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REFLECTION PRINCIPLES AND LARGE CARDINALS

Bakalářská práce

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Prohlašuji, že jsem bakalářkou práci vypracoval samostatně a že jsem uvedl všechny použité prameny a literaturu.

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Abstract

Práce zkoumá vztah tzv. principů reflexe a velkých kardinálů. Lévy ukázal, že v ZFC platí tzv. věta o reflexi a dokonce, že věta o reflexi je ekvivalentní schématu nahrazení a axiomu nekonečna nad teorií ZFC bez axiomu nekonečna a schématu nahrazení. Tedy lze na větu o reflexi pohlížet jako na svého druhu axiom nekonečna. Práce zkoumá do jaké míry a jakým způsobem lze větu o reflexi zobecnit a jaký to má vliv na existenci tzv. velkých kardinálů. Práce definuje nedosažitelné, Mahlovy a nepopsatelné kardinály a ukáže, jak je lze zavést pomocí reflexe. Přirozenou limitou kardinálů získaných reflexí jsou kardinály nekonzistentní s L. Práce nabídne intuitivní zdůvodněn, proč tomu tak je.

Abstract

This thesis aims to examine relations between so called ”Reflection Principles” and Large cardinals. Lévy has shown that Reflection Theorem is a sound theorem of ZFC and it is equivalent to Replacement Scheme and the Axiom of Infinity. From this point of view, Reflection theorem can be seen a specific version of an Axiom of Infinity. This paper aims to examine the Reflection Principle and its generalisations with respect to existence of Large Cardinals. This thesis will establish Inaccessible, Mahlo and Indescribable cardinals and their definition via reflection. A natural limit of Large Cardinals obtained via reflection are cardinals inconsistent with L. The thesis will offer an intuitive explanation of why this is the case.

Contents

1 Introduction

1.1 Motivation and Origin

> The Universe of sets cannot be uniquely characterized (i. e. distinguished from all its initial elements) by any internal structural property of the membership relation in it, which is expressible in any logic of finite of transfinite type, including infinitary logics of any cardinal order.

— Kurt Gödel [10]

To understand why do need reflection in the first place, let's think about infinity for a moment. In the intuitive sense, infinity is an upper limit of all numbers. But for centuries, this was merely a philosophical concept, closely bound to religious and metaphysical way of thinking, considered separate from numbers used for calculations or geometry. It was a rather vague concept. In ancient Greece, Aristotle's response to famous Zeno's paradoxes introduced the distinction between actual and potential infinity. He argued, that potential infinity is (in today's words) well defined, as opposed to actual infinity, which remained a vague incoherent concept. He didn't think it's possible for infinity to inhabit a bounded place in space or time, rejecting Zeno's thought experiments as a whole. Aristotle's thoughts shaped western thinking partly due to Aquinas, who himself believed actual infinity to be more of a metaphysical concept for describing God than a mathematical property attributed to any other entity. In his Summa Theologica [1] he argues:

> A geometrician does not need to assume a line actually infinite, but takes some actually finite line, from which he subtracts whatever he finds necessary; which line he calls infinite.

Less than hundred years later, Gregory of Rimini wrote

> If God can endlessly add a cubic foot to a stone–which He can–then He can create an infinitely big stone. For He need only add one cubic foot at some time, another half an hour later, another a quarter of an hour later than that, and so on ad infinitum. He would then have before Him an infinite stone at the end of the hour.

Which is basically a Zeno's Paradox made plausible with God being the actor. In contrast to Aquinas' position, Gregory of Rimini theoretically constructs an object with actual infinite magnitude that is essentially different from God.

[1] Part I, Question 7, Article 3, Reply to Objection 1

Even later, in the 17th century, pushing the property of infinitness from the Creator to his creation, Nature, Leibniz wrote to Foucher in 1962:

> I am so in favor of the actual infinite that instead of admitting that Nature abhors it, as is commonly said, I hold that Nature makes frequent use of it everywhere, in order to show more effectively the perfections of its Author. Thus I believe that there is no part of matter which is not, I do not say divisible, but actually divided; and consequently the least particle ought to be considered as a world full of an infinity of different creatures.

But even though he used potential infinity in what would become foundations of modern Calculus and argued for actual infinity in Nature, Leibniz refused the existence of an infinite, thinking that Galileo's Paradoxon[2] is in fact a contradiction. The so called Galileo's Paradoxon is an observation Galileo Galilei made in his final book "Discourses and Mathematical Demonstrations Relating to Two New Sciences". He states that if all numbers are either squares and non–squares, there seem to be less squares than there is all numbers. On the other hand, every number can be squared and every square has it's square root. Therefore, there seem to be as many squares as there are all numbers. Galileo concludes, that the idea of comparing sizes makes sense only in the finite realm.

> Salviati: So far as I see we can only infer that the totality of all numbers is infinite, that the number of squares is infinite, and that the number of their roots is infinite; neither is the number of squares less than the totality of all the numbers, nor the latter greater than the former; and finally the attributes "equal," "greater," and "less," are not applicable to infinite, but only to finite, quantities. When therefore Simplicio introduces several lines of different lengths and asks me how it is possible that the longer ones do not contain more points than the shorter, I answer him that one line does not contain more or less or just as many points as another, but that each line contains an infinite number.

Leibniz insists in part being smaller than the whole saying

> Among numbers there are infinite roots, infinite squares, infinite cubes. Moreover, there are as many roots as numbers. And there are as many squares as roots. Therefore there are as many squares

[2] zneni galileova paradoxu

> as numbers, that is to say, there are as many square numbers as there are numbers in the universe. Which is impossible. Hence it follows either that in the infinite the whole is not greater than the part, which is the opinion of Galileo and Gregory of St. Vincent, and which I cannot accept; or that infinity itself is nothing, i.e. that it is not one and not a whole.

TODO Hegel–strucne?

TODO Cantor

TODO mene teologie, vice matematiky

TODO definovat pojmy (trida etc)

TODO neni V v nejakem smyslu porad potencialni nekonecno, zatimco mnoziny vetsi nez omega jsou aktualni? nebo jsou potencialni protoze se staveji pres indukci, od spoda.

In his work, he defined transfinite numbers to extend existing natural number structure so it contains more objects that behave like natural numbers and are based on an object (rather a meta-object) that doesn't explicitly exist in the structure, but is closely related to it. This is the first instance of reflection. This paper will focus on taking this principle a step further, extending Cantor's (or Zermelo–Fraenkel's, to be more precise) universe so it includes objects so big, they could be considered the universe itself, in a certain sense.

TODO dal asi smazat

The original idea behind reflection principles probably comes from what could be informally called "universality of the universe". The effort to precisely describe the universe of sets was natural and could be regarded as one of the impulses for formalization of naive set theory. If we try to express the universe as a set $\{x|x = x\}$, a paradox appears, because either our set is contained in itself and therefore is contained in a set (itself again), which contradicts the intuitive notion of a universe that contains everything but is not contained itself.

TODO ???

If there is an object containing all sets, it must not be a set itself. The notion of class seems inevitable. Either directly the ways for example the Bernays–Gödel set theory, we will also discuss later in this paper, does in, or on a meta–level like the Zermelo–Fraenkel set theory, that doesn't refer to them in the axioms but often works with the notion of a universal class. duet Another obstacle of constructing a set of all sets comes from Georg Cantor, who proved that the set of all subsets of a set (let x be the set and $\mathscr{P}(()x)$ its powerset) is strictly larger that x. That would turn every aspiration to finally establish an universal set into a contradictory infinite regression.[3]. We will use V to denote the class of all sets. From previous thoughts we can easily argue, that it is impossible to construct a property that holds for V and no set and is neither paradoxical like $\{x|x = x\}$ nor trivial. Previous observation can be transposed to a rather naive formulation of the reflection principle:

Reflection Any property which holds in V already holds in some initial segment of V.

To avoid vagueness of the term "property", we could informally reformulate the above statement into a schema:

For every first-order formula[4] φ holds in $V \leftrightarrow \varphi$ holds in some initial segment of V.

