

1 Univerzita Karlova v Praze, Filozofická fakulta
2 Katedra logiky

3 MIKULÁŠ MRVA

4 REFLECTION PRINCIPLES AND LARGE
5 CARDINALS
6 Bakalářská práce

7 Vedoucí práce: Mgr. Radek Honzík, Ph.D.

8 2016

¹⁰ Prohlašuji, že jsem bakalářskou práci vypracoval samostatně a že jsem uvedl
¹¹ všechny použité prameny a literaturu.

¹² V Praze 22. května 2016

Abstract

Práce zkoumá vztah tzv. principů reflexe a velkých kardinálů. Lévy ukázal, že v ZFC platí tzv. věta o reflexi a dokonce, že věta o reflexi je ekvivalentní schématu nahrazení a axiomu nekonečna nad teorií ZFC bez axiomu nekonečna a schématu nahrazení. Tedy lze na větu o reflexi pohlížet jako na svého druhu axiom nekonečna. Práce zkoumá do jaké míry a jakým způsobem lze větu o reflexi zobecnit a jaký to má vliv na existenci tzv. velkých kardinálů. Práce definuje nedosažitelné, Mahlovy a nepopsatelné kardinály a ukáže, jak je lze zavést pomocí reflexe. Přirozenou limitou kardinálů získaných reflexí jsou kardinály nekonzistentní s L. Práce nabídne intuitivní zdůvodnění, proč tomu tak je.

Abstract

This thesis aims to examine relations between the so called Reflection Principles and Large cardinals. Lévy has shown that the Reflection Theorem is a sound theorem of ZF and it is equivalent to the Replacement Scheme and the Axiom of Infinity. From this point of view, Reflection theorem can be seen a specific version of an Axiom of Infinity. This paper aims to examine the Reflection Principle and its generalisations with respect to the existence of Large Cardinals. This thesis will establish the Inaccessible, Mahlo and Indescribable cardinals and show how can those be defined via reflection. A natural limit of Large Cardinals obtained via reflection are cardinals inconsistent with L. This thesis will offer an intuitive explanation of why this holds.

39 **Contents**

1 Introduction

1.1 Motivation and Origin

“The Universe of sets cannot be uniquely characterised (i. e. distinguished from all its initial elements) by any internal structural property of the membership relation in it, which is expressible in any logic of finite or transfinite type, including infinitary logics of any cardinal order.”

— Kurt Gödel [?]

1.2 Notation and Terminology

1.2.1 The Language of Set Theory

This text assumes the knowledge of basic terminology and some results from first-order predicate logic.¹ All proofs are based on [?] unless explicitly stated otherwise. Notable amount of material is also drawn from [?] and [?].

We will now shortly review the basic notions that allow us to define the *Zermelo–Fraenkel* set theory.

When we talk about *class*, we have the notion of definable class in mind. If $\varphi(x, p_1, \dots, p_n)$ is a formula in the language of set theory, we call

$$A = \{x : \varphi(x)\} \tag{1.1}$$

a class of all sets satisfying $\varphi(x)$ in a sense that

$$x \in A \leftrightarrow \varphi(x) \tag{1.2}$$

Given classes A, B , one can easily define the elementary set operations such as $A \cap B, A \cup B, A \setminus B, \bigcup A$, see the first part of [?] for details. Axioms are the tools by which we can decide whether a particular class is “small enough” to be considered a set². A class that fails to be considered a set is called a *proper class*.

We will often write “ M is a limit ordinal”, it should always be clear that this can be rewritten as a formula that was introduced earlier.

¹todo odkaz na pripadny zdroj? svejdar? neco en?

²“Small enough” means that it doesn’t introduce a paradox similar to Russell’s.

1.2.2 The Axioms

Definition 1.1 (*The Existence of a Set*)

$$\exists x(x = x) \quad (1.3)$$

Definition 1.2 (*Axiom of Extensionality*)

$$\forall x, y(x = y \leftrightarrow \forall z(z \in x \leftrightarrow z \in y)) \quad (1.4)$$

Definition 1.3 (*Axiom Schema of Specification*)

The following yields an axiom for every first-order formula $\varphi(x, p_1, \dots, p_n)$ with no free variables other than x, p_1, \dots, p_n .

$$\forall x, p_1, \dots, p_n \exists y \forall z(z \in y \leftrightarrow z \in x \ \& \ \varphi(z, p_1, \dots, p_n)) \quad (1.5)$$

We will now provide two definitions that are not axioms, but will be helpful in establishing some axioms in a more comprehensible way.

Definition 1.4 ($x \subseteq y, x \subset y$)

$$x \subseteq y \leftrightarrow (\forall z(z \in x \rightarrow z \in y)) \quad (1.6)$$

$$x \subset y \leftrightarrow x \subseteq y \ \& \ x \neq y \quad (1.7)$$

We read $x \subseteq y$ as x is a subset of y and $x \subset y$ as x is a proper subset of y .

Definition 1.5 (*Empty Set*) For an arbitrary set x , the empty set, represented by the symbol " \emptyset ", is defined by the following formula:

$$(\forall y(y \in x) \rightarrow (y \in \emptyset \leftrightarrow \neg(y = y))) \quad (1.8)$$

\emptyset is a set due to Specification. While the empty set could also be defined by the formula $\forall y(y \in \emptyset \leftrightarrow \neg(y = y))$, the version we use is Δ_0 , which we will find useful later. The two definitions yield the same set for every x given because of Extensionality.

Definition 1.6 (*Axiom of Pairing*)

$$\forall x, y \exists z \forall q(q \in z \leftrightarrow q = x \vee q = y) \quad (1.9)$$

Definition 1.7 (*Axiom of Union*)

$$\forall x \exists y \forall z(z \in y \leftrightarrow \exists q(z \in q \ \& \ q \in x)) \quad (1.10)$$

Now we can introduce more axioms.

85 **Definition 1.8** (*Axiom of Foundation*)

$$\forall x(x \neq \emptyset \rightarrow (\exists y \in x)(x \cap y = \emptyset)) \quad (1.11)$$

86 **Definition 1.9** (*Axiom of Powerset*)

$$\forall x \exists y \forall z (z \in y \leftrightarrow z \subseteq x) \quad (1.12)$$

87 **Definition 1.10** (*Axiom of Infinity*)

$$\exists x(\emptyset \in x \ \& \ (\forall y \in x)(y \cup \{y\} \in x)) \quad (1.13)$$

88 *The least set satisfying this is denoted “ ω ”.*

89 Let us introduce a few more definitions that will make the two remaining
90 axioms more comprehensible.

91 **Definition 1.11** (*Powerset Function*)

92 *Given a set x , the powerset of x , denoted $\mathcal{P}(x)$ and satisfying ??, is defined*
93 *as follows:*

$$\mathcal{P}(x) \stackrel{\text{def}}{=} \{y : y \subseteq x\} \quad (1.14)$$

94 **Definition 1.12** (*Function*)

95 *Given arbitrary first-order formula $\varphi(x, y, p_1, \dots, p_n)$, we say that φ is a func-*
96 *tion iff*

$$\forall x, y, z, p_1, \dots, p_n (\varphi(x, y, p_1, \dots, p_n) \ \& \ \varphi(x, z, p_1, \dots, p_n) \rightarrow y = z) \quad (1.15)$$

97 When a $\varphi(x, y)$ is a function, we also write the following:

$$f(x) = y \leftrightarrow \varphi(x, y) \quad (1.16)$$

98 Alternatively, $f = \{\langle x, y \rangle : \varphi(x, y)\}$ is a class.

99 **Definition 1.13** (*Domain of a Function*)

100 *Let f be a function. We call the domain of f the set of all sets for which f*
101 *yields a value. We use “ $\text{Dom}(f)$ ” to refer to this set.*

$$x \in \text{Dom}(f) \leftrightarrow \exists y(f(x) = y) \quad (1.17)$$

102 We say “ f is a function on A ”, A being a class, if $A = \text{dom}(f)$.

103 **Definition 1.14** (*Range of a Function*)

104 *Let f be a function. We call the range of f the set of all sets that are images*
105 *of other sets via f . We use “ $\text{Rng}(f)$ ” to refer to this set.*

$$x \in \text{Rng}(f) \leftrightarrow \exists y(f(y) = x) \quad (1.18)$$

106 We say that f is a *function into* A , A being a class, if $\text{rng}(f) \subseteq A$. We say
 107 that f is a *function onto* A if $\text{rng}(f) = A$. We say a function f is a *one to one*
 108 *function*, iff

$$(\forall x_1, x_2 \in \text{dom}(f))(f(x_1) = f(x_2) \rightarrow x_1 = x_2) \quad (1.19)$$

109 We say that f is a *bijection* iff it is a one to one function that is onto.

110 Note that $\text{Dom}(f)$ and $\text{Rng}(f)$ are not definitions in a strict sense, they
 111 are in fact definition schemas that yield definitions for every function f given.
 112 Also note that they can be easily modified for φ instead of f , with the only
 113 difference being the fact that it is then defined only for those φ s that are
 114 functions, which must be taken into account. This is worth noting as we will
 115 use the notions of *function* and *formula* interchangeably.

116 **Definition 1.15** (*Function Defined For All Ordinals*)

117 We say a function f is defined for all ordinals, this is sometimes written
 118 $f : \text{Ord} \rightarrow A$ for any class A , if $\text{Dom}(f) = \text{Ord}$. Alternatively,

$$(\forall \alpha \in \text{Ord})(\exists y \in A)(f(\alpha) = y) \quad (1.20)$$

119 And now for the axioms.

120 **Definition 1.16** (*Axiom Schema of Replacement*)

121 The following is an axiom for every first-order formula $\varphi(x, p_1, \dots, p_n)$ with
 122 no free variables other than x, p_1, \dots, p_n .

$$“\varphi \text{ is a function}” \rightarrow \forall x \exists y \forall z (z \in y \leftrightarrow (\exists q \in x)(\varphi(x, y, p_1, \dots, p_n))) \quad (1.21)$$

123 **Definition 1.17** (*Choice*)

$$\begin{aligned} 124 \quad & \forall x \exists f ((f \text{ is a choice function with } \text{dom}(f) = x \setminus \{\emptyset\}) \\ & \& \forall y ((y \in x \& y \neq \emptyset) \rightarrow f(y) \in y)) \end{aligned} \quad (1.22)$$

125 We will refer to the axioms by their name, written in italic type, e.g.
 126 *Foundation* refers to the Axiom of Foundation. Now we need to define the
 127 set theories to be used in the article.

