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REFLECTION PRINCIPLES AND LARGE CARDINALS

Bakalářská práce

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Prohlašuj, že jsem bakalářkou práci vypracoval samostatně a že jsem uvedl
všechny použité prameny a literaturu.

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Abstract

Práce zkoumá vztah tzv. principů reflexe a velkých kardinálů. Lévy ukázal, že v ZFC platí tzv. věta o reflexi a dokonce, že věta o reflexi je ekvivalentní schématu nahrazení a axiomu nekonečna nad teorií ZFC bez axiomu nekonečna a schématu nahrazení. Tedy lze na větu o reflexi pohlížet jako na svého druhu axiom nekonečna. Práce zkoumá do jaké míry a jakým způsobem lze větu o reflexi zobecnit a jaký to má vliv na existenci tzv. velkých kardinálů. Práce definuje nedosažitelné, Mahlovy a nepopsatelné kardinály a ukáže, jak je lze zavést pomocí reflexe. Přirozenou limitou kardinálů získaných reflexí jsou kardinály nekonzistentní s L. Práce nabídne intuitivní zdůvodněn, proč tomu tak je.

Abstract

This thesis aims to examine relations between so called "Reflection Principles" and Large cardinals. Lévy has shown that Reflection Theorem is a sound theorem of ZFC and it is equivalent to Replacement Scheme and the Axiom of Infinity. From this point of view, Reflection theorem can be seen a specific version of an Axiom of Infinity. This paper aims to examine the Reflection Principle and it's generalisations with respect to existence of Large Cardinals. This thesis will establish Inaccessible, Mahlo and Indescribable cardinals and their definition via reflection. A natural limit of Large Cardinals obtained via reflection are cardinals inconsistent with L. The thesis will offer an intuitive explanation of why this is the case.

Contents

1 Introduction

1.1 Motivation and Origin

> The Universe of sets cannot be uniquely characterized (i. e. distinguished from all its initial elements) by any internal structural property of the membership relation in it, which is expressible in any logic of finite of transfinite type, including infinitary logics of any cardinal order.

> — Kurt Gödel [**?**]

To understand why do need reflection in the first place, let's think about infinity for a moment. In the intuitive sense, infinity is an upper limit of all numbers. But for centuries, this was merely a philosophical concept, closely bound to religious and metaphysical way of thinking, considered separate from numbers used for calculations or geometry. It was a rather vague concept. In ancient Greece, Aristotle's response to famous Zeno's paradoxes introduced the distinction between actual and potential infinity. He argued, that potential infinity is (in today's words) well defined, as opposed to actual infinity, which remained a vague incoherent concept. He didn't think it's possible for infinity to inhabit a bounded place in space or time, rejecting Zeno's thought experiments as a whole. Aristotle's thoughts shaped western thinking partly due to Aquinas, who himself believed actual infinity to be more of a metaphysical concept for describing God than a mathematical property attributed to any other entity. In his Summa Theologica [1] he argues:

> A geometrician does not need to assume a line actually infinite, but takes some actually finite line, from which he subtracts whatever he finds necessary; which line he calls infinite.

Less than hundred years later, Gregory of Rimini wrote

> If God can endlessly add a cubic foot to a stone–which He can–then He can create an infinitely big stone. For He need only add one cubic foot at some time, another half an hour later, another a quarter of an hour later than that, and so on ad infinitum. He would then have before Him an infinite stone at the end of the hour.

Which is basically a Zeno's Paradox made plausible with God being the actor. In contrast to Aquinas' position, Gregory of Rimini theoretically constructs an object with actual infinite magnitude that is essentially different from God.

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[1] Part I, Question 7, Article 3, Reply to Objection 1

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Even later, in the 17th century, pushing the property of infinitness from the Creator to his creation, Nature, Leibniz wrote to Foucher in 1962:

> I am so in favor of the actual infinite that instead of admitting that Nature abhors it, as is commonly said, I hold that Nature makes frequent use of it everywhere, in order to show more effectively the perfections of its Author. Thus I believe that there is no part of matter which is not, I do not say divisible, but actually divided; and consequently the least particle ought to be considered as a world full of an infinity of different creatures.

But even though he used potential infinity in what would become foundations of modern Calculus and argued for actual infinity in Nature, Leibniz refused the existence of an infinite, thinking that Galileo's Paradoxon[2] is in fact a contradiction. The so called Galileo's Paradoxon is an observation Galileo Galilei made in his final book "Discourses and Mathematical Demonstrations Relating to Two New Sciences". He states that if all numbers are either squares and non–squares, there seem to be less squares than there is all numbers. On the other hand, every number can be squared and every square has it's square root. Therefore, there seem to be as many squares as there are all numbers. Galileo concludes, that the idea of comparing sizes makes sense only in the finite realm.

> Salviati: So far as I see we can only infer that the totality of all numbers is infinite, that the number of squares is infinite, and that the number of their roots is infinite; neither is the number of squares less than the totality of all the numbers, nor the latter greater than the former; and finally the attributes "equal," "greater," and "less," are not applicable to infinite, but only to finite, quantities. When therefore Simplicio introduces several lines of different lengths and asks me how it is possible that the longer ones do not contain more points than the shorter, I answer him that one line does not contain more or less or just as many points as another, but that each line contains an infinite number.

Leibniz insists in part being smaller than the whole saying

> Among numbers there are infinite roots, infinite squares, infinite cubes. Moreover, there are as many roots as numbers. And there are as many squares as roots. Therefore there are as many squares

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[2]zneni galileova paradoxu

109   as numbers, that is to say, there are as many square numbers as
110   there are numbers in the universe. Which is impossible. Hence it
111   follows either that in the infinite the whole is not greater than the
112   part, which is the opinion of Galileo and Gregory of St. Vincent,
113   and which I cannot accept; or that infinity itself is nothing, i.e.
114   that it is not one and not a whole.

115   TODO Hegel–strucne?
116   TODO Cantor
117   TODO mene teologie, vice matematiky
118   TODO definovat pojmy (trida etc)
119   TODO neni V v nejakem smyslu porad potencialni nekonecno, zatimco
120 mnoziny vetsi nez omega jsou aktualni? nebo jsou potencialni protoze se
121 staveji pres indukci, od spoda.

122   In his work, he defined transfinite numbers to extend existing natural
123 number structure so it contains more objects that behave like natural num-
124 bers and are based on an object (rather a meta-object) that doesn't explicitly
125 exist in the structure, but is closely related to it. This is the first instance
126 of reflection. This paper will focus on taking this principle a step further,
127 extending Cantor's (or Zermelo–Fraenkel's, to be more precise) universe so
128 it includes objects so big, they could be considered the universe itself, in
129 a certain sense.

130   TODO dal asi smazat

131   The original idea behind reflection principles probably comes from what
132 could be informally called "universality of the universe". The effort to pre-
133 cisely describe the universe of sets was natural and could be regarded as one
134 of the impulses for formalization of naive set theory. If we try to express
135 the universe as a set $\{x|x=x\}$, a paradox appears, because either our set
136 is contained in itself and therefore is contained in a set (itself again), which
137 contradicts the intuitive notion of a universe that contains everything but is
138 not contained itself.

139   TODO ???

140   If there is an object containing all sets, it must not be a set itself. The
141 notion of class seems inevitable. Either directly the ways for example the
142 Bernays–Gödel set theory, we will also discuss later in this paper, does in, or
143 on a meta–level like the Zermelo–Fraenkel set theory, that doesn't refer to
144 them in the axioms but often works with the notion of a universal class. duet
145 Another obstacle of constructing a set of all sets comes from Georg Cantor,
146 who proved that the set of all subsets of a set (let A be the set and $\mathscr{P}(()A$
147 its powerset) is strictly larger that A. That would turn every aspiration to

finally establish an universal set into a contradictory infinite regression.[3]. We will use V to denote the class of all sets. From previous thoughts we can easily argue, that it is impossible to construct a property that holds for V and no set and is neither paradoxical like $\{x|x = x\}$ nor trivial. Previous observation can be transposed to a rather naive formulation of the reflection principle:

Reflection Any property which holds in V already holds in some initial segment of V.

