# FLOP – FREE LIST OF OPEN PROBLEMS

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## 1. Notation

| | |
|---|---|
| $\mathbb{C}\langle x_1,\dots,x_{\mathtt{g}}\rangle$ | free $\mathbb{C}$-algebra generated by $x_1,\dots,x_{\mathtt{g}}$ |
| $\mathbb{C}\langle\!\langle x_1,\dots,x_{\mathtt{g}}\rangle\!\rangle$ | free skew field (nc rationals) of $\mathbb{C}\langle x_1,\dots,x_{\mathtt{g}}\rangle$ |
| $\mathbb{C}\langle\!\langle\mathbb{x}\rangle\!\rangle_0$ | algebra of rationals regular at 0 |
| $\mathbb{C}\langle\!\langle\mathbb{x}\leftrightarrow\mathbb{y}\rangle\!\rangle$ | universal skew field of fractions of $\mathbb{C}\langle\!\langle\mathbb{x}\rangle\!\rangle \otimes \mathbb{C}\langle\!\langle\mathbb{y}\rangle\!\rangle$ |

sec:Elem_Mats

## 2. Factoring Invertible Matrices over $\mathbb{C}\langle\mathbb{x}\rangle \otimes \mathbb{C}\langle\mathbb{y}\rangle$

**Problem 2.1.** Suppose $A \in M_n(\mathbb{C}\langle\mathbb{x}\rangle \otimes \mathbb{C}\langle\mathbb{y}\rangle)$. If $A$ is invertible over the $\mathbb{C}$-algebra $\mathbb{C}\langle\mathbb{x}\rangle \otimes \mathbb{C}\langle\mathbb{y}\rangle$ then do there exist $D \in M_n(\mathbb{C})$ and $E_1,\dots,E_k \in M_n(\mathbb{C}\langle\mathbb{x}\rangle \otimes \mathbb{C}\langle\mathbb{y}\rangle)$ such that $D$ is diagonal, each $E_i$ is an elementary matrix and $A = DE_1\dots E_k$?

**Remark 2.2.** The above problem is false when $n=2$. Cohn gave the following matrix that is not a product of elementary matrices over $\mathbb{C}[x]\otimes\mathbb{C}[y]$:

$$\begin{pmatrix} 1\otimes 1 + x\otimes y & 1\otimes y^2 \\ -x^2\otimes 1 & 1\otimes 1 - x\otimes y \end{pmatrix}.$$

The above counterexample is nonexistent when our matrices are over $\mathbb{C}\langle\mathbb{x}\rangle$ instead. The proof of this uses results from SuslinCohn06 [Coh06].

**Theorem 2.3.** *If $A \in M_n(\mathbb{C}\langle\mathbb{x}\rangle)$ is invertible, then there exist $D \in M_n(\mathbb{C})$ and $E_1,\dots,E_k \in M_n(\mathbb{C}\langle\mathbb{x}\rangle)$ such that $D$ is diagonal, each $E_i$ is an elementary matrix and $A = DE_1\dots E_k$.*

The Cohn counterexample is essentially the only issue:

**Theorem 2.4** (Suslin's Stability Theorem)**.** *Suppose $A \in M_n(\mathbb{C}[t_1,\dots,t_g])$ where $n\geq 3$. If $\det(A)=1$ then there exist $E_1,\dots,E_k \in M_n(\mathbb{C}[t_1,\dots,t_g])$ such that $A = E_1\dots E_k$.*

An "algorithmic" proof of the above theorem can be found in SuslinPW95 [PW95].

**Remark 2.5.** An idea related to this is the notion of Jacobian Tame, see sec:Jac Tame 4.

## References


[Coh06] P.M. Cohn. *Free Ideal Ring and Localizations in General Rings*. Cambridge University Press, 2006.