128 **Definition 1.18** (S)

129 We call S an *axiomatic theory* in the language $\mathcal{L} = \{=, \in\}$ with exactly the
 130 following axioms:

- 131 (i) Existence of a set (see ??)
- 132 (ii) Extensionality (see ??)
- 133 (iii) Specification (see ??)

- 134 (iv) Foundation (see ??)
- 135 (v) Pairing (see ??)
- 136 (vi) Union (see ??)
- 137 (vii) Powerset (see ??)

138 **Definition 1.19** (ZF)

139 We call ZF an axiomatic theory in the language $\mathcal{L} = \{=, \in\}$ that contains
 140 all the axioms of S in addition to the following:

- 141 (i) Replacement schema (see ??)
 - 142 (ii) Infinity (see ??)
- 143 Existence of a set is usually left out because it is a consequence of infinity.

144 **Definition 1.20** (ZFC)

145 ZFC is an axiomatic theory in the language $\mathcal{L} = \{=, \in\}$ that contains all the
 146 axioms of ZF plus Choice (??).

147

148 **1.2.3 The Transitive Universe**

149 **Definition 1.21** (Transitive Class)

150 We say a class A is transitive iff

$$(\forall x \in A)(x \subseteq A) \quad (1.23)$$

151 **Definition 1.22** (Well Ordered Class) A class A is said to be well ordered
 152 by \in iff the following hold:

- 153 (i) $(\forall x \in A)(x \not\subseteq x)$ (Antireflexivity)
- 154 (ii) $(\forall x, y, z \in A)(x \in y \ \& \ y \in z \rightarrow x \in z)$ (Transitivity)
- 155 (iii) $(\forall x, y \in A)(x = y \vee x \in y \vee y \in x)$ (Linearity)
- 156 (iv) $(\forall x \subseteq A)(x \neq \emptyset \rightarrow (\exists y \in x)(\forall z \in x)(z = y \vee z \in y))$ (Existence of the
 157 least element)

158 **Definition 1.23** (Ordinal Number)

159 A set x is said to be an ordinal number if it is transitive and well-ordered
 160 by \in .

161 For the sake of brevity, we usually just say “ x is an ordinal”. Note that
 162 “ x is an ordinal” is a well-defined formula in the language of set theory,
 163 since ?? is a first-order formula and ?? is in fact a conjunction of four first-
 164 order formulas. Ordinals will be usually denoted by lower case greek letters,
 165 starting from the beginning of the alphabet: $\alpha, \beta, \gamma, \dots$. Given two different
 166 ordinals α, β , we will write $\alpha < \beta$ for $\alpha \in \beta$, see Lemma 2.11 in [?] for
 167 technical details.

168 **Definition 1.24** (*Non-Zero Ordinal*) We say an ordinal α is non-zero iff
 169 $\alpha \neq \emptyset$.

170 **Definition 1.25** (*Successor Ordinal*)
 171 Consider the following function defined for all ordinals. Let β be an arbitrary
 172 ordinal. We call S the successor function.

$$S(\beta) = \beta \cup \{\beta\} \quad (1.24)$$

173 An ordinal α is called a successor ordinal iff there is an ordinal β , such that
 174 $\alpha = S(\beta)$. We also write $\alpha = \beta + 1$.

175 **Definition 1.26** (*Limit Ordinal*)
 176 A non-zero ordinal α is called a limit ordinal iff it is not a successor ordinal.

177 **Definition 1.27** (*Ord*)
 178 The class of all ordinal numbers, which we will denote “Ord”³ is the proper
 179 class defined as follows.

$$x \in \text{Ord} \leftrightarrow x \text{ is an ordinal} \quad (1.25)$$

180 **Definition 1.28** (*Von Neumann’s Hierarchy*)
 181 The Von Neumann’s Hierarchy is a collection of sets indexed by elements of
 182 Ord, defined recursively in the following way:

(i)

$$V_0 = \emptyset \quad (1.26)$$

(ii)

$$V_{\alpha+1} = \mathcal{P}(V_\alpha) \text{ for any ordinal } \alpha \quad (1.27)$$

(iii)

$$V_\lambda = \bigcup_{\beta < \lambda} V_\beta \text{ for a limit ordinal } \lambda \quad (1.28)$$

183 We will also refer to the Von Neumann’s Hierarchy as Von Neumann’s Uni-
 184 verse or the Cumulative Hierarchy.

185 **Definition 1.29** (*Rank*)
 186 Given a set x , we say that the rank of x (written as $\text{rank}(x)$) is the least
 187 ordinal α such that $x \in V_{\alpha+1}$

188 Due to Regularity, every set has a rank.⁴

189 **Definition 1.30** (*Order-type*)
 190 Given an arbitrary well-ordered set x , we say that an ordinal α is the order-
 191 type of x iff x and α are isomorphic.

192

³Other authors use “On”, we will stick to the notation used in [?]

⁴See chapter 6 of [?] for details.

1.2.4 Cardinal Numbers

Definition 1.31 (Cardinality)

Given a set x , let the cardinality of x , written $|x|$, be defined as the smallest ordinal number such that there is a one to one mapping from x to α .

Definition 1.32 (Aleph function)

Let ω be the set defined by ω . We will recursively define the function \aleph for all ordinals.

(i) $\aleph_0 = \omega$

(ii) $\aleph_{\alpha+1}$ is the least cardinal larger than \aleph_α ⁵

(iii) $\aleph_\lambda = \bigcup_{\beta < \lambda} \aleph_\beta$ for a limit ordinal λ

If $\kappa = \aleph_\alpha$ and α is a successor ordinal, we call κ a successor cardinal. If α is a limit ordinal, we call κ a limit cardinal.

Definition 1.33 (Cardinal number)

(i) A set x is called a finite cardinal iff $x \in \omega$.

(ii) A set is called an infinite cardinal iff there is an ordinal α such that $\aleph_\alpha = x$

(iii) A set is called a cardinal iff it is either a finite cardinal or an infinite cardinal.

We say κ is an uncountable cardinal iff it is an infinite ordinal and $\aleph_0 < \kappa$. Infinite cardinals will be notated by lowercase greek letters from the middle of the alphabet, e.g. κ, μ, ν, \dots ⁶

For formal details as well as why every set can be well-ordered assuming *Choice*, and therefore has a cardinality, see [?].

Definition 1.34 (Sequence)

We say that a function $\varphi(x, y)$ is a sequence iff there is an ordinal α such that $\text{dom}(\varphi) = \alpha$. In other words, a function is called a sequence if it is defined exactly for every ordinal from 0 to some α . We then say it is an α -sequence. We usually write $\langle \beta_i : i \in \alpha \rangle$ or $\langle \beta_0, \beta_1, \dots \rangle$ when referring to a sequence, ξ_i denote the elements of $\text{rng}(\varphi)$ for every $i \in \text{dom}(\varphi)$.

Definition 1.35 (Cofinal Subset)

Given a class A , we say that $B \subseteq A$ is cofinal in A iff

$$(\forall x \in A)(\exists y \in B)(x \in y) \quad (1.29)$$

⁵“The least cardinal larger than \aleph_α ” is sometimes notated as \aleph_α^+

⁶Except λ which is preferably used for limit ordinals.

225 **Definition 1.36** (*Cofinality of a Limit Ordinal*)

226 *Let λ be a limit ordinal. We say that the cofinality of λ is κ iff κ is the least*
 227 *cardinal, such that there is a cofinal κ -sequence $\langle \beta_\xi : \xi < \kappa \rangle$, such that*

$$\sup(\{\beta_\xi : \xi < \kappa\}) = \lambda \quad (1.30)$$

228 *We write $cf(\lambda) = \kappa$.*

229 **Definition 1.37** (*Regular Cardinal*)

230 *We say a cardinal κ is regular iff $cf(\kappa) = \kappa$*

231 **Definition 1.38** (*Strong Limit Cardinal*)

232 *We say that an ordinal κ is a strong limit cardinal if it is a limit cardinal*
 233 *and*

$$(\forall \alpha \in \kappa)(\mathcal{P}(\alpha) \in \kappa) \quad (1.31)$$

234 **Definition 1.39** (*Generalised Continuum Hypothesis*)

235

$$\aleph_{\alpha+1} = \mathcal{P}(\aleph_\alpha) \quad (1.32)$$

236 *If GCH holds (for example in Gödel's L , see chapter 3), the notions of limit*
 237 *cardinal and strong limit cardinal are equivalent.*

238

239 1.2.5 Relativisation and Absoluteness

240 **Definition 1.40** (*Relativization*)

241 *Let M be a class, $R \subseteq M \times M$ and let $\varphi(p_1, \dots, p_n)$ be a first-order formula*
 242 *with no free variables besides p_1, \dots, p_n . The relativization of φ to M and R*
 243 *is the formula, written as $\varphi^{M,R}(p_1, \dots, p_n)$, defined in the following inductive*
 244 *manner:*

- 245 (i) $(x \in y)^{M,R} \leftrightarrow R(x, y)$
- 246 (ii) $(x = y)^{M,R} \leftrightarrow x = y$
- 247 (iii) $(\neg \varphi)^{M,R} \leftrightarrow \neg \varphi^{M,R}$
- 248 (iv) $(\varphi \ \& \ \psi)^{M,R} \leftrightarrow \varphi^{M,R} \ \& \ \psi^{M,R}$
- 249 (v) $(\varphi \vee \psi)^{M,R} \leftrightarrow \varphi^{M,R} \vee \psi^{M,R}$
- 250 (vi) $(\varphi \rightarrow \psi)^{M,R} \leftrightarrow \varphi^{M,R} \rightarrow \psi^{M,R}$
- 251 (vii) $(\exists x \varphi(x))^{M,R} \leftrightarrow (\exists x \in M) \varphi^{M,R}(x)$
- 252 (viii) $(\forall x \varphi(x))^{M,R} \leftrightarrow (\forall x \in M) \varphi^{M,R}(x)$