To avoid vagueness of the term "property", we could informally reformulate the above statement into a schema:

For every first-order formula[4] φ holds in $V \leftrightarrow \varphi$ holds in some initial segment of V.

Interested reader should note that this is a theorem scheme rather than a single theorem. [5]

1.2   A few historical remarks on reflection

Reflection made it's first in set-theoretical appearance in Gödel's proof of GCH in L (citace Kanamori ? Lévy and set theory), but it was around even earlier as a concept. Gödel himself regarded it as very close to Russel's reducibility axiom (an earlier equivalent of the axiom schema of Zermelo's separation). Richard Montague then studied reflection properties as a tool for verifying that Replacement is not finitely axiomatizable (citace?). a few years later Lévy proved in [**?**] the equivalence of reflection with Axiom of infinity together with Replacement in proof we shall examine closely in chaper 2.

TODO co dal? recent results?

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[3]An intuitive analogy of this *reductio ad infinitum* is the status of ω, which was originally thought to be an unreachable absolute, only to become starting point of Cantor's hierarchy of sets growing beyond all boundaries around the end of the 19th century

[4]this also works for finite sets of formulas [**?**, p. 168]

[5]If there were a single theorem stating "for any formula φ that holds in V there is an initial segment of V where φ also holds", we would obtain the following contradiction with the second Gödel's theorem: In ZFC, any finite group of axioms of ZFC holds in some initial segment of the universe. If we take the largest of those initial segments it is still strictly smaller than the universe and thus we have, via compactness, constructed a model of ZFC within ZFC. That is, of course a harsh contradiction. This also leads to an elegant way to prove that ZFC is not finitely axiomatizable.

1.3   Reflection in Platonism and Structuralism

TODO cite "reflection in a structuralist setting"

TODO veci o tom, ze reflexe je ok protoze reflektuje veci ktere objektivne plati, protoze plati pro V ...

TODO souvislost s kompaktnosti, hranice formalich systemu nebo alespon ZFC

1.4   Notation and terminology

1. *Reflection* je obecne reflexe (jaka presne?)
2. *Reflection*$_1$ je reflexe prvoradovych formuli TODO presna formulace!
3. etc...

V a V_α odkazuji k Von Neumannove hierarchii (pro jistotu)

Vypsat axiomy ZFC a jake formulace pouzivam
Replacement, *Replacement*$_2$ atd *Subsets*

Definition 1.1 *(*Extensionality*)*

$$\text{Extensionality} \leftrightarrow TODO \tag{1.1}$$

Definition 1.2 *(*Foundation*)*

$$\text{Foundation} \leftrightarrow TODO \tag{1.2}$$

Definition 1.3 *(*Pairing*)*

$$\text{Pairing} \leftrightarrow TODO \tag{1.3}$$

Definition 1.4 *(*Union*)*

$$\text{Union} \leftrightarrow TODO \tag{1.4}$$

Definition 1.5 *(*Powerset*)*

$$\text{Powerset} \leftrightarrow TODO \tag{1.5}$$

Definition 1.6 *(*Specification*)*

$$\text{Specification} \leftrightarrow TODO \tag{1.6}$$

204 **Definition 1.7** *(*Infinity*)*

205

$$\text{Infinity} \leftrightarrow TODO \tag{1.7}$$

206 **Definition 1.8** *(*Replacement*)*

207

$$\text{Replacement} \leftrightarrow TODO \tag{1.8}$$

208 **Definition 1.9** *(*Choice*)*

209

$$\text{Choice} \leftrightarrow TODO \tag{1.9}$$

210

211 **Definition 1.10** (S)
212 *TODO*

213 **Definition 1.11** (ZF)
214 *TODO*

215 **Definition 1.12** (ZFC)
216 *TODO*

217 **Definition 1.13** (ZFC$_2$)
218 *TODO*

219

220 TODO definice druhoradoveho splnovani
221 TODO funkce

222

223 Asi vsechno budeme delat v ZFC, nic bychom neziskali, pokud ne.

2 Levy's first-order reflection

2.1 Introduction

This section will try to present Lévy's proof of a general reflection principle being equivalent to Replacement and Infinity under ZF minus Replacement and Infinity. We will first introduce a few axioms and definitions that were a different in Lévy's paper[?], but are equivalent to today's terms. We will write them in contemporary notation, our aim is the result, not history of set theory notation.

Please note that Lévy's paper was written in a period when Set theory was oriented towards semantics, which means that everything was done in a model. All proofs were a model that of ZFC was V_κ (notated as $R(\kappa)$ at the time) for some cardinal κ, which means that κ is a inaccessible cadinal. Please bear in mind that this is vastly different from saying that there is an inaccessible κ inside the model. This V_κ is also referred to as $Scm^{\mathsf{Q}}(u)$, which means that u is a standard complete model of an undisclosed axiomatic set theory Q formulated in the "non-simple applied first order functional calculus", which is second-order theory is today's terminology, we are allowed to quantify over functions and thus get rid of axiom schemes. (Note that Lévy always speaks of "the axiom of replacement"). Besides placeholder set theory Q, and ZF, which the reader should be familiar with, theories Z, S, and SF are used in the text. Z is ZF minus replacement, S is ZF minus replacement and infinity, and finally SF is ZF minus infinity. The axiom of *Subsets* is an older name for the axiom scheme of specification (and it's not a scheme since we are now working in second order logic). Also note that universal quantifier does not appear, $\forall x \varphi(x)$ would be written as $(x)\varphi(x)$, the symbol for negation is "$\sim$", we will use "$\neg$" the whole time.

TODO nebudeme tady pouzivat ZFC, ale jenom ZF. (jenom v tehle kapitole)

2.2 Lévy's Original Paper

The following are a few definitions that are used in Lévy's original article. [6]

Definition 2.1 *(Relativization)[?, Definition 12.6]*
Let M be a class, E a binary relation on M and let $\varphi(x_1, \ldots, x_n)$ be a first-order formula with n parameters. The relativization of φ to M and E is the

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[6]While some of them won't be of much use in this paper, they will provide extremely helpful when reading the original article as set theory notation and terminology has evolved in the last 50 years considerably.

257 *formula written as*

$$\varphi^{M,E}(x_1, \ldots, x_n) \tag{2.10}$$

258 *Defined in the following inductive manner:*

$$
\begin{aligned}
(x \in y)^{M,E} &\leftrightarrow xEx \\
(x = y)^{M,E} &\leftrightarrow x = y \\
(\neg\varphi)^{M,E} &\leftrightarrow \neg\varphi^{M,E} \\
(\varphi \ \& \ \psi)^{M,E} &\leftrightarrow \varphi^{M,E} \ \& \ \psi^{M,E} \\
(\exists x\varphi)^{M,E} &\leftrightarrow (\exists x \in M)\varphi^{M,E}
\end{aligned}
\tag{2.11}
$$

259   Next two definitions are not used in contemporary set theory, but they
260 illustrate 1960's set theory mind-set and they are used heavily in Lévy's text,
261 so we will include and explain them for clarity. Generally in this chapter, Q
262 stands for an undisclosed axiomatic set theory, u is usually a model, coun-
263 terpart of today's V^7, E is a relation that serves as $\in$ in the given model.

