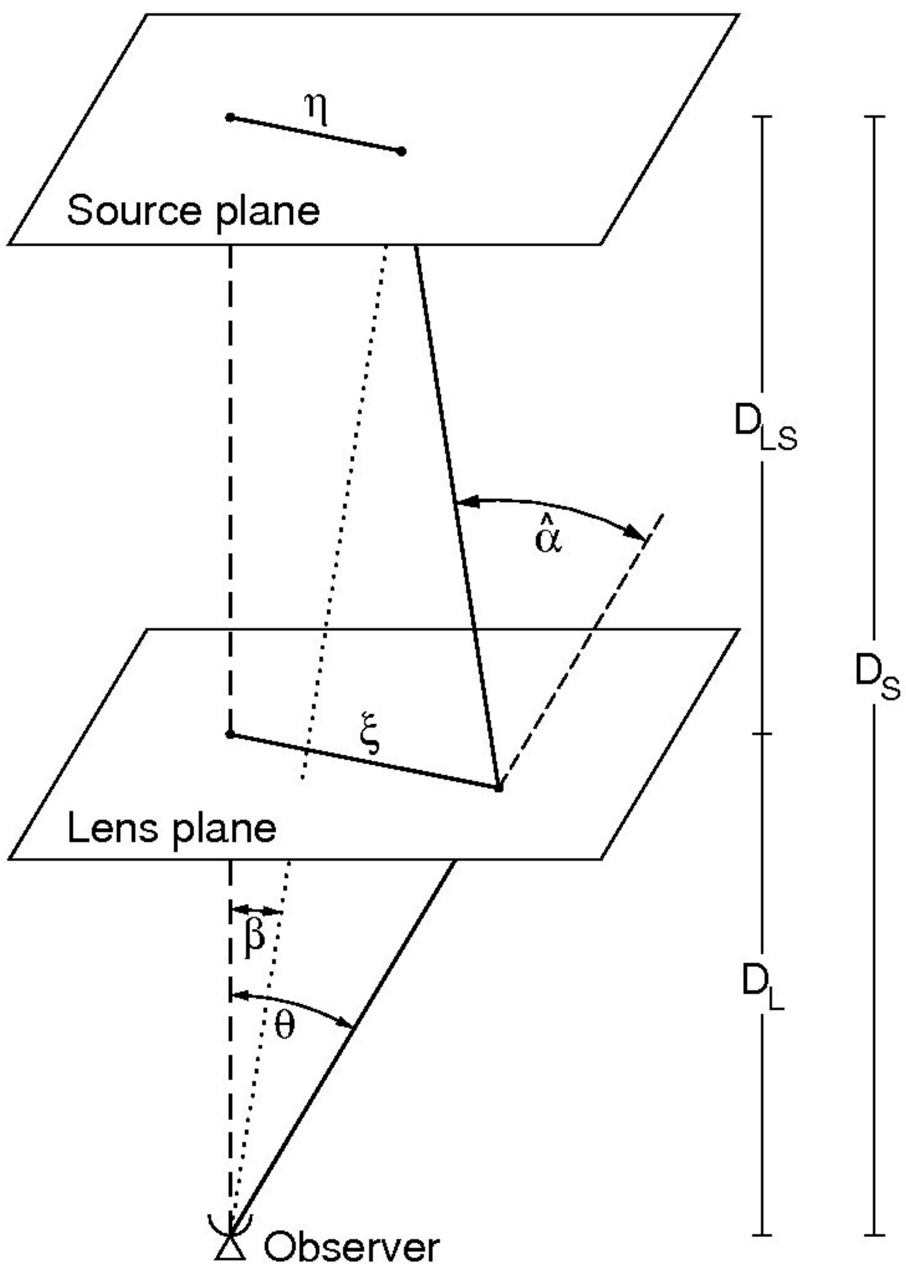
GRAVITATIONAL LENSING

4 - LENS MAPPING, MAGNIFICATION

Massimo Meneghetti

AA 2017-2018

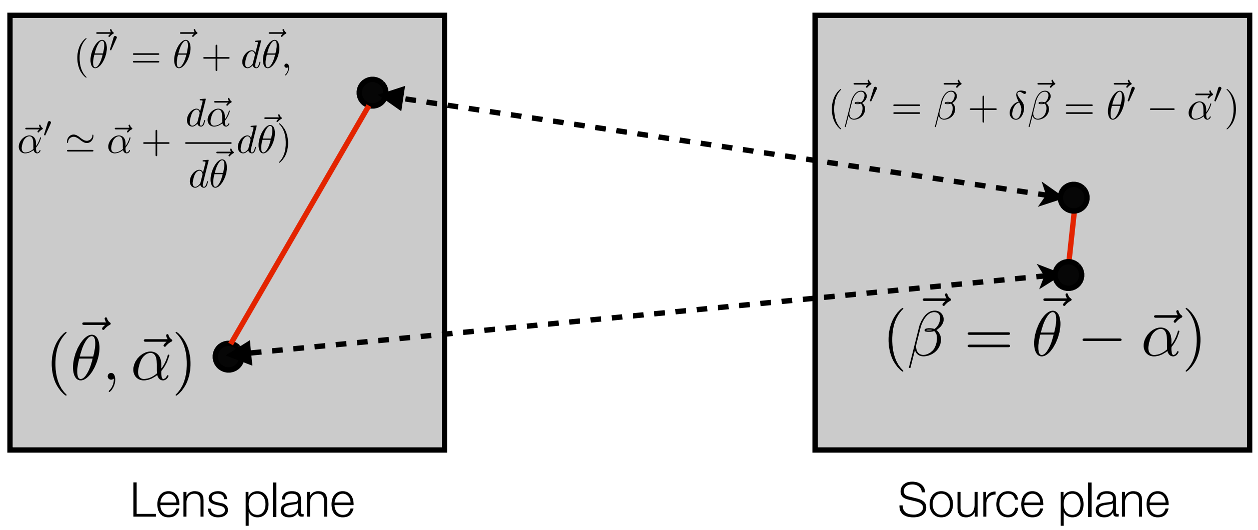
LENS MAPPING (FIRST ORDER)



- *we derived the lens equation*

$$\vec{\beta} = \vec{\theta} - \frac{D_{LS}}{D_S}\hat{\vec{\alpha}}(\vec{\theta}) = \vec{\theta} - \vec{\alpha}(\vec{\theta})$$

- *Assuming that the d.a. does not vary significantly over the scale dΘ:*



$$(\vec{\beta}' - \vec{\beta}) = \left(I - \frac{d\vec{\alpha}}{d\vec{\theta}}\right)(\vec{\theta}' - \vec{\theta}) = A(\vec{\theta}' - \vec{\theta})$$

LENS MAPPING (FIRST ORDER)

$$A \equiv \frac{\partial \vec{\beta}}{\partial \vec{\theta}} = \left(\delta_{ij} - \frac{\partial \alpha_i(\vec{\theta})}{\partial \theta_j} \right) = \left(\delta_{ij} - \frac{\partial^2 \hat{\Psi}(\vec{\theta})}{\partial \theta_i \partial \theta_j} \right)$$

*A is called "**the lensing Jacobian**": it is a symmetric second rank tensor describing the first order mapping between lens and source planes.*

This tensor can be written as the sum of an isotropic part, proportional to its trace, and an anisotropic, traceless part.

$$A_{iso,i,j} = \frac{1}{2} \mathrm{Tr} A \delta_{i,j}$$

$$A_{aniso,i,j} = A_{i,j} - \frac{1}{2} \mathrm{Tr} A \delta_{i,j}$$

ANISOTROPIC PART

$$\left(A-\frac{1}{2}\mathrm{tr}A\cdot I\right)_{ij} = \delta_{ij}-\hat{\Psi}_{ij}-\frac{1}{2}(1-\hat{\Psi}_{11}+1-\hat{\Psi}_{22})\delta_{ij}$$

$$= -\hat{\Psi}_{ij}+\frac{1}{2}(\hat{\Psi}_{11}+\hat{\Psi}_{22})\delta_{ij}$$

$$= \begin{pmatrix} -\frac{1}{2}(\hat{\Psi}_{11}-\hat{\Psi}_{22}) & -\hat{\Psi}_{12} \\ -\hat{\Psi}_{12} & \frac{1}{2}(\hat{\Psi}_{11}-\hat{\Psi}_{22}) \end{pmatrix}$$

$$\boxed{\frac{\partial^2\hat{\Psi}(\vec{\theta})}{\partial\theta_i\partial\theta_j}\equiv\hat{\Psi}_{ij}}$$

*Introducing the **shear**:*

$$\gamma_1 = \frac{1}{2}(\hat{\Psi}_{11}-\hat{\Psi}_{22})$$

$$\gamma_2 = \hat{\Psi}_{12}=\hat{\Psi}_{21}\,,$$

$$\Gamma = \begin{pmatrix} \gamma_1 & \gamma_2 \\ \gamma_2 & -\gamma_1 \end{pmatrix}$$

Symmetric, trace-less tensor with eigenvalues:

$$\pm\sqrt{\gamma_1^2+\gamma_2^2}=\pm\gamma$$

ISOTROPIC PART

$$\frac{1}{2}\mathrm{tr}A \cdot I = \left[1 - \frac{1}{2}(\hat{\Psi}_{11} + \hat{\Psi}_{22})\right]\delta_{ij}$$

$$= \left(1 - \frac{1}{2}\triangle\hat{\Psi}\right)\delta_{ij} = (1 - \kappa)\delta_{ij}$$

Remember: $\triangle_\theta \Psi(\vec{\theta}) = 2\kappa(\vec{\theta})$

THE SHEAR IS NOT A VECTOR!

$$\det A = (1 - \kappa - \gamma)(1 - \kappa + \gamma)$$
$$\gamma = \sqrt{\gamma_1^2 + \gamma_2^2}$$

There is thus an orthogonal coordinate transformation $R(\varphi)$, a rotation by an angle φ, which brings the Jacobian matrix into diagonal form.

Generally, the Jacobian matrix transforms as

$$A \to A' = R(\varphi)^T A R(\varphi)$$

This shows that the shear components transform under coordinate rotations as

$$\gamma_1 \to \gamma_1' = \gamma_1 \cos(2\varphi) + \gamma_2 \sin(2\varphi)$$
$$\gamma_2 \to \gamma_2' = -\gamma_1 \sin(2\varphi) + \gamma_2 \cos(2\varphi)$$

i.e. unlike a vector! Since the shear components are mapped onto each other after rotations of $\varphi = \pi$ rather than $\varphi = 2\pi$, they form a so-called spin-2 field.

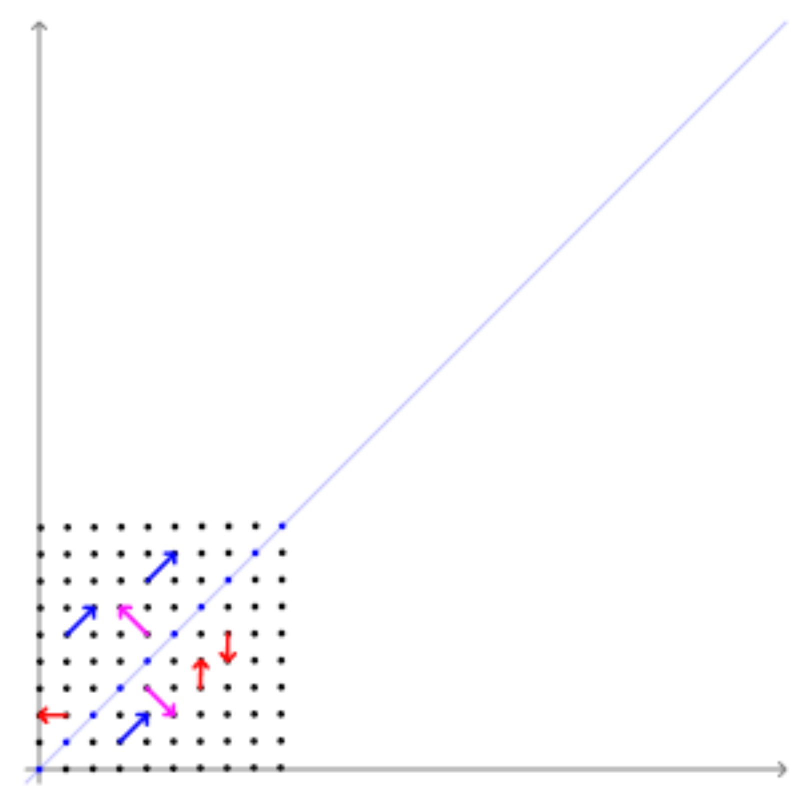
LENSING JACOBIAN

$$
\begin{aligned}
A &= \begin{pmatrix} 1-\kappa-\gamma_1 & -\gamma_2 \\ -\gamma_2 & 1-\kappa+\gamma_1 \end{pmatrix} \\
&= (1-\kappa)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \gamma \begin{pmatrix} \cos 2\phi & \sin 2\phi \\ \sin 2\phi & -\cos 2\phi \end{pmatrix}
\end{aligned}
$$

Lens mapping at first order is a linear application, distorting areas.

