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REFLECTION PRINCIPLES AND LARGE CARDINALS

Bakalářská práce

Vedoucí práce: Mgr. Radek Honzík, Ph.D.

2016

Mé poděkování patří Mgr. Radkovi Honzíkovi Ph.D. za trpělivé vedení při psaní této bakalářské práce.

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V Praze 14. srpna 2016

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Abstract

Práce zkoumá vztah tzv. principů reflexe a velkých kardinálů. Lévy ukázal, že v ZFC platí tzv. věta o reflexi a dokonce, že věta o reflexi je ekvivalentní schématu nahrazení a axiomu nekonečna nad teorií ZFC bez axiomu nekonečna a schématu nahrazení. Tedy lze na větu o reflexi pohlížet jako na svého druhu axiom nekonečna. Práce zkoumá do jaké míry a jakým způsobem lze větu o reflexi zobecnit a jaký to má vliv na existenci tzv. velkých kardinálů. Práce definuje nedosažitelné, Mahlovy a nepopsatelné kardinály a ukáže, jak je lze zavést pomocí reflexe. Přirozenou limitou kardinálů získaných reflexí jsou kardinály nekonzistentní s L. Práce nabídne intuitivní zdůvodněn, proč tomu tak je.

Abstract

This thesis aims to examine the relation between the so called Reflection Principles and Large Cardinals. Lévy has shown that the Reflection Theorem is a sound theorem of ZFC and it is equivalent to the Replacement Schema and the Axiom of Infinity. From this point of view, Reflection theorem can be seen a specific version of an Axiom of Infinity. This paper aims to examine the Reflection Principle and its generalisations with respect to the existence of Large Cardinals. This thesis will establish the Inaccessible, Mahlo and Indescribable cardinals and show how can those be defined via reflection. A natural limit of Large Cardinals obtained via reflection are cardinals inconsistent with L. This thesis will offer an intuitive explanation of why this holds.

Contents

1 Introduction

The central point of this thesis is the so called *reflection principle*, which could be informally expressed like this:

> *For every property that holds in the universe of all sets, there is a set in which this property holds.*

Clearly, this formulation is rather vague and we should be extremely cautious when dealing with the word "property". One problem that immediately comes to mind is that "being the set of all sets" must not be considered a property in this sense, otherwise we run into the well–known paradox of Russell. This is a well–known problem that exemplifies the fact that reflection is a phenomenon that is closely connected to the very foundations of mathematics. This is also emphasised by the fact that the very first explicit use of reflection in a mathematical proof can be found in Gödel's paper *The Consistency of the Axiom of Choice and of the Generalised Continuum Hypothesis with the Axioms of Set Theory*[1] that deals with the consistency of the *generalised continuum hypothesis*, which is a question that played an important part in the development of set theory in the 20^{th} century. Furthermore, Lévy's article *Axiom Schemata of Strong Infinity in Axiomatic Set Theory*, that is a cornerstone of this thesis is concerned primarily with the so called *strong axioms (or axiom schemata) of infinity*, which are axioms or axiom schemata that imply the existence of the set of all natural numbers. This assertion is called the *Axiom of Infinity*[2], but they also imply the existence of larger sets whose existence can not be proved in the current theory[3].

As we will show in chapter 2, reflection is closely related to the *Axiom Schema of Replacement*, which was the subject of philosophical debates because it wasn't included in the original axiomatic set theory proposed by Zermelo and unlike other axioms in the *Zermelo–Fraenkel set theory*, its presence is not justified from the iterative conception of a set, but rather from its usefulness. Unlike *Replacement Schema*, reflection is not so easily questioned from a platonist[4] point of view, but it may be formulated in two different was. The two following informal interpretations of reflections

[1] See [Gödel and Brown, 1940].

[2] For a rigorous definition, see definition **1.10** later in this section.

[3] For the purposes of this thesis, unless stated otherwise, this will be the *Zermelo–Fraenkel set theory*, that is formally established in definition **1.21**.

[4] According to *Stanford Encyclopedia of Philosophy*, "mathematical platonism is the metaphysical view that there are abstract mathematical objects whose existence is independent of us and our language, thought, and practices. Just as electrons and planets exist independently of us, so do numbers and sets. And just as statements about electrons and planets are made true or false by the objects with which they are concerned and

are based on [Hellman, 2010]. Their purpose is to illustrate the difference between a platonist and a structuralist[5] approach towards reflection.

> "The true situation (in the universe of sets) is reflected in arbitrarily high level of the cumulative hierarchy."

> "We're interested in structures so large that certain attempts to describe them fail to distinguish them from various proper initial segments–hence small fragments–of them."

We won't spend more time with the philosophy of mathematics as it is outside the scope of this thesis, it is only worth noting that the author usually thinks of reflection in the latter sense which may be (sic) reflected in the way this thesis is written.

After introducing the elementary theoretical tools required for this task in the rest of this chapter, in chapter 2, we will review the *Reflection Theorem* that originally formulated by Richard Montague in 1961[6] and extended by Azriel Lévy in his aforementioned article and then restate it in a way that is more in line with today's set theory. This part of the thesis deals with the fact that when the term "property" is restricted to first–order formulas in the language of set theory, it does not behave like a axiom of strong infinity, but it is equivalent to the *Axiom of Infinity* and *Replacement Schema*, which is one of the key set–forming principles in the *Zermelo–Fraenkel set theory.*

It is in chapter 3 where will examine some large cardinal properties and in a manner similar to Lévy's article, we will introduce axiom schemata that come from reflection and lead towards *inaccessible* and *Mahlo cardinals.* We will briefly argue that Mahlo's operation exhausts large cardinals reachable via reflection from below and introduce indescribable cardinals, which are also based on reflection, but lead us into higher–order logic. We will introduce

these objects' perfectly objective properties, so are statements about numbers and sets. Mathematical truths are therefore discovered, not invented."

[5]According to wikipedia, "Structuralism is a theory in the philosophy of mathematics that holds that mathematical theories describe structures of mathematical objects. Mathematical objects are exhaustively defined by their place in such structures. Consequently, structuralism maintains that mathematical objects do not possess any intrinsic properties but are defined by their external relations in a system."

[6]Note that Lévy's paper was published in 1960, a year before Montague's, but Lévy refers to Montague and not vice versa. While this may seem confusing, it is because Montague gave a lecture on this topic at a conference at the Cornell University in 1957. It is also interesting that Lévy's article refers for Montague's reflection to a publication by Montague and Solomon Feferman called *The method of arithmetization and some of its applications* which was never finished. This is explained by Solomon Feferman in [Feferman, 2008].

weakly inaccessible cardinals and show that they are also based on reflection and examine their relation to the cardinals presented earlier. Finally, we will examine Gödel's constructible universe and see whether the large cardinals we have introduced are compatible with the *Axiom of Constructibility*, an assertion that every set is definable.

1.1 Notation and Terminology

1.1.1 The Language of Set Theory

This text assumes the knowledge of basic terminology and some results from first–order predicate logic, see any entry–level like [Hamilton, 1988]. For this reason, we won't introduce the notions of *language*, *function symbol*, *predicate*, *term*, *model* and *interpretation* that are used in definition **1.42**.

All proofs are based on [Jech, 2006] unless explicitly stated otherwise. Notable amount of inspiration is also drawn from [Kanamori, 2003] and [Drake, 1974].

We will now shortly review the basic notions that allow us to define the *Zermelo–Fraenkel* set theory.

When we talk about a *class*, we have the notion of a definable class in mind. If $\varphi(x, p_1, \ldots, p_n)$ is a formula in the language of set theory, we call

$$A = \{x : \varphi(x, p_1, \ldots, p_n)\} \tag{1.1}$$

a class of all sets satisfying $\varphi(x, p_1, \ldots, p_n)$ in a sense that

$$x \in A \leftrightarrow \varphi(x, p_1, \ldots, p_n) \tag{1.2}$$

for some $p_1, \ldots, p_n$. Given classes A, B, one can easily define the elementary set operations such as $A \cap B$, $A \cup B$, $A \setminus B$, $\bigcup A$, see the first chapter of [Jech, 2006] for details. Axioms are the tools by which we can decide whether a particular class is "small enough" to be considered a set[7]. A class that fails to be considered a set is called a *proper class*.

We will often write something like "M is a limit ordinal", it should always be clear that this can be rewritten as a formula that was introduced earlier. Tuples are notated as $\langle a, b\rangle$.

[7] "Small enough" means that it doesn't lead to a paradox similar to the famous Russell's paradox.

1.1.2 The Axioms

Definition 1.1 *(The Existence of a Set)*

$$\exists x(x = x) \tag{1.3}$$

Definition 1.2 *(Axiom of Extensionality)*

$$\forall x, y(x = y \leftrightarrow \forall z(z \in x \leftrightarrow z \in y)) \tag{1.4}$$

Definition 1.3 *(Axiom Schema of Specification)*
The following yields an axiom for every first–order formula $\varphi(x, p_1, \ldots, p_n)$ with no free variables other than $x, p_1, \ldots, p_n$.

$$\forall x, p_1, \ldots, p_n \exists y \forall z(z \in y \leftrightarrow z \in x \,\&\, \varphi(z, p_1, \ldots, p_n)) \tag{1.5}$$

We will now provide two definitions that are not axioms, but will be helpful in establishing the next axioms in a more comprehensible way.

Definition 1.4 *($x \subseteq y$, $x \subset y$)*

$$x \subseteq y \leftrightarrow (\forall z \in x) z \in y \tag{1.6}$$

$$x \subset y \leftrightarrow x \subseteq y \,\&\, x \neq y \tag{1.7}$$

We read $x \subseteq y$ as x is a subset of y *and $x \subset y$ as* x is a proper subset of y.

Definition 1.5 *(Empty Set) For an arbitrary set x, the empty set, represented by the symbol "$\emptyset$", is the set defined by the following formula:*

$$(\forall y \in x)(y \in \emptyset \leftrightarrow \neg(y = y)) \tag{1.8}$$

Clearly $\emptyset$ is a set due to Specification Schema*, there is only one such set due to the* Axiom of Extensionality*, no matter which x is chosen.*

Definition 1.6 *(Axiom of Pairing)*

$$\forall x, y \exists z \forall q(q \in z \leftrightarrow q = x \vee q = y) \tag{1.9}$$

Definition 1.7 *(Axiom of Union)*

$$\forall x \exists y \forall z(z \in y \leftrightarrow \exists q(z \in q \,\&\, q \in x)) \tag{1.10}$$

Definition 1.8 *(Axiom of Foundation)*

$$\forall x(x \neq \emptyset \rightarrow (\exists y \in x)(x \cap y = \emptyset)) \tag{1.11}$$

Definition 1.9 *(Axiom of Power Set)*

$$\forall x \exists y \forall z (z \in y \leftrightarrow z \subseteq x) \tag{1.12}$$

Definition 1.10 *(Axiom of Infinity)*

$$\exists x (\emptyset \in x \;\&\; (\forall y \in x)(y \cup \{y\} \in x)) \tag{1.13}$$

The least set satisfying (1.13) is denoted ω*.*

Definition 1.11 *(Function)*
Given an arbitrary first–order formula $\varphi(x, y, p_1, \ldots, p_n)$*, we say that* φ *is a function iff*

$$\forall x, y, z, p_1, \ldots, p_n (\varphi(x, y, p_1, \ldots, p_n) \;\&\; \varphi(x, z, p_1, \ldots, p_n) \rightarrow y = z) \tag{1.14}$$

When a $\varphi(x, y)$ is a function, we also write the following:

$$\varphi(x, y) \text{ iff } f(x) = y \tag{1.15}$$

Alternatively, $f = \{\langle x, y \rangle : \varphi(x, y)\}$ is a class.

Let us introduce a few more definitions that will make the two remaining axioms more comprehensible.

Definition 1.12 *(Power Set Function)*
Given a set x*, the* power set of x*, denoted* $\mathscr{P}(x)$ *and satisfying the definition* **1.9** *is defined as follows:*

$$\mathscr{P}(x) = \{y : y \subseteq x\} \tag{1.16}$$

Definition 1.13 *(Domain of a Function)*
Let f *be a function. We call the* domain of f *the class of all sets for which* f *is defined. We use "Dom(*f*)" to refer to this set.*

$$\forall x (x \in Dom(f) \leftrightarrow \exists y (f(x) = y)) \tag{1.17}$$

We say "f is a function on A", A being a class, if $A = dom(f)$.

Definition 1.14 *(Range of a Function)*
Let f *be a function. We call the* range of f *the set of all sets that are images of other sets via* f*. We use "Rng(*f*)" to refer to this set.*

$$\forall x (x \in Rng(f) \leftrightarrow \exists y (f(y) = x)) \tag{1.18}$$

We say that f is *a function into A*, A being a class, iff $rng(f) \subseteq A$. We say that f is *a function onto A* iff $rng(f) = A$. We say a function f is a *one to one function*, iff

$$(\forall x_1, x_2 \in dom(f))(f(x_1) = f(x_2) \to x_1 = x_2) \tag{1.19}$$

We say that f is a bijection iff it is a one to one function that is onto.

Note that *Dom(f)* and *Rng(f)* are not definitions in a strict sense, they are in fact definition schemas that yield definitions for every function f given. Also note that they can be easily modified for φ instead of f, with the only difference being the fact that it is then defined only for those φs that are functions, which must be taken into account. This is worth noting as we will use the notions of *function* and *formula* interchangably.

Definition 1.15 *(Function Defined For All Ordinals)*
We say a function f is defined for all ordinals, *this is sometimes written $f : Ord \to A$ for any class A, if $Dom(f) = Ord$. Alternatively,*

$$(\forall \alpha \in Ord)(\exists y \in A)(f(\alpha) = y)) \tag{1.20}$$

Definition 1.16 *(Axiom Schema of Replacement)*
The following is an axiom for every first–order formula $\varphi(x, p_1, \dots, p_n)$ with no free variables other than $x, p_1, \dots, p_n$.

$$\text{“}\varphi \textit{ is a function}'' \to \forall x \exists y \forall z (z \in y \leftrightarrow (\exists q \in x)(\varphi(x, y, p_1, \dots, p_n))) \tag{1.21}$$

Definition 1.17 *(Choice function)*
We say that a function f is a choice function on x *iff*

$$dom(f) = x \setminus \{\emptyset\}) \;\&\; (\forall y \in dom(f))(f(y) \in y) \tag{1.22}$$

Definition 1.18 *(Axiom of Choice)*
For every set x there is a function f that is a choice function on x.

One might be unsettled by the fact that this definition quantifies over functions, which are generally classes, but in this particular case, since $dom(f) = x$ and x is a set, f is also a set due to *Replacement*[8].

