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REFLECTION PRINCIPLES AND LARGE CARDINALS

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Prohlašuj, že jsem bakalářkou práci vypracoval samostatně a že jsem uvedl všechny použité prameny a literaturu.

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Abstract

Práce zkoumá vztah tzv. principů reflexe a velkých kardinálů. Lévy ukázal, že v ZFC platí tzv. věta o reflexi a dokonce, že věta o reflexi je ekvivalentní schématu nahrazení a axiomu nekonečna nad teorií ZFC bez axiomu nekonečna a schématu nahrazení. Tedy lze na větu o reflexi pohlížet jako na svého druhu axiom nekonečna. Práce zkoumá do jaké míry a jakým způsobem lze větu o reflexi zobecnit a jaký to má vliv na existenci tzv. velkých kardinálů. Práce definuje nedosažitelné, Mahlovy a nepopsatelné kardinály a ukáže, jak je lze zavést pomocí reflexe. Přirozenou limitou kardinálů získaných reflexí jsou kardinály nekonzistentní s L. Práce nabídne intuitivní zdůvodněn, proč tomu tak je.

Abstract

This thesis aims to examine relations between so called "Reflection Principles" and Large cardinals. Lévy has shown that Reflection Theorem is a sound theorem of ZFC and it is equivalent to Replacement Scheme and the Axiom of Infinity. From this point of view, Reflection theorem can be seen a specific version of an Axiom of Infinity. This paper aims to examine the Reflection Principle and it's generalisations with respect to existence of Large Cardinals. This thesis will establish Inaccessible, Mahlo and Indescribable cardinals and their definition via reflection. A natural limit of Large Cardinals obtained via reflection are cardinals inconsistent with L. The thesis will offer an intuitive explanation of why this is the case.

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1 Introduction

1.1 Motivation and Origin

> The Universe of sets cannot be uniquely characterized (i. e. distinguished from all its initial elements) by any internal structural property of the membership relation in it, which is expressible in any logic of finite of transfinite type, including infinitary logics of any cardinal order.

— Kurt Gödel [10]

To understand why do need reflection in the first place, let's think about infinity for a moment. In the intuitive sense, infinity is an upper limit of all numbers. But for centuries, this was merely a philosophical concept, closely bound to religious and metaphysical way of thinking, considered separate from numbers used for calculations or geometry. It was a rather vague concept. In ancient Greece, Aristotle's response to famous Zeno's paradoxes introduced the distinction between actual and potential infinity. He argued, that potential infinity is (in today's words) well defined, as opposed to actual infinity, which remained a vague incoherent concept. He didn't think it's possible for infinity to inhabit a bounded place in space or time, rejecting Zeno's thought experiments as a whole. Aristotle's thoughts shaped western thinking partly due to Aquinas, who himself believed actual infinity to be more of a metaphysical concept for describing God than a mathematical property attributed to any other entity. In his Summa Theologica [1] he argues:

> A geometrician does not need to assume a line actually infinite, but takes some actually finite line, from which he subtracts whatever he finds necessary; which line he calls infinite.

Less than hundred years later, Gregory of Rimini wrote

> If God can endlessly add a cubic foot to a stone–which He can–then He can create an infinitely big stone. For He need only add one cubic foot at some time, another half an hour later, another a quarter of an hour later than that, and so on ad infinitum. He would then have before Him an infinite stone at the end of the hour.

Which is basically a Zeno's Paradox made plausible with God being the actor. In contrast to Aquinas' position, Gregory of Rimini theoretically constructs an object with actual infinite magnitude that is essentially different from God.

[1] Part I, Question 7, Article 3, Reply to Objection 1

Even later, in the 17th century, pushing the property of infinitness from the Creator to his creation, Nature, Leibniz wrote to Foucher in 1962:

> I am so in favor of the actual infinite that instead of admitting that Nature abhors it, as is commonly said, I hold that Nature makes frequent use of it everywhere, in order to show more effectively the perfections of its Author. Thus I believe that there is no part of matter which is not, I do not say divisible, but actually divided; and consequently the least particle ought to be considered as a world full of an infinity of different creatures.

But even though he used potential infinity in what would become foundations of modern Calculus and argued for actual infinity in Nature, Leibniz refused the existence of an infinite, thinking that Galileo's Paradoxon[2] is in fact a contradiction. The so called Galileo's Paradoxon is an observation Galileo Galilei made in his final book "Discourses and Mathematical Demonstrations Relating to Two New Sciences". He states that if all numbers are either squares and non–squares, there seem to be less squares than there is all numbers. On the other hand, every number can be squared and every square has it's square root. Therefore, there seem to be as many squares as there are all numbers. Galileo concludes, that the idea of comparing sizes makes sense only in the finite realm.

> Salviati: So far as I see we can only infer that the totality of all numbers is infinite, that the number of squares is infinite, and that the number of their roots is infinite; neither is the number of squares less than the totality of all the numbers, nor the latter greater than the former; and finally the attributes "equal," "greater," and "less," are not applicable to infinite, but only to finite, quantities. When therefore Simplicio introduces several lines of different lengths and asks me how it is possible that the longer ones do not contain more points than the shorter, I answer him that one line does not contain more or less or just as many points as another, but that each line contains an infinite number.

Leibniz insists in part being smaller than the whole saying

> Among numbers there are infinite roots, infinite squares, infinite cubes. Moreover, there are as many roots as numbers. And there are as many squares as roots. Therefore there are as many squares

[2] zneni galileova paradoxu

> as numbers, that is to say, there are as many square numbers as there are numbers in the universe. Which is impossible. Hence it follows either that in the infinite the whole is not greater than the part, which is the opinion of Galileo and Gregory of St. Vincent, and which I cannot accept; or that infinity itself is nothing, i.e. that it is not one and not a whole.

TODO Hegel–strucne?

TODO Cantor

TODO mene teologie, vice matematiky

TODO definovat pojmy (trida etc)

TODO neni V v nejakem smyslu porad potencialni nekonecno, zatimco mnoziny vetsi nez omega jsou aktualni? nebo jsou potencialni protoze se staveji pres indukci, od spoda.

In his work, he defined transfinite numbers to extend existing natural number structure so it contains more objects that behave like natural numbers and are based on an object (rather a meta-object) that doesn't explicitly exist in the structure, but is closely related to it. This is the first instance of reflection. This paper will focus on taking this principle a step further, extending Cantor's (or Zermelo–Fraenkel's, to be more precise) universe so it includes objects so big, they could be considered the universe itself, in a certain sense.

TODO dal asi smazat

The original idea behind reflection principles probably comes from what could be informally called "universality of the universe". The effort to precisely describe the universe of sets was natural and could be regarded as one of the impulses for formalization of naive set theory. If we try to express the universe as a set $\{x|x = x\}$, a paradox appears, because either our set is contained in itself and therefore is contained in a set (itself again), which contradicts the intuitive notion of a universe that contains everything but is not contained itself.

TODO ???

If there is an object containing all sets, it must not be a set itself. The notion of class seems inevitable. Either directly the ways for example the Bernays–Gödel set theory, we will also discuss later in this paper, does in, or on a meta–level like the Zermelo–Fraenkel set theory, that doesn't refer to them in the axioms but often works with the notion of a universal class. duet Another obstacle of constructing a set of all sets comes from Georg Cantor, who proved that the set of all subsets of a set (let A be the set and $\mathscr{P}(()A)$ its powerset) is strictly larger that A. That would turn every aspiration to finally establish an universal set into a contradictory infinite regression.[3]. We will use V to denote the class of all sets. From previous thoughts we can easily argue, that it is impossible to construct a property that holds for V and no set and is neither paradoxical like $\{x|x = x\}$ nor trivial. Previous observation can be transposed to a rather naive formulation of the reflection principle:

Reflection Any property which holds in V already holds in some initial segment of V.

