GRAVITATIONAL LENSING

5 – FLEXION, TIME DELAYS (1)

Massimo Meneghetti

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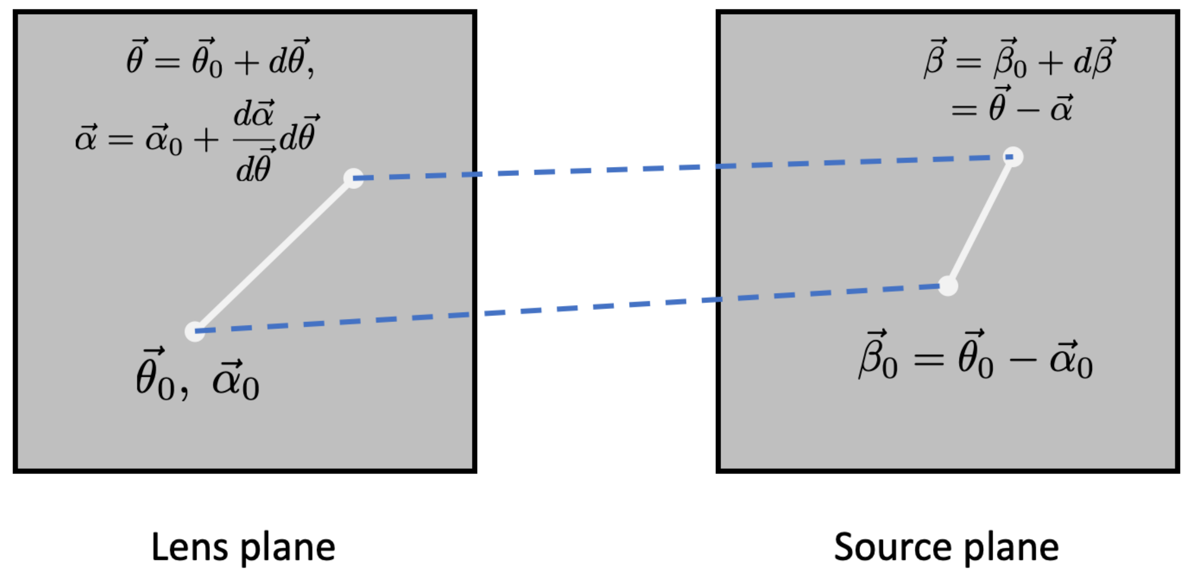
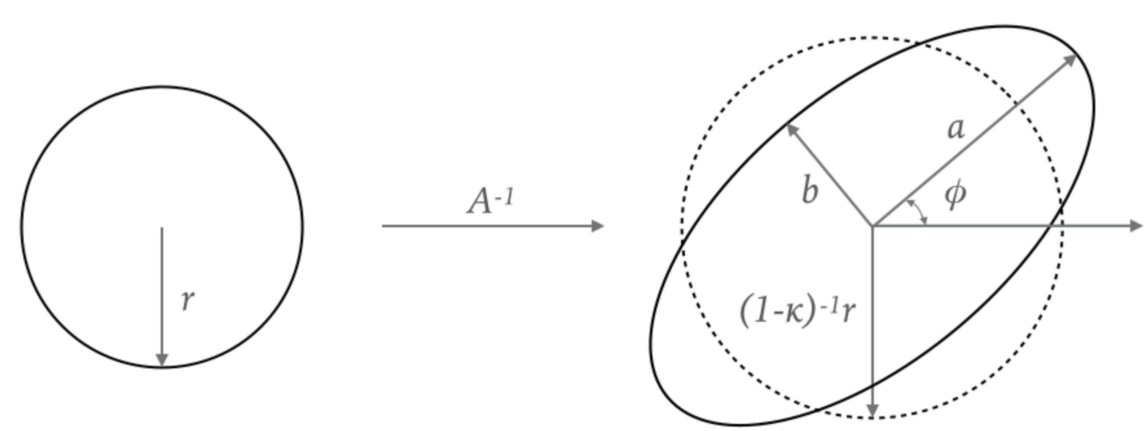
LENS MAPPING AT FIRST ORDER



$$(\vec{\beta} - \vec{\beta}_0) = \left(I - \frac{d\vec{\alpha}(\vec{\theta})}{d\vec{\theta}} \right) (\vec{\theta} - \vec{\theta}_0) \quad A \equiv \frac{d\vec{\beta}}{d\vec{\theta}} = \left(\delta_{ij} - \frac{\partial \alpha_i(\vec{\theta})}{\partial \theta_j} \right) = \left(\delta_{ij} - \frac{\partial^2 \hat{\Psi}(\vec{\theta})}{\partial \theta_i \partial \theta_j} \right) = \left(\delta_{ij} - \hat{\Psi}_{ij} \right)$$

$$\vec{\beta}_0 = (0,0) \qquad \vec{\theta}_0 = (0,0)$$

$$\beta_i \simeq \sum_j A_{ij} \theta_j = \sum_j \frac{\partial \beta_i}{\partial \theta_j} \theta_j$$



