2$$

- *Assuming that the d.a. does not vary significantly over the scale dΘ:*



$$(\vec{\beta}\,' - \vec{\beta}) = \left(I - \frac{d\vec{\alpha}(\vec{\theta})}{d\vec{\theta}}\right)(\vec{\theta}\,' - \vec{\theta})$$

$$A \equiv \frac{d\vec{\beta}}{d\vec{\theta}} = \left(\delta_{ij} - \frac{\partial\alpha_i(\vec{\theta})}{\partial\theta_j}\right) = \left(\delta_{ij} - \frac{\partial^2\hat{\Psi}(\vec{\theta})}{\partial\theta_i\partial\theta_j}\right) = \left(\delta_{ij} - \hat{\Psi}_{ij}\right)$$

LENS MAPPING (FIRST ORDER)

$$A \equiv \frac{d\overrightarrow{\beta}}{d\overrightarrow{\theta}} = \left(\delta_{ij} - \frac{\partial \alpha_i(\overrightarrow{\theta})}{\partial \theta_j}\right) = \left(\delta_{ij} - \frac{\partial^2 \hat{\Psi}(\overrightarrow{\theta})}{\partial \theta_i \partial \theta_j}\right) = \left(\delta_{ij} - \hat{\Psi}_{ij}\right) = \begin{pmatrix} 1 - \hat{\Psi}_{11} & -\hat{\Psi}_{12} \\ -\hat{\Psi}_{12} & 1 - \hat{\Psi}_{22} \end{pmatrix}$$

*A is called "**the lensing Jacobian**": it is a symmetric second rank tensor describing the first order mapping between lens and source planes.*

This tensor can be written as the sum of an isotropic part, proportional to its trace, and an anisotropic, traceless part.

$$A_{iso,ij} = \frac{1}{2}\mathrm{Tr}A\delta_{ij}$$

$$A_{aniso,ij} = A_{ij} - \frac{1}{2}\mathrm{Tr}A\delta_{ij}$$

ISOTROPIC PART

$$A_{iso,ij} = \frac{1}{2}\mathrm{Tr}A\delta_{ij} = \left[1 - \frac{1}{2}(\hat{\Psi}_{11} + \hat{\Psi}_{22})\right]\delta_{ij}$$

$$= \left(1 - \frac{1}{2}\Delta_\theta\hat{\Psi}\right)\delta_{ij} = (1-\kappa)\delta_{ij}$$

ANISOTROPIC PART

$$A_{aniso,ij} = A_{ij} - \frac{1}{2}\mathrm{Tr}A\delta_{ij} = \delta_{ij} - \hat{\Psi}_{ij} - \frac{1}{2}(1 - \hat{\Psi}_{11} + 1 - \hat{\Psi}_{22})\delta_{ij}$$

$$= -\hat{\Psi}_{ij} + \frac{1}{2}(\hat{\Psi}_{11} + \hat{\Psi}_{22})\delta_{ij} = \begin{pmatrix} -\hat{\Psi}_{11} & -\hat{\Psi}_{12} \\ -\hat{\Psi}_{12} & -\hat{\Psi}_{22} \end{pmatrix} + \begin{pmatrix} \frac{1}{2}(\hat{\Psi}_{11} + \hat{\Psi}_{22}) & 0 \\ 0 & \frac{1}{2}(\hat{\Psi}_{11} + \hat{\Psi}_{22}) \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{1}{2}(\hat{\Psi}_{11} - \hat{\Psi}_{22}) & -\hat{\Psi}_{12} \\ -\hat{\Psi}_{12} & \frac{1}{2}(\hat{\Psi}_{11} - \hat{\Psi}_{22}) \end{pmatrix} = -\Gamma$$

*Introducing the **shear components**:* $\gamma_1 = \frac{1}{2}(\hat{\Psi}_{11} - \hat{\Psi}_{22}) \qquad \gamma_2 = \hat{\Psi}_{12} = \hat{\Psi}_{21}$

*The **shear** is a symmetric, trace-less 2x2 tensor* $\Gamma = \begin{pmatrix} \gamma_1 & \gamma_2 \\ \gamma_2 & -\gamma_1 \end{pmatrix}$

We define the shear module: $\gamma = \sqrt{\gamma_1^2 + \gamma_2^2}$

The determinant is $\det\Gamma = -(\gamma_1^2 + \gamma_2^2) = -\gamma^2$ *Eigenvalues* $\pm\gamma$

JACOBIAN EIGENVALUES

$$A = A_{iso,ij} + A_{aniso,ij} = (1-\kappa)\delta_{ij} - \Gamma_{ij}$$

$$= (1-\kappa)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} \gamma_1 & \gamma_2 \\ \gamma_2 & -\gamma_1 \end{pmatrix}$$

$$= \begin{pmatrix} 1-\kappa-\gamma_1 & -\gamma_2 \\ -\gamma_2 & 1-\kappa+\gamma_1 \end{pmatrix}$$

$$\det A = (1-\kappa-\gamma)(1-\kappa+\gamma)$$

There is thus an orthogonal coordinate transformation $R(\varphi)$, a rotation by an angle φ, which brings the Jacobian matrix into diagonal form.

This is the same transformation that brings the shear in a diagonal form…This is the rotation that aligns the axes of the reference frame with the eigenvectors of Γ.

SHEAR TRANSFORMATION

Under rotations, tensors (like the shear and the Jacobian) transform as

$$\Gamma' = R(\varphi)^T \Gamma R(\varphi)$$

This implies that the shear components transform as

$$\gamma_1' = \gamma_1 \cos 2\varphi + \gamma_2 \sin 2\varphi$$
$$\gamma_2' = -\gamma_1 \sin 2\varphi + \gamma_2 \cos 2\varphi$$

i.e. unlike a vector! Since the shear components are mapped on themselves after rotations by multiples of π, they form a so-called spin-2 field. Vectors, on the contrary, are invariant under rotations by angles that are multiples of 2π and have spin 1

SHEAR EIGENVECTORS

The rotation we seek is the one for which

$$\Gamma' = \begin{pmatrix} \gamma & 0 \\ 0 & -\gamma \end{pmatrix} = R(\varphi)^T \begin{pmatrix} \gamma_1 & \gamma_2 \\ \gamma_2 & -\gamma_1 \end{pmatrix} R(\varphi)$$

The formula above shows that

$$\gamma_1' = \gamma_1 \cos 2\varphi + \gamma_2 \sin 2\varphi = \gamma$$

$$\gamma_2' = -\gamma_1 \sin 2\varphi + \gamma_2 \cos 2\varphi = 0$$

From which we obtain that

$$\gamma_1 = \gamma \cos 2\varphi$$
$$\gamma_2 = \gamma \sin 2\varphi$$

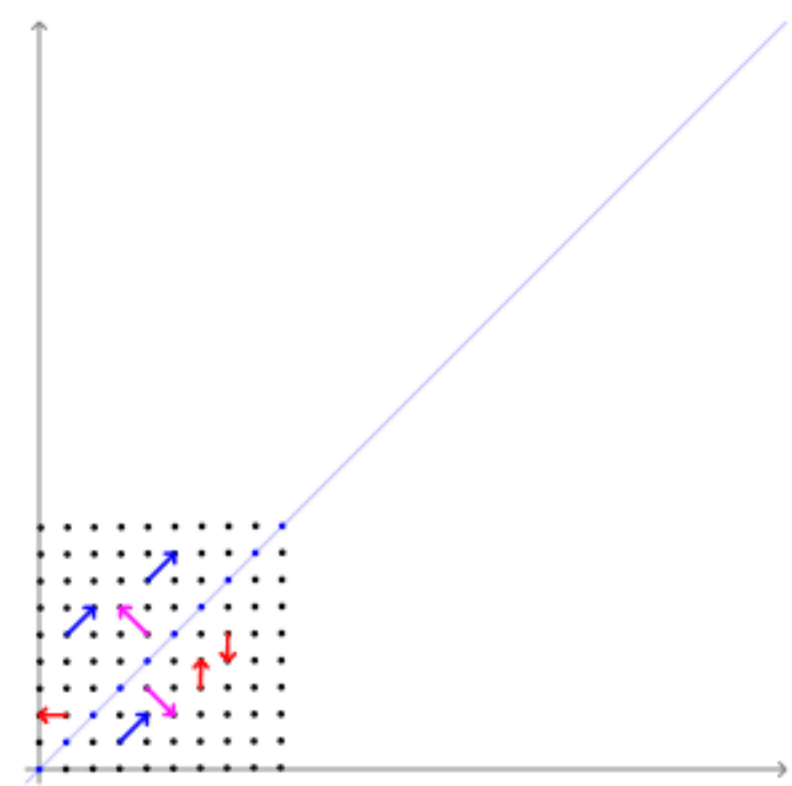
LENSING JACOBIAN

$$A = \begin{pmatrix} 1-\kappa-\gamma_1 & -\gamma_2 \\ -\gamma_2 & 1-\kappa+\gamma_1 \end{pmatrix} = (1-\kappa)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \gamma \begin{pmatrix} \cos 2\varphi & \sin 2\varphi \\ \sin 2\varphi & -\cos 2\varphi \end{pmatrix}$$

Lens mapping at first order is a linear application, distorting areas.

Distortion directions are given by the ***eigenvectors*** *of A or Γ.*

Distortion amplitudes in these directions are given by the ***eigenvalues****.*



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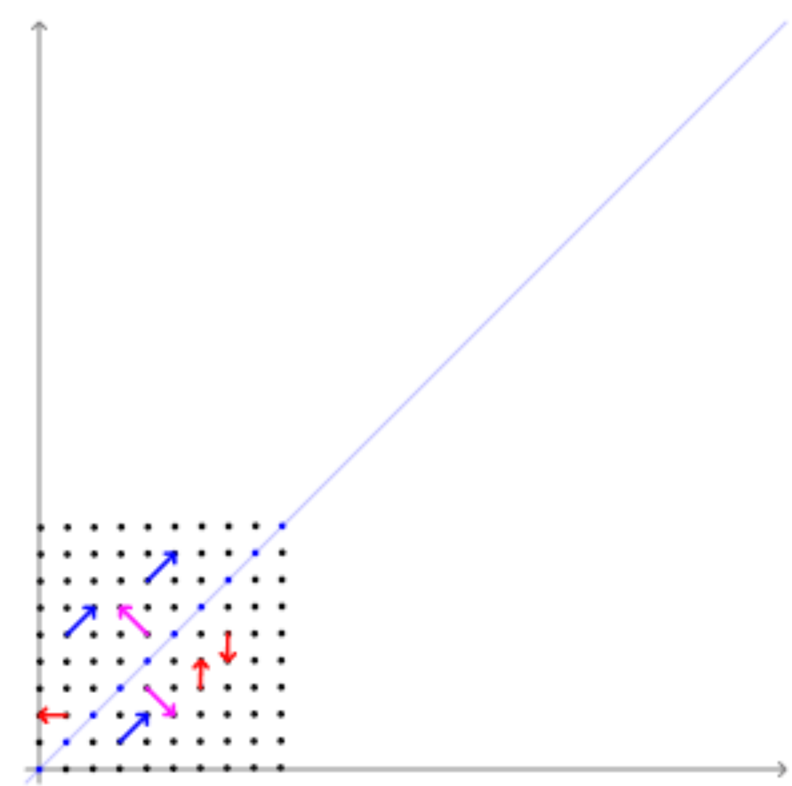

EXAMPLE: FIRST ORDER DISTORTION OF A CIRCULAR SOURCE

$$\beta_1^2 + \beta_2^2 = \beta^2$$

In the reference frame where A is diagonal:

$$\begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} 1-\kappa-\gamma & 0 \\ 0 & 1-\kappa+\gamma \end{pmatrix} \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}$$

$$\begin{aligned} \beta_1 &= (1-\kappa-\gamma)\theta_1 \\ \beta_2 &= (1-\kappa+\gamma)\theta_2 \end{aligned}$$

$$\beta^2 = \beta_1^2 + \beta_2^2 = (1-\kappa-\gamma)^2\theta_1^2 + (1-\kappa+\gamma)^2\theta_2^2$$