253 *When $R = \in \cap (M \times M)$, we usually write φ^M instead of $\varphi^{M,R}$. When we talk*
 254 *about $\varphi^M(p_1, \dots, p_n)$, it is understood that $p_1, \dots, p_n \in M$. We will also use*
 255 *$M \models \varphi(p_1, \dots, p_n)$ and $\varphi^M(p_1, \dots, p_n)$ interchangeably.*

Definition 1.41 (*Absoluteness*) Given a transitive class M , we say a formula φ is absolute in M if for all $p_1, \dots, p_n \in M$

$$\varphi^M(p_1, \dots, p_n) \leftrightarrow \varphi(p_1, \dots, p_n) \quad (1.33)$$

Definition 1.42 (*Hierarchy of First-Order Formulas*)

- (I) A first-order formula φ is Δ_0 iff it is logically equivalent to a first-order formula φ' satisfying any of the following:
- (i) φ' contains no quantifiers
 - (ii) y is a set, ψ is a Δ_0 formula, and φ' is either $(\exists x \in y)\psi(y)$ or $(\forall x \in y)\psi(y)$.
 - (iii) ψ_1, ψ_2 are Δ_0 formulas and φ' is any of the following: $\psi_1 \vee \psi_2$, $\psi_1 \& \psi_2$, $\psi_1 \rightarrow \psi_2$, $\neg\psi_2$,
- (II) If a formula is Δ_0 it is also Σ_0 and Π_0
- (III) A formula φ is $\Pi_n + 1$ if it is logically equivalent to a formula φ' such that $\varphi' = \forall x\psi$ where ψ is a Σ_n -formula for any $n < \omega$.
- (IV) A formula φ is $\Sigma_n + 1$ if it is logically equivalent to a formula φ' such that $\varphi' = \forall x\psi$ where ψ is a Π_n -formula for any $n < \omega$.

Note that we can use the pairing function so that for $\forall p_1, \dots, p_n \psi(p_1, \dots, p_n)$, there is a logically equivalent formula of the form $\forall x\psi'(x)$.

Lemma 1.43 (Δ_0 absoluteness) Let φ be a Δ_0 formula, then φ is absolute in any transitive class M .

Proof. This will be proven by induction over the complexity of a given Δ_0 formula φ . Let M be an arbitrary transitive class.

Atomic formulas are always absolute by the definition of relativisation, see ???. Suppose that Δ_0 formulas ψ_1 and ψ_2 are absolute in M . Then from relativization, $(\psi_1 \& \psi_2)^M \leftrightarrow \psi_1^M \& \psi_2^M$, which is, from the induction hypothesis, equivalent to $\psi_1 \& \psi_2$. The same holds for \vee, \rightarrow, \neg .

Suppose that a Δ_0 formula ψ is absolute in M . Let y be a set and let $\varphi = (\exists x \in y)\psi(x)$. From relativization, $(\exists x\psi(x))^M \leftrightarrow (\exists x \in M)\psi^M(x)$. Since the hypotheses makes it clear that $\psi^M \leftrightarrow \psi$, we get $((\exists x \in y)\psi(x))^M \leftrightarrow (\exists x \in y \cap M)\psi(x)$, which is the equivalent of $\varphi^M \leftrightarrow \varphi$. The same applies to $\varphi = (\forall x \in y)\psi(x)$. \square

Lemma 1.44 (*Downward Absoluteness*)

Let φ be a Π_1 formula and M a transitive class. Then the following holds:

$$(\forall p_1, \dots, p_n \in M)(\varphi(p_1, \dots, p_n) \rightarrow \varphi(p_1, \dots, p_n)^M) \quad (1.34)$$

289 *Proof.* Since $\varphi(p_1, \dots, p_n)$ is Π_1 , there is a Δ_0 formula $\psi(p_1, \dots, p_n, x)$ such
 290 that $\varphi = \forall x \psi(p_1, \dots, p_n, x)$. From relativization and lemma ??, $\varphi^M(p_1, \dots, p_n) \leftrightarrow$
 291 $(\forall x \in M) \psi(p_1, \dots, p_n, x)$.

292 Assume that for $p_1, \dots, p_n \in M$ fixed, that $\forall x \psi(p_1, \dots, p_n, x)$ holds, but
 293 $(\forall x \in M) \psi(p_1, \dots, p_n, x)$ does not. Therefore $\exists x \neg \psi(p_1, \dots, p_n, x)$, which
 294 contradicts $\forall x \psi(p_1, \dots, p_n, x)$. \square

295 **Lemma 1.45** (*Upward Absoluteness*)

296 *Let φ be a Σ_1 formula and M a transitive class. Then the following holds:*

$$(\forall p_1, \dots, p_n \in M)(\varphi^M(p_1, \dots, p_n) \rightarrow \varphi(p_1, \dots, p_n)) \quad (1.35)$$

297 *Proof.* Since $\varphi(p_1, \dots, p_n)$ is Σ_1 , there is a Δ_0 formula $\psi(p_1, \dots, p_n, x)$ such
 298 that $\varphi = \exists x \psi(p_1, \dots, p_n, x)$. From relativization and lemma ??, $\varphi^M(p_1, \dots, p_n) \leftrightarrow$
 299 $(\exists x \in M) \psi(p_1, \dots, p_n, x)$.

300 Assume that for $p_1, \dots, p_n \in M$ fixed, that $(\exists x \in M) \psi(p_1, \dots, p_n, x)$
 301 holds, but $\exists x \psi(p_1, \dots, p_n, x)$ does not. This is an obvious contradiction. \square

302 1.2.6 More Functions

303 **Definition 1.46** (*Strictly Increasing Function*)

304 *A function $f : \text{Ord} \rightarrow \text{Ord}$ is said to be strictly increasing iff*

$$\forall \alpha, \beta \in \text{Ord} (\alpha < \beta \rightarrow f(\alpha) < f(\beta)). \quad (1.36)$$

305 **Definition 1.47** (*Continuous Function*)

306 *A function $f : \text{Ord} \rightarrow \text{Ord}$ is said to be continuous iff*

$$\lambda \text{ is limit} \rightarrow f(\lambda) = \bigcup_{\alpha < \lambda} f(\alpha). \quad (1.37)$$

307 **Definition 1.48** (*Normal Function*)

308 *A function $f : \text{Ord} \rightarrow \text{Ord}$ is said to be normal iff it is strictly increasing*
 309 *and continuous.*

310 **Definition 1.49** (*Fixed Point*)

311 *We say x is a fixed point of a function f iff $x = f(x)$.*

312 **Definition 1.50** (*Unbounded Class*)

313 *We say a class A is unbounded iff*

$$\forall x (\exists y \in A)(x < y) \quad (1.38)$$

314 **Definition 1.51** (*Limit Point*)

315 *Given a class $x \subseteq \text{Ord}$, we say that $\alpha \neq \emptyset$ is a limit point of x iff*

$$\alpha = \bigcup (x \cap \alpha) \quad (1.39)$$

316 **Definition 1.52** (*Closed Class*)

317 *We say a class $A \subseteq \text{Ord}$ is closed iff it contains all its limit points.*

318 **Definition 1.53** (*Club set*)

319 *For a regular uncountable cardinal κ , a set $x \subset \kappa$ is a closed unbounded*
 320 *subset, abbreviated as a club set, iff x is both closed and unbounded in κ .*

321 **Definition 1.54** (*Stationary set*)

322 *For a regular uncountable cardinal κ , we say a set $A \subset \kappa$ is stationary in κ*
 323 *iff it intersects every club subset of κ .*

324 1.2.7 Structure, Substructure and Embedding

325 Structures will be denoted $\langle M, \in, R \rangle$ where M is a domain, \in stands for the
 326 standard membership relation, it is assumed to be restricted to the domain⁷,
 327 $R \subseteq M$ is a relation on the domain. When R is not needed, we can as well
 328 only write M instead of $\langle M, \in \rangle$.

329 **Definition 1.55** (*Elementary Embedding*)

330 *Given the structures $\langle M_0, \in, R \rangle$, $\langle M_1, \in, R \rangle$ and a one-to-one function $j :$
 331 $M_0 \rightarrow M_1$, we say j is an elementary embedding of M_0 into M_1 , we write
 332 $j : M_0 \prec M_1$, when the following holds for every formula $\varphi(p_1, \dots, p_n)$ and
 333 every $p_1, \dots, p_n \in M_0$:*

$$\langle M_0, \in, R \rangle \models \varphi(p_1, \dots, p_n) \leftrightarrow \langle M_1, \in, R \rangle \models \varphi(j(p_1), \dots, j(p_n)) \quad (1.40)$$

334 **Definition 1.56** (*Elementary Substructure*)

335 *Given the structures $\langle M_0, \in, R \rangle$, $\langle M_1, \in, R \rangle$ and a one-to-one function $j :$
 336 $M_0 \rightarrow M_1$ such that $j : M_0 \prec M_1$, we say that M_0 is an elementary sub-
 337 structure of M_1 , denoted as $M_0 \prec M_1$, iff j is an identity on M_0 . In other
 338 words*

$$\langle M_0, \in, R \rangle \models \varphi(p_1, \dots, p_n) \leftrightarrow \langle M_1, \in, R \rangle \models \varphi(p_1, \dots, p_n) \quad (1.41)$$

339 *for $p_1, \dots, p_n \in M_0$*

⁷To be totally explicit, we should write $\langle M, \in \cap M \times M, R \rangle$

2 Levy's First-Order Reflection

2.1 Lévy's Original Paper

This section is based on Lévy's paper *Axiom Schemata of Strong Infinity in Axiomatic Set Theory*, [?]. It presents Lévy's general reflection principle and its equivalence to *Replacement* and *Infinity* under S^8 .