Interested reader should note that this is a theorem scheme rather than a single theorem. [5]

1.2 A few historical remarks on reflection

Reflection made its first in set-theoretical appearance in Gödel's proof of GCH in L (citace Kanamori ? Lévy and set theory), but it was around even earlier as a concept. Gödel himself regarded it as very close to Russel's reducibility axiom (an earlier equivalent of the axiom schema of Zermelo's separation). Richard Montague then studied reflection properties as a tool for verifying that Replacement is not finitely axiomatizable (citace?). a few years later Lévy proved in [2] the equivalence of reflection with Axiom of infinity together with Replacement in proof we shall examine closely in chaper 2.

TODO co dal? recent results?

[3]An intuitive analogy of this *reductio ad infinitum* is the status of ω, which was originally thought to be an unreachable absolute, only to become starting point of Cantor's hierarchy of sets growing beyond all boundaries around the end of the 19^{th} century

[4]this also works for finite sets of formulas [4, p. 168]

[5]If there were a single theorem stating "for any formula φ that holds in V there is an initial segment of V where φ also holds", we would obtain the following contradiction with the second Gödel's theorem: In ZFC, any finite group of axioms of ZFC holds in some initial segment of the universe. If we take the largest of those initial segments it is still strictly smaller than the universe and thus we have, via compactness, constructed a model of ZFC within ZFC. That is, of course a harsh contradiction. This also leads to an elegant way to prove that ZFC is not finitely axiomatizable.

1.3 Reflection in Platonism and Structuralism

TODO cite "reflection in a structuralist setting"

TODO veci o tom, ze reflexe je ok protoze reflektuje veci ktere objektivne plati, protoze plati pro V ...

TODO souvislost s kompaktnosti, hranice formalich systemu nebo alespon ZFC

1.4 Notation and terminology

1.4.1 The Language of Set Theory

We are about to define basic set-theoretical terminology on which the rest of this thesis will be built. For Chapter 2, the underlying theory will be the *Zermelo –Fraenkel* set theory with the Axiom of Choice (ZFC), a first-order set theory in the language $\mathscr{L} = \{=, \in\}$, which will be sometimes referred to as *the language of set theory*. In Chapter 3[6], we shall always make it clear whether we are in first-order ZFC or second-order ZFC_2, which will be precisely defined later in this chapter. When in second-order theory, we will usually denote type 1 variables, which are elements of the domain of discourse[7] by lower-case letters, mostly $u, v, w, x, y, z, p_1, p_2, p_3, \ldots$ while type 2 variables, which represent n-ary relations of the domain of discourse for any natural number n, are usually denoted by upper-case letters A, B, C, X, Y, Z. Note that those may be used both as relations and functions, see the definition of a function below.[8]

TODO uppercase M is a set!

TODO "M is a limit ordinal" je ve skutecnosti formule, nekam to sem napis!

The informal notions of *class* and *property* will be used throughout this thesis. They both represent formulas with respect to the domain of discourse. If $\varphi(x, p_1, \ldots, p_n)$ is a formula in the language of set theory, we call

$$A = \{x : \varphi(x, p_1, \ldots, p_n)\} \tag{1.1}$$

a class of all sets satisfying $\varphi(x, p_1, \ldots, p_n)$ in a sense that

$$x \in A \leftrightarrow \varphi(x, p_1, \ldots, p_n) \tag{1.2}$$

One can easily define for classes A, B the operations like $A \cap B$, $A \cup B$, $A \setminus C$, $\bigcup A$, but it is elementary and we won't do it here, see the first part of

[6]TODO bude jich vic? Chapter 4 taky?

[7]co je "domain of discourse"?

[8]TODO ref?

[4] for technical details. The following axioms are the tools by which decide whether particular classes are in fact sets. A class that fails to be considered a set is called a *proper class*.

1.4.2 The Axioms

Definition 1.1 *(The existence of a set)*

$$\exists x(x = x) \tag{1.3}$$

The above axiom is usually not used because it can be deduced from the axiom of *Infinity* (see below), but since we will be using set theories that omit *Infinity*, this will be useful.

Definition 1.2 *(Extensionality)*

$$\forall x, y(\forall z(z \in x \leftrightarrow z \in y) \leftrightarrow x = y) \tag{1.4}$$

Definition 1.3 *(Specification)*
The following is a schema for every first-order formula $\varphi(x, p_1, \ldots, p_n)$ with no free variables other than $x, p_1, \ldots, p_n$.

$$\forall x, p_1, \ldots, p_n \exists y \forall z(z \in y \leftrightarrow (z \in x \,\&\, \varphi(z, p_1, \ldots, p_n))) \tag{1.5}$$

We will now provide two definitions that are not axioms, but will be helpful in establishing some of the other axioms in a more intuitive way.

Definition 1.4 *($x \subseteq y$, $x \subset y$)*

$$x \subseteq y \leftrightarrow \forall z(z \in x \to z \in y) \tag{1.6}$$

$$x \subset y \leftrightarrow x \subseteq y \,\&\, x \neq y \tag{1.7}$$

Definition 1.5 *(Empty set)*

$$\emptyset \overset{\mathsf{def}}{=} \{x : x \neq x\} \tag{1.8}$$

To make sure that $\emptyset$ is a set, note that there exists at least one set y from 1.1, then consider the following alternative definition.

$$\emptyset' \overset{\mathsf{def}}{=} \{x : \varphi(x) \,\&\, x \in y\} \text{ where } y\ \varphi \text{ is the formula "} x \neq x \text{".} \tag{1.9}$$

It should be clear that $\emptyset' = \emptyset$.[9]

Now we can introduce more axioms.

[9] For details, see page 8 in [4].

Definition 1.6 *(Foundation)*

$$\forall x(x \neq \emptyset \to \exists(y \in x)(\forall z \neg(z \in y \;\&\; z \in x))) \tag{1.10}$$

Definition 1.7 *(Pairing)*

$$\forall x, y \exists z \forall q(q \in z \leftrightarrow q \in z \vee q \in y) \tag{1.11}$$

Definition 1.8 *(Union)*

$$\forall x \exists y \forall z(z \in x \leftrightarrow \exists q(z \in q \;\&\; q \in x)) \tag{1.12}$$

Definition 1.9 *(Powerset)*

$$\forall x \exists y \forall z(z \subseteq x \leftrightarrow z \in y) \tag{1.13}$$

Definition 1.10 *(Infinity)*

$$\exists x(\forall y \in x)(y \cup \{y\} \in x) \tag{1.14}$$

Let us introduce a few more definitions that will make the two remaining axioms more comprehensible.

Definition 1.11 *(Function)*
Given arbitrary first-order formula $\varphi(x, y, p_1, \dots, p_n)$, we say that φ is a function iff

$$\forall x, y, z, p_1, \dots, p_n(\varphi(x, y, p_1, \dots, p_n) \;\&\; \varphi(x, z, p_1, \dots, p_n) \to y = z) \tag{1.15}$$

When a $\varphi(x, y)$ is a function, we also write the following:

$$f(x) = y \leftrightarrow \varphi(x, y) \tag{1.16}$$

Note that this f is in fact a formula

TODO $f = \{(x, y) : \varphi(x, y)\}$!!! f muze byt mnozina i trida! [10]

Definition 1.12 *(Dom(f))*
Let f be a function. We read the following as "Dom(f) is the domain of f".

$$Dom(f) \stackrel{\text{def}}{=} \{x : \exists y(f(x) = y)\} \tag{1.17}$$

We say "f is a function on A", A being a class, if $A = dom(f)$.

[10] This can also be done for φs with more than two free variables by either setting $f(x, p_1, \dots p_n) = y \leftrightarrow \varphi(x, yp_1, \dots, p_n)$ or saying that φ codes more functions, determined by the various parameters, so $f(x) = y \leftrightarrow \varphi(x, y, t_1, \dots, t_n)$ for given terms $t_1, \dots, t_n$.

Definition 1.13 *(Rng(f))*
Let f be a function. We read the following as ”Rng(f) is the range of f”.

$$Rng(f) \stackrel{\text{def}}{=} \{x : \exists y(f(x) = y)\} \tag{1.18}$$

We say that f is i function into A, A being a class, if $rng(f) \subseteq A$.

Note that *Dom(f)* and *Rng(f)* are not definitions in a strict sense, they are in fact definition schemas that yield definitions for every function f given. Also note that they can be easily modified for φ instead of f, with the only difference that then it is defined only for those φs that are functions.

Definition 1.14 *(Powerset)*
TODO

And now for the axioms.