Definition 1.19 (S)
We call S an axiomatic theory in the language $\mathscr{L} = \{=, \in\}$ with exactly the following axioms:

[8] If the underlying theory includes of implies *Replacement*.

(i) Existence of a Set *(see definition* **1.1***)*
(ii) Axiom of Extensionality *(see definition* **1.2***)*
(iii) Axiom of Specification *(see definition* **1.3***)*
(iv) Axiom of Foundation *(see definition* **1.8***)*
(v) Axiom of Pairing *(see definition* **1.6***)*
(vi) Axiom of Union *(see definition* **1.7***)*
(vii) Axiom of Power Set *(see definition* **1.9***)*

Definition 1.20 (ZF)
We call ZF *an axiomatic theory in the language* $\mathscr{L} = \{=, \in\}$ *that contains all the axioms of* S *in addition to the following:*
(i) Axiom of Replacement *schema (see definition* **1.16***)*
(ii) Axiom of Infinity *(see definition* **1.10***)*
Existence of a Set *is usually left out because it is a consequence of the* Axiom of Infinity*.*

Definition 1.21 (ZFC)
ZFC *is an axiomatic theory in the language* $\mathscr{L} = \{=, \in\}$ *that contains all the axioms of* ZF *plus* Choice*, see definition* **1.18***).*

1.1.3 The Transitive Universe

Definition 1.22 *(Transitive Class)*
We say a class A is transitive *iff*

$$(\forall x \in A)(x \subseteq A). \tag{1.23}$$

Definition 1.23 *(Well–Ordered Class) A class A is said to be* well–ordered by $\in$ *iff the following hold:*
(i) $(\forall x \in A)(x \notin x)$ *(Antireflexivity)*
(ii) $(\forall x, y, z \in A)(x \in y \ \&\ y \in z \to x \in z)$ *(Transitivity)*
(iii) $(\forall x \subseteq A)(x \neq \emptyset \to (\exists y \in x)(\forall z \in x)(z = y \lor z \in y)))$ *(Existence of the least element)*

Definition 1.24 *(Ordinal Number)*
A set x is said to be an ordinal number *if it is* transitive *and* well–ordered by $\in$*.*

For the sake of brevity, we usually just say "x is an *ordinal*". Note that "x is an ordinal" is a well–defined formula in the language of set theory, since transitivity is defined in definition **1.22** via a first–order formula and well–ordering[9] is in fact a conjunction of four first–order formulas. Ordinals will be usually denoted by lower case greek letters, starting from the beginning of the alphabet: $\alpha, \beta, \gamma, \ldots$. Given two different ordinals α, β, we will write $\alpha < \beta$ for $\alpha \in \beta$, see Lemma 2.11 in [Jech, 2006] for technical details.

Definition 1.25 *(Non–Zero Ordinal)*
We say an ordinal α is non–zero *iff $\alpha \neq \emptyset$.*

Definition 1.26 *(Successor Ordinal)*
Consider the following function defined for all ordinals. Let β be an arbitrary ordinal. We call S the successor function.

$$S(\beta) = \beta \cup \{\beta\} \tag{1.24}$$

An ordinal α is called a successor ordinal *iff there is an ordinal β, such that $\alpha = S(\beta)$. We also write $\alpha = \beta + 1$.*

Definition 1.27 *(Limit Ordinal)*
A non–zero ordinal α is called a limit ordinal *iff it is not a successor ordinal.*

Definition 1.28 *(Ord)*
The class of all ordinal numbers*, which we will denote "Ord"*[10] *is the proper class defined as follows.*

$$x \in Ord \leftrightarrow x \text{ is an ordinal} \tag{1.25}$$

Definition 1.29 *(Von Neumann's Hierarchy)*
The Von Neumann's hierarchy *is a collection of sets indexed by the elements of Ord, defined recursively in the following way:*

(i)

$$V_0 = \emptyset, \tag{1.26}$$

(ii)

$$V_{\alpha+1} = \mathscr{P}(V_\alpha) \text{ for any ordinal } \alpha, \tag{1.27}$$

(iii)

$$V_\lambda = \bigcup_{\beta<\lambda} V_\beta \text{ for a limit ordinal } \lambda, \tag{1.28}$$

(iv)

$$V = \bigcup_{\alpha \in Ord} V_\alpha. \tag{1.29}$$

[9] See definition **1.23**.

[10] Some authors use "*On*" instead of "*Ord*", we will stick to the notation used in [Jech, 2006].

We will also refer to the Von Neumann's hierarchy *as* Von Neumann's universe *or the* cumulative hierarchy*. This definition is only correct in a theory that contains or implies* Replacement Schema*. Even though V is sometimes also used for the universal class that contains all sets, in this thesis, it will always mean the V defined above.*

Definition 1.30 *(Rank)*
Given a set x, we say that the rank of x (written as $rank(x)$) is the least ordinal α such that $x \in V_{\alpha+1}$

Due to *Axiom of Regularity*, every set has a rank.[11] The Von Neumann's hierarchy defined above can also be defined by the fact that every V_α is a set of all set with rank less than α.

Definition 1.31 *(Order–type)*
Given an arbitrary well–ordered set x, we say that an ordinal α is the order–type *of x iff x and α are isomorphic.*

1.1.4 Cardinal Numbers

Definition 1.32 *(Cardinality)*
Given a set x, let the cardinality of x, written $|x|$, be defined as the smallest ordinal number such that there is a one to one mapping from x onto α.

Definition 1.33 *(Aleph function)*
Let ω be the least set satisfying the Axiom of Infinity*. We will recursively define the function $\aleph$ for all ordinals.*
(i) $\aleph_0 = \omega$,
(ii) $\aleph_{\alpha+1}$ *is the least cardinal larger than* $\aleph_\alpha$[12],
(iii) $\aleph_\lambda = \bigcup_{\beta<\lambda} \aleph_\beta$ *for a limit ordinal* λ.
If $\kappa = \aleph_\alpha$ and α is a successor ordinal, we call κ a successor cardinal*. If α is a limit ordinal, we call κ a* limit cardinal.

Definition 1.34 *(Cardinal number)*

(i) *A set x is called a* finite cardinal *iff $x \in \omega$.*
(ii) *A set is called an* infinite cardinal *iff there is an ordinal α such that $\aleph_\alpha = x$.*
(iii) *A set is called a* cardinal *iff it is either a* finite cardinal *or an* infinite cardinal.

[11] See chapter 6 of [Jech, 2006] for details.
[12] "The least cardinal larger than $\aleph_\alpha$" is sometimes notated as $\aleph_\alpha^+$.

We say κ is an uncountable cardinal iff it is an infinite ordinal and $\aleph_0 < \kappa$. Infinite cardinals will be notated by lowercase greek letters from the middle of the alphabet, e.g. $\kappa, \mu, \nu, \ldots$ with the possible exception of λ, which is next to κ in the greek alphabet, but is also sometimes used to denote limit ordinals.

For formal details as well as why every set can be well–ordered assuming the *Axiom of Choice*, and therefore has a cardinality, see [Jech, 2006].

Definition 1.35 *(Sequence)*
We say that a function $\varphi(x, y)$ is a sequence *iff there is an ordinal α such that $dom(\varphi) = \alpha$. In other words, a function is called a sequence if it is defined exactly for every ordinal from below some α. We then say it is an α–sequence. We usually write $\langle \beta_i : i \in \alpha \rangle$ or $\langle \beta_0, \beta_1, \ldots \rangle$ when referring to a sequence, for every $i \in dom(\varphi)$, β_i then denotes the respective elements of $rng(\varphi)$.*

Definition 1.36 *(Cofinal Subset)*
Given a class A of ordinals, we say that $B \subseteq A$ is cofinal in A *iff*

$$(\forall x \in A)(\exists y \in B)(x \in y). \tag{1.30}$$

Definition 1.37 *(Cofinality of a Limit Ordinal)*
Let λ be a limit ordinal. We say that the cofinality *of λ is κ iff κ is the least ordinal, such that there is a cofinal κ–sequence $\langle \beta_\xi : \xi < \kappa \rangle$ satisfying*

$$sup(\{\beta_\xi : \xi < \kappa\}) = \lambda. \tag{1.31}$$

We write $cf(\lambda) = \kappa$.

Note that $cf(\alpha)$ is alway a cardinal[13].

Definition 1.38 *(Regular Cardinal)*
We say an infinite cardinal κ is regular *iff $cf(\kappa) = \kappa$.*

Definition 1.39 *(Strong Limit Cardinal)*
We say that an ordinal κ is a strong limit cardinal *if it is a* limit cardinal *and*

$$(\forall \alpha \in \kappa)(|\mathscr{P}(\alpha)| \in \kappa). \tag{1.32}$$

[13] If $cf(\alpha)$ is not a cardinal, so $|cf(\alpha)| < cf(\alpha)$, then there is a mapping from $|cf(\alpha)|$ onto $cf(\alpha)$. But then the range of this mapping is a cofinal subset of $cf(\alpha)$ that is strictly smaller than $cf(\alpha)$.

Definition 1.40 *(Generalised Continuum Hypothesis)*

$$(\forall \alpha \in Ord)(\aleph_{\alpha+1} = |\mathscr{P}(\aleph_\alpha)|) \tag{1.33}$$

If GCH holds (for example in Gödel's L, see chapter 3), the notions of limit cardinal and strong limit cardinal are equivalent.

1.1.5 Relativisation and Absoluteness

Definition 1.41 *(Relativization)*
Let M be a class, $R \subseteq M \times M$ and let $\varphi(p_1, \dots, p_n)$ be a first–order formula with no free variables besides $p_1, \dots, p_n$. The relativization of φ to M and R *is the formula, written as $\varphi^{M,R}$, defined in the following inductive manner:*

(i) $(x \in y)^{M,R} \leftrightarrow R(x, y)$,
(ii) $(x = y)^{M,R} \leftrightarrow x = y$,
(iii) $(\neg\varphi)^{M,R} \leftrightarrow \neg\varphi^{M,R}$,
(iv) $(\varphi \,\&\, \psi)^{M,R} \leftrightarrow \varphi^{M,R} \,\&\, \psi^{M,R}$,
(v) $(\varphi \vee \psi)^{M,R} \leftrightarrow \varphi^{M,R} \vee \psi^{M,R}$,
(vi) $(\varphi \to \psi)^{M,R} \leftrightarrow \varphi^{M,R} \to \psi^{M,R}$,
(vii) $(\exists x\varphi(x))^{M,R} \leftrightarrow (\exists x \in M)\varphi^{M,R}(x)$,
(viii) $(\forall x\varphi(x))^{M,R} \leftrightarrow (\forall x \in M)\varphi^{M,R}(x)$.

When $R = \in \cap (M \times M)$, we usually write φ^M instead of $\varphi^{M,R}$. When we talk about $\varphi^M(p_1, \dots, p_n)$, it is understood that $p_1, \dots, p_n \in M$.

Definition 1.42 *(Satisfaction in a Structure)*
Let M be a set and R a binary relation on M. Let $Terms$ be the set of all terms, let $e : Terms \to M$ be any evaluation function. Let φ be a first–order formula in the language of set theory.

We say that φ holds in $\langle M, R\rangle$ under the evaluation e, *we write* $\langle M, R\rangle \models \varphi[e]$, *iff any of the following hold:*

(i) *φ is the formula "$s = t$", s, t are terms, both $e(s)$ and $e(t)$ are defined, and $e(s) = e(t)$.*
(ii) *φ is the formula "$s \in t$", s, t are terms, both $e(s)$ and $e(t)$ are defined, and the pair $\langle e(s), e(t)\rangle$ is in R.*
(iii) *φ is the formula "$\neg\psi$" and not $\langle M, R\rangle \models \psi[e]$*
(iv) *φ is the formula "$\psi_1 \,\&\, \psi_2$" and both $\langle M, R\rangle \models \psi_1[e]$ and $\langle M, R\rangle \models \psi_2[e]$.*
(v) *φ is the formula "$\psi_1 \vee \psi_2$" and either $\langle M, R\rangle \models \psi_1[e]$ or $\langle M, R\rangle \models \psi_2[e]$.*
(vi) *φ is the formula "$\psi_1 \to \psi_2$" and either not $\langle M, R\rangle \models \psi_1[e]$ or $\langle M, R\rangle \models \psi_2[e]$.*

(vii) *φ is the formula "$\psi_1 \to \psi_2$" and either not $\langle M, R\rangle \models \psi_1[e]$ or $\langle M, R\rangle \models \psi_2[e]$.*

(viii) *φ is the formula "$\forall x_1 \psi$" and $\langle M, R\rangle \models \psi[e']$ for every e' that differs from e only in the value of x_1.*

(ix) *φ is the formula "$\forall x_1 \psi$" and $\langle M, R\rangle \models \psi[e']$ for every e' that differs from e only in the value of x_1.*

(x) *φ is the formula "$\exists x_1 \psi$" and $\langle M, R\rangle \models \psi[e']$ for some e' that differs from e only in the value of x_1.*

If φ is a sentence, we also write $\langle M, R\rangle \models \varphi$. If φ is not a sentence, the universal closure of φ is assumed to be used instead of φ if no evaluation is explicitly metioned.

Note that we say that M is a set.

We will use $\langle M, R\rangle \models \varphi(p_1, \ldots, p_n)$ and $\varphi^M(p_1, \ldots, p_n)$ interchangably.

Definition 1.43 *(Absoluteness)*
Given a transitive class M, we say a formula φ is absolute in M *if for all $p_1, \ldots, p_n \in M$*

$$\varphi^M(p_1, \ldots, p_n) \leftrightarrow \varphi(p_1, \ldots, p_n) \tag{1.34}$$

Definition 1.44 *(Hierarchy of First–Order Formulas)*

(I) *A first–order formula φ is Δ_0 iff it is logically equivalent to a first–order formula φ' satisfying any of the following:*
  - *(i)* *φ' contains no quantifiers*
  - *(ii)* *y is a set, ψ is a Δ_0–formula, and φ' is either $(\exists x \in y)\psi(y)$ or $(\forall x \in y)\psi(y)$.*
  - *(iii)* *ψ_1, ψ_2 are Δ_0–formulas and φ' is any of the following: $\psi_1 \vee \psi_2$, $\psi_1 \,\&\, \psi_2$, $\psi_1 \to \psi_2$, $\neg\psi_2$,*

(II) *If a formula is Δ_0 it is also Σ_0 and Π_0*

(III) *A formula φ is $\Pi_n + 1$ if it is logically equivalent to a formula φ' such that $\varphi' = \forall x \psi$ where ψ is a Σ_n–formula for any $n < \omega$.*

(IV) *A formula φ is $\Sigma_n + 1$ if it is logically equivalent to a formula φ' such that $\varphi' = \forall x \psi$ where ψ is a Π_n–formula for any $n < \omega$.*

Lemma 1.45 *(Δ_0 absoluteness)*
Let φ be a Δ_0–formula, then φ is absolute in any transitive class M.