*Distortion directions are given by the **eigenvectors** of A.*

*Distortion amplitudes in these directions are given by the **eigenvalues**.*



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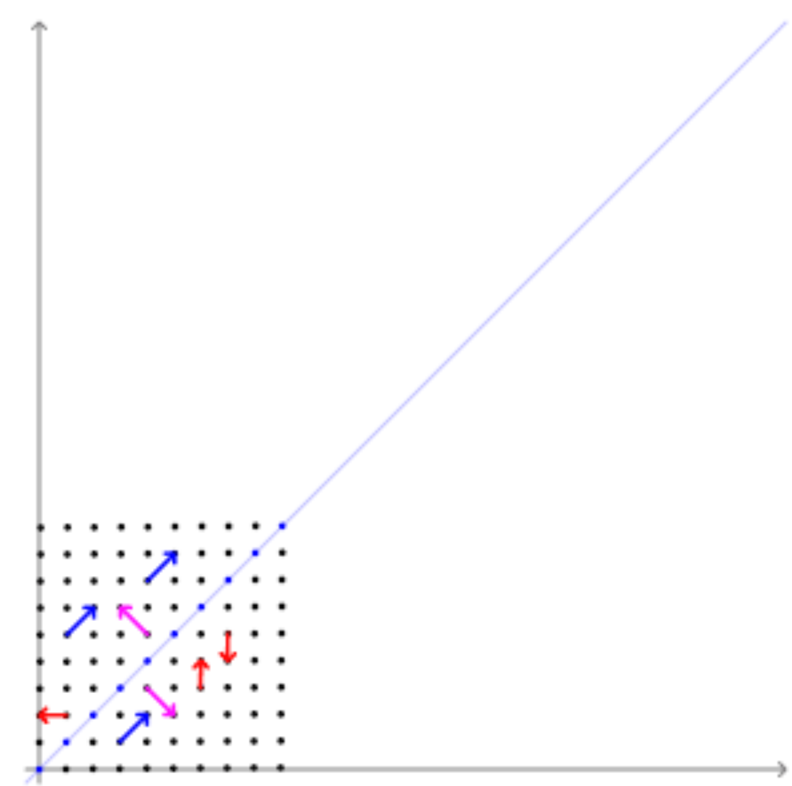

EXAMPLE: FIRST ORDER DISTORTION OF A CIRCULAR SOURCE

$$\beta_1^2 + \beta_2^2 = \beta^2$$

In the reference frame where A is diagonal:

$$\begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} 1-\kappa-\gamma & 0 \\ 0 & 1-\kappa+\gamma \end{pmatrix} \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}$$

$$\begin{aligned} \beta_1 &= (1-\kappa-\gamma)\theta_1 \\ \beta_2 &= (1-\kappa+\gamma)\theta_2 \end{aligned}$$

$$\beta^2 = \beta_1^2 + \beta_2^2 = (1-\kappa-\gamma)^2\theta_1^2 + (1-\kappa+\gamma)^2\theta_2^2$$

This is the equation of an ellipse with semi-axes:

$$a = \frac{\beta}{1-\kappa-\gamma} \qquad b = \frac{\beta}{1-\kappa+\gamma}$$

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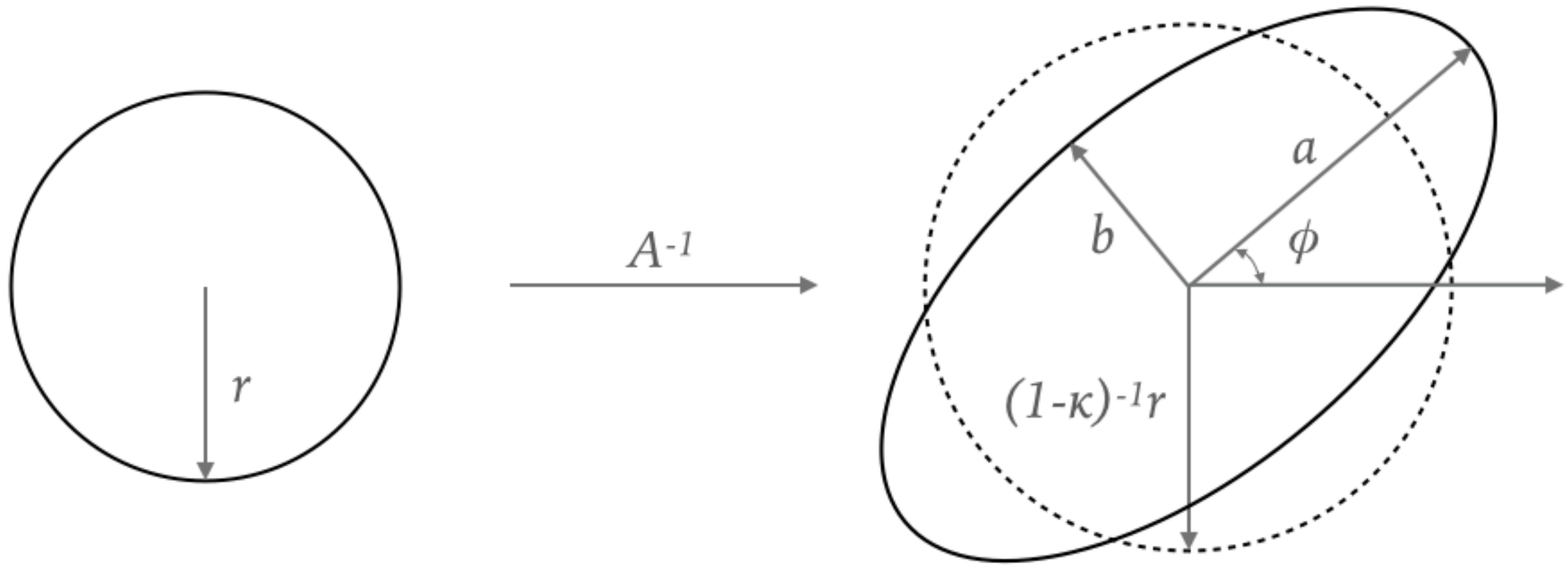

convergence: *responsible for isotropic expansion or contraction*

shear: *responsible for anisotropic distortion*

Ellipticity: $$e = \frac{a-b}{a+b} = \frac{\gamma}{1-\kappa} = g$$

ON THE SPIN-2 NATURE OF SHEAR: QUIZ

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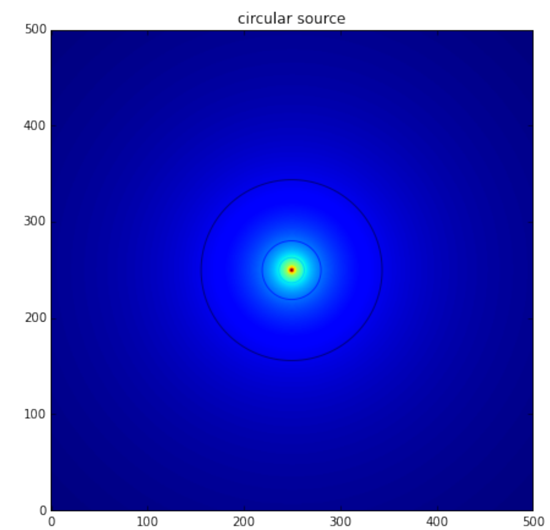
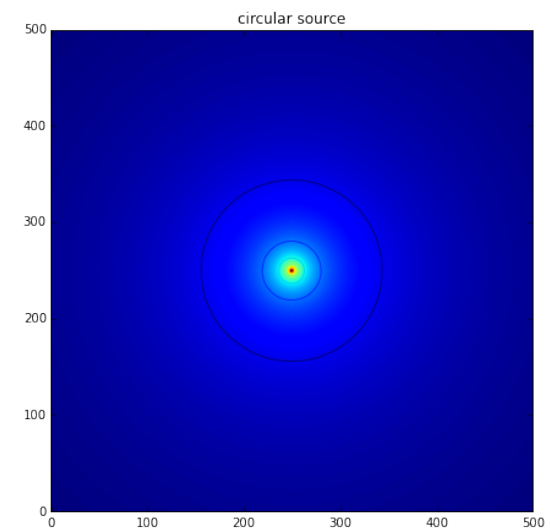
➤ Let's consider a circular source



ON THE SPIN-2 NATURE OF SHEAR: QUIZ

- Let’s consider a circular source
- How is it distorted if we apply a pure convergence trasformation?



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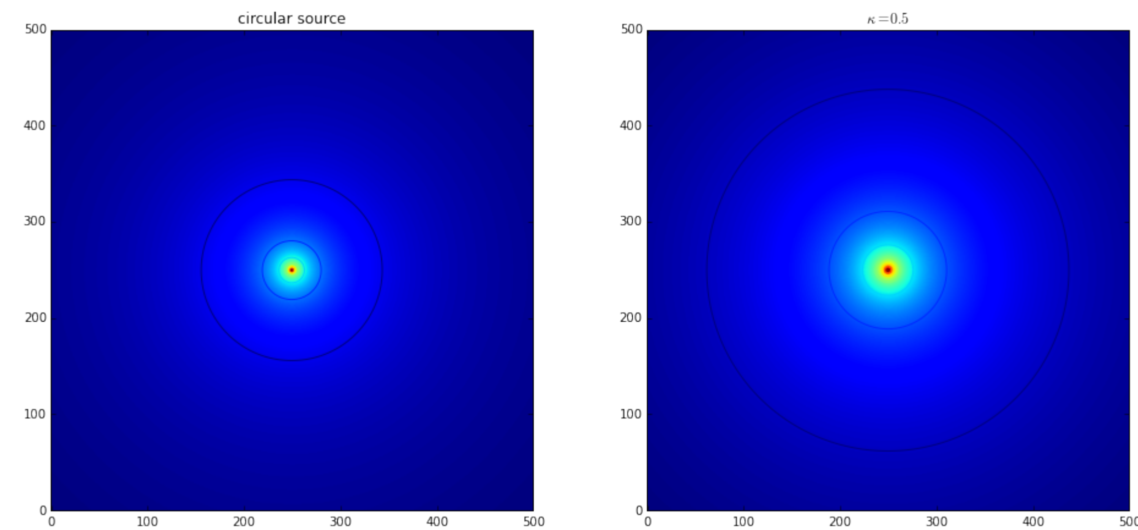

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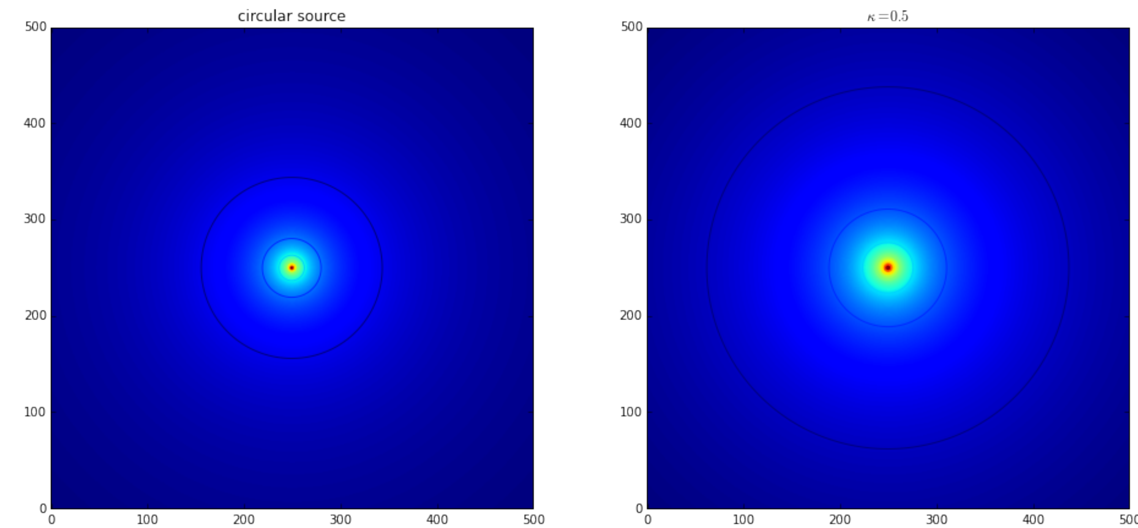
- Now, assume that $\gamma_1 > 0$ and $\gamma_2 = 0$. How is the image distorted?



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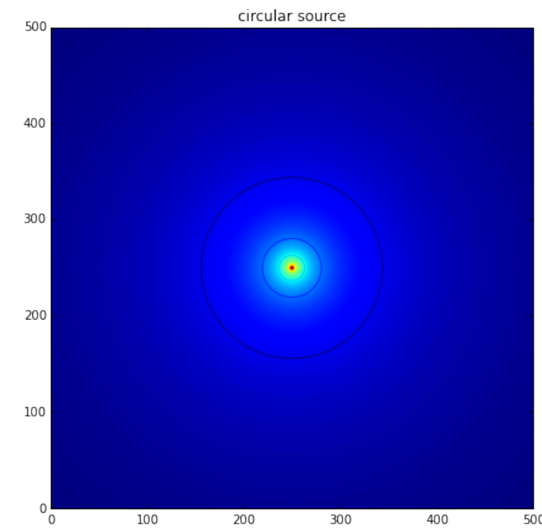

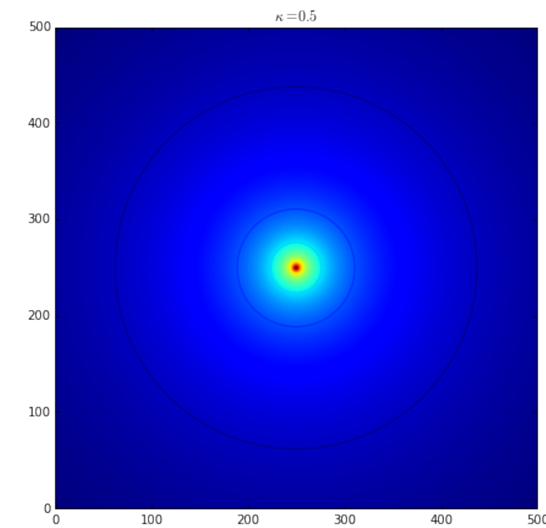

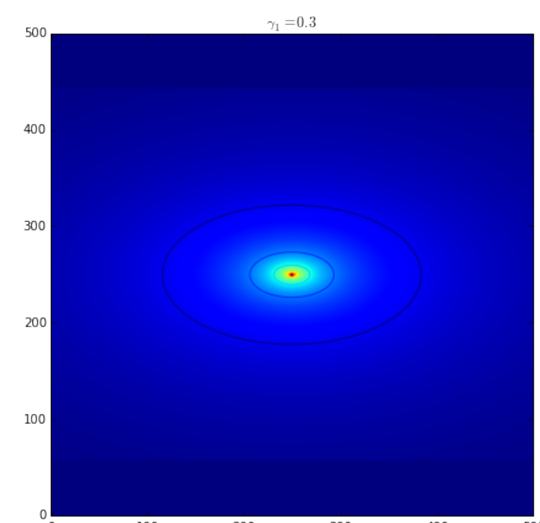

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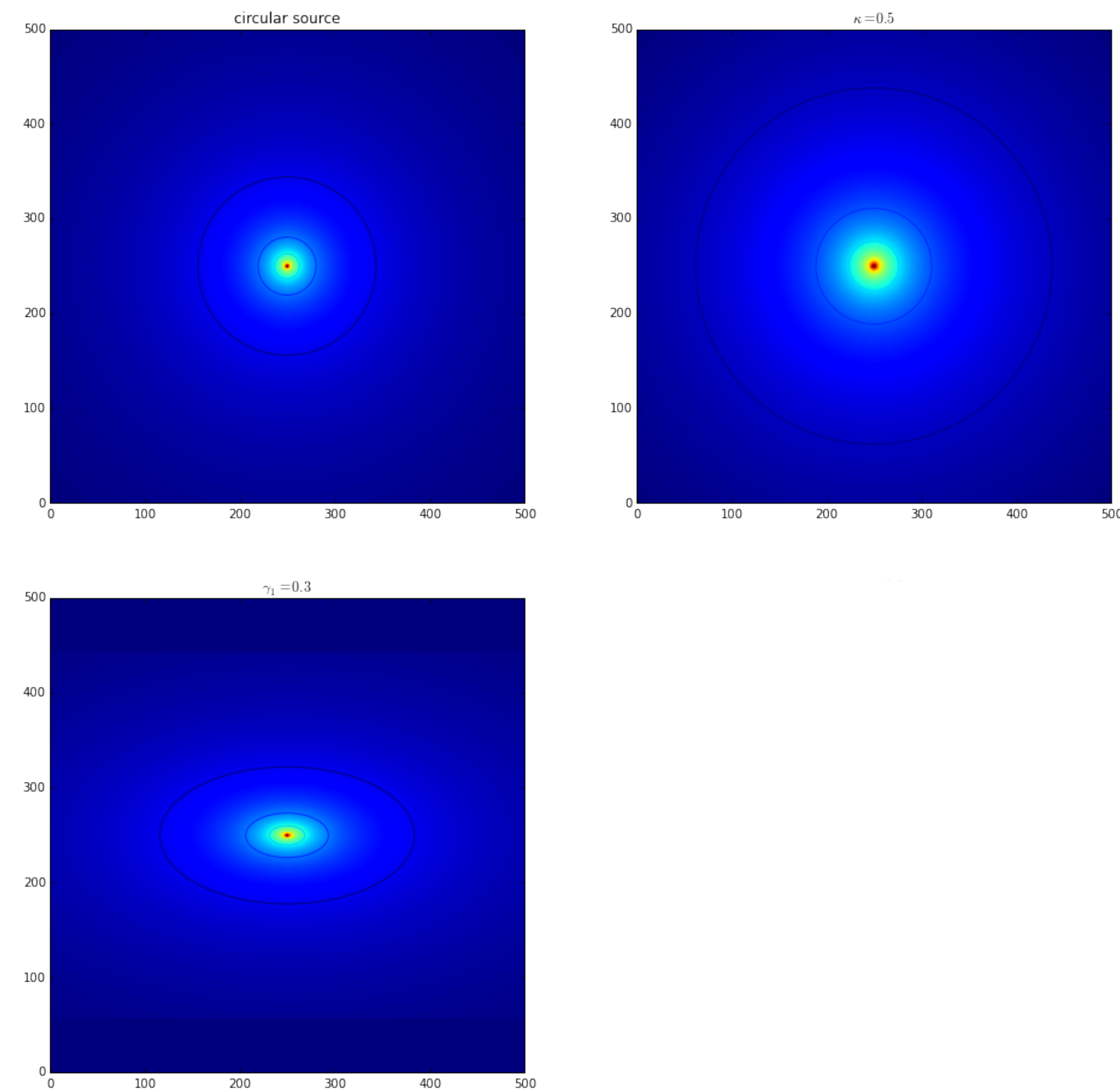

