

FLOP – FREE LIST OF OPEN PROBLEMS – August 9, 2020

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1. NOTATION

$\mathbb{C}\langle x_1, \dots, x_g \rangle$	free \mathbb{C} -algebra generated by x_1, \dots, x_g
$\mathbb{C}\langle\!\!\langle x_1, \dots, x_g \rangle\!\!\rangle$	free skew field (nc rationals) of $\mathbb{C}\langle x_1, \dots, x_g \rangle$
$\mathbb{C}\langle\!\!\langle \mathbf{x} \rangle\!\!\rangle_0$	algebra of rationals regular at 0
$\mathbb{C}\langle\!\!\langle \mathbf{x} \leftrightarrow \mathbf{y} \rangle\!\!\rangle$	universal skew field of fractions of $\mathbb{C}\langle\!\!\langle \mathbf{x} \rangle\!\!\rangle \otimes \mathbb{C}\langle\!\!\langle \mathbf{y} \rangle\!\!\rangle$

2. FACTORING INVERTIBLE MATRICES OVER $\mathbb{C}\langle \mathbf{x} \rangle \otimes \mathbb{C}\langle \mathbf{y} \rangle$

Problem 2.1. Suppose $A \in M_n(\mathbb{C}\langle \mathbf{x} \rangle \otimes \mathbb{C}\langle \mathbf{y} \rangle)$. If A is invertible over the \mathbb{C} -algebra $\mathbb{C}\langle \mathbf{x} \rangle \otimes \mathbb{C}\langle \mathbf{y} \rangle$ then do there exist $D \in M_n(\mathbb{C})$ and $E_1, \dots, E_k \in M_n(\mathbb{C}\langle \mathbf{x} \rangle \otimes \mathbb{C}\langle \mathbf{y} \rangle)$ such that D is diagonal, each E_i is an elementary matrix and $A = DE_1 \dots E_k$?

Remark 2.2. This problem is false when $n = 2$. Cohn gave the following matrix that is not a product of elementary matrices over $\mathbb{C}[x] \otimes \mathbb{C}[y]$:

$$\begin{pmatrix} 1 \otimes 1 + x \otimes y & 1 \otimes y^2 \\ -x^2 \otimes 1 & 1 \otimes 1 - x \otimes y \end{pmatrix}.$$

The above counterexample is nonexistent when our matrices are over $\mathbb{C}\langle \mathbf{x} \rangle$ instead of $\mathbb{C}\langle \mathbf{x} \rangle \otimes \mathbb{C}\langle \mathbf{y} \rangle$. The proof of this uses results from [Coh06].

Theorem 2.3. *If $A \in M_n(\mathbb{C}\langle \mathbf{x} \rangle)$ is invertible, then there exist $D \in M_n(\mathbb{C})$ and $E_1, \dots, E_k \in M_n(\mathbb{C}\langle \mathbf{x} \rangle)$ such that D is diagonal, each E_i is an elementary matrix and $A = DE_1 \dots E_k$.*

The Cohn counterexample is essentially the only issue:

Theorem 2.4 (Suslin’s Stability Theorem). *Suppose $A \in M_n(\mathbb{C}[t_1, \dots, t_g])$ where $n \geq 3$. If $\det(A) = 1$ then there exist $E_1, \dots, E_k \in M_n(\mathbb{C}[t_1, \dots, t_g])$ such that $A = E_1 \dots E_k$.*

An “algorithmic” proof of the above theorem can be found in [PW95].

Remark 2.5. An idea related to this is the notion of Jacobian Tame, see 4.

REFERENCES

- [Coh06] P.M. Cohn. *Free Ideal Ring and Localizations in General Rings*. Cambridge University Press, 2006.
- [PW95] H.J. Park and C. Woodburn. An algorithmic proof of Suslin’s stability theorem for polynomial rings. *Journal of Algebra*, 178(1):277 – 298, 1995.