To avoid vagueness of the term "property", we could informally reformulate the above statement into a schema:

For every first-order formula[4] φ holds in $V \leftrightarrow \varphi$ holds in some initial segment of V.

Interested reader should note that this is a theorem scheme rather than a single theorem. [5]

1.2 A few historical remarks on reflection

Reflection made it's first in set-theoretical appearance in Gödel's proof of GCH in L (citace Kanamori ? Lévy and set theory), but it was around even earlier as a concept. Gödel himself regarded it as very close to Russel's reducibility axiom (an earlier equivalent of the axiom schema of Zermelo's separation). Richard Montague then studied reflection properties as a tool for verifying that Replacement is not finitely axiomatizable (citace?). a few years later Lévy proved in [2] the equivalence of reflection with Axiom of infinity together with Replacement in proof we shall examine closely in chaper 2.

TODO co dal? recent results?

[3]An intuitive analogy of this *reductio ad infinitum* is the status of ω, which was originally thought to be an unreachable absolute, only to become starting point of Cantor's hierarchy of sets growing beyond all boundaries around the end of the 19^{th} century

[4]this also works for finite sets of formulas [4, p. 168]

[5]If there were a single theorem stating "for any formula φ that holds in V there is an initial segment of V where φ also holds", we would obtain the following contradiction with the second Gödel's theorem: In ZFC, any finite group of axioms of ZFC holds in some initial segment of the universe. If we take the largest of those initial segments it is still strictly smaller than the universe and thus we have, via compactness, constructed a model of ZFC within ZFC. That is, of course a harsh contradiction. This also leads to an elegant way to prove that ZFC is not finitely axiomatizable.

1.3 Reflection in Platonism and Structuralism

TODO cite "reflection in a structuralist setting"

TODO veci o tom, ze reflexe je ok protoze reflektuje veci ktere objektivne plati, protoze plati pro V ...

TODO souvislost s kompaktnosti, hranice formalich systemu nebo alespon ZFC

1.4 Notation (??) TODO

1. *Reflection* je obecne reflexe (jaka presne)
2. $\textit{Reflection}_1$ je reflexe prvoradovych formuli
3. $\textit{Reflection}_2$ je reflexe druhoradovych formuli
4. etc...

V a V_α odkazuji k Von Neumannove hierarchii (pro jistotu)

2 Levy's first-order reflection

2.1 Introduction

This section will try to present Lévy's proof of a general reflection principle being equivalent to Replacement and Infinity under ZF minus Replacement and Infinity. We will first introduce a few axioms and definitions that were a different in Lévy's paper[2], but are equivalent to today's terms. We will write them in contemporary notation, our aim is the result, not history of set theory notation.

Please note that Lévy's paper was written in a period when Set theory was oriented towards semantics, which means that everything was done in a model. All proofs were theodel that of ZFC was V_α (notated as $R(\alpha)$ at the time) for some cardinal α, which means that α is a inaccessible cadinal. Please bear in mind that this is vastly different from saying that there is an inaccessible α inside the model. This V_α is also referred to as $Scm^{\mathsf{Q}}(u)$, which means that u ($u = V_\alpha$ in our case) is a standard complete model of an undisclosed axiomatic set theory Q formulated in the "non-simple applied first order functional calculus", which is second-order theory is today's terminology, we are allowed to quantify over functions and thus get rid of axiom schemes. (Note that Lévy always speaks of "the axiom of replacement"). Besides placeholder set theory Q and ZF, which the reader should be familiar with, theories Z, S, and SF are used in the text. Z is ZF minus replacement, S is ZF minus replacement and infinity, and finally SF is ZF minus infinity. "The axiom of subsets" is an older name for the axiom scheme of specification (and it's not a scheme since we are now working in second order logic). Also note that universal quantifier does not appear, $\forall x \varphi(x)$ would be written as $(x)\varphi(x)$, the symbol for negation is "$\sim$".

2.2 Lévy's Original Paper

The following are a few definitions that are used in Lévy's original article. [6]

Definition 2.1 *Relativization[4, Definition 12.6]*
Let M be a class, E a binary relation on M and let $\varphi(x_1, \ldots, x_n)$ be a formula. The relativization of φ to M and E is the formula

$$\varphi^{M,E}(x_1, \ldots, x_n) \tag{2.1}$$

[6] While some of them won't be of much use in this paper, they will provide extremely helpful when reading the original article as set theory notation and terminology has evolved in the last 50 years considerably.

Defined in the following inductive manner:

$$\begin{aligned}(x \in y)^{M,E} &\leftrightarrow xEx \\ (x = y)^{M,E} &\leftrightarrow x = y \\ (\neg\varphi)^{M,E} &\leftrightarrow \neg\varphi^{M,E} \\ (\varphi \ \&\ \psi)^{M,E} &\leftrightarrow \varphi^{M,E} \ \&\ \psi^{M,E} \\ (\exists x\varphi)^{M,E} &\leftrightarrow (\exists x \in M)\varphi^{M,E}\end{aligned} \tag{2.2}$$

Next two definitions are not used in contemporary set theory, but they illustrate 1960's set theory mind-set and they are used heavily in Lévy's text, so we will include and explain them for clarity. Generally in this chapter, Q stands for an undisclosed axiomatic set theory, u is usually a model, counterpart of today's V[7], e is a relation that serves as $\in$ in the given model.

Definition 2.2 *Standard model of a set theory*
We say the u is a standard model of Q *with ša membership relation e, written as* $Sm^{\mathsf{Q}}(u)$*, if both of the following hold*
(i) $(x, y) \in e \leftrightarrow y \in u \ \&\ x \in y$
(ii) $y \in u \ \&\ x \in y \rightarrow x \in u$

Definition 2.3 *Standard complete model of a set theory*
We say that that u is a standard complete model of a set theory Q *with a membership relation e if:*
(i) u is a transitive set with respect to $\in$
(ii) $\forall e((x, y) \in e \leftrightarrow (y \in u \ \&\ x \in y) \ \&\ Sm^{\mathsf{Q}}(u, e))$
this is written as $Scm^{\mathsf{Q}}(u)$*.*

Definition 2.4 *Cardinal inaccessible with respect to* Q

$$In^{\mathsf{Q}}(\kappa) \ = \ Scm^{\mathsf{Q}}(V_\kappa) \tag{2.3}$$

This definition is more general than the usual one[8], we will often write $In(\kappa)$ as a shorthand for $In^{\mathsf{ZF}}(\kappa)$.

The following is a principle of complete reflection over ZF.

[7]Which is of course not referred to as a model, but it is used in a similar fashion, in this case the term "model" was a metamathematical notion because it was not based on any underlying structure of theory. It can be easily formalized in any set theory, but it's not helpful for our case.

[8]Which says that a cardinal κ is inaccessible iff it is a strong limit regular cardinal.

Definition 2.5 $N(\varphi)$

$$\exists u(Scm^{\mathsf{ZF}}(u) \ \& \ \forall x_1, \ldots, x_n(x_1, \ldots, x_n \in u \to \varphi \leftrightarrow \varphi^u)) \tag{2.4}$$

where φ is a formula which does not contain free variables except $x_1, \ldots, x_n$.

Note that this by (**??**) equivalent to $\exists u(In^{\mathsf{ZF}}(u) \ \& \ \forall x_1, \ldots, x_n(x_1, \ldots, x_n \in u \to \varphi \leftrightarrow \varphi^u))$, where $In(\alpha)$ is equivalent to the standard notion of inaccessibility.