Proof. This will be proved by induction over the complexity of a given Δ_0–formula φ. Let M be an arbitrary transitive class.

As M is transitive, atomic formulas are always absolute by the definition of relativisation, see definition **1.41**. Suppose that Δ_0–formulas ψ_1 and ψ_2

are absolute in M. Then from relativization, $(\psi_1 \ \& \ \psi_2)^M \leftrightarrow \psi_1^M \ \& \ \psi_2^M$, which is equivalent to $\psi_1 \ \& \ \psi_2$ from the induction hypothesis. The same holds for $\vee, \rightarrow$ and $\neg$.

Suppose that a Δ_0–formula ψ is absolute in M. Let y be a set and let $\varphi = (\exists x \in y)\psi(x)$. From relativization, $(\exists x \psi(x))^M \leftrightarrow (\exists x \in M)\psi^M(x)$. Since the induction hypothesis makes it clear that $\psi^M \leftrightarrow \psi$, we get

$$((\exists x \in y)\psi(x))^M \leftrightarrow (\exists x \in y \cap M)\psi(x)^M \leftrightarrow (\exists x \in y \cap M)\psi(x), \tag{1.35}$$

which is equivalent to $\varphi^M \leftrightarrow \varphi$. Note that from transitivity of M, is $x \in M$ and $x \in y$, it is the case that $x \in y \cap M$. The same argument applies to $\varphi = (\forall x \in y)\psi(x)$. □

Lemma 1.46 *(Downward Absoluteness)*
Let φ be a Π_1–formula and M a transitive class. Then the following holds:

$$(\forall p_1, \ldots, p_n \in M)(\varphi(p_1, \ldots, p_n) \rightarrow \varphi(p_1, \ldots, p_n)^M) \tag{1.36}$$

Proof. Since $\varphi(p_1, \ldots, p_n)$ is Π_1, there is a Δ_0–formula $\psi(p_1, \ldots, p_n, x)$ such that $\varphi = \forall x \psi(p_1, \ldots, p_n, x)$. From relativization and lemma **1.45**,

$$\varphi^M(p_1, \ldots, p_n) \leftrightarrow (\forall x \in M)\psi(p_1, \ldots, p_n, x). \tag{1.37}$$

Assume that for $p_1, \ldots, p_n \in M$ fixed, that $\forall x \psi(p_1, \ldots, p_n, x)$ holds, but $(\forall x \in M)\psi(p_1, \ldots, p_n, x)$ does not. Therefore $\exists x \neg\psi(p_1, \ldots, p_n, x)$, which contradicts $\forall x \psi(p_1, \ldots, p_n, x)$. □

Lemma 1.47 *(Upward Absoluteness)*
Let φ be a Σ_1–formula and M a transitive class. Then the following holds:

$$(\forall p_1, \ldots, p_n \in M)(\varphi^M(p_1, \ldots, p_n) \rightarrow \varphi(p_1, \ldots, p_n)) \tag{1.38}$$

Proof. Since $\varphi(p_1, \ldots, p_n)$ is Σ_1, there is a Δ_0–formula $\psi(p_1, \ldots, p_n, x)$ such that $\varphi = \exists x \psi(p_1, \ldots, p_n, x)$. From relativization and lemma **1.45**,

$$\varphi^M(p_1, \ldots, p_n) \leftrightarrow (\exists x \in M)\psi(p_1, \ldots, p_n, x). \tag{1.39}$$

Assume that for $p_1, \ldots, p_n \in M$ fixed, that $(\exists x \in M)\psi(p_1, \ldots, p_n, x)$ holds, but $\exists x \psi(p_1, \ldots, p_n, x)$ does not. This is an obvious contradiction. □

1.1.6 More Functions

Definition 1.48 *(Strictly Increasing Function)*
A function $f : Ord \to Ord$ is said to be strictly increasing *iff*

$$(\forall \alpha, \beta \in Ord)(\alpha < \beta \to f(\alpha) < f(\beta)). \tag{1.40}$$

Definition 1.49 *(Continuous Function)*
A function $f : Ord \to Ord$ is said to be continuous *iff*

$$\text{“}\lambda \textit{ is limit”} \to f(\lambda) = \bigcup_{\alpha<\lambda} f(\alpha). \tag{1.41}$$

Definition 1.50 *(Normal Function)*
A function $f : Ord \to Ord$ is said to be normal *iff it is* strictly increasing *and* continuous.

Definition 1.51 *(Fixed Point)*
We say x is a fixed point of a function f iff $x = f(x)$.

Definition 1.52 *(Unbounded Class)*
We say a class A of ordinals is unbounded iff

$$\forall x(\exists y \in A)(x < y). \tag{1.42}$$

Definition 1.53 *(Limit Point)*
Given a class $x \subseteq Ord$, we say that $\alpha \neq \emptyset$ is a limit point of x iff

$$\alpha = \bigcup (x \cap \alpha) \tag{1.43}$$

Definition 1.54 *(Closed Class)*
We say a class $A \subseteq Ord$ is closed iff it contains all its limit points.

Definition 1.55 *(Club set)*
For a regular uncountable cardinal κ, a set $x \subset \kappa$ is a closed unbounded *subset, abbreviated as a* club set, *iff x is both closed and unbounded in κ.*

Definition 1.56 *(Stationary set)*
For a regular uncountable cardinal κ, we say a set $A \subset \kappa$ is stationary in κ iff it intersects every club subset of κ.

1.1.7 Structure, Substructure and Embedding

Structures will be denoted $\langle M, \in, R\rangle$ where M is a domain, $\in$ stands for the standard membership relation, it is assumed to be restricted to the domain[14], $R \subseteq M$ is an unary relation on the domain.

Definition 1.57 *(Elementary Embedding)*
Given the structures $\langle M_0, \in, R\rangle$, $\langle M_1, \in, R\rangle$ and a one–to–one function $j : M_0 \to M_1$, we say j is an elementary embedding *of M_0 into M_1, we write $j : M_0 \prec M_1$, when the following holds for every formula $\varphi(p_1, \ldots, p_n)$ and every $p_1, \ldots, p_n \in M_0$:*

$$\langle M_0, \in, R\rangle \models \varphi(p_1, \ldots, p_n) \leftrightarrow \langle M_1, \in, R\rangle \models \varphi(j(p_1), \ldots, j(p_n)) \tag{1.44}$$

Definition 1.58 *(Elementary Substructure)*
Given the structures $\langle M_0, \in, R\rangle$, $\langle M_1, \in, R\rangle$ and a one–to–one function $j : M_0 \to M_1$ such that $j : M_0 \prec M_1$, we say that M_0 is an elementary substructure *of M_1, denoted as $M_0 \prec M_1$, iff j is an identity on M_0. In other words*

$$\langle M_0, \in, R\rangle \models \varphi(p_1, \ldots, p_n) \leftrightarrow \langle M_1, \in, R\rangle \models \varphi(p_1, \ldots, p_n) \tag{1.45}$$

for $p_1, \ldots, p_n \in M_0$

[14]To be totally explicit, we should write $\langle M, \in \cap M \times M, R\rangle$.

2 Lévy's First–Order Reflection

2.1 Lévy's Original Paper

This section is based on Lévy's paper *Axiom Schemata of Strong Infinity in Axiomatic Set Theory*, [Lévy, 1960]. It presents Lévy's *principle of complete reflection*and its equivalence to the *Replacement Schema* and *Axiom of Infinity* under S^{15}.

First, we should point out that set theory has changed over the last 66 years and show a few notable differences. One might be confused by the fact that Lévy treats the *Axiom of Subsets*, which we call *Axiom Schema of Specification*, as a single axiom rather than a schema. He even takes the conjunction of all axioms of ZF and treats it like a formula. This is possible because the underlying logic calculus is different. Lévy works with set theories formulated in the *non–simple applied first order functional calculus*. The calculus works with two kinds of variables, one for sets and the other for functions. It contains a substitution rule for functional variables, but doesn't quantify over them, so it is not full second–order logic, see the beginning of *Chapter IV* in [Church, 1996] for details.We will use the usual first–order axiomatization of ZFC as seen in [Jech, 2006]. It should also be noted that the logical connectives look different. The symbol used nowaday for an universal quantifier does not appear, $\forall x\varphi(x)$ was be written as $(x)\varphi(x)$. The symbol for negation is "$\sim$", implication is written as "$\supset$" and equivalence is "$\equiv$". We will use standard notation with "$\neg$", "$\rightarrow$" and "$\leftrightarrow$" respectively when presenting Lévy's results.

This subsection uses ZF instead of the usual ZFC as the underlying theory.

Definition 2.1 *(Standard Complete Model of a Set Theory)*
Let Q be an arbitrary axiomatic set theory. We say that u is a standard complete model of Q iff
(i) $(\forall\sigma \in \mathsf{Q})(\langle \mathsf{u}, \in\rangle \models \sigma)$,
(ii) "u is transitive".
We write $Scm^{\mathsf{Q}}(u)$.

Definition 2.2 *(Cardinals Inaccessible With Respect to* Q)
Let Q be an arbitrary axiomatic set theory. We say that a cardinal κ is inaccessible with respect to theory Q iff

$$Scm^{\mathsf{Q}}(V_\kappa). \tag{2.46}$$

[15] See definition **1.19**.

We write $In^{\mathsf{Q}}(\kappa)$.[16]

Definition 2.3 *(Inaccessible Cardinal With Respect to* ZF*)*
When a cardinal κ is inaccessible with respect to ZF*, we only say that it is* inaccessible. We write $In(\kappa)$ instead of $In^{\mathsf{ZF}}(\kappa)$.

The above definition of inaccessibles is used because it doesn't require the *Axiom of Choice*.

For the definition of relativization, see definition **1.41**. The notation used by Lévy is "$Rel(u, \varphi)$", we will stick to "φ^u".

Definition 2.4 *(N)*
The following is the Axiom Schema of Complete Reflection Over ZF*, denoted N. For every first–order formula φ in the language of set theory with no free variables except for $p_1, \ldots, p_n$, the following is an instance of schema N:*

$$\exists u(Scm^{\mathsf{ZF}}(u) \ \& \ (\forall p_1, \ldots, p_n \in u)(\varphi \leftrightarrow \varphi^u)). \tag{2.47}$$

Definition 2.5 *(N')*
For any arbitrary first–order formulas $\varphi_1, \ldots, \varphi_m$ in the language of set theory with no free variables except for $p_1, \ldots, p_n$, the following is an instance of schema N':

$$\exists u(z \in u \ \& \ Scm^{\mathsf{ZF}}(u) \ \& \ (\forall p_1, \ldots, p_n \in u)(\varphi_1 \leftrightarrow \varphi_1^u) \ \& \ \ldots \ \& \ \varphi_m \leftrightarrow \varphi_m^u)). \tag{2.48}$$

Definition 2.6 *(N")*
For an arbitrary first–order formulas $\varphi_1, \ldots, \varphi_m$ in the language of set theory with no free variables except for $p_1, \ldots, p_n$, the following is an instance of schema N'':

$$\exists u(Scm^{\mathsf{ZF}}(u) \ \& \ (\forall p_1, \ldots, p_n \in u)(\varphi_1 \leftrightarrow \varphi_1^u) \ \& \ \ldots \ \& \ \varphi_m \leftrightarrow \varphi_m^u)). \tag{2.49}$$

Let S be an axiomatic set theory defined in definition **1.19**.
This is *Theorem 2* in [Lévy, 1960]

Lemma 2.7 *($N \leftrightarrow N'' \leftrightarrow N'$)*
The schemas N, N' and N'' are equivalent under S.

[16]To be able to define V_κ, we need to work in a logic that contains the *Replacement Schema* or any of its equivalents. It should be noted that we don't work in an arbitrary theory Q, but in ZF, which contains the *Replacement Schema*.

We will execute this proof in the theory ZF, but the reader should note that we have neither used the *Replacement Schema* nor the *Axiom of Infinity*, so for schemas similar to N, N', N'' but with "$Scm^{\mathsf{S}}(u)$" instead of "$Scm^{\mathsf{ZF}}(u)$", the proof works equally well.
Proof.

Clearly, $N' \to N'' \to N$.

Now, assuming N and given the formulas $\varphi_1, \ldots, \varphi_n$, we will prove N''. Consider the following formula:

$$\psi = \bigvee_{i=1}^{t} t = i \;\&\; \varphi_i. \tag{2.50}$$

We will take advantage of the fact that natural numbers are defined by atomic formulas and therefore absolute in transitive structures. From N, we get such u that

$$Scm^{\mathsf{ZF}}(u) \;\&\; (\forall p_1, \ldots, p_n \in u)(\bigvee_{i=1}^{t} t = i \;\&\; \varphi_i \leftrightarrow \bigvee_{i=1}^{t} t = i \;\&\; \varphi_i^u). \tag{2.51}$$

This already satisfies N''. In order to prove N' from N'', let's add two more formulas. Given $p_1, \ldots, p_n$, we denote

$$\varphi_{m+1} = \exists u(z \in u \;\&\; Scm^{\mathsf{ZF}}(u) \;\&\; (\forall p_1, \ldots, p_n \in u)(\bigvee_{i=1}^{m} \varphi_i = \varphi_i^u)), \tag{2.52}$$

$$\varphi_{m+2} = \forall z \varphi_{m+1}. \tag{2.53}$$

So, by N'', we have a set u that satisfies $Scm^{\mathsf{ZF}}(u)$ as well as the following:

$$(\forall p_1, \ldots, p_n \in u)(\varphi_i \leftrightarrow \varphi_i^u) \text{ for } 1 \leq i \leq m, \tag{2.54}$$

$$z \in u \to \varphi_{m+1} \;\leftrightarrow \varphi_{m+1}^u, \tag{2.55}$$

$$\varphi_{m+2} \leftrightarrow \varphi_{m+2}^u. \tag{2.56}$$

By $Scm^{\mathsf{ZF}}(u)$ and (2.54), we get $(\forall z \in u)\varphi_{m+1}$, so together with (2.55), we get $(\forall z \in u)\varphi_{m+1}^u$, exactly φ_{m+2}^u, so by (2.56) we get φ_{m+2}. But φ_{m+2} is exactly the instance of N' we were looking for. □

Definition 2.8 *(N_0)*
Axiom schema N_0 is similar to axoim schema N defined above, but with S instead of ZF. For every φ, a first–order fomula in the language of set theory with no free variables except $p_1, \ldots, p_n$, the following is an instance of N_0:

$$\exists u(Scm^{\mathsf{S}}(u) \;\&\; (\forall p_1, \ldots, p_n \in u)(\varphi \leftrightarrow \varphi^u)). \tag{2.57}$$

We will now show that in S, N_0 implies both the *Replacement Schema* and the *Axiom of Infinity*.