LENS MAPPING AT FIRST ORDER

$$\beta_i \simeq \sum_j A_{ij}\theta_j = \sum_j \frac{\partial \beta_i}{\partial \theta_j}\theta_j$$

LENS MAPPING AT FIRST ORDER

$$\beta_i \simeq \sum_j A_{ij}\theta_j = \sum_j \frac{\partial \beta_i}{\partial \theta_j}\theta_j + \frac{1}{2}\sum_j \sum_k \frac{\partial^2 \beta_i}{\partial \theta_j \partial \theta_k}\theta_j \theta_k$$

LENS MAPPING AT FIRST ORDER

$$\beta_i \simeq \sum_j A_{ij}\theta_j = \sum_j \frac{\partial\beta_i}{\partial\theta_j}\theta_j + \frac{1}{2}\sum_j\sum_k \frac{\partial^2\beta_i}{\partial\theta_j\partial\theta_k}\theta_j\theta_k$$

The second order term in the lens equation can be expressed in terms of a 2x2x2 tensor D

$$A_{ij} = \frac{\partial\beta_i}{\partial\theta_j} \qquad D_{ijk} = \frac{\partial^2\beta_i}{\partial\theta_j\partial\theta_k} = \frac{\partial A_{ij}}{\partial\theta_k}$$

$$D_{ij1} = \begin{pmatrix} -2\gamma_{1,1} - \gamma_{2,2} & -\gamma_{2,1} \\ -\gamma_{2,1} & -\gamma_{2,2} \end{pmatrix} \qquad D_{ij2} = \begin{pmatrix} -\gamma_{2,1} & -\gamma_{2,2} \\ -\gamma_{2,2} & 2\gamma_{1,2} - \gamma_{2,1} \end{pmatrix}$$

The elements of D are first derivatives of the shear

$$\gamma_{i,j} = \frac{\partial\gamma_i}{\partial\theta_j}$$

Example: calculate D_{111}

$$D_{ij1} = \begin{pmatrix} \dfrac{\partial A_{11}}{\partial \theta_1} & \dfrac{\partial A_{12}}{\partial \theta_1} \\[2ex] \dfrac{\partial A_{21}}{\partial \theta_1} & \dfrac{\partial A_{22}}{\partial \theta_2} \end{pmatrix} \qquad A_{ij} = \begin{pmatrix} 1-\psi_{11} & -\psi_{12} \\[2ex] -\psi_{21} & 1-\psi_{22} \end{pmatrix}$$

$$= \begin{pmatrix} -\psi_{111} & -\psi_{121} \\[4ex] -\psi_{211} & -\psi_{221} \end{pmatrix}$$

Remember: $\quad \gamma_1 = \frac{1}{2}(\psi_{11} - \psi_{22}) \qquad \gamma_2 = \psi_{12} = \psi_{21}$

$$\gamma_{1,1} = \frac{\partial \gamma_1}{\partial \theta_1} = \frac{1}{2}(\psi_{111} - \psi_{221}) \qquad \gamma_{2,2} = \frac{\partial \gamma_2}{\partial \theta_2} = \psi_{122}$$

Then, we see that: $\quad 2\gamma_{1,1} + \gamma_{2,2} = \psi_{111}$

$$\Rightarrow \qquad D_{111} = -\psi_{111} = -2\gamma_{1,1} - \gamma_{2,2}$$

COMPLEX NOTATION

In the lensing literature, it is common to use the complex notation to represent vectors or pseudo-vectors

$$\vec{v} = (v_1, v_2) \rightarrow v = v_1 + iv_2$$

For example:

$$\vec{\alpha} = (\alpha_1, \alpha_2) \rightarrow \alpha = \alpha_1 + i\alpha_2$$

$$\bar{\gamma} = (\gamma_1, \gamma_2) \rightarrow \gamma = \gamma_1 + i\gamma_2$$

we can also define complex differential operators:

$$\partial = \partial_1 + i\partial_2$$

$$\partial^\dagger = \partial_1 - i\partial_2$$

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we can also define complex differential operators:

$$\partial = \partial_1 + i\partial_2$$ *Spin raising operator*

$$\partial^\dagger = \partial_1 - i\partial_2$$ *Spin lowering operator*

COMPLEX NOTATION

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$$\partial\hat{\Psi} = \partial_1\hat{\Psi} + i\partial_2\hat{\Psi} = \alpha_1 + i\alpha_2 = \alpha$$