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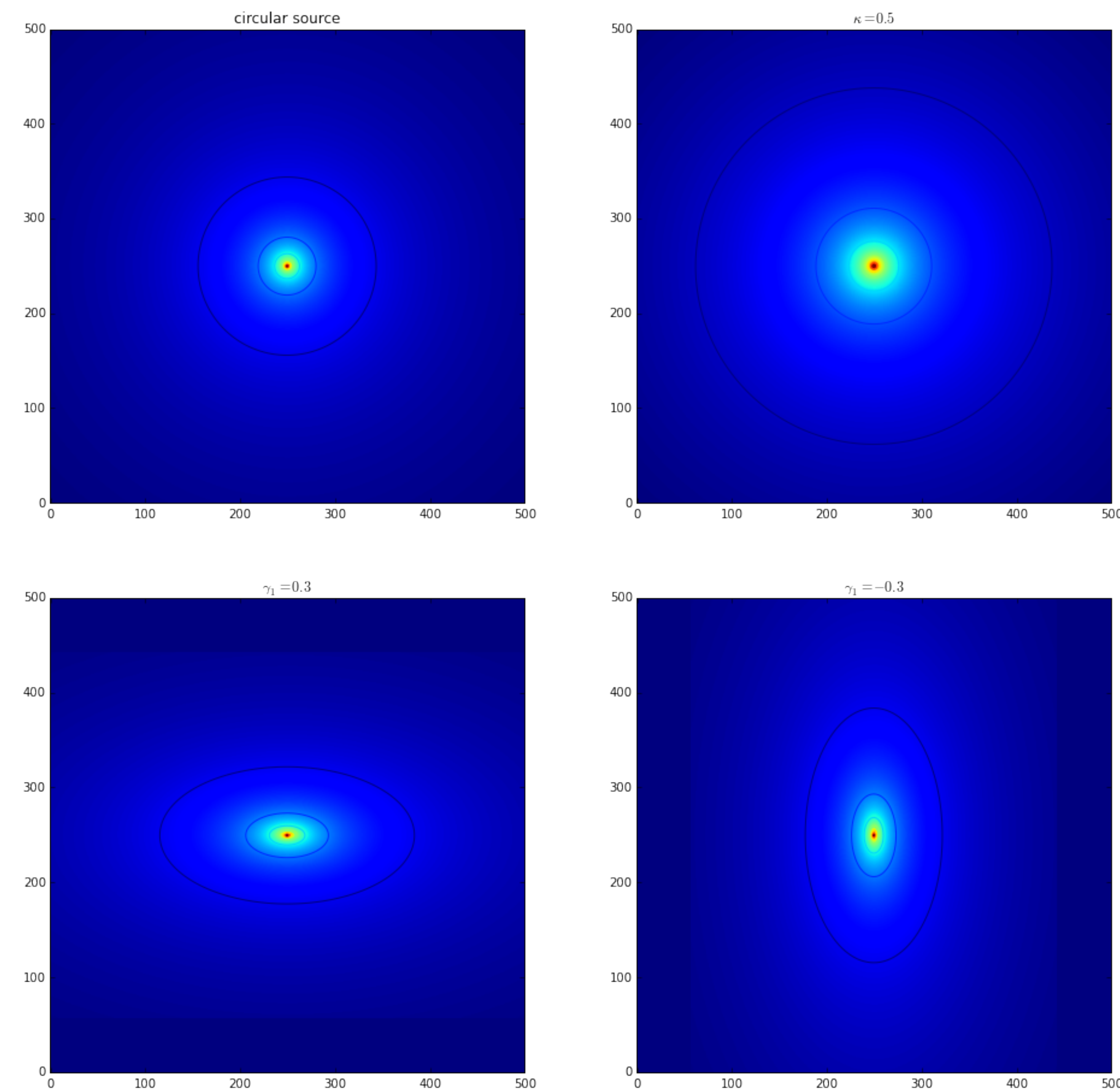

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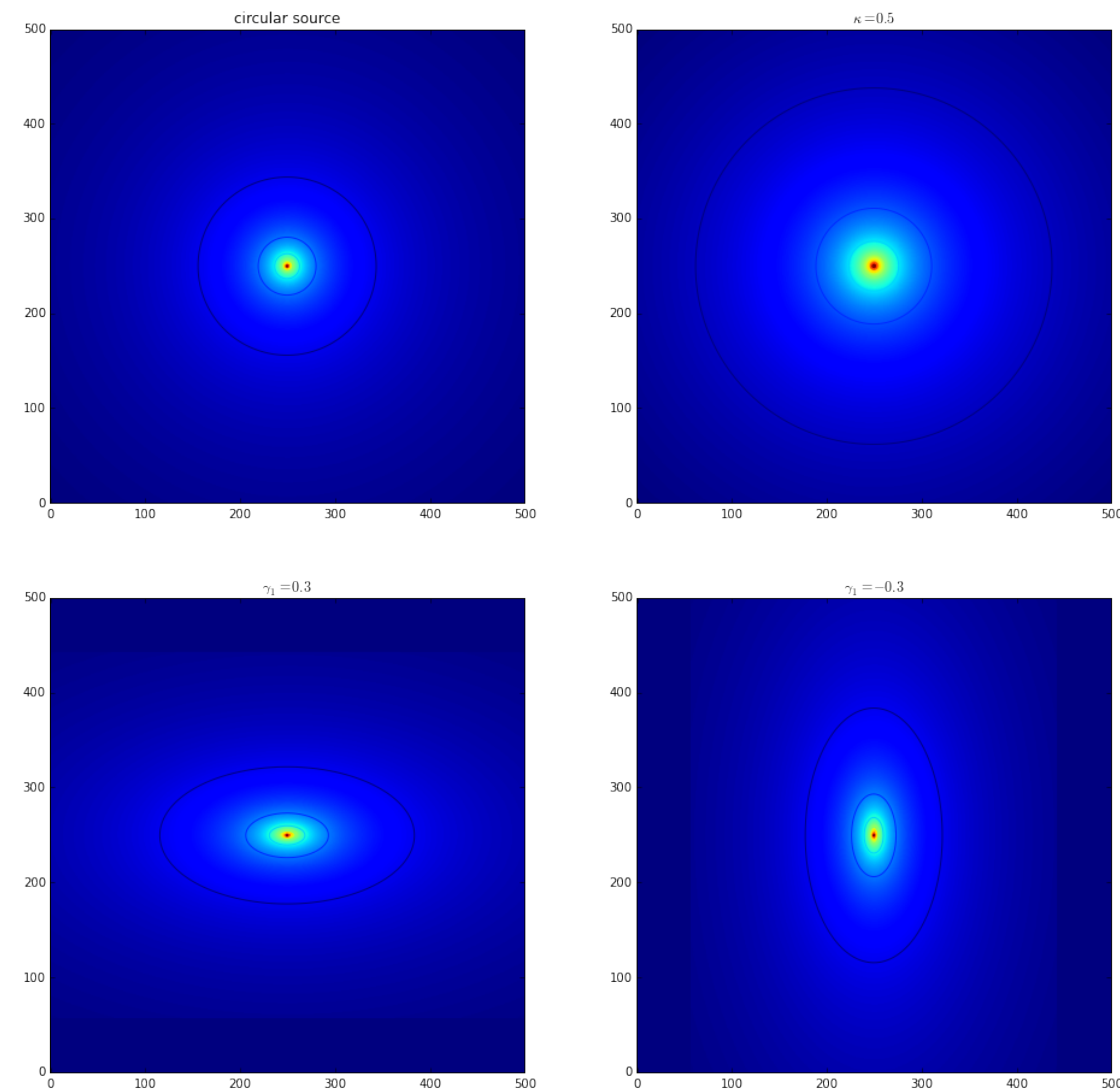

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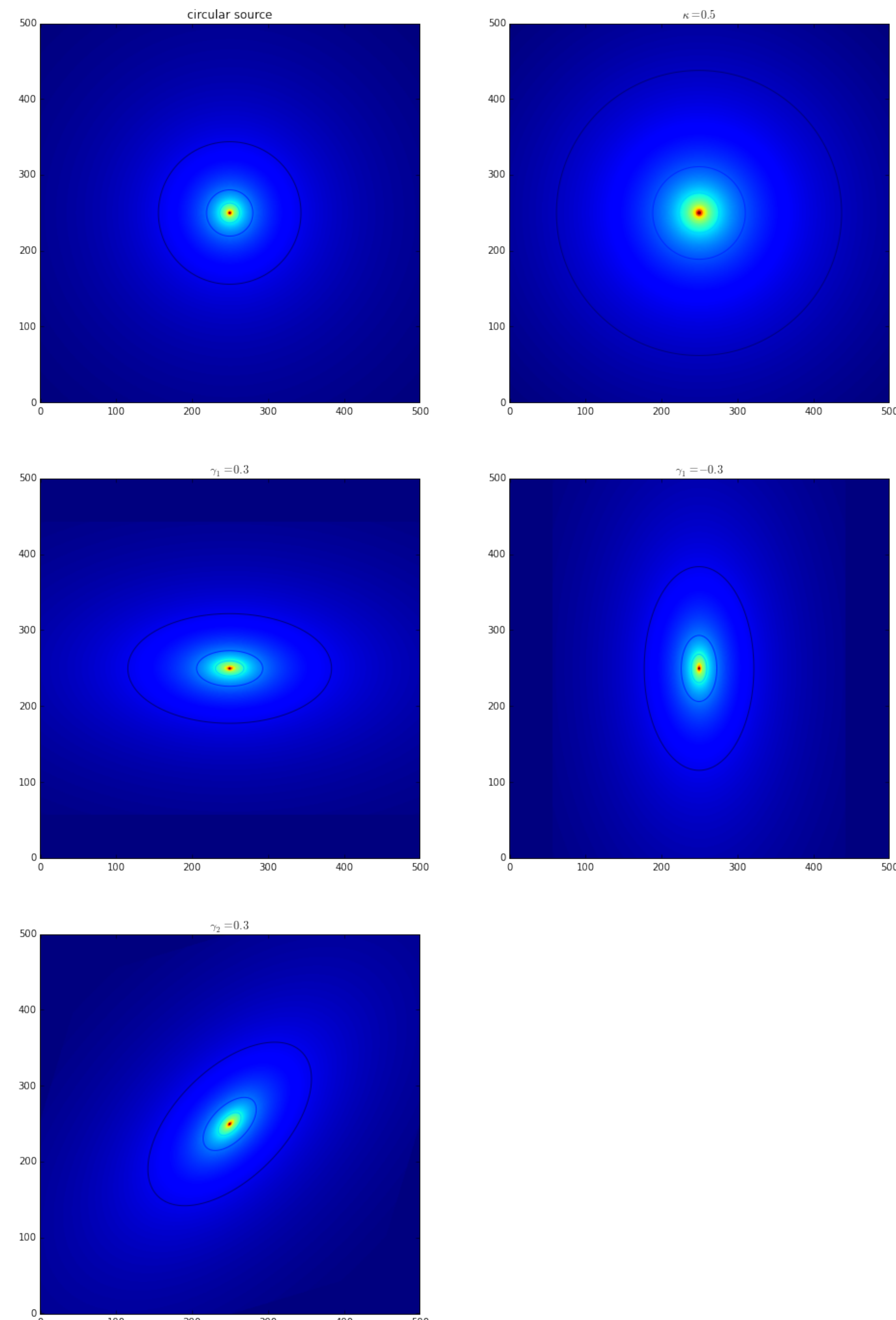