First, we should point out that set theory has changed over the last 66 years and show a few notable, albeit only formal, differences. Firstly, when reading Lévy's article, one should bear in mind that while the author often speaks about a model of ZF, usually denoted u , it doesn't necessarily mean that there is a set u that is a model of ZF⁹, we are nowadays used to using the notion of universal class V in similar sense, even though independently from a particular axiomatic set theory. The theory ZF is practically identical to the theory we have established in (??), the differences are only formal. One might be confused by the fact that Lévy treats the *Subsets* axiom, which we call *Specification*, as a single axiom rather than a schema. He even takes the conjunction of all axioms of ZF and treats it like a formula. This is possible because the underlying logic calculus is different. Lévy works with set theories formulated in the *non-simple applied first order functional calculus*, see Chapter IV in [?] for details. For now, we only need to know that the calculus contains a substitution rule for functional variables. This way, *Subsets* is de facto a schema even though it sometimes treated as a single formula¹⁰. It should also be noted that the logical connectives look different. The now usual symbol for an universal quantifier does not appear, $\forall x\varphi(x)$ would be written as $(x)\varphi(x)$. The symbol for negation is " \sim ", implication is written as " \supset " and equivalence is " \equiv ". We will use standard notation with " \neg ", " \rightarrow " and " \leftrightarrow " respectively when presenting Lévy's results.

This subsection uses ZF instead of the usual ZFC as the underlying theory.

Definition 2.1 (*Standard Complete Model of a Set Theory*)

Let Q be an arbitrary axiomatic set theory. We say that u is a standard complete model of Q iff

- (i) $(\forall \sigma \in Q)(u \models \sigma)$
- (ii) $\forall y(y \in u \rightarrow y \subset u)$

We write $Scm^Q(u)$.

⁸See definition (??).

⁹This is indeed impossible to prove in ZF due to Gödel's Incompleteness.

¹⁰This way, the conjunction of all axioms is then in fact an axiom schema.

374 **Definition 2.2** (*Cardinals Inaccessible With Respect to Q*)

375 *Let Q be an arbitrary axiomatic set theory. We say that a cardinal κ is*
 376 *inaccessible with respect to theory Q iff*

$$Scm^Q(V_\kappa) \quad (2.42)$$

377 *We write $In^Q(\kappa)$*

378 **Definition 2.3** (*Inaccessible Cardinal With Respect to ZF*)

379 *When a cardinal κ is inaccessible with respect to ZF, we only say that it is*
 380 *inaccessible. We write $In(\kappa)$.*

$$In(\kappa) \leftrightarrow In^{ZF}(\kappa) \quad (2.43)$$

381 The above definition of inaccessibles is used because it doesn't require *Choice*.

382 For the definition of relativization, see (??). The notation used by Lévy
 383 is " $Rel(u, \varphi)$ ", we will stick to " φ^u ".

384 **Definition 2.4** (*N*)

385 *The following is an axiom schema of complete reflection over ZF, denoted as*
 386 *N. For every first-order formula φ in the language of set theory with no free*
 387 *variables except for p_1, \dots, p_n , the following is an instance of schema N.*

$$\exists u(Scm^{ZF}(u) \ \& \ (\forall p_1, \dots, p_n \in u)(\varphi \leftrightarrow \varphi^u)) \quad (2.44)$$

388 Let S be an axiomatic set theory defined in (??).

389 **Definition 2.5** (*N₀*)

390 *Axiom schema N₀ is similar to N defined above, but with S instead of ZF.*
 391 *For every φ , a first-order fomula in the language of set theory with no free*
 392 *variables except p_1, \dots, p_n , the following is an instance of N₀.*

$$\exists u(Scm^S(u) \ \& \ (\forall p_1, \dots, p_n \in u)(\varphi \leftrightarrow \varphi^u)) \quad (2.45)$$

393 We will now show that in S, N₀ implies both *Replacement* and *Infinity*.

394

395 Let N₀ be defined as in (??), for *Infinity* see (??).

396 **Theorem 2.6** *In S, the axiom schema N₀ implies Infinity.*

397 *Proof.* Let $\varphi = \forall x \exists y (y = x \cup \{x\})$. This clearly holds in S because given a
 398 set x , there is a set $y = x \cup \{x\}$ obtained via *Pairing* and *Union*. From N₀,
 399 there is a set u such that φ^u holds. This u satisfies the conditions required
 400 by *Infinity*. \square

401 Lévy proves this theorem in a different way. He argues that for an arbitrary
 402 formula φ , N_0 gives us $\exists u Scm^S(u)$ and this u already satisfies *Infinity*.
 403 To do this, we would need to prove lemma (??) now, which would make
 404 second half of this chapter quite confusing.

405
 406 Let S be a set theory defined in (??), N_0 a schema defined in (??) and
 407 *Replacement* a schema defined in (??).

408 **Theorem 2.7** *In S , axiom the schema N_0 implies Replacement.*

409 *Proof.* Let $\varphi(x, y, p_1, \dots, p_n)$ be a formula with no free variables except
 410 x, y, p_1, \dots, p_n . Let χ be an instance of the *Replacement* schema for the
 411 φ given. We want to verify that χ holds in S with N_0 .

$$\begin{aligned} \chi = \forall x, y, z (\varphi(x, y, p_1, \dots, p_n) \ \& \ \varphi(x, z, p_1, \dots, p_n) \rightarrow y = z) \\ \rightarrow \forall x \exists y \forall z (z \in y \leftrightarrow (\exists q \in x) (\varphi(x, y, p_1, \dots, p_n))) \end{aligned} \quad (2.46)$$

412 Now consider the following formulas.

- 413 (i) $(\forall x, y, p_1, \dots, p_n \in u) (\varphi \leftrightarrow \varphi^u)$
- 414 (ii) $(\forall x, p_1, \dots, p_n \in u) (\exists y \varphi \leftrightarrow (\exists y \varphi)^u)$
- 415 (iii) $(\forall x, p_1, \dots, p_n \in u) (\chi \leftrightarrow \chi^u)$
- 416 (iv) $\forall x, p_1, \dots, p_n \chi \leftrightarrow (\forall x, p_1, \dots, p_n \chi)^u$

417 The above formulas are instances of the N_0 schema for φ , $\exists y \varphi$, χ and the
 418 universal closure of χ respectively. By N_0 , there exists a set u where all four
 419 formulas hold.¹¹ From relativization, $(\exists y \varphi)^u$ is equivalent to $(\exists y \in u) \varphi^u$,
 420 together with (i) and (ii), we get

$$(\forall x, p_1, \dots, p_n \in u) ((\exists y \in u) \varphi \leftrightarrow \exists y \varphi) \quad (2.47)$$

421 If φ is a function, then for every $x \in u$, which is also $x \subset u$ since $Scm^S(u)$
 422 and therefore u is transitive, it maps elements of x into u . From the *Speci-*
 423 *fication*, we can find y , a set of all images of the elements of x . That gives
 424 us $x, p_1, \dots, p_n \in u \rightarrow \chi$. By (iii) we get that $x, p_1, \dots, p_n \in u \rightarrow \chi^u$
 425 holds. The universal closure of this formula is $\forall x, p_1, \dots, p_n (x, p_1, \dots, p_n \in$
 426 $u \rightarrow \chi^u)$ which is equivalent to $(\forall x, p_1, \dots, p_n \in u) (\chi)^u$, which is exactly
 427 $(\forall x, p_1, \dots, p_n \chi)^u$. From (iv), $\forall x, p_1, \dots, p_n \chi$ holds. \square

428 What we have just proven is only a single theorem from Lévy's afore-
 429 mentioned article, we will introduce other interesting results, mostly related
 430 to Mahlo and inaccessible cardinals, later in their appropriate context in
 431 chapter 3.

¹¹Despite the fact that N_0 is defined for one formula, we have just used it for four at once. To make this formally possible, we can either prove that N_0 is equivalent to a more general version for any finite number of formulas or we can reflect their conjunction and argue that if $u \models \varphi_1 \ \& \ \dots \ \& \ \varphi_n$, then $(u \models \varphi_1), \dots, (u \models \varphi_n)$.

2.2 Contemporary Restatement

We will now introduce and prove a theorem that is called Lévy's Reflection in contemporary set theory. The only difference is that while Lévy originally reflects a formula φ from V to a set u which is a *standard complete model of* S , we say that there is a V_λ for a limit λ that reflects φ . Those two conditions are equivalent due to lemma (??).

Lemma 2.8 *Let $\varphi_1, \dots, \varphi_n$ be first-order formulas in the language of set theory, all with m free variables¹².*

(i) *For each set M_0 there is such set M that $M_0 \subset M$ and the following holds for every i , $1 \leq i \leq n$:*

$$\exists x \varphi_i(p_1, \dots, p_{m-1}, x) \rightarrow (\exists x \in M) \varphi_i(p_1, \dots, p_{m-1}, x) \quad (2.48)$$

for every $p_1, \dots, p_{m-1} \in M$.

(ii) *Furthermore, there is a limit ordinal λ such that $M_0 \subset V_\lambda$ and the following holds for each i , $1 \leq i \leq n$:*

$$\exists x \varphi_i(p_1, \dots, p_{m-1}, x) \rightarrow (\exists x \in V_\lambda) \varphi_i(p_1, \dots, p_{m-1}, x) \quad (2.49)$$

for every $p_1, \dots, p_{m-1} \in M$.

(iii) *Assuming Choice, there is M , $M_0 \subset M$ such that (??) holds for every M , $i \leq n$ and $|M| \leq |M_0| \cdot \aleph_0$.*

Proof. We will simultaneously prove statements (i) and (ii), denoting M^T the transitive set required by part (ii). Steps in the construction of M^T that are not explicitly included are equivalent to steps for M .

Let us first define an operation $H_i(p_1, \dots, p_{m-1})$ that yields the set of x 's with minimal rank¹³ satisfying $\varphi_i(p_1, \dots, p_{m-1}, x)$ for p_1, \dots, p_{m-1} and for every i , $1 \leq i \leq n$.

$$H_i(p_1, \dots, p_n) = \{x \in C_i : (\forall z \in C)(\text{rank}(x) \leq \text{rank}(z))\} \quad (2.50)$$

for each $1 \leq i \leq n$, where

$$C_i = \{x : \varphi_i(p_1, \dots, p_{m-1}, x)\} \text{ for } 1 \leq i \leq n \quad (2.51)$$

¹²For formulas with a different number of free variables, take for m the highest number of parameters among those formulas. Add spare parameters to every formula that has less than m parameters in a way that preserves the last parameter, which we will denote x . E.g. let φ'_i be the a formula with k parameters, $k < m$. Let us set $\varphi_i(p_1, \dots, p_{m-1}, x) = \varphi'_i(p_1, \dots, p_{k-1}, x)$, notice that the parameters p_k, \dots, p_{m-1} are not used.