Definition 2.2 *(Standard model of a set theory)*
265 *Let Q be a axiomatic set theory in first-order logic. We say the the a class u*
266 *is a standard model of Q with respect to a membership relation E, written as*
267 *$Sm^{\mathsf{Q}}(u)$, iff both of the following hold*
268   *(i) $(x, y) \in E \leftrightarrow y \in u \ \& \ x \in y$*
269   *(ii) $y \in u \ \& \ x \in y \rightarrow x \in u$*

Definition 2.3 *Standard complete model of a set theory*
271 *Let Q and E be like in ??. We say that that u is a standard complete model*
272 *of Q with respect to a membership relation E iff both of the following hold*
273   *(i) u is a transitive set with respect to $\in$*
274   *(ii) $\forall E((x, y) \in E \leftrightarrow (y \in u \ \& \ x \in y) \ \& \ Sm^{\mathsf{Q}}(u, E))$*
275 *this is written as $Scm^{\mathsf{Q}}(u)$.*

Definition 2.4 *(Inaccessible cardinal with respect to Q)*
277 *Let Q be an axiomatic first-order set theory. We say that a cardinal κ is*
278 *inaccessible with respect to Q, we write $In^{\mathsf{Q}}(\kappa)$, iff*

$$Scm^{\mathsf{Q}}(V_\kappa). \tag{2.12}$$

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[7]Which is of course not referred to as a model, but it is used in a similar fashion, in this case the term "model" was a metamathematical notion because it was not based on any underlying structure of theory. It can be easily formalized in any set theory, but it's not helpful for our case.

Definition 2.5 *(Inaccessible cardinal with respect to* ZF*)*

When a cardinal κ *is inaccessible with respect to* ZF*, we only say that it is inaccessible. In the abbreviated version, we just leave out the superscript.*

$$In(\kappa) \leftrightarrow In^{\mathsf{ZF}}(\kappa) \tag{2.13}$$

Definition 2.6 *(N)*

The following is an axiom schema of complete reflection over ZF*, denoted as* N.

$$N \leftrightarrow \exists u(Scm^{\mathsf{ZF}}(u) \ \& \ \forall x_1, \ldots, x_n(x_1, \ldots, x_n \in u \rightarrow \varphi \leftrightarrow \varphi^u)) \tag{2.14}$$

where φ *is a formula which contains no free variables except for* $x_1, \ldots, x_n$.

Definition 2.7 *(N_0)*

If we substitute ZF *for* S*, which is* ZF *minus* Replacement *and* Infinity*, we obtain what will now be called* N_0.

$$N_0 \leftrightarrow \exists u(Scm^{\mathsf{S}}(u) \ \& \ \forall x_1, \ldots, x_n(x_1, \ldots, x_n \in u \rightarrow \varphi \leftrightarrow \varphi^u)) \tag{2.15}$$

where φ *is a formula which contains no free variables except for* $x_1, \ldots, x_n$.

Once we have established the definitions, it's time to prove something interesting.

2.3 $\mathsf{S} \models (\mathsf{N}_0 \leftrightarrow$ *Replacement* & *Infinity*)

Let N_0 be defined as in **??**, for *Infinity* see **??**.

Theorem 2.8 *In* S*, the schema* N_0 *implies* Infinity.

Proof. For any φ, N_0 gives us $\exists u Scm^{\mathsf{S}}(u)$, which means that there is a set u that is identical to V_α for some alpha, so $\exists \alpha Scm^{\mathsf{S}}(V_\alpha)$. We don't know the exact size of this α, but we know that $\alpha \geq \omega$, otherwise α would be finite, therefore not closed under the powerset operation, which would contradict *Powerset*. In order to prove that it is a model of S, we would need to verify all axioms of S. We have already shown that ω is closed under the powerset operation. Foundation, extensionality and comprehension are clear from the fact that we work in ZF[8], pairing is clear from the fact, that given two sets x, y, they have ranks α, β, without loss of generality we can assume that

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[8]We only need to verify axioms that provide means of constructing larger sets from smaller to make sure they don't exceed ω. Since ω is an initial segment of ZF, the axiom scheme of specification can't be broken, the same holds for foundation and extensionality.

304 $\alpha \le \beta$, which means that $x \in V_\alpha \in V_\beta$, therefore V_β is a set that satisfies the
305 paring axiom: it contains both x and B.
306   Note that this implies that any (strong) limit cardinal is a model of S.
307   We now want to prove that V_α leads to existence of an inductive set,
308 which is a set that satisfies $\exists A(\emptyset \in A \,\&\, \forall x \in A \,((x \cup \{x\}) \in A))$. If we can
309 find a way to construct V_ω from any V_α satisfying $\alpha \ge \omega$, we are done. Since
310 ω is the least limit ordinal, all we need is the following

$$\bigcap \{V_\kappa \mid \forall \lambda(\lambda < \kappa \to \exists \mu(\lambda < \mu < \kappa))\} \tag{2.16}$$

311 because V_κ is a transitive set for every κ, thus the intersection is non-empty
312 unless empty set satisfies the property or the set of V_κs is itself empty. $\square$
313
314   Let N_0 be defined as in **??**, for *Replacement* see **??**.

315 **Theorem 2.9** *In* S*, the schema* N_0 *implies* Replacement.

316 *Proof.* Let $\varphi(v, w, x_1, \ldots, x_n)$ be a formula wth no free variables except
317 $v, w, x_1, \ldots, x_n$ where n is any natural number. Let χ be an instance of
318 replacement schema for this φ which is what we want to prove:

$$\begin{aligned} \chi = \forall r, s, t(\varphi(r, s, x_1, \ldots, x_n) \,\&\, \varphi(r, t, x_1, \ldots, x_n) \to s = t) \\ \to \forall x \exists y \forall w(w \in y \leftrightarrow \exists v(v \in x \,\&\, \varphi(v, w, x_1, \ldots, x_n))) \end{aligned} \tag{2.17}$$

319   We can deduce the following from N_0:
320   (i) $x_1, \ldots, x_n, v, w \in u \to (\varphi \leftrightarrow \varphi^u)$
321   (ii) $x_1, \ldots, x_n, v \in u \to (\exists w \varphi \leftrightarrow (\exists w \varphi)^u)$
322   (iii) $x_1, \ldots, x_n, x \in u \to (\chi \leftrightarrow \chi^u)$
323   (iv) $\forall x_1, \ldots, x_n \forall x(\chi \leftrightarrow (\forall x_1, \ldots, x_n \forall x \chi)^u)$
324   It is easy to see that (**i**), (**ii**), (**iii**) are the instances of N_0 for φ, $\exists w \varphi$ and
325 χ respectively. From relativization we also know that $(\exists w \varphi)^u$ is equivalent
326 to $\exists w(w \in u \,\&\, \varphi^u)$. Therefore (**ii**) is equivalent to

$$x_1, \ldots, x_n, v \in u \to (\exists w(w \in u \,\&\, \varphi^u)). \tag{2.18}$$

327   If φ is a function[9], then for every $x \in u$, which is also $x \subset u$ by the
328 transitivity of $Scm^\mathsf{S}(u)$, it maps elements of x onto u. From the axiom scheme
329 of comprehension[10], we can find y, a set of all images of elements of x. That
330 gives us $x_1, \ldots, x_n, x \in u \to \chi$. By (iii) we get $x_1, \ldots, x_n, x \in u \to \chi^u$, the
331 universal closure of this formula is $(\forall x_1, \ldots, x_n \forall x \chi)^u$, which together with

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[9] $\forall r, s, t(\varphi(r, s) \,\&\, \varphi(r, t) \to r = t)$
[10] Lévy's uses its equivalent, axiom of subsets

332 (iv) yields $\forall x_1, \ldots, x_n \forall x \chi$. By the means of specification we end up with χ,
333 Q.E.D. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \square$

334   What we have just proven is just a single theorem from said article, we
335 will introduce other interesting propositions, mostly related to the existence
336 of large cardinals, later in their appropriate context in chapter 3.

337 2.4 Contemporary restatement

338 We will now prove what is also Lévy's reflection theorem, but a little stronger,
339 rephrased with more up to date set theory. The main difference is, that while
340 Lévy reflects φ from V into a set u that is a "standard complete model of
341 S"[11], we say that there is a V_α that reflects φ.

342   We will prove the equivalence of $Reflection_1$ with *Replacement* and *In-
343 finity* in two parts. First, we will show that $Reflection_1$ is a theorem of
344 ZF, then the second implication which proves *Infinity* and *Replacement* from
345 $Reflection_1$ in S.

346   The following lemma is usually done in more parts, the first being with one
347 formula and the other with n. We will only state and prove the generalised
348 version for n formulas, knowing that $n = 1$ is just a specific case and the
349 proof is exactly the same.

