Variable coefficients

Heterogeneity is a key element of porous media

Continuous eqn.: $\quad -\nabla \cdot [K(x)\nabla h] = f_s$

Discrete eqn.: $\quad -\underline{\underline{D}} * [\underline{\underline{Kd}} * \underline{\underline{G}}\,\underline{h}] = \underline{f_s}$

Size of $\underline{\underline{Kd}}$ matrix?

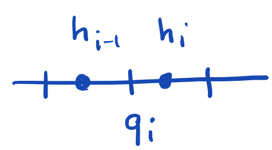
$$\underset{Nx\cdot(Nx+1)}{\underline{\underline{D}}} \qquad \underset{(Nx+1)\cdot(Nx+1)}{\underline{\underline{Kd}}} \qquad \underset{(Nx+1)\cdot Nx}{\underline{\underline{G}}}$$

$\Rightarrow$ $\underline{\underline{Kd}}$ is $(Nx+1)$ by $(Nx+1)$ matrix associated with faces

Entries into $\underline{\underline{Kd}}$ matrix?

Darcy flux: $\quad q = -K\,\nabla h$

$$\underline{q} = -\underline{\underline{Kd}} * \underline{\underline{G}} * \underline{h}$$



$$q_i = -K_{i-\frac{1}{2}}\,\frac{h_i - h_{i-1}}{\Delta x}$$

where $K_{i-\frac{1}{2}}$ is mean of K_{i-1} and K_i

$\Rightarrow$ $\underline{\underline{Kd}}$ must multiply every entry of $\underline{\underline{G}}*\underline{h}$ with the mean of K on interface if we had an $(N_x+1)\cdot N_x$ vector $\underline{Kmean}$

we could simply write: $q = -\underline{Kmean} .* \underbrace{\underline{\underline{G}}*\underline{h}}_{\underline{dh}}$

$\underbrace{}_{\text{eleleument wise multiplication}}$

But to form $\underline{\underline{L}} = -\underline{\underline{D}} * \underline{\underline{Kd}} * \underline{\underline{G}}$ we need to write

$\underline{Kmean} .* \underline{dh}$ as $\underline{\underline{Kd}} * \underline{dh}$.

$\Rightarrow$ simply place $\underline{Kmean}$ on diagonal of $\underline{\underline{Kd}}$.

The appropriate average depends on problem:

1) Hydraulic conductivity $\rightarrow$ harmonic mean

because $K(x)$ is often discontinuous (layering)

$\Rightarrow$ flow across layers (from one cell into the next)

$$K_{i-1} = \frac{2}{\frac{1}{K_{i-1}} + \frac{1}{K_i}} \qquad (\Delta l_{i-1} = \Delta l_i)$$

2) Unconfined flow: need to average h to faces

$h(x)$ is smooth function $\rightarrow$ arithmetic average.

Implementation

1) Generate mean matrix $\underline{\underline{M}}$ so that

$\underline{\text{Kmean}} = \underline{\underline{M}} * \underline{K}$ arithmetic mean via matrix vector multiplication

Note: $\underline{\underline{M}}$ must have same shape & fill pattern as $\underline{\underline{G}}$, because it takes values from cell centers and computes mean of faces

$$\boxed{\underline{\underline{M}} = \frac{2}{\Delta x} |\underline{\underline{G}}|}$$

Note: This sets mean on bnd to zero. O.K.

Similarly we can compute harmonic mean as

$$\boxed{\text{Kmean} = 1./(\underline{\underline{M}} * (1./\underline{K}))}$$