¹³Rank is defined in (??)

455

456 Next, let's construct M from given M_0 by induction.

$$M_{i+1} = M_i \cup \bigcup_{j=0}^n \{H_j(p_1, \dots, p_{m-1}) : p_1, \dots, p_{m-1} \in M_i\} \quad (2.52)$$

457 In other words, in each step we include into the construction the elements
 458 satisfying $\varphi(p_1, \dots, p_{m-1}, x)$ for p_1, \dots, p_{m-1} from the previous step. For
 459 statement (ii), this is the only part that differs from (i). To end up with a
 460 transitive M , we need to extend every step to its transitive closure
 461 closure of M_{i+1} from (i). In other words, let γ be the smallest ordinal such
 462 that

$$(M_i^T \cup \bigcup_{j=0}^n \{\bigcup \{H_j(p_1, \dots, p_{m-1}) : p_1, \dots, p_{m-1} \in M_i\}\}) \subset V_\gamma \quad (2.53)$$

463 Then the incremental step is

$$M_{i+1}^T = V_\gamma \quad (2.54)$$

464 and the final M is obtained by joining the previous steps.

$$M = \bigcup_{i=0}^{\infty} M_i, \quad M^T = \bigcup_{i=0}^{\infty} M_i^T = V_\lambda \text{ for some limit } \lambda. \quad (2.55)$$

465

466 We have yet to finish part (iii). Let's try to construct a set M' that
 467 satisfies the same conditions like M but is kept as small as possible. As-
 468 suming the Axiom of Choice, we can modify the construction so that the
 469 cardinality of M' is at most $|M_0| \cdot \aleph_0$. Note that the size of M in the previous
 470 construction is determined by the size of M_0 and, most importantly, by the
 471 size of $H_i(p_1, \dots, p_{m-1})$ for every i , $1 \leq i \leq n$ in individual iterations of
 472 the construction. Since (i) only ensures the existence of an x that satisfies
 473 $\varphi_i(p_1, \dots, p_{m-1}, x)$ for any i , $1 \leq i \leq n$, we only need to add one x for ev-
 474 ery set of parameters but $H_i(u_1, \dots, u_{m-1})$ can be arbitrarily large. Let F be
 475 a choice function on $\mathcal{P}(M')$. Also let $h_i(p_1, \dots, p_{m-1}) = F(H_i(p_1, \dots, p_{m-1}))$
 476 for i , where $1 \leq i \leq n$, which means that h is a function that outputs an x
 477 that satisfies $\varphi_i(p_1, \dots, p_{m-1}, x)$ for i such that $1 \leq i \leq n$ and has minimal
 478 rank among all such sets. The induction step needs to be redefined to

$$M'_{i+1} = M'_i \cup \bigcup_{j=0}^n \{h_j(p_1, \dots, p_{m-1}) : p_1, \dots, p_{m-1} \in M'_i\} \quad (2.56)$$

479 This way, the amount of elements added to M'_{i+1} in each step of the con-
 480 struction is the same as the amount of m -tuples of parameters that yielded
 481 elements not included in M'_i . It is easy to see that if M_0 is finite, M' is
 482 countable because it was constructed as a countable union of sets that are
 483 themselves at most countable. If M_0 is countable or larger, the cardinality
 484 of M' is equal to the cardinality of M_0 .¹⁴ Therefore $|M'| \leq |M_0| \cdot \aleph_0$ \square

485 **Theorem 2.9** (*Lévy's first-order reflection theorem*)

486 Let $\varphi(p_1, \dots, p_n)$ be a first-order formula.

487 (i) For every set M_0 there exists a set M such that $M_0 \subset M$ and the
 488 following holds:

$$\varphi^M(p_1, \dots, p_n) \leftrightarrow \varphi(p_1, \dots, p_n) \quad (2.57)$$

489 for every $p_1, \dots, p_n \in M$.

490 (ii) For every set M_0 there is a transitive set M , $M_0 \subset M$ such that the
 491 following holds:

$$\varphi^M(p_1, \dots, p_n) \leftrightarrow \varphi(p_1, \dots, p_n) \quad (2.58)$$

492 for every $p_1, \dots, p_n \in M$.

493 (iii) For every set M_0 there is a limit ordinal λ such that $M_0 \subset V_\lambda$ and the
 494 following holds:

$$\varphi^{V_\lambda}(p_1, \dots, p_n) \leftrightarrow \varphi(p_1, \dots, p_n) \quad (2.59)$$

495 for every $p_1, \dots, p_n \in M$.

496 (iv) Assuming Choice, for every set M_0 there is M such that $M_0 \subset M$ and
 497 $|M| \leq |M_0| \cdot \aleph_0$ and the following holds:

$$\varphi^M(p_1, \dots, p_n) \leftrightarrow \varphi(p_1, \dots, p_n) \quad (2.60)$$

498 for every $p_1, \dots, p_n \in M$.

499 *Proof.* Let's now prove (i) for given φ via induction by complexity. We
 500 can safely assume that φ contains no quantifiers besides " \exists " and no logical
 501 connectives other than " \neg " and " $\&$ ". Let $\varphi_1, \dots, \varphi_n$ be all subformulas of φ .
 502 Then there is a set M , obtained by the means of lemma (??), for all of the
 503 formulas $\varphi_1, \dots, \varphi_n$.

¹⁴It can not be smaller because $|M'_{i+1}| \geq |M'_i|$ for every i . It may not be significantly larger because the maximum of elements added is the number of n -tuples in M'_i , which is of the same cardinality as M'_i .

Let's first consider atomic formulas in the form of either $x_1 = x_2$ or $x_1 \in x_2$. It is clear from relativisation¹⁵ that (??) holds for both cases, $(x_1 = x_2)^M \leftrightarrow (x_1 = x_2)$ and $(x_1 \in x_2)^M \leftrightarrow (x_1 \in x_2)$.

We now want to verify the inductive step. First, take $\varphi = \neg\varphi'$. From relativization, we get $(\neg\varphi')^M \leftrightarrow \neg(\varphi'^M)$. Because the induction hypothesis tells us that $\varphi'^M \leftrightarrow \varphi'$, the following holds:

$$(\neg\varphi')^M \leftrightarrow \neg(\varphi'^M) \leftrightarrow \neg\varphi' \quad (2.61)$$

The same holds for $\varphi = \varphi_1 \& \varphi_2$. From the induction hypothesis, we know that $\varphi_1^M \leftrightarrow \varphi_1$ and $\varphi_2^M \leftrightarrow \varphi_2$, which together with relativization for formulas in the form of $\varphi_1 \& \varphi_2$ gives us

$$(\varphi_1 \& \varphi_2)^M \leftrightarrow \varphi_1^M \& \varphi_2^M \leftrightarrow \varphi_1 \& \varphi_2 \quad (2.62)$$

Let's now examine the case when $\varphi = \exists x\varphi'(p_1, \dots, p_n, x)$. The induction hypothesis tells us that $\varphi'^M(p_1, \dots, p_n, x) \leftrightarrow \varphi'(p_1, \dots, p_n, x)$, so, together with above lemma (??), the following holds:

$$\begin{aligned} & \varphi(p_1, \dots, p_n, x) \\ & \leftrightarrow \exists x\varphi'(p_1, \dots, p_n, x) \\ & \leftrightarrow (\exists x \in M)\varphi'(p_1, \dots, p_n, x) \\ & \leftrightarrow (\exists x \in M)\varphi'^M(p_1, \dots, p_n, x) \\ & \leftrightarrow (\exists x\varphi'(p_1, \dots, p_n, x))^M \\ & \leftrightarrow \varphi^M(p_1, \dots, p_n, x) \end{aligned} \quad (2.63)$$

Which is what we wanted to prove for part (i).

We now need to verify that the same holds for any finite number of formulas $\varphi_1, \dots, \varphi_n$. This has in fact been already done since lemma (??) gives us a set M for any finite amount of formulas and given M_0 . We can therefore find a set M for the union of all of their subformulas. When we obtain such M , it should be clear that it also reflects every formula in $\varphi_1, \dots, \varphi_n$.

Since V_λ is a transitive set, by proving (iii) we also satisfy (ii). To do so, we only need to look at part (ii) of lemma (??). All of the above proof also holds for $M = V_{\text{lambda}}$.

¹⁵See (??). This only holds for relativization to $M, \in \cap M \times M$, not M, R for an arbitrary R .

528 To finish part (iv), we take M of size $\leq |M_0| \cdot \aleph_0$, which exists due to
 529 part (iii) of lemma (??), the rest being identical. \square

530
 531 Let \mathbf{S} be a set theory defined in (??), for ZFC see definition (??).

532 The two following lemmas are based on [?][Chapter 3, Theorem 1.2].

533 **Lemma 2.10** *If M is a transitive set, then $M \models \text{Extensionality}$.*

534 *Proof.* Given a transitive set M , we want to show that the following holds.

$$M \models \forall x, y (x = y \leftrightarrow \forall z (z \in x \leftrightarrow z \in y)) \quad (2.64)$$

535 Given arbitrary $x, y \in M$, we want to prove that $M \models (x = y \leftrightarrow \forall z (z \in$
 536 $x \leftrightarrow z \in y))$. This is equivalent to $M \models x = y$ iff $M \models \forall z (z \in x \leftrightarrow z \in y)$,
 537 which is the same as $x = y$ iff $M \models \forall z (z \in x \leftrightarrow z \in y)$.

538 So all elements of x are also elements of y in M , and vice versa. Because
 539 M is transitive, all elements of x and y are in M , so $M \models \forall z (z \in x \leftrightarrow z \in y)$
 540 holds iff x and y contain the same elements and are therefore equal. \square

541 **Lemma 2.11** *If M is a transitive set, then $M \models \text{Foundation}$.*

542 *Proof.* We want to prove the following:

$$M \models \forall x (x \neq \emptyset \rightarrow (\exists y \in x)(x \cap y = \emptyset)) \quad (2.65)$$

543 Given an arbitrary non-empty $x \in M$ let's show that $M \models (\exists y \in x)(x \cap$
 544 $y = \emptyset)$.

545 Because M is transitive, every element of x is an element of M . Take for
 546 y the element of x with the lowest rank¹⁶. It should be clear that there is no
 547 $z \in y$ such that $z \in x$, because then $\text{rank}(z) < \text{rank}(y)$, which would be a
 548 contradiction. \square

549 Let \mathbf{S} be a set theory as defined in (??).

550 **Lemma 2.12** *The following holds for every λ .*

$$“\lambda \text{ is a limit ordinal}” \rightarrow V_\lambda \models \mathbf{S} \quad (2.66)$$

551 *Proof.* Given an arbitrary limit ordinal λ , we will verify the axioms of \mathbf{S} one
 552 by one.