350 **Lemma 2.10**   *Let $\varphi_1, \ldots, \varphi_n$ be formulas with m parameters[12].*
351   *(i) For each set M_0 there is such M that $M_0 \subset M$ and the following holds*
352   *for every $i \le n$:*

$$\exists x \varphi_i(u_1, \ldots, u_{m-1}, x) \to (\exists x \in M)\varphi_i(u_1, \ldots, u_{m-1}, x) \qquad (2.19)$$

353   *for every $u_1, \ldots, u_{m-1} \in M$.*
354   *(ii) Furthermore there is an ordinal α such that $M_0 \subset V_\alpha$ and the following*
355   *holds for each $i \le n$:*

$$\exists x \varphi_i(u_1, \ldots, u_{m-1}, x) \to (\exists x \in V_\alpha)\varphi_i(u_1, \ldots, u_{m-1}, x) \qquad (2.20)$$

356   *for every $u_1, \ldots, u_{m-1} \in M$.*

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[11] Any limit ordinal is in fact a model of S, we shall pay more attention to that in a moment.

[12] For formulas with a different number of parameters, take for m the highest number of parameters among given formulas. Add spare parameters to the other formulas so that x remains the last parameter. That can be done in a following manner: Let φ_i' be the a formula with k parameters, $k < m$. Let us set $\varphi_i(u_1, \ldots, u_{m-1}, x) = \varphi_i'(u_1, \ldots, u_{k-1}, u_k, \ldots, u_{m-1}, x)$, notice that $u_k, \ldots, u_{m-1}$ are the aforementioned spare variables.

357 *(iii) Assuming* Choice, *there is* M, $M_0 \subset M$ *such that* **??** *holds for every*
358   M, $i \le n$ *and* $|M| \le |M_0| \cdot \aleph_0$.

359 *Proof.* We will simultaneously prove statements (**i**) and (**ii**), denoting M^T
360 the transitive set required by part (**ii**). Unless explicitly stated otherwise for
361 specific steps, it is thought to be equivalent to M.
362   Let us first define operation $H(u_1, \ldots, u_{m-1})$ that gives us the set of
363 x's with minimal rank satisfying $\varphi_i(u_1, \ldots, u_{m-1}, x)$ for given parameters
364 $u_1, \ldots, u_{m-1}$ for every $i \le n$.

$$H_i(u_1, \ldots, u_n) = \{x \in C_i : (\forall z \in C)(rank(x) \le rank(z))\} \qquad (2.21)$$

365 for each $i \le n$, where

$$C_i = \{x : \varphi_i(u_1, \ldots, u_{m-1}, x)\} \text{ for } i \le n \qquad (2.22)$$

366
367   Next, let's construct M from given M_0 by induction.

$$M_{i+1} = M_i \cup \bigcup_{j=0}^{n} \bigcup \{H_j(u_1, \ldots, u_{m-1}) : u_1, \ldots, u_{m-1} \in M_i\} \qquad (2.23)$$

368 In other words, in each step we add the elements satisfying $\varphi(u_1, \ldots, u_{m-1}, x)$
369 for all parameters that were either available earlier or were added in the
370 previous step. For statement (**ii**), this is the only part that differs from (i).
371 Let us take for each step transitive closure of M_{i+1} from (i). In other words,
372 let γ be the smallest ordinal such that

$$(M_i^T \cup \bigcup_{j=0}^{n} \{\bigcup \{H_j(u_1, \ldots, u_{m-1}) : u_1, \ldots, u_{m-1} \in M_i\}\}) \subset V_\gamma \qquad (2.24)$$

373 Then the incremetal step is like so:

$$M_{i+1}^T = V_\gamma \qquad (2.25)$$

374 The final M is obtained by joining all incremental steps together.

$$M = \bigcup_{i=0}^{\infty} M_i, \ M^T = \bigcup_{i=0}^{\infty} M_i^T \qquad (2.26)$$

375
376   We have yet to finish part (**iii**). Let's try to construct a set M' that
377 satisfies the same conditions like M but is kept as small as possible. Assuming

15

the Axiom of Choice, we can modify the process so that cardinality of M' is at most $|M_0| \cdot \aleph_0$. Note that the size of M' is determined by the size of M_0 an, most importantly, by the size of $H_i(u_1, \ldots, u_{m-1})$ for any $i \leq n$ in individual levels of the construction. Since the lemma only states existence of some x that satisfies $\varphi_i(u_1, \ldots, u_{m-1}, x)$ for any $i \leq n$, we only need to add one x for every set of parameters but $H_i(u_1, \ldots, u_{m-1})$ can be arbitrarily large. Since Axiom of Choice ensures that there is a choice function, let F be a choice function on $\mathscr{P}(()M')$. Also let $h_i(u_1, \ldots, u_{m-1}) = F(H_i(u_1, \ldots, u_{m-1}))$ for $i \leq n$, which means that h is a function that outputs an x that satisfies $\varphi_i(u_1, \ldots, u_{m-1}, x)$ for $i \leq n$ and has minimal rank among all such witnesses. The induction step needs to be redefined to

$$M'_{i+1} = M'_i \cup \bigcup_{j=0}^{n} \{H_j(u_1, \ldots, u_{m-1}) : u_1, \ldots, u_{m-1} \in M'_i\} \tag{2.27}$$

In every step, the amount of elements added in M'_{i+1} is equivalent to the amount of sets of parameters the yielded elements not included in M'_i. So the cardinality of M'_{i+1} exceeds the cardinality of M'_i only for finite M'_i. It is easy to see that if M_0 is finite, M' is countable because it was built from countable union of finite sets. If M_0 is countable or larger, cardinaly of M' is equal to the cardinality of M_0.[13] Therefore $|M'| \leq |M_0| \cdot \aleph_0$ □

And now for the theorem itself

Theorem 2.11 *First-order Reflection*
Let $\varphi(x_1, \ldots, x_n)$ be a first-order formula.
  (i) For every set M_0 there exists M such that $M_0 \subset M$ and the following holds:

$$\varphi^M(x_1, \ldots, x_n) \leftrightarrow \varphi(x_1, \ldots, x_n) \tag{2.28}$$

      for every $x_1, \ldots, x_n$.
  (ii) For every set M_0 there is a transitive set M, $M_0 \subset M$ such that the following holds:

$$\varphi^M(x_1, \ldots, x_n) \leftrightarrow \varphi(x_1, \ldots, x_n) \tag{2.29}$$

      for every $x_1, \ldots, x_n$.
  (iii) For every set M_0 there is α such that $M_0 \subset V_\alpha$ and the following holds:

$$\varphi^{V_\alpha}(x_1, \ldots, x_n) \leftrightarrow \varphi(x_1, \ldots, x_n) \tag{2.30}$$

      for every $x_1, \ldots, x_n$.

---

[13]It can not be smaller because $|M'_{i+1}| \geq |M'_i|$ for every i. It may not be significantly larger because the maximum of elements added is the number of n-tuples in M'_i, which is of the same cardinality is M'_i.

(iv) *Assuming the Axiom of Choice, for every set M_0 there is M such that $M_0 \subset M$ and $|M| \leq |M_0| \cdot \aleph_0$ and the following holds:*

$$\varphi^M(x_1, \ldots, x_n) \leftrightarrow \varphi(x_1, \ldots, x_n) \tag{2.31}$$

for every $x_1, \ldots, x_n$.