[PW95] H.J. Park and C. Woodburn. An algorithmic proof of Suslin's stability theorem for polynomial rings. *Journal of Algebra*, 178(1):277 – 298, 1995.


sec:Luroth

## 3. Free Lüroth Theorem

**Problem 3.1** (Free Lüroth Theorem)**.** Suppose $\boldsymbol{k}$ is an algebraically closed infinite field and $x_1, \ldots, x_g$ are freely noncommuting indeterminates ($g \geq 2$). If $\boldsymbol{k} \subsetneq D \subsetneq \boldsymbol{k}\langle\!\!\!<\! x_1, \ldots, x_{\mathtt{g}} \!>\!\!\!\rangle$ is a subfield, then do there exist $q_1, \ldots, q_{\mathtt{h}} \in \mathbb{C}\langle\!\!\!<\! x_1, \ldots, x_{\mathtt{g}} \!>\!\!\!\rangle$ such that $D = \mathbb{C}\langle\!\!\!<\! q_1, \ldots, q_{\mathtt{h}} \!>\!\!\!\rangle$?

In other words, is every non-trivial subfield of $\mathbb{C}\langle\!\!\!<\! \mathtt{x} \!>\!\!\!\rangle$ a free skew field?

**Theorem 3.2** (Lüroth's Theorem)**.** *Suppose $\boldsymbol{k}$ is a field. If $\boldsymbol{k} \subsetneq D \subsetneq \boldsymbol{k}(t)$ is a subfield, then there exists $q \in \boldsymbol{k}(t)$ such that $D = \boldsymbol{k}(q(t))$.*

**Remark 3.3.** If $\boldsymbol{k}$ is algebraically closed and infinite, then Lüroth's Theorem holds as well for $\boldsymbol{k}(t_1, t_2)$.

On the other hand, there are counterexamples to Lüroth's Theorem for $\boldsymbol{k}(t_1, t_2, t_3)$.

**Remark 3.4.** Schofield ref? shows that if $f, g \in \mathbb{C}\langle\!\!\!<\! \mathtt{x} \!>\!\!\!\rangle$, then either $[f, g] = 0$ or $\mathbb{C}\langle\!\!\!<\! f, g \!>\!\!\!\rangle$ is free.

sec:Jac Tame

## 4. Jacobian Tame

An automorphism $\tau$ of the free algebra $\mathbb{C}\langle x_1, \ldots, x_{\mathtt{g}}\rangle$ is **elementary** if $\tau : x_i \mapsto x_i$ for $i \neq j$ and $\tau : x_j \mapsto cx_j + f$, where $c \in \mathbb{C}$ and $f \in \mathbb{C}\langle x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_{\mathtt{g}}\rangle$. We say an automorphism is **tame** if it is a composition of elementary automorphisms. If the automorphism is not tame, then it is **wild**.

In JacTame07 [Umi07] it is shown that the Anick automorphism,

$$\delta(x, y, z) = (x + z(xz - zy), y + (xz - zy)z, z) \in \mathbb{C}\langle x, y, z\rangle$$

is wild. Looking at it, the Anick automorphism doesn't seem particularly "wild," so maybe this can be improved? Perhaps it is something about Jacobian matrices?

**Definition 4.1.** We say an automorphism $\tau$ of $\mathbb{C}\langle x_1, \ldots, x_{\mathtt{g}}\rangle$ is **Jacobian Tame** if $J_\tau = DE_1 \ldots E_k$, where $D \in M_{\mathtt{g}}(\mathbb{C}\langle \mathtt{x}'\rangle^{opp} \otimes \mathbb{C}\langle \mathtt{x}\rangle)$ is diagonal and $E_1, \ldots, E_k \in M_{\mathtt{g}}(\mathbb{C}\langle \mathtt{x}'\rangle^{opp} \otimes \mathbb{C}\langle \mathtt{x}\rangle)$ are elementary matrices.

If such a factorization does not exist, then we say $\tau$ is **Jacobian wild**.

**Problem 4.2.** Are there any Jacobian wild automorphisms of the free algebra? The only known (to me) example of a wild automorphism of $\mathbb{C}\langle x, y, z\rangle$ is Jacobian tame (this is explained below).

In the commutative case there are no Jacobian wild automorphisms. Every automorphism of $\mathbb{C}[t_1, t_2]$ is tame, hence Jacobian tame. If $\phi$ is an automorphism of $\mathbb{C}[t_1, \ldots, t_{\mathtt{g}}]$ with $\mathtt{g} > 2$, then its Jacobian matrix $J_\phi \in M_{\mathtt{g}}(\mathbb{C}[t_1, \ldots, t_{\mathtt{g}}, t'_1, \ldots, t'_{\mathtt{g}}])$ factors into such a product by Suslin's Stability Theorem (see sec:Elem Mats 2), thus is Jacobian tame.