2.3 $\mathsf{S} \models$ *Reflection* $\leftrightarrow$ (*Replacement* & *Infinity*)

Definition 2.6 $N_0(\varphi)$

$$\exists u(Scm^{\mathsf{S}}(u) \ \& \ \forall x_1, \ldots, x_n(x_1, \ldots, x_n \in u \to \varphi \leftrightarrow \varphi^u)) \tag{2.5}$$

where φ is a formula which does not contain free variables except $x_1, \ldots, x_n$.

Note that the only difference between N and N_0 is the set theory used.

Theorem 2.7 *In S, the schema N_0 implies the Axiom of Infinity.*

Proof. For any φ, N_0 gives us $\exists u Scm^{\mathsf{S}}(u)$, which means that there is a set u that is identical to V_α for some alpha, so $\exists \alpha Scm^{\mathsf{S}}(V_\alpha)$. We don't know the exact size of this α, but we know that $\alpha \geq \omega$, otherwise α would be finite, therefore not closed under the powerset operation, which would contradict the axiom of powersets. In order to prove that it is a model of S, we would need to verify all axioms of S. We have already shown that ω is closed under the powerset operation. Foundation, extensionality and comprehension are clear from the fact that we work in ZF[9], pairing is clear from the fact, that given two sets A, B, they have ranks a, b, without loss of generality we can assume that $a \leq b$, which means that $A \in V_a \in V_b$, therefore V_b is a set that satisfies the paring axiom: it contains both A and B.

Note that any limit cardinal is a model of S.

We now want to prove that V_α leads to existence of an inductive set, which is a set that satisfies $\exists A(\emptyset \in A \ \& \forall x \in A \ ((x \cup \{x\}) \in A))$. If we can find a way to construct V_ω from any V_α satisfying $\alpha \geq \omega$, we are done. Since ω is the least limit ordinal, all we need is the following

$$\bigcap \{V_\kappa \mid \forall \lambda(\lambda < \kappa \to \exists \mu(\lambda < \mu < \kappa))\} \tag{2.6}$$

[9]We only need to verify axioms that provide means of constructing larger sets from smaller to make sure they don't exceed ω. Since ω is an initial segment of ZF, the axiom scheme of specification can't be broken, the same holds for foundation and extensionality.

because V_κ is a transitive set for every κ, thus the intersection is non-empty unless empty set satisfies the property or the set of V_κs is itself empty. $\square$

Theorem 2.8 *In* S*, the schema* N_0 *implies Replacement schema.*

Proof. TODO vysvetlit! (podle contemporary verze)

Let $\varphi(v,w)$ be a formula wth no free variables except $v, w, x_1, \dots, x_n$ where n is any natural number. Let χ be an instance of replacement schema for this φ:

$$\chi = \forall r,s,t(\varphi(r,s)\&\varphi(r,t) \to s = t) \to \forall x\exists y\forall w(w \in y \leftrightarrow \exists v(v \in x\&\varphi(v,w))) \tag{2.7}$$

We can deduce the following from N_0:

(i) $x_1, \dots, x_n, v, w \in u \to (\varphi \leftrightarrow \varphi^u)$

(ii) $x_1, \dots, x_n, v \in u \to (\exists w\varphi \leftrightarrow (\exists w\varphi)^u)$

(iii) $x_1, \dots, x_n, x \in u \to (\chi \leftrightarrow \chi^u)$

(iv) $\forall x_1, \dots, x_n\forall x(\chi \leftrightarrow (\forall x_1, \dots, x_n\forall x\chi)^u)$

Note that (i), (ii), (iii) are obtained from instances of N_0 for φ, $\exists w\varphi$ and χ respectively. From relativization we also know that $(\exists w\varphi)^u$ is equivalent to $\exists w(w \in u\&\varphi^u)$. Therefore (ii) is equivalent to $x_1, \dots, x_n, v \in u \to (\exists w(w \in u\&\varphi^u))$.

If φ is a function $(\forall r,s,t(\varphi(r,s)\&\varphi(r,t) \to r = t))$, then for every $x \in u$, which is also $x \subset u$ by $Scm^{\mathsf{S}}(u)$, it maps elements of x onto u. From the axiom scheme of comprehension[10], we can find a set of all images of elements of x. Let's call it y. That gives us $x_1, \dots, x_n, x \in u \to \chi$. By (iii) we get $x_1, \dots, x_n, x \in u \to \chi^u$, closure of this formula is $(\forall x_1, \dots, x_n\forall x\chi)^u$, which together with (iv) yields $\forall x_1, \dots, x_n\forall x\chi$. By the means of specification we end up with χ, which is all we need for now. $\square$

What we have just proven in only a single theorem form said article, we will introduce other interesting propositions later in this thesis in their appropriate context[11].

2.4 Contemporary restatement

We will now prove what is also Lévy's reflection theorem, but a little stronger, rephrased with more up to date set theory. The main difference is, that while

[10] axiom of subsets in Levy's version

[11] See chapter 3

Lévy reflects φ from V into a set u that is a "standard complete model of S"[12], we say that there is a V_α that reflects φ.

We will prove the equivalence of $Reflection_1$ with *Replacement* and *Infinity* in two parts. First, we will show that $Reflection_1$ is a theorem of ZF, then the second implication which proves *Infinity* and *Replacement* from $Reflection_1$ in S.

The following lemma is usually done in more parts, the first being with one formula and the other with n. We will only state and prove the generalised version for n formulas, knowing that $n = 1$ is just a specific case and the proof is exactly the same.

Lemma 2.9 *Lemma Let $\varphi_1, \ldots, \varphi_n$ be any formulas with m parameters*[13].

(i) For each set M_0 there is such M that $M_0 \subset M$ and the following holds for every $i \leq n$:

$$\exists x \varphi_i(u_1, \ldots, u_{m-1}, x) \rightarrow (\exists x \in M)\varphi_i(u_1, \ldots, u_{m-1}, x) \tag{2.8}$$

for every $u_1, \ldots, u_{m-1} \in M$.

(ii) Furthermore there is an ordinal α such that $M_0 \subset V_\alpha$ and the following holds for each $i \leq n$:

$$\exists x \varphi_i(u_1, \ldots, u_{m-1}, x) \rightarrow (\exists x \in V_\alpha)\varphi_i(u_1, \ldots, u_{m-1}, x) \tag{2.9}$$

for every $u_1, \ldots, u_{m-1} \in M$.

(iii) Assuming Choice*, there is M, $M_0 \subset M$ such that 2.8 holds for every M, $i \leq n$ and $|M| \leq |M_0| \cdot \aleph_0$.*

Proof. We will simultaneously prove statements (**i**) and (**ii**), denoting M^T the transitive set required by part (ii). Unless explicitly stated otherwise for specific steps, it is thought to be equivalent to M.

Let us first define operation $H(u_1, \ldots, u_{m-1})$ that gives us the set of x's with minimal rank satisfying $\varphi_i(u_1, \ldots, u_{m-1}, x)$ for given parameters $u_1, \ldots, u_{m-1}$ for every $i \leq n$.

$$H_i(u_1, \ldots, u_n) = \{x \in C_i : (\forall z \in C)(rank(x) \leq rank(z))\} \tag{2.10}$$

[12] Any limit ordinal is in fact a model of S, we shall pay more attention to that in a moment.