Proof. Let's prove (i) for one formula φ via induction by complexity first. We can safely assume that φ contains no quantifiers besides $\exists$ and no logical connectives other than $\neg$ and $\&$. Assume that this M is obtained from lemma **??**. The fact, that atomic formulas are reflected in every M comes directly from definition of relativization and the fact that they contain no quantifiers.[14] The same holds for formulas in the form of $\varphi = \neg\varphi'$. Let us recall the definition of relativization for those formulas in **??**.

$$(\neg\varphi_1)^M \leftrightarrow \neg(\varphi_1^M) \tag{2.32}$$

Because we can assume from induction that $\varphi'^M \leftrightarrow \varphi'$, the following holds:

$$(\neg\varphi')^M \leftrightarrow \neg(\varphi'^M) \leftrightarrow \neg\varphi' \tag{2.33}$$

The same holds for $\varphi = \varphi_1 \& \varphi_2$. From the induction hypothesis we know that $\varphi_1^M \leftrightarrow \varphi_1$ and $\varphi_2^M \leftrightarrow \varphi_2$, which together with relativization for formulas in the form of $\varphi_1 \& \varphi_2$ gives us

$$(\varphi_1 \& \varphi_2)^M \leftrightarrow \varphi_1^M \& \varphi_2^M \leftrightarrow \varphi_1 \& \varphi_2 \tag{2.34}$$

Let's now examine the case when from the induction hypethesis, M reflects $\varphi'(u_1, \ldots, u_n, x)$ and we are interested in $\varphi = \exists x \varphi'(u_1, \ldots, u_n, x)$. The induction hypothesis tells us that

$$\varphi'^M(u_1, \ldots, u_n, x) \leftrightarrow \varphi'(u_1, \ldots, u_n, x) \tag{2.35}$$

so, together with above lemma **??**, the following holds:

$$\begin{aligned} &\varphi(u_1, \ldots, u_n, x) \\ \leftrightarrow\ &\exists x \varphi'(u_1, \ldots, u_n, x) \\ \leftrightarrow\ &(\exists x \in M)\varphi'(u_1, \ldots, u_n, x) \\ \leftrightarrow\ &(\exists x \in M)\varphi'^M(u_1, \ldots, u_n, x) \\ \leftrightarrow\ &(\exists x \varphi'(u_1, \ldots, u_n, x))^M \\ \leftrightarrow\ &\varphi^M(u_1, \ldots, u_n, x) \end{aligned} \tag{2.36}$$

---

[14]Note that this does not hold generally for relativizations to M, E, but only for relativization to $M, \in$, which is our case.

425 Which is what we have needed to prove:

426

427 So far we have proven part (i) of this theorem for one formula φ, we only
428 need to verify that the same holds for any finite number of formulas. This
429 has in fact been already done since lemma **??** gives us M for any (finite)
430 amount of formulas. We can than use the induction above to verify that it
431 reflects each of the formulas individually.

432

433 Now we want to verify other parts of our theorem. Since V_α is a transitive
434 set, by proving (iii) we also satisfy (**ii**). To do so, we only need to look at
435 part (**ii**) of lemma **??**. All of the above proof also holds for $M = V_\alpha$.
436 To finish part (iv), we take M of size $\leq |M_0| \cdot \aleph_0$, which exists due to part
437 (iii) of lemma **??**, the rest being identical. $\square$

438

439 **Theorem 2.12** Reflection *is equivalent to* Infinity & Replacement *under*
440 *ZFC minus* Infinity & Replacement

441

442 *Proof.* Since **??** already gives one side of the implication, we are only inter-
443 ested in showing the converse which we shall do in two parts:
444   *Reflection* $\rightarrow$ *Infinity*
445 Let us first find a formula to be reflected that requires a set M at least
446 as large as V_ω. Let us consider the following formula:

$$\varphi'(x) = \forall \lambda(\lambda < x \rightarrow \exists \mu(\lambda < \mu < x)) \tag{2.37}$$

447 Because φ says "there is a limit ordinal", if it holds for some x, the Infinity
448 axiom is very easy to satisfy. But we know that there are limit ordinals in
449 ZF, therefore $\varphi = \exists x \varphi'(x)$ is a valid statement. *Reflection* then gives us a set
450 M in which φ^M holds, that is, a set that contains a limit ordinal. So the set
451 of off limit ordinals is non-empty and because ordinals are well-founded, it
452 has a minimal element. Let's call it μ.

$$\mu = \bigcap \{V_\kappa \ : \ \forall \lambda(\lambda < \kappa \rightarrow \exists \mu(\lambda < \mu < \kappa))\} \tag{2.38}$$

453 We can see that μ is the least limit ordinal and therefore it satisfies *Infinity*.

454

455   *Reflection* $\rightarrow$ *Replacement*
456 Given a formula $\varphi(x, y, u_1, \ldots, u_n)$, we can suppose that it is reflected in
457 any M [15] What we want to obtain is the following:

$$\forall x, y, z(\varphi(x, y, u_1, \ldots, u_n) \& \varphi(x, z, u_1, \ldots, u_n) \rightarrow y = z) \rightarrow \tag{2.39}$$

---

[15]Which means that for $x, y, u_1, \ldots, u_n \in M$, $\varphi^M(x, y, u_1, \ldots, u_n) \leftrightarrow \varphi(x, y, u_1, \ldots, u_n)$.

$$\rightarrow \forall X \exists Y \forall y \ (y \in Y \leftrightarrow \exists x(\varphi(x, y, u_1, \dots, u_n) \ \& \ x \in X)) \qquad (2.40)$$

We do also know that $x, y \in M$, in other words for every X, $Y =$ $\{y \mid \varphi(x, y, u_1, \dots, u_n)\}$ we know that $X \subset M$ and $Y \subset M$, which, together with the comprehension schema[16] implies that Y, the image of X over φ, is a set. Which is exactly the Replacement Schema we hoped to obtain. $\qquad \square$

We have shown that *Reflection* for first-order formulas, *Reflection*$_1$ is a theorem of ZF, which means that it won't yield us any large cardinals. We have also shown that it can be used instead of the Axiom of Infinity and Replacement Scheme, but ZF + *Reflection*$_1$ is a conservative extension of ZF. Besides being a starting point for more general and powerful statements, it can be used to show that ZF is not finitely axiomatizable. That is because *Reflection* gives a model to any finite number of (consistent) formulas. So if $\varphi_1, \dots, \varphi_n$ for any finite n would be the axioms of ZF, *Reflection* would always contain a model of itself, which would in turn contradict the Second Gödel's Theorem[17]. Notice that, in a way, reflection is complementary to compactness. Compactness argues that given a set of sentences, if every finite subset yields a model, so does the whole set. Reflection, on the other hand, says that while the whole set has no model in the underlying theory, every finite subset does have one.

Also, notice how reflection can be used in ways similar to upward Löwenheim–Skolem theorem. Since Reflection extends any set M_0 into a model of given formulas $\varphi_1, \dots, \varphi_n$, we can choose the lower bound of the size of M by appropriately chocing M_0.

In the next section, we will try to generalize *Reflection* in a way that transcends ZF and finally yields some large cardinals.

---

[16]Called the axiom of subsets in Levy's proof.
[17]See chapter **??** for further details.

3 Reflecting To Large Cardinals

In this chapter we aim to examine stronger reflection properties in order to reach cardinals unavailable in ZF. Like we said in the first chapter, the variety of reflection principles comes from the fact that there are many way to formalize "properties of the universal class". It is not always obvious what properties hold for V because, unlike Lévy's approach, not much attention is paid to what exactly is this V, and, more importantly, there are many ways to formalize the notion of property. We have shown that reflecting properties as first-order formulas doesn't allow us to leave ZF. We will broaden the class of admissible properties to be reflected and see whether there is a natural limit in the height or width on the reflected universe and also see that no matter how far we go, the universal class is still as elusive as it is when seen from S. That is because for every process for obtaining larger sets such as for example the powerset operation in ZF, this process can't reach V and thus, from reflection, there is an initial segment of V that can't be reached via said process.