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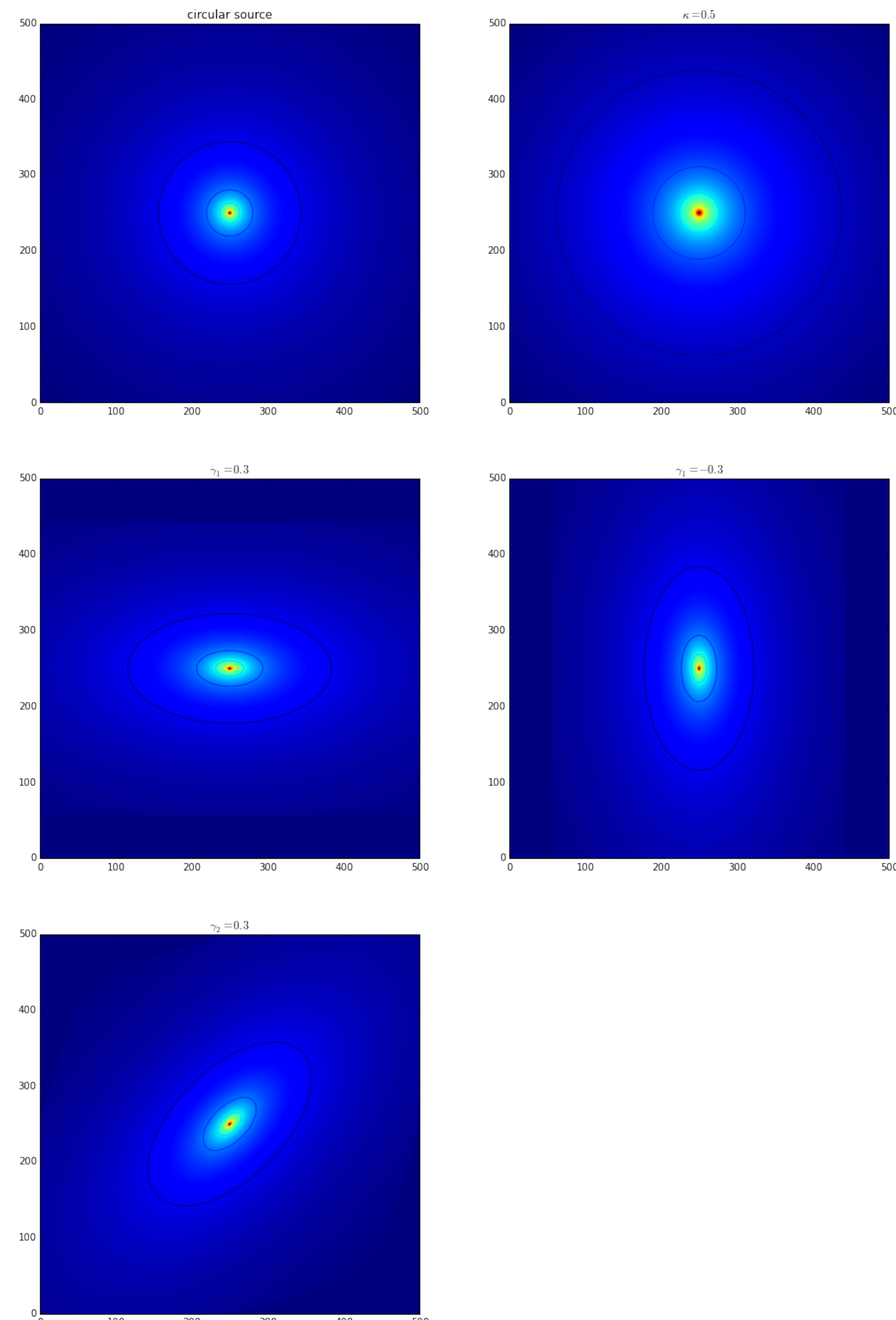

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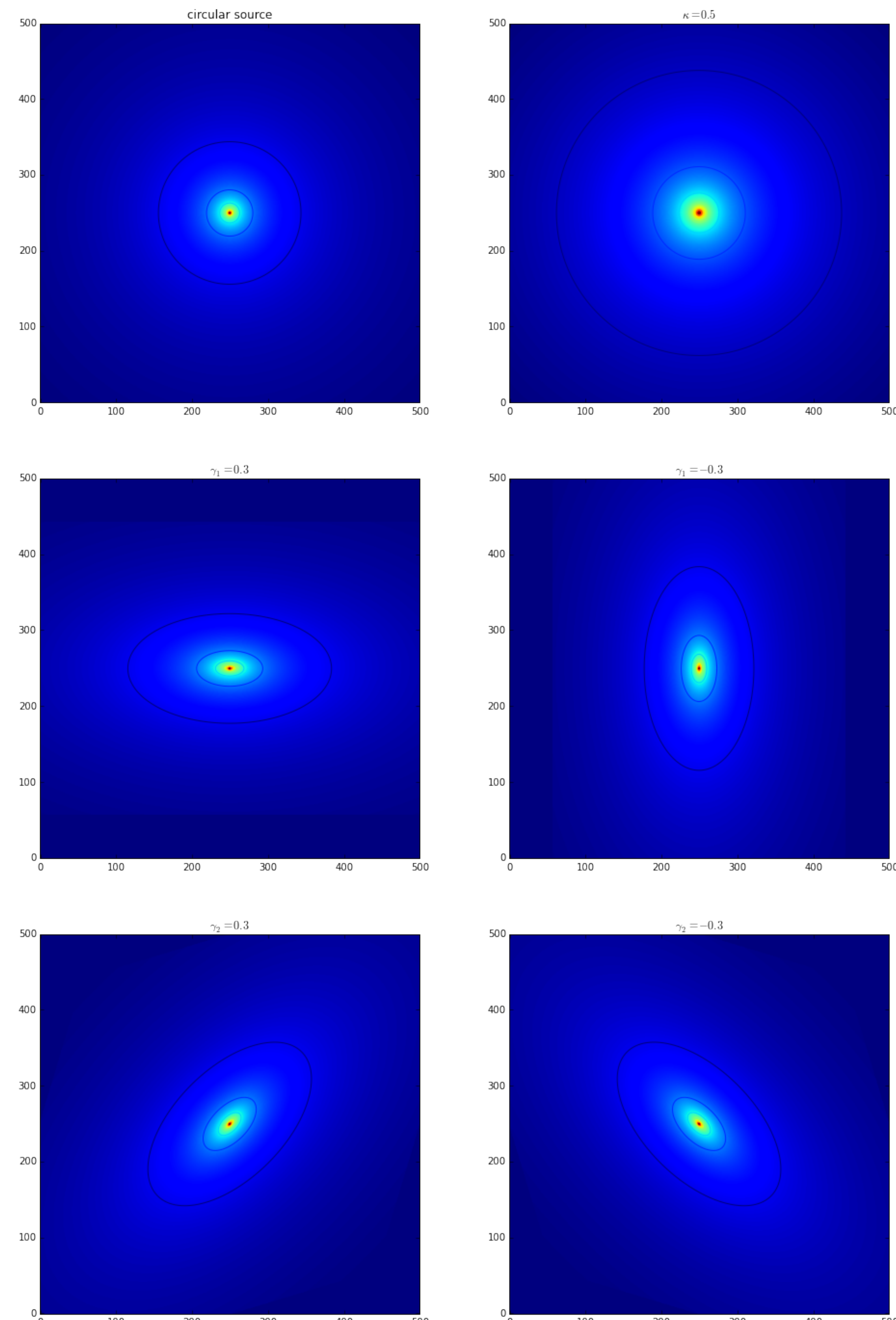
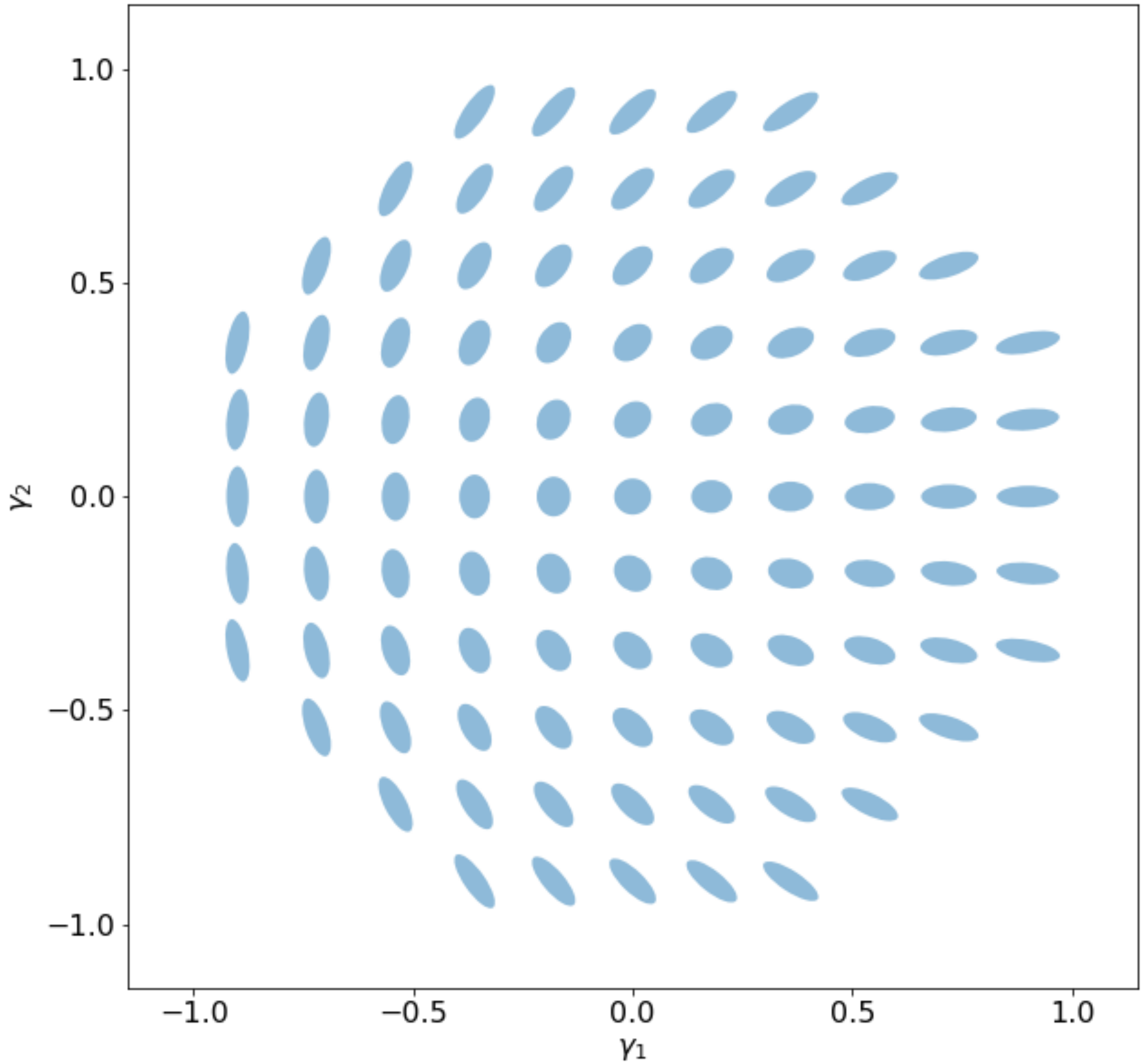


SHEAR DISTORTIONS



DEPENDENCE ON REDSHIFT

We have seen that the lensing potential, the deflection angle, the convergence, the shear… depend on a combination of distances.

For example, the lensing potential is:

$$\hat{\Psi}(\vec{\theta}) = \frac{D_{\mathrm{LS}}}{D_{\mathrm{L}} D_{\mathrm{S}}} \frac{2}{c^2} \int \Phi(D_{\mathrm{L}}\vec{\theta}, z) \mathrm{d}z$$

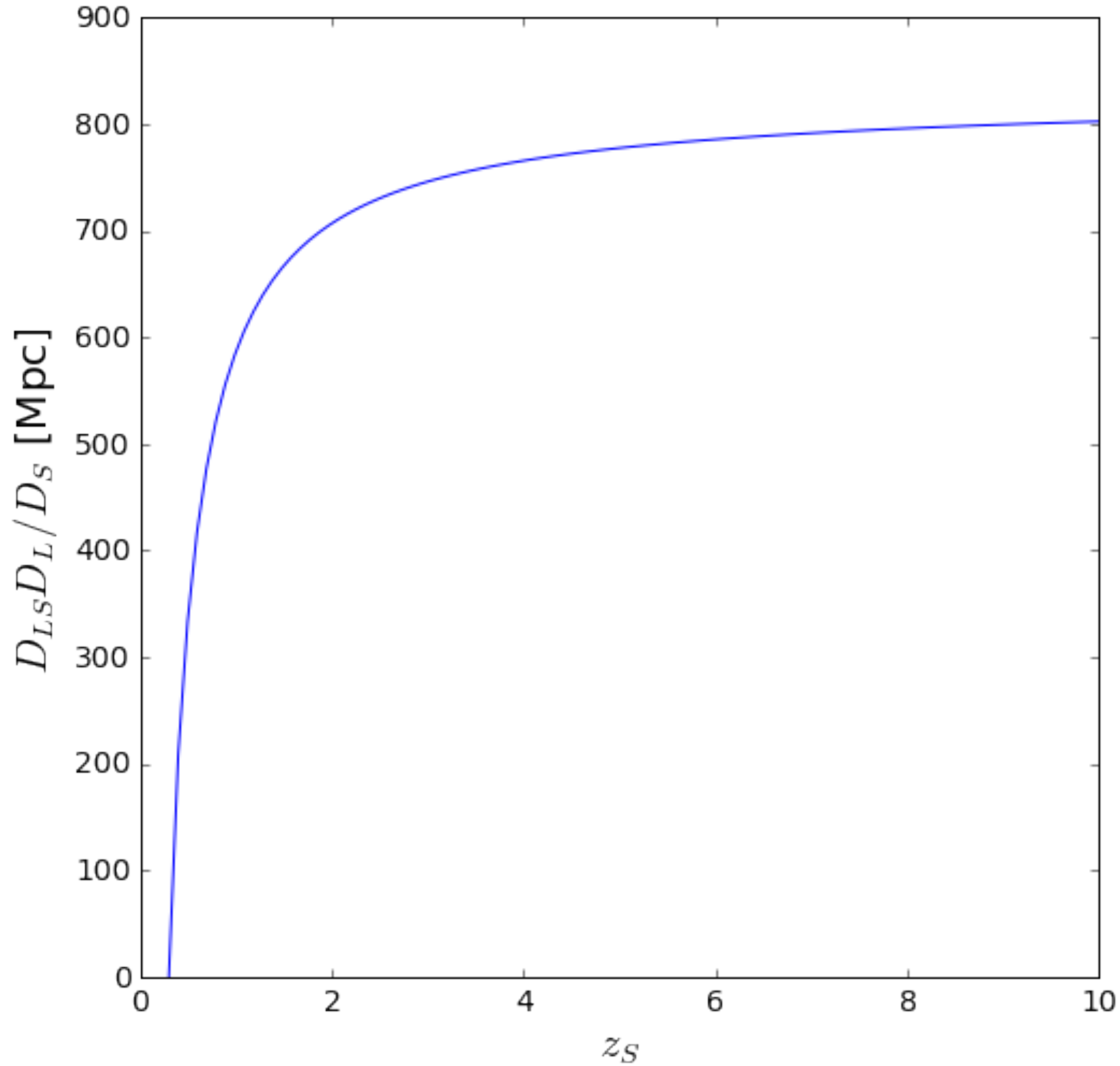
Every spatial derivative of Ψ introduces a factor D_L.

The distance ratio $D_{LS}D_L/D_S$ is called "lensing distance".