[13] For formulas with different number of parameters take for m the highest number of parameters among given formulas. Add spare parameters to the other formulas so that x remains the last parameter. That can be done in a following manner: Let φ'_i be the a formula with k parameters, $k < m$. Let us set $\varphi_i(u_1, \ldots, u_{m-1}, x) = \varphi'_i(u_1, \ldots, u_{k-1}, u_k, \ldots, u_{m-1}, x)$, notice that $u_k, \ldots, u_{m-1}$ are spare variables added just for formal simplicity.

for each $i \leq n$, where

$$C_i = \{x : \varphi_i(u_1, \dots, u_{m-1}, x)\} \text{ for } i \leq n \tag{2.11}$$

Next, let's construct M from given M_0 by induction.

$$M_{i+1} = M_i \cup \bigcup_{j=0}^{n} \bigcup \{H_j(u_1, \dots, u_{m-1}) : u_1, \dots, u_{m-1} \in M_i\} \tag{2.12}$$

In other words, in each step we add the elements satisfying $\varphi(u_1, \dots, u_{m-1}, x)$ for all parameters that were either available earlier or were added in the previous step. For statement (ii), this is the only part that differs from (i). Let us take for each step transitive closure of M_{i+1} from (i). In other words, let γ be the smallest ordinal such that

$$(M_i^T \cup \bigcup_{j=0}^{n} \{\bigcup \{H_j(u_1, \dots, u_{m-1}) : u_1, \dots, u_{m-1} \in M_i\}\}) \subset V_\gamma \tag{2.13}$$

Then the incremetal step is like so:

$$M_{i+1}^T = V_\gamma \tag{2.14}$$

The final M is obtained by joining all incremental steps together.

$$M = \bigcup_{i=0}^{\infty} M_i, \;\; M^T = \bigcup_{i=0}^{\infty} M_i^T \tag{2.15}$$

We have yet to finish part (**iii**). Let's try to construct a set M' that satisfies the same conditions like M but is kept as small as possible. Assuming the Axiom of Choice, we can modify the process so that cardinality of M' is at most $|M_0| \cdot \aleph_0$. Note that the size of M' is determined by the size of M_0 an, most importantly, by the size of $H_i(u_1, \dots, u_{m-1})$ for any $i \leq n$ in individual levels of the construction. Since the lemma only states existence of some x that satisfies $\varphi_i(u_1, \dots, u_{m-1}, x)$ for any $i \leq n$, we only need to add one x for every set of parameters but $H_i(u_1, \dots, u_{m-1})$ can be arbitrarily large. Since Axiom of Choice ensures that there is a choice function, let F be a choice function on $\mathscr{P}(()M')$. Also let $h_i(u_1, \dots, u_{m-1}) = F(H_i(u_1, \dots, u_{m-1}))$ for $i \leq n$, which means that h is a function that outputs an x that satisfies $\varphi_i(u_1, \dots, u_{m-1}, x)$ for $i \leq n$ and has minimal rank among all such witnesses. The induction step needs to be redefined to

$$M'_{i+1} = M'_i \cup \bigcup_{j=0}^{n} \{H_j(u_1, \dots, u_{m-1}) : u_1, \dots, u_{m-1} \in M'_i\} \tag{2.16}$$

In every step, the amount of elements added in M'_{i+1} is equivalent to the amount of sets of parameters the yielded elements not included in M'_i. So the cardinality of M'_{i+1} exceeds the cardinality of M'_i only for finite M'_i. It is easy to see that if M_0 is finite, M' is countable because it was built from countable union of finite sets. If M_0 is countable or larger, cardinaly of M' is equal to the cardinality of M_0.[14] Therefore $|M'| \leq |M_0| \cdot \aleph_0$ ☐

And now for the theorem itself

Theorem 2.10 *First-order Reflection*
Let $\varphi(x_1, \ldots, x_n)$ be a first-order formula.

(i) For every set M_0 there exists M such that $M_0 \subset M$ and the following holds:

$$\varphi^M(x_1, \ldots, x_n) \leftrightarrow \varphi(x_1, \ldots, x_n) \tag{2.17}$$

for every $x_1, \ldots, x_n$.

(ii) For every set M_0 there is a transitive set M, $M_0 \subset M$ such that the following holds:

$$\varphi^M(x_1, \ldots, x_n) \leftrightarrow \varphi(x_1, \ldots, x_n) \tag{2.18}$$

for every $x_1, \ldots, x_n$.

(iii) For every set M_0 there is α such that $M_0 \subset V_\alpha$ and the following holds:

$$\varphi^{V_\alpha}(x_1, \ldots, x_n) \leftrightarrow \varphi(x_1, \ldots, x_n) \tag{2.19}$$

for every $x_1, \ldots, x_n$.

(iv) Assuming the Axiom of Choice, for every set M_0 there is M such that $M_0 \subset M$ and $|M| \leq |M_0| \cdot \aleph_0$ and the following holds:

$$\varphi^M(x_1, \ldots, x_n) \leftrightarrow \varphi(x_1, \ldots, x_n) \tag{2.20}$$

for every $x_1, \ldots, x_n$.

Proof. Let's prove (i) for one formula φ via induction by complexity first. We can safely assume that φ contains no quantifiers besides $\exists$ and no logical connectives other than $\neg$ and &. Assume that this M is obtained from lemma 2.9. The fact, that atomic formulas are reflected in every M comes directly from definition of relativization and the fact that they contain no

[14]It can not be smaller because $|M'_{i+1}| \geq |M'_i|$ for every i. It may not be significantly larger because the maximum of elements added is the number of n-tuples in M'_i, which is of the same cardinality is M'_i.

quantifiers.[15] The same holds for formulas in the form of $\varphi = \neg\varphi'$. Let us recall the definition of relativization for those formulas in 2.1.

$$(\neg\varphi_1)^M \leftrightarrow \neg(\varphi_1^M) \tag{2.21}$$

Because we can assume from induction that $\varphi'^M \leftrightarrow \varphi'$, the following holds:

$$(\neg\varphi')^M \leftrightarrow \neg(\varphi'^M) \leftrightarrow \neg\varphi' \tag{2.22}$$

The same holds for $\varphi = \varphi_1 \ \& \ \varphi_2$. From the induction hypothesis we know that $\varphi_1^M \leftrightarrow \varphi_1$ and $\varphi_2^M \leftrightarrow \varphi_2$, which together with relativization for formulas in the form of $\varphi_1\& \ \varphi_2$ gives us

$$(\varphi_1 \ \& \ \varphi_2)^M \leftrightarrow \varphi_1^M \ \& \ \varphi_2^M \leftrightarrow \varphi_1 \ \& \ \varphi_2 \tag{2.23}$$

Let's now examine the case when from the induction hypethesis, M reflects $\varphi'(u_1, \ldots, u_n, x)$ and we are interested in $\varphi = \exists x\varphi'(u_1, \ldots, u_n, x)$. The induction hypothesis tells us that

$$\varphi'^M(u_1, \ldots, u_n, x) \leftrightarrow \varphi'(u_1, \ldots, u_n, x) \tag{2.24}$$

so, together with above lemma 2.9, the following holds:

$$\begin{aligned}
\varphi(u_1, \ldots, u_n, x) \\
\leftrightarrow \exists x\varphi'(u_1, \ldots, u_n, x) \\
\leftrightarrow (\exists x \in M)\varphi'(u_1, \ldots, u_n, x) \\
\leftrightarrow (\exists x \in M)\varphi'^M(u_1, \ldots, u_n, x) \\
\leftrightarrow (\exists x\varphi'(u_1, \ldots, u_n, x))^M \\
\leftrightarrow \varphi^M(u_1, \ldots, u_n, x)
\end{aligned} \tag{2.25}$$

Which is what we have needed to prove:

So far we have proven part (i) of this theorem for one formula φ, we only need to verify that the same holds for any finite number of formulas. This has in fact been already done since lemma 2.9 gives us M for any (finite) amount of formulas. We can than use the induction above to verify that it reflects each of the formulas individually.