This is the equation of an ellipse with semi-axes:

$$a = \frac{\beta}{1-\kappa-\gamma} \qquad b = \frac{\beta}{1-\kappa+\gamma}$$

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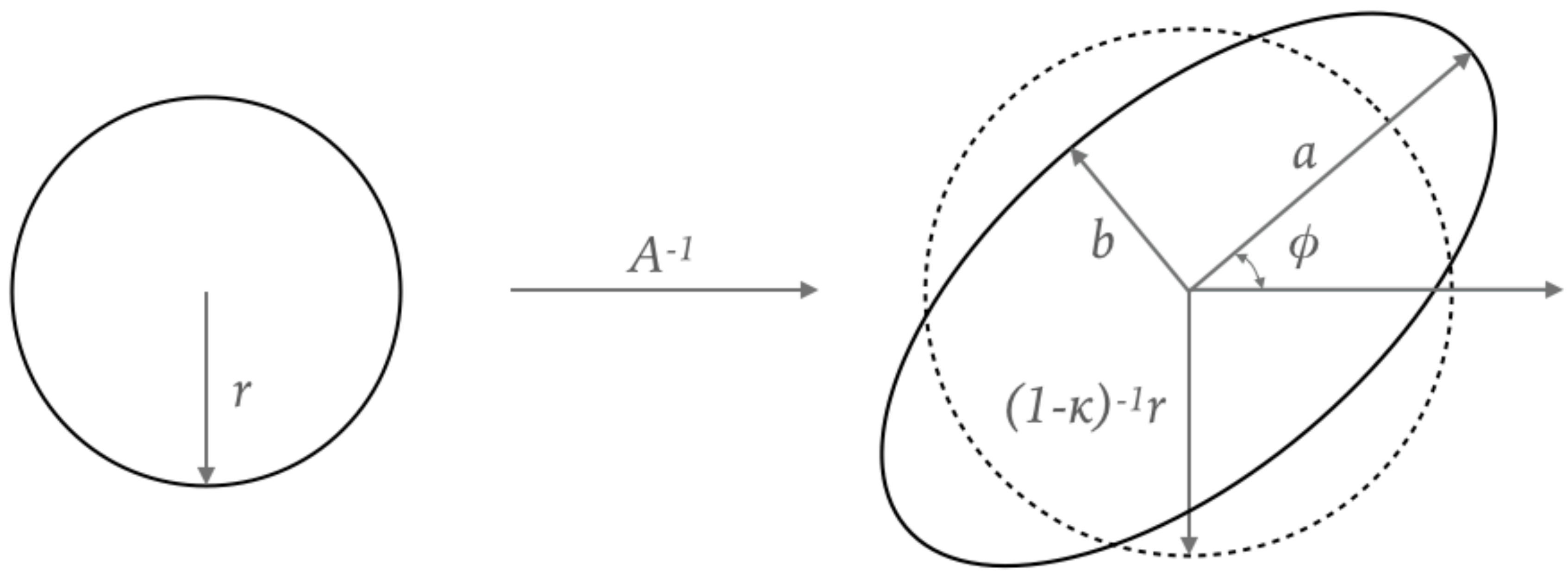

***convergence**: responsible for isotropic expansion or contraction*

***shear**: responsible for anisotropic distortion*

Ellipticity: $$e = \frac{a-b}{a+b} = \frac{\gamma}{1-\kappa} = g$$

ON THE SPIN-2 NATURE OF SHEAR: QUIZ

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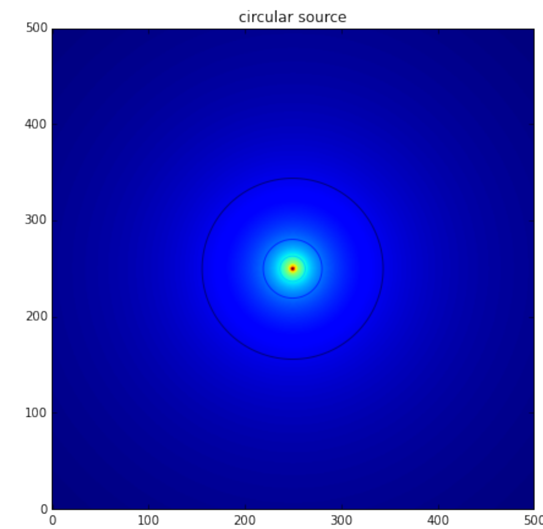

ON THE SPIN-2 NATURE OF SHEAR: QUIZ

- Let's consider a circular source
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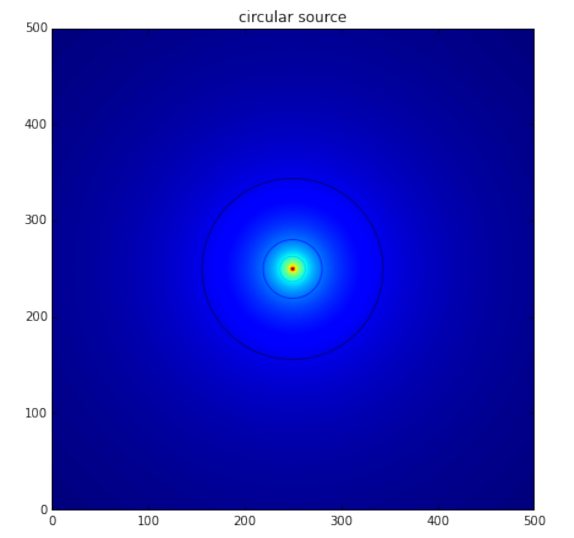

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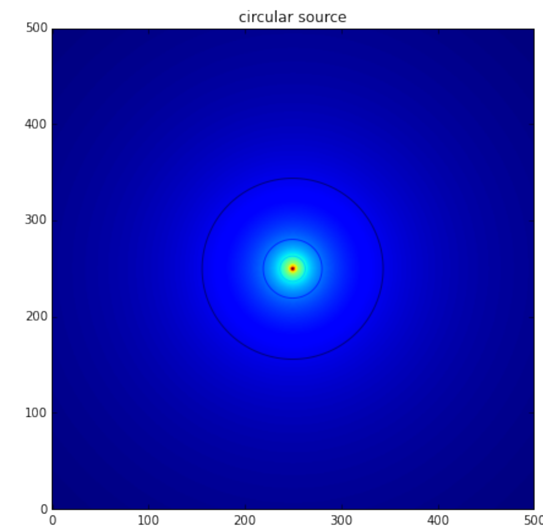

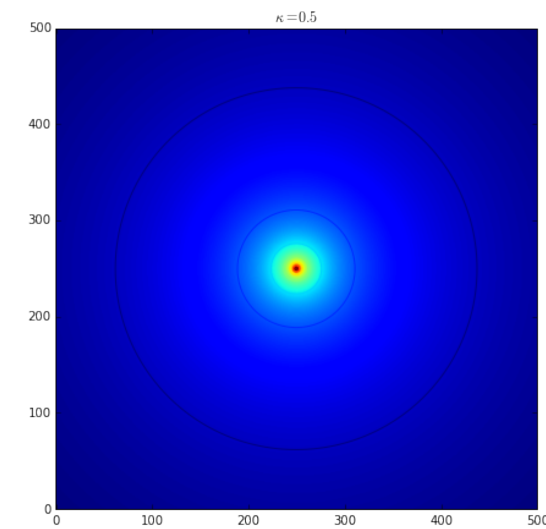

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- Let's consider a circular source
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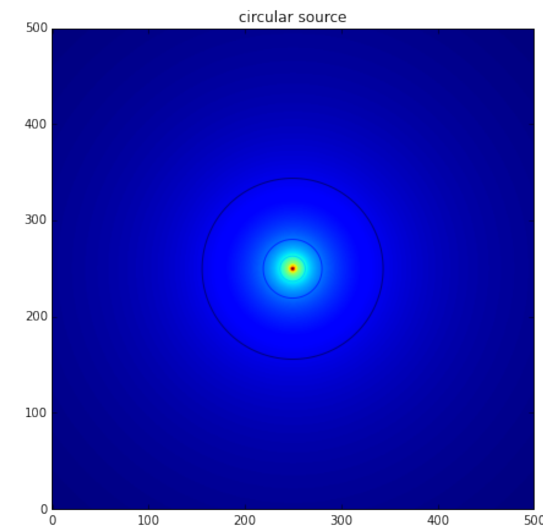
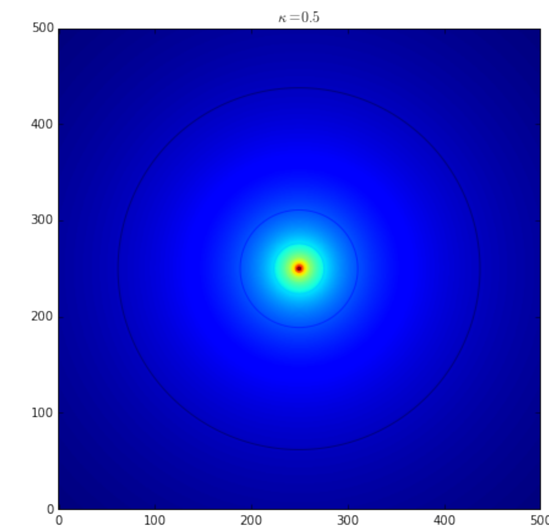
- Now, assume that $\gamma_1>0$ and $\gamma_2=0$. How is the image distorted?





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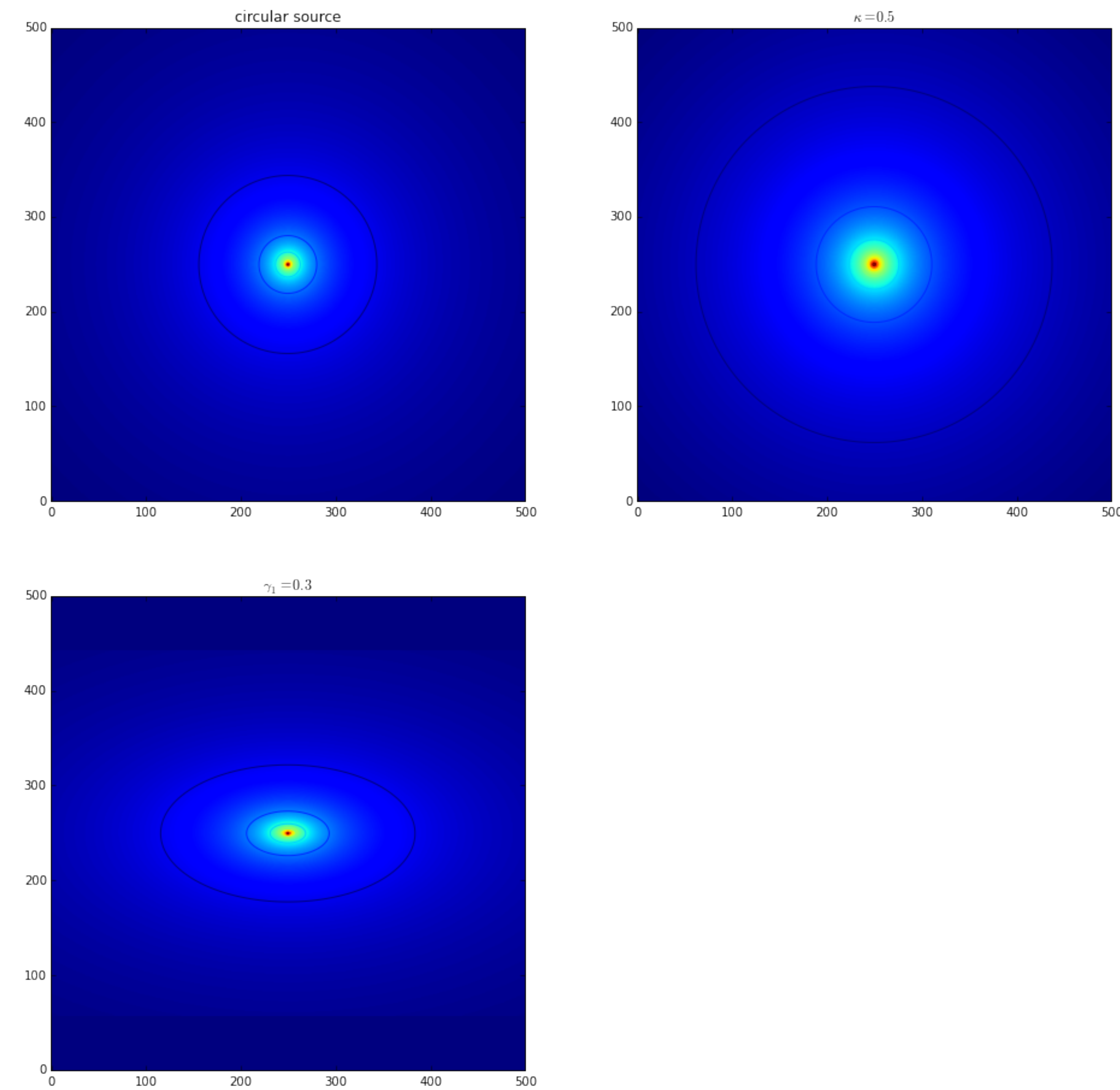

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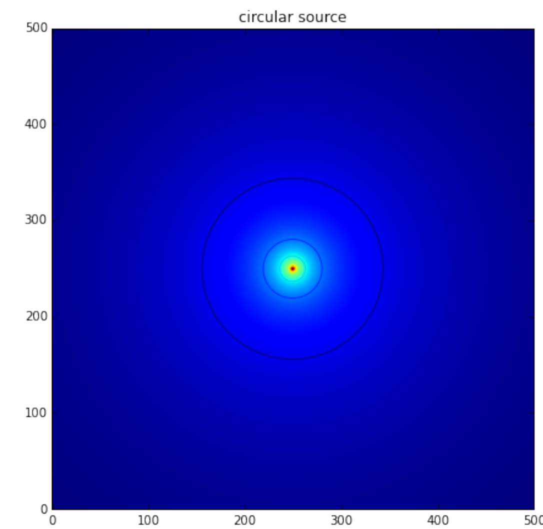