Definition 1.15 *(Replacement)*
The following is a schema for every first-order formula $\varphi(x, p_1, \ldots, p_n)$ with no free variables other than $x, p_1, \ldots, p_n$.

$$\text{”}\varphi \textit{ is a function”} \rightarrow \forall x \exists y \forall z(z \in y \leftrightarrow (\exists q \in x)(\varphi(x, y, p_1, \ldots, p_n)) \tag{1.19}$$

Definition 1.16 *(Choice)*
This is also a schema. For every A, a family of non-empty sets[11], such that $\emptyset \notin S$, there is a function f such that for every $x \in A$

$$f(x) \in x \tag{1.20}$$

We will refer the axioms by their name, written in italic type, e.g. *Foundation* refers to the Axiom of Foundation. Now we need to define some basic set theories to be used in the article. There will be others introduce in Chapter 3, but those will usually be defined just by appending additional axioms or schemata to one of the following.

Definition 1.17 (S)
We call S *a set theory with the following axioms:*
(i) Existence of a set *(see 1.1)*
(ii) Extensionality *(see 1.2)*
(iii) Specification *(see 1.3)*
(iv) Foundation *(see 1.6)*
(v) Pairing *(see 1.7)*

[11] We say a class A is a ”family of non-empty sets” iff there is B such that $A \subseteq \mathscr{P}(B)$

(vi) Union *(see 1.8)*
(vii) Powerset *(see 1.9)*

Definition 1.18 (ZF)
We call ZF *a set theory that contains all the axioms of the theory* S[12] *in addition to the following*
(i) Replacement *schema (see 1.15)*
(ii) Infinity *(see 1.10)*

Definition 1.19 (ZFC)
ZFC *is a theory that contains all the axioms of* ZF *plus* Choice *(1.16).*

1.4.3 The transitive universe

Definition 1.20 *(Transitive class)*
We say a class A is transitive *iff*

$$\forall x(x \in A \to x \subseteq A) \tag{1.21}$$

Definition 1.21 *Well Ordered Class A class A is said to be* well ordered by $\in$ *iff the following hold:*
(i) $(\forall x \in A)(x \notin x)$ *(Antireflexivity)*
(ii) $(\forall x, y, z \in A)(x \in y \;\&\; y \in z \to x \in z)$ *(Transitivity)*
(iii) $(\forall x, y \in A)(x = y \vee x \in y \vee y \in x)$ *(Linearity)*
(iv) $(\forall x)(x \subseteq A \;\&\; x \neq \emptyset \to (\exists y \in x)(\forall z \in x)(z = y \vee z \in y)))$

Definition 1.22 *(Ordinal number)*
A set x is said to be an ordinal number*, also known as an* ordinal*, if it is* transitive *and* well-ordered by $\in$.

For the sake of brevity, we usually just say "x is an *ordinal*". Note that "x is an ordinal" is a well-defined formula, since 1.20 is a formula and 1.21 is in fact a conjunction of four formulas. Ordinals will be usually denoted by lower case greek letters, starting from the beginning: $\alpha, \beta, \gamma, \ldots$. Given two different ordinals α, β, we will write $\alpha < \beta$ for $\alpha \in \beta$, see [4]Lemma 2.11 for technical details.

[12]With the exception of *Existence of a set*

Definition 1.23 *(Successor Ordinal)*
Consider the following operation

$$\beta + 1 \stackrel{\text{def}}{=} \beta \cup \{\beta\} \tag{1.22}$$

An ordinal α is called a successor ordinal *iff there is an ordinal β, such that $\alpha = \beta + 1$*

Definition 1.24 *(Limit Ordinal)*
A non-zero ordinal α[13] *is called a* limit ordinal *iff it is not a successor ordinal.*

Definition 1.25 *(Ord)*
The class of all ordinal numbers*, which we will denote Ord*[14] *be the following class:*

$$Ord \stackrel{\text{def}}{=} \{x : x \textit{ is an ordinal}\} \tag{1.23}$$

The following construction will be often referred to as the *Von Neumann's Hierarchy*, sometimes also the *Von Neumann's Universe*.

Definition 1.26 *(Von Neumann's Hierarchy)*
The Von Neumann's Hierarchy *is a collection of sets indexed by elements of Ord, defined recursively in the following way:*

(i)

$$V_0 = \emptyset \tag{1.24}$$

(ii)

$$V_{\alpha+1} = \mathscr{P}(V_\alpha) \textit{ for any ordinal } \alpha \tag{1.25}$$

(iii)

$$V_\lambda = \bigcup_{\beta<\lambda} V_\beta \textit{ for a limit ordinal } \lambda \tag{1.26}$$

Definition 1.27 *(Rank)*
Given a set x, we say that the rank of x (written as $rank(x)$) is the least ordinal α such that

$$x \in V_{\alpha+1} \tag{1.27}$$

Due to *Regularity*, every set has a rank.[15]

Definition 1.28 *(ω)*

$$\omega \stackrel{\text{def}}{=} \bigcap \{x : xisalimitordinal)\} \tag{1.28}$$

[13] $\alpha \neq \emptyset$
[14] It is sometimes denoted On, but we will stick to the notation in [4]
[15] See chapter 6 of [4] for details.

1.4.4 Cardinal numbers

Definition 1.29 *(Cardinality)*
Given a set x, let the cardinality of x, written $|x|$, be defined as the smallest ordinal number such that there is an injective mapping from x to α.

For formal details as well as why every set can be well-ordered assuming *Choice*, see [4].

Definition 1.30 *(Aleph function)*
Let ω be the set defined by **??**. *We will recursively define the function $\aleph$ for all ordinals.*

(i) $\aleph_0 = \omega$
(ii) $\aleph_{\alpha+1}$ *is the least cardinal larger than* $\aleph_\alpha$[16]
(iii) $\aleph_\lambda = \bigcup_{\beta<\lambda} \aleph_\beta$ *for a limit ordinal* λ

Definition 1.31 *(Cardinal number)*
We say a set x is a cardinal number*, usually called a* cardinal*, if either $x \in \omega$*

Cardinals will be notated by lower-case greek letters starting from $\kappa, \lambda, \mu, \ni, \ldots$[17].

Definition 1.32 *(Cofinality)*
Let λ be a limit ordinal. The cofinality *of λ, written $cf(\lambda)$, is the least limit ordinal α such that there is an increasing α-sequence*[18] $\langle \lambda_\beta : \beta < \alpha \rangle$ *with* $lim_{\beta\to\alpha}\lambda_\beta = \lambda$.

Definition 1.33 *(Limit Cardinal)*
We say that a cardinal κ is a limit cardinal *if*

$$(\exists\alpha \in Ord)(\kappa = \aleph_\alpha) \tag{1.29}$$

Definition 1.34 *(Strong Limit Cardinal)*
We say that an ordinal κ is a strong limit cardinal *if it is a* limit cardinal *and*

$$\forall\alpha(\alpha \in \kappa \to \mathscr{P}(\alpha) \in \kappa) \tag{1.30}$$

Definition 1.35 *(Generalised Continuum Hypothesis)*

$$\aleph_{\alpha+1} = 2^{\aleph_\alpha} \tag{1.31}$$

If *GCH* holds (for example in Gödel's L, see chapter 3), the notions of a limit cardinal and a strong limit cardinal are equivalent.

[16] "The least cardinal larger than $\aleph_\alpha$" is sometimes notated as $\aleph_\alpha^+$
[17] λ is also sometimes used for limit ordinals, the distinction should be clear from the context.
[18] TODO def α-sequence

1.4.5 Relativisation

Definition 1.36 *(Relativization)*
Let M be a class, R a binary relation on M and let $\varphi(p_1, \ldots, p_n)$ be a first-order formula with n parameters. The relativization of φ to M and R *is the formula, written as $\varphi^{M,R}(p_1, \ldots, p_n)$, defined in the following inductive manner:*

(i) $(x \in y)^{M,R} \leftrightarrow R(x, y)$
(ii) $(x = y)^{M,R} \leftrightarrow x = y$
(iii) $(\neg\varphi)^{M,R} \leftrightarrow \neg\varphi^{M,R}$
(iv) $(\varphi \ \& \ \psi)^{M,R} \leftrightarrow \varphi^{M,R} \ \& \ \psi^{M,R}$
(v) $(\exists x\varphi)^{M,R} \leftrightarrow (\exists x \in M)\varphi^{M,R}$

1.4.6 More functions

TODO def $f : Ord \to Ord$, asi u powersetu.