553 (i) *The existence of a set* comes from the fact that V_λ is a non-empty set
 554 because limit ordinal is non-zero by definition.

555 (ii) *Extensionality* holds from (??).

¹⁶Rank is defined in (??).

556 (iii) *Foundation* holds from (??).

557 (iv) *Union*:

558 Given any $x \in V_\lambda$, we want verify that $y = \bigcup x$ is also in V_λ . Note that
 559 $y = \bigcup x$ is a Δ_0 -formula.

$$y = \bigcup x \leftrightarrow (\forall z \in y)(\exists q \in x)z \in q \ \& \ (\forall z \in x)(\forall q \in z)q \in y \quad (2.67)$$

560 So by lemma (??)

$$y = \bigcup x \leftrightarrow V_\lambda \models y = \bigcup x \quad (2.68)$$

561 (v) *Pairing*:

562 Given two sets $x, y \in V_\lambda$, we want to show that $z = \{x, y\}$ is also an
 563 element of V_λ .

$$z = \{x, y\} \leftrightarrow x \in z \ \& \ y \in z \ \& \ (\forall q \in z)(q = x \vee q = y) \quad (2.69)$$

564 So $z = \{x, y\}$ is a Δ_0 -formula, and thus by lemma (??) it holds that

$$z = \{x, y\} \leftrightarrow V_\lambda \models z = \{x, y\} \quad (2.70)$$

565 (vi) *Powerset*:

566 Given any $x \in V_\lambda$, we want to make sure that $\mathcal{P}(x) \in V_\lambda$. Let $\varphi(y)$ denote
 567 the formula $y \in \mathcal{P}(x) \leftrightarrow y \subset x$. according to definition of subset (??),
 568 $y \subset x$ is Δ_0 , so for any given $x, y \in V_\lambda$, $y = \mathcal{P}(x) \leftrightarrow V_\lambda \models y = \mathcal{P}(x)$.
 569 Because λ is limit and $\text{rank}(\mathcal{P}(x)) = \text{rank}(x) + 1$, if $\mathcal{P}(x) \in V_\lambda$ for every
 570 $x \in V_\lambda$.

571 (vii) *Specification*:

572 Given a first-order formula φ , we want to show the following:

$$V_\lambda \models \forall x, p_1, \dots, p_n, \exists y \forall z (z \in y \leftrightarrow z \in x \ \& \ \varphi(z, p_1, \dots, p_n)) \quad (2.71)$$

573 Given any x along with parameters p_1, \dots, p_n in V_λ , we set

$$y = \{z \in x : \varphi^{V_\lambda}(z, p_1, \dots, p_n)\} \quad (2.72)$$

574 From transitivity of V_λ and the fact that $y \subset x$ and $x \in V_\lambda$, we know that
 575 $y \in V_\lambda$, so $V_\lambda \models \forall z (z \in y \leftrightarrow z \in x \ \& \ \varphi(z, p_1, \dots, p_n))$.
 576 □

577 **Definition 2.13** (*First-Order Reflection Schema*)

578 For every first-order formula φ , the following is an axiom:

$$\forall M_0 \exists M (M_0 \subseteq M \ \& \ (\varphi(p_1, \dots, p_n) \leftrightarrow \varphi(p_1, \dots, p_n)^M)) \quad (2.73)$$

579 We will refer to this axiom schema as First-order reflection.

580 Let *Infinity* and *Replacement* be as defined in (??) and (??) respectively.

581 **Theorem 2.14** First-order reflection is equivalent to Infinity & Replace-
582 ment under S.

583 *Proof.* Since (??) already gives us one side of the implication, we are only
584 interested in showing the converse which we shall do in two parts:

585 *First-order reflection \rightarrow Infinity* This is done exactly like (??). We pick
586 for φ the formula $(\forall y \in x)(y \cup \{y\} \in x)$, $M_0 = \{\emptyset\}$. From (??), there is a set
587 M that satisfies φ , so there is an inductive set. We have picked M_0 so that
588 $\emptyset \in M$ obviously holds and M is the witness for

$$\exists x(\emptyset \in x \ \& \ (\forall y \in x)(y \cup \{y\} \in x)) \quad (2.74)$$

589 which is exactly (??).

590 *First-order reflection \rightarrow Replacement*

591 Let's first point out that while *First-order reflection* gives us a set for
592 one formula, we can generalize it to hold for any finite number of formulas.
593 We will show how is it done for two formulas, which is what we will use in
594 this proof. Given two first-order formulas φ, ψ , we can suppose that there
595 are formulas φ' and ψ' that are equivalent to φ and ψ respectively, but their
596 free variables are different ¹⁷. Let $\xi = \varphi \ \& \ \psi$, given any M_0 , we can find a
597 M such that $\xi \leftrightarrow \xi^M$. It is easy to see that from relativisation, the following
598 holds:

$$\varphi \ \& \ \psi \leftrightarrow \varphi' \ \& \ \psi' \leftrightarrow \xi \leftrightarrow \xi^M \leftrightarrow (\varphi' \ \& \ \psi')^M \leftrightarrow \varphi'^M \ \& \ \psi'^M \leftrightarrow \varphi^M \ \& \ \psi^M \quad (2.75)$$

600 Now given a function $\varphi(x, y)$, we know from *First-order reflection* that
601 for every M_0 , there is a set M such that $M_0 \subseteq M$ and both

$$(\forall x, y \in M)(\varphi(x, y) \leftrightarrow \varphi^M(x, y)) \quad (2.76)$$

602 and

$$(\forall x, y \in M)(\exists y \varphi(x, y) \leftrightarrow (\exists y \varphi(x, y))^M) \quad (2.77)$$

603 hold, the latter being equivalent to

$$(\forall x, y \in M)(\exists y \varphi(x, y) \leftrightarrow (\exists y \in M) \varphi^M(x, y)) \quad (2.78)$$

¹⁷This is plausible since we can for example substitute all free variables in φ' for x_0, x_2, x_4, \dots and use x_1, x_3, x_5, \dots for free variables in ψ' , the resulting formulas will be equivalent.

604 Therefore

$$(\forall x, y \in M)(\exists y \varphi(x, y) \leftrightarrow (\exists y \in M)\varphi(x, y)) \quad (2.79)$$

605 holds too. That means that we have a set M such that for every $x \in M$, if
606 φ is defined for x , $(\exists y \in M)\varphi(x, y)$.

607 To show that *Replacement* holds for this particular φ , we need to verify
608 that given a set M_0 , $M'_0 = \{y : (\exists x \in M_0)\varphi(x, y)\}$ is also a set. But since
609 $M_0 \subseteq M$ and because given any $x \in M$, there is $y \in M$ satisfying $\varphi(x, y)$,
610 the following is a set due to *Specification*:

$$M'_0 = \{y : (\exists x \in M_0)\varphi(x, y)\} = \{y \in M : (\exists x \in M_0)\varphi(x, y)\} \quad (2.80)$$

611

□

612

613 We have shown that *Reflection* for first-order formulas, *First-order reflec-*
614 *tion* is a theorem of ZFC. We have also shown that it can be used instead of
615 the *Infinity* and *Replacement* scheme, but ZFC + *First-order reflection* is
616 a conservative extension of ZF. Besides being a starting point for more gen-
617 eral and powerful statements, it can be used to show that ZF is not finitely
618 axiomatizable. This follows from the fact that *Reflection* gives a model to
619 any consistent finite set of formulas. So if $\varphi_1, \dots, \varphi_n$ would be the axioms
620 of ZFC, *Reflection* would prove that every model of ZFC contains a smaller
621 model of ZFC, which would in turn contradict the Second Gödel's Theorem¹⁸.

622 It is also worthwhile to note that, in a way, Reflection is dual to compact-
623 ness. Compactness says that given a set of sentences, if every finite subset
624 yields a model, so does the whole set. Reflection, on the other hand, says
625 that while the whole set has no model in the underlying theory, every finite
626 subset has a model.

627 Furthermore, *Reflection* can be used in ways similar to upward Löwenheim–Skolem
628 theorem. Since Reflection extends any set M_0 into a model of given formulas
629 $\varphi_1, \dots, \varphi_n$, we can choose the lower bound of the size of M by appropriately
630 choosing M_0 .

631 In the next section, we will try to generalize *Reflection* in a way that
632 transcends ZF and yields some large cardinals.

¹⁸See chapter ?? for further details.

3 Reflection And Large Cardinals

3.1 Regular Fixed-Point Axioms

Lévy's article mentions various schemata that are not instances of reflection per se, but deal with fixed points of normal ordinal functions. We will introduce them and show that they are equivalent to *First-Order Reflection*¹⁹.

Lemma 3.1 (*Fixed-point lemma for normal functions*)

Let f be a normal function defined for all ordinals²⁰. Then all of the following hold:

- (i) $\forall \lambda$ ("λ is a limit ordinal" \rightarrow " $f(\lambda)$ is a limit ordinal")
- (ii) $\forall \alpha (\alpha \leq f(\alpha))$
- (iii) $\forall \alpha \exists \beta (\alpha < \beta \ \& \ f(\beta) = \alpha)$
- (iv) The fixed points of f form a closed unbounded class.²¹

Proof. Let f be a normal function defined for all ordinals.

- (i) Suppose λ is a limit ordinal. For an arbitrary ordinal $\alpha < \lambda$, the fact that f is strictly increasing means that $f(\alpha) < f(\lambda)$ and for any ordinal β , satisfying $\alpha < \beta < \lambda$, $f(\alpha) < f(\beta) < f(\lambda)$. We know that there is such β from limitness of λ . Because f is continuous and λ is limit, $f(\lambda) = \bigcup_{\gamma < \lambda} f(\gamma)$. That means that if λ is limit, so is $f(\lambda)$.