Throughout this I will be using the transposed Jacobian matrix: $J_\tau \in M_{\mathtt{g}}(\mathbb{C}\langle \mathbb{x}' \rangle^{opp} \otimes \mathbb{C}\langle \mathbb{x} \rangle)$ where the $i^{\text{th}}$ column of $J_\tau$ corresponds to the derivatives of $\tau_i$.

It turns out that the Anick automorphism

$$\delta(x,y,z) = (x + z(xz - zy), y + (xz - zy)z, z) \in \mathbb{C}\langle x, y, z\rangle$$

has a Jacobian matrix that can be written as a product of elementary matrices. The Jacobian matrix of $\delta$ is

$$J_\delta = \begin{pmatrix} 1\otimes 1 + z'\otimes z & 1\otimes z^2 & 0 \\ -(z')^2\otimes 1 & 1 - z'\otimes z & 0 \\ \zeta_1 & \zeta_2 & 1 \end{pmatrix}$$

where $\zeta_1 = 1\otimes xz + x'z'\otimes 1 - 1\otimes zy - z'\otimes y$ and $\zeta_2 = x\otimes z + z'x'\otimes 1 - 1\otimes yz - y'z'\otimes 1$. Let $E_{i,j}(\alpha) = I + \alpha e_{i,j}$ ($e_{i,j}$ has a 1 in the $i,j$ entry and 0's elsewhere) and observe

$$J_\delta E_{3,1}(-\zeta_1)E_{3,2}(-\zeta_2) = \begin{pmatrix} 1\otimes 1 + z'\otimes z & 1\otimes z^2 & 0 \\ -(z')^2\otimes 1 & 1 - z'\otimes z & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

In 1966, Cohn proved that

$$\mathfrak{C} = \begin{pmatrix} 1\otimes 1 + z'\otimes z & 1\otimes z^2 \\ -(z')^2\otimes 1 & 1 - z'\otimes z \end{pmatrix}$$

cannot be written as a product of elementary matrices over $\mathbb{F}[z'\otimes 1, 1\otimes z]$. This is a crucial aspect of Umirbaev's ([Umi07]) proof that the Anick automorphism is wild. However, Park and Woodburn [PW95] give a decomposition of $\left(\begin{smallmatrix} \mathfrak{C} & 0 \\ 0 & 1 \end{smallmatrix}\right)$ into a product of elementary matrices:

$$\begin{aligned} J_\delta E_{3,1}(-\zeta_1)E_{3,2}(-\zeta_2) &= \begin{pmatrix} 1\otimes 1 + z'\otimes z & 1\otimes z^2 & 0 \\ -(z')^2\otimes 1 & 1 - z'\otimes z & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= E_{2,3}(-z'\otimes 1)E_{1,3}(1\otimes z)E_{3,2}(1\otimes z)E_{3,1}(z'\otimes 1) \\ &\quad E_{2,3}(z'\otimes 1)E_{1,3}(-1\otimes z)E_{3,2}(-1\otimes z)E_{3,1}(-z'\otimes 1). \end{aligned}$$

Thus

$$\begin{aligned} J_\delta =& E_{2,3}(-z'\otimes 1)E_{1,3}(1\otimes z)E_{3,2}(1\otimes z)E_{3,1}(z'\otimes 1) \\ & E_{2,3}(z'\otimes 1)E_{1,3}(-1\otimes z)E_{3,2}(-1\otimes z)E_{3,1}(-z'\otimes 1)E_{3,2}(\zeta_2)E_{3,1}(\zeta_1). \end{aligned}$$

Thus, the Jacobian matrix of $\delta$ factors as a product of elementary matrices even though it is wild.

I am not yet sure where to go with **Jacobian Tame** since I cannot think of any nice properties that such automorphisms would satisfy.

Naturally, a positive answer to the Problem in 2 would show that every automorphism is Jacobian Tame. On the other hand, a negative answer to the Problem in 2 would only serve to complicate things since a Jacobian matrix will certainly look quite different from a generic matrix in $\mathrm{GL}_{\mathtt{g}}(\mathbb{C}\langle \mathbb{x}' \rangle^{opp} \otimes \mathbb{C}\langle \mathbb{x} \rangle)$.