Definition 1.37 *(Strictly increasing function)*
A function $f : Ord \to Ord$ is said to be strictly increasing *iff*

$$\forall\alpha, \beta \in Ord(\alpha < \beta \to f(\alpha) < f(\beta)). \tag{1.32}$$

Definition 1.38 *(Continuous function)*
A function $f : Ord \to Ord$ is said to be continuous *iff*

$$\alpha \textit{ is limit} \to f(\lambda) = \bigcup_{\alpha<\lambda} f(\alpha). \tag{1.33}$$

Definition 1.39 *(Normal function)*
A function $f : Ord \to Ord$ is said to be normal *if it is* strictly increasing *and* continuous.

Definition 1.40 *Fixed point*
We say α is a fixed point of ordinal function f when $\alpha = f(\alpha)$.

1.4.7 Higher-Order Logic

Since we will utilise some basic tools of set theories formalized in second- and occassionally higher-order logic, we need to establish the basics here. This part is heavily inspired by Preliminaries from [**?**].

TODO viz kanamori p. 6

TODO proc se neda formalizovat obecne splnovani ve vyssich radech? cite?

While higher-order satisfaction relation for proper classes is unformalizable[19],we can formalize satisfaction in a structure. For the rest of this chapter, let D be a domain of such structure.

TODO druhoradove splnovani?

Definition 1.41 *(Hierarchy of formulas)*
Let φ be a formula. ((v logice radu n)) Π^m_n *und* Σ^m_n

Lemma 1.42 *Δ_0 formulas are absolute in transitive sets, in other words, let φ be a first-order Δ_0 formula and let M be a transitive class.*

$$\varphi \leftrightarrow \varphi^M \tag{1.34}$$

Definition 1.43 (ZFC_2)
TODO ?

TODO nenechat do patricne kapitoly? asi jo.

Definition 1.44 *(*Reflection$_1$*)*

$$ASD \tag{1.35}$$

2 Levy's first-order reflection

2.1 Lévy's Original Paper

This section will try to present Lévy's proof of a general reflection principle being equivalent to *Replacement* and *Infinity* under ZF minus *Replacement* and *Infinity* from his 1960 paper *Axiom Schemata of Strong Infinity in Axiomatic Set Theory*[20].

When reading said article, one should bear in mind that it was written in a period when set theory was semantically oriented, so while there are many statements about a model of ZF, usually denoted u, this is equivalent to today's universal class V, so it doesn't necessarily mean that there is a set u

[19] TODO CITE KDE? Tarski nebo tak neco?
[20] [2]

that is a model of ZF. We will review the notion of a standard complete model used by Lévy throughout the paper in a moment. Let's first say that the set theory ZF was formulated in the "non-simple applied first order functional calculus", is

TODO viz A. Church nebo tak neco.

The axioms are equivalent to those defined in 1.18, except for the *Axiom of Subsets*, which is just a different name for *Specification*. Besides ZF and S, defined in 1.18 and yrefdef:s respectively, the set theories theories Z, and SF are used in the text. Z is ZF minus replacement, SF is ZF minus *Infinity*. Also note that universal quantifier does not appear, $\forall x\varphi(x)$ would be written as $(x)\varphi(x)$. The symbol for negation is "$\sim$", implication is written as "$\supset$" and equivalence is "$\equiv$", we will use "$\neg$", "$\rightarrow$" and "$\leftrightarrow$".

Next two definitions are not used in contemporary set theory, but they illustrate 1960's set theory mind-set and they are used heavily in Lévy's text, so we will include and explain them for clarity. Generally, in this chapter, Q stands for an arbitrary axiomatic set theory used for general definitions, u is usually a model of Q, counterpart of today's V.

This subsection uses ZF instead of the usual ZFC as the underlying theory.

Definition 2.1 *(Standard model of a set theory)*
Let Q *be a axiomatic set theory in first-order logic. We say the the a class u is a standard model of* Q *with respect to a membership relation E, written as $Sm^{\mathsf{Q}}(u)$, iff both of the following hold*

(i) $(x, y) \in E \leftrightarrow y \in u \;\&\; x \in y$

(ii) $y \in u \;\&\; x \in y \rightarrow x \in u$

Definition 2.2 *Standard complete model of a set theory*
Let Q *and E be like in 2.1. We say that that u is a standard complete model of* Q *with respect to a membership relation E iff both of the following hold*

(i) u is a transitive set with respect to $\in$

(ii) $\forall E((x, y) \in E \leftrightarrow (y \in u \;\&\; x \in y) \;\&\; Sm^{\mathsf{Q}}(u, E))$

this is written as $Scm^{\mathsf{Q}}(u)$.

Definition 2.3 *(Inaccessible cardinal with respect to* Q*)*
Let Q *be an axiomatic first-order set theory. We say that a cardinal κ is inaccessible with respect to* Q*, we write $In^{\mathsf{Q}}(\kappa)$.*

$$In^{\mathsf{Q}}(\kappa) \stackrel{\text{def}}{=} Scm^{\mathsf{Q}}(V_\kappa). \tag{2.36}$$

Definition 2.4 *(Inaccessible cardinal with respect to* ZF*)*
When a cardinal κ is inaccessible with respect to ZF*, we only say that it is inaccessible. We write $In(\kappa)$.*

$$In(\kappa) \stackrel{\text{def}}{=} In^{\mathsf{ZF}}(\kappa) \tag{2.37}$$

The above definition of inaccessibles is used because it doesn't require *Choice*.

For the definition of relativization, see 1.36. The syntax used by Lévy is $Rel(u, \varphi)$, we will use φ^u, which is more usual these days.

Definition 2.5 *(N)*
The following is an axiom schema of complete reflection over ZF*, denoted as* N*.*

$$N \stackrel{\text{def}}{=} \exists u(Scm^{\mathsf{ZF}}(u) \;\&\; \forall x_1, \ldots, x_n(x_1, \ldots, x_n \in u \to \varphi \leftrightarrow \varphi^u)) \tag{2.38}$$

where φ *is a formula which contains no free variables except for* $x_1, \ldots, x_n$*.*

Definition 2.6 *(N_0)*
With S *instead of* ZF*we obtain what will now be called* N_0*.*

$$N_0 \stackrel{\text{def}}{=} \exists u(Scm^{\mathsf{S}}(u) \;\&\; \forall x_1, \ldots, x_n(x_1, \ldots, x_n \in u \to \varphi \leftrightarrow \varphi^u)) \tag{2.39}$$

where φ *is a formula which contains no free variables except for* $x_1, \ldots, x_n$*.*

2.2 $\mathsf{S} \models (\mathsf{N}_0 \leftrightarrow$ *Replacement* & *Infinity*)

Let S be a set theory defined in 1.17.

Lemma 2.7 *The following holds for every* u*.*

$$\text{"}u \text{ is a limit ordinal"} \leftrightarrow Scm^{\mathsf{S}}(u) \tag{2.40}$$

Proof. TODO !

—-

In order to prove that it is a model of S, we would need to verify all axioms of S. We have already shown that ω is closed under the powerset operation. Foundation, extensionality and comprehension are clear from the fact that we work in ZF[21], pairing is clear from the fact, that given two sets x, y, they have ranks α, β, without loss of generality we can assume that $\alpha \leq \beta$, which means that $x \in V_\alpha \in V_\beta$, therefore V_β is a set that satisfies the paring axiom: it contains both x and B.

□

Let N_0 be defined as in 2.6, for *Infinity* see 1.10.

Theorem 2.8 *In* S*, the schema* N_0 *implies* Infinity*.*

[21] We only need to verify axioms that provide means of constructing larger sets from smaller to make sure they don't exceed ω. Since ω is an initial segment of ZF, the axiom scheme of specification can't be broken, the same holds for foundation and extensionality.

Proof. Lévy skips this proof because it seems too obvious to him, but let's do it here for plasticity. For an arbitrary φ, N_0 gives us $\exists u Scm^{\mathsf{S}}(u)$, but from lemma 2.7, we know that this u is a limit ordinal. This u already satisfies *Infinity.* □

Let N_0 be defined as in 2.6, for *Replacement* see 1.15, S is again the set theory defined in 1.17.