[15]Note that this does not hold generally for relativizations to M, E, but only for relativization to $M, \in$, which is our case.

Now we want to verify other parts of our theorem. Since V_α is a transitive set, by proving (iii) we also satisfy (ii). To do so, we only need to look at part (ii) of lemma 2.9. All of the above proof also holds for $M = V_\alpha$.

To finish part (iv), we take M of size $\leq |M_0| \cdot \aleph_0$, which exists due to part (iii) of lemma 2.9, the rest being identical. ☐

Theorem 2.11 Reflection *is equivalent to* Infinity & Replacement *under ZFC minus* Infinity & Replacement

Proof. Since 2.10 already gives one side of the implication, we are only interested in showing the converse which we shall do in two parts:

Reflection → Infinity

Let us first find a formula to be reflected that requires a set M at least as large as V_ω. Let us consider the following formula:

$$\varphi'(x) = \forall\lambda(\lambda < x \rightarrow \exists\mu(\lambda < \mu < x)) \tag{2.26}$$

Because φ says "there is a limit ordinal", if it holds for some x, the Infinity axiom is very easy to satisfy. But we know that there are limit ordinals in ZF, therefore $\varphi = \exists x\varphi'(x)$ is a valid statement. *Reflection* then gives us a set M in which φ^M holds, that is, a set that contains a limit ordinal. So the set of off limit ordinals is non-empty and because ordinals are well-founded, it has a minimal element. Let's call it μ.

$$\mu = \bigcap\{V_\kappa \ : \ \forall\lambda(\lambda < \kappa \rightarrow \exists\mu(\lambda < \mu < \kappa))\} \tag{2.27}$$

We can see that μ is the least limit ordinal and therefore it satisfies *Infinity.*

Reflection → Replacement

Given a formula $\varphi(x, y, u_1, \ldots, u_n)$, we can suppose that it is reflected in any M [16] What we want to obtain is the following:

$$\forall x, y, z(\varphi(x, y, u_1, \ldots, u_n)\&\varphi(x, z, u_1, \ldots, u_n) \rightarrow y = z) \rightarrow \tag{2.28}$$

$$\rightarrow \forall X\exists Y\forall y \ (y \in Y \leftrightarrow \exists x(\varphi(x, y, u_1, \ldots, u_n) \ \& \ x \in X)) \tag{2.29}$$

We do also know that $x, y \in M$, in other words for every X, $Y = \{y \mid \varphi(x, y, u_1, \ldots, u_n)\}$ we know that $X \subset M$ and $Y \subset M$, which, together

[16] Which means that for $x, y, u_1, \ldots, u_n \in M$, $\varphi^M(x, y, u_1, \ldots, u_n) \leftrightarrow \varphi(x, y, u_1, \ldots, u_n)$.

with the comprehension schema[17] implies that Y, the image of X over φ, is a set. Which is exactly the Replacement Schema we hoped to obtain. $\square$

We have shown that *Reflection* for first-order formulas, $\mathit{Reflection}_1$ is a theorem of ZF, which means that it won't yield us any large cardinals. We have also shown that it can be used instead of the Axiom of Infinity and Replacement Scheme, but ZF + $\mathit{Reflection}_1$ is a conservative extension of ZF. Besides being a starting point for more general and powerful statements, it can be used to show that ZF is not finitely axiomatizable. That is because *Reflection* gives a model to any finite number of (consistent) formulas. So if $\varphi_1, \ldots, \varphi_n$ for any finite n would be the axioms of ZF, *Reflection* would always contain a model of itself, which would in turn contradict the Second Gödel's Theorem[18]. Notice that, in a way, reflection is complementary to compactness. Compactness argues that given a set of sentences, if every finite subset yields a model, so does the whole set. Reflection, on the other hand, says that while the whole set has no model in the underlying theory, every finite subset does have one.

Also, notice how reflection can be used in ways similar to upward Löwenheim–Skolem theorem. Since Reflection extends any set M_0 into a model of given formulas $\varphi_1, \ldots, \varphi_n$, we can choose the lower bound of the size of M by appropriately chocing M_0.

In the next section, we will try to generalize *Reflection* in a way that transcends ZF and finally yields some large cardinals.

[17] Called the axiom of subsets in Levy's proof.

[18] See chapter 3.3 for further details.

3 Reflecting To Large Cardinals

In this chapter we aim to examine stronger reflection properties in order to reach cardinals unavailable in ZF. Like we said in the first chapter, the variety of reflection principles comes from the fact that there are many way to formalize "properties of the universal class". It is not always obvious what properties hold for V because, unlike Lévy's approach, not much attention is paid to what exactly is this V, and, more importantly, there are many ways to formalize the notion of property. We have shown that reflecting properties as first-order formulas doesn't allow us to leave ZF. We will broaden the class of admissible properties to be reflected and see whether there is a natural limit in the height or width on the reflected universe and also see that no matter how far we go, the universal class is still as elusive as it is when seen from S. That is because for every process for obtaining larger sets such as for example the powerset operation in ZF, this process can't reach V and thus, from reflection, there is an initial segment of V that can't be reached via said process.

To see why this is important, let's dedicate a few lines to the intuition behind the notions of limitness, regularity and inaccessibility in a manner strongly influenced by [8]. To see why limit and strongly limit cardinals are worth mentioning, note that they are "limit" not only in a sense of being a supremum of an ordinal sequence, they also show that a certain way of obtaining bigger objects from smaller ones limited in terms of possibilities. $\aleph_\lambda$ is a limit cardinal iff there is no α such that $\aleph_{\alpha+1} = \aleph_\lambda$. Strongly limit cardinals point to the limits of the powerset operation. It has been too obvious so far, so let's look at the regular cardinals in this manner. Regular cardinals are those that cannot be[19], expressed as a supremum of smaller amount of smaller objects[20]. More precisely, κ is regular if there is no way to define it as u union of less than κ ordinals, all smaller than κ. So unless there already is a set of size κ, *Replacement* is useless in determining whether κ is really a set. Note that assuming *Choice*, successor cardinals are always regular, so most[21] limit cardinals are singular cardinals. So if one is traversing the class of all cardinals upwards, successor steps are still sets thanks to the powerset axiom while singular limits cardinal are not proper classes because they are images of smaller sets via *Replacement*. Regular cardinals are, in a way, limits of how far can we get by taking limits of increasing sequences of ordinals obtained via *Replacement*.

[19] Assuming *Choice*.

[20] Just like ω can not be expressed as a supremum of a finite set consisting solely of finite numbers.

[21] All provable to exist in ZF

That all being said, it is easy to see that no cardinals in ZF are both strongly limit and regular beucase there is no way in ZF to ensure they are sets and not proper classes. The only exception to this rule is $\aleph_0$ which need a special axiom for itself to exist. It should now be obvious why the fact that κ is inaccessible implies that $\kappa = \aleph_\kappa$.[22]

We will also examine the connection between reflection principles and fixed points of ordinal functions in a manner proposed by Lévy in [2]. We will also see that, like Lévy [2] has proposed, there is a meaningful way to extend the relation between S and ZF into a hierarchy of axiomatic set theories. Those are the three lines of thinking that we will find are in fact different facets of the same gem, especially in the section devoted to Inaccessible and Mahlo cardinals.

3.1 Fixed-point phenomena and axioms

This small chapter is dedicated to

Lévy's article mentions various schemata that are not instances of reflection themselves. We will mention them because they are equivalent to N_0 and because they are fixed-point theorems, which we will find useful later in this thesis.