## References


[PW95] H.J. Park and C. Woodburn. An algorithmic proof of Suslin's stability theorem for polynomial rings. *Journal of Algebra*, 178(1):277 – 298, 1995.

[Umi07] U. U. Umirbaev. The Anick automorphism of free associative algebras. *J. Reine Angew. Math.*, 605:165–178, 2007.

sec:TPFSF

## 5. Tensor Product of Free Skew Fields

**Problem 5.1.** Classify the invertible elements of $\mathbb{C}\langle\!\!\langle \mathbb{x} \rangle\!\!\rangle \otimes \mathbb{C}\langle\!\!\langle \mathbb{y} \rangle\!\!\rangle$. The expectation is that if $Q$ and $Q^{-1}$ are in $\mathbb{C}\langle\!\!\langle \mathbb{x} \rangle\!\!\rangle \otimes \mathbb{C}\langle\!\!\langle \mathbb{y} \rangle\!\!\rangle$, then $Q = r \otimes s$, for some nonzero elements $r \in \mathbb{C}\langle\!\!\langle \mathbb{x} \rangle\!\!\rangle$ and $s \in \mathbb{C}\langle\!\!\langle \mathbb{y} \rangle\!\!\rangle$.

The following was proven in Swe70 [Swe70] using techniques that seemingly don't translate well to the noncommutative setting.

**Theorem 5.2.** *Suppose $A$ and $B$ are commutative domains over an algebraically closed field $\boldsymbol{k}$ and $\boldsymbol{k}$ is algebraically closed in $A$ and $B$. If $z \in A \otimes B$ is invertible, then $z = a \otimes b$ for some invertible elements $a \in A$ and $b \in B$.*

The following two simplifications have been proven using Complex Analysis and Realization Theory:

**Proposition 5.3.** *Suppose $r \in \mathbb{C}\langle\!\!\langle \mathbb{x} \rangle\!\!\rangle$ and $s \in \mathbb{C}\langle\!\!\langle \mathbb{y} \rangle\!\!\rangle$. If $(1 \otimes 1 - r \otimes s)^{-1} \in \mathbb{C}\langle\!\!\langle \mathbb{x} \rangle\!\!\rangle \otimes \mathbb{C}\langle\!\!\langle \mathbb{y} \rangle\!\!\rangle$ then either $r$ or $s$ is constant.*

*Suppose $r_1, \ldots, r_{\mathtt{h}} \in \mathbb{C}\langle\!\!\langle \mathbb{x} \rangle\!\!\rangle$, $C\langle\!\!\langle s_1 \ldots, s_{\mathtt{h}} \rangle\!\!\rangle$ is isomorphic as a skew field to $\mathbb{C}\langle\!\!\langle w_1, \ldots, w_{\mathtt{h}} \rangle\!\!\rangle$. If $(1 \otimes 1 - \sum_{i=1}^{k} r_i \otimes s_i)^{-1} \in \mathbb{C}\langle\!\!\langle \mathbb{x} \rangle\!\!\rangle \otimes \mathbb{C}\langle\!\!\langle \mathbb{y} \rangle\!\!\rangle$ then $r_1, \ldots, r_k$ are all constant.*

The tensor product membership problem is strongly related to the rational automorphism problem as well, although that may require an understanding of domains.

### References

[Swe70] Moss Eisenberg Sweedler. A units theorem applied to Hopf algebras and Amitsur cohomology. *American Journal of Mathematics*, 92(1):259–271, 1970.

sec:RatAuts

## 6. Rational Automorphisms

**Problem 6.1.** Suppose $\mathbb{r} \in (\mathbb{C}\langle\!\!\langle x_1, \ldots, x_{\mathtt{g}} \rangle\!\!\rangle)^{\mathtt{g}}$. Find an evaluation criterion that is necessary and sufficient for the induced map $\rho : \mathbb{C}\langle\!\!\langle x_1, \ldots, x_{\mathtt{g}} \rangle\!\!\rangle \to \mathbb{C}\langle\!\!\langle x_1, \ldots, x_{\mathtt{g}} \rangle\!\!\rangle$ $(\rho(x_i) = \mathbb{r}_i)$ to be an automorphism.