Both the shear and the convergence, being second derivatives of the lensing potential, scale as the lensing distance

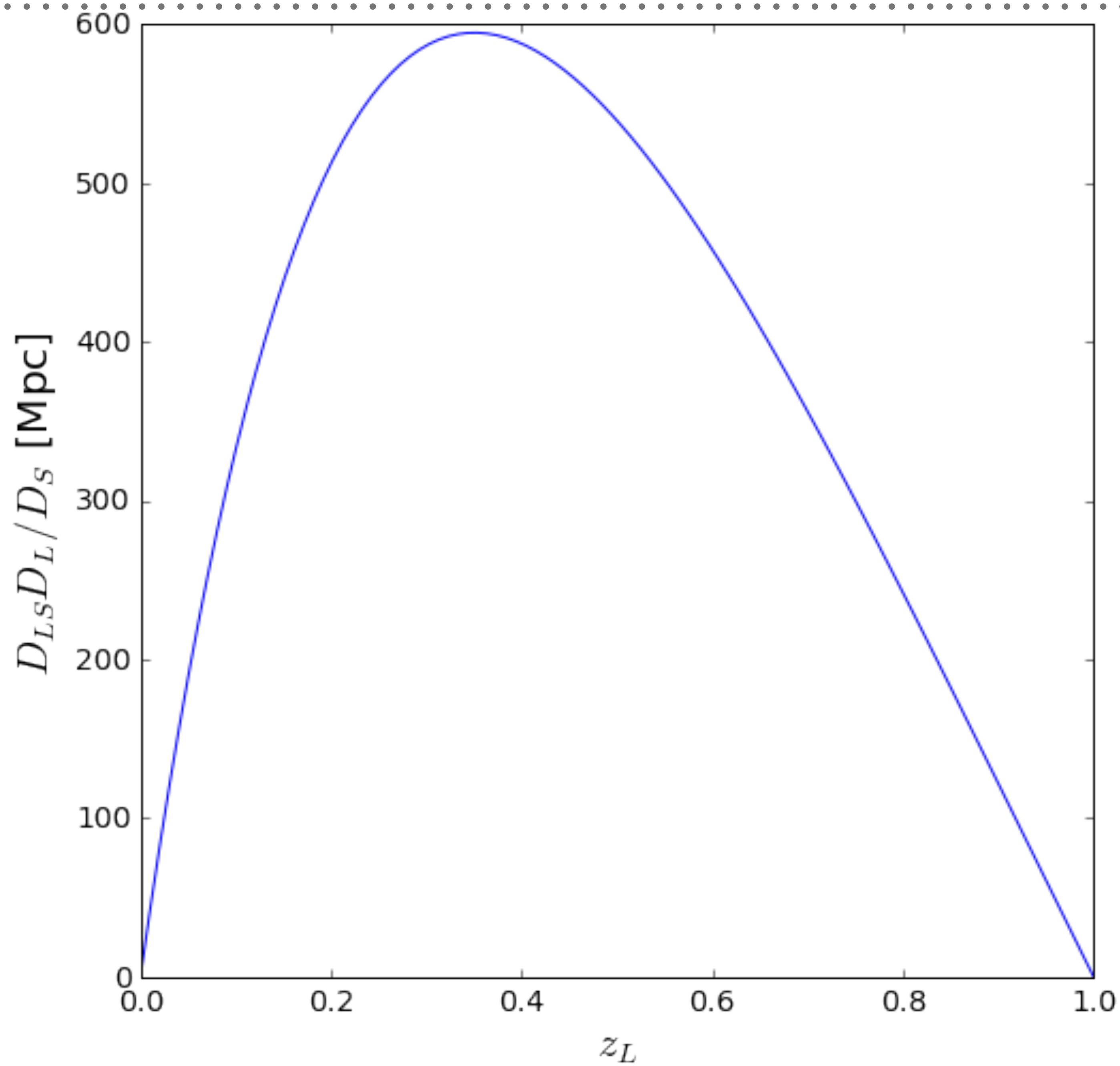
$$\Sigma_{cr} = \frac{c^2}{4\pi G} \frac{D_S}{D_{LS} D_L}$$

HOW DOES THE LENSING DISTANCE SCALE WITH SOURCE REDSHIFT?



Note that if the lensing distance grows, the critical surface density decreases, the convergence and the shear grow!

HOW DOES THE LENSING DISTANCE SCALE WITH LENS REDSHIFT?



The lensing distance peaks at ~half way between the source and the observer, meaning that there is an optimal distance where the lens produces its largest effects.

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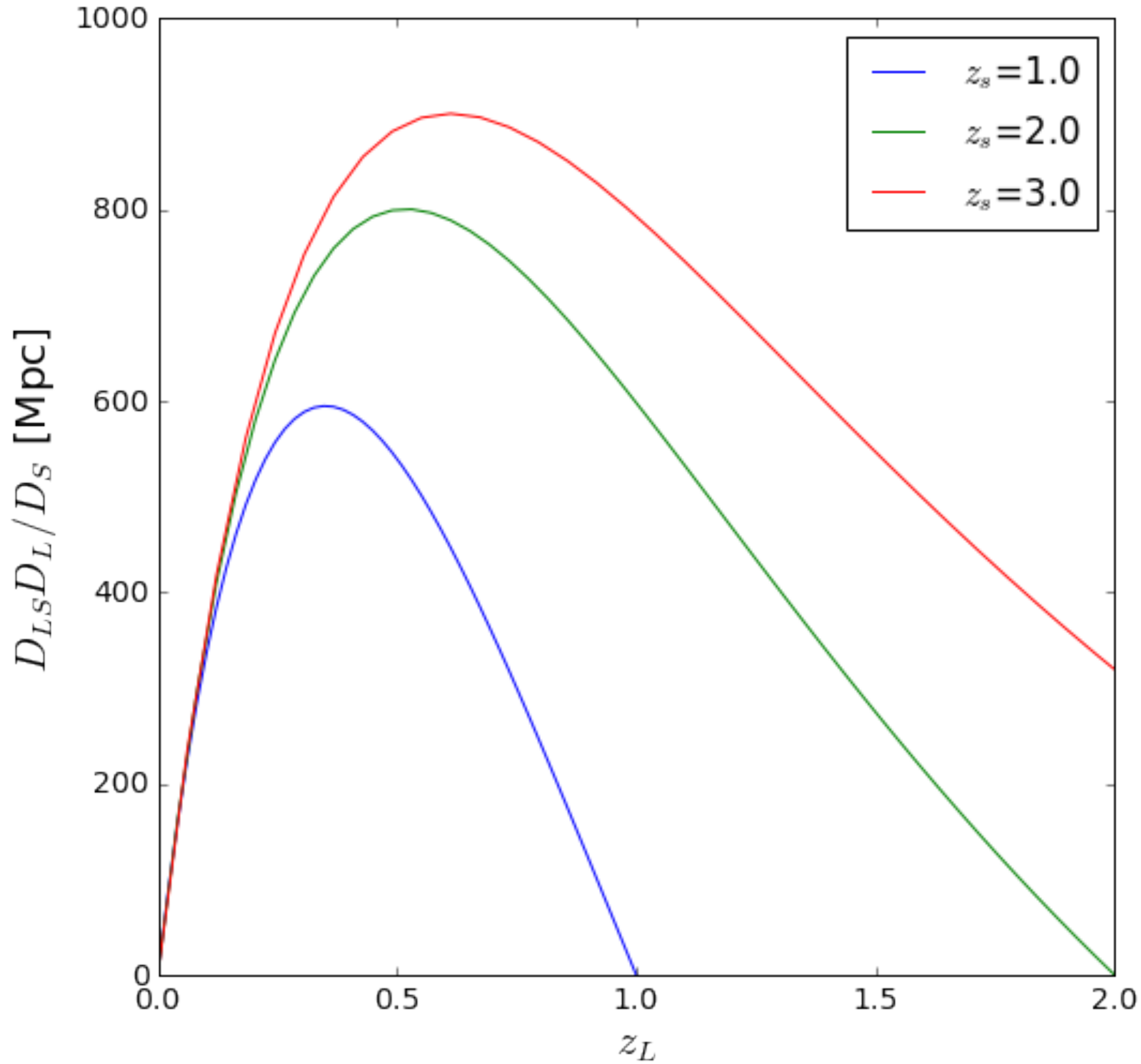


Of course, the peak moves to larger distances as the distance to the source increases.

CONSERVATION OF SURFACE BRIGHTNESS

The source ***surface brightness*** *is*

$$I_\nu = \frac{dE}{dt dA d\Omega d\nu}$$

In phase space, the radiation emitted is characterized by the density

$$f(\vec{x}, \vec{p}, t) = \frac{dN}{d^3x d^3p}$$

In absence of photon creations or absorptions, f is conserved (Liouville theorem)

$$dN = \frac{dE}{h\nu} = \frac{dE}{cp}$$

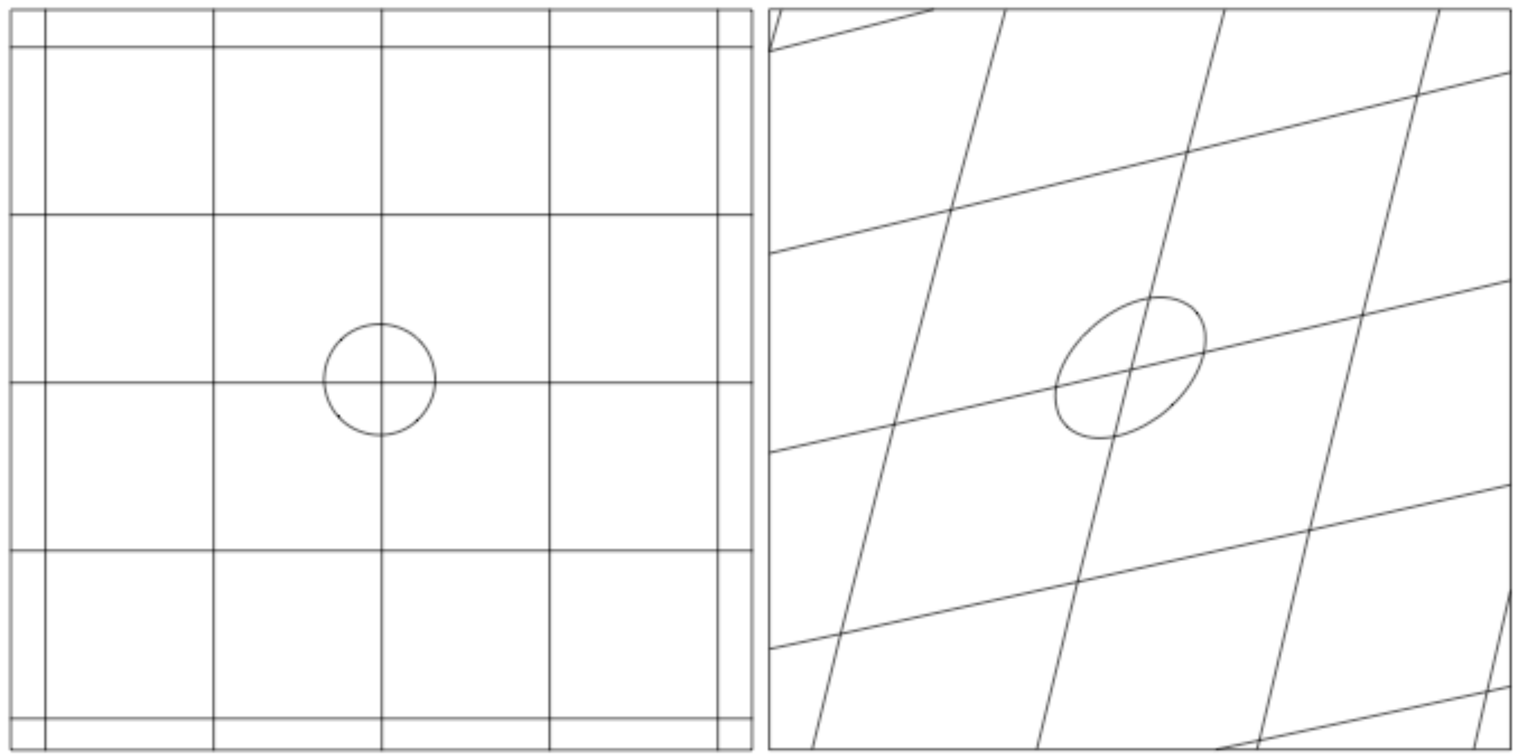
$$d^3x = c dt dA$$

$$d^3\vec{p} = p^2 dp d\Omega$$

$$f(\vec{x}, \vec{p}, t) = \frac{dN}{d^3x d^3p} = \frac{dE}{hcp^3 dA dt d\nu d\Omega} = \frac{I_\nu}{hcp^3}$$

Since GL does not involve creation or absorption of photons, neither it changes the photon momenta (achromatic!), surface brightness is conserved!

MAGNIFICATION



Kneib & Natarajan (2012)

$$F_\nu = \int_I I_\nu(\vec{\theta}) d^2\theta = \int_S I_\nu^S[\vec{\beta}(\vec{\theta})]\mu d^2\beta$$

Lensing changes the amount of photons (flux) we receive from the source by changing the solid angle the source subtends

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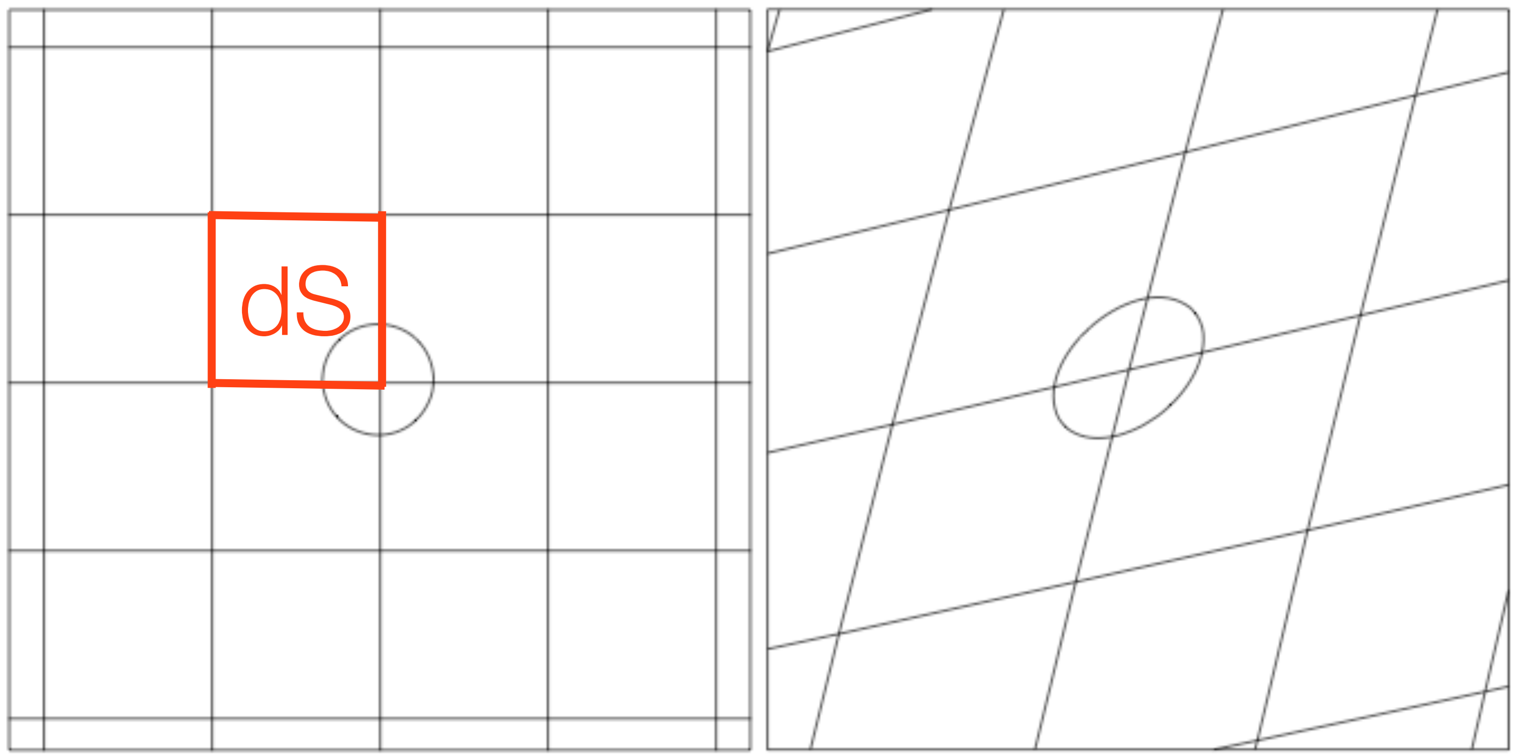