3. FREE LÜROTH THEOREM

Problem 3.1 (Free Lüroth Theorem). Suppose \mathbf{k} is an algebraically closed infinite field and x_1, \dots, x_g are freely noncommuting indeterminates ($g \geq 2$). If $\mathbf{k} \subsetneq D \subsetneq \mathbf{k}\langle x_1, \dots, x_g \rangle$ is a subfield, then do there exist $q_1, \dots, q_h \in \mathbb{C}\langle x_1, \dots, x_g \rangle$ such that $D = \mathbb{C}\langle q_1, \dots, q_h \rangle$?

In other words, is every non-trivial subfield of $\mathbb{C}\langle \mathbf{x} \rangle$ a free skew field?

Theorem 3.2 (Lüroth’s Theorem). Suppose \mathbf{k} is a field. If $\mathbf{k} \subsetneq D \subsetneq \mathbf{k}(t)$ is a subfield, then there exists $q \in \mathbf{k}(t)$ such that $D = \mathbf{k}(q(t))$.

Remark 3.3. If \mathbf{k} is algebraically closed and infinite, then Lüroth’s Theorem holds as well for $\mathbf{k}(t_1, t_2)$.

On the other hand, there are counterexamples to Lüroth’s Theorem for $\mathbf{k}(t_1, t_2, t_3)$.

Remark 3.4. Schofield [Sch85] shows that if $f, g \in \mathbb{C}\langle \mathbf{x} \rangle$, then either $[f, g] = 0$ or $\mathbb{C}\langle f, g \rangle$ is free.

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- [Sch85] Schofield, A. H. Representations of rings over skew fields. *London Mathematical Society Lecture Note Series*, 1985.

4. JACOBIAN TAME

An automorphism τ of the free algebra $\mathbb{C}\langle x_1, \dots, x_g \rangle$ is **elementary** if $\tau : x_i \mapsto x_i$ for $i \neq j$ and $\tau : x_j \mapsto cx_j + f$, where $c \in \mathbb{C}$ and $f \in \mathbb{C}\langle x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_g \rangle$. We say an automorphism is **tame** if it is a composition of elementary automorphisms. If the automorphism is not tame, then it is **wild**.

In [Umi07] it is shown that the Anick automorphism,

$$\delta(x, y, z) = (x + z(xz - zy), y + (xz - zy)z, z) \in \mathbb{C}\langle x, y, z \rangle$$

is wild. Looking at it, the Anick automorphism doesn’t seem particularly “wild,” so maybe this can be improved? Perhaps it is something about Jacobian matrices?

Definition 4.1. We say an automorphism τ of $\mathbb{C}\langle x_1, \dots, x_g \rangle$ is **Jacobian Tame** if $J_\tau = DE_1 \dots E_k$, where $D \in M_g(\mathbb{C}\langle \mathbf{x}' \rangle^{opp} \otimes \mathbb{C}\langle \mathbf{x} \rangle)$ is diagonal and $E_1, \dots, E_k \in M_g(\mathbb{C}\langle \mathbf{x}' \rangle^{opp} \otimes \mathbb{C}\langle \mathbf{x} \rangle)$ are elementary matrices.

If such a factorization does not exist, then we say τ is **Jacobian wild**.

Problem 4.2. Are there any Jacobian wild automorphisms of the free algebra? The only known (to me) example of a wild automorphism of $\mathbb{C}\langle x, y, z \rangle$ is Jacobian tame (this is explained below).

In the commutative case there are no Jacobian wild automorphisms. Every automorphism of $\mathbb{C}[t_1, t_2]$ is tame, hence Jacobian tame. If ϕ is an automorphism of $\mathbb{C}[t_1, \dots, t_g]$ with $g > 2$, then its Jacobian matrix $J_\phi \in M_g(\mathbb{C}[t_1, \dots, t_g, t'_1, \dots, t'_g])$

factors into such a product by Suslin’s Stability Theorem (see 2), thus is Jacobian tame.