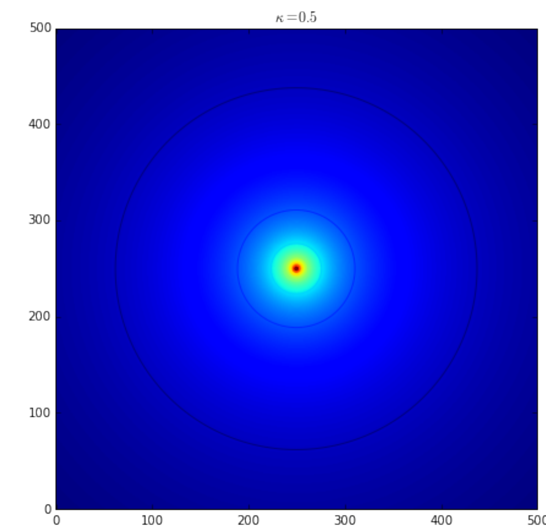

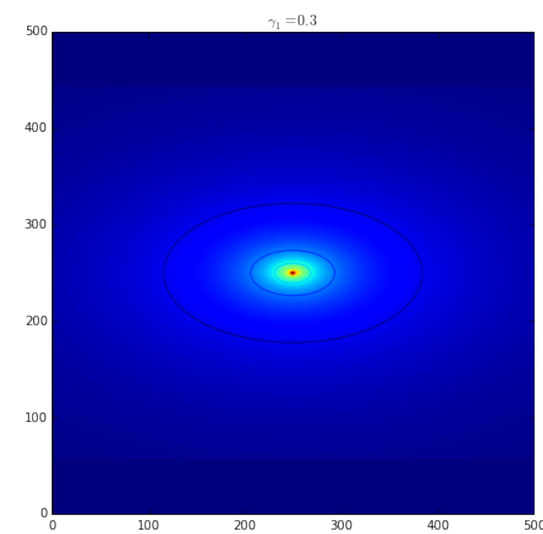

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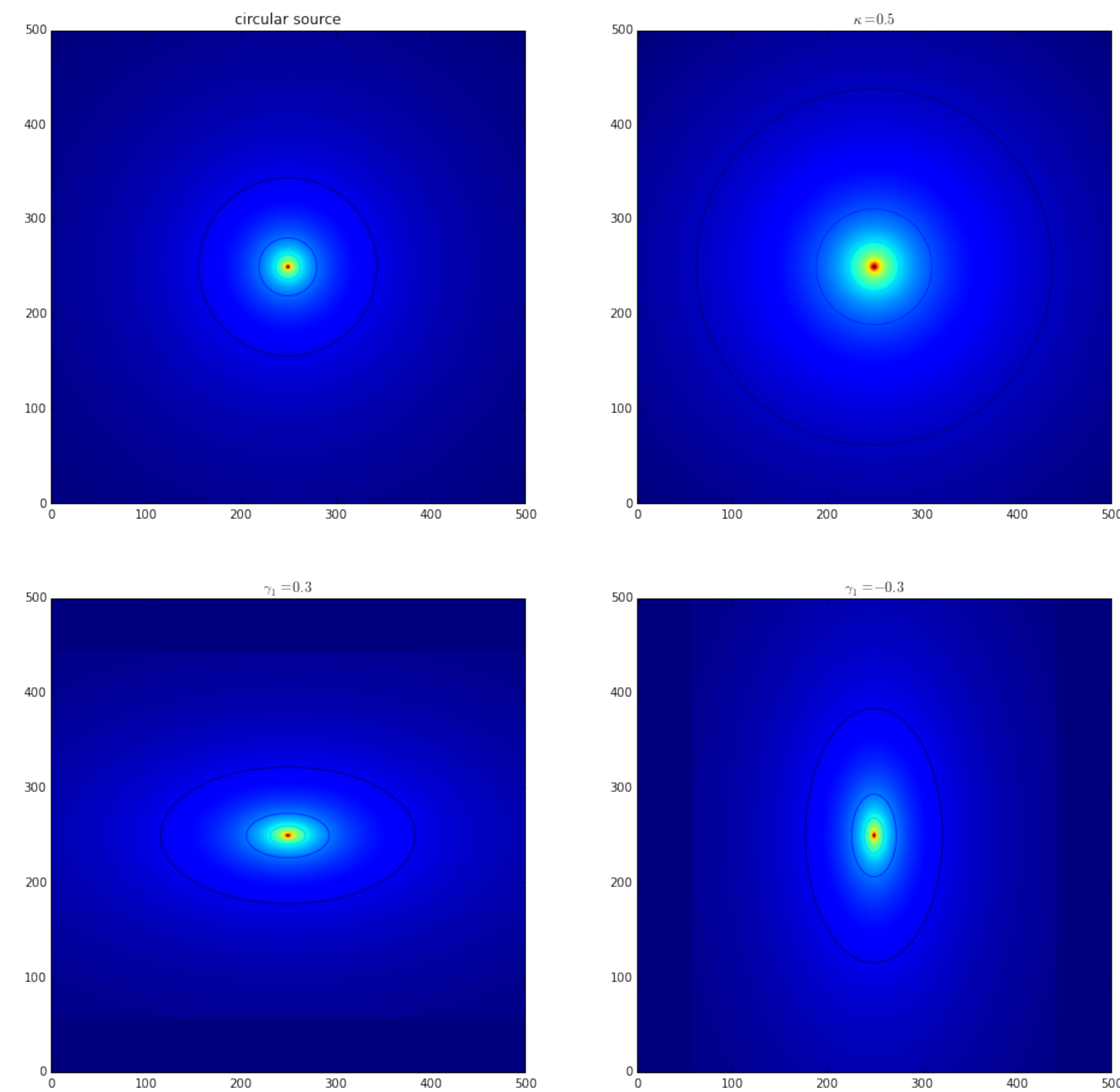

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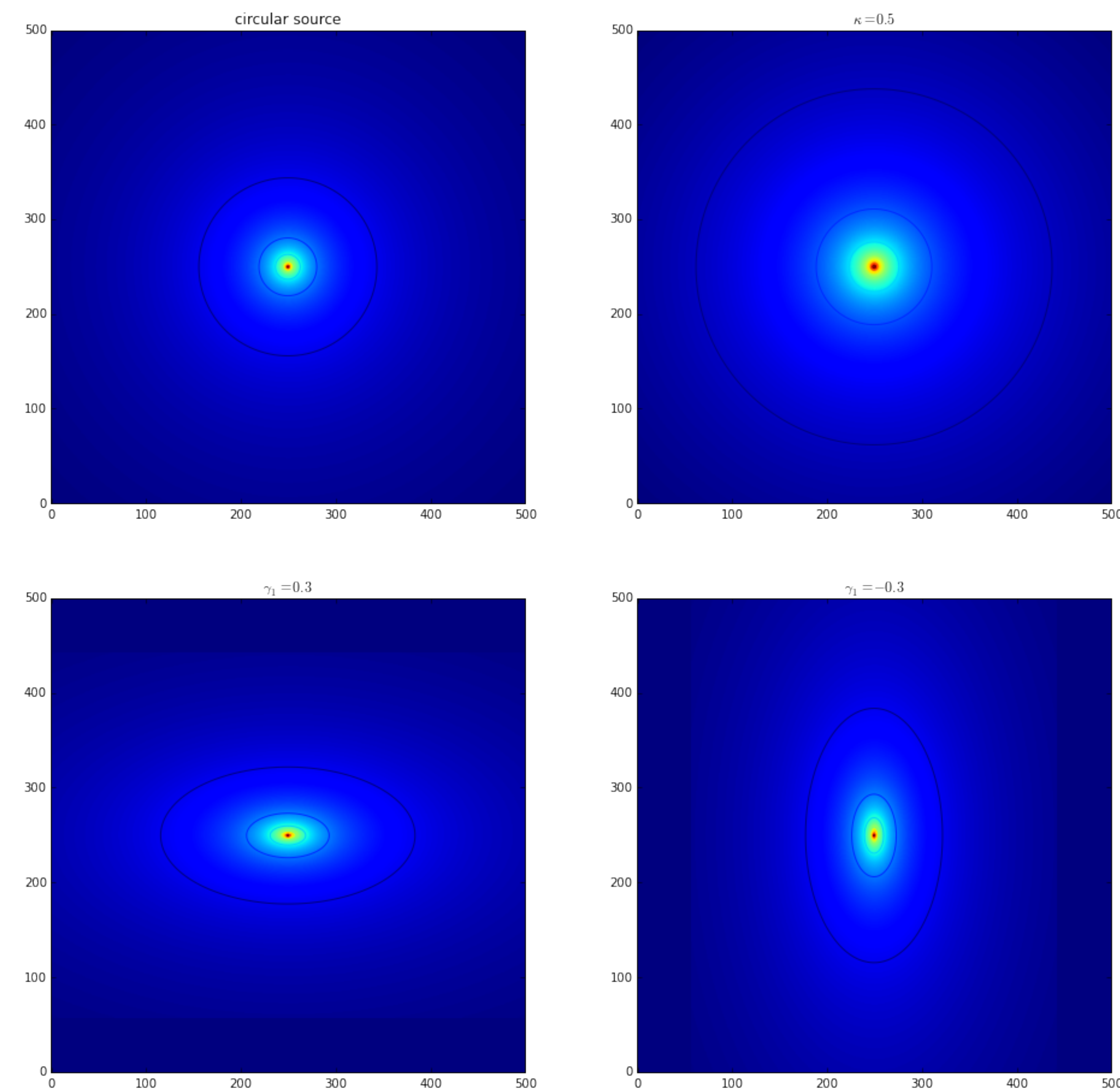

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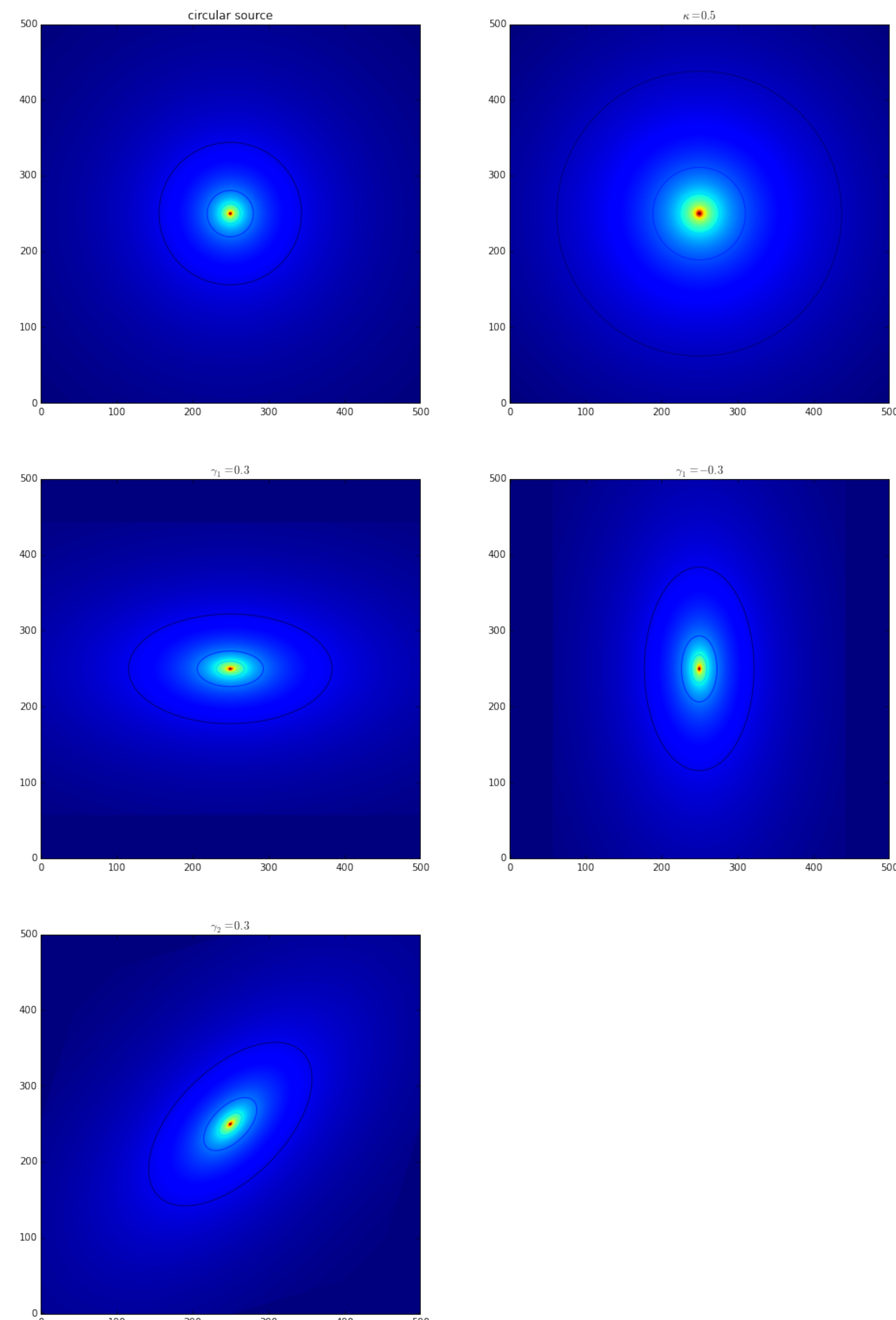