Definition 3.1 *Strictly increasing function*
A function $F(\alpha)$ defined on the ordinal number into the ordinal numbers is said to be strictly increasing *if* $\forall \alpha, \beta \in On(\alpha < \beta \rightarrow F(\alpha) < F(\beta))$.

Definition 3.2 *Continuous function*
A function $F(\alpha)$ defined on the ordinal number into the ordinal numbers is said to be continuous *if for any limit α, $F(\alpha) = lim_{\beta < \ \alpha} F(\beta)$.*

Alternatively, a function F is continuous iff for limit $lambda$, $F(\lambda) = supF(\alpha) : \alpha < \lambda$.

Definition 3.3 *Normal function*
A function $F(\alpha)$ defined on the ordinal numbers into the ordinal numbers is said to be normal *if it is* strictly increasing *and* continuous

Definition 3.4 *Normal function on a set Let α be an ordinal. A function $f : \delta \rightarrow \alpha$ is a normal function on α if it is increasing, continuous and its range is unbounded in α.*

[22] This doesn't work backwards, the first fixed point of the $\aleph$ function is the limit of $\{\aleph_0, \aleph_{\aleph_0}, \aleph_{\aleph_{\aleph_0}}, \ldots\}$ is singular since the sequence has countably many elements.

Definition 3.5 *Fixed point*
We say α is a fixed point of ordinal function f when $\alpha = f(\alpha)$.

Lévy ([2]) proposes those axioms as equivalent to one on his reflection principles.

Definition 3.6 *M Every normal function defined for all ordinals has at least one inaccessible number in its range.*

Definition 3.7 *M' Every normal function defined for all ordinals has at least one fixed point which is inaccessible.*

Definition 3.8 *M'' Every normal function defined for all ordinals has arbitrarily great fixed points which are inaccessible.*

The following axiom is proposed by Drake in [3].

Definition 3.9 *F Every normal function for all ordinals has a regular fixed point.*

Theorem 3.10

$$F \leftrightarrow M \leftrightarrow M' \leftrightarrow M'' \tag{3.30}$$

Proof. One can find the proof of $M \leftrightarrow M' \leftrightarrow M''$ in [2], *Theorem 1.*

□

3.2 Reflecting Second-order Formulas

To see that there is a way to transcend ZF, let us briefly show how a model of ZF can be obtained in ZF_2+"second-order reflection"[23]. This will be more closely examined in section 3.3.

We know that ZF can not be finitely axiomatized in first-order formulas, however if Replacement and Comprehension schemes can be substituted by second-order formulas, ZF becomes ZF_2, which is finitely axiomatizable in second-order logic. Therefore if we take second-order reflection into consideration, we can obtain a set M that is a model of ZF_2. For now, we have left out the details of how exactly is first-order reflection generalised into stronger statements and how second-order axiomatization of ZF looks like as we will examine those problems closely in the following pages.

Lower-case letters represent first-order variables and upper-case P represents a second-order variable. [9]

[23] ZF_2 is an axiomatization of ZF in second-order formulas, to be more rigorously established later.

Definition 3.11 Replacement$_2$

$$\forall P(\forall x, y, z(P(x,y)\&P(x,z) \to y = z) \to \\ \to (\forall x \exists y \forall z (z \in y \leftrightarrow \exists w \in x(P(w,z))))) \quad (3.31)$$

We will denote this axiom Replacement$_2$.

Definition 3.12 Specification$_2$

$$\forall P \forall x \exists y \forall z \, (z \in y \leftrightarrow [z \in x \& P(z,x)]) \quad (3.32)$$

Definition 3.13 ZF_2
Let ZF_2 *be a theory with all axioms identical with the axioms of* ZF *with the exception of* Replacement *and* Specification *schemes, which are replaced with* Replacement$_2$ *and* Specification$_2$ *respectively.*

3.3 Inaccessibility

Definition 3.14 *(limit cardinal) kappa is a* limit cardinal *if it is* $\aleph_\alpha$ *for some limit ordinal* α*.*

Definition 3.15 *(strong limit cardinal) kappa is a* strong limit cardinal *if for every* $\lambda < \kappa$, $2^\lambda < \kappa$

The two above definition become equivalent when we assume GCH.

Definition 3.16 *(weak inaccessibility) An uncountable cardinal* κ *is* weakly inaccessible $\leftrightarrow$ *it is* regular *and* limit.

Definition 3.17 *(inaccessibility) An uncountable cardinal* κ *is* inaccessible *(written* $In(\alpha)$*)* $\leftrightarrow$ *it is* regular *and* strongly limit.

We will now show that the above notion is equivalent to the definition Levy uses in [2], which is, in more contemporary notation, the following:

Theorem 3.18 *The following are equivalent:*
1. κ *in inaccessible*
2. $\langle V_\kappa, \in \rangle \models \mathsf{ZFC}$

Proof. Let's first prove that if κ is inaccessible, it is a model of ZFC. We will do that by verifying the axioms of ZFC just like Kanamori does it in in [1, 1.2] and Drake in [3, Chapter 4].

(i) *Extensionality*:

$$V_\kappa \models \forall x, y(\forall z(z \in x \leftrightarrow z \in y) \to x = y) \tag{3.33}$$

We need to prove that, given two sets that are equal in V, they are equal in V_κ, in other words, that the *Extensionality* formula is reflected, that is

$$V_\kappa \models \forall x, y \in V_\kappa(\forall z \in V_\kappa(z \in x \leftrightarrow z \in y) \to x = y) \tag{3.34}$$

But that comes from transitivity. If x and y are in V_κ their members are also in V_κ.

(ii) *Foundation*:

$$V_\kappa \models \forall x(\exists z(z \in x) \to \exists z(z \in x \ \&\ \forall u \neg(u \in z \ \&\ u \in x))) \tag{3.35}$$

The argument for *Foundation* is almost identical to the one for *Extensionality*. For any set $x \in V_\kappa$, transitivity of V_κ makes sure that every element of x is also an element of V_κ and the same holds for the elements of elements of x et cetera. So statements about those elements are absolute between any transitive structures. V and V_κ are both transitive therefore *Foundation* holds and so does its relativisation to V_κ, $Foundation^{V_\kappa}$.

(iii) *Powerset*:

$$V_\kappa \models \forall x \exists y \forall z(z \subseteq x \to z \in y). \tag{3.36}$$

If we take x, an element of V_κ, $\mathscr{P}(()x)$ has to be an element of V_κ to, because it is transitive and a strong limit cardinal.

(iv) *Pairing*:

$$V_\kappa \models \forall x, y \exists z(x \in z \wedge y \in z). \tag{3.37}$$

Pairing holds from similar argument like above: let x and y be elements of V_κ, so there are ordinals $\alpha, \beta < \kappa$ such that $x \in V_\alpha$, $y \in V_\beta$. Without any loss of generality, suppose $\alpha < \beta$, threfore $V_\alpha \subset V_\beta$ which, from transitivity of the cumulative hierarchy, means that $x \in V_\beta$, then $\{x, y\} \in V_{\beta+1}$ which is still in V_κ because it is a strong limit cardinal.

(v) *Union*

$$V_\kappa \models \forall x\, \exists y\, \forall z\, \forall w((w \in z \wedge z \in x) \to w \in y). \tag{3.38}$$

We want to see that for every $x \in V_\kappa$, this is equivalent to

$$V_\kappa \models \forall x \in V_\kappa, \exists y \in V_\kappa\, \forall z \in V_\kappa\, \forall w \in V_\kappa((w \in z \wedge z \in x) \to w \in y). \tag{3.39}$$

Since V_κ is transitive, if $x \in V_\kappa$, all of its elements as well as their elements are in V_κ. To see that they also form a set themselves we only need to remember that V_κ is limit and therefore if α is the least ordinal such that $x \in V_\alpha$, $\bigcup x \in V_{\alpha+1}$.