Throughout this I will be using the transposed Jacobian matrix: $J_\tau \in M_g(\mathbb{C}\langle \mathbf{x}' \rangle^{opp} \otimes \mathbb{C}\langle \mathbf{x} \rangle)$ where the i^{th} column of J_τ corresponds to the derivatives of τ_i .

It turns out that the Anick automorphism

$$\delta(x, y, z) = (x + z(xz - zy), y + (xz - zy)z, z) \in \mathbb{C}\langle x, y, z \rangle$$

has a Jacobian matrix that can be written as a product of elementary matrices. The Jacobian matrix of δ is

$$J_\delta = \begin{pmatrix} 1 \otimes 1 + z' \otimes z & 1 \otimes z^2 & 0 \\ -(z')^2 \otimes 1 & 1 - z' \otimes z & 0 \\ \zeta_1 & \zeta_2 & 1 \end{pmatrix}$$

where $\zeta_1 = 1 \otimes xz + x'z' \otimes 1 - 1 \otimes zy - z' \otimes y$ and $\zeta_2 = x \otimes z + z'x' \otimes 1 - 1 \otimes yz - y'z' \otimes 1$. Let $E_{i,j}(\alpha) = I + \alpha e_{i,j}$ ($e_{i,j}$ has a 1 in the i, j entry and 0’s elsewhere) and observe

$$J_\delta E_{3,1}(-\zeta_1) E_{3,2}(-\zeta_2) = \begin{pmatrix} 1 \otimes 1 + z' \otimes z & 1 \otimes z^2 & 0 \\ -(z')^2 \otimes 1 & 1 - z' \otimes z & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

In 1966, Cohn proved that

$$\mathfrak{C} = \begin{pmatrix} 1 \otimes 1 + z' \otimes z & 1 \otimes z^2 \\ -(z')^2 \otimes 1 & 1 - z' \otimes z \end{pmatrix}$$

cannot be written as a product of elementary matrices over $\mathbb{F}[z' \otimes 1, 1 \otimes z]$. This is a crucial aspect of Umirbaev’s ([Umi07]) proof that the Anick automorphism is wild. However, Park and Woodburn [PW95] give a decomposition of $\begin{pmatrix} \mathfrak{C} & 0 \\ 0 & 1 \end{pmatrix}$ into a product of elementary matrices:

$$\begin{aligned} J_\delta E_{3,1}(-\zeta_1) E_{3,2}(-\zeta_2) &= \begin{pmatrix} 1 \otimes 1 + z' \otimes z & 1 \otimes z^2 & 0 \\ -(z')^2 \otimes 1 & 1 - z' \otimes z & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= E_{2,3}(-z' \otimes 1) E_{1,3}(1 \otimes z) E_{3,2}(1 \otimes z) E_{3,1}(z' \otimes 1) \\ &\quad E_{2,3}(z' \otimes 1) E_{1,3}(-1 \otimes z) E_{3,2}(-1 \otimes z) E_{3,1}(-z' \otimes 1). \end{aligned}$$

Thus

$$\begin{aligned} J_\delta &= E_{2,3}(-z' \otimes 1) E_{1,3}(1 \otimes z) E_{3,2}(1 \otimes z) E_{3,1}(z' \otimes 1) \\ &\quad E_{2,3}(z' \otimes 1) E_{1,3}(-1 \otimes z) E_{3,2}(-1 \otimes z) E_{3,1}(-z' \otimes 1) E_{3,2}(\zeta_2) E_{3,1}(\zeta_1). \end{aligned}$$

Thus, the Jacobian matrix of δ factors as a product of elementary matrices even though it is wild.

Naturally, a positive answer to the Problem in 2 would show that every automorphism is Jacobian Tame. On the other hand, a negative answer to the Problem in 2 would only serve to complicate things since a Jacobian matrix will certainly look quite different from a generic matrix in $\text{GL}_g(\mathbb{C}\langle \mathbf{x}' \rangle^{opp} \otimes \mathbb{C}\langle \mathbf{x} \rangle)$.