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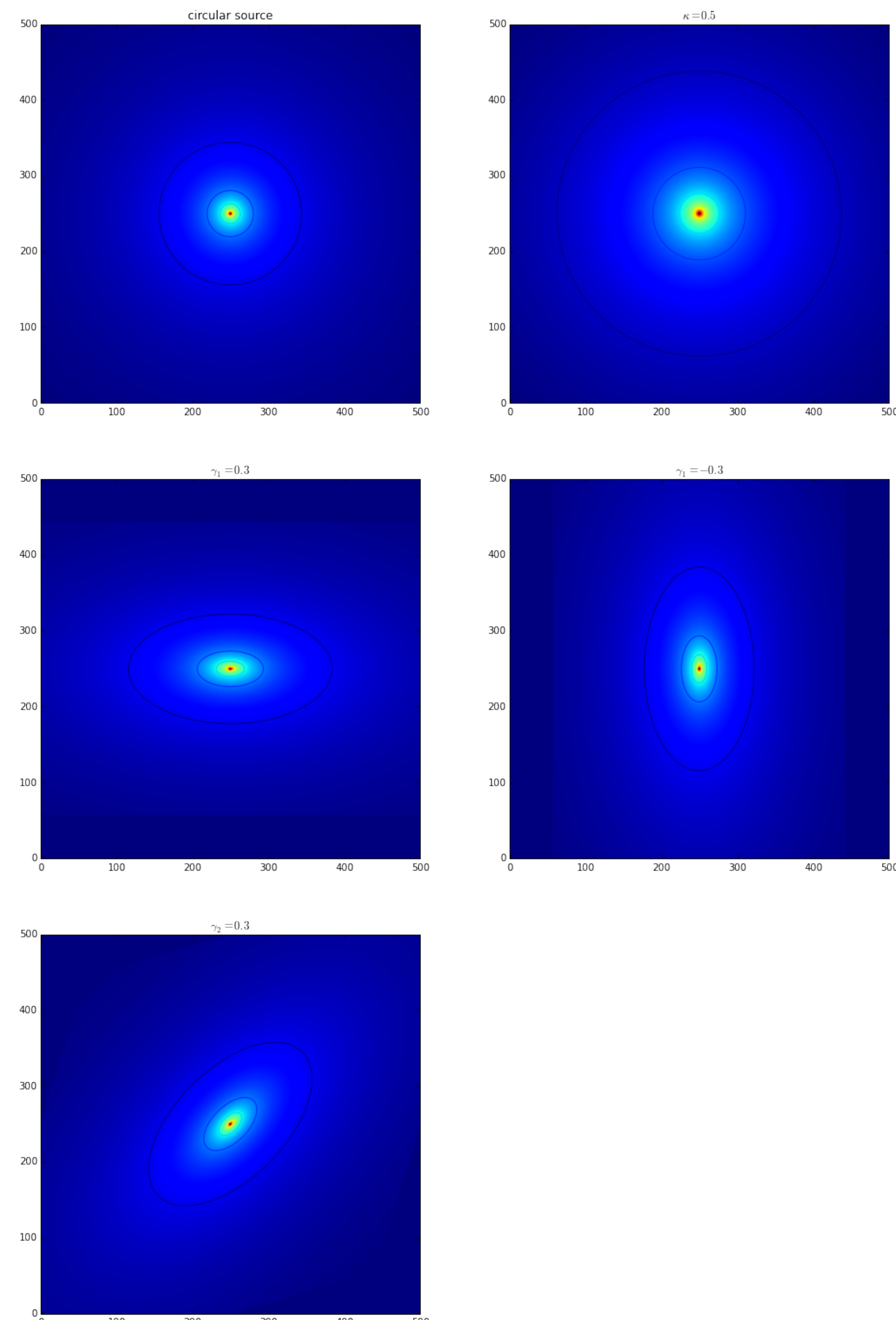

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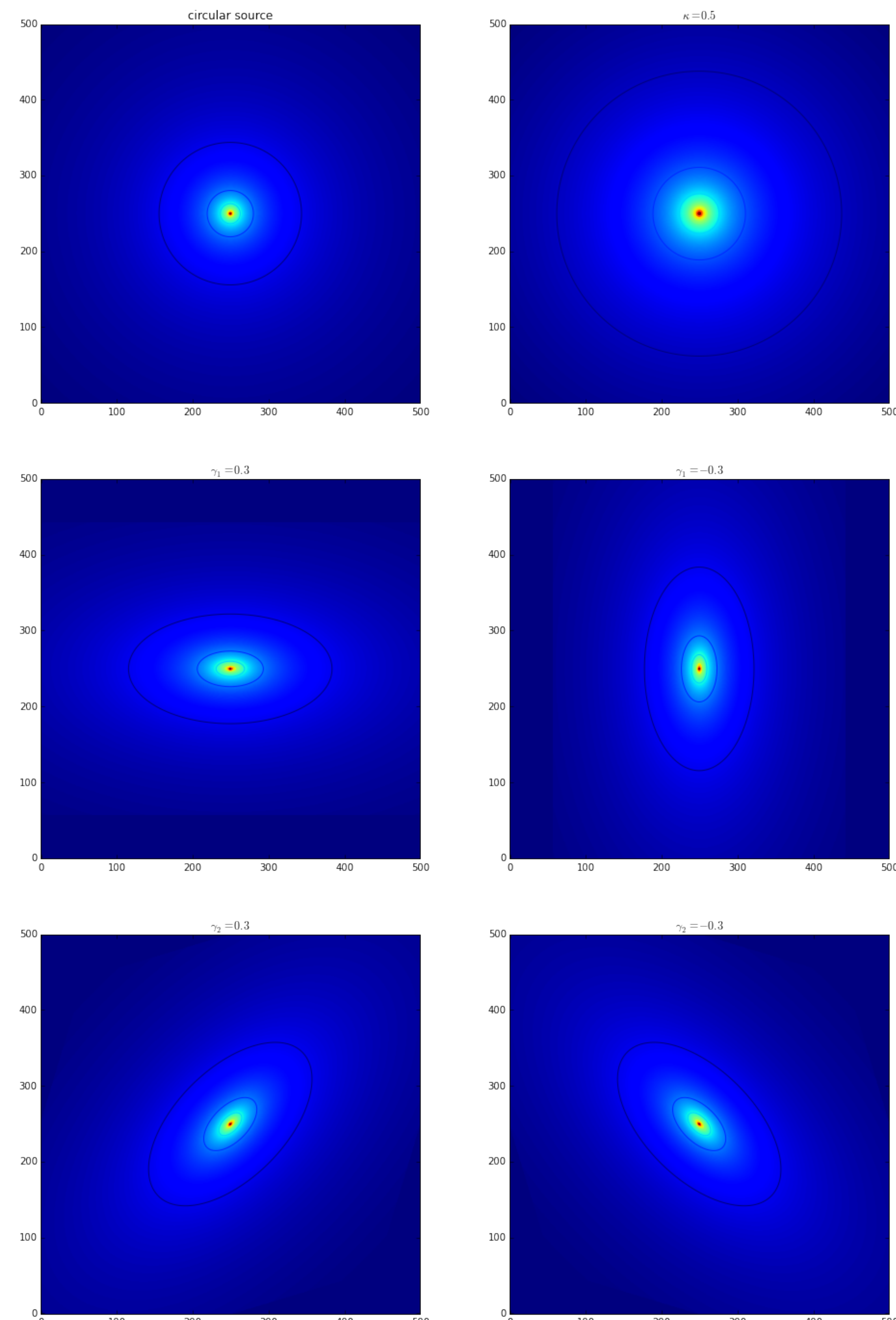
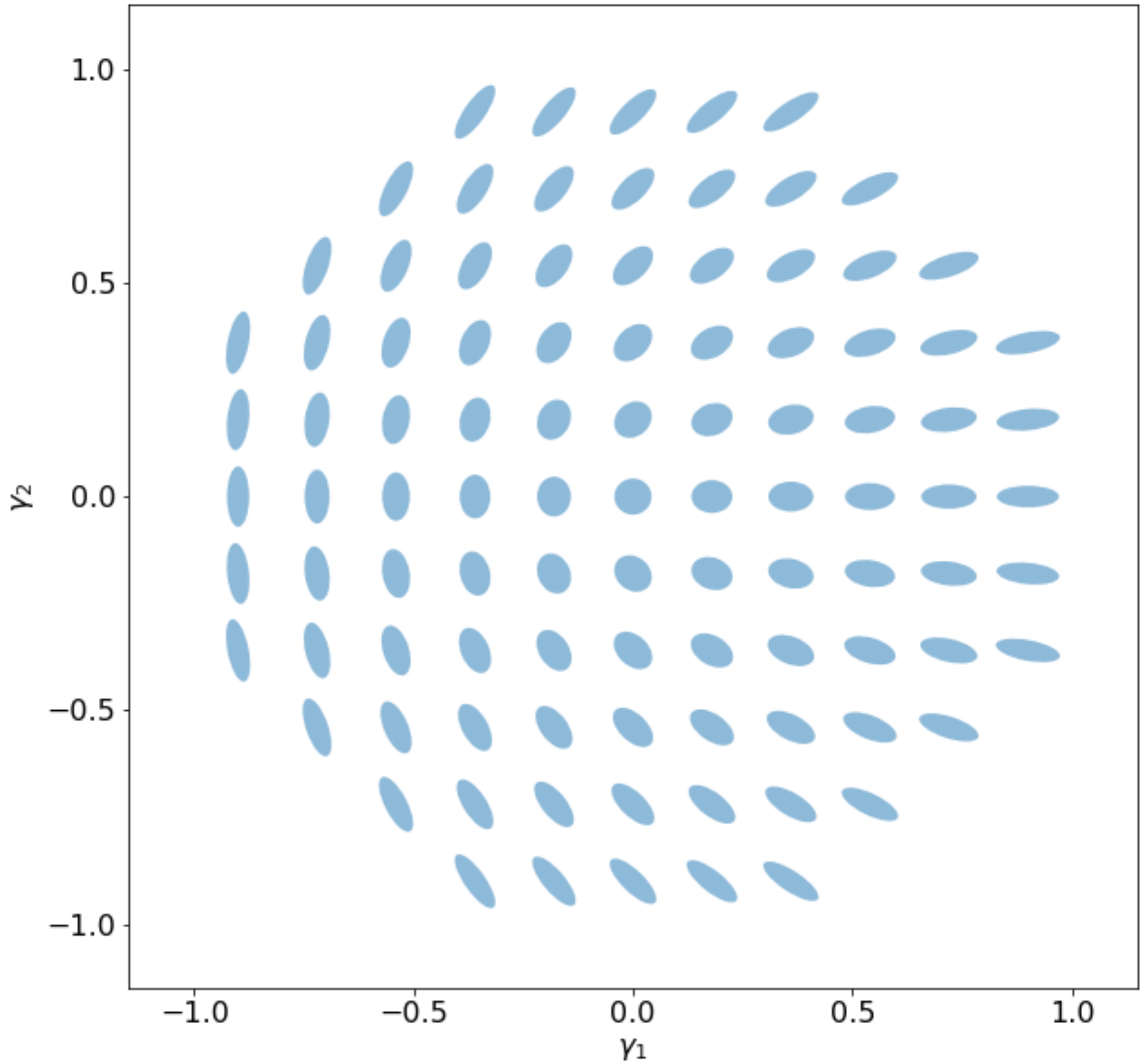


SHEAR DISTORTIONS



DEPENDENCE ON REDSHIFT

We have seen that the lensing potential, the deflection angle, the convergence, the shear… depend on a combination of distances.