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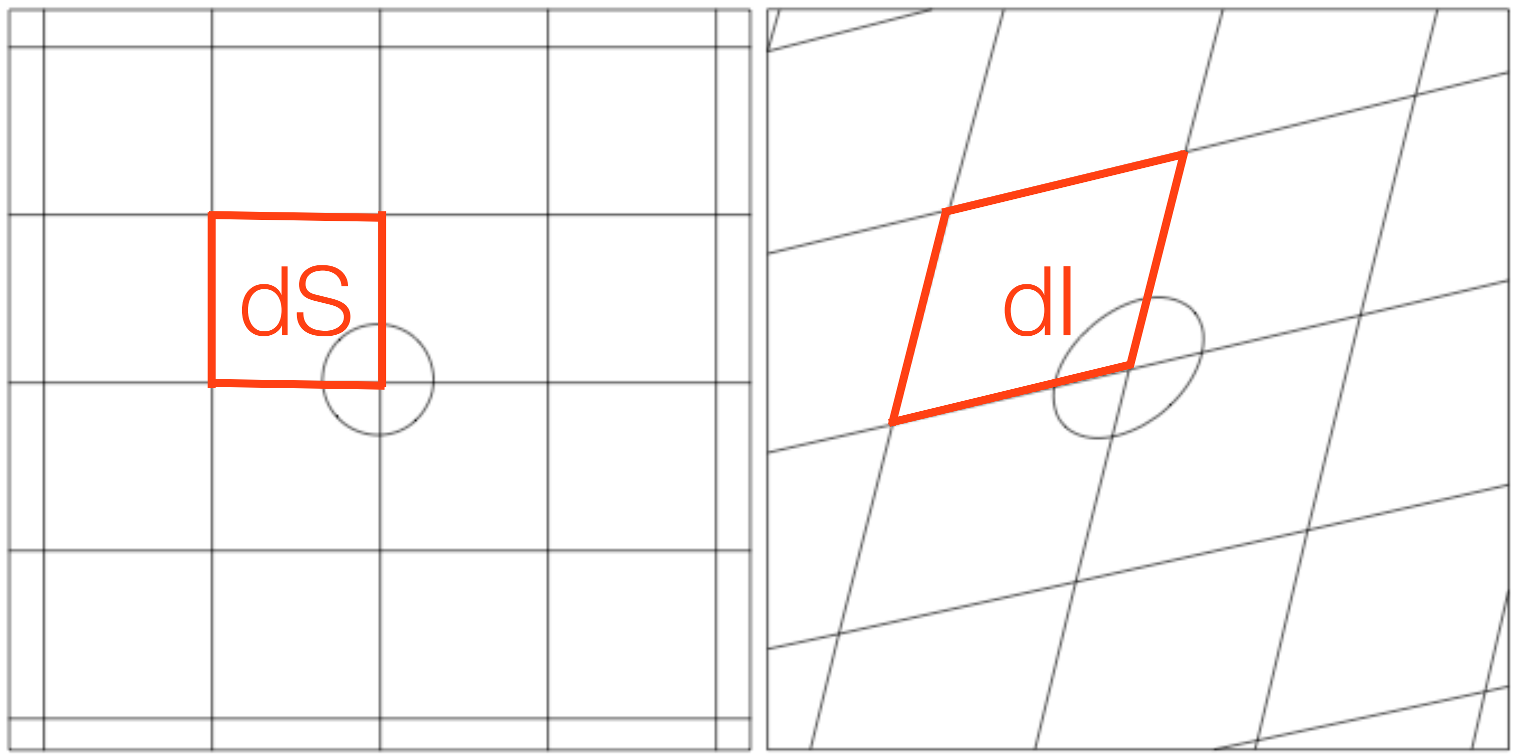

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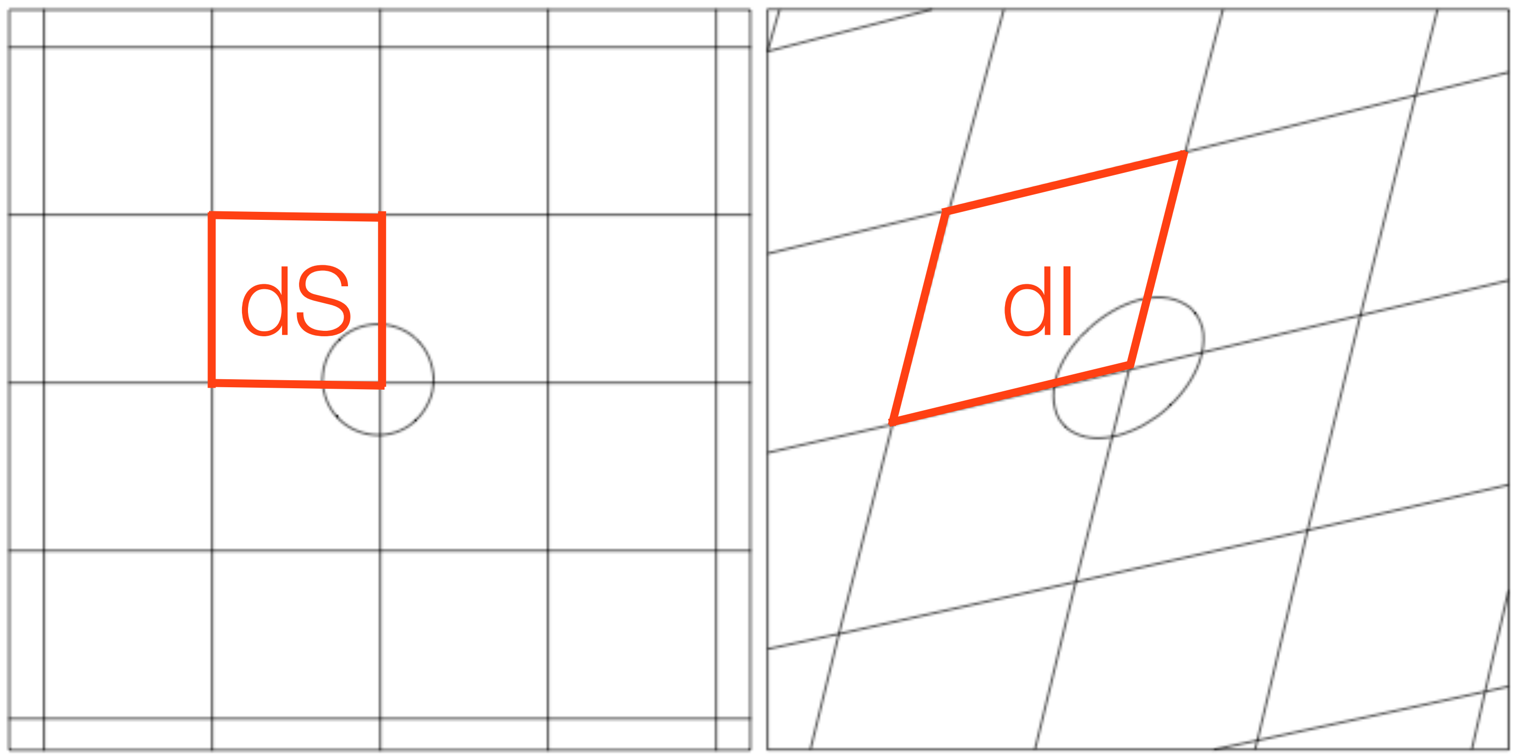

Kneib & Natarajan (2012)

$$\boxed{\mu = \frac{dI}{dS} = \frac{\delta\theta^2}{\delta\beta^2} = \det A^{-1}}$$

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Lensing changes the amount of photons (flux) we receive from the source by changing the solid angle the source subtends

CRITICAL LINES AND CAUSTICS

Both convergence and shear are functions of position on the lens plane:

$\kappa = \kappa(\vec{\theta})$

$\gamma = \gamma(\vec{\theta})$

The determinant of the lensing Jacobian is

$$\det A = (1 - \kappa - \gamma)(1 - \kappa + \gamma) = \mu^{-1}$$

The ***critical lines*** *are the lines where the eigenvalues of the Jacobian are zero:*

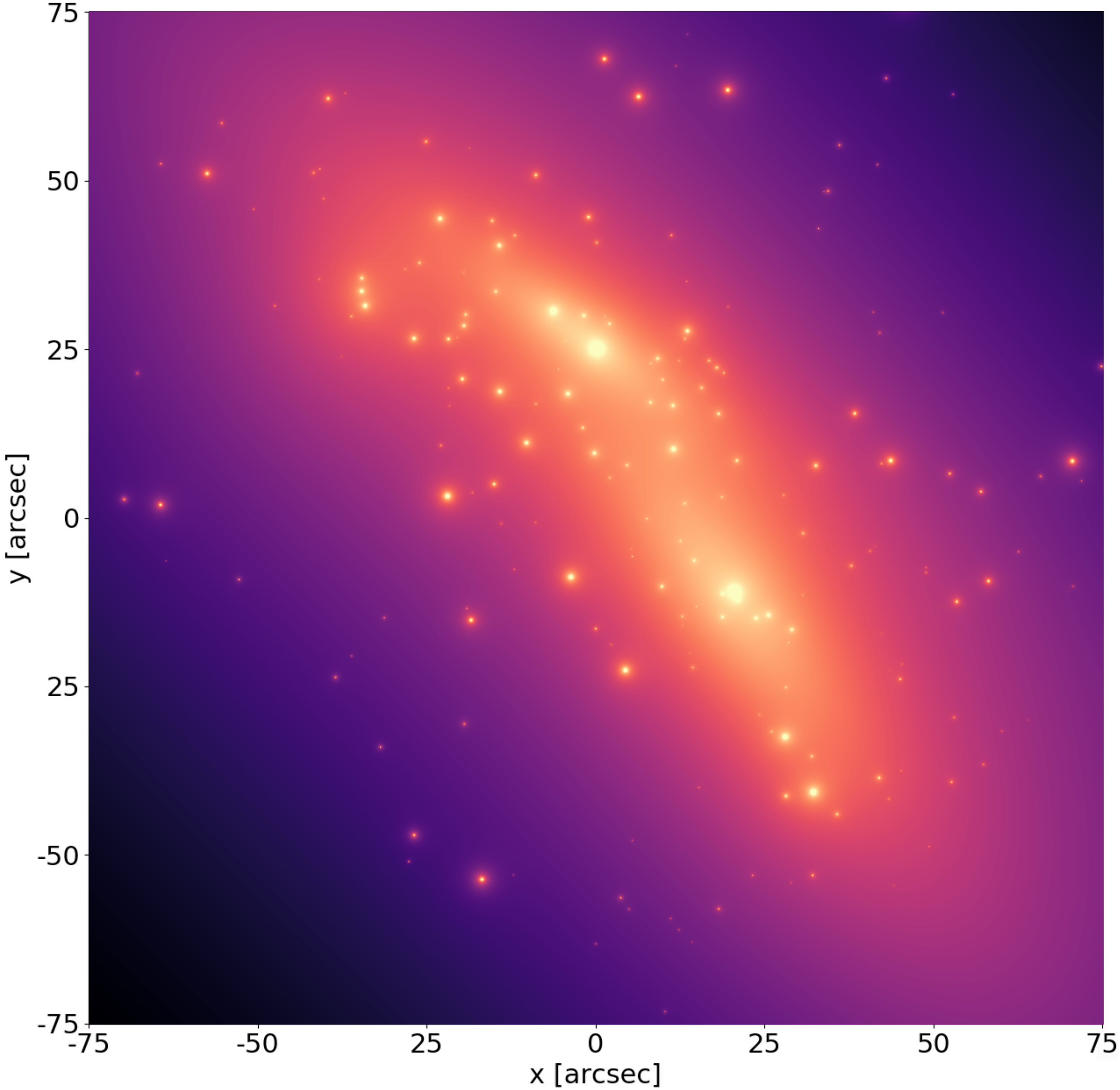
$(1 - \kappa - \gamma) = 0$ *tangential critical line*

$(1 - \kappa + \gamma) = 0$ *radial critical line*

Along these lines the magnification diverges!