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- [PW95] H.J. Park and C. Woodburn. An algorithmic proof of Suslin’s stability theorem for polynomial rings. *Journal of Algebra*, 178(1):277 – 298, 1995.
- [Umi07] U. U. Umirbaev. The Anick automorphism of free associative algebras. *J. Reine Angew. Math.*, 605:165–178, 2007.

5. TENSOR PRODUCT OF FREE SKEW FIELDS

Problem 5.1. Classify the invertible elements of $\mathbb{C}\langle x \rangle \otimes \mathbb{C}\langle y \rangle$. The expectation is that if Q and Q^{-1} are in $\mathbb{C}\langle x \rangle \otimes \mathbb{C}\langle y \rangle$, then $Q = r \otimes s$, for some nonzero elements $r \in \mathbb{C}\langle x \rangle$ and $s \in \mathbb{C}\langle y \rangle$.

The following was proven in [Swe70] using techniques that seemingly don't translate well to the noncommutative setting.

Theorem 5.2. *Suppose A and B are commutative domains over an algebraically closed field k and k is algebraically closed in A and B . If $z \in A \otimes B$ is invertible, then $z = a \otimes b$ for some invertible elements $a \in A$ and $b \in B$.*

The following two simplifications have been proven using Complex Analysis and Realization Theory:

Proposition 5.3. *Suppose $r \in \mathbb{C}\langle x \rangle$ and $s \in \mathbb{C}\langle y \rangle$. If $(1 \otimes 1 - r \otimes s)^{-1} \in \mathbb{C}\langle x \rangle \otimes \mathbb{C}\langle y \rangle$ then either r or s is constant.*

Suppose $r_1, \dots, r_h \in \mathbb{C}\langle x \rangle$, $\mathbb{C}\langle s_1, \dots, s_h \rangle$ is isomorphic as a skew field to $\mathbb{C}\langle w_1, \dots, w_h \rangle$. If $(1 \otimes 1 - \sum_{i=1}^h r_i \otimes s_i)^{-1} \in \mathbb{C}\langle x \rangle \otimes \mathbb{C}\langle y \rangle$ then r_1, \dots, r_h are all constant.

The tensor product membership problem is strongly related to the rational automorphism problem as well, although that may require an understanding of domains.

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[Swe70] Moss Eisenberg Sweedler. A units theorem applied to Hopf algebras and Amitsur cohomology. *American Journal of Mathematics*, 92(1):259–271, 1970.

6. RATIONAL AUTOMORPHISMS

Problem 6.1. Suppose $r \in (\mathbb{C}\langle x_1, \dots, x_g \rangle)^g$. Find an evaluation criterion that is necessary and sufficient for the induced map $\rho : \mathbb{C}\langle x_1, \dots, x_g \rangle \rightarrow \mathbb{C}\langle x_1, \dots, x_g \rangle$ ($\rho(x_i) = r_i$) to be an automorphism.

An evaluation criterion is some condition on r when we treat it as a function. For example, injective, surjective, etc.

Conjecture 6.2. Suppose $r \in (\mathbb{C}\langle x_1, \dots, x_g \rangle)^g$. The following are equivalent:

- (1) there exists a free, Euclidean open and Euclidean dense set Ω such that $r|_{\Omega}$ is injective;
- (2) J_r is an invertible element of $M_g(\mathbb{C}\langle x' \rangle^{opp} \otimes \mathbb{C}\langle x \rangle)$;
- (3) the induced map $\rho : \mathbb{C}\langle x \rangle \rightarrow \mathbb{C}\langle x \rangle$ given by $\rho(x_i) = r_i$ is an automorphism.

This is simply a best guess conjecture at the moment. Clearly, (3) \Rightarrow (1), (2). Condition (1) is seemingly strange, but the naïve attempt of requiring injective on its domain is insufficient.