For example:

$$\hat{\Psi}(\vec{\theta}) = \frac{D_{LS}}{D_L D_S}\frac{2}{c^2}\int_0^\infty \Phi(D_L\vec{\theta})dz$$

$$\vec{\alpha}(\vec{\theta}) = \vec{\nabla}\hat{\Psi}(\vec{\theta}) = \frac{4G}{c^2}\frac{D_{LS}}{D_L D_S}\int \frac{\vec{\theta}-\vec{\theta}'}{|\vec{\theta}-\vec{\theta}'|^2}\Sigma(\vec{\theta}')d\theta'^2$$

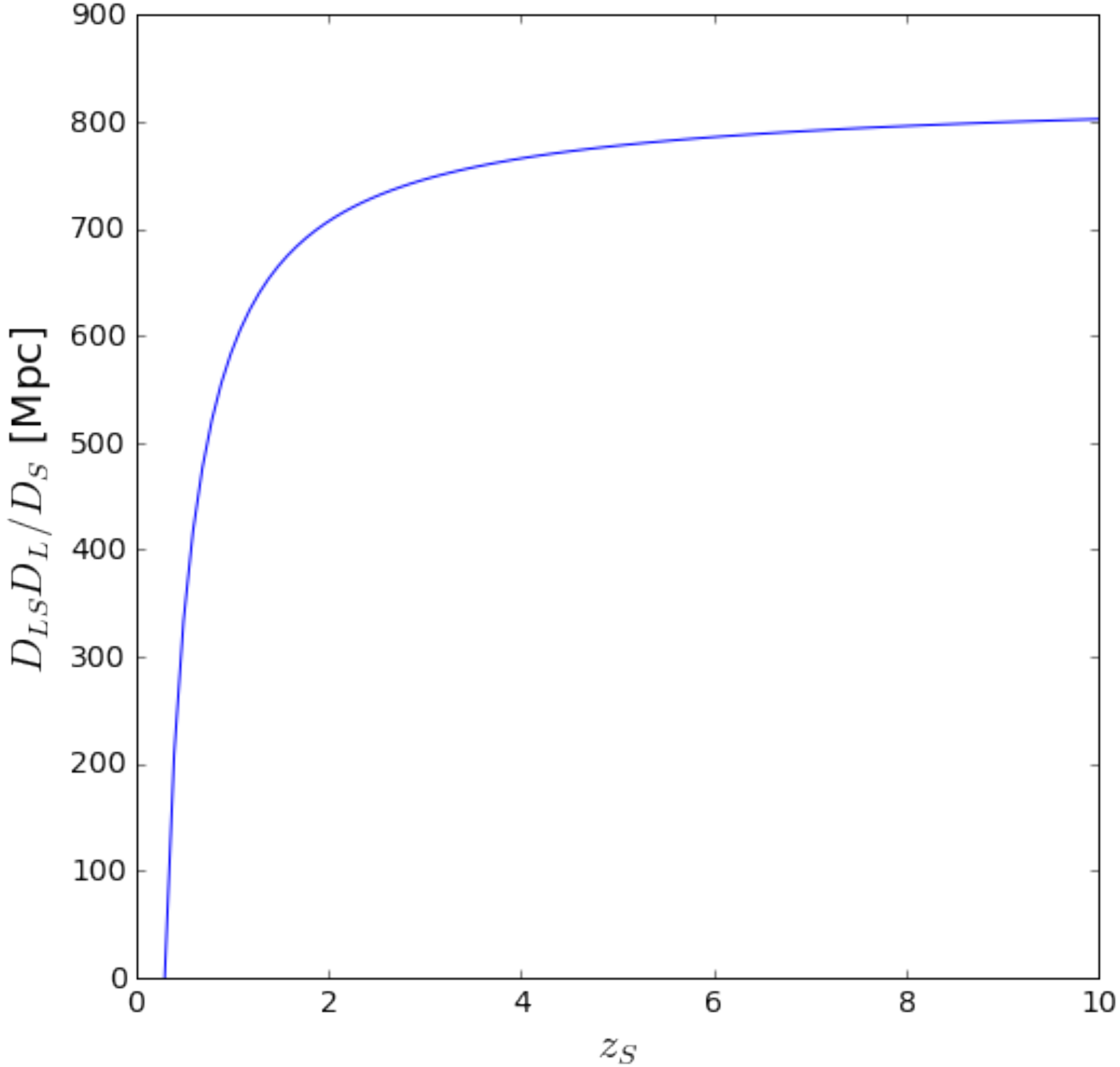
$$\kappa(\vec{\theta}) = \frac{\Sigma(\vec{\theta})}{\Sigma_{cr}} = \frac{1}{2}\Delta_\theta\hat{\Psi}(\vec{\theta}) \qquad \gamma_1 = \frac{1}{2}(\hat{\Psi}_{11} - \hat{\Psi}_{22}) \qquad \gamma_2 = \hat{\Psi}_{12} = \hat{\Psi}_{21} \qquad \Sigma_{cr} = \frac{c^2}{4\pi G}\frac{D_S}{D_L D_{LS}}$$

Every spatial derivative of Ψ introduces a factor D_L.

The distance ratio $D_{LS}D_L/D_S$ is called "lensing distance".

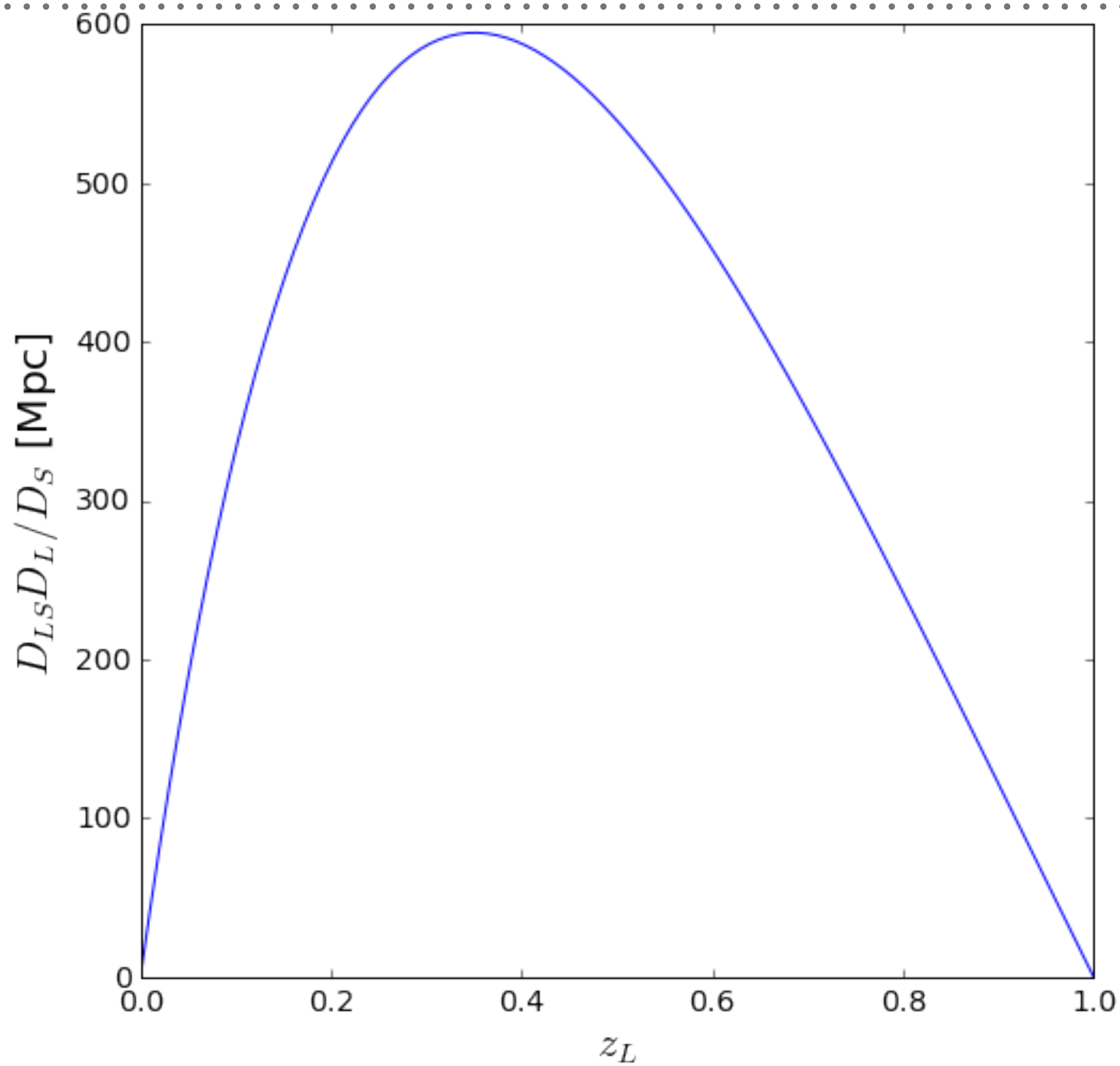
Both the shear and the convergence, being second derivatives of the lensing potential, scale as the lensing distance

HOW DOES THE LENSING DISTANCE SCALE WITH SOURCE REDSHIFT?



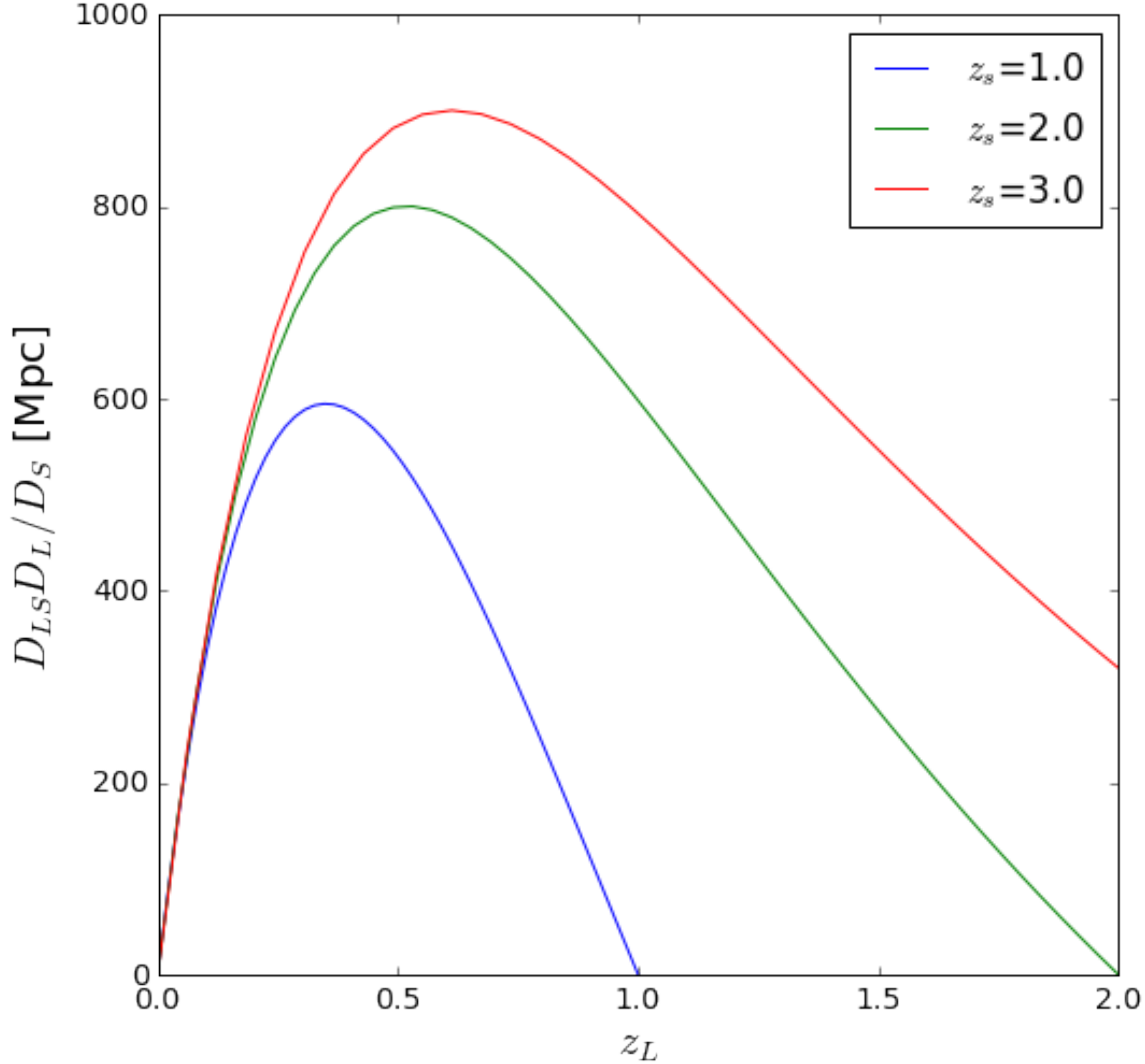
Note that if the lensing distance grows, the critical surface density decreases, the convergence and the shear grow!

HOW DOES THE LENSING DISTANCE SCALE WITH LENS REDSHIFT?



The lensing distance peaks at ~half way between the source and the observer, meaning that there is an optimal distance where the lens produces its largest effects.

HOW DOES THE LENSING DISTANCE SCALE WITH LENS REDSHIFT?



Of course, the peak moves to larger distances as the distance to the source increases.