Let N_0 be defined as in definition **2.8**, for the *Axiom of Infinity* see definition **1.10**.

Theorem 2.9 *In* S, the axiom schema N_0 implies the *Axiom of Infinity*.

Proof. Let $\varphi = \forall x \exists y(y = x \cup \{x\})$. This clearly holds in S because given a set x, there is a set $y = x \cup \{x\}$ obtained via *Axiom of Pairing* and *Axiom of Union*. Since the sets obtained via these axioms are definable via Δ_0–formulas, they are absolute in transitive structures thanks to Lemma **1.45**. From N_0, there is a set u such that $Scm^{\mathsf{S}}(u)$ and φ^u holds. This u satisfies the conditions required by the *Axiom of Infinity*. □

Lévy proves this theorem in a different way. He argues that for an arbitrary formula φ, N_0 gives us $\exists u Scm^{\mathsf{S}}(u)$ and this u already satisfies the *Axiom of Infinity*. To do this, we would need to prove lemma **2.15** earlier on, we will do that later in this chapter.

Let S be a set theory defined in definition **1.19**, N_0 a schema defined in definition **2.8** and the *Replacement Schema* a schema defined in definition **1.16**.

Theorem 2.10 *In* S, the axiom schema N_0 implies the *Replacement Schema*.

Proof. Let $\varphi(x, y, p_1, \ldots, p_n)$ be a formula with no free variables except for $x, y, p_1, \ldots, p_n$. Let a set x be given and let χ be an instance of the the *Replacement Schema* schema for the φ given. We want to verify in S that given a formula φ, the instance of N_0 for φ implies χ.

$$\begin{aligned}\chi = \forall x', y', z(\varphi(x', y', p_1, \ldots, p_n) \;\&\; \varphi(x', z, p_1, \ldots, p_n) \to y' = z') \\ \to \exists y \forall z(z \in y \leftrightarrow (\exists q \in x)(\varphi(x, y, p_1, \ldots, p_n)))\end{aligned} \tag{2.58}$$

Since it can be shown that N_0 is equivalent to N_0' similar to N' in lemma **2.7**, there is a set u such that $Scm^{\mathsf{S}}(u)$, $x \in u$ and all of the following hold:

(i) $\varphi \leftrightarrow \varphi^u$

(ii) $\exists y \varphi \leftrightarrow (\exists y \varphi)^u$

From relativization, $(\exists y \varphi)^u$ is equivalent to $(\exists y \in u)\varphi^u$, together with (i) and (ii), we get

$$(\exists y \in u)\varphi \leftrightarrow \exists y \varphi \tag{2.59}$$

If φ is a function, it maps the elements of x, which are also elements of u due to transitivity of u, to elements of u. From *Specification Schema*,

$$y = \{z \in u : (\exists q \in x)\varphi(q, z, p_1, \ldots, p_n)\} \tag{2.60}$$

is a set and a subset of u. Since $Scm^{\mathsf{S}}(u)$ holds and $y \subset u$, then also $\mathscr{P}(y) \subset u$, so $y \in u$. That means we have satisfied the *Replacement Schema* – given a function and a set, we have proved that the image of the set via the given function is again a set. □

2.2 Contemporary Restatement

We will now introduce and prove a theorem that is called *Lévy's Reflection* or *Lévy–Montague Reflection* in contemporary set theory. The only difference is that while Lévy originally reflects a formula φ from the universe of all sets to a set u which is a *standard complete model of* S, we say that there is a V_λ for a limit λ that reflects φ. Those two conditions are equivalent due to lemma **2.15**.

Lemma 2.11 *Let $\varphi_1, \dots, \varphi_n$ be first–order formulas in the language of set theory, all with m free variables* [17].

(i) For each set M_0 there is such set M that $M_0 \subset M$ and the following holds for every i, $1 \leq i \leq n$:

$$\exists x \varphi_i(p_1, \dots, p_{m-1}, x) \rightarrow (\exists x \in M)\varphi_i(p_1, \dots, p_{m-1}, x) \tag{2.61}$$

for every $p_1, \dots, p_{m-1} \in M$.

(ii) Furthermore, there is a limit ordinal λ such that $M_0 \subset V_\lambda$ and the following holds for each i, $1 \leq i \leq n$:

$$\exists x \varphi_i(p_1, \dots, p_{m-1}, x) \rightarrow (\exists x \in V_\lambda)\varphi_i(p_1, \dots, p_{m-1}, x) \tag{2.62}$$

for every $p_1, \dots, p_{m-1} \in M$.

(iii) Assuming Choice, *there is M, $M_0 \subset M$ such that (2.61) holds for every M, $i \leq n$ and $|M| \leq |M_0| \cdot \aleph_0$.*

Proof. We will simultaneously prove statements (i) and (ii), denoting M^T the transitive set required by part (ii). Steps in the construction of M^T that are not explicitly included are equivalent to steps for M.

Let us first define an operation $H_i(p_1, \dots, p_{m-1})$ that yields the set of x's with minimal rank[18] satisfying $\varphi_i(p_1, \dots, p_{m-1}, x)$ for $p_1, \dots, p_{m-1}$ and for every i, $1 \leq i \leq n$.

[17] For formulas with a different number of free variables, take for m the highest number of parameters among those formulas. Add spare parameters to every formula that has less than m parameters in a way that preserves the last parameter, which we will denote x. E.g. let φ'_i be the a formula with k parameters, $k < m$. Let us set $\varphi_i(p_1, \dots, p_{m-1}, x) = \varphi'_i(p_1, \dots, p_{k-1}, x)$, notice that the parameters $p_k, \dots, p_{m-1}$ are not used.

[18] Rank is defined in definition **1.30**.

$$H_i(p_1, \ldots, p_n) = \{x \in C_i : (\forall z \in C)(rank(x) \leq rank(z))\} \tag{2.63}$$

for each $1 \leq i \leq n$, where

$$C_i = \{x : \varphi_i(p_1, \ldots, p_{m-1}, x)\} \text{ for } 1 \leq i \leq n \tag{2.64}$$

Next, let's construct M from given M_0 by induction.

$$M_{i+1} = M_i \cup \bigcup_{j=0}^{n} \bigcup \{H_j(p_1, \ldots, p_{m-1}) : p_1, \ldots, p_{m-1} \in M_i\} \tag{2.65}$$

In other words, in each step we include into the construction the elements satisfying $\varphi(p_1, \ldots, p_{m-1}, x)$ for $p_1, \ldots, p_{m-1}$ from the previous step. For statement (ii), this is the only part that differs from (i). To end up with a transitive M, we need to extend every step to its transitive closure transitive closure of M_{i+1} from (i). In other words, let γ be the smallest ordinal such that

$$(M_i^T \cup \bigcup_{j=0}^{n} \{\bigcup \{H_j(p_1, \ldots, p_{m-1}) : p_1, \ldots, p_{m-1} \in M_i\}\}) \subset V_\gamma \tag{2.66}$$

Then the incremental step is

$$M_{i+1}^T = V_\gamma \tag{2.67}$$

and the final M is obtained by joining the previous steps.

$$M = \bigcup_{i=0}^{\infty} M_i, \;\; M^T = \bigcup_{i=0}^{\infty} M_i^T = V_\lambda \text{ for some limit } \lambda. \tag{2.68}$$

We have yet to finish part (iii). Let's try to construct a set M' that satisfies the same conditions like M but is kept as small as possible. Assuming the Axiom of Choice, we can modify the construction so that the cardinality of M' is at most $|M_0| \cdot \aleph_0$. Note that the size of M in the previous construction is determined by the size of M_0 and, most importantly, by the size of $H_i(p_1, \ldots, p_{m-1})$ for every i, $1 \leq i \leq n$ in individual iterations of the construction. Since (i) only ensures the existence of an x that satisfies $\varphi_i(p_1, \ldots, p_{m-1}, x)$ for any i, $1 \leq i \leq n$, we only need to add one x for every set of parameters but $H_i(u_1, \ldots, u_{m-1})$ can be arbitrarily large. Let F be a choice function on $\mathscr{P}(M')$. Also let $h_i(p_1, \ldots, p_{m-1}) = F(H_i(p_1, \ldots, p_{m-1}))$ for i, where $1 \leq i \leq n$, which means that h is a function that outputs an x that satisfies $\varphi_i(p_1, \ldots, p_{m-1}, x)$ for

i such that $1 \leq i \leq n$ and has minimal rank among all such sets. The induction step needs to be redefined to

$$M'_{i+1} = M'_i \cup \bigcup_{j=0}^{n} \{h_j(p_1, \ldots, p_{m-1}) : p_1, \ldots, p_{m-1} \in M'_i\} \tag{2.69}$$

This way, the amount of elements added to M'_{i+1} in each step of the construction is the same as the amount of m–tuples of parameters that yielded elements not included in M'_i. It is easy to see that if M_0 is finite, M' is countable because it was constructed as a countable union of sets that are themselves at most countable. If M_0 is countable or larger, the cardinality of M' is equal to the cardinality of M_0.[19] Therefore $|M'| \leq |M_0| \cdot \aleph_0$ □

Theorem 2.12 *(Lévy's first–order reflection theorem)*
Let $\varphi(p_1, \ldots, p_n)$ be a first–order formula.

(i) For every set M_0 there exists a set M such that $M_0 \subset M$ and the following holds:

$$\varphi^M(p_1, \ldots, p_n) \leftrightarrow \varphi(p_1, \ldots, p_n) \tag{2.70}$$

for every $p_1, \ldots, p_n \in M$.

(ii) For every set M_0 there is a transitive set M, $M_0 \subset M$ such that the following holds:

$$\varphi^M(p_1, \ldots, p_n) \leftrightarrow \varphi(p_1, \ldots, p_n) \tag{2.71}$$

for every $p_1, \ldots, p_n \in M$.

(iii) For every set M_0 there is a limit ordinal λ such that $M_0 \subset V_\lambda$ and the following holds:

$$\varphi^{V_\lambda}(p_1, \ldots, p_n) \leftrightarrow \varphi(p_1, \ldots, p_n) \tag{2.72}$$

for every $p_1, \ldots, p_n \in M$.

(iv) Assuming Choice, *for every set M_0 there is M such that $M_0 \subset M$ and $|M| \leq |M_0| \cdot \aleph_0$ and the following holds:*

$$\varphi^M(p_1, \ldots, p_n) \leftrightarrow \varphi(p_1, \ldots, p_n) \tag{2.73}$$

for every $p_1, \ldots, p_n \in M$.

Proof. Let's now prove (i) for a given φ via induction by complexity. We can safely assume that φ contains no quantifiers besides "$\exists$" and no logical connectives other than "$\neg$" and "$\&$". Let $\varphi_1, \ldots, \varphi_n$ be all subformulas of φ.

[19] It can not be smaller because $|M'_{i+1}| \geq |M'_i|$ for every i. It may not be significantly larger because the maximum of elements added is the number of n–tuples in M'_i, which is of the same cardinality as M'_i.

Then there is a set M, obtained by the means of lemma **2.11**, for all of the formulas $\varphi_1, \ldots, \varphi_n$.

Let's first consider atomic formulas in the form of either $x_1 = x_2$ or $x_1 \in x_2$. It is clear from relativisation[20] that (2.70) holds for both cases, $(x_1 = x_2)^M \leftrightarrow (x_1 = x_2)$ and $(x_1 \in x_2)^M \leftrightarrow (x_1 \in x_2)$.

We now want to verify the inductive step. First, take $\varphi = \neg\varphi'$. From relativization, we get $(\neg\varphi')^M \leftrightarrow \neg(\varphi'^M)$. Because the induction hypothesis tells us that $\varphi'^M \leftrightarrow \varphi'$, the following holds:

$$(\neg\varphi')^M \leftrightarrow \neg(\varphi'^M) \leftrightarrow \neg\varphi' \tag{2.74}$$

The same holds for $\varphi = \varphi_1 \;\&\; \varphi_2$. From the induction hypothesis, we know that $\varphi_1^M \leftrightarrow \varphi_1$ and $\varphi_2^M \leftrightarrow \varphi_2$, which together with relativization for formulas in the form of $\varphi_1 \;\&\; \varphi_2$ gives us

$$(\varphi_1 \;\&\; \varphi_2)^M \leftrightarrow \varphi_1^M \;\&\; \varphi_2^M \leftrightarrow \varphi_1 \;\&\; \varphi_2 \tag{2.75}$$

Let's now examine the case when $\varphi = \exists x\varphi'(p_1, \ldots, p_n, x)$. The induction hypothesis tells us that $\varphi'^M(p_1, \ldots, p_n, x) \leftrightarrow \varphi'(p_1, \ldots, p_n, x)$, so, together with above lemma **2.11**, the following holds:

$$\begin{gathered}
\varphi(p_1, \ldots, p_n, x) \\
\leftrightarrow \exists x\varphi'(p_1, \ldots, p_n, x) \\
\leftrightarrow (\exists x \in M)\varphi'(p_1, \ldots, p_n, x) \\
\leftrightarrow (\exists x \in M)\varphi'^M(p_1, \ldots, p_n, x) \\
\leftrightarrow (\exists x\varphi'(p_1, \ldots, p_n, x))^M \\
\leftrightarrow \varphi^M(p_1, \ldots, p_n, x)
\end{gathered} \tag{2.76}$$

Which is what we wanted to prove for part (i).

We now need to verify that the same holds for any finite number of formulas $\varphi_1, \ldots, \varphi_n$. This has in fact been already done since lemma **2.11** gives us a set M for any finite amount of formulas and given M_0. We can therefore find a set M for the union of all of their subformulas. When we obtain such M, it should be clear that it also reflects every formula in $\varphi_1, \ldots, \varphi_n$.

Since V_λ is a transitive set, by proving (iii) we also satisfy (ii). To do so, we only need to look at part (ii) of lemma **2.11**. All of the above proof also holds for $M = V_lambda$.

[20]See definition **1.41**. This only holds for relativization to $M, \in \cap M \times M$, as opposed to M, R for an arbitrary R.

To finish part (iv), we take M of size $\leq |M_0| \cdot \aleph_0$, which exists due to part (iii) of lemma **2.11**, the rest being identical. $\square$

Let S be a set theory defined in definition **1.19**, for ZFC see definition **1.21**.

The two following lemmas are based on [Drake, 1974], *Chapter 3, Theorem 1.2*.

