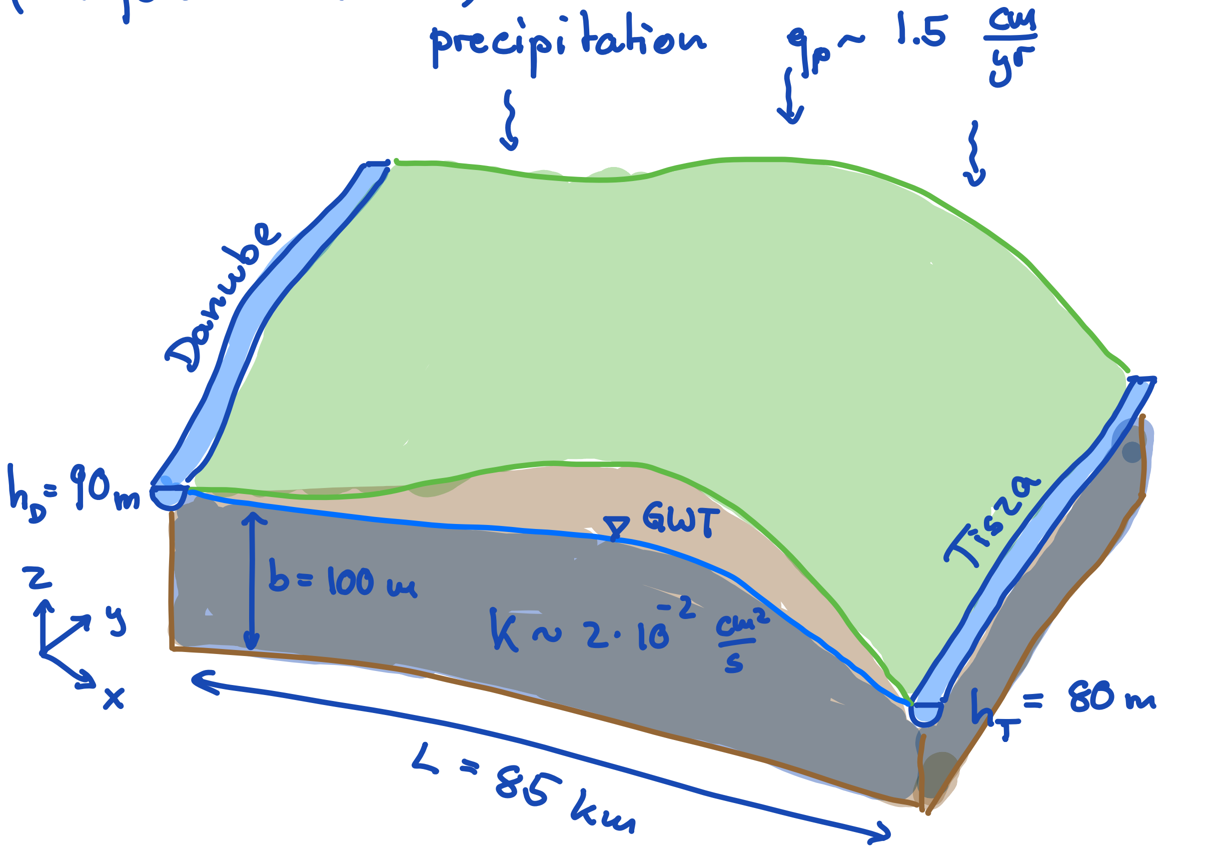
Groundwater recharge between two rivers

(Sanford et al. 2001)



Aquifer aspect ratio: $b/L = \frac{100}{85\,000} = \frac{1}{850} \sim 0.001$

$\Rightarrow$ flow is practically 1D in horizontal direction

This can be seen from a scaling analysis of the continuity equation.

Introduce characteristic scales:

$$x_D = \frac{x}{L} \qquad z_D = \frac{z}{b} \qquad q_{x,D} = \frac{q_x}{q_{x,c}} \qquad q_{z,D} = \frac{q_z}{q_{z,c}}$$

substitute into continuity, eg.. $q_x = q_{x,c}\, q_{x,D}$

$$\nabla \cdot q = \frac{\partial q_x}{\partial x} + \frac{\partial q_z}{\partial z} = \frac{q_{x,c}}{L} \frac{\partial q_{x,D}}{\partial x_D} + \frac{q_{z,c}}{b} \frac{\partial q_{z,D}}{\partial z_D} = 0$$

collect terms

$$\frac{\partial q_{x,D}}{\partial x_D} + \underbrace{\frac{q_{z,c} L}{q_{x,c} b}}_{\Pi} \frac{\partial q_{z,D}}{\partial z_{,D}} = 0$$

dimensionless parameter

Set $\Pi = 1$ to get relation between fluxes

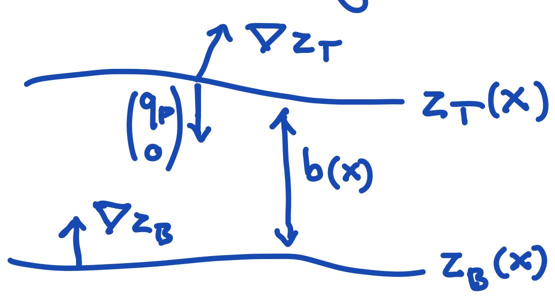
$$\boxed{q_{z,c} = \frac{b}{L} q_{x,c} \ll q_{x,c}}$$