Example 6.3. Let $r(x, y) = (x, y - x^2y)$. This induces a rational automorphism, however $r(1, \alpha) = r(1, \beta)$, hence r is not injective on its domain (in fact, the points where it is not injective are exactly the points where r^{-1} is not defined). If $\Omega = \{(X, Y) \in M_n(\mathbb{C})^2 : \det(I_n - X^2) \neq 0\}$, then Ω is free, Euclidean open and dense and r is injective on Ω .

Let us see a slightly harder example.

Example 6.4. Let $\mathfrak{r}(x, y) = (x, y - xyx)$. Naturally \mathfrak{r} is not injective on \mathbb{C}^2 since $\mathfrak{r}(1, \alpha) = \mathfrak{r}(1, \beta)$. Moreover, \mathfrak{r} does not induce a rational automorphism and this fact is a bit harder to see.

The first observed reason why is found by using formal power series. Since the derivative of \mathfrak{r} is invertible on some free neighborhood of 0, it must have a local inverse that we can find as a power series. It turns out that

$$f(x, y) = (x, \sum_{n=0}^{\infty} x^n y x^n)$$

is the formal power series representation of \mathfrak{r}^{-1} . If one tries to write down a realization for f , then a contradiction is eventually attained showing that f is not a rational power series. Thus, \mathfrak{r} does not induce a rational automorphism. This doesn't give us too much information at the moment, since it reveals little about the injectivity of \mathfrak{r} .

Our alternative approach is to use the idea behind the conception of hyporationals: the matrix identity $\mathbf{vec}(AXB) = \mathbf{vec}(X)(A^T \otimes B)$. Let $f = (\mathfrak{r}^{-1})_2$. Since $f(x, y)$ satisfies the equation $f(x, y) = y + xf(x, y)x$, we evaluate on a pair of matrices (X, Y) (when it makes sense) and note we have

$$f(X, Y) = Y + Xf(X, Y)X.$$

Taking the vectorization of both sides and rearranging, we have

$$\mathbf{vec}(f(X, Y))(I_n \otimes I_n - X^T \otimes X) = \mathbf{vec}(I_n)(I_n \otimes Y).$$

Multiplying on the right by an inverse we see

$$\mathbf{vec}(f(X, Y)) = \mathbf{vec}(I_n)(I_n \otimes Y) (I_n \otimes I_n - X^T \otimes X)^{-1}.$$

Thus, for any matrix X where $(I_n \otimes I_n - X^T \otimes X)$ is invertible we have (X, Y) is in the domain of \mathfrak{r}^{-1} . As pointed out in 5, the function $(1 \otimes 1 - x' \otimes x)^{-1} \notin \mathbb{C}\langle x' \rangle \otimes \mathbb{C}\langle x \rangle$. However, a consequence of its (first) proof is that there is no free Euclidean open and Euclidean dense set upon which $(1 \otimes 1 - x' \otimes x)$ is invertible. Thus, \mathfrak{r} fails to satisfy requirement (1) from the Conjecture and \mathfrak{r} does not induce a rational automorphism.

To see why $q = (1 \otimes 1 - x' \otimes x)$ fails to be injective on a “big” set, suppose Ω is any nonempty free open set upon which q is invertible. If λ and μ are any eigenvalues of X then $\lambda\mu$ is an eigenvalue of $X^T \otimes X$ and $1 - \lambda\mu$ is an eigenvalue of $q(X^T, X)$. Hence, if $X \in \Omega$, then for each eigenvalue λ of X , λ^{-1} is not an eigenvalue of any matrix in Ω . Thus, the eigenvalues of $q(X^T, X)$, taken over all $X \in \Omega$ partition the complex plane.

For any $X \in M_n(\mathbb{C})$ let $\sigma(X)$ denote its set of eigenvalues and for any $U \subset M_n(\mathbb{C})$ let $\sigma(U) = \cup_{X \in U} \sigma(X)$. Since Ω is assumed to be open, $\Omega[n]$ is open and $\sigma(\Omega[n])$ must contain an open set. Hence, if $\lambda \in \sigma(\Omega[n])$ is nonzero, then there exists an open set W containing λ^{-1} such that $W \cap \sigma(\Omega[n]) = \emptyset$.