Theorem 2.9 *In* S*, the schema* N_0 *implies* Replacement.

Proof. Let $\varphi(x, y, p_1, \ldots, p_n)$ be a formula with no free variables except $x, y, p_1, \ldots, p_n$ for an arbitrary natural number n.

$$\begin{aligned} \chi = \forall x, y, z(\varphi(x, y, p_1, \ldots, p_n) \,\&\, \varphi(x, z, p_1, \ldots, p_n) \rightarrow y = z) \\ \rightarrow \forall x \exists y \forall z(z \in y \leftrightarrow \exists q(q \in x \,\&\, \varphi(q, z, p_1, \ldots, p_n))) \end{aligned} \tag{2.41}$$

Let χ be an instance of *Replacement* schema for given φ. Let the following formulas be instances of the N_0 schema for formulas φ, $\exists y\varphi$, χ and $\forall x, p_1, \ldots, p_n\chi$ respectively:

We can deduce the following from N_0:

(i) $x, y, p_1, \ldots, p_n \in u \rightarrow (\varphi \leftrightarrow \varphi^u)$
(ii) $x, p_1, \ldots, p_n \in u \rightarrow (\exists y\varphi \leftrightarrow (\exists y\varphi)^u)$
(iii) $x, p_1, \ldots, p_n \in u \rightarrow (\chi \leftrightarrow \chi^u)$
(iv) $\forall x, p_1, \ldots, p_n(\chi \leftrightarrow (\forall x, p_1, \ldots, p_n\chi)^u)$

From relativization, we also know that $(\exists y\varphi)^u$ is equivalent to $(\exists y \in u)\varphi^u$. Therefore (**ii**) is equivalent to

$$x, p_1, \ldots, p_n \in u \rightarrow (\exists y \in u)\varphi^u. \tag{2.42}$$

If φ is a function[22], then for every $x \in u$, which is also $x \subset u$ by the transitivity of $Scm^{\mathsf{S}}(u)$, it maps elements of x onto u. From the axiom scheme of comprehension[23], we can find y, a set of all images of elements of x. That gives us $x, p_1, \ldots, p_n \in u \rightarrow \chi$. By (**iii**) we get $x, p_1, \ldots, p_n \in u \rightarrow \chi^u$, the universal closure of this formula is $(\forall x, p_1, \ldots, p_n\chi)^u$, which together with (**iv**) yields $\forall x, p_1, \ldots, p_n\chi$. Via universal instantiation, we end up with χ. We have inferred replacement for a given arbitrary formula. □

What we have just proven is just a single theorem from the above mentioned article by Lévy, we will introduce other interesting propositions, mostly related to the existence of large cardinals, later in their appropriate context in chapter 3.

[22] See definition 1.11

[23] Lévy uses its equivalent, axiom of subsets

2.3 Contemporary restatement

We will now prove what is also Lévy's first-order reflection theorem, but rephrased with up to date set theory terminology. The main difference is, that while Lévy reflects φ from V to a set u that is a "standard complete model of S", we say that there is a V_α for a limit α that reflects φ. We will argue that those are equivalent.[24]

Definition 2.10 *(Reflection$_1$)*
Let $\varphi(p_1, \ldots, p_n)$ be a first-order formula in the language of set theory. Than the following holds for any such φ.

$$\forall M_0 \exists M(M_0 \subseteq M \;\&\; (\varphi^M(p_1, \ldots, p_n) \leftrightarrow \varphi(p_1, \ldots, p_n))) \tag{2.43}$$

Note that this restatement on Lévy's N_0 from the previous chapter, see definition **??**. We prefer to call it *Reflection$_1$* so it complies with how other axioms and schemata are called. [25] Note that the subscript 1 refers to the fact that $\varphi(p_1, \ldots, p_n)$ is a first-order formula, and since we're using the work "reflection" in less strict meaning throughout this thesis, distinguishing between the two just by using italic font face for the schema might cause confusion.

We will now prove the equivalence of *Reflection$_1$* with *Replacement* and *Infinity* in S in two parts. First, we will show that N_0 is a theorem of ZFC, then we shall show that the second implication, which proves *Infinity* and *Replacement* from N_0, also holds.

The following lemma is usually done in more parts, the first being for one formula, the other for n formulas. We will only state and prove the more general version for n formulas, knowing that setting $n = 1$ turns it to a specific version.

Lemma 2.11 *Let $\varphi_1, \ldots, \varphi_n$ be formulas with m parameters*[26]*.*

(i) For each set M_0 there is such set M that $M_0 \subset M$ and the following holds for every i, $1 \leq i \leq n$:

$$\exists x \varphi_i(p_1, \ldots, p_{m-1}, x) \rightarrow (\exists x \in M)\varphi_i(p_1, \ldots, p_{m-1}, x) \tag{2.44}$$

for every $p_1, \ldots, p_{m-1} \in M$.

[24] TODO nekde na to bude lemma!

[25] We will not use the name N_0, because it might be confusing to work N_0 and M_0 where M_0 is a set and N_0 is an axiom schema.

[26] For formulas with a different number of parameters, take for m the highest number of parameters among those formulas. Add spare parameters to every formula that has less than m parameters in a way that preserves the last parameter, which we will denote x. E.g. let φ'_i be the a formula with k parameters, $k < m$. Let us set $\varphi_i(p_1, \ldots, p_{m-1}, x) \stackrel{\mathsf{def}}{=} \varphi'_i(p_1, \ldots, p_{k-1}, x)$, notice that the parameters $p_k, \ldots, p_{m-1}$ are not used.

(ii) *Furthermore there is an ordinal α such that $M_0 \subset V_\alpha$ and the following holds for each i, $1 \leq i \leq n$:*

$$\exists x \varphi_i(p_1, \ldots, p_{m-1}, x) \to (\exists x \in V_\alpha)\varphi_i(p_1, \ldots, p_{m-1}, x) \tag{2.45}$$

for every $p_1, \ldots, p_{m-1} \in M$.

(iii) *Assuming* Choice, *there is M, $M_0 \subset M$ such that 2.44 holds for every M, $i \leq n$ and $|M| \leq |M_0| \cdot \aleph_0$.*

Proof. We will simultaneously prove statements (**i**) and (**ii**), denoting M^T the transitive set required by part (**ii**). Unless explicitly stated otherwise for specific steps, it is thought to be equivalent to M.

Let us first define operation $H(p_1, \ldots, p_{m-1})$ that gives us the set of x's with minimal rank[27] satisfying $\varphi_i(p_1, \ldots, p_{m-1}, x)$ for given parameters $p_1, \ldots, p_{m-1}$ for every i such that $1 \leq i \leq n$.

$$H_i(p_1, \ldots, p_n) = \{x \in C_i : (\forall z \in C)(rank(x) \leq rank(z))\} \tag{2.46}$$

for each $1 \leq i \leq n$, where

$$C_i = \{x : \varphi_i(p_1, \ldots, p_{m-1}, x)\} \text{ for } 1 \leq i \leq n \tag{2.47}$$

Next, let's construct M from given M_0 by induction.

$$M_{i+1} = M_i \cup \bigcup_{j=0}^{n} \bigcup \{H_j(p_1, \ldots, p_{m-1}) : p_1, \ldots, p_{m-1} \in M_i\} \tag{2.48}$$

In other words, in each step we add the elements satisfying $\varphi(p_1, \ldots, p_{m-1}, x)$ for all parameters that were either available earlier or were added in the previous step. For statement (**ii**), this is the only part that differs from (**i**). Let us take for each step transitive closure of M_{i+1} from (**i**). In other words, let γ be the smallest ordinal such that

$$(M_i^T \cup \bigcup_{j=0}^{n} \{\bigcup \{H_j(p_1, \ldots, p_{m-1}) : p_1, \ldots, p_{m-1} \in M_i\}\}) \subset V_\gamma \tag{2.49}$$

Then the incremetal step is like so:

$$M_{i+1}^T = V_\gamma \tag{2.50}$$

[27] Rank is defined in 1.27

The final M is obtained by joining all the individual steps.

$$M = \bigcup_{i=0}^{\infty} M_i, \;\; M^T = \bigcup_{i=0}^{\infty} M_i^T = V_\alpha \tag{2.51}$$ [28]