- (ii) This step will be proven using the transfinite induction. Since f is defined for all ordinals, there is an ordinal α such that $f(\emptyset) = \alpha$ and because \emptyset is the least ordinal, (ii) holds for \emptyset .

Suppose (ii) holds for some β from the induction hypothesis. It then holds for $\beta + 1$ because f is strictly increasing.

For a limit ordinal λ , suppose (ii) holds for every $\alpha < \lambda$. (i) implies that $f(\lambda)$ is also limit, so there is a strictly increasing κ -sequence $\langle \alpha_0, \alpha_1, \dots \rangle$ for some κ such that $\lambda = \bigcup_{i < \kappa} \alpha_i$. Because f is strictly increasing, the κ -sequence $\langle f(\alpha_0), f(\alpha_1), \dots \rangle$ is also strictly increasing, the induction hypothesis implies that $\alpha_i \leq f(\alpha_i)$ for each $i \leq \kappa$. Thus, $\lambda \leq f(\lambda)$.

- (iii) For a given ordinal α , let there be an ω -sequence $\langle \alpha_0, \alpha_1, \dots \rangle$, such that $\alpha_0 = \alpha$ and $\alpha_{i+1} = f(\alpha_i)$ for each $i < \omega$. This sequence is strictly increasing because so is f . Now, there's a limit ordinal $\beta = \bigcup_{i < \omega} \alpha_i$, we want to show that this is the fixed point. So $f(\beta) = f(\bigcup_{i < \omega} \alpha_i) = \bigcup_{i < \omega} f(\alpha_i)$ because f is continuous. We have defined the above sequence so that $\beta, \bigcup_{i < \omega} f(\alpha_i) = \bigcup_{i < \omega} \alpha_{i+1}$, which means we are done, since $\bigcup_{i < \omega} \alpha_{i+1} = \bigcup_{i < \omega} \alpha_i = \beta$.

¹⁹For definition, see (??).

²⁰For the definition of normal function, see (??).

²¹See (??.) for the definition of closed class, (??) for the definition of unboundedness.

(iv) The class of fixed points of f is obviously unbounded by (iii). It remains to show that it is closed, this is based on [?], chapter 4. Let Y be a non-empty set of fixed points of f such that $\bigcup Y \notin Y$. Since f is defined on ordinals, Y is a set of ordinals, so $\bigcup Y$ is an ordinal because a supremum of a set of ordinals is an ordinal. $\bigcup Y$ is a limit ordinal. If it were a successor ordinal, suppose that $\alpha + 1 = \bigcup Y$, then $\alpha \in \bigcup Y$, which means that there is some x such that $\alpha \in x \in Y$. But the least such x is $\alpha + 1$, so $\bigcup Y \in Y$.
 Note that $\alpha < \bigcup Y$ iff $\exists \xi \in Y (\alpha < \xi)$. Since f is defined for all ordinals and $\bigcup Y$ is a limit ordinal, $f(\bigcup Y) = \bigcup_{\alpha \in Y} f(\alpha)$, but because Y is a set of fixed points of f , $f(\bigcup Y) = \bigcup_{\alpha \in Y} \alpha = \bigcup Y$, so $\bigcup Y$ is also a limit point of Y .

□

Lemma 3.2 *Let α be a limit ordinal. Then the following hold:*

- (i) *If C is a club set in α , then there is an ordinal β and a normal function $f : \beta \rightarrow \alpha$ such that $\text{rng}(f) = C$. We say that f enumerates C .*
- (ii) *If β is an ordinal and f is a normal function such that $f : \beta \rightarrow \alpha$ and $\text{rng}(f)$ is unbounded in α , then $\text{rng}(f)$ is a closed unbounded set in α .*

This proof comes from (<http://euclid.colorado.edu/~monkd/m6730/gradsets09.pdf> TODO cite!) *Proof.*

- (i) Let β be the order-type²² of C , let f be the isomorphism from β onto C . Since $C \subseteq \alpha$, f is also an increasing function from β into α . In order to be continuous, let γ be a limit ordinal under β , let $\epsilon = \bigcup_{\delta < \gamma} f(\delta)$. We want to verify that $f(\gamma) = \epsilon$. Since ϵ is a limit ordinal, we only need to show that $C \cap \epsilon$ is unbounded in ϵ .
 Take $\zeta < \epsilon$. Then there is a $\delta < \gamma$ such that $\zeta < f(\delta)$. Since γ is limit, $\delta + 1 < \gamma$ and also $f(\delta + 1) < f(\gamma)$, we know that $f(\delta) \in C \cap \epsilon$. But that means that $C \cap \epsilon$ is unbounded in ϵ , so $\epsilon \in C$. We have also shown that ϵ is closed unbounded in the image of γ over f . Therefore, $f(\gamma) = \epsilon = \bigcup_{\delta < \gamma} f(\delta)$, so f is normal.

- (ii) TODO (potrebuj to?)

□ It

should be clear that while this lemma works with club subsets of an ordinal, we can formulate analogous statement for club classes, which then yields a normal function defined for all ordinals, with the only exception that there is no such β because f is a function from Ord to Ord .

²²See definition (??).

704 **Definition 3.3** (Axiom Schema M_1)

705 “Every normal function defined for all ordinals has at least one inaccessible
706 number in its range.”

707 Lévy uses “ M ” to refer to this axiom but since we also use “ M ” for sets and
708 models, for example in (??), we will call the above axiom “*Axiom Schema*
709 M_1 ” to avoid confusion.

710 Let $\varphi(x, y, p_1, \dots, p_n)$ be a first-order formula with no free variables be-
711 sides x, y, p_1, \dots, p_n . The following is equivalent to *Axiom* M_1 .

$$\begin{aligned} & \text{“}\varphi \text{ is a normal function”} \ \& \ \forall x(x \in \text{Ord} \rightarrow \exists y(\varphi(x, y, p_1, \dots, p_n))) \rightarrow \\ & \rightarrow \exists y(\exists x \varphi(x, y, p_1, \dots, p_n) \ \& \ cf(y) = y \ \& \ (\forall x \in \kappa)(\exists y \in \kappa)(x > y)) \end{aligned} \quad (3.81)$$

712 **Definition 3.4** (Axiom Schema M_2)

713 “Every normal function defined for all ordinals has at least one fixed point
714 which is inaccessible.”

715 **Definition 3.5** (Axiom Schema M_3)

716 “Every normal function defined for all ordinals has arbitrarily great fixed
717 points which are inaccessible.”

718 Similar axiom is proposed in [?].

719 **Definition 3.6** (Axiom Schema F)

720 “Every normal function has a regular fixed point.”

721 **Lemma 3.7** Let f be a normal function defined for all ordinals.

722 (i) There is a normal function g_1 defined for all ordinals that enumerates
723 the class $\{\alpha : f(\alpha) = \alpha \ \& \ \alpha \in \text{Ord}\}$.

724 (ii) There is a normal function g_2 defined for all ordinals that enumerates
725 the class $\{\lambda : \text{“}f(\lambda) \text{ is a strong limit cardinal.”}\}$.

726 *Proof.* We know that (ii) holds from lemma (??) and lemma (??).

727 For (i), It should be clear that there is no largest strong limit ordinal ν ,
728 because the limit of $\nu, \mathcal{P}(\nu), \mathcal{P}(\mathcal{P}(\nu)), \dots$ is again a limit ordinal. The class
729 of limit ordinals is closed because a limit of strong limit ordinals is clearly
730 always a strong limit ordinal. Let h be a function enumerating limit ordinals
731 which exists from lemma (??). Then $g_1(\alpha) = f(h(\alpha))$ for every ordinal α is
732 normal and defined for all ordinals. \square

733 The following is *Theorem 1* in [?], the parts dealing with *Axiom Schema*
734 F come from [?].

735 **Theorem 3.8** *The following are all equivalent:*

- 736 (i) Axiom Schema M_1
- 737 (ii) Axiom Schema M_2
- 738 (iii) Axiom Schema M_3
- 739 (iv) Axiom Schema F

740 *Proof.* It is clear that *Axiom Schema M_3* is a stronger version of *Axiom*
 741 *Schema M_2* , which is in turn a stronger version of both *Axiom Schema M_1*
 742 and *Axiom Schema F_1* .

743 We will now prove that *Axiom Schema $F \rightarrow$ Axiom Schema M_2* . Lemma
 744 (??) tells us that given a normal function f defined for all ordinals, there is
 745 a normal function g_1 defined for all ordinals that enumerates the fixed-points
 746 of f . There is also a function g_2 that enumerates the strong limit ordinals in
 747 $\text{rng}(f)$. By *Axiom Schema F* , g_2 has a regular fixed-point κ , which is also a
 748 strong limit ordinal, so

$$f(\kappa) = g_2(\kappa) = \kappa \text{ and } \kappa \text{ is inaccessible.} \quad (3.82)$$

749 So every normal function d.f.a.o. has a regular fixed-point.

750 We have yet to show *Axiom Schema $M_1 \rightarrow$ Axiom Schema M_3* . Again
 751 by lemma (??), there is a normal function g defined for all ordinals that
 752 enumerates the fixed points of f . Let $h_\alpha(\beta) = g(\alpha + \beta)$ for any given ordinal
 753 α , then h_α is a normal function defined for all ordinals. Then, given an
 754 arbitrary α , from *Axiom Schema M_1* , there is a β such that $\gamma = h_\alpha(\beta)$ is
 755 inaccessible. Because $\gamma = g(\alpha + \beta)$, $f(\gamma) = \gamma$. Since $\alpha \leq f'(\alpha)$ for any
 756 ordinal α and any normal function f' , we know that $\alpha \leq \alpha + \gamma \leq \gamma$, so γ is
 757 inaccessible and arbitrarily large, depending on the choice of α . \square

758 But how do those schemata relate to reflection? Let's introduce a stronger
 759 version of *First-order reflection schema* from the previous chapter to see it
 760 more clearly. But in order to do this, we must establish the inaccessible
 761 cardinal first.

762 3.2 Inaccessible Cardinal

763 **Definition 3.9** *An uncountable cardinal κ is inaccessible iff it is regular*
 764 *and strongly limit. We write $\text{In}(\kappa)$ to say that κ is an inaccessible cardinal.*

765 An uncountable cardinal that is regular and limit is called a *weakly in-*
 766 *accessible cardinal*, we will only use the (strongly) inaccessible cardinal, but
 767 most of the results are similar for weakly inaccessible, including higher types
 768 of ordinals that will be presented later in this chapter.