An evaluation criterion is some condition on $\mathbb{r}$ when we treat it as a function. For example, injective, surjective, etc.

conj:rats auts conj

**Conjecture 6.2.** Suppose $\mathbb{r} \in (\mathbb{C}\langle\!\!\langle x_1, \ldots, x_{\mathtt{g}} \rangle\!\!\rangle)^{\mathtt{g}}$. The following are equivalent:

(1) there exists a free, Euclidean open and Euclidean dense set $\Omega$ such that $\mathbb{r}|_\Omega$ is injective;
(2) $J_{\mathbb{r}}$ is an invertible element of $M_{\mathtt{g}}(\mathbb{C}\langle\!\!\langle \mathbb{x}' \rangle\!\!\rangle^{opp} \otimes \mathbb{C}\langle\!\!\langle \mathbb{x} \rangle\!\!\rangle)$;
(3) the induced map $\rho : \mathbb{C}\langle\!\!\langle \mathbb{x} \rangle\!\!\rangle \to \mathbb{C}\langle\!\!\langle \mathbb{x} \rangle\!\!\rangle$ given by $\rho(x_i) = \mathbb{r}_i$ is an automorphism.

This is simply a best guess conjecture at the moment. Clearly, $(3) \Rightarrow (1), (2)$. Condition (1) is seemingly strange, but the naïve attempt of requiring injective on its domain is insufficient.

**Example 6.3.** Let $\mathbb{r}(x, y) = (x, y - x^2 y)$. This induces a rational automorphism, however $\mathbb{r}(1, \alpha) = \mathbb{r}(1, \beta)$, hence $\mathbb{r}$ is not injective on its domain (in fact, the points where it is not injective are exactly the points where $\mathbb{r}^{-1}$ is not defined). If $\Omega = \left\{(X, Y) \in M_n(\mathbb{C})^2 : \det(I_n - X^2) \neq 0\right\}$, then $\Omega$ is free, Euclidean open and dense and $\mathbb{r}$ is injective on $\Omega$.

Let us see a slightly harder example.

**Example 6.4.** Let $\mathbb{r}(x, y) = (x, y - xyx)$. Naturally $\mathbb{r}$ is not injective on $\mathbb{C}^2$ since $\mathbb{r}(1, \alpha) = \mathbb{r}(1, \beta)$. Moreover, $\mathbb{r}$ does not induce a rational automorphism and this fact is a bit harder to see.

The first observed reason why is found by using formal power series. Since the derivative of $\mathbb{r}$ is invertible on some free neighborhood of 0, it must have a local inverse that we can find as a power series. It turns out that

$$f(x, y) = (x, \sum_{n=0}^{\infty} x^n y x^n)$$

is the formal power series representation of $\mathbb{r}^{-1}$. If one tries to write down a realization for $f$, then a contradiction is eventually attained showing that $f$ is not a rational power series. Thus, $\mathbb{r}$ does not induce a rational automorphism. This doesn't give us too much information at the moment, since it reveals little about the injectivity of $\mathbb{r}$.

Our alternative approach is to use the idea behind the conception of hyporationals: the matrix identity $\mathbf{vec}(AXB) = \mathbf{vec}(X)(A^T \otimes B)$. Let $f = (\mathbb{r}^{-1})_2$. Since $f(x, y)$ satisfies the equation $f(x, y) = y + xf(x, y)x$, we evaluate on a pair of matrices $(X, Y)$ (when it makes sense) and note we have

$$f(X, Y) = Y + Xf(X, Y)X.$$

Taking the vectorization of both sides and rearranging, we have

$$\mathbf{vec}(f(X, Y))(I_n \otimes I_n - X^T \otimes X) = \mathbf{vec}(I_n)(I_n \otimes Y).$$

Multiplying on the right by an inverse we see

$$\mathbf{vec}(f(X, Y)) = \mathbf{vec}(I_n)(I_n \otimes Y)\left(I_n \otimes I_n - X^T \otimes X\right)^{-1}.$$

Thus, for any matrix $X$ where $(I_n \otimes I_n - X^T \otimes X)$ is invertible we have $(X, Y)$ is in the domain of $\mathbb{r}^{-1}$. As pointed out in 5, the function $(1 \otimes 1 - x' \otimes x)^{-1} \notin \mathbb{C}\langle\!\langle x' \rangle\!\rangle \otimes \mathbb{C}\langle\!\langle x \rangle\!\rangle$. However, a consequence of its (first) proof is that there is no free Euclidean open and Euclidean dense set upon which $(1 \otimes 1 - x' \otimes x)$ is invertible. Thus, $\mathbb{r}$ fails to satisfy requirement (1) from the Conjecture and $\mathbb{r}$ does not induce a rational automorphism.

