aMI Handin assignments

Sander Jespersen, Mathias Melgaard Andersen, Anders Roland Nielsen,
Andreas Berre Eriksen, Kent Munthe Caspersen & Mikkel Alexander Madsen

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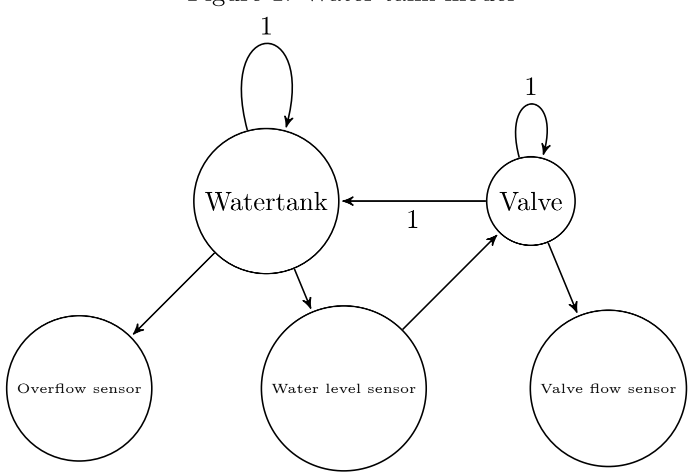
1 Exercise 1

1.1 Baysian network

Figure 1: Water tank model



1.2 Suitable conditional probability distribution

Hvis ventil er åben fanger den stortset altid der er flow, hvis den er lukket laver den oftere fejl.

- Water tank:

	0	1	2	3	4	5	6	7	8	9
L0	0.02	0.95	0.95	0.96	0.96	0.99	0.99	0.99	0.99	0.99
L10	0	0.98	0	0.0038	0	0.033	0.0016	$7.97e^{-5}$	$3.94e^{-6}$	$1.95e^{-7}$
L20	0	0	0.049	0	0.033	0.0016	$7.97e^{-5}$	$3.94e^{-6}$	$1.95e^{-7}$	$9.66e^{-9}$
L30	0	0	0	0.045	0	0	0	0	0	0
Overflow	0	0	0	0	0.012	0.011	0.011	0.011	0.011	0.011

	10	11	12	13	14
L0	0.99	0.99	0.99	0.99	0.99
L10	$9.66e^{-9}$	$4.78e^{-10}$	$2.37e^{-11}$	\$ 1.17e^{-12}\$	\$ 5.80e^{-14}\$
L20	$4.78e^{-10}$	$2.37e^{-11}$	\$ 1.17e^{-12}\$	\$ 5.80e^{-14}\$	\$ 2.87e^{-15}\$
L30	0	0	0	0	0
Overflow	0.011	0.011	0.011	0.011	0.011

- Valve:

	0	1	2	3	4	5	6	7	8
Open	0.02	0.95	0.96	0.97	0.96	0.95	0.94	0.93	0.92
Closed	0.98	0.039	0.035	0.0017	8.47e^{-5}	4.19e^{-6}	2.076e^{-7}	1.028e^{-8}	5.087e^{-10}
Open Broken	0	0.0002	0.0097	0.019	0.029	0.038	0.048	0.057	0.067
Closed Broken	0	0.0098	0.01	0.011	0.011	0.011	0.011	0.011	0.011

	9	10	11	12	13	14
Open	0.91	0.90	0.9	0.89	0.88	0.87
Closed	$2.52e^{-11}$	1.25e^{-12}	6.17e^{-14}	3.054e^{-15}	1.51e^{-16}	7.48e^{-18}
Open Broken	0.076	0.085	0.094	0.1	0.11	0.12
Closed Broken	0.011	0.011	0.011	0.01	0.01	0.011

- Overflow sensor:

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
water	0.05	0.05	0.05	0.05	0.06	0.06	0.06	0.06	0.06	0.06	0.06	0.06	0.06	0.06	0.06
nowater	0.95	0.95	0.95	0.95	0.94	0.94	0.94	0.94	0.94	0.94	0.94	0.94	0.94	0.94	0.94

- Water level sensor:

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
L0	0.90	0.06	0.86	0.86	0.86	0.86	0.89	0.89	0.89	0.90	0.89	0.89	0.89	0.89	0.89
L10	0.05	0.88	0.05	0.05	0.05	0.08	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05
L20	0.03	0.04	0.07	0.03	0.06	0.03	0.03	0.03	0.03	0.03	0.03	0.03	0.03	0.03	0.03
L30	0.02	0.02	0.02	0.06	0.03	0.03	0.03	0.03	0.03	0.03	0.03	0.03	0.03	0.03	0.03

- Valve flow sensor:

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
Flow	0.07	0.94	0.95	0.98	0.98	0.98	0.98	0.98	0.98	0.98	0.98	0.98	0.98	0.98	0.98
No flow	0.93	0.06	0.05	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02

1.3 Simulation

We assume that only the controller has access to the water level sensor, and thus we can not manually read the water level through the sensor.

2 Exercise 2

2.1 Forward-backward algorithm

The changes we did to the given code was as follows:

- In the HMM class we added a backwardMessages
- We initialised it to NaN when a HMM object was created

```
backwardMessages;

function obj = HMM(priorModel, transModel, sensorModel)
  ...
  obj.backwardMessages = NaN;
end
```

The code below is our implementation of the backward part of the algorithm. The linebreak in the code is not present in the actual code but was done to fit on the page.

```
function obj = backward(obj, data)
  totalTime = length(data);

  obj.backwardMessages=zeros(obj.noHidden,totalTime+1);

  obj.backwardMessages(:,totalTime+1) = 1;
    for t=totalTime:-1:1,
      obj.backwardMessages(:,t)
      = obj.transModel*obj.sensorModel{data(t)}*obj.backwardMessages(:,t+1);
      obj.backwardMessages(:,t)
      = obj.backwardMessages(:,t)./sum(obj.backwardMessages(:,t));
    end
end
```

The final step of combining backward and forward has not been implemented at this given time but, the result of running the backwards function on the given demo that forward was run on gives the following result:

0.6469	0.5923	0.3763	0.6533	0.6273	1.0000
0.3531	0.4077	0.6237	0.3467	0.3727	1.0000

2.2 HMM for exercise 1

```
Trans =
[ 0.8, 0.2;
  0.2, 0.8 ];
Prio = [ 0.6, 0.4 ]';
Sens =
[ 0.02, 0.21;
  0.18, 0.49;
  0.08, 0.09;
  0.72, 0.21 ]';

% 1=yes+red, 2=yes+not red,  3=no+red, 4=no+not red
Dat = [ 4, 2, 1 ];

newhmm = HMM(Prio, Trans, Sens);
newhmm = newhmm.forward(Dat);
newhmm = newhmm.backward(Dat);

disp('Forward:');
disp(newhmm.forwardMessages);
disp('Backward:');
disp(newhmm.backwardMessages);
```

2.3 Implementation of HMM

- Forward:

0.8372	0.4643	0.0804
0.1628	0.5357	0.9196

- Backward:

0.5325	0.2661	0.2522	1.0000
0.4675	0.7339	0.7478	1.0000

3 Exercise 3

3.1 Umbrella

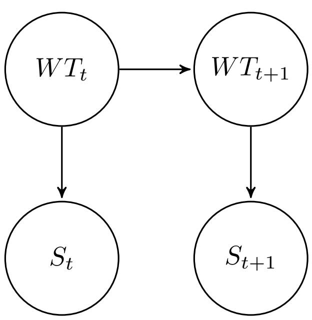
- By calculating the likelihood of the models correctness and the one with the highest likelihood is the most reliable model:
  - $0.7 \cdot 0.7 \cdot 0.7 \cdot 0.3 \cdot 0.7 \cdot 0.7 \cdot 0.3 \cdot 0.3 \cdot 0.7 = 0.003176523$
  - $0.6 \cdot 0.6 \cdot 0.6 \cdot 0.4 \cdot 0.8 \cdot 0.8 \cdot 0.2 \cdot 0.4 \cdot 0.8 = 0.003538944$
- Matlab

```
function p = SSP(obj, sequence)
  p = 1;
  for t=2:length(sequence),
    transition = obj.transModel(sequence(t-1),sequence(t));
    p = p * transition;
  end
end
```

- MATLAB gave us the same results as the manual calculations of the likelihood.