(vi) *Replacement, Infinity* We know that those hold from 2.11.

We will now show that if a set is a model of ZFC, it is in fact an inaccessible cardinal. So let V_κ be a model of ZFC which means that it is closed under the powerset operation, in other words:

$$\forall\lambda(\lambda < \kappa \to 2^\lambda < \kappa) \tag{3.40}$$

which is exactly the definition of strong limitness. κ is regular from the following argument by contradiction:
Let us suppose for a moment that κ is singular. Therefore there is an ordinal $\alpha < \kappa$ and a function $F : \alpha \to \kappa$ such that the range of F in unbounded in κ, in other words, $F[\alpha] \subseteq V_\kappa$ and $sup(F[\alpha]) = \kappa$. In order to achieve the desired contradiction, we need to see that it is the case that $F[\alpha] \in V_\kappa$. Let $\varphi(x, y)$ be the following first-order formula:

$$F(x) = y \tag{3.41}$$

Then there is an instance of Axiom Schema of Replacement that states the following:

$$\begin{gathered}(\forall x, y, z(\varphi(x, y)\&\varphi(x, z) \to y = z)) \to \\ \to (\forall x\exists y\forall z(z \in y \leftrightarrow \exists w(\varphi(w, z))))\end{gathered} \tag{3.42}$$

Which in turn means that there is a set $y = F[\alpha]$ and $y \in V_\kappa$, which is the contradiction with $sup(y) = \kappa$ we are looking for. □

The same holds for ZF_2, the proof is very similar.

Theorem 3.19

$$V_\kappa \models \mathsf{ZF}_2 \leftrightarrow \kappa \textit{ is inaccessible} \tag{3.43}$$

Proof. κ is a strong limit cardinal because from ZF_2 and *Powerset* we know that for every $\lambda < \kappa$, we know that $2^\lambda < \kappa$.

κ is also regular, because otherwise there would be an ordinal α and a function $F : \alpha \to \kappa$ with a range unbounded in κ. *Replacement*^{2} gives us a set $y = F[\alpha]$, so $y \in V_\kappa$, which contradicts the fact that $sup(y) = \kappa$. It can not be the case that $\kappa \in V_\kappa$.

The other direction is exactly like the first part of above theorem 3.18. □

This is how the existence of an inaccessible cardinal is established in [2].

Definition 3.20 N

$$\exists u(In(\alpha) \ \& \ \forall x_1, \ldots, x_n(x_1, \ldots, x_n \in u \to (\varphi \leftrightarrow \varphi^u))) \tag{3.44}$$

It is interesting to see that the above schema yields the first inaccessible cardinal if we take for φ the conjunction of all axioms of ZF_2.

To see that inaccessible cardinal can be also obtained by a fixed-point axiom (or a scheme if were in first-order logic), see the following theorem by Lévy, we won't repeat the proof here, it is available in [2, Theorem 3],

Theorem 3.21

$$M \leftrightarrow N \tag{3.45}$$

We have transcended ZF, but that is just a start. Naturally, we could go on and consider the next inaccessible cardinal, which is inaccessible with respect to the theory $\mathsf{ZF} + \exists\kappa(\kappa \models \mathsf{ZF})$. But let's try to find a faster way up, informally at first.

Since we can find an inaccessible set larger than any chosen set M_0, it is clear that there are arbitrarily large inaccessible cardinals in V, they are "unbounded"[24] in V. If V were a cardinal, we could say that there are V inaccesible cardinals less than V, but this statement of course makes no sense in set theory as is because V is not a set. But being more careful, we could find a property that can be formalized in second-order logic and reflect it to an initial segment of V. That would allow us to construct large cardinals more efficiently than by adding inaccessibles one by one. The property we are looking for ought to look like something like this:

$$\begin{array}{r} \kappa \text{ is an inaccessible cardinal and} \\ \text{there are } \kappa \text{ inaccessible cardinals } \mu \ < \ \kappa \end{array} \tag{3.46}$$

This is in fact a fixed-point type of statement. We shall call those cardinals hyper-inaccessible. Now consider the following definition.

[24] The notion is formaly defined for sets, but the meaning should be obvious.

Definition 3.22 *0-inaccessible cardinal*
A cardinal κ is 0-inaccessible if it is inaccessible.

We can define *α-weakly-inaccessible* cardinals analogously with the only difference that those are limit, not strongly limit.

Definition 3.23 *α-hyper-inaccessible cardinal*
For any ordinal α, κ is called α-inaccessible, if κ is inaccessible and for each β ¡ α, the set of β-inaccessible cardinals less than κ is unbounded in κ.

Because κ is inaccessible and therefore regular, the number of β-inaccessibles below κ is equal to κ. We have therefore successfully formalized the above vague notion of hyper-inaccessible cardinal into a hierarchy of α-inaccessibles.

Let's now consider iterating this process over again. Since, informally, V would be α-inaccessible for any α, this property of the universal class could possibly be reflected to an initial segment, the smallest of those will be the first hyper-inaccessible cardinal. Such κ is larger than any α-inaccessible since from regularity of κ, for given $\alpha < \kappa$, κ is κ-th α-hyper-inaccessible cardinal. It is in fact "inaccessible" via α-inaccessibility.

Definition 3.24 *Hyper-inaccessible cardinal*
κ is called the hyper-inaccessible, also 0-hyper-inaccessible, cardinal if it is α-inaccessible for every $\alpha < \kappa$.

Definition 3.25 *α-hyper-inaccessible cardinal*
For any ordinal α, κ is called α-hyper-inaccessible cardinal if for each ordinal $\beta < \alpha$, the set of β-hyper-inaccessible cardinals less the κ is inbounded in κ.

Obviously we could go on and iterate it ad libitum, but the nomenclature would be increasingly confusing. A smarter way to accomplish the same goal is carried out in the following section.

3.4 Mahlo Cardinals

While the previous chapter introduced us to a notion of inaccessibility and the possibility of iterating it ad libitum in new theories, there is an even faster way to travel upwards in the cumulative hierarchy, that was proposed by Paul Mahlo in his papers (see [5], [6] and [7]) at the very beginning of the 20th century, and which can be easily reformulated using *(Reflection)*. To see how Lévy's initial statement of reflection was influenced by Mahlo's work, refer to section 2.2. The aim of the following paragraphs is to give an intuitive explanation of the idea behind Mahlo's hierarchy of cardinals, all claims made here ought to be stated formally later in the very same chapter.

At the very end of section 3.3, we have tried to establish the notion of hyper-inaccessibility and iterate it to yield even larger large cardinals. In order to avoid too bulky cardinal names, let's try a different route and establish those cardinals directly via reflection.

The following two definitions come from [8] and while they are rather informal, we will find them very helpful for understanding the Mahlo cardinals.

Definition 3.26 *Fixed-point property*
For any $\psi(x, u_1, \ldots, u_n)$ which is any property of ordinals, we say that a property φ is a fixed-point property *if φ has the form*

$$\begin{aligned} &x \textit{ is an inaccessible cardinal and} \\ &\textit{there are } x \textit{ ordinals less than } x \textit{ that have the property } \psi(x, u_1, \ldots, u_n). \end{aligned} \tag{3.47}$$

Definition 3.27 *Fixed-point reflection*
If φ is a fixed-point property that holds for V, it also holds for some V_α, an initial segment of V.