To see why $q = (1 \otimes 1 - x' \otimes x)$ fails to be injective on a "big" set, suppose $\Omega$ is any nonempty free open set upon which $q$ is invertible. If $\lambda$ and $\mu$ are any eigenvalues of $X$ then $\lambda\mu$ is an eigenvalue of $X^T \otimes X$ and $1 - \lambda\mu$ is an eigenvalue of $q(X^T, X)$. Hence, if $X \in \Omega$, then for each eigenvalue $\lambda$ of $X$, $\lambda^{-1}$ is not an eigenvalue of any matrix in $\Omega$. Thus, the eigenvalues of $q(X^T, X)$, taken over all $X \in \Omega$ partition the complex plane.

For any $X \in M_n(\mathbb{C})$ let $\sigma(X)$ denote its set of eigenvalues and for any $U \subset M_n(\mathbb{C})$ let $\sigma(U) = \cup_{X \in U} \sigma(X)$. Since $\Omega$ is assumed to open, $\Omega[n]$ is open and $\sigma(\Omega[n])$ must contain an open set. Hence, if $\lambda \in \sigma(\Omega[n])$ is nonzero, then there exists an open set $W$ containing $\lambda^{-1}$ such that $W \cap \sigma(\Omega[n]) = \varnothing$.

However, $\sigma^{-1}(W)$ contains an open set of matrices, thus $\Omega$ cannot be free, open and dense.

Condition (1) is currently a best guess (the density was used to invoke complex analytic methods). Understanding the domains of elements of $\mathbb{C}\langle\!\langle \mathbb{x}' \leftrightarrow \mathbb{x} \rangle\!\rangle$ seems to be quite important.

sec:LiftingAuts

## 7. Lifting Automorphisms

In the noncommutative setting, a free polynomial mapping $\mathbb{p}$ is injective iff it is invertible iff its Jacobian matrix is invertible (over tensor product of polynomial algebras) iff $(D\mathbb{p}(\boldsymbol{y})[\boldsymbol{x}], \boldsymbol{y})$ is injective.

If $p \in \mathbb{C}[\mathbb{t}]^{\mathrm{g}} = \mathbb{C}[t_1, \dots, t_{\mathrm{g}}]^{\mathrm{g}}$ has an invertible Jacobian matrix, then the polynomial in $2\mathrm{g}$ variables $P(\boldsymbol{t})[\boldsymbol{s}] = (Dp(\boldsymbol{t})[\boldsymbol{s}], \boldsymbol{t})$ is invertible. Exactly as one would expect, we see that

$$P(\boldsymbol{t})[\boldsymbol{s}] = (\boldsymbol{s}, \boldsymbol{t}) \begin{pmatrix} J_p(\boldsymbol{t}) & 0 \\ 0 & I_{\mathrm{g}} \end{pmatrix}$$

where $(\boldsymbol{s}, \boldsymbol{t})$ are taken as a row vector (and $J_p$ is potentially the transposed Jacobian to make this work) and we let

$$\mathcal{J}(\boldsymbol{t}) = \begin{pmatrix} J_p(\boldsymbol{t}) & 0 \\ 0 & I_{\mathrm{g}} \end{pmatrix}$$

and note that $\mathcal{J}$ is invertible. Assume $\mathrm{g} \geq 3$. Hence Suslin's Stability Theorem says that $J_p(t)$ factors as a product of elementaries hence $\mathcal{J}(\boldsymbol{t}) = DE_1 \dots E_k$, where $D$ is a diagonal matrix in $M_{2\mathrm{g}}(\mathbb{C})$ (and is of the form $D' \oplus I_{\mathrm{g}}$) and each $E_i \in M_{2\mathrm{g}}(\mathbb{C}[\mathbb{t}])$ is elementary (and is of the form $E_i' \oplus I_{\mathrm{g}}$). Thus, we obtain elementary automorphisms $\delta$ and $\varepsilon_1, \dots, \varepsilon_k$ given by

$$\delta(\boldsymbol{s}, \boldsymbol{t}) = (\boldsymbol{s}, \boldsymbol{t})D \quad \text{ and } \quad \varepsilon_i(\boldsymbol{s}, \boldsymbol{t}) = (\boldsymbol{s}, \boldsymbol{t})E_i.$$