We have yet to finish part (**iii**). Let's try to construct a set M' that satisfies the same conditions like M but is kept as small as possible. Assuming the Axiom of Choice, we can modify the process so that the cardinality of M' is at most $|M_0| \cdot \aleph_0$. Note that the size of M' is determined by the size of M_0 and, most importantly, by the size of $H_i(p_1, \ldots, p_{m-1})$ for any i, $1 \leq i \leq n$ in individual levels of the construction. Since the lemma only states existence of some x that satisfies $\varphi_i(p_1, \ldots, p_{m-1}, x)$ for any $1 \leq i \leq n$, we only need to add one x for every set of parameters but $H_i(u_1, \ldots, u_{m-1})$ can be arbitrarily large. Since Axiom of Choice ensures that there is a choice function, let F be a choice function on $\mathscr{P}(M')$. Also let $h_i(p_1, \ldots, p_{m-1}) = F(H_i(p_1, \ldots, p_{m-1}))$ for i, where $1 \leq i \leq n$, which means that h is a function that outputs an x that satisfies $\varphi_i(p_1, \ldots, p_{m-1}, x)$ for i such that $1 \leq i \leq n$ and has minimal rank among all such witnesses. The induction step needs to be redefined to

$$M'_{i+1} = M'_i \cup \bigcup_{j=0}^{n} \{H_j(p_1, \ldots, p_{m-1}) : p_1, \ldots, p_{m-1} \in M'_i\} \tag{2.52}$$

This way, the amount of elements added to M'_{i+1} in each step of the construction is the same as the amount of sets of parameters that yielded elements not included in M'_i. It is easy to see that if M_0 is finite, M' is countable because it was constructed as a countable union of finite sets. If M_0 is countable or larger, the cardinality of M' is equal to the cardinality of M_0.[29] Therefore $|M'| \leq |M_0| \cdot \aleph_0$ □

Theorem 2.12 *(Lévy's first-order reflection theorem)*
Let $\varphi(p_1, \ldots, p_n)$ be a first-order formula.

(i) For every set M_0 there exists M such that $M_0 \subset M$ and the following holds:

$$\varphi^M(p_1, \ldots, p_n) \leftrightarrow \varphi(p_1, \ldots, p_n) \tag{2.53}$$

for every $p_1, \ldots, p_n \in M$.

[29] It can not be smaller because $|M'_{i+1}| \geq |M'_i|$ for every i. It may not be significantly larger because the maximum of elements added is the number of n-tuples in M'_i, which is of the same cardinality is M'_i.

(ii) For every set M_0 there is a transitive set M, $M_0 \subset M$ such that the following holds:

$$\varphi^M(p_1, \ldots, p_n) \leftrightarrow \varphi(p_1, \ldots, p_n) \tag{2.54}$$

for every $p_1, \ldots, p_n \in M$.

(iii) For every set M_0 there is α such that $M_0 \subset V_\alpha$ and the following holds:

$$\varphi^{V_\alpha}(p_1, \ldots, p_n) \leftrightarrow \varphi(p_1, \ldots, p_n) \tag{2.55}$$

for every $p_1, \ldots, p_n \in M$.

(iv) Assuming Choice*, for every set M_0 there is M such that $M_0 \subset M$ and $|M| \leq |M_0| \cdot \aleph_0$ and the following holds:*

$$\varphi^M(p_1, \ldots, p_n) \leftrightarrow \varphi(p_1, \ldots, p_n) \tag{2.56}$$

for every $p_1, \ldots, p_n \in M$.

Proof. Before we start, note that the following holds for any set M if φ is an atomic formula, as a direct consequence of relativisation to $M, \in$[30].

$$\varphi \leftrightarrow \varphi^M \tag{2.57}$$

Let's now prove (**i**) for given φ via induction by complexity. We can safely assume that φ contains no quantifiers besides "$\exists$" and no logical connectives other than "$\neg$" and "&". Let $\varphi_1, \ldots, \varphi_n$ be all subformulas of φ. Then there is a set M, obtained by the means of lemma 2.11, for all of the formulas $\varphi_1, \ldots, \varphi_n$.

We know that $\psi \leftrightarrow \psi^M$ for atomic ψ, we need to verify that it won't fail in the inductive step. Let us consider $\psi = \neg\psi'$ along with the definition of relativization for those formulas in 1.36.

$$(\neg\psi')^M \leftrightarrow \neg(\psi'^M) \tag{2.58}$$

Because the induction hypothesis says that 2.53 holds for every subformula of ψ, we can assume that $\psi'^M \leftrightarrow \psi'$, therefore the following holds:

$$(\neg\psi')^M \leftrightarrow \neg(\psi'^M) \leftrightarrow \neg\psi' \tag{2.59}$$

The same holds for $\psi = \psi_1 \,\&\, \psi_2$. From the induction hypothesis, we know that $\psi_1^M \leftrightarrow \psi_1$ and $\psi_2^M \leftrightarrow \psi_2$, which together with relativization for formulas in the form of $\psi_1 \,\&\, \psi_2$ gives us

$$(\psi_1 \,\&\, \psi_2)^M \leftrightarrow \psi_1^M \,\&\, \psi_2^M \leftrightarrow \psi_1 \,\&\, \psi_2 \tag{2.60}$$

[30]See **??**. Also note that this works for relativization to $M, \in$, not M, E where E is an arbitrary membership relation on M.

Let's now examine the case when from the induction hypothesis, M reflects $\psi'(p_1,\dots,p_n,x)$ and we are interested in $\psi = \exists x\psi'(p_1,\dots,p_n,x)$. The induction hypothesis tells us that

$$\varphi'^M(p_1,\dots,p_n,x) \leftrightarrow \psi'(p_1,\dots,p_n,x) \tag{2.61}$$

so, together with above lemma 2.11, the following holds:

$$\begin{gathered}
\psi(p_1,\dots,p_n,x)\\
\leftrightarrow \exists x\psi'(p_1,\dots,p_n,x)\\
\leftrightarrow (\exists x\in M)\psi'(p_1,\dots,p_n,x)\\
\leftrightarrow (\exists x\in M)\psi'^M(p_1,\dots,p_n,x)\\
\leftrightarrow (\exists x\psi'(p_1,\dots,p_n,x))^M\\
\leftrightarrow \psi^M(p_1,\dots,p_n,x)
\end{gathered} \tag{2.62}$$

Which is what we have needed to prove. 2.53 holds for all subformulas $\varphi_1,\dots,\varphi_n$ of a given formula φ.

So far we have proven part (**i**) of this theorem for one formula φ, we only need to verify that the same holds for any finite number of formulas. This has in fact been already done since lemma 2.11 gives us M for any (finite) amount of formulas, we can find a set M for the union of all of their subformulas. We can than use the induction above to verify that M reflects each of the formulas individually iff it reflects all of its subformulas.

Since V_α is a transitive set, by proving (**iii**) we also satisfy (**ii**). To do so, we only need to look at part (**ii**) of lemma 2.11. All of the above proof also holds for $M = V_\alpha$.

To finish part (**iv**), we take M of size $\le |M_0|\cdot\aleph_0$, which exists due to part (**iii**) of lemma 2.11, the rest being identical. ☐

Let S be a set theory defined in 1.17, for ZFC see 1.19.

Lemma 2.13 *Let M be a set. Then the following holds:*

$$\mathsf{ZFC} \models (\mathsf{M} \models \mathsf{S}) \leftrightarrow \text{"}M \textit{ is a limit cardinal}\text{"} \tag{2.63}$$

Proof. For the left-to-right direction, we shall verify that if M is a model of S, it necessarily is a limit cardinal. From *Powerset*[31], we know that for

[31] 1.9.

any $x \in M$, $\mathscr{P}(x) \in M$. But that is already the definition of a strong limit cardinal[32].

For the converse, we need to see that if there is a limit ordinal α, such that $V_\alpha = M$, the axioms of S hold M.

(i) *Existence of a set* (see 1.1)
There obviously is a set $x \in M$

(ii) *Extensionality* (see 1.2)
Since $Extensionality^M$ is a Δ_0 formula, it holds in any transitive class by 1.42.