$\Rightarrow$ vertical flux is negligible

Assume $q_z = 0 \Rightarrow \frac{\partial h}{\partial z} = 0 \Rightarrow h(x)$

Darcy: $q_x = -K \frac{\partial h}{\partial x}$

Vertical integration



$b(x) = z_T(x) - z_B(x)$

$$\int_{z_B}^{z_T} \nabla \cdot q(x)\, dz = ?$$

Leibnitz integral rule:

$$\int_{z_B}^{z_T} \nabla \cdot \underline{q} = \nabla \cdot \int_{z_B}^{z_T} \underline{q}\, dz + \left(\underline{q} \cdot \nabla z_B\right|_{z_B} - \left(\underline{q} \cdot \nabla z_T\right)_{z_T}$$

assume: 1) $\underline{q} \approx \begin{pmatrix} q_x \\ 0 \end{pmatrix} \Rightarrow \underline{q} \neq \underline{q}(z)$

$$\Rightarrow \int_{z_B}^{z_T} \underline{q}\, dz = (z_T - z_B)\, \underline{q}(x) = b(x)\, \underline{q}(x)$$

2) bottom of aquifer is impermeable

$$\underline{q} \cdot \nabla z_B \big|_{z_B} = 0$$

3) slope of top of aquifer is small

$$\underline{q} \cdot \nabla z_T \big|_{z_T} \approx -q_p \quad \text{recharge}$$

Substitute

$$\int_{z_B}^{z_T} \nabla \cdot \underline{q}\, dz = -\tilde{\nabla} \cdot [b K \tilde{\nabla} h] = q_p \qquad \tilde{\nabla} \text{ horiz. op.}$$

or in 1D

$$\boxed{-\frac{\partial}{\partial x}\left[b K \frac{\partial h}{\partial x}\right] = q_p}$$

Note: In unconfined aquifer $b = h \Rightarrow$ non-linear

here we assume a confined aquifer

Simplified example problem:

PDE: $-\frac{d}{dx}\left[bK\frac{dh}{dx}\right] = q_P \qquad x \in [0, L]$

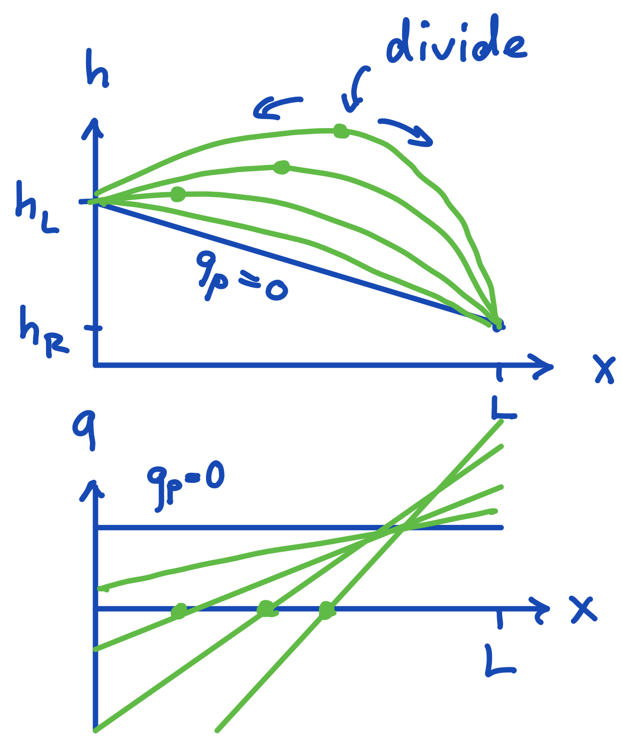
BC: $h(0) = h_D \qquad h(L) = h_T$

Integrate twice to obtain analytic solution

$$\boxed{\begin{aligned} h &= h_D + \left(\frac{h_T - h_D}{L} + \frac{q_P L}{2bK}\right)x - \frac{q_P}{2bK}x^2 \\ q &= \frac{q_P}{b}\left(x - \frac{L}{2}\right) - \frac{K}{L}(h_T - h_D) \end{aligned}}$$

$\Rightarrow$ solve numerically!

Sketch of solution:



As recharge increases a "groundwater divide" forms that separates water flowing to the Danube and the Tisza rivers.