From spin-0 scalar field to spin-1 vector field (deflection angle)

COMPLEX NOTATION

$$\partial\hat{\Psi} = \partial_1\hat{\Psi} + i\partial_2\hat{\Psi} = \alpha_1 + i\alpha_2 = \alpha$$

From spin-0 scalar field to spin-1 vector field (deflection angle)

$$\partial^\dagger\partial = \partial_1^2 + \partial_2^2 = \Delta$$

$$\partial^\dagger\partial\hat{\Psi} = \Delta\hat{\Psi} = 2\kappa$$

From spin-1 vector field to spin-0 scalar field

COMPLEX NOTATION

$$\partial\hat{\Psi} = \partial_1\hat{\Psi} + i\partial_2\hat{\Psi} = \alpha_1 + i\alpha_2 = \alpha$$

From spin-0 scalar field to spin-1 vector field (deflection angle)

$$\partial^\dagger\partial = \partial_1^2 + \partial_2^2 = \Delta$$
$$\partial^\dagger\partial\hat{\Psi} = \Delta\hat{\Psi} = 2\kappa$$

From spin-1 vector field to spin-0 scalar field

$$\begin{aligned}\frac{1}{2}\partial\partial\hat{\Psi} &= \frac{1}{2}\partial\alpha = \frac{1}{2}(\partial_1 + i\partial_2)(\alpha_1 + i\alpha_2)\\ &= \frac{1}{2}[(\partial_1\alpha_1 - \partial_2\alpha_2) + i(\partial_2\alpha_1 + \partial_1\alpha_2)]\\ &= \gamma_1 + i\gamma_2 = \gamma\end{aligned}$$

The shear is a spin-2 field

COMPLEX NOTATION

$$
\begin{aligned}
F &= \frac{1}{2}\partial\partial^{\dagger}\partial\hat{\Psi} = \partial\kappa \\
G &= \frac{1}{2}\partial\partial\partial\hat{\Psi} = \partial\gamma
\end{aligned}
$$

COMPLEX NOTATION

$$\begin{aligned} F &= \frac{1}{2}\partial\partial^\dagger\partial\hat{\Psi} = \partial\kappa \\ G &= \frac{1}{2}\partial\partial\partial\hat{\Psi} = \partial\gamma \end{aligned}$$

Spin-1

Spin-3

COMPLEX NOTATION

$$F = \frac{1}{2}\partial\partial^\dagger\partial\hat{\Psi} = \partial\kappa$$ *Spin-1*

$$G = \frac{1}{2}\partial\partial\partial\hat{\Psi} = \partial\gamma$$ *Spin-3*

$$F = F_1 + iF_2 = (\gamma_{1,1} + \gamma_{2,2}) + i(\gamma_{2,1} - \gamma_{1,2})$$

$$G = G_1 + iG_2 = (\gamma_{1,1} - \gamma_{2,2}) + i(\gamma_{2,1} + \gamma_{1,2})$$

$$D_{ij1} = \begin{pmatrix} -2\gamma_{1,1} - \gamma_{2,2} & -\gamma_{2,1} \\ -\gamma_{2,1} & -\gamma_{2,2} \end{pmatrix} \qquad D_{ij2} = \begin{pmatrix} -\gamma_{2,1} & -\gamma_{2,2} \\ -\gamma_{2,2} & 2\gamma_{1,2} - \gamma_{2,1} \end{pmatrix}$$

$$D_{111} = -2\gamma_{11} - \gamma_{22} = -\frac{1}{2}(3F_1 + G_1)$$
$$D_{211} = D_{121} = D_{112} = -\gamma_{21} = -\frac{1}{2}(F_2 + G_2)$$
$$D_{122} = D_{212} = D_{221} = -\gamma_{22} = -\frac{1}{2}(F_1 - G_1)$$
$$D_{222} = 2\gamma_{12} - \gamma_{21} = -\frac{1}{2}(3F_2 - G_2)$$