Composing shows us

$$\varepsilon_k(\varepsilon_{k-1}(\boldsymbol{s}, \boldsymbol{t})) = \varepsilon_{k-1}(\boldsymbol{s}, \boldsymbol{t})E_k(\boldsymbol{t}) = (\boldsymbol{s}, \boldsymbol{t})E_{k-1}E_k.$$

Thus, $F = \varepsilon_k \circ \cdots \circ \varepsilon_1 \circ \delta$ and $F$ is a tame automorphism. Hence, $F$ lifts to $\mathcal{F}$, an automorphism of $\mathbb{C}\langle \mathbb{x}, \mathbb{y} \rangle$ that is linear in $\mathbb{x}$. The (probably extremely difficult) question is, of course, can we choose $\mathcal{F}$ so that it is a derivative as well?

**Problem 7.1.** Suppose $\mathcal{F} = \alpha_1 \circ \dots \alpha_k \in (\mathbb{C}\langle \mathbb{x}, \mathbb{y} \rangle)^{\mathrm{g}}$ where each $\alpha_i$ is an elementary automorphism of the form $(\alpha_i'(\boldsymbol{y})[\boldsymbol{x}], \boldsymbol{y})$ with $\alpha_i'(\boldsymbol{y})[\boldsymbol{x}]$ an $\boldsymbol{x}$-linear polynomial. If $\mathcal{F}$ agrees with the derivative of $p \in \mathbb{C}[\mathbb{t}]^{\mathrm{g}}$ on $\mathbb{C}^{\mathrm{g}}$ then can we find invertible $\beta_1, \dots, \beta_{k+1} \in (\mathbb{C}\langle \mathbb{x}, \mathbb{y} \rangle)^{\mathrm{g}}$ (with $\beta_i = (\beta_i'(\boldsymbol{y})[\boldsymbol{x}], \boldsymbol{y})$) such that

$$\mathbb{p} = \beta_1 \circ \alpha_1 \circ \dots \alpha_k \circ \beta_{k+1}$$

and $\mathbb{p}$ agrees with the derivative of $p$ on $\mathbb{C}^{\mathrm{g}}$ and $\mathbb{p}$ is a derivative?

General Warning: Lifting an invertible derivative to the derivative of a free automorphism strays very close to the Jacobian conjecture. Be warned that proving that an invertible derivative always lifts to the derivative of a free automorphism implies the Jacobian conjecture, thus it is either false (and probably hard to show it is false) or it is incredibly difficult.

**Conjecture 7.2** (Jacobian Conjecture '39)**.** Suppose $p : \mathbb{C}^{\mathrm{g}} \to \mathbb{C}^{\mathrm{g}}$ is a polynomial. If the Jacobian matrix $J_p$ has a nonzero constant determinant (equivalently, $J_p$ is invertible over $\mathbb{C}[t]$), then $p$ is invertible.

Open for 80+ years so very very difficult.

A more mild form of the problem is to try and do this for the Nagata automorphism.

**Remark 7.3.** A potential way to generate a counter-example is to find a polynomial whose derivative is a Cohn-type matrix:

$$I_{\mathrm{g}} + a\boldsymbol{v} \cdot (v_j \boldsymbol{e}_i - v_i \boldsymbol{e}_j)$$

where $\boldsymbol{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_{\mathrm{g}} \end{pmatrix} \in (\mathbb{C}[t_1, \ldots, t_{\mathrm{g}}])^{\mathrm{g}}$, $i < j \in \{1, \ldots, \mathrm{g}\}$, $a \in \mathbb{C}[\mathbb{t}]$ and $\boldsymbol{e}_i$ is the $i^{\text{th}}$ standard basis (row) vector. With $\mathrm{g} = 2$, we get

$$I_2 + a \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \cdot \left( \begin{pmatrix} v_2 & -v_1 \end{pmatrix} \right) = \begin{pmatrix} 1 + av_1v_2 & -av_1^2 \\ av_2^2 & 1 - av_1v_2 \end{pmatrix}.$$