CONSERVATION OF SURFACE BRIGHTNESS

*The source **surface brightness** is*

$$I_\nu = \frac{dE}{dtdAd\Omega d\nu}$$

In phase space, the radiation emitted is characterized by the density

$$f(\vec{x},\vec{p},t) = \frac{dN}{d^3x d^3p}$$

In absence of photon creations or absorptions, f is conserved (Liouville theorem)

$$dN = \frac{dE}{h\nu} = \frac{dE}{cp}$$

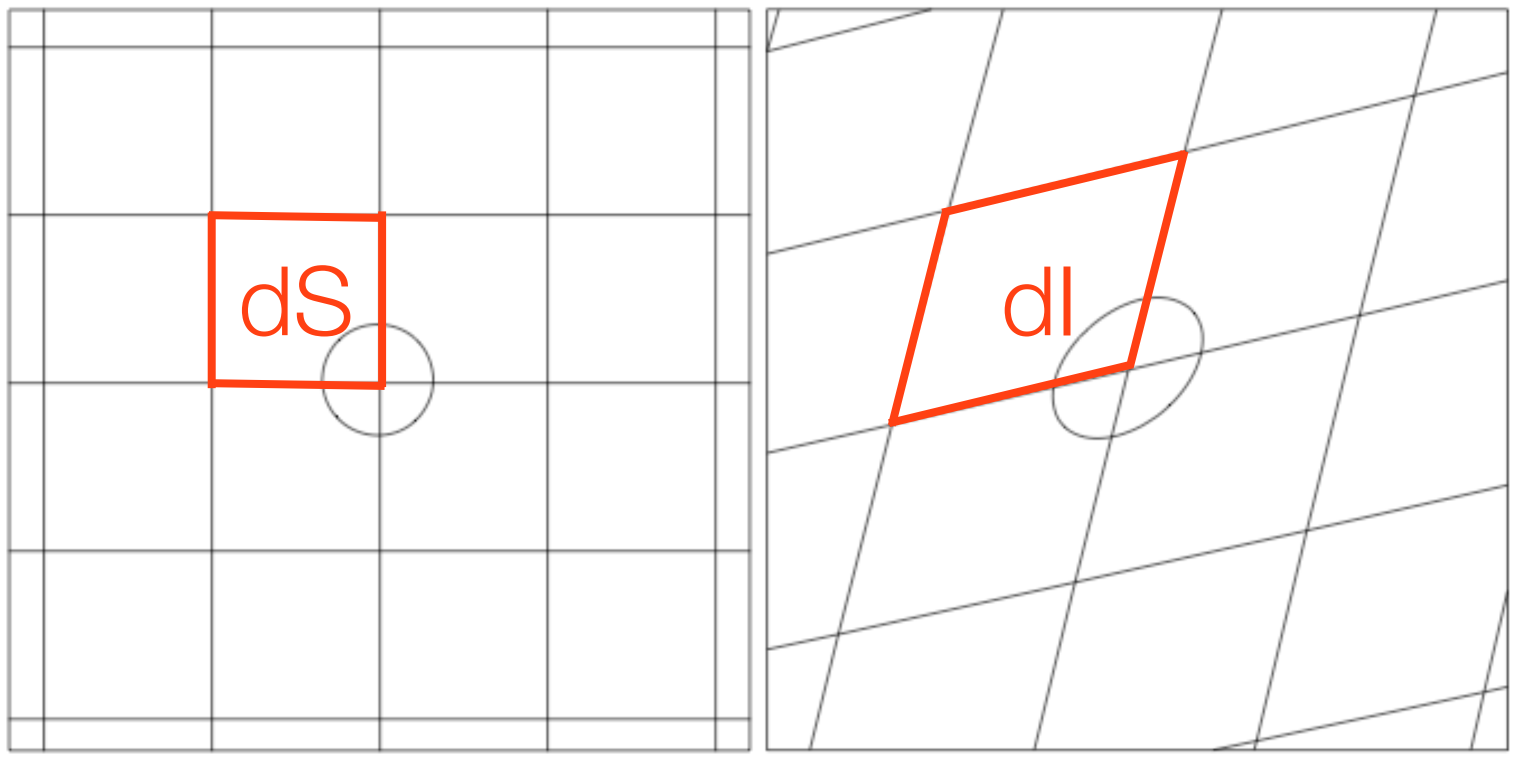
$$d^3x = cdtdA$$

$$d^3\vec{p} = p^2 dp d\Omega$$

$$f(\vec{x},\vec{p},t) = \frac{dN}{d^3x d^3p} = \frac{dE}{hcp^3 dAdtd\nu d\Omega} = \frac{I_\nu}{hcp^3}$$

Since GL does not involve creation or absorption of photons, neither it changes the photon momenta (achromatic!), surface brightness is conserved!

MAGNIFICATION



Kneib & Natarajan (2012)

$$\mu(\vec{\theta}) = \frac{dI}{dS} = \frac{d^2\theta}{d^2\beta} = \det A^{-1}(\vec{\theta})$$

$$F_\nu = \int_I I_\nu(\vec{\theta})d^2\theta = \int_S I_\nu^S[\vec{\beta}(\vec{\theta})]\mu[\vec{\beta}(\vec{\theta})]d^2\beta$$

Lensing changes the amount of photons (flux) we receive from the source by changing the solid angle the source subtends

CRITICAL LINES AND CAUSTICS

Both convergence and shear are functions of position on the lens plane:

$\kappa = \kappa(\vec{\theta})$

$\gamma = \gamma(\vec{\theta})$

The determinant of the lensing Jacobian is

$$\det A(\vec{\theta}) = [1 - \kappa(\vec{\theta}) - \gamma(\vec{\theta})][1 - \kappa(\vec{\theta}) + \gamma(\vec{\theta})]$$

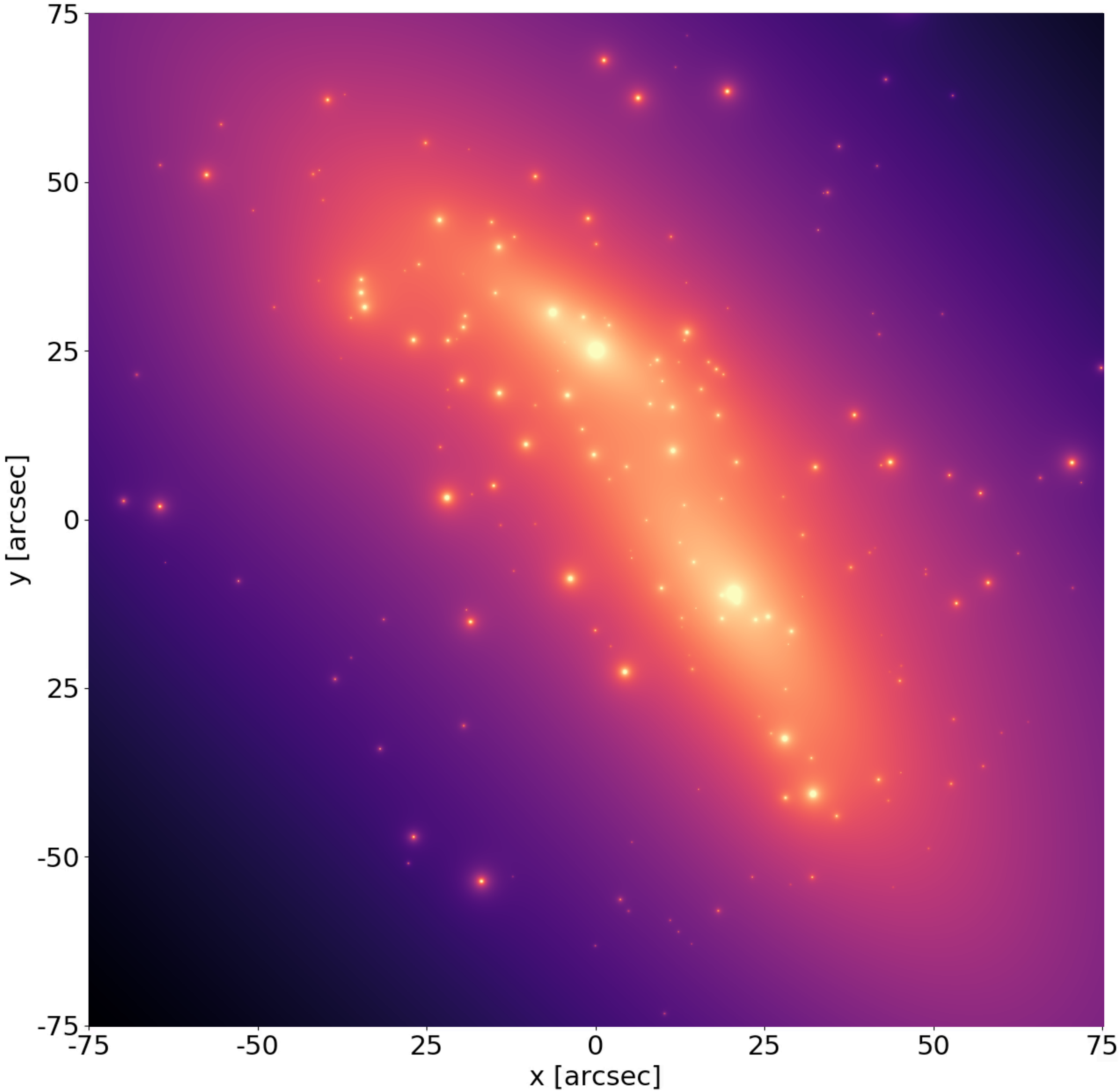
*The **critical lines** are the lines made of the points where the eigenvalues of the Jacobian are zero:*

$$1 - \kappa(\vec{\theta}_t) - \gamma(\vec{\theta}_t) = 0 \qquad \textit{tangential critical line}$$
$$1 - \kappa(\vec{\theta}_r) + \gamma(\vec{\theta}_r) = 0 \qquad \textit{radial critical line}$$

Along these lines the magnification diverges!

*Via the lens equations, they are mapped into the **caustics**…* $\vec{\beta}_{t,r} = \vec{\theta}_{t,r} - \vec{\alpha}(\vec{\theta}_{t,r})$

Model of MACS0416 by Caminha, MM, et al. (2016)



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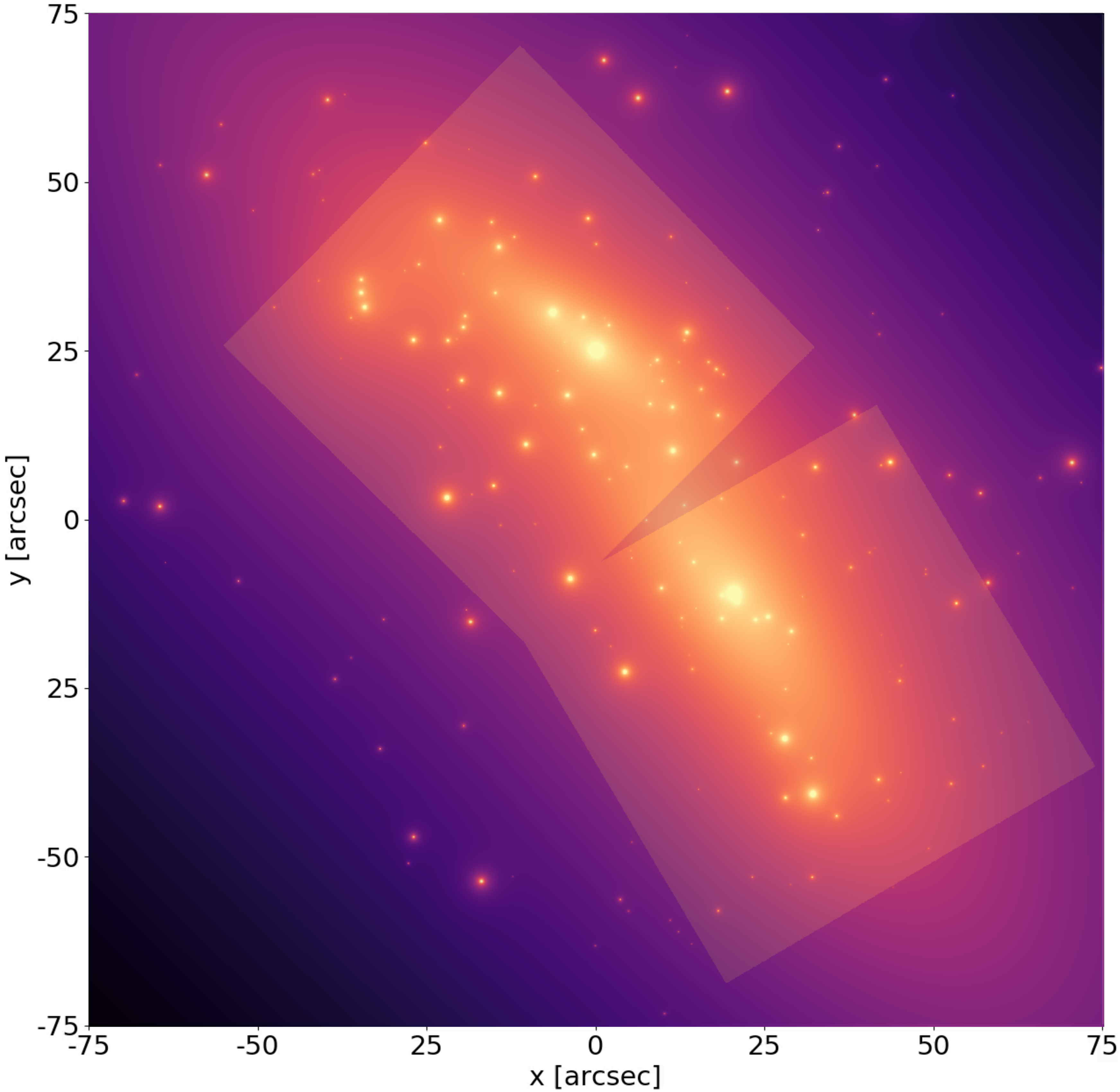