Example: derivation of $\mathcal{F}$

$$\mathcal{F} = \partial\kappa \qquad \kappa = \frac{1}{2}\left(\psi_{11} + \psi_{22}\right) \qquad \gamma_1 = \frac{1}{2}\left(\psi_{11} - \psi_{22}\right)$$

$$\gamma_2 = \psi_{12} = \psi_{21}$$

$$\mathcal{F} = \frac{\partial\kappa}{\partial\theta_1} + i\frac{\partial\kappa}{\partial\theta_2} = \frac{1}{2}\left(\psi_{111} + \psi_{221}\right) + i\,\frac{1}{2}\left(\psi_{112} + \psi_{222}\right)$$

Note that:

$$\gamma_{1,1} = \frac{1}{2}\left(\psi_{111} - \psi_{221}\right) \qquad \gamma_{2,1} = \psi_{121}$$

$$\gamma_{1,2} = \frac{1}{2}\left(\psi_{112} - \psi_{222}\right) \qquad \gamma_{2,2} = \psi_{122}$$

$$\Rightarrow \frac{1}{2}\left(\psi_{111} + \psi_{221}\right) = \frac{1}{2}\left(\psi_{111} - \psi_{221} + 2\psi_{221}\right) = \gamma_{1,1} + \gamma_{2,2}$$

$$\frac{1}{2}\left(\psi_{112} + \psi_{222}\right) = \frac{1}{2}\left(2\psi_{112} - \psi_{112} + \psi_{222}\right) = \gamma_{2,1} - \gamma_{1,2}$$

$$\Rightarrow \mathcal{F} = \left(\gamma_{1,1} + \gamma_{2,2}\right) + i\left(\gamma_{2,1} - \gamma_{1,2}\right)$$

Notebook 5: building a app to visualise the effects of convergence, shear, and flexion

GRAVITATIONAL TIME DELAY

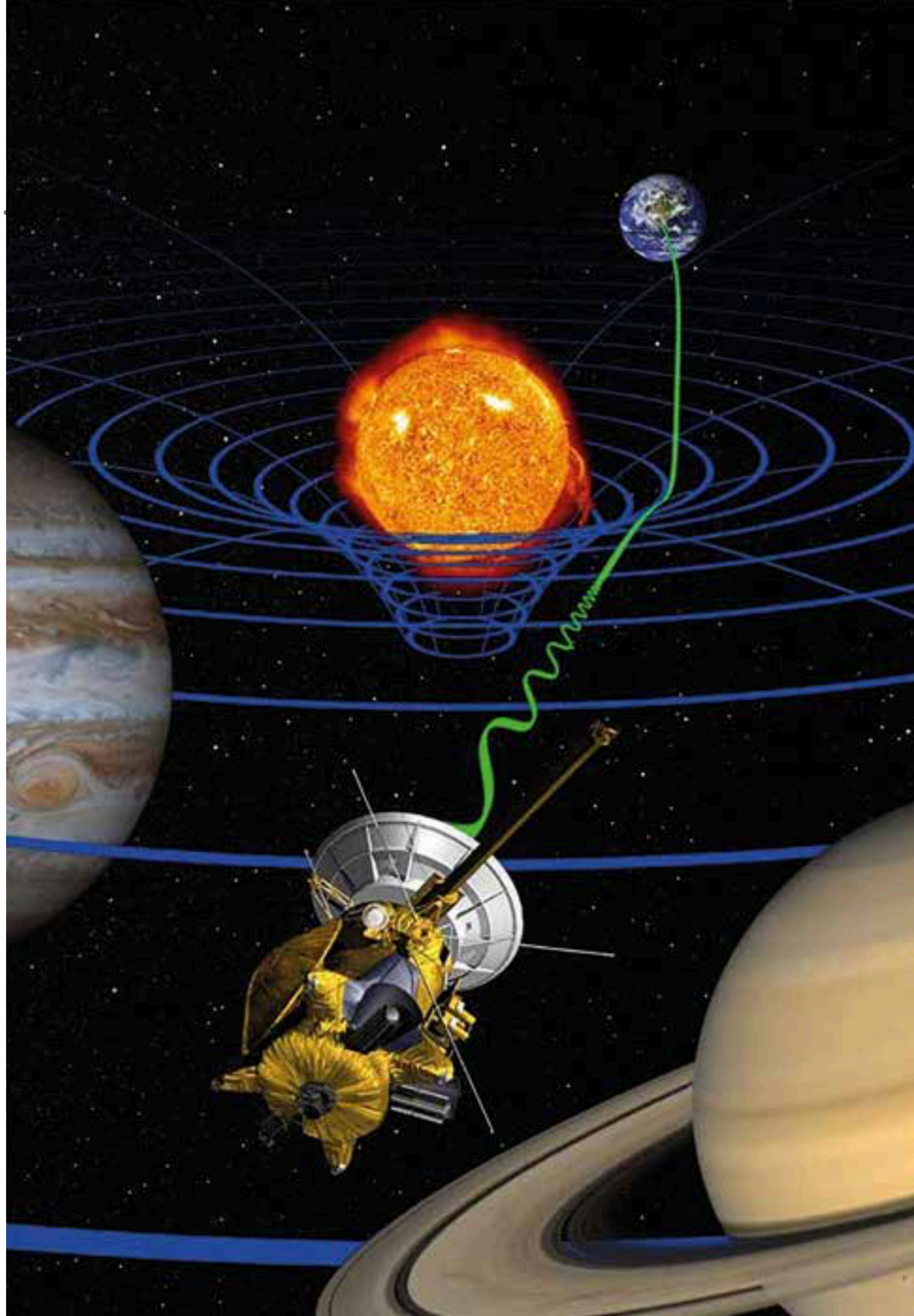
- In a lensing phenomenon, light travels with an effective velocity $c' < c$. As seen, this implies an effective refractive index $n > 1$
- The effective refractive index is expressed in terms of the Newtonian potential
- If we compare the travel times of two photons, one traveling at velocity c and the other at velocity c', we notice that the second accumulates a time delay t_{grav}
- This time delay is called *gravitational* time delay, or *Shapiro* time delay (Shapiro, 1964)