Lemma 2.13 *If M is a transitive set, then $\langle M, \in \rangle \models$* Axiom of Extensionality*.*

Proof. Given a transitive set M, we want to show that the following holds.

$$\langle M, \in \rangle \models \forall x, y(x = y \leftrightarrow \forall z(z \in x \leftrightarrow z \in y)) \tag{2.77}$$

Given arbitrary $x, y \in M$, we want to prove that,

$$\langle M, \in \rangle \models (x = y \leftrightarrow \forall z(z \in x \leftrightarrow z \in y)). \tag{2.78}$$

According to definition **1.42**, this is equivalent to

$$\langle M, \in \rangle \models x = y \text{ iff } \langle M, \in \rangle \models \forall z(z \in x \leftrightarrow z \in y), \tag{2.79}$$

which is the same as

$$x = y \text{ iff } \langle M, \in \rangle \models \forall z(z \in x \leftrightarrow z \in y). \tag{2.80}$$

So all elements of x are also elements of y in M, and vice versa. Because M is transitive, all elements of x and y are in M, so

$$\langle M, \in \rangle \models \forall z(z \in x \leftrightarrow z \in y) \tag{2.81}$$

holds iff x and y contain the same elements and are therefore equal. $\square$

Lemma 2.14 *If M is a transitive set, then $\langle M, \in \rangle \models$* Axiom of Foundation*.*

Proof. We want to prove the following:

$$\langle M, \in \rangle \models \forall x(x \neq \emptyset \rightarrow (\exists y \in x)(x \cap y = \emptyset)) \tag{2.82}$$

Given an arbitrary non–empty $x \in M$ let's show that

$$\langle M, \in \rangle \models (\exists y \in x)(x \cap y = \emptyset). \tag{2.83}$$

Because M is transitive, every element of x is an element of M. Take for y the element of x with the lowest rank[21]. It should be clear that there is no $z \in y$ such that $z \in x$, because then $rank(z) < rank(y)$, which would be a contradiction. $\square$

Let S be a set theory as defined in definition **1.19**.

[21]Rank is defined in definition **1.30**.

Lemma 2.15 *The following holds for every λ:*

$$\text{“}\lambda \textit{ is a limit ordinal”} \rightarrow \langle V_\lambda, \in\rangle \models \mathsf{S}. \tag{2.84}$$

Proof. Given an arbitrary limit ordinal λ, we will verify the axioms of S one by one.

(i) *The existence of a set* comes from the fact that V_λ is a non–empty set because a limit ordinal is non–zero by definition.

(ii) Axiom of Extensionality holds from lemma **2.13**.

(iii) *Axiom of Foundation* holds from lemma **2.14**.

(iv) *Axiom of Union*:
Given any $x \in V_\lambda$, we want verify that $y = \bigcup x$ is also in V_λ. Note that $y = \bigcup x$ is a Δ_0–formula.

$$y = \bigcup x \text{ iff } (\forall z \in y)(\exists q \in x) z \in q \;\&\; (\forall z \in x)(\forall q \in z) q \in y \tag{2.85}$$

So by lemma **1.45**

$$y = \bigcup x \text{ iff } \langle V_\lambda, \in\rangle \models y = \bigcup x \tag{2.86}$$

(v) *Axiom of Pairing*:
Given two sets $x, y \in V_\lambda$, we want to show that $z = \{x, y\}$ is also an element of V_λ.

$$z = \{x, y\} \text{ iff } x \in z \;\&\; y \in z \;\&\; (\forall q \in z)(q = x \vee q = y) \tag{2.87}$$

So $z = \{x, y\}$ is a Δ_0–formula, and thus by lemma **1.45** it holds that

$$z = \{x, y\} \text{ iff } \langle V_\lambda, \in\rangle \models z = \{x, y\} \tag{2.88}$$

(vi) *Axiom of Power Set*:
Given any $x \in V_\lambda$, we want to make sure that $\mathscr{P}(x) \in V_\lambda$. Let $\varphi(y)$ denote the formula $y \in \mathscr{P}(x) \leftrightarrow y \subset x$. According to definition **1.4**, $y \subset x$ is Δ_0, so for any given $x, y \in V_\lambda$,

$$y = \mathscr{P}(x) \leftrightarrow \langle V_\lambda, \in\rangle \models y = \mathscr{P}(x). \tag{2.89}$$

Because λ is limit and $rank(\mathscr{P}(x)) = rank(x) + 1$, we know that $\mathscr{P}(x) \in V_\lambda$ for every $x \in V_\lambda$.

(vii) *Specification Schema*:
Given a first–order formula φ, we want to show the following:

$$\langle V_\lambda, \in\rangle \models \forall x, p_1, \ldots, p_n \exists y \forall z (z \in y \leftrightarrow z \in x \;\&\; \varphi(z, p_1, \ldots, p_n)) \tag{2.90}$$

Given any x along with parameters $p_1, \ldots, p_n \in V_\lambda$, we set

$$y = \{z \in x : \varphi^{V_\lambda}(z, p_1, \ldots, p_n)\} \tag{2.91}$$

From transitivity of V_λ and the fact that $y \subset x$ and $x \in V_\lambda$, we know that $y \in V_\lambda$, so

$$\langle V_\lambda, \in\rangle \models \forall z(z \in y \leftrightarrow z \in x \,\&\, \varphi(z, p_1, \ldots, p_n)). \tag{2.92}$$

□

Definition 2.16 *(First–Order Reflection Schema)*
For every first–order formula φ, the following is an axiom:

$$\forall M_0 \exists M(M_0 \subseteq M \,\&\, (\varphi(p_1, \ldots, p_n) \leftrightarrow \varphi(p_1, \ldots, p_n)^M)). \tag{2.93}$$

We will refer to this axiom schema as First–Order Reflection Schema.

Let the *Axiom of Infinity* and the *Replacement Schema* be as defined in definition **1.10** and definition **1.16** respectively.

Theorem 2.17 First–Order Reflection Schema *is equivalent to the* Axiom of Infinity & *the* Replacement Schema *under* S.

Proof. Since theorem **2.12** already gives us one side of the implication, we are only interested in showing the converse which we shall do in two parts:

(i) *First–Order Reflection Schema* $\rightarrow$ **the***Axiom of Infinity*
This is done exactly like theorem **2.9**. We pick for φ the formula $(\forall y \in x)(y \cup \{y\} \in x)$, $M_0 = \{\emptyset\}$. From definition **2.16**, there is a set M that satisfies φ, so there is an inductive set. We have picked M_0 so that $\emptyset \in M$ obviously holds and M is the witness for

$$\exists x(\emptyset \in x \,\&\, (\forall y \in x)(y \cup \{y\} \in x)) \tag{2.94}$$

which is exactly definition **1.10**.

(ii) *First–Order Reflection Schema* $\rightarrow$ *Replacement Schema*
Let's first point out that while *First–Order Reflection Schema* gives us a set for one formula, we can generalise it to hold for any finite number of formulas. We will show how is it done for two formulas, which is what we will use in this proof. Given two first–order formulas φ, ψ, we can suppose that there are formulas φ' and ψ' that are equivalent to φ and ψ respectively,

but their free variables are different [22]. Let $\xi = \varphi \;\&\; \psi$, given any M_0, we can find a M such that $\xi \leftrightarrow \xi^M$. It is easy to see that from relativisation, the following holds:

$$\varphi \;\&\; \psi \leftrightarrow \varphi' \;\&\; \psi' \leftrightarrow \xi \leftrightarrow \xi^M \leftrightarrow (\varphi' \;\&\; \psi')^M \leftrightarrow \varphi'^M \;\&\; \psi'^M \leftrightarrow \varphi^M \;\&\; \psi^M \tag{2.95}$$

Now given a function $\varphi(x, y)$, we know from *First–Order Reflection Schema* that for every M_0, there is a set M such that $M_0 \subseteq M$ and both

$$(\forall x, y \in M)(\varphi(x, y) \leftrightarrow \varphi^M(x, y)) \tag{2.96}$$

and

$$(\forall x, y \in M)(\exists y \varphi(x, y) \leftrightarrow (\exists y \varphi(x, y))^M) \tag{2.97}$$

hold, the latter being equivalent to

$$(\forall x, y \in M)(\exists y \varphi(x, y) \leftrightarrow (\exists y \in M)\varphi^M(x, y)). \tag{2.98}$$

Therefore

$$(\forall x, y \in M)(\exists y \varphi(x, y) \leftrightarrow (\exists y \in M)\varphi(x, y)) \tag{2.99}$$

holds too. That means that we have a set M such that for every $x \in M$, if φ is defined for x, then $(\exists y \in M)\varphi(x, y)$.

To show that the *Replacement Schema* holds for this particular φ, we need to verify that given a set M_0, $M_0' = \{y : (\exists x \in M_0)\varphi(x, y)\}$ is also a set. But since $M_0 \subseteq M$ and because given any $x \in M$, there is $y \in M$ satisfying $\varphi(x, y)$, the following is a set due to *Specification Schema*:

$$M_0' = \{y : (\exists x \in M_0)\varphi(x, y)\} = \{y \in M : (\exists x \in M_0)\varphi(x, y)\} \tag{2.100}$$

□

We have shown that reflection for first–order formulas, is a theorem of ZFC. We have also shown that it can be used instead of the the *Axiom of Infinity* and the *Replacement Schema* scheme, but ZFC + *First–Order Reflection Schema* is no stronger than ZFC. Besides being a starting point for more general and powerful statements, it can be used to show that ZFC is not finitely axiomatizable. This follows from the fact that reflection yields an inner model to any consistent finite set of formulas that hold in V. So if $\varphi_1, \ldots, \varphi_n$ would be the axioms of ZFC, reflection would prove that every model of ZFC contains a smaller model of ZFC, which would in turn contradict the Second Gödel's Theorem.

[22] This is plausible since we can for example substitute all free variables in φ' for $x_0, x_2, x_4, \ldots$ and use $x_1, x_3, x_5, \ldots$ for free variables in ψ', the resulting formulas will be logically equivalent.

It is also worthwhile to note that, in a way, reflection is dual to compactness. Compactness says that given a set of sentences, if every finite subset yields a model, so does the whole set. Reflection, on the other hand, says that while the whole set has no model in the underlying theory, every finite subset has a model.

Furthemore, reflection can be used in ways similar to upward Löwenheim–Skolem theorem. Since Reflection extends any set M_0 into a model of given formulas $\varphi_1, \ldots, \varphi_n$, we can choose the lower bound of the size of M by appropriately choosing M_0.

In the next section, we will try to generalise reflection in a way that transcends ZFC and yields some large cardinals.

3 Reflection And Large Cardinals

3.1 Regular Fixed–Point Axioms

Lévy's article mentions various schemata that are not instances of reflection per se, but deal with fixed points of normal ordinal functions. After proving a helpful lemma, we will introduce them and show that they are equivalent to *First–Order Reflection Schema*[23].

Lemma 3.1 *(Fixed–Point Lemma for Normal Functions)*
Let f be a normal function defined for all ordinals[24]*. Then all of the following hold:*
(i) $\forall\lambda($*"λ is a limit ordinal"* $\rightarrow$ *"f(λ) is a limit ordinal"*$)$
(ii) $\forall\alpha(\alpha \leq f(\alpha))$
(iii) $\forall\alpha\exists\beta(\alpha < \beta \,\&\, f(\beta) = \beta)$
(iv) The fixed points of f form a closed unbounded class.[25]

Proof. Let f be a normal function defined for all ordinals.
(i) Suppose λ is a limit ordinal. For an arbitrary ordinal $\alpha < \lambda$, the fact that f is strictly increasing means that $f(\alpha) < f(\lambda)$ and for any ordinal β, satisfying $\alpha < \beta < \lambda$, $f(\alpha) < f(\beta) < f(\lambda)$. We know that there is such β from limitness of λ. Because f is continuous and λ is limit, $f(\lambda) = \bigcup_{\gamma<\lambda} f(\gamma)$.Therefore λ is limit, so is $f(\lambda)$.
(ii) This step will be proved using the transfinite induction. Since f is defined for all ordinals, there is an ordinal α such that $f(\emptyset) = \alpha$ and because $\emptyset$ is the least ordinal, (ii) holds for $\emptyset$.
Suppose (ii) holds for some β from the induction hypothesis. It then holds for $\beta + 1$ because f is strictly increasing.
For a limit ordinal λ, suppose (ii) holds for every $\alpha < \lambda$. (i) implies that $f(\lambda)$ is also limit, so there is a strictly increasing κ–sequence $\langle\alpha_0, \alpha_1, \ldots\rangle$ for some κ such that $\lambda = \bigcup_{i<\kappa} \alpha_i$. Because f is strictly increasing, the κ–sequence $\langle f(\alpha_0), f(\alpha_1), \ldots\rangle$ is also strictly increasing, in then holds from the induction hypothesis that $\alpha_i \leq f(\alpha_i)$ for each $i \leq \kappa$. Thus, $\lambda \leq f(\lambda)$.
(iii) For an arbitrary α, let there be an ω–sequence $\langle\alpha_0, \alpha_1, \ldots\rangle$, such that $\alpha_0 = \alpha$ and $\alpha_{i+1} = f(\alpha_i)$ for each $i < \omega$. This sequence is stricly increasing because so is f. Now, there's a limit ordinal $\beta = \bigcup_{i<\omega} \alpha_i$, we want to show

[23] For the definition, see definition **2.16**.
[24] For the definition of normal function, see definition **1.50**.
[25] See definition **1.54** for the definition of a closed class, definition **1.52** for the definition of unboundedness.

that this is a fixed point of f. Because f is continuous,

$$f(\beta) = f(\bigcup_{i<\omega} \alpha_i) = \bigcup_{i<\omega} f(\alpha). \tag{3.101}$$

We have defined the above sequence so that

$$f(\beta) = \bigcup_{i<\omega} f(\alpha) = \bigcup_{i<\omega} \alpha_{i+1}, \tag{3.102}$$

which means we are done, since

$$\bigcup_{i<\omega} \alpha_{i+1} = \bigcup_{i<\omega} \alpha_i = \beta. \tag{3.103}$$

(iv) The class of fixed points of f is obviously unbounded because in (iii), we start with an arbitrary ordinal. It remains to show that it is closed, this is based on [Drake, 1974], *chapter 4*. Let Y be a non–empty set of fixed points of f such that $\bigcup Y \notin Y$. Since f is defined on ordinals, Y is a set of ordinals, so $\bigcup Y$ is an ordinal. $\bigcup Y$ is a limit ordinal. If it were a successor ordinal, suppose that $\alpha + 1 = \bigcup Y$, then $\alpha \in \bigcup Y$, which would mean that there is some x such that $\alpha \in x \in Y$. But the least such x is $\alpha + 1$, so $\bigcup Y \in Y$.
Note that $\alpha < \bigcup Y$ iff $\exists \xi \in Y(\alpha < \xi)$. Since f is defined for all ordinals and $\bigcup Y$ is a limit ordinal, $f(\bigcup Y) = \bigcup_{\alpha \in Y} f(\alpha)$, but because Y is a set of fixed points of f,

$$f(\bigcup Y) = \bigcup_{\alpha \in Y} f(\alpha) = \bigcup Y, \tag{3.104}$$

so $\bigcup Y$ is a limit point of Y.