3.2 Water tank

- Kalman Filter



1. $WT_{t+1} = \mathcal{N}(WT_t, 1)$
2. $S_t = \mathcal{N}(WT_t, 1.5)$

- Filtered estimates:

$$\mu_{t+1} = \frac{(1+1) \cdot 44 + 1.5 \cdot 50}{1+1+1.5} = 46.57 \qquad \sigma^2_{t+1} = \frac{(1+1) \cdot 1.5}{1+1+1.5} = 0.857$$

$$\mu_{t+2} = \frac{(0.857+1) \cdot 56 + 1.5 \cdot 46.57}{0.857+1+1.5} = 51.786 \qquad \sigma^2_{t+2} = \frac{(0.857+1) \cdot 1.5}{0.857+1+1.5} = 0.0.83$$

$$\mu_{t+3} = \frac{(0.83+1) \cdot 49 + 1.5 \cdot 51.786}{0.83+1+1.5} = 50.25 \qquad \sigma^2_{t+3} = \frac{(0.83+1) \cdot 1.5}{0.83+1+1.5} = 0.82$$

$$\mu_{t+4} = \frac{(0.82+1) \cdot 51 + 1.5 \cdot 50.25}{0.82+1+1.5} = 50.66 \qquad \sigma^2_{t+4} = \frac{(0.82+1) \cdot 1.5}{0.82+1+1.5} = 0.82$$

$$x = \frac{(x+1)\cdot 1.5}{x+1+1.5}$$
$$x = \frac{(x+1)\cdot 1.5}{x+2.5}$$
$$x(x+2.5) = 1.5x + 1.5$$
$$x^2 + 2.5x = 1.5x + 1.5$$
$$x^2 + x - 1.5 = 0$$

d = 1^2 - 4 · 1 · -1.5 = 7
$x = \frac{-1\pm\sqrt{7}}{2\cdot 1} = \begin{cases} x = \frac{-1+\sqrt{7}}{2\cdot 1} = 0.823 \\ x = \frac{-1-\sqrt{7}}{2\cdot 1} = -1.823 \end{cases}$

4 Exercise 4

4.1 Incomplete observations

- The Initial values and information:

$$P(R_0) = (0.5, 0.5)$$
$$P(R_t \mid R_{t-1} = t) = (0.7, 0.3)$$
$$P(R_t \mid R_{t-1} = f) = (0.3, 0, 7)$$
$$P(U_t \mid R_t = t) = (0.9, 0.1)$$
$$P(U_t \mid R_t = f) = (0.2, 0.8)$$

$$S_1 = (U, \neg U, U)$$
$$S_2 = (U, \neg U, \neg U)$$

- Forward and backward where C is the normalisation constant:

$$P(R_0 = i \mid S) = \alpha_i(i)\beta_i(i) \cdot C$$
$$P(R_0 \mid S_1) = (0.4827, 0.0745) \cdot C$$
$$P(R_0 \mid S_2) = (0.4692, 0.0775) \cdot C$$
$$P(R_0 \mid S) = (0.9519, 0.1520) \cdot C = (0.8623, 0.1377)$$

Next step on the first observations S_1

$$P(R_{t-1}, R_t \mid S) = \alpha(t-1)P(R_t \mid R_{t-1})P(U_T \mid R_t)\beta(t)$$
$$P(R_1, R_2 \mid S_1) = \alpha(1)P(R_2 \mid R_1)P(U_2 \mid R_2)\beta(2)$$

$$forward_0 \cdot backward_1 \cdot Trans \cdot Sensor = x$$
$$0.8182 \cdot 0.3695 \cdot 0.7 \cdot 0.1 = 0.0212$$
$$0.8182 \cdot 0.6305 \cdot 0.3 \cdot 0.8 = 0.1256$$
$$0.1818 \cdot 0.3695 \cdot 0.3 \cdot 0.1 = 0.0020$$
$$0.1818 \cdot 0.6395 \cdot 0.7 \cdot 0.8 = 0.0642$$

$$forward_1 \cdot backward_2 \cdot Trans \cdot Sensor = x$$
$$0.1738 \cdot 0.6273 \cdot 0.7 \cdot 0.9 = 0.0687$$
$$0.1738 \cdot 0.3737 \cdot 0.3 \cdot 0.2 = 0.0039$$
$$0.8268 \cdot 0.6273 \cdot 0.3 \cdot 0.9 = 0.1400$$
$$0.8268 \cdot 0.3737 \cdot 0.7 \cdot 0.2 = 0.0433$$