$$n = 1 - \frac{2\Phi}{c^2}$$

$$\begin{aligned} t_{grav} &= \int \frac{dz}{c'} - \int \frac{dz}{c} \\ &= \frac{1}{c} \int (n-1) dz \\ &= -\frac{2}{c^3} \int \Phi dz \end{aligned}$$

SHAPIRO DELAY AS A TEST OF GR

- Send a radio signal towards another planet (Mercury, Venus, Mars) behind the sun
- Measure time needed to the signal to come back after being reflected
- Measurement done in 2003 with the Cassini spacecraft
- Delay is few ~100 microseconds
- GR confirmed at the level of 0.002%



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&= \frac{1}{c} \int (n-1) dz \\
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\end{aligned}
$$

$$\hat{\Psi}(\vec{\theta}) = \frac{D_{LS}}{D_L D_S} \frac{2}{c^2} \int \Phi(\vec{\theta}, z) dz$$

$$t_{grav} = -\frac{1}{c} \frac{D_S D_L}{D_{LS}} \hat{\Psi}$$

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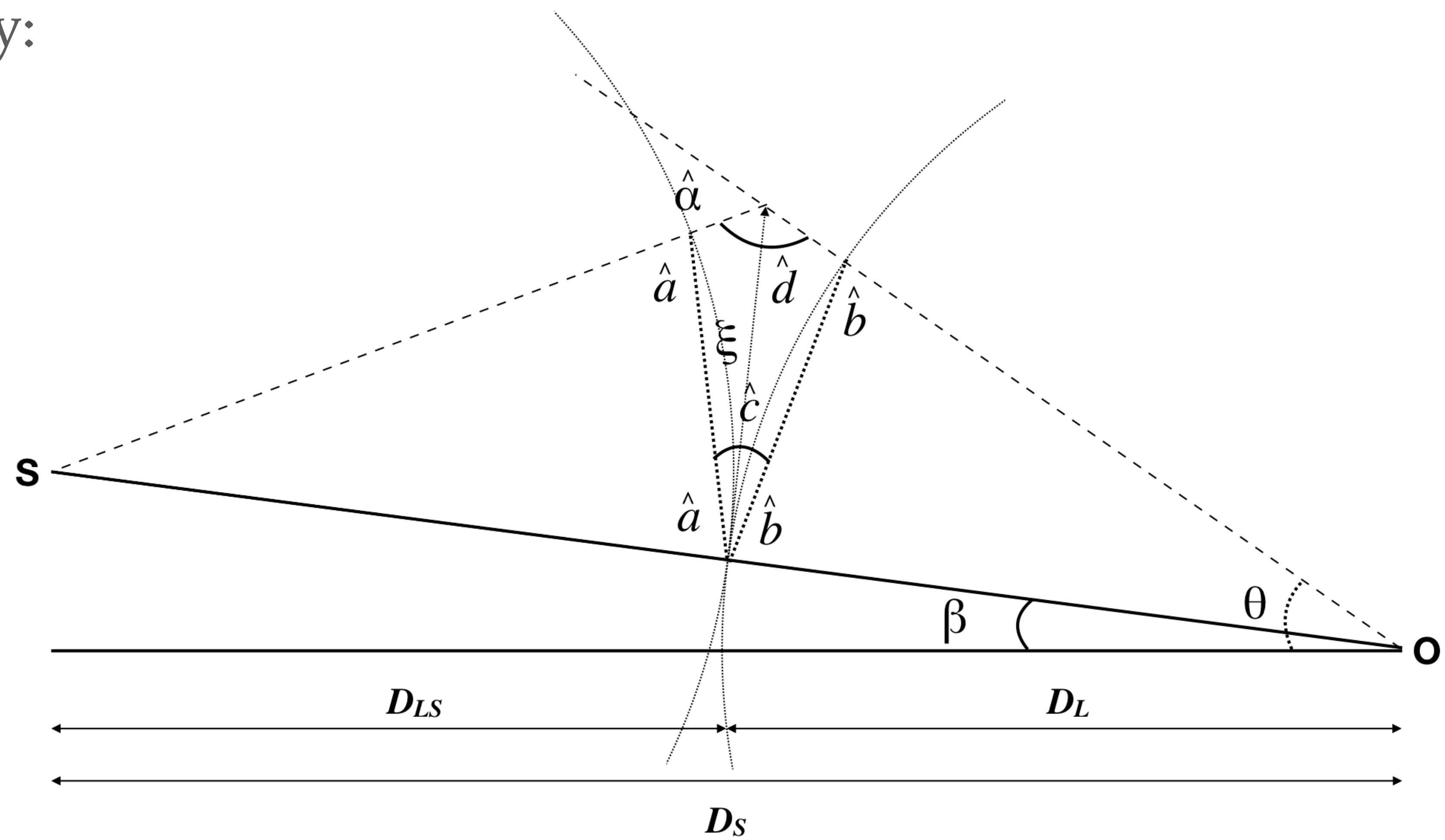
$$t_{grav} = -\frac{1}{c}\frac{D_S D_L}{D_{LS}}\hat{\Psi}$$

This time delay does not account yet for the different path of photons!

GEOMETRICAL TIME DELAY

- We need to combine the gravitational time delay to the so called *geometrical* time delay:

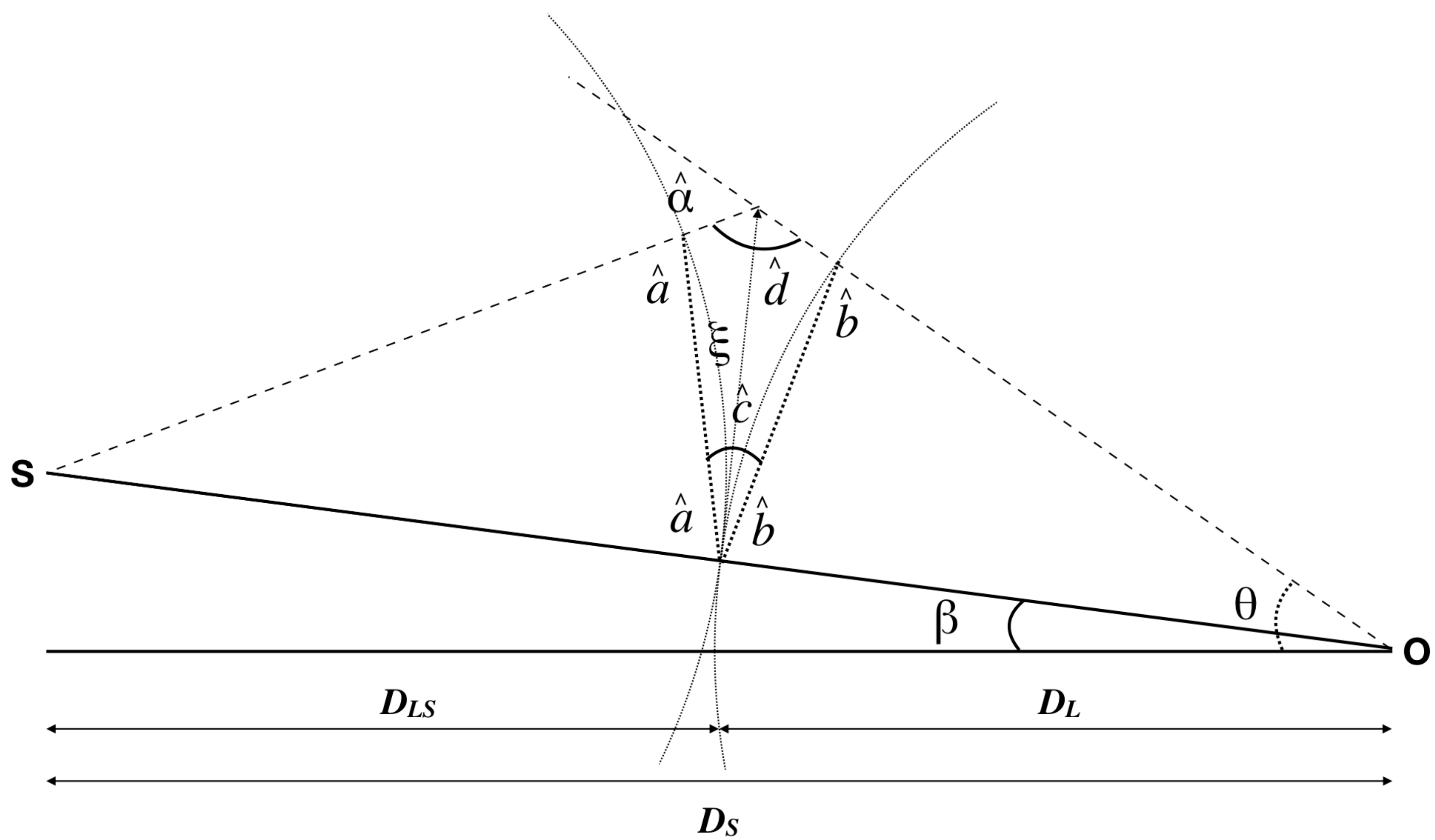


$$(\pi - \hat{a}) + (\pi - \hat{b}) + \hat{c} + \hat{d} = 2\pi \Rightarrow \hat{a} + \hat{b} = \hat{c} + \hat{d}$$

$$\hat{a} + \hat{b} + \hat{c} = \pi \Rightarrow 2\hat{c} = \pi - \hat{d}$$