To see why this is important, let's dedicate a few lines to the intuition behind the notions of limitness, regularity and inaccessibility in a manner strongly influenced by [?]. To see why limit and strongly limit cardinals are worth mentioning, note that they are "limit" not only in a sense of being a supremum of an ordinal sequence, they also show that a certain way of obtaining bigger objects from smaller ones limited in terms of possibilities. $\aleph_\lambda$ is a limit cardinal iff there is no α such that $\aleph_{\alpha+1} = \aleph_\lambda$. Strongly limit cardinals point to the limits of the powerset operation. It has been too obvious so far, so let's look at the regular cardinals in this manner. Regular cardinals are those that cannot be[18], expressed as a supremum of smaller amount of smaller objects[19]. More precisely, κ is regular if there is no way to define it as u union of less than κ ordinals, all smaller than κ. So unless there already is a set of size κ, *Replacement* is useless in determining whether κ is really a set. Note that assuming *Choice*, successor cardinals are always regular, so most[20] limit cardinals are singular cardinals. So if one is traversing the class of all cardinals upwards, successor steps are still sets thanks to the powerset axiom while singular limits cardinal are not proper classes because they are images of smaller sets via *Replacement*. Regular cardinals are, in a way, limits of how far can we get by taking limits of increasing sequences of ordinals obtained via *Replacement*.

---

[18]Assuming *Choice*.

[19]Just like ω can not be expressed as a supremum of a finite set consisting solely of finite numbers.

[20]All provable to exist in ZF

520   That all being said, it is easy to see that no cardinals in ZF are both
521 strongly limit and regular beucase there is no way in ZF to ensure they are
522 sets and not proper classes. The only exception to this rule is $\aleph_0$ which need
523 a special axiom for itself to exist. It should now be obvious why the fact that
524 κ is inaccessible implies that $\kappa = \aleph_\kappa$.[21]
525   We will also examine the connection between reflection principles and
526 fixed points of ordinal functions in a manner proposed by Lévy in [?]. We will
527 also see that, like Lévy [?] has proposed, there is a meaningful way to extend
528 the relation between S and ZF into a hierarchy of axiomatic set theories.
529 Those are the three lines of thinking that we will find are in fact different
530 facets of the same gem, especially in the section devoted to Inaccessible and
531 Mahlo cardinals.

532 3.1   Fixed-point phenomena and axioms

533 This small chapter is dedicated to
534   Lévy's article mentions various schemata that are not instances of reflec-
535 tion themselves. We will mention them because they are equivalent to N_0
536 and because they are fixed-point theorems, which we will find useful later in
537 this thesis.

538 **Definition 3.1** *Strictly increasing function*
539 *A function $F(\alpha)$ defined on the ordinal number into the ordinal numbers is*
540 *said to be* strictly increasing *if $\forall \alpha, \beta \in On(\alpha < \beta \rightarrow F(\alpha) < F(\beta))$.*

541 **Definition 3.2** *Continuous function*
542 *A function $F(\alpha)$ defined on the ordinal number into the ordinal numbers is*
543 *said to be* continuous *if for any limit α, $F(\alpha) = lim_{\beta < \ \alpha} F(\beta)$.*

544 Alternatively, a function F is continuous iff for limit *lambda*, $F(\lambda) = supF(\alpha) : \alpha < \lambda$.

545 **Definition 3.3** *Normal function*
546 *A function $F(\alpha)$ defined on the ordinal numbers into the ordinal numbers is*
547 *said to be* normal *if it is* strictly increasing *and* continuous

548 **Definition 3.4** *Normal function on a set Let α be an ordinal. A function*
549 *$f : \delta \rightarrow \alpha$ is a normal function on α if it is increasing, continuous and its*
550 *range is unbounded in α.*

---

[21]This doesn't work backwards, the first fixed point of the $\aleph$ function is the limit of
$\{\aleph_0, \ \aleph_{\aleph_0}, \ \aleph_{\aleph_{\aleph_0}}, \ \ldots\}$ is singular since the sequence has countably many elements.

Definition 3.5 *Fixed point*
We say α is a fixed point of ordinal function f when $\alpha = f(\alpha)$.

Lévy ([?]) proposes those axioms as equivalent to one on his reflection principles.

Definition 3.6 *M Every normal function defined for all ordinals has at least one inaccessible number in its range.*

Definition 3.7 *M' Every normal function defined for all ordinals has at least one fixed point which is inaccessible.*

Definition 3.8 *M'' Every normal function defined for all ordinals has arbitrarily great fixed points which are inaccessible.*

The following axiom is proposed by Drake in [?].

Definition 3.9 *F Every normal function for all ordinals has a regular fixed point.*

Theorem 3.10
$$F \leftrightarrow M \leftrightarrow M' \leftrightarrow M'' \tag{3.41}$$

Proof. One can find the proof of $M \leftrightarrow M' \leftrightarrow M''$ in [?], *Theorem 1.*

□

3.2 Reflecting Second-order Formulas

To see that there is a way to transcend ZF, let us briefly show how a model of ZF can be obtained in $ZF_2 +$ "second-order reflection"[22]. This will be more closely examined in section ??.

We know that ZF can not be finitely axiomatized in first-order formulas, however if Replacement and Comprehension schemes can be substituted by second-order formulas, ZF becomes ZF_2, which is finitely axiomatizable in second-order logic. Therefore if we take second-order reflection into consideration, we can obtain a set M that is a model of ZF_2. For now, we have left out the details of how exactly is first-order reflection generalised into stronger statements and how second-order axiomatization of ZF looks like as we will examine those problems closely in the following pages.

Lower-case letters represent first-order variables and upper-case P represents a second-order variable. [?]

---

[22]ZF_2 is an axiomatization of ZF in second-order formulas, to be more rigorously established later.

Definition 3.11 Replacement$_2$

$$\forall P(\forall x, y, z(P(x,y)\&P(x,z) \rightarrow y = z) \rightarrow \\ \rightarrow (\forall x \exists y \forall z(z \in y \leftrightarrow \exists w \in x(P(w,z))))) \tag{3.42}$$

We will denote this axiom Replacement$_2$.

Definition 3.12 Specification$_2$

$$\forall P \forall x \exists y \forall z \, (z \in y \leftrightarrow [z \in x \& P(z,x)]) \tag{3.43}$$

Definition 3.13 ZF$_2$
Let ZF$_2$ *be a theory with all axioms identical with the axioms of* ZF *with the exception of* Replacement *and* Specification *schemes, which are replaced with* Replacement$_2$ *and* Specification$_2$ *respectively.*

3.3 Inaccessibility

Definition 3.14 *(limit cardinal) kappa is a* limit cardinal *if it is* $\aleph_\alpha$ *for some limit ordinal* α.

Definition 3.15 *(strong limit cardinal) kappa is a* strong limit cardinal *if for every* $\lambda < \kappa$, $2^\lambda < \kappa$

The two above definition become equivalent when we assume GCH.

Definition 3.16 *(weak inaccessibility) An uncountable cardinal* κ *is* weakly inaccessible $\leftrightarrow$ *it is* regular *and* limit.

Definition 3.17 *(inaccessibility) An uncountable cardinal* κ *is* inaccessible *(written* $In(\alpha)$*)* $\leftrightarrow$ *it is* regular *and* strongly limit.

We will now show that the above notion is equivalent to the definition Levy uses in [**?**], which is, in more contemporary notation, the following:

Theorem 3.18 *The following are equivalent:*
    1. κ in inaccessible
    2. $\langle V_\kappa, \in \rangle \models$ ZFC

Proof. Let's first prove that if κ is inaccessible, it is a model of ZFC. We will do that by verifying the axioms of ZFC just like Kanamori does it in in [**?**, 1.2] and Drake in [**?**, Chapter 4].

23

(i) *Extensionality*:

$$V_\kappa \models \forall x, y(\forall z(z \in x \leftrightarrow z \in y) \rightarrow x = y) \qquad (3.44)$$

We need to prove that, given two sets that are equal in V, they are equal in V_κ, in other words, that the *Extensionality* formula is reflected, that is

$$V_\kappa \models \forall x, y \in V_\kappa(\forall z \in V_\kappa(z \in x \leftrightarrow z \in y) \rightarrow x = y) \qquad (3.45)$$

But that comes from transitivity. If x and y are in V_κ their members are also in V_κ.