This gives us the new transition model:

$$P(R_t = T \mid R_{t-1} = T) = 0.0212 + 0.0687 = 0.0899$$
$$P(R_t = F \mid R_{t-1} = T) = 0.0687 + 0.0039 = 0.1295$$
$$P(R_t = T \mid R_{t-1} = F) = 0.0020 + 0.1400 = 0.1420$$
$$P(R_t = F \mid R_{t-1} = F) = 0.0642 + 0.0433 = 0.1075$$

	T	F
T	0.0899	0.1295
F	0.1420	0.1075

And new sensor model, where Umbrella is true:

$$P(R_1 \mid S_{1_{top}}) = 0.8182 \cdot 0.5900 = 0.4827$$
$$P(R_1 \mid S_{1_{bot}}) = 0.1818 \cdot 0.4100 = 0.0745$$
$$P(R_3 \mid S_{1_{top}}) = 0.7251 \cdot 0.6273 = 0.4549$$
$$P(R_3 \mid S_{1_{bot}}) = 0.2749 \cdot 0.3727 = 0.2702$$
$$P(R_1 \mid S_{2_{top}}) = 0.8182 \cdot 0.5735 = 0.4692$$
$$P(R_1 \mid S_{2_{bot}}) = 0.1818 \cdot 0.4264 = 0.0775$$

Summation of top and bottom values respectively:

$$0.4827 + 0.4549 + 0.4692 = 1.4068$$
$$0.0745 + 0.2702 + 0.0775 = 0.4222$$

And where Umbrella is false:

$$P(R_2 \mid S_{1_{top}}) = 0.1738 \cdot 0.3695 = 0.0642$$
$$P(R_2 \mid S_{1_{bot}}) = 0.8262 \cdot 0.6305 = 0.5209$$
$$P(R_2 \mid S_{2_{top}}) = 0.1738 \cdot 0.3247 = 0.0564$$
$$P(R_2 \mid S_{2_{bot}}) = 0.8262 \cdot 0.6753 = 0.5579$$
$$P(R_3 \mid S_{2_{top}}) = 0.0683 \cdot 0.3444 = 0.0235$$
$$P(R_3 \mid S_{2_{bot}}) = 0.9317 \cdot 0.6556 = 0.6108$$

Summation of top and bottom values respectively:

$$0.0642 + 0.0564 + 0.0235 = 0.1441$$
$$0.5209 + 0.5579 + 0.6108 = 1.6896$$

New table after values have been normalised:

	T	F
T	0.9071	0.0929
F	0.1999	0.8001

5 Exercise 5

5.1 Slide 19 example

The formula: $\gamma = 0.9$

$$U(x,y) = R(x,y) + \gamma \cdot max_{a\in\{left,right,up,down\}} \sum_{s'\in S} P(s' \mid a,s)U^2(s')$$

$$U(2,1) = \begin{array}{r} -0.1 + 0.9 \cdot max(0.7 \cdot -0.63 + 0.1 \cdot -5.17 + 0.1 \cdot -0.19 + 0.1 \cdot -0.19, \\ 0.7 \cdot -5.17 + 0.1 \cdot -0.63 + 0.1 \cdot -0.19 + 0.1 \cdot -0.19, \\ 0.7 \cdot -0.19 + 0.1 \cdot -0.63 + 0.1 \cdot -5.17 + 0.1 \cdot -0.19, \\ 0.7 \cdot -0.19 + 0.1 \cdot -0.63 + 0.1 \cdot -5.17 + 0.1 \cdot -0.19) \end{array}$$

$$U(2,1) = -0.1 + 0.9 \cdot max(-0.996, -3.72, -0.732, -0.732)$$

$$U(2,1) = -0.7588$$

	1	2	3
1	3.42	6.23	10
2	-0.76	-1.07	5.24
3	-0.35	-0.77	2.79

5.2 Policy iteration