(iii) *Specification* (see 1.3)
TODO

(iv) *Foundation* (see 1.6)
$Foundation^M$ is also a Δ_0 formula, so it holds by 1.42 since M is transitive because it is a cardinal.

(v) *Pairing* (see 1.7)
TODO

(vi) *Union* (see 1.8
TODO

(vii) *Powerset* (see 1.9)
TODO

□

Let *Infinity* and *Replacement* be as defined in 1.10 and 1.15 respectively.

Theorem 2.14 Reflection*1* *is equivalent to* Infinity & Replacement *under* S.

Proof. Since 2.12 already gives us one side of the implication, we are only interested in showing the converse which we shall do in two parts:

TODO N_0 prepsat zpatky na *Reflection*$_1$

$\mathbf{N_0} \rightarrow$ *Infinity* From N_0 (**??**), we know that for any first-order formula φ and a set M_0, there is a M such that $M_0 \subseteq M$ and $\varphi^M \leftrightarrow \varphi$. Let's pick *Powerset* for φ, then by N_0 there is a set that satisfies *Powerset*, ergo there is a strong limit cardinal, which in turn satisfies *Infinity*.

Reflection $\rightarrow$ *Replacement*

Given a formula $\varphi(x, y, p_1, \ldots, p_n)$, we can suppose that it is reflected in any M [33] What we want to obtain is the following:

$$\forall x, y, z(\varphi(x, y, p_1, \ldots, p_n) \ \& \ \varphi(x, z, p_1, \ldots, p_n) \rightarrow y = z) \rightarrow\rightarrow \forall X \exists Y \forall y \ (y \in Y \leftrightarrow \exists x(\varphi(x, y, p_1, . \tag{2.64}$$

[32] see **??**

[33] Which means that for $x, y, p_1, \ldots, p_n \in M$, $\varphi^M(x, y, p_1, \ldots, p_n) \leftrightarrow \varphi(x, y, p_1, \ldots, p_n)$.

We do also know that $x, y \in M$, in other words for every X, $Y = \{y \mid \varphi(x, y, p_1, \ldots, p_n)\}$ and we know that $X \subset M$ and $Y \subset M$, which, together with the comprehension schema implies that Y, the image of X over φ, is a set. □

We have shown that *Reflection* for first-order formulas, *Reflection*$_1$ is a theorem of ZF, which means that it won't yield us any large cardinals. We have also shown that it can be used instead of the *Infinity* and *Replacement* scheme, but ZF + *Reflection*$_1$ is a conservative extension of ZF. Besides being a starting point for more general and powerful statements, it can be used to show that ZF is not finitely axiomatizable. That follows from the fact that *Reflection* gives a model to any finite number of (consistent) formulas. So if $\varphi_1, \ldots, \varphi_n$ for any finite n would be the axioms of ZF, *Reflection* would always contain a model of itself, which would in turn contradict the Second Gödel's Theorem[34]. Notice that, in a way, reflection is complementary to compactness. Compactness argues that given a set of sentences, if every finite subset yields a model, so does the whole set. Reflection, on the other hand, says that while the whole set has no model in the underlying theory, every finite subset does have one.

Also, notice how reflection can be used in ways similar to upward Löwenheim–Skolem theorem. Since Reflection extends any set M_0 into a model of given formulas $\varphi_1, \ldots, \varphi_n$, we can choose the lower bound of the size of M by appropriately chocing M_0.

In the next section, we will try to generalize *Reflection* in a way that transcends ZF and finally yields some large cardinals.

3 Reflection And Large Cardinals

In this chapter we aim to examine stronger reflection properties in order to reach cardinals unavailable in ZFC. Like we said in the first chapter, the variety of reflection principles comes from the fact that there are many way to formalize "properties of the universal class". It is not always obvious what properties hold for V because, (TODO Tarski) We have shown that reflecting properties as first-order formulas doesn't allow us to leave ZFC. We will broaden the class of admissible properties to be reflected and see whether there is a natural limit in the height or width on the reflected universe and also see that no matter how far we go, the universal class is still as elusive as it is when seen from S. That is because for every process for obtaining larger sets such as for example the powerset operation in ZFC, this process

[34] See chapter 3.3 for further details.

can't reach V and thus, from reflection, there is an initial segment of V that can't be reached via said process.

To see why this is important, let's dedicate a few lines to the intuition behind the notions of limitness, regularity and inaccessibility in a manner strongly influenced by [8]. To see why limit and strongly limit cardinals are worth mentioning, note that they are "limit" not only in a sense of being a supremum of an ordinal sequence, they also show that a certain way of obtaining larger sets from smaller ones is limited. We will see that all of the alternatives offered in this thesis are in a sense limited. $\aleph_\lambda$ is a limit cardinal iff there is no α such that $\aleph_{\alpha+1} = \aleph_\lambda$. Strongly limit cardinals point to the limits of the powerset operation. It has been too obvious so far, so let's look at the regular cardinals in this manner. Regular cardinals are those that cannot be[35], expressed as a supremum of smaller amount of smaller objects[36]. More precisely, κ is regular if there is no way to define it as a union of less than κ ordinals, all smaller than κ. So unless there already is a set of size κ, *Replacement* is useless in determining whether κ is really a set. Note that assuming *Choice*, successor cardinals are always regular, so most[37] limit cardinals are singular cardinals. So if one is traversing the class of all cardinals upwards, successor steps are still sets thanks to the powerset axiom while singular limits cardinal are not proper classes because they are suprema of images of smaller sets via *Replacement*. Regular cardinals are, in a way, limits of how far can we get by taking limits of increasing sequences of ordinals obtained via *Replacement*.

TODO prepsat – regularita a replacement, proc reflexe dava silnejsi veci

That all being said, it is easy to see that no cardinals in ZFC are both strongly limit and regular because there is no way to ensure they are sets and not proper classes in ZFC. The only exception to this rule is $\aleph_0$ which needs *Infinity* to exist. It should now be obvious why the fact that κ is inaccessible implies that $\kappa = aleph_\kappa$.[38]

We will also examine the connection between reflection principles and (regular) fixed points of ordinal functions in a manner proposed by Lévy in [2]. We will also see that, like Lévy has proposed in the same paper, there is a meaningful way to extend the relation between S and ZFC into a hierarchy of stronger axiomatic set theories.

[35] Assuming *Choice*.

[36] Just like ω can not be expressed as a supremum of a finite set consisting solely of finite numbers.

[37] All provable to exist in ZFC

[38] This doesn't work backwards, the least fixed point of the $\aleph$ function is the limit of $\{\aleph_0, \aleph_{\aleph_0}, \aleph_{\aleph_{\aleph_0}}, \ldots\}$, it is singular since the sequence has countably many elements.

3.1 Regular Fixed-Point Axioms

Lévy's article mentions various schemata that are not instances of reflection per se. We will mention them because they are equivalent to *Reflection$_1$*[39].

Definition 3.1 *(*Axiom M_1*)*
"Every normal function defined for all ordinals has at least one inaccessible number in its range."

Lévy uses "M" to refer to this axiom but since we also use "M" for sets and models, for example in 2.10, we will call the above axiom "*Axiom M_1*" to avoid confusion.

Now we will express *Axiom M_1* to formula to make it clear that it is an axiom scheme and the same can be done with *Axiom M'_1* as well as *Axiom M''_1* introduced immediately afterwards. Since it is an axiom schema and we will later dive into second-order logic, we may also want to refer to *Axiom M_2* as opposed *Axiom M_1*, the former being a single second-order sentence obtained by the obvious modification of *Axiom M_1*.[40]

Let $\varphi(x, y, p_1, \ldots, p_n)$ be a first-order formula with no free variables besides $x, y, p_1, \ldots, p_n$. The following is equivalent to *Axiom M_1*.

$$\begin{gathered}\text{"}\varphi \text{ is a normal function" } \& \ \forall x(x \in Ord \rightarrow \exists y(\varphi(x, y, p_1, \ldots, p_n))) \rightarrow \\ \rightarrow \exists y(\exists x \varphi(x, y, p_1, \ldots, p_n) \ \& \ cf(y) = y \ \& \ (\forall x \in \kappa)(\exists y \in \kappa)(x > y))\end{gathered} \tag{3.65}$$

[41]

Definition 3.2 *(Axiom M'_1)*
Every normal function defined for all ordinals has at least one fixed point which is inaccessible.