However, $\sigma^{-1}(W)$ contains an open set of matrices, thus Ω cannot be free, open and dense.

Condition (1) is currently a best guess (the density was used to invoke complex analytic methods). Understanding the domains of elements of $\mathbb{C}\langle x' \leftrightarrow x \rangle$ seems to be quite important.

7. RATIONAL DERIVATIVES - **SOLVED!**

Originally posed by James Pascoe to Meric Augat in March of 2020.

Let $\mathfrak{x} = \{x_1, \dots, x_g\}$ and $\mathfrak{h} = \{h_1, \dots, h_g\}$ be a sets of freely noncommuting indeterminates. Let $\mathbb{C}\langle\mathfrak{x}\rangle$ denote the formal power series in \mathfrak{x} and let $\mathbb{C}\langle\mathfrak{x}\rangle_0$ denote the rational formal power series. If $S \in \mathbb{C}\langle\mathfrak{x}\rangle$ and w is a word, then $[S, w]$ is the coefficient of w appearing in S . For any word w and series $S \in \mathbb{C}\langle\mathfrak{x}\rangle$, we let $w^{-1}S = \sum_{v \in \langle\mathfrak{x}\rangle} [S, vw]v$.

Proposition 7.1. *Suppose $f \in \mathbb{C}\langle\mathfrak{x}\rangle$. If $Df(\mathbf{x})[\mathbf{h}] \in \mathbb{C}\langle\mathfrak{x}, \mathfrak{h}\rangle_0$ then $f \in \mathbb{C}\langle\mathfrak{x}\rangle_0$.*

Proof. For each $1 \leq i \leq g$ we let $\partial_i f := Df(\mathbf{x})[0, \dots, 0, h_i, 0, \dots, 0]$ and note $\partial_i f \in \mathbb{C}\langle\mathfrak{x}, \mathfrak{h}\rangle_0$. Next, we write

$$f = c_0 + \sum_{i=1}^g x_i f_i$$

where each $f_i \in \mathbb{C}\langle\mathfrak{x}\rangle$. Hence,

$$\partial_i f = h_i f_i + \sum_{j=1}^g x_j \partial_i f_j \in \mathbb{C}\langle\mathfrak{x}, \mathfrak{h}\rangle_0$$

and it follows that $h_i^{-1} \partial_i f_i = f_i \in \mathbb{C}\langle\mathfrak{x}, \mathfrak{h}\rangle_0$ since $\partial_i f_i$ is contained in a stable submodule. Therefore, $f = c_0 + \sum_{i=1}^g x_i f_i \in \mathbb{C}\langle\mathfrak{x}\rangle_0$ since each f_i is rational. \square

The argument below shows it for generalized series. Notation and ideas are pulled from [Vol18]. Notably, if v is a word, $L_v S = \sum_{w \in \langle\mathfrak{x}\rangle} [S, vw]w$.

Remark 7.2. Let $\mathcal{A} = M_n(\mathbb{C})$ for some n . If $f \in \mathcal{A}\langle\mathfrak{x}\rangle$ then

$$f = f_0 + \sum_{i=1}^g L_{x_i} f$$

and

$$Df(\mathbf{x})[\mathbf{h}] = \sum_{i=1}^g L_{h_i} Df(\mathbf{x})[\mathbf{h}] + \sum_{i=1}^g L_{x_i} Df(\mathbf{x})[\mathbf{h}].$$

If $f_i(\mathbf{x}, h_i) = L_{h_i} Df(\mathbf{x})[\mathbf{h}]$, then $f_i(\mathbf{x}, x_i) = L_{x_i} f$. Hence, if $Df(\mathbf{x})[\mathbf{h}]$ is rational, then so is $L_{h_i} Df(\mathbf{x})[\mathbf{h}]$ and consequently so is $L_{x_i} f$. Therefore, f is rational since it is a sum of rational series.

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