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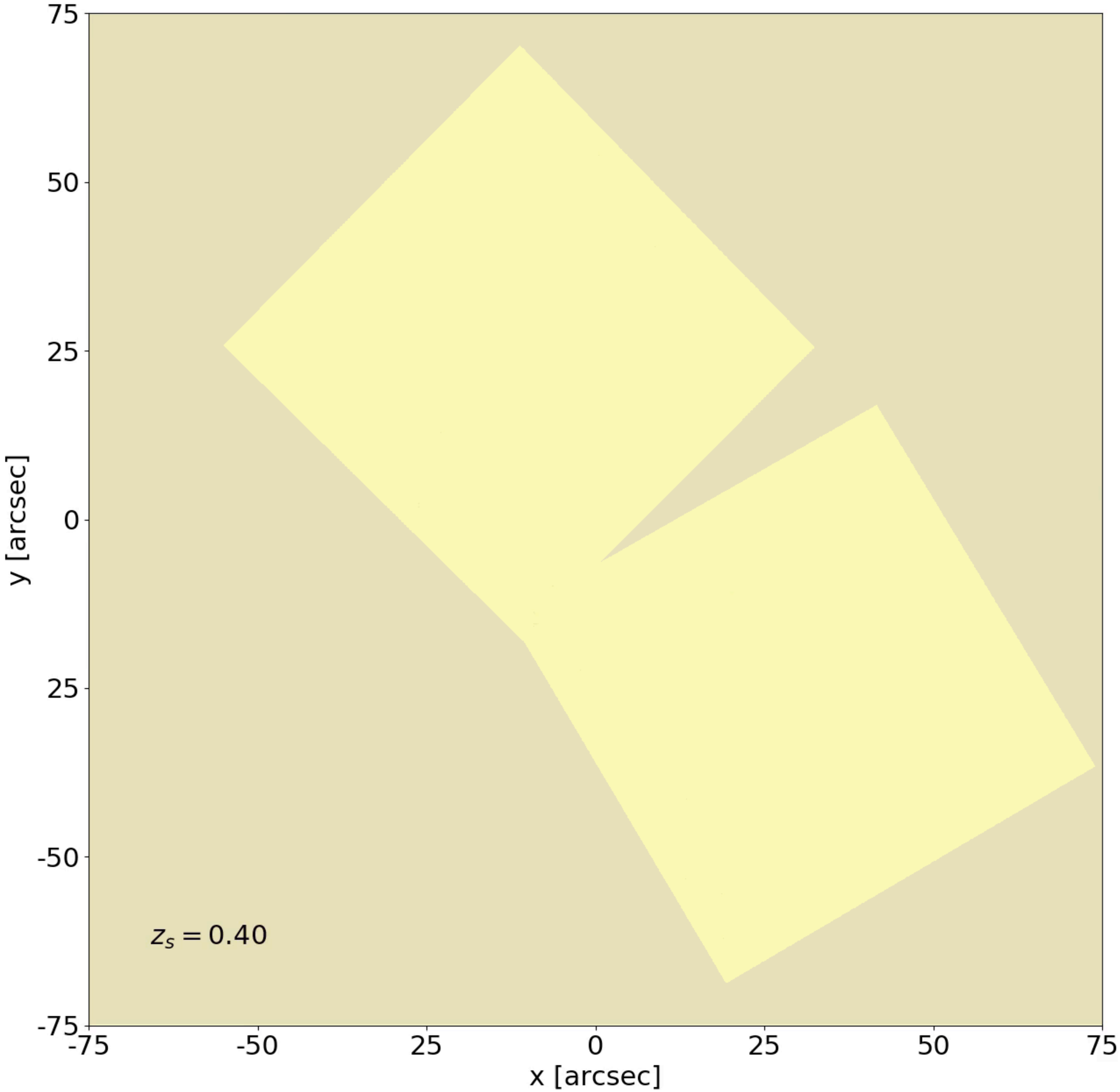
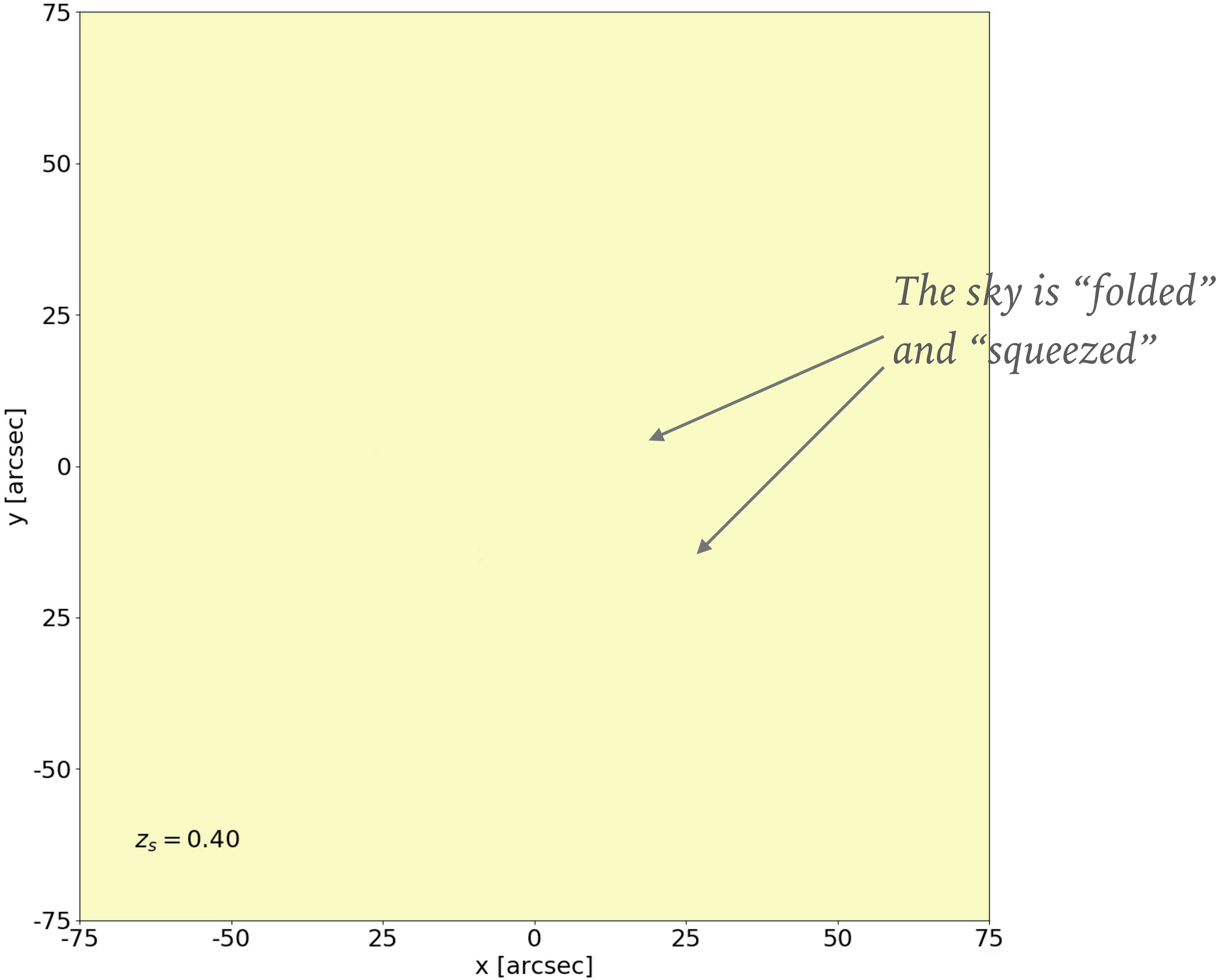
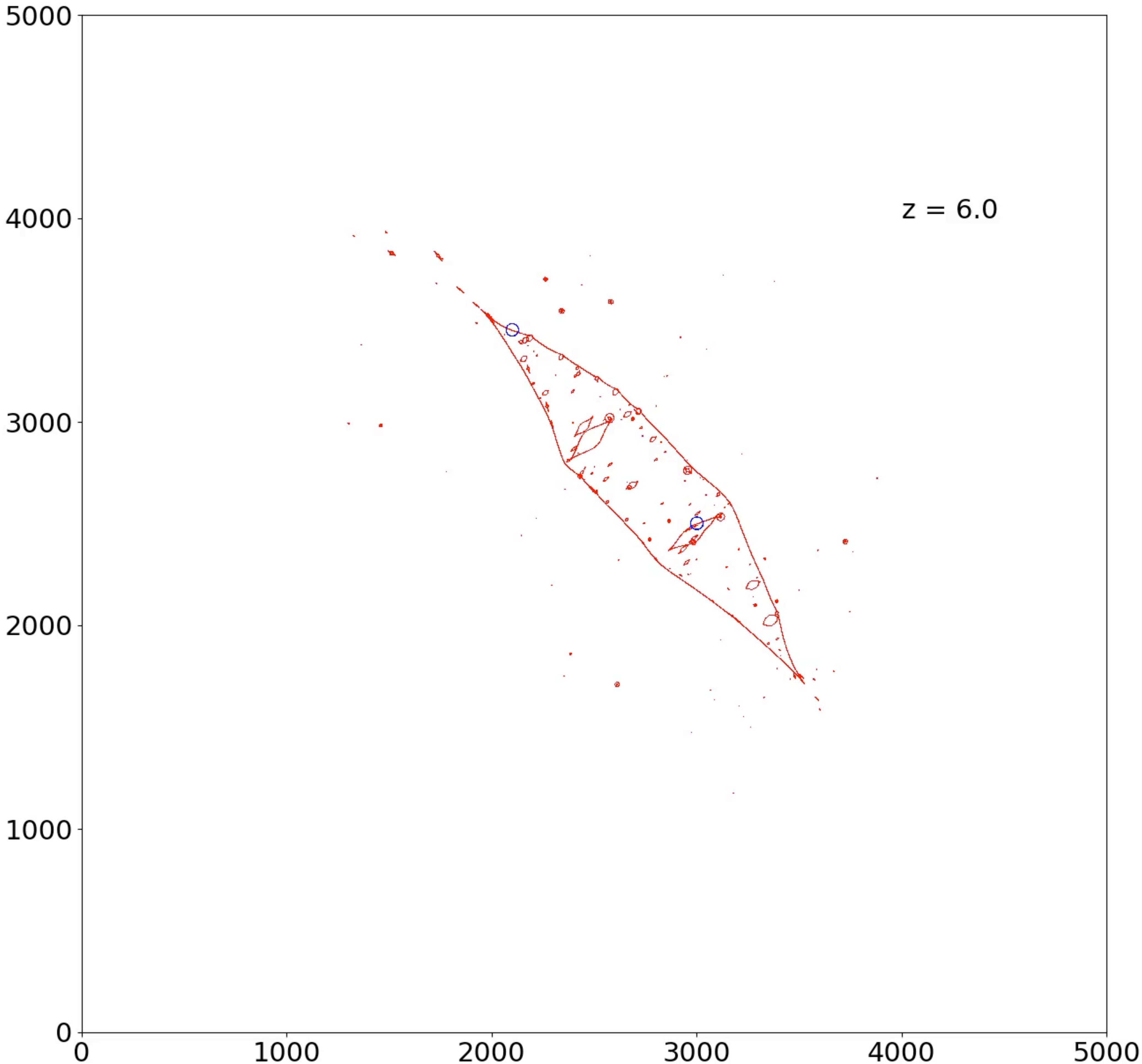


Model of MACS0416 by Caminha, MM, et al. (2016)





z = 6.0
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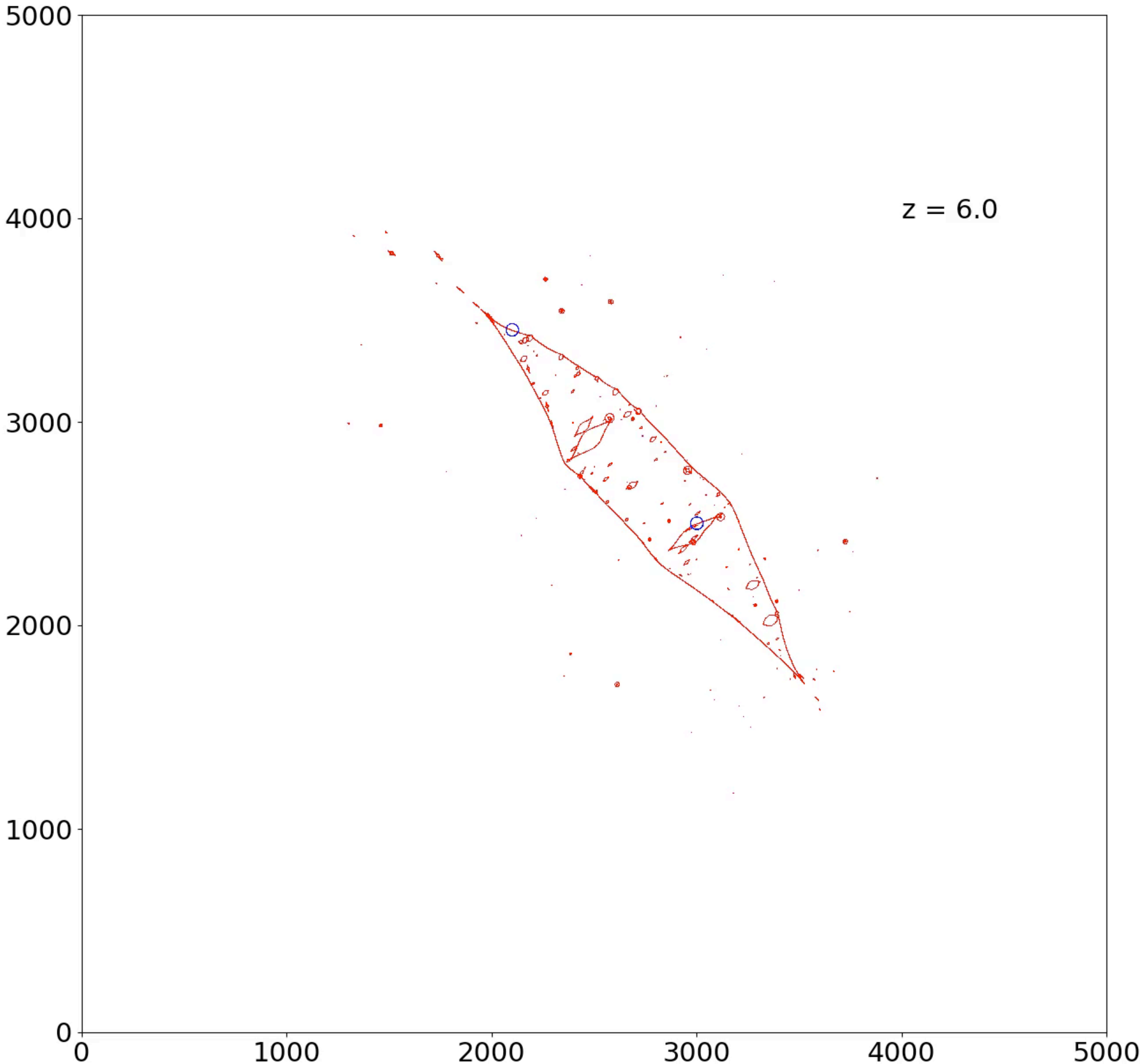
4000
3000
2000
1000
0
0
1000
2000
3000
4000
5000