$$\hat{\alpha} + \hat{d} = \pi \Rightarrow \hat{d} = \pi - \hat{\alpha}$$

GEOMETRICAL TIME DELAY



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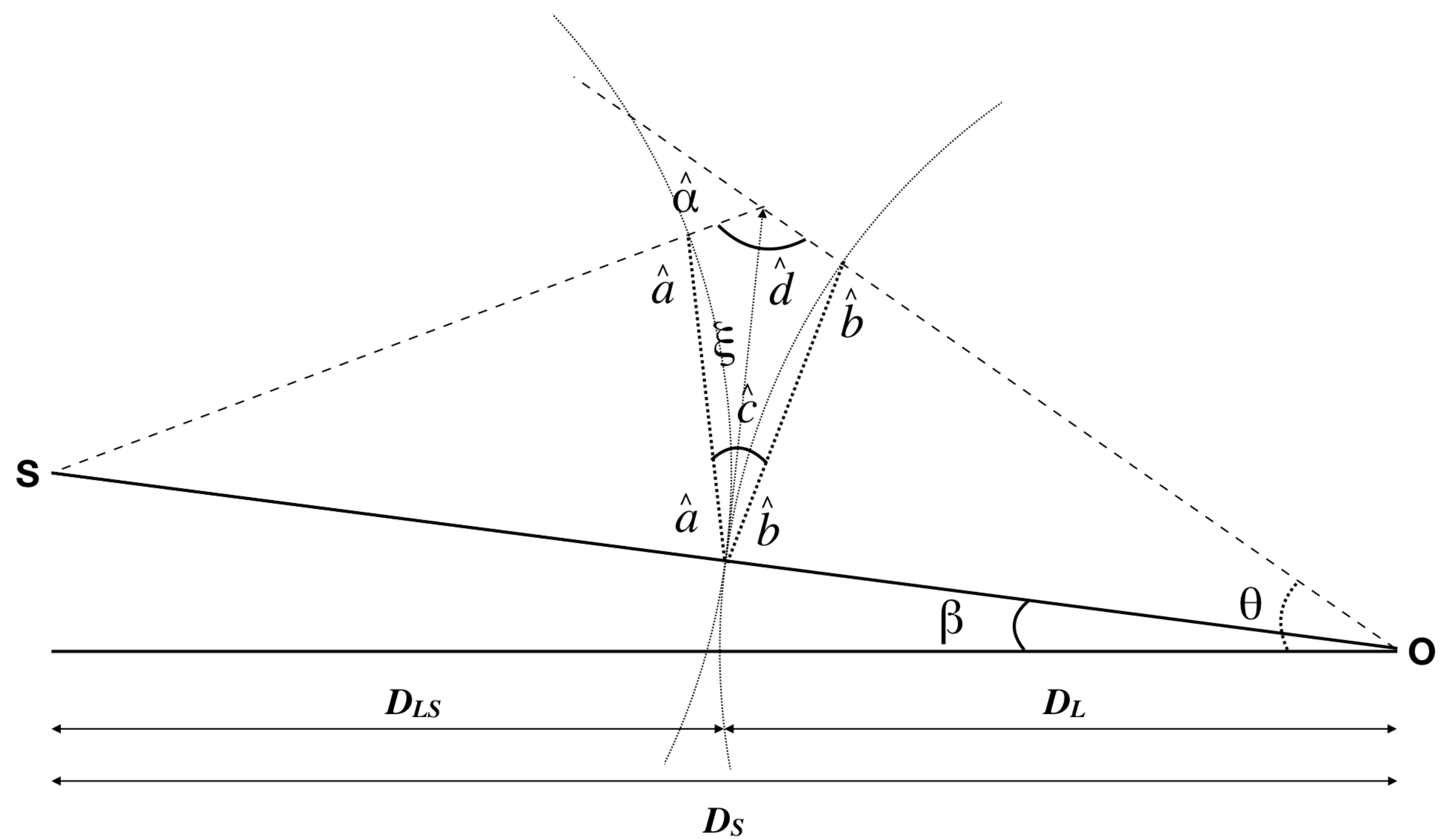
$$\hat{\alpha} + \hat{d} = \pi \Rightarrow \hat{d} = \pi - \hat{\alpha}$$