(ii) *Foundation*:

$$V_\kappa \models \forall x(\exists z(z \in x) \rightarrow \exists z(z \in x \ \& \ \forall u \neg(u \in z \ \& \ u \in x))) \qquad (3.46)$$

The argument for *Foundation* is almost identical to the one for *Extensionality*. For any set $x \in V_\kappa$, transitivity of V_κ makes sure that every element of x is also an element of V_κ and the same holds for the elements of elements of x et cetera. So statements about those elements are absolute between any transitive structures. V and V_κ are both transitive therefore *Foundation* holds and so does its relativisation to V_κ, $Foundation^{V_\kappa}$.

(iii) *Powerset*:

$$V_\kappa \models \forall x \exists y \forall z(z \subseteq x \rightarrow z \in y). \qquad (3.47)$$

If we take x, an element of V_κ, $\mathscr{P}(()x)$ has to be an element of V_κ to, because it is transitive and a strong limit cardinal.

(iv) *Pairing*:

$$V_\kappa \models \forall x, y \exists z(x \in z \wedge y \in z). \qquad (3.48)$$

Pairing holds from similar argument like above: let x and y be elements of V_κ, so there are ordinals $\alpha, \beta < \kappa$ such that $x \in V_\alpha$, $y \in V_\beta$. Without any loss of generality, suppose $\alpha < \beta$, threfore $V_\alpha \subset V_\beta$ which, from transitivity of the cumulative hierarchy, means that $x \in V_\beta$, then $\{x, y\} \in V_{\beta+1}$ which is still in V_κ because it is a strong limit cardinal.

(v) *Union*

$$V_\kappa \models \forall x \, \exists y \, \forall z \, \forall w((w \in z \wedge z \in x) \rightarrow w \in y). \qquad (3.49)$$

We want to see that for every $x \in V_\kappa$, this is equivalent to

$$V_\kappa \models \forall x \in V_\kappa, \exists y \in V_\kappa \, \forall z \in V_\kappa \, \forall w \in V_\kappa((w \in z \wedge z \in x) \rightarrow w \in y).$$
$$(3.50)$$

Since V_κ is transitive, if $x \in V_\kappa$, all of its elements as well as their elements are in V_κ. To see that they also form a set themselves we only need to remember that V_κ is limit and therefore if α is the least ordinal such that $x \in V_\alpha$, $\bigcup x \in V_{\alpha+1}$.

(vi) *Replacement, Infinity* We know that those hold from **??**.

We will now show that if a set is a model of ZFC, it is in fact an inaccessible cardinal. So let V_κ be a model of ZFC which means that it is closed under the powerset operation, in other words:

$$\forall \lambda (\lambda < \kappa \to 2^\lambda < \kappa) \tag{3.51}$$

which is exactly the definition of strong limitness. κ is regular from the following argument by contradiction:
Let us suppose for a moment that κ is singular. Therefore there is an ordinal $\alpha < \kappa$ and a function $F : \alpha \to \kappa$ such that the range of F in unbounded in κ, in other words, $F[\alpha] \subseteq V_\kappa$ and $sup(F[\alpha]) = \kappa$. In order to achieve the desired contradiction, we need to see that it is the case that $F[\alpha] \in V_\kappa$. Let $\varphi(x, y)$ be the following first-order formula:

$$F(x) = y \tag{3.52}$$

Then there is an instance of Axiom Schema of Replacement that states the following:

$$\begin{aligned}(\forall x, y, z(\varphi(x,y)\&\varphi(x,z) \to y = z)) \to \\ \to (\forall x \exists y \forall z (z \in y \leftrightarrow \exists w(\varphi(w, z))))\end{aligned} \tag{3.53}$$

Which in turn means that there is a set $y = F[\alpha]$ and $y \in V_\kappa$, which is the contradiction with $sup(y) = \kappa$ we are looking for. $\square$

The same holds for ZF_2, the proof is very similar.

Theorem 3.19
$$V_\kappa \models \mathsf{ZF}_2 \leftrightarrow \kappa \text{ is inaccessible} \tag{3.54}$$

Proof. κ is a strong limit cardinal because from ZF_2 and *Powerset* we know that for every $\lambda < \kappa$, we know that $2^\lambda < \kappa$.

κ is also regular, because otherwise there would be an ordinal α and a function $F : \alpha \to \kappa$ with a range unbounded in κ. *Replacement*[2] gives us a set $y = F[\alpha]$, so $y \in V_\kappa$, which contradicts the fact that $sup(y) = \kappa$. It can not be the case that $\kappa \in V_\kappa$.

666

667   The other direction is exactly like the first part of above theorem **??**. □

668

669   This is how the existence of an inaccessible cardinal is established in [**?**].

670 **Definition 3.20** N

671

$$\exists u(In(\alpha) \ \& \ \forall x_1, \ldots, x_n(x_1, \ldots, x_n \in u \to (\varphi \leftrightarrow \varphi^u))) \tag{3.55}$$

672 It is interesting to see that the above schema yields the first inaccessible
673 cardinal if we take for φ the conjunction of all axioms of ZF_2.

674

675   To see that inaccessible cardinal can be also obtained by a fixed-point
676 axiom (or a scheme if were in first-order logic), see the following theorem by
677 Lévy, we won't repeat the proof here, it is available in [**?**, Theorem 3],

Theorem 3.21
$$M \leftrightarrow N \tag{3.56}$$

678   We have transcended ZF, but that is just a start. Naturally, we could
679 go on and consider the next inaccessible cardinal, which is inaccessible with
680 respect to the theory $\mathsf{ZF} + \exists \kappa(\kappa \models \mathsf{ZF})$. But let's try to find a faster way up,
681 informally at first.
682   Since we can find an inaccessible set larger than any chosen set M_0, it
683 is clear that there are arbitrarily large inaccessible cardinals in V, they are
684 "unbounded"[23] in V. If V were a cardinal, we could say that there are V
685 inaccesible cardinals less than V, but this statement of course makes no sense
686 in set theory as is because V is not a set. But being more careful, we could
687 find a property that can be formalized in second-order logic and reflect it to
688 an initial segment of V. That would allow us to construct large cardinals
689 more efficiently than by adding inaccessibles one by one. The property we
690 are looking for ought to look like something like this:

$$\kappa \text{ is an inaccessible cardinal and}$$
$$\text{there are } \kappa \text{ inaccessible cardinals } \mu \ < \ \kappa \tag{3.57}$$

691 This is in fact a fixed-point type of statement. We shall call those cardinals
692 hyper-inaccessible. Now consider the following definition.

693

694 **Definition 3.22** 0-*inaccessible cardinal*
695 *A cardinal κ is* 0-*inaccessible if it is inaccessible.*

---

[23]The notion is formaly defined for sets, but the meaning should be obvious.

We can define α-*weakly-inaccessible* cardinals analogously with the only difference that those are limit, not strongly limit.

Definition 3.23 α-*hyper-inaccessible cardinal*
For any ordinal α, κ is called α-inaccessible, if κ is inaccessible and for each β ¡ α, the set of β-inaccessible cardinals less than κ is unbounded in κ.

Because κ is inaccessible and therefore regular, the number of β-inaccessibles below κ is equal to κ. We have therefore successfully formalized the above vague notion of hyper-inaccessible cardinal into a hierarchy of α-inaccessibles.

Let's now consider iterating this process over again. Since, informally, V would be α-inaccessible for any α, this property of the universal class could possibly be reflected to an initial segment, the smallest of those will be the first hyper-inaccessible cardinal. Such κ is larger than any α-inaccessible since from regularity of κ, for given $\alpha < \kappa$, κ is κ-th α-hyper-inaccessible cardinal. It is in fact "inaccessible" via α-inaccessibility.

Definition 3.24 *Hyper-inaccessible cardinal*
κ *is called the hyper-inaccessible, also 0-hyper-inaccessible, cardinal if it is α-inaccessible for every $\alpha < \kappa$.*

Definition 3.25 α-*hyper-inaccessible cardinal*
For any ordinal α, κ is called α-hyper-inaccessible cardinal if for each ordinal $\beta < \alpha$, the set of β-hyper-inaccessible cardinals less the κ is inbounded in κ.

Obviously we could go on and iterate it ad libitum, but the nomenclature would be increasingly confusing. A smarter way to accomplish the same goal is carried out in the following section.