□

Lemma 3.2 *Let α be a limit ordinal. Then the following hold: If C is a club subset of α, then there is an ordinal β and a normal function $f : \beta \to \alpha$ such that $rng(f) = C$. We say that f* enumrates *C.*

This proof in inspired by [Monk, 2011].
Proof. Let β be the order–type[26] of C and let f be the isomorphism from β onto C. Since $C \subseteq \alpha$, f is an increasing function from β into α. To show that f is continuous, let γ be a limit ordinal below β, let $\epsilon = \bigcup_{\delta<\gamma} f(\delta)$. We want to verify that $f(\gamma) = \epsilon$. Since ϵ is a limit ordinal, we only need to show that $C \cap \epsilon$ is inbounded in ϵ.

[26] See definition **1.31**.

Take $\zeta < \epsilon$. Then there is a $\delta < \gamma$ such that $\zeta < f(\delta)$. Since γ is limit, $\delta + 1 < \gamma$ and also $f(\delta + 1) < f(\gamma)$, we know that $f(\delta) \in C \cap \epsilon$. But that means that $C \cap \epsilon$ is unbounded in ϵ, so $\epsilon \in C$. We have also shown that ϵ is closed unbounded in the image of γ over f. Therefore, $f(\gamma) = \epsilon = \bigcup_{\delta < \gamma} f(\delta)$, so f is normal. □

It should be clear that while this lemma works with club subsets of an ordinal, we can formulate analogous statement for club classes, which then yields a normal function defined for all ordinals, with the only exception that there is no such β is an the beginning of the above proof because f is then a function from Ord to Ord and proper classes have no order–type.

Definition 3.3 *(*Axiom Schema M_1*)*
"Every normal function defined for all ordinals has at least one inaccessible number in its range."

Lévy uses "M" to refer to this axiom but since we also use "M" for sets and models, for example in definition **2.12**, we will call the above axiom "*Axiom Schema M_1*" to avoid confusion.

In order to be able to meaningfully work with this schema, we must clarify what it actually states. Because we are working in first–order logic, and a *normal function defined for all ordinals* is a proper class, we can not quantify over functions that are not sets. Instead, we will think of *Axiom Schema M_1* as schema that, given a formula φ, states "If φ is a normal function defined for all ordinals, then φ has at least one inaccessible number in its range"[27]. We will approach the following two axiom schemata in a similar manner.

Definition 3.4 *(Axiom Schema M_2)*
"Every normal function defined for all ordinals has at least one fixed point which is inaccessible."

Definition 3.5 *(Axiom Schema M_3)*
"Every normal function defined for all ordinals has arbitrarily great fixed points which are inaccessible."

Similar axiom is proposed in [Drake, 1974].

[27]More formally, let $\varphi(x, y, p_1, \dots, p_n)$ be a first–order formula with no free variables besides $x, y, p_1, \dots, p_n$. The following is equivalent to *Axiom M_1*.

$$\begin{aligned}&\text{"}\varphi \text{ is a normal function" } \& \ \forall x(x \in Ord \rightarrow \exists y(\varphi(x, y, p_1, \dots, p_n))) \rightarrow \\ &\rightarrow \exists y(\exists x \varphi(x, y, p_1, \dots, p_n) \ \& \ cf(y) = y \ \& \ (\forall x \in \kappa)(\exists y \in \kappa)(x > y))\end{aligned} \tag{3.105}$$

Definition 3.6 *(Axiom Schema F)*
"Every normal function has a regular fixed point."

Lemma 3.7 *Let f be a normal function defined for all ordinals.*
(i) There is a is normal function g_1 defined for all ordinals that enumerates the class $\{\alpha : f(\alpha) = \alpha\}$.
(ii) There is a is normal function g_2 defined for all ordinals that enumerates the class $\{\lambda :$ "$f(\lambda)$ is a strong limit cardinal."$\}$.

Proof. We know that (ii) holds from lemma **3.1** and lemma **3.2**.

Clearly, there is no largest strong limit ordinal ν, because the limit of $\langle \nu, \mathscr{P}(\nu), \mathscr{P}(\mathscr{P}(\nu)), \ldots \rangle$ is again a limit ordinal. The class of strong limit ordinals is closed because a limit of strong limit ordinals of is always a strong limit ordinal. Let h be a function enumerating limit ordinals that exists from lemma **3.2**. Then $g_1(\alpha) = f(h(\alpha))$ for every ordinal α is normal and defined for all ordinals. □

The following is *Theorem 1* in [Lévy, 1960], the parts dealing with *Axiom Schema F* come from [Drake, 1974].

Theorem 3.8 *The following are all equivalent:*
(i) Axiom Schema M_1,
(ii) Axiom Schema M_2,
(iii) Axiom Schema M_3,
(iv) Axiom Schema F.

Proof. It is clear that *Axiom Schema M_3* is a stronger version of *Axiom Schema M_2*, which is in turn a stronger version of both *Axiom Schema M_1* and *Axiom Schema F_1*.

We will now prove that *Axiom Schema F → Axiom Schema M_2*. Lemma **3.7** tells us that given a normal function f defined for all ordinals, there is a normal function g_1 defined for all ordinals that enumerates the fixed points of f. There is also a function g_2 that enumerates the strong limit ordinals in $rng(f)$. By *Axiom Schema F*, g_2 has a regular fixed point κ, which is also a strong limit ordinal, so

$$f(\kappa) = g_2(\kappa) = \kappa \text{ and } \kappa \text{ is inaccessible.} \tag{3.106}$$

So every normal function defined for all ordinals has a regular fixed point.

We have yet to show that *Axiom Schema M_1 → Axiom Schema M_3*. Again by lemma **3.7**, there is a normal function g defined for all ordinals that enumerates the fixed points of f. Let $h_\alpha(\beta) = g(\alpha + \beta)$ for any given ordinal α, then h_α is a normal function defined for all ordinals. Then, given an arbitrary α, from *Axiom Schema M_1*, there is a β such that $\gamma = h_\alpha(\beta)$ is inaccessible. Because

$\gamma = g(\alpha+\beta)$, thus $f(\gamma) = \gamma$. Since $\alpha \leq f'(\alpha)$ for any ordinal α and any normal function f', we know that $\alpha \leq \alpha + \gamma \leq \gamma$, so γ is inaccessible and arbitrarily large, depending on the choice of α. □

To see how those schemata relate to reflection, let's introduce a stronger version of *First–Order Reflection Schema*[28] from the previous chapter. But in order to do this, we must establish the inaccessible cardinal first.

3.2 Inaccessible Cardinal

Definition 3.9 *An uncountable cardinal κ is* inaccessible *iff it is* regular *and* strongly limit. *We write $In(\kappa)$ to say that κ is an inaccessible cardinal.*

An uncountable cardinal that is regular and limit is called a *weakly inaccessible cardinal*, we will only use the (strongly) inaccessible cardinal, but most of the results are similar for weakly inaccessibles, including higher types of ordinals that will be presented later in this chapter.

Theorem 3.10 *Let κ be an inaccessible cardinal.*

$$\langle V_\kappa, \in \rangle \models \text{ZFC} \tag{3.107}$$

We will prove this theorem in a way similar to [Kanamori, 2003].

Proof. Most of this is already done in lemma **2.15**, we only need to verify that *Replacement* and *Infinity* axioms hold in V_κ.

Infinity holds because κ is uncountable, so $\omega \in V_\kappa$.

To verify *Replacement*, let x be an element of V_κ and f a function from x to V_κ. Let $y = \{z \in V_\kappa : (\exists q \in x) f(q) = z\}$, so $y \subset V_\kappa$, it remains to show that $y \in V_\kappa$. Because f is a function, we know that $|y| \leq |x| \leq \kappa$. But since κ is regular, $\{rank(z) : z \in y\} \subseteq \alpha$ for some $\alpha < \kappa$, and so $x \in V_{\alpha+1} \in V_\kappa$. Therefore $y \in V_\kappa$. □

Definition 3.11 *(Inaccessible Reflection Schema)*
For every first–order formula φ, the following is an axiom:

$$\forall M_0 \exists \kappa (M_0 \subseteq V_\kappa \;\&\; In(\kappa) \;\&\; (\varphi(p_1, \ldots, p_n) \leftrightarrow \varphi(p_1, \ldots, p_n)^{V_\kappa})) \tag{3.108}$$

We will refer to this axiom schema as Inaccessible Reflection Schema. *Note that M is a set, even though we often use upper–case letters for classes. This is due to fact that "M" is used in the same meaning in theorem* **2.12**.

[28] See definition **2.16**.

We have added the requirement that α is inaccessible, which trivially means that there is an inaccessible cardinal. By taking appropriate M_0, it can be shown that in a theory that includes the *Inaccessible Reflection Schema*, there is a closed unbounded class of inaccessible cardinals. Since we know that for an inaccessible κ, V_κ is a model of ZFC, *Inaccessible Reflection Schema* is equivalent to

$$\forall M_0 \exists \kappa (M_0 \subseteq V_\kappa \,\&\, \langle V_\kappa, \in\rangle \models \mathsf{ZFC} \,\&\, (\varphi(\mathsf{p_1}, \ldots, \mathsf{p_n}) \leftrightarrow \varphi(\mathsf{p_1}, \ldots, \mathsf{p_n})^{\mathsf{V}_\kappa})) \tag{3.109}$$

because we have proved in the last section that for an inaccessible κ,

$$\langle V_\kappa, \in\rangle \models \mathsf{ZFC}. \tag{3.110}$$

Theorem 3.12 Inaccessible Reflection Schema *is equivalent to* Axiom schema F.

This is *Theorem 4.1* in chapter 4 of [Drake, 1974], also equivalent to *Theorerem 3* in [Lévy, 1960].

Proof. Let's start by showing that *Inaccessible Reflection Schema* implies *Axiom schema F*. It should be clear from previous results that we can reflect two formulas to a single set, for example by taking the conjunction of universal closures of the formulas.

Given a normal function f defined for all ordinals, we want to show that it has a regular fixed point. For any ordinal α, there is an ordinal κ such that

$$\alpha < \kappa \,\&\, In(\kappa) \,\&\, (\forall \gamma, \delta \in V_\kappa)(f(\gamma) = \delta \leftrightarrow (f(\gamma) = \delta)^{V_\kappa}) \tag{3.111}$$

and

$$\alpha < \kappa \,\&\, In(\kappa) \,\&\, \forall \gamma \exists \delta (f(\gamma) = \delta) \leftrightarrow (\forall \gamma \exists \delta f(\gamma) = \delta)^{V_\kappa}. \tag{3.112}$$

Since V_κ is the set of all sets of rank less than κ and since every ordinal is the rank of itself, there is an inaccessible ordinal κ such that

$$(\forall \gamma < \kappa)(\exists \delta < \kappa)(f^{V_\kappa}(\gamma) = \delta). \tag{3.113}$$

We also know that $f(\gamma) = \delta$ iff $(f(\gamma) = \delta)^{V_\kappa}$. Now since κ is a limit ordinal and f is continuous we get

$$f(\kappa) = \bigcup_{\gamma < \kappa} f^{V_\kappa}(\gamma) = \bigcup_{\gamma < \kappa} f(\gamma). \tag{3.114}$$

From (3.113) and the fact that f is increasing, we know that $\kappa \leq \bigcup_{\gamma<\kappa} f(\gamma) \leq \kappa$. Therefore κ is an inaccessible fixed point of f.

For the opposite direction, it suffices to show that since there is an inaccessible cardinal due to *Axiom schema* F, given a first–order formula φ, there is an arbitrarily large inaccessible cardinal κ for which

$$\varphi \leftrightarrow \varphi^{V_\kappa}. \tag{3.115}$$

Note that the arbitrary size of κ means given an arbitrary ordinal α, there is a κ satisfying $\alpha \in \kappa$ and (3.115). In the previous chapter, in theorem **2.12**, we have shown that we can easily obtain a limit ordinal satisfying (3.115). Note that since for any set M_0, there is such α that $M_0 \subseteq V_\alpha$, there is a closed unbounded class of sets satisfying (3.115), which are levels in the cumulative hierarchy, so there is a club class of κs satisfying (3.115).

Let f be a normal function defined for all ordinals that enumerates this club class, there is such f by lemma **3.2**. Let g be the function that enumerates strong limit ordinals in $rng(f)$, there is one by lemma **3.7**. Then g has a regular fixed point κ, which is also a regular fixed point of f, so (3.115) holds for κ. $\square$

Definition 3.13 *(*ZMC)
We will call ZMC *an axiomatic set theory that contains all axioms and schemas of* ZFC *together with* Axiom Schema M_1.

We have decided to call it ZMC, because Lévy uses ZM, derived from ZF, which is more intuitive, but we also need the axiom of choice, thus, ZMC.

As a sidenote, we should note that ZMC is extension of ZFC, which is in turn an extension of S. This way, reflection can be seen as a natural continuation of the *Axiom of Infinity* and *Replacement Schema*.

3.3 Mahlo Cardinals

We have shown that ZMC contains arbitrarily large inaccessible cardinals. To return to reflection–style argument, is there a set that satisfies this property? To be able to properly answer this question, we have to formulate the notion of "containing arbitrarily large cardinals" more carefully. While we have previously used club sets, this is not an option in this case because inaccessibles don't form a club class in ZMC[29].

We have shown earlier in this chapter that there is a simple relation between normal functions defined for all ordinals and closed unbounded classes. We will now use a similar approach utilising normal functions. By saying that for a class of ordinals C, a normal function f has at least one element of C in its range, we say that C is stationary. Or, as Drake writes in [Drake, 1974] when dealing

[29]Note that cofinality of the limit of the first ω inaccessibles is ω, which makes is singular.

with the class of inaccessible cardinals, and a cardinal κ, in which inaccessibles are stationary:

> " The class of inaccessible cardinals is so rich that there are members κ of the class such that no normal function on κ can avoid this class; however we climb though κ, provided we are continuous at limits (so that we are enumerating a closed subset of κ), we shall eventually have to hit an inaccessible."

Definition 3.14 *(Mahlo Cardinal)*
We say that κ is a Mahlo Cardinal *iff it is an inaccessible cardinal and the set $\{\lambda < \kappa : \lambda$ is inaccessible$\}$ is stationary in κ.*

Alternatively, κ is Mahlo iff $\langle V_\kappa, \in\rangle \models$ ZMC as shown above, this is also sometimes written as *Ord is Mahlo*. There are also *weakly Mahlo cardinals*, that are defined via weakly inaccessible cardinals below them, Mahlo cardinals are then also called *strongly Mahlo* to highlight the difference, but we will only use the term *Mahlo cardinal*.