Via the lens equations, they are mapped into the ***caustics****…*

Model of MACS0416 by Caminha, MM, et al. (2016)



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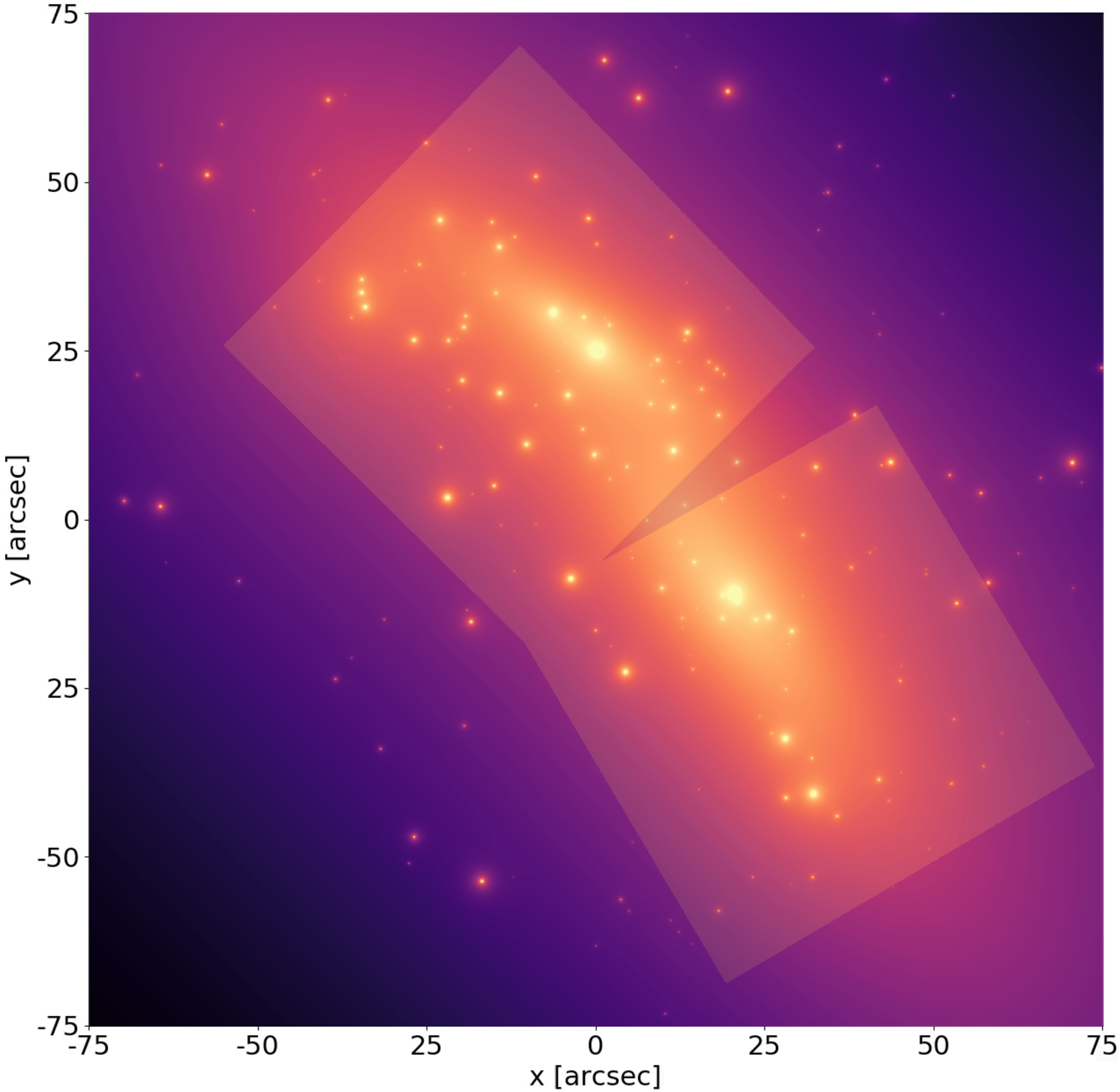

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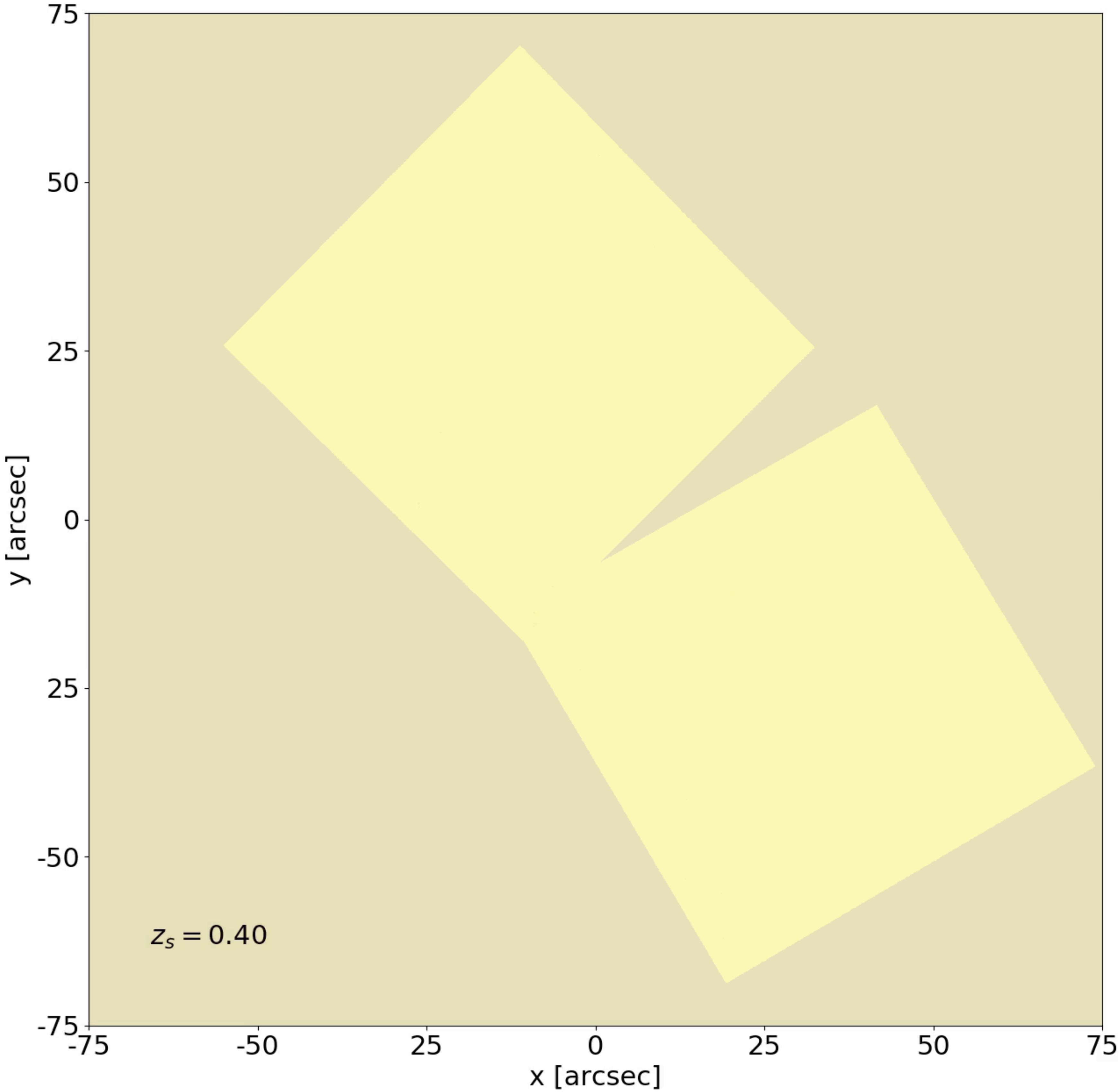

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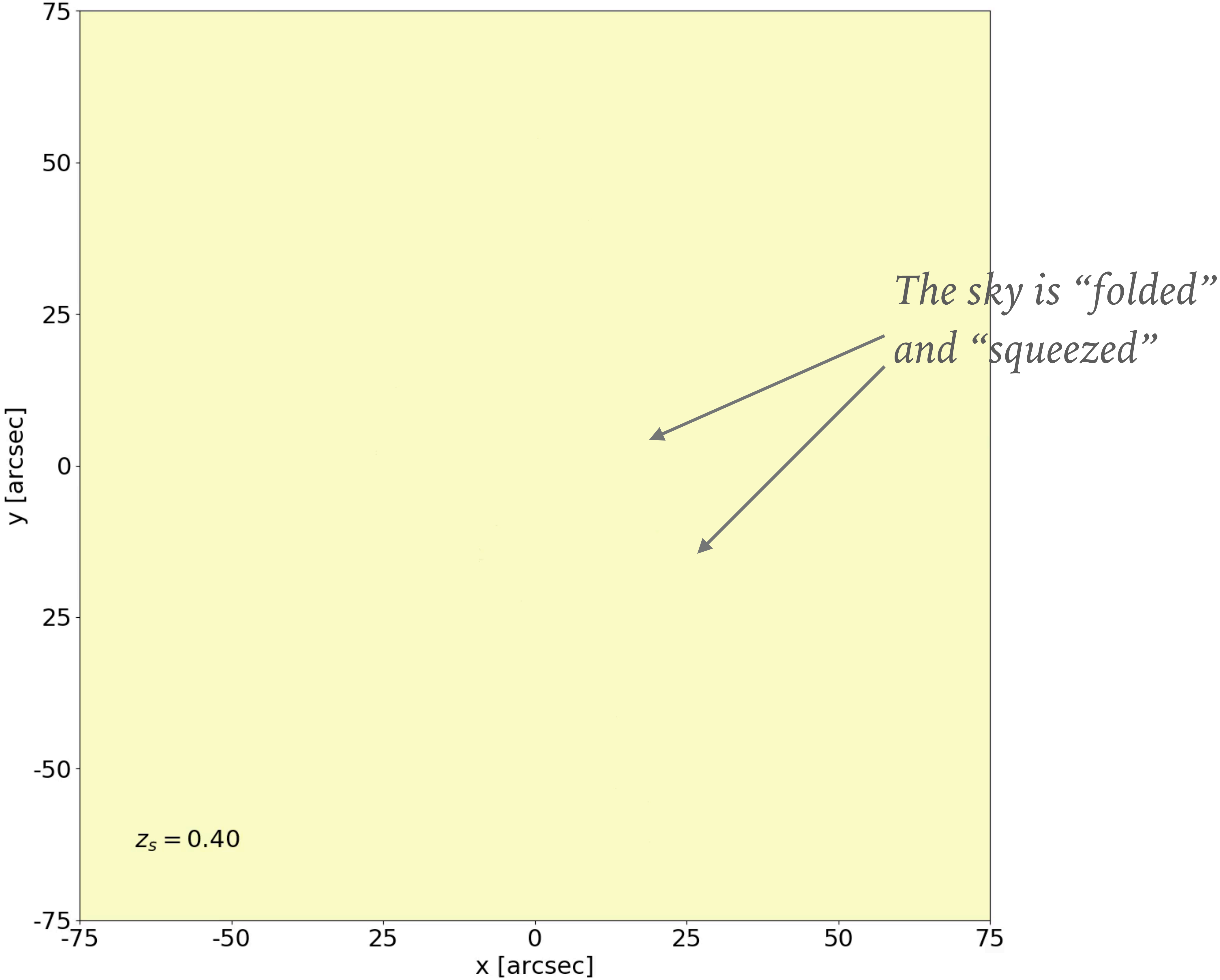
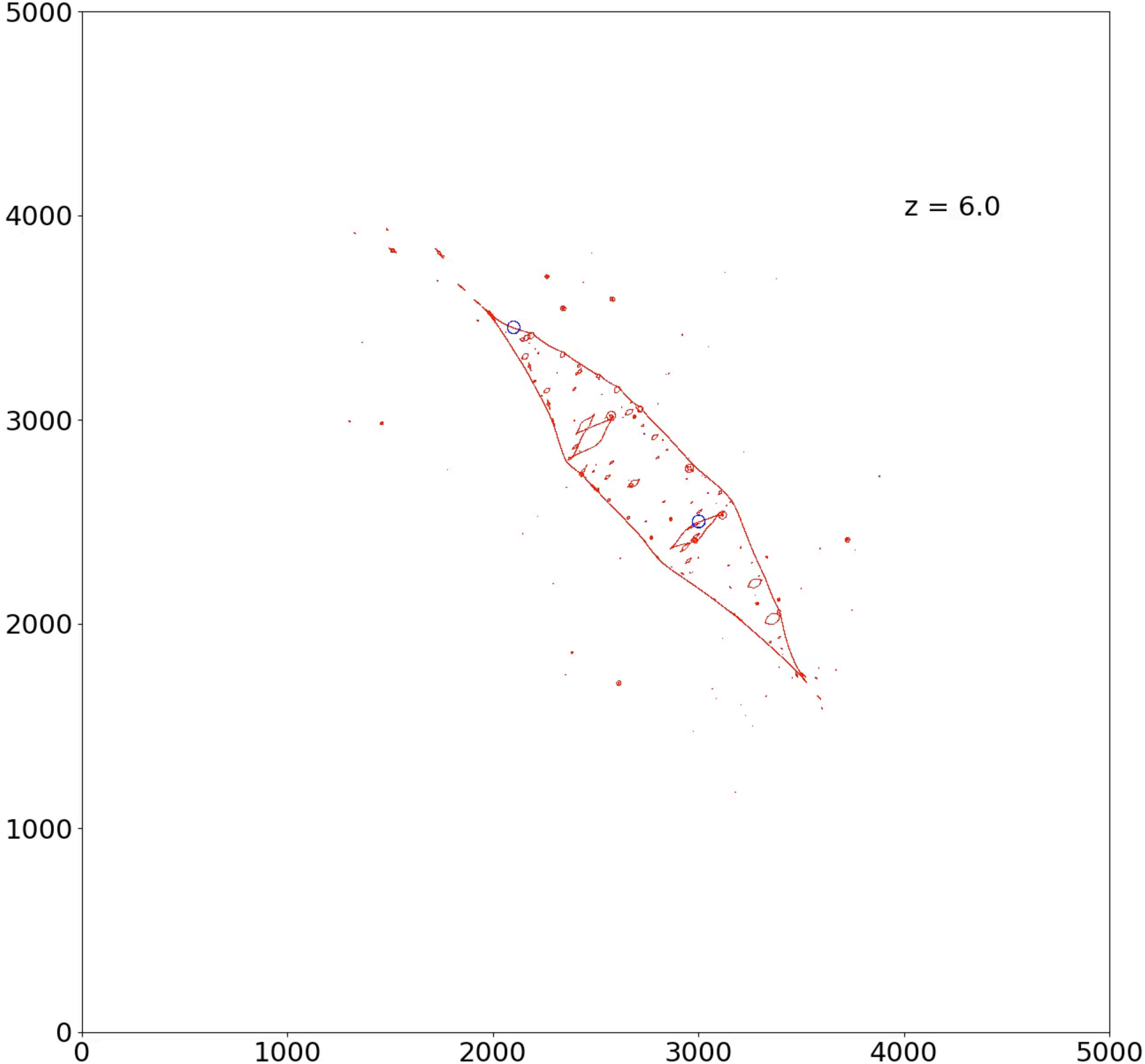
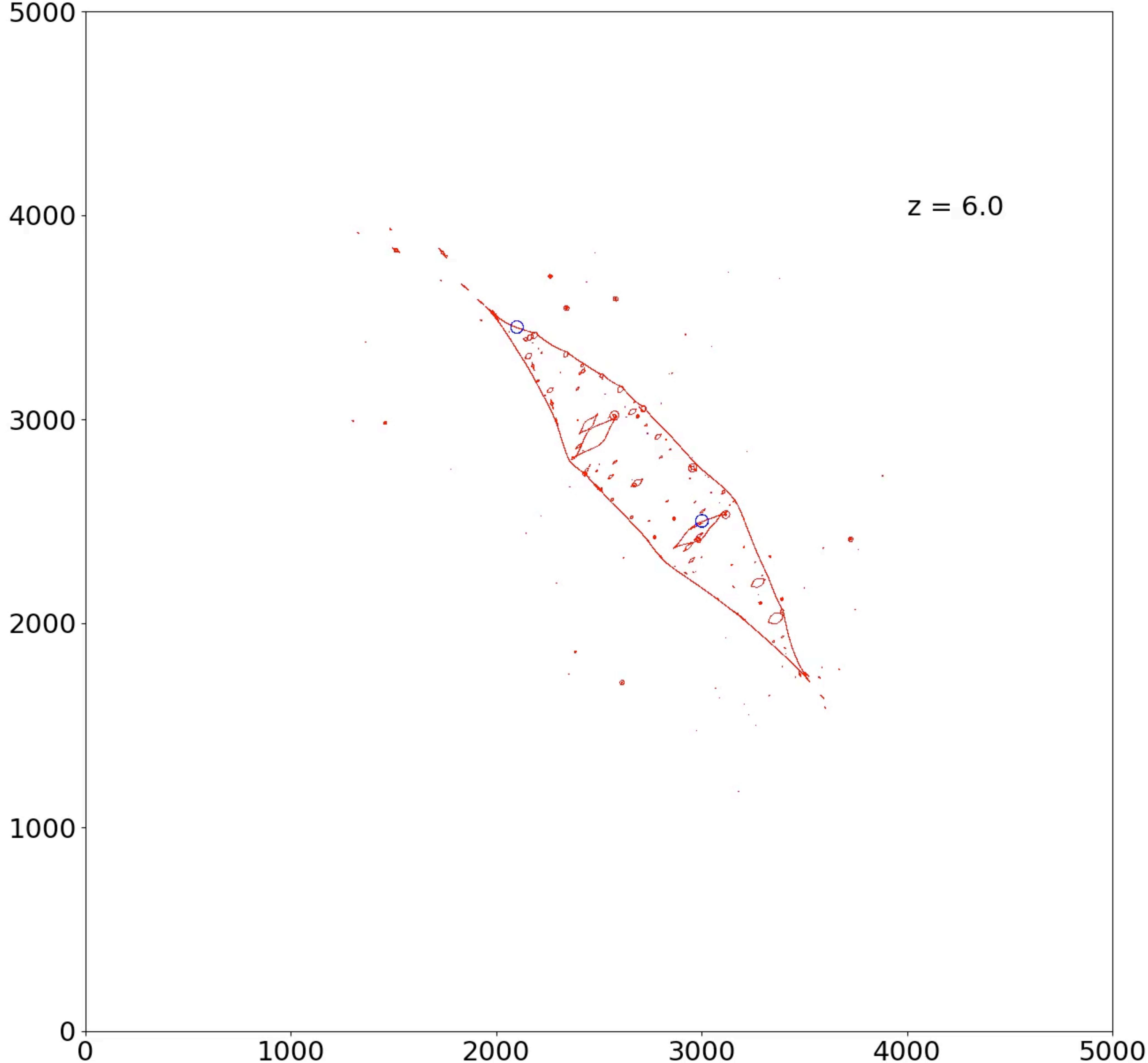