3.4 Mahlo Cardinals

While the previous chapter introduced us to a notion of inaccessibility and the possibility of iterating it ad libitum in new theories, there is an even faster way to travel upwards in the cumulative hierarchy, that was proposed by Paul Mahlo in his papers (see [**?**], [**?**] and [**?**]) at the very beginning of

the 20th century, and which can be easily reformulated using *(Reflection)*. To see how Lévy's initial statement of reflection was influenced by Mahlo's work, refer to section **??**. The aim of the following paragraphs is to give an intuitive explanation of the idea behind Mahlo's hierarchy of cardinals, all claims made here ought to be stated formally later in the very same chapter.

At the very end of section **??**, we have tried to establish the notion of hyper-inaccessibility and iterate it to yield even larger large cardinals. In order to avoid too bulky cardinal names, let's try a different route and establish those cardinals directly via reflection.

The following two definitions come from [**?**] and while they are rather informal, we will find them very helpful for understanding the Mahlo cardinals.

Definition 3.26 *Fixed-point property*
For any $\psi(x, u_1, \ldots, u_n)$ which is any property of ordinals, we say that a property φ is a fixed-point property *if φ has the form*

$$x \text{ is an inaccessible cardinal and}$$
$$\text{there are } x \text{ ordinals less than } x \text{ that have the property } \psi(x, u_1, \ldots, u_n).$$

$$(3.58)$$

Definition 3.27 *Fixed-point reflection*
If φ is a fixed-point property that holds for V, it also holds for some V_α, an initial segment of V.

Obviously those are in on way rigorous definitions because we have no idea what $\psi(x, u_1, \ldots, u_n)$ looks like. Let's try to restate the same idea in a useful way. But first, let's show that the formal counterpart of the idea of containing "enough" ordinals with a property is the notion of stationary set.

Definition 3.28 *Supremum*
Given A a set of ordinals, the supremum of A, denoted $sup(A)$, is the least upper bound of A.

$$sup(A) \;=\; \bigcup A \qquad\qquad (3.59)$$

where α is an ordinal.

Definition 3.29 *Limit point*
Given A, a set of ordinals and an ordinal α, we say that α is a limit point *of A if $sup(A \cap \alpha) \;=\; \alpha$*

Definition 3.30 *Club set*
For a regular uncountable κ, a set $A \subset \kappa$ is a closed unbounded *subset (often abbreviated as a* club set*) iff A is both closed, which means it contains all it's limit points, and unbounded, which means that for every β ¡ κ there is a $\beta' \in \alpha$ such that $\beta < \beta' < \kappa$.*

Definition 3.31 *Stationary set*
For a regular uncountable κ, a set $A \subset \kappa$ is stationary if it intersects every club subset of κ.

Theorem 3.32 *The intersection of fewer than κ^{24} club subsets of κ is a club set.*

For proof, see [**?**, Theorem 8.3]

Definition 3.33 *Weakly Mahlo Cardinal*
κ is weakly Mahlo $\leftrightarrow$ *it is a regular limit ordinal and the set of all regular ordinals less then κ is stationary in κ*

Definition 3.34 *Mahlo Cardinal*
κ is a Mahlo Cardinal *iff it is an inaccessible cardinal and the set of all inaccessible ordinals less then κ is stationary in κ.*

It is interesting to note, that weakly-Mahlo cardinals are fixed points of α-weakly inaccessible cardinals, so if κ is weakly mahlo, .. viz Kanamori Proposition 1.1
    Analogously,

Definition 3.35 *α-Mahlo Cardinal*
κ is a α-Mahlo Cardinal iff it is an α-inaccessible cardinal and the set of all α-inaccessible ordinals less then κ is stationary in κ.

    In other words, κ is a mahlo cardinal if it is inaccessible and every club set in κ contains an inaccessible cardinal. This is exactly the notion of fixed-point reflection we were trying to show earlier.

    [**?**]

Definition 3.36 *The folllowing definitions are equivalent:*

---

[24]κ is again a regular uncountable cardinal and it will always be when we will be talking about club sets.

791   *(i) κ is Mahlo*
792   *(ii) κ is weakly Mahlo and strong limit*
793   *(iii) The set $\{\lambda < \kappa : \lambda$ is inaccessible$\}$ is stationary in κ.*
794   *(iv) Every normal function on κ has an inaccessible fixed point.*

795   *Proof.* (**i**) $\leftrightarrow$ (**ii**) Let κ_1 be a mahlo cardinal and let κ_2 be a strong limit
796   weakly Mahlo cardinal. We know from the definitions that the set $\{\lambda <$
797   $\kappa : \lambda$ is inaccessible$\}$ is stationary in both κ_1 and κ_2, the only difference
798   being that κ_1 is a strongly limit cardinal, but κ_2 would be limit from weak
799   Mahloness, wasn't it for the fact that it is also strong limit. This eliminates
800   the only difference between them and therefore κ_1 is also strong limit weakly
801   Mahlo cardinal and κ_2 is Mahlo.
802

803   (**i**) $\rightarrow$ (**iii**) We know that κ is uncountable, regular, strong limit and that
804   the set $S = \{\lambda < \kappa : \lambda$ is regular$\}$ is stationary in κ. We want to prove
805   that $S' = \{\lambda < \kappa : \lambda$ is inaccessible$\}$ is thus also stationary in κ.
806   Since stationary set intersects every club set in κ, let C be any such set.
807   Let $D = \{\lambda < \kappa : \lambda$ is strong limit$\}$. D is a club set because TODO.
808   Since intersection of less than κ club sets is a club set, $C \cap D \neq \emptyset$.
809   TODO proc $\lambda = S \cap C \cap D$ je inaccessible?
810   (**iii**) $\rightarrow$ (**iv**)
811   TODO jak to dela Levy?
812   (**iv**) $\rightarrow$ (**i**)
813   TODO jak to dela Levy?
814   range kazde normalni funkce je club v On. (nevadi ze On je trida?)
815   co treba lemma ze pevne body tvori taky club set
816   mozna by stacilo ze jsou unbounded, tedy kazda normalni funkce ma
817   libovolne velke pevne body.   □
818

819   TODO obdoba pro α-Mahlo kardinaly?
820   TODO κ is hyper-Mahlo iff κ is inaccessible and the set $\{\lambda < \kappa :$
821   λ is Mahlo$\}$ is stationary in κ.

822 3.5   Indescribality

823

824   TODO indescribable – reflecting indescribability – we can't reach V by a
825   Σ_1^1 formula, so there's some initial segment V_α that is also unreachable (we
826   say indescribable) by the means of a ... formula
827   TODO co je "partition property"?
828   TODO pak dk. ekvivalenci

TODO Kanamori 6.3

Definition 3.37 *A cardinal κ is* weakly compact *if it is uncountable and satisfies the partition property $\kappa \rightarrow (\kappa)^2$*

opsano z jecha!
   TODO definice pres nepopsatelnost, ekvivalence

3.6   Bernays–Gödel Set Theory

TODO Plagiat – prepsat a vysvetlit
   TODO

3.7   Reflection and the constructible universe

TODO reflektovat muzeme jenom kardinaly konzistentni s V=L, proc?
   TODO Plagiat – prepsat a vysvetlit
   L was introduced by Kurt Gödel in 1938 in his paper *The Consistency of the Axiom of Choice and of the Generalised Continuum Hypothesis* and denotes a class of sets built recursively in terms of simpler sets, somewhat similar to Von Neumann universe V. Assertion of their equality, $V = L$, is called the *axiom of constructibility.* The axiom implies GCH and therefore also AC and contradicts the existence of some of the large cardinals, our goal is to decide whether those introduced earlier are among them.
   On order to formally establish this class, we need to formalize the notion of definability first:
   TODO zduvodneni

   TODO kratka diskuse jestli refl implikuje transcendenci na L - polemika, nazor - V=L a slaba kompaktnost a dalsi

   TODO asi nekde bude meritelny kardinal

4 Conclusion

TODO na konec

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