Mahlo cardinals are related to reflection principles in an interesting way. Note that given a formula φ, *First–Order Reflection Schema* gives us a club set of ordinals α such that V_α reflects φ, all below the first inaccessible cardinal. We have then used a different reflection schema to obtain arbitrarily high inaccessible cardinals κ such that V_κ refpects φ. Now we have a cardinal in which this reflection schema holds, so we are in fact reflecting reflection. Beware that this is done rather informally, because *Axiom Schema M_1* is a countable set of axioms, which can not be reflected via the schemas introduced so far. One way to deal with this would be to extend reflection for second– and possibly higher–order formulas, but we would have to be very careful with the notion of satisfaction. For now, let us explore where can stationary sets take us because as we have shown, their connection to reflection is quite clear.

What would happen if we strengthened *Axiom Schema M_1* to say that every normal function has a Mahlo cardinal in its range?

Definition 3.15 *(hyper–Mahlo cardinal)*
We say that κ is a hyper–Mahlo cardinal *iff it is inaccessible and the set*

$$\{\lambda < \kappa : \lambda \text{ is Mahlo}\} \tag{3.116}$$

is stationary in κ.

Definition 3.16 *(hyper–hyper–Mahlo cardinal)*
We say that κ is a hyper–hyper–Mahlo cardinal *iff it is inaccessible and the set*

$$\{\lambda < \kappa : \lambda \text{ is hyper–Mahlo}\} \tag{3.117}$$

is stationary in κ.

It is clear that one can continue in this direction, but the nomenclature gets increasingly confusing even if we rewrite them as *hyper$^\alpha$–Mahlo cardinals* instead of repeating the prefix. To see there is a more elegant way to reach those cardinals, we will now establish an operation that elegantly exhausts all such cardinals.

Definition 3.17 *(Mahlo Operation)*
Let A be a class of ordinals. Let

$$H(A) = \{\alpha \in A : A \cap \alpha \text{ is stationary in } \alpha\}. \tag{3.118}$$

We call H the Mahlo's operation.

If we pick for A the class of all inaccessible cardinals, $H(A)$ is the class of Mahlo cardinals. It is easy to see that is A is the class of all α–Mahlo cardinals, $H(A)$ is the class of $\alpha+1$–Mahlo cardinals, $H(H(A))$ is the class of $\alpha+2$–Mahlo cardinals and so on.

Definition 3.18 *(Iterated Mahlo Operation)*
Let A be a class of ordinals. We shall extend the Mahlo operation in the following way:
(i) $H^0(A) = A$,
(ii) $H^{\alpha+1}(A) = H(H^\alpha(A))$,
(iii) $H^\lambda(A) = \bigcap_{\alpha< \ \lambda} H^\alpha(X)$ *for limit λ.*

Clearly if A is the class of inaccessibles, $H^\alpha(A)$ is the class of α–Mahlo cardinals. To get to hyper–Mahlo cardinals, we can diagonalise the operation.

Definition 3.19 *(Diagonal Mahlo Operation)*
Let A be a class of ordinals. Then the diagonal Mahlo operation *is defined as follows:*

$$H^\Delta(A) = \{\alpha \in Ord : \forall\beta < \alpha(\alpha \in H^\beta(X))\}. \tag{3.119}$$

We can further diagonalise the diagonal version and continue this process ad libitum in order to reach all large cardinals accessible *from below*. To see what is meant by *from below*, note that the approach that led us to the *Mahlo operation* was taking a property, for example regularity, that is already available in our current theory, e.g. ZFC, and making an assertion of the height of the universe such that there are "enough" other ordinals holding this property in a sense that a normal function defined on ordinals inevitably has at least one such ordinal in its range.

3.4 Indescribable Cardinals

Indescribability is another approach towards large cardinals that is based on reflection. We will briefly introduce the basic definitions and show that it yield large cardinals, but most of them are not reachable from below in a sense established at the end of previous subsection.

Most of the results presented in this subchapter come from [Kanamori, 2003].

Since this chapter uses higher–order logic, we need to introduce the hierarchy of formulas first.

Definition 3.20 *(Higher–Order Variables)*
Let M be a structure and D its domain. In first–order logic, variables range over individuals, that is, over elements of D. We shall call those type 1 variables *for the purposes of higher–order logic. Type 2 variables then range over collections, that is, the elements of $\mathscr{P}(D)$. Generally, type n variables are defined for any $n \in \omega$ such that they range over $\mathscr{P}^{n-1}(D)$.*

We will use lowercase latin letters for type 1 variables for backward compatibility with first–order logic, type 2 variables will be represented by uppercase letters, mostly P, X, Y, Z, higher–order variables won't be needed in this thesis. If we wanted to define satisfaction for second–order formulas in a model $\langle V_\alpha, \in\rangle$ that we have often used in this thesis, type 2 variables would be interpreted to range over a set is isomorphic to $V_{\alpha+1}$[30].

Definition 3.21 *(Full Prenex Normal Form)*
We say a formula is in the prenex normal form *if it is written as a block of quantifiers followed by a quantifier–free part.*
We say a formula is in the full prenex normal form *if it is written in* prenex normal form *and if there are type $n+1$ quantifiers, they are written before type n quantifiers.*

It is an elementary that every formula is equivalent to a formula in the full prenex normal form.

Definition 3.22 *(Hierarchy of Formulas)*
Let φ be a formula in the prenex formal form.
(i) We say φ is a Δ^0_0–formula if it contains only bounded quantifiers.
(ii) We say φ is a Σ^0_0–formula or a Π^0_0–formula if it is a Δ^0_0–formula.

[30] It might be useful to keep a separate version instead of using $V_{\alpha+1}$ so that we can distinguish between sets and classes that turn out to have the same extension. See [Koellner, 2009] for details.

(iii) *We say φ is a Π_0^{m+1}–formula if it is a Π_n^m– or Σ_n^m–formula for any $n \in \omega$ or if it is a Π_n^m– or Σ_n^m–formula with additional free variables of type $m+1$.*
(iv) *We say φ is a Σ_0^m–formula if it is a Π_0^m–formula.*
(v) *We say φ is a Σ_n^m+1–formula if it is of a form $\exists P_1, \ldots, P_i \psi$ for any non–zero i, where ψ is a Π_n^m–formula and $P_1, \ldots, P_i$ are type $m+1$ variables.*
(vi) *We say φ is a Π_n^m+1–formula if it is of a form $\forall P_1, \ldots, P_i \psi$ for any non–zero i, where ψ is a Σ_n^m–formula and $P_1, \ldots, P_i$ are type $m+1$ variables.*

Definition 3.23 *(Describability)*
We say an ordinal α is described by a sentence φ in the language $\mathscr{L}$ with relation symbols $P_1, \ldots, P_n$ given iff

$$\langle V_\alpha, \in, P_1, \ldots, P_n \rangle \models \varphi \tag{3.120}$$

but for every $\beta < \alpha$

$$\langle V_\beta, \in, P_1 \cap V_\beta, \ldots, P_n \cap V_\beta \rangle \not\models \varphi. \tag{3.121}$$

For the definition of a Π_n^m–formula and a Σ_n^m–formula, see definition **3.22**.

Definition 3.24 *(Π_n^m–Indescribable Cardinal)*
We say that κ is Π_n^m–indescribable iff it is not described by any Π_n^m–formula.

Definition 3.25 *(Σ_n^m–Indescribable Cardinal)*
We say that κ is Σ_n^m–indescribable iff it is not described by any Σ_n^m–formula.

To see that this notion is based in reflection, let us recall the opening quote of this thesis by Gödel which says *"The Universe of sets cannot be uniquely characterised (i. e. distinguished from all its initial elements) by any internal structural property of the membership relation on it."*. A cardinal κ is Π_n^m–indescribable[31] iff every Π_n^m–formula fails to describe V_κ and describes an initial segment instead. In a sense, V_κ reflects the "property"[32] of indescribability of the universal class with respect to certain classes of formulas.

Lemma 3.26 *Let κ be a cardinal, then the following holds for any $n \in \omega$. κ is Π_n^1–indescribable iff κ is Σ_{n+1}^1–indescribable.*

[31] This holds for Σ_n^m–formulas alike.
[32] In this case, we are not using the word to refer to a definable class, but on a meta level to refer to a property expressible in the natural language, hence the quotation marks.

Proof. The forward direction is obvious, we can always add a spare quantifier over a type 2 variable to turn a Π^1_n formula φ into a $\exists P\varphi$ which is then a Σ^1_{n+1}–formula.[33]

To prove the opposite direction, suppose that $\langle V_\kappa, \in\rangle \models \exists X\varphi(X)$ where X is a type 2 variable and φ is a Π^1_n–formula with one free variable of type 2. This means that there is a set $S \subseteq V_\kappa$ that is a witness of $\exists X\varphi(X)$, in other words, $\varphi[S]$ holds. We can replace every occurence of X in φ by a new predicate symbol S, this allows us to say that κ is Π^1_n–indescribable (with respect to $\langle V_\kappa, \in, R, S\rangle$).[34] □

The above lemma makes it clear that, without the loss of generality, we can suppose that all formulas with no higher than type 2 variables are Π^1_n–formulas.

Lemma 3.27 *If κ is an inaccessible cardinal and given $R \subseteq V_\kappa$, then the following is a club set in κ:*

$$\{\alpha \in \kappa : \langle V_\alpha, \in, R \cap V_\alpha\rangle \prec \langle V_\kappa, \in, R\rangle\}. \tag{3.122}$$

Proof. To see that (3.122) is closed, let us recall that a $A \subseteq \kappa$ is closed iff for every ordinal α such that $\emptyset < \alpha < \kappa$, it holds that if $A \cap \alpha$ is unbounded in α then $\alpha \in A$. Since κ is an inaccessible cardinal, thus strong limit, it is closed under limits of sequences of ordinals smaller than κ. In order to verify that it is unbounded, we will use a recursively defined κ–sequence $\langle \alpha_0, \alpha_1, \ldots\rangle$ to build $\langle V_\alpha, \in, R \cap V_\alpha\rangle$, an elementary substructure of $\langle V_\kappa, \in, R\rangle$ such that $\alpha > \alpha_0$ for an arbitrary ordinal $\alpha_0 < \kappa$. Let us fix one such α_0. Given α_n, α_{n+1} is defined as the least β, $\alpha_n \leq \beta$ that satisfies the following for any formula φ for $p_1, \ldots, p_m \in V_{\alpha_n}, m \in \omega$:

$$\begin{gathered}\text{If } \langle V_\kappa, \in, R\rangle \models \exists x\varphi(p_1, \ldots, p_n),\\ \text{then } \exists x \in V_\beta \text{ such that } \langle V_\kappa, \in, R\rangle \models \varphi(x, p_1, \ldots, p_n).\end{gathered} \tag{3.123}$$

Let $\alpha = \bigcup_{n<\omega} \alpha_n$. Then

$$\langle V_\alpha, \in, R \cap V_\alpha\rangle \prec \langle V_\kappa, \in, R\rangle, \tag{3.124}$$

in other words, for any φ with given arbitrary parameters $p_1, \ldots, p_n \in V_\alpha$, it holds that

$$\langle V_\alpha, \in, R \cap V_\alpha\rangle \models \varphi(p_1, \ldots, p_n) \leftrightarrow \langle V_\kappa, \in, R\rangle \models \varphi(p_1, \ldots, p_n). \tag{3.125}$$

Which should be clear from the construction of α. □

[33] Note that unlike in previous sections, it is worth noting that φ is now a sentence so we don't have to worry whether P is free in φ.

[34] A different yet interesting approach is taken by Tate in [Tait, 2005]. He states that for $n \geq 0$, a formula of order $\leq n$ is called a Π^n_0 and a Σ^n_0 formula. Then a Π^n_{m+1} is a formula of form $\forall Y\psi(Y)$ where ψ is a Σ^n_m formula and Y is a variable of type n. Finally, a Σ^n_{m+1} is the negation of a Π^n_m formula. So the above holds ad definitio.

Theorem 3.28 *Let κ be an ordinal. The following are equivalent.*
(i) κ is inaccessible.
(ii) κ is Π^1_0–indescribable.

Note that Π^1_0 formulas are those that contain zero unbound quantifiers over type–2 variables, they are in fact first–order formulas, but with additional type 2 free variables allowed.
Proof. Π^1_0–sentences contain type 2 variables, but only type 1 quantifiers. We want to prove that κ is an inaccessible cardinal iff whenever a formula tries to describe κ in the sense of definition **3.23**, the formula fails to do so and describes a initial segment thereof instead. We have already shown in theorem **3.10** that there is no way to climb the cumulative hierarchy to the height of an inaccesible cardinal via first–order formulas in ZFC. We will now prove that adding unqantified type 2 variables does not make it possible, note that all of the axiom schemata used in the previous chapter can be rewritten to use a type 2 variable instead of a given function.

For (i)$\rightarrow$(ii), suppose that κ is inaccessible.

Then there is, by lemma **3.27** a club set of ordinals α such that V_α is an elementary substructure of V_κ. For κ to be Π^1_0–indescribable, we need to make sure that given an arbitrary Π^1_0–formula φ satisfied in the structure $\langle V_\kappa, \in, R\rangle$, there is an ordinal $\alpha < \kappa$, such that $\langle V_\alpha, \in, R \cap V_\alpha\rangle \models \varphi$. But this follows from the definition of elementary substructure.

For (ii)$\rightarrow$(i), suppose κ is not inaccessible, so it is either singular, or there is a cardinal $\nu < \kappa$ such that $\kappa \leq \mathscr{P}(\nu)$ or $\kappa = \omega$.

Suppose κ is singular. Then there is a cardinal $\nu < \kappa$ and a function $f : \nu \rightarrow \kappa$ such that $rng(f)$ is cofinal in κ. Since $f \subseteq V_\kappa$, we can add f as a relation to the language. We can do the same with $\{\nu\}$. That means $\langle V_\kappa, \in, P_1, P_2\rangle$ with $P_1 = f, P_2 = \{\nu\}$ is a structure. Let

$$\varphi = (P_1 \neq \emptyset \ \& \ rng(P_1) = P_2)^{35}. \tag{3.126}$$

Since for every $\alpha < \nu$, $P_1 \cap V_\alpha = \emptyset$, φ is false and therefore describes κ. That contradicts the fact that κ was supposed to be Π^1_0–indescribable, but φ is a first–order formula.