z = 6.0
5000
4000
3000
2000
1000
0
0
1000
2000
3000
4000
5000



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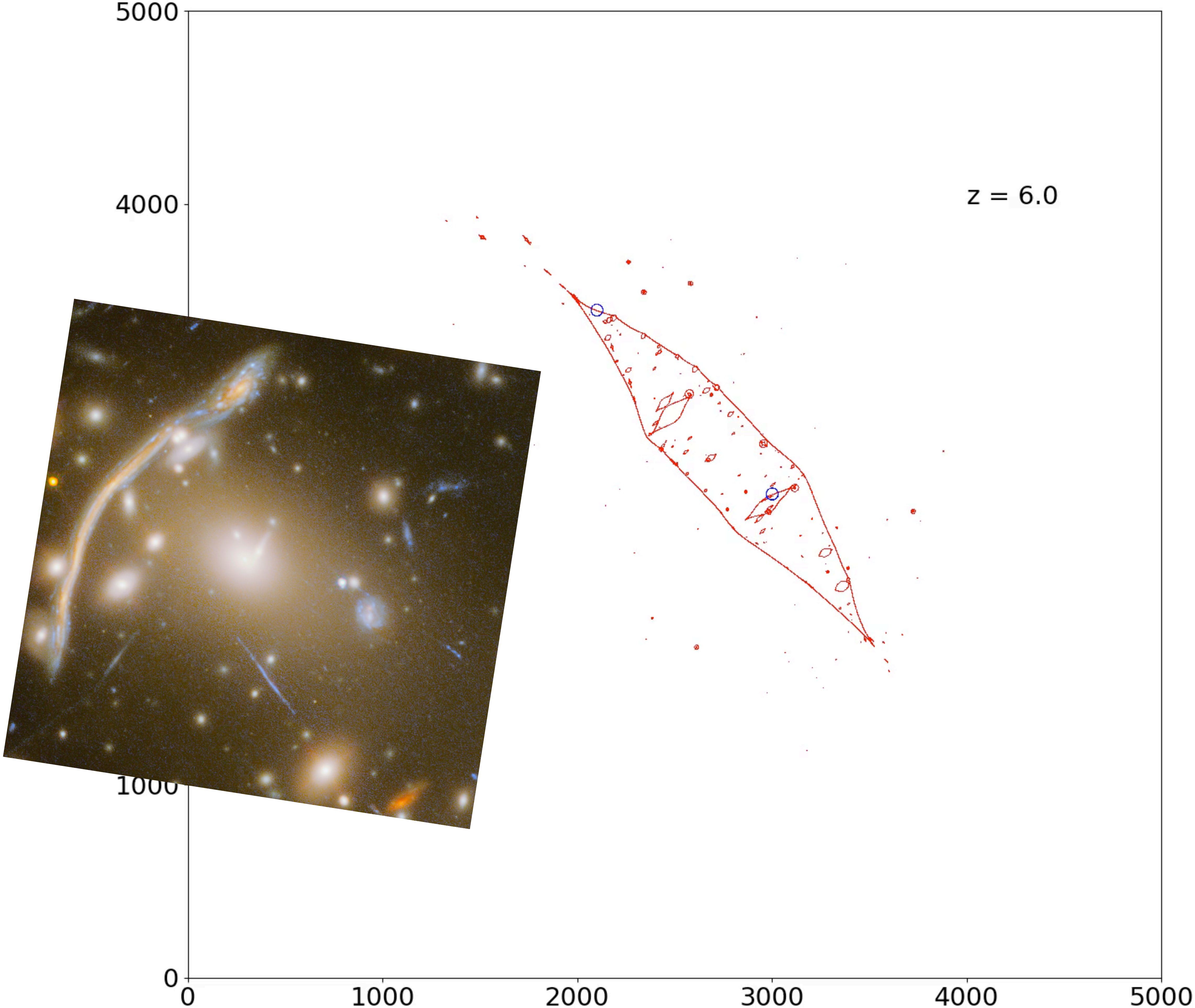

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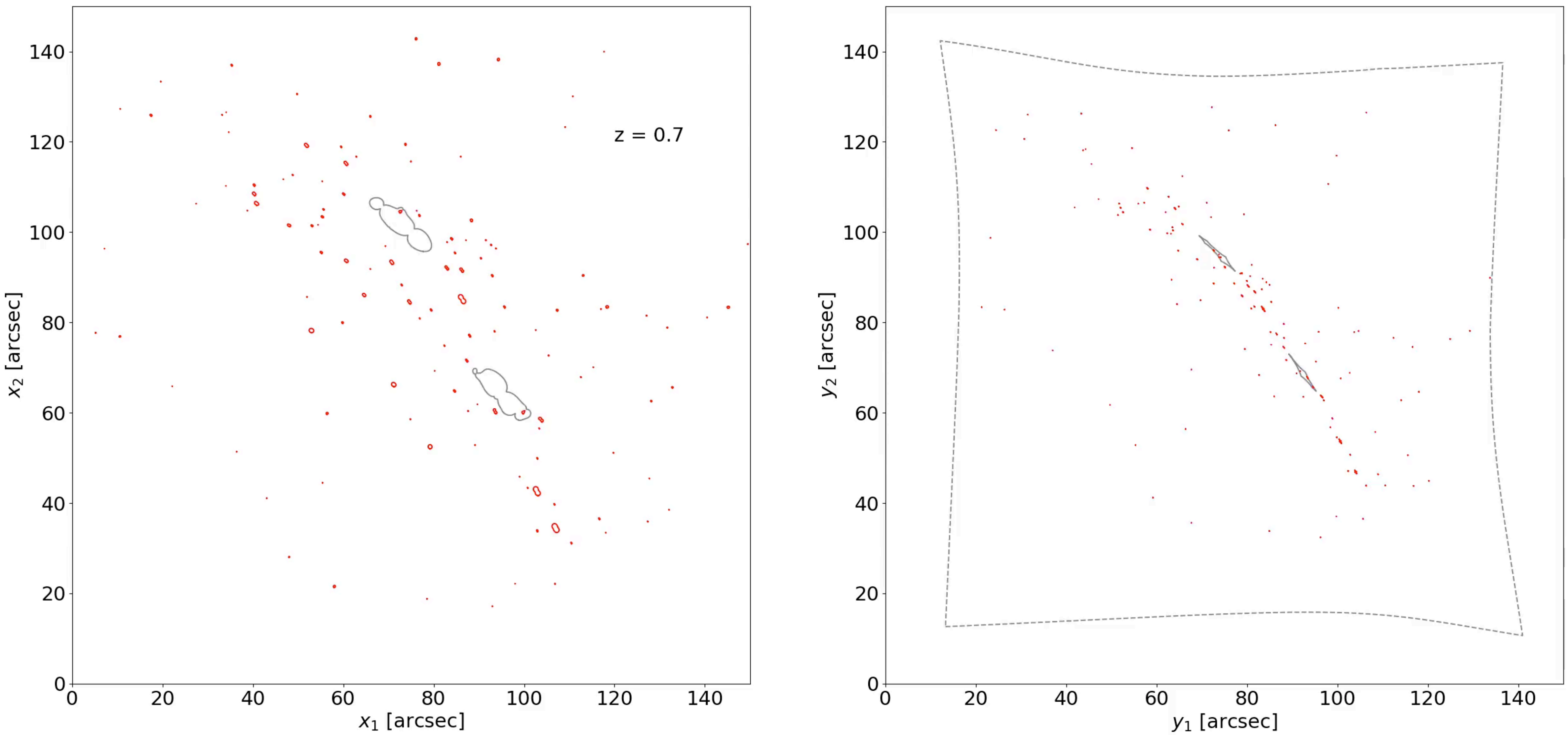
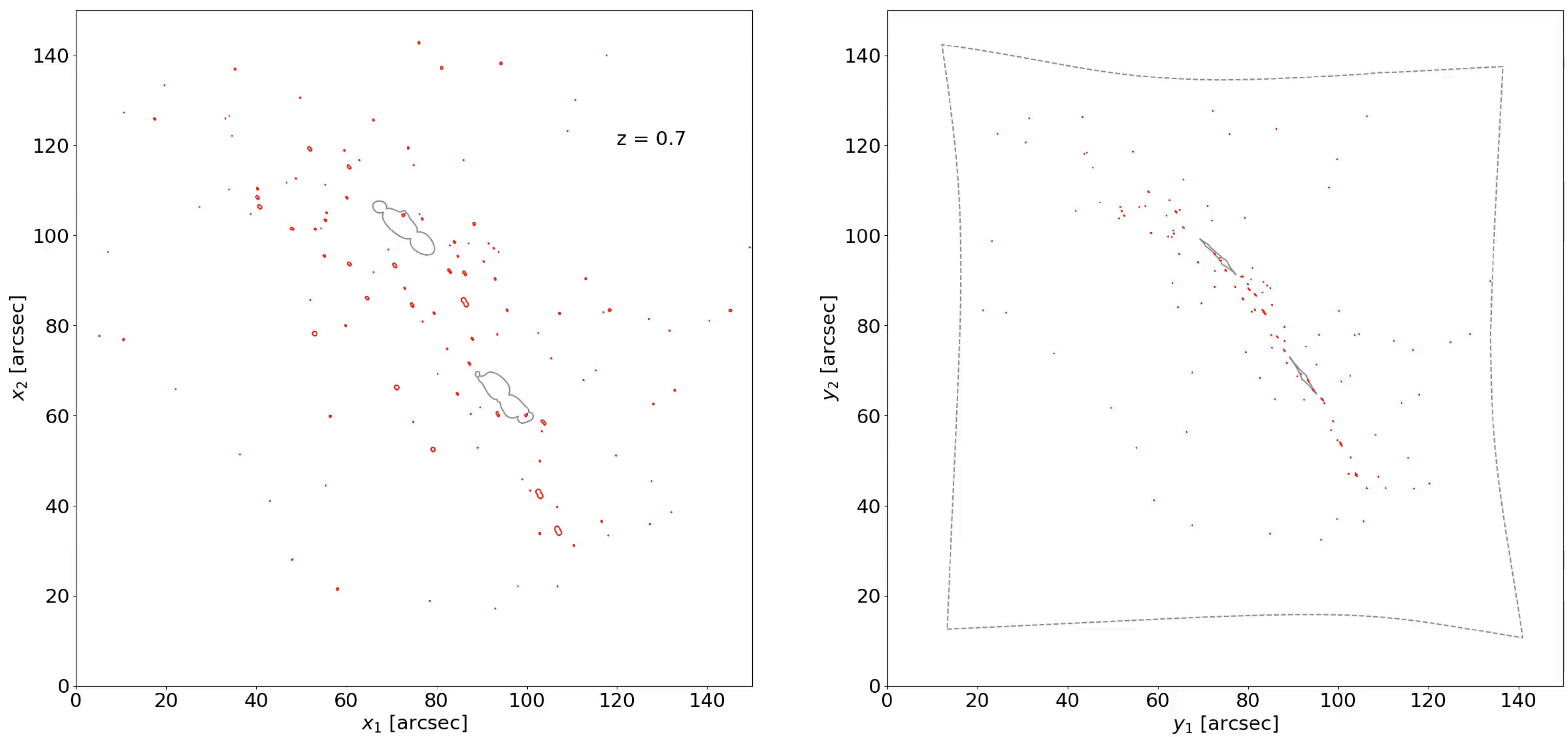


multiple images of two different sources at different redshift constrain the "family ratio"

$$\Xi(z_{\mathrm{d}}, z_{\mathrm{s}_1}, z_{\mathrm{s}_2}) = \frac{D_{\mathrm{ds}_1} D_{\mathrm{s}_2}}{D_{\mathrm{s}_1} D_{\mathrm{ds}_2}}$$

which in turn depends on cosmological parameters (Soucail et al. 2004; Jullo et al. 2010)



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