Can this be a Jacobian?

sec:RatDeriv

## 8. Rational Derivatives - SOLVED!

Let $\mathbb{x} = \{x_1, \ldots, x_{\mathrm{g}}\}$ and $\mathbb{h} = \{h_1, \ldots, h_{\mathrm{g}}\}$ be a sets of freely noncommuting indeterminates. Let $\mathbb{C}\langle\langle\mathbb{x}\rangle\rangle$ denote the formal power series in $\mathbb{x}$ and let $\mathbb{C}\langle\!\!\langle\mathbb{x}\rangle\!\!\rangle_0$ denote the rational formal power series. If $S \in \mathbb{C}\langle\langle\mathbb{x}\rangle\rangle$ and $w$ is a word, then $[S, w]$ is the coefficient of $w$ appearing in $S$. For any word $w$ and series $S \in \mathbb{C}\langle\langle\mathbb{x}\rangle\rangle$, we let $w^{-1}S = \sum_{v \in \langle\mathbb{x}\rangle} [S, wv]v$.

**Proposition 8.1.** *Suppose $f \in \mathbb{C}\langle\langle\mathbb{x}\rangle\rangle$. If $Df(\boldsymbol{x})[\boldsymbol{h}] \in \mathbb{C}\langle\!\!\langle\mathbb{x}, \mathbb{h}\rangle\!\!\rangle_0$ then $f \in \mathbb{C}\langle\!\!\langle\mathbb{x}\rangle\!\!\rangle_0$.*

*Proof.* For each $1 \leq i \leq \mathrm{g}$ we let $\partial_i f := Df(\boldsymbol{x})[0, \ldots, 0, h_i, 0, \ldots, 0]$ and note $\partial_i f \in \mathbb{C}\langle\!\!\langle\mathbb{x}, \mathbb{h}\rangle\!\!\rangle_0$. Next, we write

$$f = c_0 + \sum_{i=1}^{\mathrm{g}} x_i f_i$$

where each $f_i \in \mathbb{C}\langle\langle\mathbb{x}\rangle\rangle$. Hence,

$$\partial_i f = h_i f_i + \sum_{j=1}^{\mathrm{g}} x_j \partial_i f_j \in \mathbb{C}\langle\!\!\langle\mathbb{x}, \mathbb{h}\rangle\!\!\rangle_0$$

and it follows that $h_i^{-1}\partial_i f_i = f_i \in \mathbb{C}\langle\!\!\langle\mathbb{x}, \mathbb{h}\rangle\!\!\rangle_0$ since $\partial_i f_i$ is contained in a stable submodule. Therefore, $f = c_0 + \sum_{i=1}^{\mathrm{g}} x_i f_i \in \mathbb{C}\langle\!\!\langle\mathbb{x}\rangle\!\!\rangle_0$ since each $f_i$ is rational. $\square$

I think the argument below shows it for generalized series.

**Remark 8.2.** Let $\mathcal{A} = M_n(\mathbb{C})$ for some $n$. If $f \in \mathcal{A}\langle\langle\mathbb{x}\rangle\rangle$ then

$$f = f_0 + \sum_{i=1}^{\mathrm{g}} L_{x_i} f$$

and

$$Df(\boldsymbol{x})[\boldsymbol{h}] = \sum_{i=1}^{\mathrm{g}} L_{h_i} Df(\boldsymbol{x})[\boldsymbol{h}] + \sum_{i=1}^{\mathrm{g}} L_{x_i} Df(\boldsymbol{x})[\boldsymbol{h}].$$

If $f_i(\boldsymbol{x}, h_i) = L_{h_i} Df(\boldsymbol{x})[\boldsymbol{h}]$, then $f_i(\boldsymbol{x}, x_i) = L_{x_i} f$. Hence, if $Df(\boldsymbol{x})[\boldsymbol{h}]$ is rational, then so is $L_{h_i} Df(\boldsymbol{x})[\boldsymbol{h}]$ and consequently so is $L_{x_i} f$. Therefore, $f$ is rational since it is a sum of rational series.