z = 6.0
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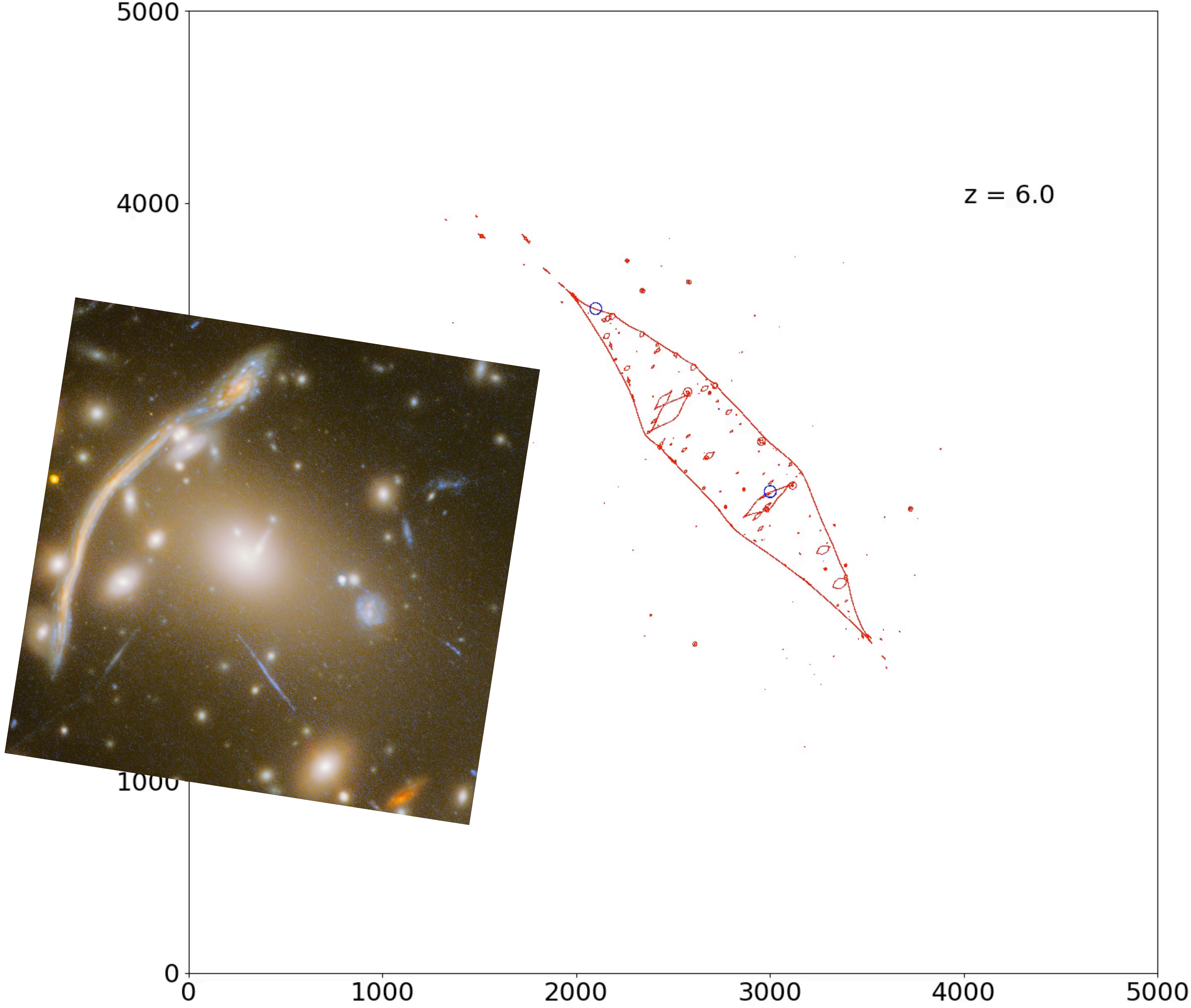

z = 6.0
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0
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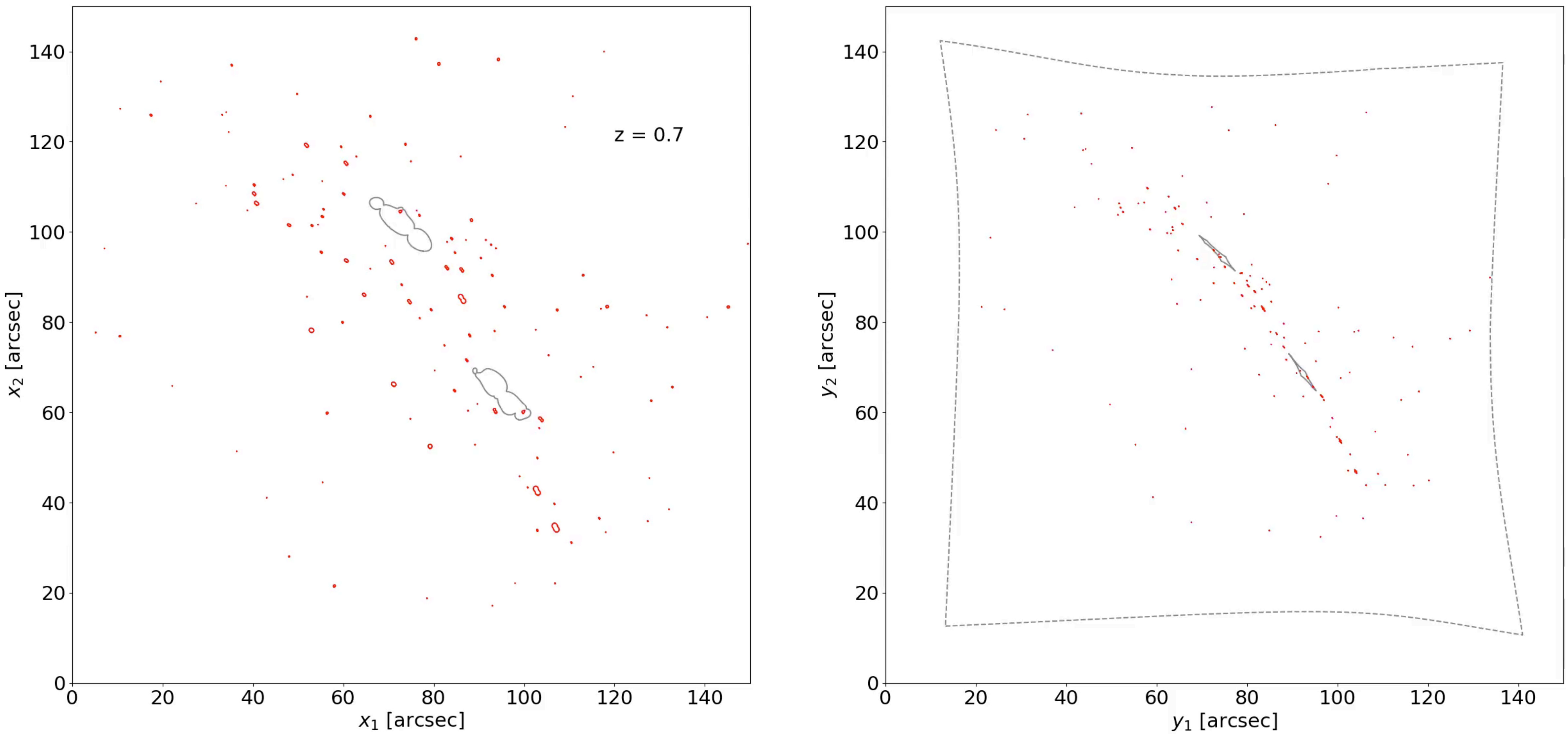
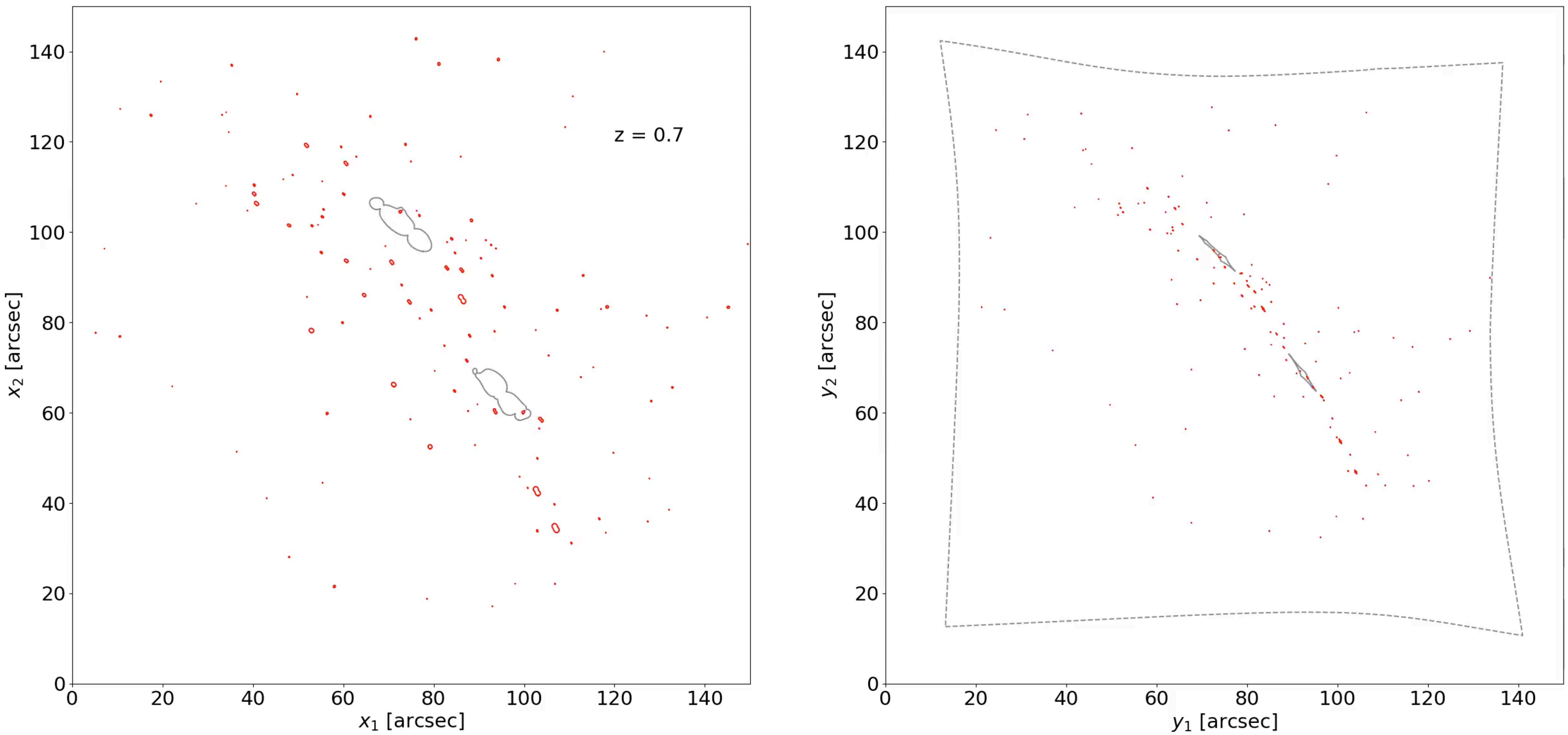


The size of the critical lines and caustics changes as a function of the source redshift! This is because the deflection angle between two redshifts changes by a factor

$$\Xi(z_L, z_{S_1}, z_{S_2}) = \frac{D_{LS_2}}{D_{S_2}} \frac{D_{S_1}}{D_{LS_1}}$$



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SAMPLED VOLUME