Suppose there is a cardinal ν satisfying $\kappa \leq \mathscr{P}(\nu)$. Let there be a function $f : \mathscr{P}(\nu) \rightarrow \kappa$ that is onto. Then, like in the previous paragraph, we can obtain a structure $\langle V_\kappa, \in, P_1, P_2\rangle$, where $P_1 = f$ like before, but this time $P_2 = \mathscr{P}(\nu)$. Again,

$$\varphi = (P_1 \neq \emptyset \ \& \ rng(P_1) = P_2) \tag{3.127}$$

describes κ.

[35] $rng(x) = y$ is a first–order formula, see definition **1.14**.

Finally, suppose $\kappa = \omega$, then the first-order sentence $\varphi = \forall x \exists y (x \in y)$ describes κ, which is a contradiction. $\square$

Generally, it should be clear that it a cardinal κ is Π^m_n–indescribable, it is also $\Pi^{m'}_{n'}$–indescribable for every $m' < m, n' < n$. By the same line of thought, if a cardinal κ satisfies the property implied by Π^m_n–indescribability, it satisfies all properties implied by $\Pi^{m'}_{n'}$–indescribability for $m' < m, n' < n$. For example, if κ is Π^m_n–indescribable for $m \geq 1$ then it is also an inaccessible cardinal.

Theorem 3.29 *If a cardinal κ is Π^1_1–indescribable, then it is a Mahlo cardinal.*

Proof. Assuming that κ is Π^1_1–indescribable, we want to prove that every club set of in κ contains an inaccessible cardinal.

Consider the following Π^1_1–sentence φ:

$$\varphi = \forall P(\text{“P is a function''} \rightarrow \forall x \exists y \forall z (z \in y \leftrightarrow (\exists q \in x)(P(x, y, p_1, \ldots, p_n)))) \\ \& \ \forall x \exists y \forall z (z \in y \leftrightarrow z \subseteq x) \tag{3.128}$$

where P is a type 2 variable and the rest are type 1 variables, “P is a function” is a first–order formula defined in definition **1.11**. As has been shown earlier in this chapter, given a cardinal μ, the following holds if and only if μ is inaccessible:

$$\langle V_\mu, \in \rangle \ \models \ \varphi. \tag{3.129}$$

Now fix an arbitrary $C \subset \kappa$, a club set in κ. We want to show that it contains an inaccessible cardinal. Since C is a subset of κ and therefore a subset of V_κ, we can use the structure $\langle V_\kappa, \in, C \rangle$ instead of $\langle V_\kappa, \in \rangle$. Then the following holds:

$$\langle V_\kappa, \in, C \rangle \ \models \ \varphi \ \& \text{ “}C\text{ is unbounded”}.^{36} \tag{3.130}$$

Note that this holds because κ is Π^1_1–indescribable, and therefore also Π^1_0–indescribable. So κ is itself inaccessible and therefore $\langle V_\kappa, \in, C \rangle \ \models \ \varphi$.

Since κ is Π^1_1–indescribable and φ & “C is unbounded” is equivalent to a Π^1_1–formula, there must be an ordinal α that satisfies

$$\langle V_\alpha, \in, C \cap V_\alpha \rangle \ \models \ \varphi \ \& \text{ “}C\text{ is unbounded”}, \tag{3.131}$$

which implies that α is inaccessible; it is regular because it reflects *Replacement* and it is limit because if α were a successor ordinal, it couldn't contain an unbounded class of ordinals.

We only need to verify that $\alpha \in C$, which is clear from the fact that C is a club set in κ and it is unbounded in α. $\square$

There is an even stronger large cardinal property implied by Π^1_1–indescribability that is based on reflection.

[36] “C is unbounded” is a first–order formula, see definition **1.52**.

Definition 3.30 *(Extension Property)*
We say a cardinal κ has the extension property *iff for all $U \subset V_\kappa$ there exists a transitive set X such that $\kappa \in X$, and a set $S \subset X$, such that $(V_\kappa, \in, U)$ is an elementary substructure of $(X, \in, S)$.*

Definition 3.31 *(Weakly Compact Cardinal)*
We say that a cardinal κ is weakly compact *iff it has the extension property.*

Theorem 3.32 *A cardinal κ is Π^1_1–indescribable iff it is weakly compact.*

For the proof, see [Kanamori, 2003]. Note that the extension property is also very similar to reflection

We will now introduce the measurable cardinal, which is not based on reflection from below in our sense, but illustrates the fact that indescribability leads to cardinals that contradict *Axiom of Constructibility*, that will be introduced right after the measurable cardinal.

Definition 3.33 *(Ultrafilter)*
Given a set x, we say $U \subset \mathscr{P}(x)$ is an ultrafilter *over x iff all of the following hold:*
(i) $\emptyset \notin U$,
(ii) $\forall y, z(y \subset x \,\&\, z \subset x \,\&\, y \subset z \,\&\, y \in U \to z \in U)$,
(iii) $(\forall y, z \in U)(y \cap z) \in U$,
(iv) $\forall y(y \subset x \to (y \in U \vee (x \setminus y) \in U))$.

Definition 3.34 *(κ–Complete Ultrafilter)*
We say that an ultrafilter U is κ–complete iff it is closed under intersection of κ–many elements. More precisely,

$$(\forall\gamma < \kappa)(\{a_\alpha : \alpha < \gamma\} \subseteq U \to \bigcup_{\alpha < \gamma} a_\alpha \in U). \tag{3.132}$$

Definition 3.35 *(Measurable Cardinal)*
We say that a cardinal κ is a measurable cardinal *iff there is a κ–complete ultrafilter over κ.*

Theorem 3.36 *Let κ be a cardinal. If κ is a measurable cardinal then the following hold:*
(i) κ *is Π^2_1–indescribable.*
(ii) Given U, a normal ultrafilter over κ, a relation $R \subseteq V_\kappa$ and a Π^2_1–formula φ such that $\langle V_\kappa, \in, R\rangle \models \varphi$, then

$$\{\alpha < \kappa : \langle V_\alpha, \in, R \cap V_\alpha\rangle \models \varphi\} \in U. \tag{3.133}$$

For a proof, see *Proposition 6.5* in [Kanamori, 2003].

3.5 The Constructible Universe

The constructible universe, denoted L, is a cumulative hierarchy of sets, presented by Kurt Gödel in his paper [Gödel and Brown, 1940]. Assertion of its equality to the *Von Neumann's hierarchy*, $V = L$, is called the *Axiom of Constructibility*. The axiom implies GCH and AC and contradicts the existence of some large cardinals, our goal is to decide whether those introduced earlier are among them.

On order to formally establish this class, we need to formalise the notion of definability first.

Definition 3.37 *(Definability)*
We say that a set X is definable *over a model $\langle M, \in \rangle$ if there is a formula φ together with parameters $p_1, \ldots, p_n \in M$ such that*

$$X = \{x : x \in M \ \&\ \langle M, \in \rangle \models \varphi(x, p_1, \ldots, p_n)\} \tag{3.134}$$

Definition 3.38 *(The Set of Definable Subsets)*
The following is a set of all definable subsets of a given set M, denoted Def(M).

$$\begin{aligned} Def(M) = \{\{y : x \in M \ \&\ \langle M, \in \rangle \models \varphi(y, u_1, \ldots, i_n)\} : \\ \varphi \text{ is a first–order formula, } p_1, \ldots, p_n \in M\} \end{aligned} \tag{3.135}$$

We will use $Def(M)$ in the following construction in the way the power set operation is used when constructing the usual Von Neumann's hierarchy of sets[37].

Definition 3.39 *(The Constructible Universe)*
The constructible universe *is a collection of sets similar to the Von Neumann's hierarchy but consisting only of definable sets.*

(i)

$$L_0 = \emptyset, \tag{3.136}$$

(ii)

$$L_{\alpha+1} = Def(L_\alpha) \text{ for any ordinal } \alpha, \tag{3.137}$$

(iii)

$$L_\lambda = \bigcup_{\alpha<\lambda} L_\alpha \text{ For a limit ordinal } \lambda, \tag{3.138}$$

(iv)

$$L = \bigcup_{\alpha \in Ord} L_\alpha. \tag{3.139}$$

[37]For that reason, some authors use $\mathscr{P}^*(M)$ instead of $Def(M)$, see section 11 of [Pinter, 2014] for one such example.

Note that while L bears very close resemblance to V, the difference is, that in every successor step of constructing V, we take every subset of V_α to be $V_{\alpha+1}$, whereas $L_{\alpha+1}$ consists only of definable subsets of L_α. Also note that L is transitive.

Theorem 3.40 *Let L be as in definition* **3.39**.

$$\langle L, \in \rangle \text{ is a model of ZFC} \tag{3.140}$$

For details, refer to Theorem 13.3 in [Jech, 2006].

Definition 3.41 *(Constructibility)*
The axiom of constructibility states that every set is constructible. It is usually denoted as $L = V$.

Without providing a proof, we will introduce two important results established by Gödel in [Gödel and Brown, 1940].

Theorem 3.42 *(Constructibility $\rightarrow$ Choice)*

$$\text{ZF} \vdash \text{Constructibility} \rightarrow \text{Axiom of Choice} \tag{3.141}$$

The GCH refers to the *Generalised Continuum Hypothesis*, see definition **1.40**.

Theorem 3.43 *(Constructibility $\rightarrow$ Generalised Continuum Hypothesis)*

$$\text{ZF} \vdash \text{Constructibility} \rightarrow \text{GCH} \tag{3.142}$$

It is worth mentioning that Gödel's proof of *Construcibility $\rightarrow$ GCH* featured the first formal use of a reflection principle. For the actual proofs, see for example [Kunen, 1983],

Since *GCH* implies that κ is a limit cardinal iff κ is a strong limit cardinal for every κ, the distinctions between inaccessible and weakly inaccessible cardinals as well as between Mahlo and weakly Mahlo cardinals vanish.

Theorem 3.44 *(Inaccessibility in L)*
Let κ be an inaccessible cardinal. Then $In(\kappa)^L$.

Proof. We want to show that the following are all true for an inaccessible cardinal κ:
(i) "κ is a cardinal"^{L},
(ii) $(\omega < \kappa)^L$,

(iii) "κ is regular"^{L},
(iv) "κ is limit"^{L}.[38]

Suppose "κ is not a cardinal"^{L} holds, then there is a cardinal μ, $\mu < \kappa$ and a function $f : \mu \to \kappa$, $f \in L$, such that "$f : \mu \to \kappa$ is onto"^{L}. But since "f is onto" is a Δ_0 formula and Δ_0 formulas are are absolute in transitive structures[39] and L is a transitive class, "f is onto"$^L \leftrightarrow$ "f is onto", this contradicts the fact that κ is a cardinal. $(\omega < \kappa)^L$ holds because $\omega \in \kappa$ and because ordinals remain ordinals in L, so $(\omega \in \kappa)^L$.

In order to see that "κ is regular"^{L}, we can repeat the argument by contradiction used to show that κ is a cardinal in L. If κ was singular, there is a $\mu < \kappa$ together with a function $f : \mu \to \kappa$ that is onto, but since "f is onto" implies "f is onto"^{L}, we have reached a contradiction with the fact that κ is regular, but singular in L.

It now suffices to show that "κ is a limit cardinal"^{L}. That means, that for any given $\lambda < \kappa$, we need to find an ordinal μ such that $\lambda < \mu < \kappa$ that is also a cardinal in L. But since cardinals remain cardinals in L by an argument with surjective functions just like above, it holds. □

Theorem 3.45 *(Mahloness in L)*
Let κ be a Mahlo cardinal. Then "κ is Mahlo"^{L}.

Proof. Let κ be a Mahlo cardinal. From the definition of Mahloness in definition **3.14**, it should be clear that we want prove that κ is inaccessible in L and

$$\text{"The set } \{\alpha : \alpha \in \kappa \;\&\; \text{'}\alpha \text{ is inaccessible'}\} \text{ is stationary in } \kappa\text{"}^L \tag{3.143}$$

Since we have shown that an inaccessible cardinals remain inaccessible in L in the previous theorem, $In(\kappa)^L$ holds.

Now consider the two following sets:

$$S = \{\alpha \in \kappa : In(\alpha)\} \tag{3.144}$$

$$T = \{\alpha \in \kappa : In(\alpha)^L\} \tag{3.145}$$

Since inaccessible cardinals are inaccessible in L from theorem **3.44**, $S \subseteq T$. So if T is stationary in κ, we are done. Suppose for contradiction that it is not the case. Therefore there is a $C \subset \kappa$ satisfying "C is a club set in κ"^{L}, but it is the case that $T \cap C = \emptyset$. But because "C is a club set in κ" is equivalent to a Δ_0 formula,

$$\text{"}C \text{ is a club set in } \kappa\text{"}^M \leftrightarrow \text{"}C \text{ is a club set in } \kappa\text{"}, \tag{3.146}$$

[38]While inaccessible cardinals are strong limit cardinals, since GCH holds in L, "κ is limit"^{L}, implies "κ is strong limit"^{L}.

[39]See lemma **1.45**.

ergo C is a club set in κ. But since it has o intersection with T, it can't have an intersection with a subset thereof, which contradicts the fact that S is stationary in κ.

κ remains Mahlo in L. □

It should be clear that the above process can be iterated over again. Since Mahlo cardinals are absolute in L, the same argument using stationary sets can be carried out for hyper–Mahlo cardinals and so on. It is clear that since a regular and an inaccessible cardinal in consistent with *Constructibility*, so should be the higher properties acquired from assuring the existence of regular, inaccessible and Mahlo fixed points of normal functions.

Theorem 3.46 *If there is a measurable cardinal, then $V \neq L$.*

This is proved in [Scott, 1961] or [Kanamori, 2003].

3.6 Conclusion

To have an intuitive idea of why apart from measurability, every large cardinal property we have established is absolute in L, let us stress that measurability is the only one that does not deal with the height of the cumulative hierarchy of sets. The assertion of the existence of an inaccessible cardinal can be informally rephrased as "The universe of all sets is so big in terms of height, there are ordinals unreachable by the power set operation"[40]. Gödel's Constructible universe deals only with the width of the universe, which is kept as small as possible, so there is no way it can be inconsistent with assertions that deal with height and have no implications in terms of width. Similarly, the Mahlo operation only deals with ordinals, therefore it's not surprising that it has no implications on width of the universe alone. This is not the case with measurability. Measurability is such a strong statement that even though it seems to explicitly speak of height only, the existence of a measurable cardinal implies the existence of non–constructible subsets of ω[41].

[40] This approach is embodied in the definition of Q–inaccessibility used by Lévy, see definition **2.2**, that can be understood as "given a set theory with some means of traversing the cumulative hierarchy upwards, a cardinal is inaccessible with respect to Q if it can't be reached by those means alone".

[41] See [Drake, 1974], p. 196.

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