769 **Theorem 3.10** *Let κ be an inaccessible cardinal.*

$$V_\kappa \models \text{ZFC} \quad (3.83)$$

770 We will prove this theorem in a way similar to [?]. *Proof.* Most of this
771 is already done in lemma (??), we only need to verify that *Replacement* and
772 *Infinity* axioms hold in V_κ .

773 *Infinity* holds because κ is uncountable, so $\omega \in V_\kappa$.

774 To verify *Replacement*, let x be an element of V_κ and f a function from
775 x to V_κ . Let $y = \{z \in V_\kappa : (\exists q \in x)f(q) = z\}$, so $y \subset V_\kappa$, it remains to show
776 that $y \in V_\kappa$. Because f is a function, we know that $|y| \leq |x| \leq \kappa$. But since
777 κ is regular, $\{\text{rank}(z) : z \in y\} \subseteq \alpha$ for some $\alpha < \kappa$, and so $x \in V_{\alpha+1} \subseteq V_\kappa$.
778 Therefore $y \in V_\kappa$. \square

779 **Definition 3.11** (*Inaccessible Reflection Schema*)

780 *For every first-order formula φ , the following is an axiom:*

$$\forall M_0 \exists \kappa (M_0 \subseteq V_\kappa \ \& \ In(\kappa) \ \& \ (\varphi(p_1, \dots, p_n) \leftrightarrow \varphi(p_1, \dots, p_n)^{V_\kappa})) \quad (3.84)$$

781 *We will refer to this axiom schema as Inaccessible reflection schema.*

782 We have added the requirement that α is inaccessible, which trivially
783 means that there is an inaccessible cardinal. By taking appropriate M_0 ,
784 it can be shown that in a theory that includes the *Inaccessible reflection*
785 *schema*, there is a closed unbounded class of inaccessible cardinals. Since we
786 know that for an inaccessible κ , V_κ is a model of ZFC, *Inaccessible reflection*
787 *schema* is equivalent to

$$\forall M_0 \exists \kappa (M_0 \subseteq V_\kappa \ \& \ V_\kappa \models \text{ZFC} \ \& \ (\varphi(p_1, \dots, p_n) \leftrightarrow \varphi(p_1, \dots, p_n)^{V_\kappa})) \quad (3.85)$$

788 because we have proven in the last section that for an inaccessible κ , $V_\kappa \models \text{ZFC}$.

789 **Theorem 3.12** *Inaccessible reflection schema is equivalent to Axiom schema*
790 *F.*

791 This is *Theorem 4.1* in chapter four of [?], also equivalent to *Theorem 3*
792 *in [?]*. *Proof.* Let's start by showing that *Inaccessible reflection schema* implies
793 *Axiom schema F*. It should be clear that we can reflect two formulas to a single
794 set, just form a new formula as a conjunction of universal closures of the two.

795 Given a normal function f defined for all ordinals, we want to show that it
796 has a regular fixed point. For any ordinal α , there is an ordinal κ such that

$$\alpha < \kappa \ \& \ In(\kappa) \ \& \ (\forall \gamma, \delta \in V_\kappa)(f(\gamma) = \delta \leftrightarrow (f(\gamma) = \delta)^{V_\kappa}) \quad (3.86)$$

797 and

$$\alpha < \kappa \ \& \ In(\kappa) \ \& \ \forall \gamma \exists \delta (f(\gamma) = \delta) \leftrightarrow (\forall \gamma \exists \delta f(\gamma) = \delta)^{V_\kappa} \quad (3.87)$$

798 Since V_κ is the set of all sets of rank less than κ and since every ordinal is the
799 rank of itself, there is an inaccessible ordinal κ such that

$$\forall \gamma < \kappa \exists \delta < \kappa (f^{V_\kappa}(\gamma) = \delta) \quad (3.88)$$

800 We also know that $f(\gamma) = \delta \leftrightarrow (f(\gamma) = \delta)^{V_\kappa}$. Now since κ is a limit ordinal
801 and f is continuous we get

$$f(\kappa) = \bigcup_{\gamma < \kappa} f^{V_\kappa}(\gamma) = \bigcup_{\gamma < \kappa} f(\gamma). \quad (3.89)$$

802 From (??) and the fact that f is increasing, we know that $\kappa \leq \bigcup_{\gamma < \kappa} f(\gamma) \leq \kappa$.
803 Therefore κ is an inaccessible fixed point of f .

804 For the opposite direction, it suffices to show that since there is an inacces-
805 sible cardinal from *Axiom schema F*, given a first-order formula φ , there is an
806 arbitrarily large inaccessible cardinal κ for which

$$\varphi \leftrightarrow V_\kappa \models \varphi. \quad (3.90)$$

807 Note that the arbitrary size of κ means given an arbitrary ordinal α , there is a
808 κ satisfying (??). In the previous chapter, in theorem (??), we have shown that
809 we can easily obtain a limit ordinal satisfying (??). Note that since for any set
810 M_0 , there is such α that $M_0 \subseteq V_\alpha$, there is a closed unbounded class of sets
811 satisfying (??), which are levels in the cumulative hierarchy, so there is a club
812 sets of κ s satisfying (??).

813 Let f be a normal function defined for all ordinals that enumerates this club
814 class, there is such by lemma (??). Let g be the function that enumerates
815 strong limit ordinals in $rng(f)$. Then g has a regular fixed point κ , which is also
816 a regular fixed point of f , so (??) holds for κ .
817 □

818 Definition 3.13 (ZMC)

819 We will call ZMC an axiomatic set theory that contains all axioms and schemas
820 of ZFC together with Axiom Schema M_1 .

821 We have decided to call it ZMC, because Lévy uses ZM, derived from ZF, which
822 is more intuitive, but we also need the axiom of choice, thus, ZMC.

3.3 Mahlo Cardinals

We have shown that ZMC contains arbitrarily large inaccessible cardinals. To return to reflection-style argument, is there a set that satisfies this property? To be able to properly answer this question, we have to formulate the notion of "containing arbitrarily large cardinals" more carefully. While we have previously used club sets, this is not an option because inaccessible cardinals don't form a club class in ZMC²³, we could try to formulate stronger versions of *Axiom Schema M_1* .

Let's shortly review what *Axiom Schema M_1* says. We have shown earlier in this chapter that there is a simple relation between normal function defined for all ordinals and closed unbounded classes. So by saying that for a class of ordinals C , a normal function f has at least one element of C in its range, we say that C is stationary. Or, as Drake puts it for C , the class of inaccessible cardinals, and a κ , in which C is stationary:

"The class of inaccessible cardinals is so rich that there are members κ of the class such that no normal function on κ can avoid this class; however we climb through κ , provided we are continuous at limits (so that we are enumerating a closed subset of κ), we shall eventually have to hit an inaccessible."

Definition 3.14 (Mahlo Cardinal)

We say that κ is a Mahlo Cardinal iff it is an inaccessible cardinal and the set $\{\lambda < \kappa : \lambda \text{ is inaccessible}\}$ is stationary in κ .

Alternatively, κ is Mahlo iff $V_\kappa \models \text{ZMC}$ as shown above, this is also sometimes written as *Ord is Mahlo*. There are also *weakly Mahlo cardinals*, that are defined via weakly inaccessible cardinal below them, Mahlo cardinals are then also called *strongly Mahlo* to highlight the difference, but we will only use the term *Mahlo cardinal*.

Mahlo cardinals are related to reflection principles in an interesting way. Note that given a formula φ , reflection gave us a club set of ordinals α such that V_α reflects φ , all below the first inaccessible cardinal. We have then used a different reflection schema to obtain arbitrarily high inaccessible cardinals. Now we have a cardinal in which this reflection schema holds, so we're in fact reflecting reflection. Beware that this is done rather informally, because *Axiom Schema M_1* is a countable set of axioms, which is too large to be reflected via the schemas introduced so far. One way to deal with this would be to extend reflection for second- and possibly higher-order formulas, but we would have to be very careful with the notion of satisfaction. For now, explore where can stationary sets take us because as we have shown, their connection to reflection is quite clear.

²³Note that cofinality of the limit of the first ω inaccessible cardinals is ω , which makes it singular.

860 What would happen if we strengthened *Axiom Schema M_1* to say that every
 861 normal function has a Mahlo cardinal in its range?

862 **Definition 3.15** (*hyper-Mahlo cardinal*)

863 We say that κ is a hyper-Mahlo cardinal iff it is inaccessible and the set $\{\lambda < \kappa :$
 864 $\lambda \text{ is Mahlo}\}$ is stationary in κ .

865 **Definition 3.16** (*hyper-hyper-Mahlo cardinal*)

866 We say that κ is a hyper-hyper-Mahlo cardinal iff it is inaccessible and the set
 867 $\{\lambda < \kappa : \lambda \text{ is hyper-Mahlo}\}$ is stationary in κ .

868 It is clear that one can continue in this direction, but the nomenclature gets
 869 increasingly overwhelming even if we introduce *hyper $^\alpha$ -Mahlo cardinals*.

870 TODO Mahlo operation

⁸⁷¹ **4 Conclusion**

References

- [Church, 1996] Church, A. (1996). *Introduction to Mathematical Logic*. Annals of Mathematics Studies. Princeton University Press.
- [Drake, 1974] Drake, F. (1974). *Set theory. An introduction to large cardinals*. Studies in Logic and the Foundations of Mathematics, Volume 76. NH.
- [Jech, 2006] Jech, T. (2006). *Set theory*. Springer monographs in mathematics. Springer, the 3rd millennium ed., rev. and expanded edition.
- [Kanamori, 2003] Kanamori, A. (2003). *The higher infinite: Large cardinals in set theory from their beginnings*. Springer Monographs in Mathematics. Springer-Verlag Berlin Heidelberg, 2 edition.
- [Lévy, 1960] Lévy, A. (1960). Axiom schemata of strong infinity in axiomatic set theory. *Pacific Journal of Mathematics*, 10.
- [Wang, 1997] Wang, H. (1997). *"A Logical Journey: From Gödel to Philosophy"*. A Bradford Book.