$$\begin{aligned} \Rightarrow \hat{c} &= \frac{\hat{\alpha}}{2} \\ &= \frac{1}{2}\frac{D_S}{D_{LS}}\alpha \\ &= \frac{1}{2}\frac{D_S}{D_{LS}}(\theta - \beta) \end{aligned}$$

GEOMETRICAL TIME DELAY



$$
\begin{aligned}
\Rightarrow \hat{c} &= \frac{\hat{\alpha}}{2} \\
&= \frac{1}{2}\frac{D_S}{D_{LS}}\alpha \\
&= \frac{1}{2}\frac{D_S}{D_{LS}}(\theta-\beta)
\end{aligned}
$$

$$
\begin{aligned}
\xi &= D_L(\theta-\beta) \\
t_{geom} &= \frac{1}{c}\xi\hat{c} \\
&= \frac{1}{2c}\frac{D_S D_L}{D_{LS}}(\theta-\beta)^2
\end{aligned}
$$

TOTAL TIME DELAY

$$t_{geom} = \frac{1}{2c}\frac{D_S D_L}{D_{LS}}(\theta - \beta)^2$$

$$t_{grav} = -\frac{1}{c}\frac{D_S D_L}{D_{LS}}\hat{\Psi}$$

$$\begin{aligned}
t_{tot} &= t_{geom} + t_{grav} \\
&= \frac{1}{2c}\frac{D_S D_L}{D_{LS}}(\theta - \beta)^2 - \frac{1}{c}\frac{D_S D_L}{D_{LS}}\hat{\Psi}(\theta) \\
&= \frac{1}{c}\frac{D_S D_L}{D_{LS}}\left[\frac{1}{2}(\theta - \beta)^2 - \hat{\Psi}(\theta)\right]
\end{aligned}$$

TOTAL TIME DELAY

$$
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&= \frac{1}{2c}\frac{D_S D_L}{D_{LS}}(\theta - \beta)^2 - \frac{1}{c}\frac{D_S D_L}{D_{LS}}\hat{\Psi}(\theta) \\
&= \frac{1}{c}\frac{D_S D_L}{D_{LS}}\left[\frac{1}{2}(\theta - \beta)^2 - \hat{\Psi}(\theta)\right]
\end{aligned}
$$

Accounting for the expansion of the universe and for the fact that this is a surface:

$$
t_{tot}(\vec{\theta}) = \frac{1 + z_L}{c}\frac{D_S D_L}{D_{LS}}\left[\frac{1}{2}(\vec{\theta} - \vec{\beta})^2 - \hat{\Psi}(\vec{\theta})\right]
$$