The methodology for solving the problem

Basically, we want $a = xF_1 + x^2F_2 + x^3F_3 + \dots$ to equal an integer for x being a rational number.

So we know that the recurrence relation for the fibonacci numbers is $F_n = F_{n-1} + F_{n-2}$. Then, using the recurrence relation $F_n = \frac{(1+\sqrt{5})^n}{2^n\sqrt{5}} + \frac{(1-\sqrt{5})^n}{2^n\sqrt{5}}$

Then if we have $A_F(x) = xF_1 + x^2F_2 + x^3F_3 + \dots$, this is equivalent to saying

$$A_F(x) = \sum_{k=1}^{\infty} x^k F_k = \sum_{k=1}^{\infty} x^k \frac{(1+\sqrt{5})^k}{2^k\sqrt{5}} + x^k \frac{(1-\sqrt{5})^k}{2^k\sqrt{5}}$$

Then the sum of the terms come out to be $\frac{1}{\sqrt{5}}(\frac{2}{2-x(1+\sqrt{5})} - \frac{2}{2-x(1-\sqrt{5})})$, or simplified out, $a = \frac{-4x}{4x^2+4x+4} = \frac{-x}{x^2+x+1} = a$.

Cross multiplying it out, we get that $-x = a(x^2+x-1)$, or $ax^2+(a+1)x-a = 0$. Since we want rational solutions, the x inputted to this must be rational, thus $\frac{-a\pm\sqrt{(a+1)^2+4a^2}}{2a}$ must be rational $\Rightarrow \sqrt{(a+1)^2+4a^2}$ must be rational, thus $(a+1)^2+4a^2 = 5a^2+2a+1$ must be a square.

Now for the less elegant part. Given the hint that the first solution is 2, followed by 15, 104, and 714, I concluded that there was a ratio between answers approximated to be 6.854, a number I updated at each stage (to reduce the number of computations at each step). Thus, given an answer, start checking at 6.854-ish * answer to find the next answer. After repeating enough times, the solution 1120149658760 is found $\square$