Obviously those are in on way rigorous definitions because we have no idea what $\psi(x, u_1, \ldots, u_n)$ looks like. Let's try to restate the same idea in a useful way. But first, let's show that the formal counterpart of the idea of containing "enough" ordinals with a property is the notion of stationary set.

Definition 3.28 *Supremum*
Given A a set of ordinals, the supremum of A, denoted sup(A), is the least upper bound of A.

$$sup(A) = \bigcup A \tag{3.48}$$

where α is an ordinal.

Definition 3.29 *Limit point*
Given A, a set of ordinals and an ordinal α, we say that α is a limit point *of A if $sup(A \cap \alpha) = \alpha$*

Definition 3.30 *Club set*
For a regular uncountable κ, a set $A \subset \kappa$ is a closed unbounded *subset (often abbreviated as a* club set*) iff A is both closed, which means it contains all it's limit points, and unbounded, which means that for every β ¡ κ there is a $\beta' \in \alpha$ such that $\beta < \beta' < \kappa$.*

Definition 3.31 *Stationary set*
For a regular uncountable κ, a set $A \subset \kappa$ is stationary if it intersects every club subset of κ.

Theorem 3.32 *The intersection of fewer than κ[25] club subsets of κ is a club set.*

For proof, see [4, Theorem 8.3]

Definition 3.33 *Weakly Mahlo Cardinal*
κ is weakly Mahlo $\leftrightarrow$ *it is a regular limit ordinal and the set of all regular ordinals less then κ is stationary in κ*

Definition 3.34 *Mahlo Cardinal*
κ is a Mahlo Cardinal *iff it is an inaccessible cardinal and the set of all inaccessible ordinals less then κ is stationary in κ.*

It is interesting to note, that weakly-Mahlo cardinals are fixed points of α-weakly inaccessible cardinals, so if κ is weakly mahlo, .. viz Kanamori Proposition 1.1

Analogously,

Definition 3.35 *α-Mahlo Cardinal*
*κ is a α-*Mahlo Cardinal *iff it is an α-inaccessible cardinal and the set of all α-inaccessible ordinals less then κ is stationary in κ.*

In other words, κ is a mahlo cardinal if it is inaccessible and every club set in κ contains an inaccessible cardinal. This is exactly the notion of fixed-point reflection we were trying to show earlier.

[3]

[25] κ is again a regular uncountable cardinal and it will always be when we will be talking about club sets.

Definition 3.36 *The folllowing definitions are equivalent:*

(i) κ is Mahlo

(ii) κ is weakly Mahlo and strong limit

(iii) The set $\{\lambda < \kappa : \lambda$ is inaccessible$\}$ is stationary in κ.

(iv) Every normal function on κ has an inaccessible fixed point.

Proof. (**i**) $\leftrightarrow$ (**ii**) Let κ_1 be a mahlo cardinal and let κ_2 be a strong limit weakly Mahlo cardinal. We know from the definitions that the set $\{\lambda < \kappa : \lambda \text{ is inaccessible}\}$ is stationary in both κ_1 and κ_2, the only difference being that κ_1 is a strongly limit cardinal, but κ_2 would be limit from weak Mahloness, wasn't it for the fact that it is also strong limit. This eliminates the only difference between them and therefore κ_1 is also strong limit weakly Mahlo cardinal and κ_2 is Mahlo.

(**i**) $\rightarrow$ (**iii**) We know that κ is uncountable, regular, strong limit and that the set $S = \{\lambda < \kappa : \lambda \text{ is regular}\}$ is stationary in κ. We want to prove that $S' = \{\lambda < \kappa : \lambda \text{ is inaccessible}\}$ is thus also stationary in κ.

Since stationary set intersects every club set in κ, let C be any such set. Let $D = \{\lambda < \kappa : \lambda \text{ is strong limit}\}$. D is a club set because TODO. Since intersection of less than κ club sets is a club set, $C \cap D \neq \emptyset$.

TODO proc $\lambda = S \cap C \cap D$ je inaccessible?

(**iii**) $\rightarrow$ (**iv**)

TODO jak to dela Levy?

(**iv**) $\rightarrow$ (**i**)

TODO jak to dela Levy?

range kazde normalni funkce je club v On. (nevadi ze On je trida?)

co treba lemma ze pevne body tvori taky club set

mozna by stacilo ze jsou unbounded, tedy kazda normalni funkce ma libovolne velke pevne body. □

TODO obdoba pro α-Mahlo kardinaly?

TODO κ is hyper-Mahlo iff κ is inaccessible and the set $\{\lambda < \kappa : \lambda \text{ is Mahlo}\}$ is stationary in κ.

3.5 Indescribality

TODO indescribable – reflecting indescribability – we can't reach V by a Σ^1_1 formula, so there's some initial segment V_α that is also unreachable (we say indescribable) by the means of a ... formula

TODO co je "partition property"?

TODO pak dk. ekvivalenci

TODO Kanamori 6.3

Definition 3.37 *A cardinal κ is* weakly compact *if it is uncountable and satisfies the partition property $\kappa \rightarrow (\kappa)^2$*

opsano z jecha!

TODO definice pres nepopsatelnost, ekvivalence

3.6 Bernays–Gödel Set Theory

TODO Plagiat – prepsat a vysvetlit

TODO

3.7 Reflection and the constructible universe

TODO reflektovat muzeme jenom kardinaly konzistentni s V=L, proc?

TODO Plagiat – prepsat a vysvetlit

L was introduced by Kurt Gödel in 1938 in his paper *The Consistency of the Axiom of Choice and of the Generalised Continuum Hypothesis* and denotes a class of sets built recursively in terms of simpler sets, somewhat similar to Von Neumann universe V. Assertion of their equality, $V = L$, is called the *axiom of constructibility*. The axiom implies GCH and therefore also AC and contradicts the existence of some of the large cardinals, our goal is to decide whether those introduced earlier are among them.

On order to formally establish this class, we need to formalize the notion of definability first:

TODO zduvodneni

TODO kratka diskuse jestli refl implikuje transcendenci na L - polemika, nazor - V=L a slaba kompaktnost a dalsi

TODO asi nekde bude meritelny kardinal

4 Conclusion

TODO na konec

References

[1] Akihiro Kanamori (auth.). *The higher infinite: Large cardinals in set theory from their beginnings*. Springer Monographs in Mathematics. Springer-Verlag Berlin Heidelberg, 2 edition, 2003.

[2] Lévy Azriel. Axiom schemata of strong infinity in axiomatic set theory. *Pacific Journal of Mathematics*, 10, 1960.

[3] Drake F. *Set theory. An introduction to large cardinals*. Studies in Logic and the Foundations of Mathematics, Volume 76. NH, 1974.

[4] Thomas Jech. *Set theory*. Springer monographs in mathematics. Springer, the 3rd millennium ed., rev. and expanded edition, 2006.

[5] P. Mahlo. Über lineare transfinite Mengen. Leipz. Ber. 63, 187-225 (1911)., 1911.

[6] P. Mahlo. Über lineare transfinite Mengen. Leipz. Ber. 63, 187-225 (1911)., 1911.

[7] P. Mahlo. Zur Theorie und Anwendung der ϱ_v-Zahlen. II. Leipz. Ber. 65, 268-282 (1913)., 1913.

[8] Rudy von Bitter Rucker. *Infinity and the mind : the science and philosophy of the infinite*. Princeton science library. Princeton University Press, 2005 ed edition, 2005.

[9] Stewart Shapiro. Principles of reflection and second-order logic. *Journal of Philosophical Logic*, 16, 1987.

[10] Hao Wang. *"A Logical Journey: From Gödel to Philosophy"*. A Bradford Book, 1997.