Definition 3.3 *(Axiom M''_1)*
"Every normal function defined for all ordinals has arbitrarily great fixed points which are inaccessible."

The following axiom is proposed by Drake in [3].

Definition 3.4 *(Axiom F_1)*
Every normal function defined for all ordinals has a regular fixed point.

[39] For definition, see 2.10

[40] Second-order set theory will be introduced in the next subsection.

[41] "φ is a normal function" is equivalent to the following first-order formula:

Lemma 3.5 *(Fixed-point lemma for normal functions)*
Let f be a normal function defined for all ordinals. The all of the following hold

(i)

$$\forall\lambda(\text{"}\lambda \textit{ is a limit ordinal"} \rightarrow \text{"}\textit{f}(\lambda) \textit{ is a limit ordinal"}) \tag{3.66}$$

(ii)

$$\forall\alpha(\alpha \leq f(\alpha)) \tag{3.67}$$

(iii)

$$\forall\alpha\exists\beta(\alpha < \beta \;\&\; f(\beta) = \beta)(f \textit{ has arbitrarily large fixed points.}) \tag{3.68}$$

Proof. Let f be a normal function.

(i) Proof of (**i**):
Suppose λ is a limit ordinal. For an arbitrary ordinal $\alpha < \lambda$, the fact that f is strictly increasing means that $f(\alpha) < f(\lambda)$ and for an ordinal β, $\beta < \alpha$, $f(\alpha) < f(\beta)$. Because f is continuous and λ limit, $f(\lambda) = \bigcup_{\alpha<\lambda} f(\alpha)$ and since $\beta < \lambda$, $f(\beta) < f(\lambda)$. So we have found $f(\beta)$ such that $f(\alpha) < f(\beta) < f(\lambda)$, therefore $f(\lambda)$ is a limit ordinal.

(ii) Let's prove (**ii**) via transfinite induction:
Since f is defined for all ordinals, there is an ordinal α such that $f(\emptyset) = \alpha$ and because $\emptyset$ is the least ordinal, (**ii**) holds for $\emptyset$.
Suppose (**ii**) holds for some β form the induction hypothesis. It the holds for $\beta + 1$ because f is strictly increasing.
For a limit ordinal λ, suppose (**ii**) holds for every $\alpha < \lambda$. (**i**) implies that $f(\lambda)$ is also limit, so there is a strictly increasing κ-sequence $\langle\alpha_0, \alpha_1, \ldots\rangle$ for some κ such that $\lambda = \bigcup_{i<\kappa} \alpha_i$. Because f is stricly increasing, the κ-sequence $\langle f(\alpha_0), f(\alpha_1), \ldots$ is also strictly increasing, the induction hypothesis implies that $\alpha_i \leq f(\alpha_i)$ for each $i \leq \kappa$. Thus, $\lambda \leq f(\lambda)\rangle$.

(iii) Proof of (**iii**):

For a given α, let there be a ω-sequence $\langle\alpha_0, \alpha_1, \ldots\rangle$, such that $\alpha_0 = \alpha$ and $\alpha_{i+1} = f(\alpha_i)$ for each $i < \omega$. This sequence is stricly increasing because so is f. Now, there's a limit ordinal $\beta = \bigcup_{i<\omega} \alpha_i$, we want to show that this is the fixed point. So $f(\beta) = f(\bigcup_{i<\omega} \alpha_i) = \bigcup_{i<\omega} f(\alpha)$ because f is continuous. We have defined the above sequence so that β, $\bigcup_{i<\omega} f(\alpha) = \bigcup_{i<\omega} \alpha_{i+1}$, which means we are done, since $\bigcup_{i<\omega} \alpha_{i+1} = \bigcup_{i<\omega} \alpha_i = \beta$.

$\square$

Theorem 3.6

$$\text{Axiom } M_1 \leftrightarrow \text{Axiom } M'_1 \leftrightarrow \text{Axiom } M''_1 \leftrightarrow \text{Axiom } F_1 \tag{3.69}$$

This is *Theorem 1* in [2]. *Proof.* It is clear that *Axiom* M''_1 is a stronger version of *Axiom* M'_1, which is in turn a stronger version of both *Axiom* M_1 and *Axiom* F_1, so the implication *Axiom* $M''_1 \rightarrow$ *Axiom* $M'_1 \rightarrow$ *Axiom* M_1 is satisfied and *Axiom* $M'_1 \rightarrow$ *Axiom* F_1 holds too.

We will now make sure that *Axiom* $M_1 \rightarrow$ *Axiom* M''_1 also holds. Let f be a normal function defined for all ordinals. Let g be a normal function that counts the fixed points of f. Lemma 3.5 implies that there arbitrarily many fixed points of f, therefore g is defined for all ordinals. Let there be another family of functions, $h_\alpha(\beta) = g(\alpha + \beta)$, obviously h_α is defined for all ordinals for every $\alpha \in Ord$ because so is g. Given an arbitrary ordinal γ, from *Axiom* M_1 we can assume that there is an ordinal δ such that such that $h_\alpha(\delta) = \kappa$, where κ is inaccessible. But since $\kappa = g(\alpha + \delta)$, κ is a fixed point of f. To show that there are arbitrarily many fixed points of f, notice that γ is arbitrary and h_γ is a normal function, so, by lemma 3.5, $(\forall \alpha \in Ord)(\alpha \leq f(\alpha)$, therefore $\gamma \leq \gamma + \alpha \leq \kappa$, in other words, there is κ above an arbitrary ordinal γ.

□

TODO nevyhodime F?

3.2 Reflecting Second-Order Formulas

To see that there is a way to transcend ZFC, let us briefly show how a model of ZFC can be obtained in ZFC_2 + "second-order reflection"[42]. This will be more closely examined in section 3.3.

We know that ZFC can not be finitely axiomatized in first-order formulas, however if *Replacement* and *Comprehension* schemes can be substituted by second-order formulas, ZFC becomes ZFC_2, which is finitely axiomatizable in second-order logic. Therefore if we take second-order reflection into consideration, we can obtain a set M that is a model of ZFC_2. For now, we have left out the details of how exactly is first-order reflection generalised into stronger statements and how second-order axiomatization of ZFC looks like as we will examine those problems closely in the following pages.

Lower-case letters represent first-order variables and upper-case P represents a second-order variable. [9]

[42] ZFC_2 is an axiomatization of ZFC in second-order formulas, to be more rigorously established later.

Definition 3.7 *(*Replacement$_2$*)*

$$\forall P(\forall x, y, z(P(x,y) \mathbin{\&} P(x,z) \to y = z) \to \to (\forall x \exists y \forall z(z \in y \leftrightarrow \exists w \in x(P(w,z))))) \tag{3.70}$$

We will denote this axiom Replacement$_2$.

Definition 3.8 *(*Specification$_2$*)*

$$\forall P \forall x \exists y \forall z\,(z \in y \leftrightarrow (z \in x \mathbin{\&} P(z,x))) \tag{3.71}$$

Definition 3.9 *(*ZFC_2)
Let ZFC_2 *be a theory with all axioms identical with the axioms of* ZFC *with the exception of* Replacement *and* Specification *schemes, which are replaced with* Replacement$_2$ *and* Specification$_2$ *respectively.*

3.3 Inaccessibility

Definition 3.10 *(limit cardinal) κ is a* limit cardinal *iff it is $\aleph_\alpha$ for some limit ordinal α.*

Definition 3.11 *(strong limit cardinal) κ is a* strong limit cardinal *iff it is a limit cardinal and for every $\lambda < \kappa$, $2^\lambda < \kappa$*

The two above definition become equivalent when we assume GCH.

Definition 3.12 *(weak inaccessibility) An uncountable cardinal κ is* weakly inaccessible *iff it is* regular *and* limit.

Definition 3.13 *(inaccessibility) An uncountable cardinal κ is* inaccessible *(written $In(\alpha)$) iff it is* regular *and* strongly limit.

We will now show that the above notion is equivalent to the definition Lévy uses in [2], which is, in more contemporary notation, the following:

Theorem 3.14 *The following are equivalent:*

1. *κ in inaccessible*
2. $\langle V_\kappa, \in \rangle \models \mathsf{ZFC}$

Proof. Let's first prove that if κ is inaccessible, it is a model of ZFC. We will do that by verifying the axioms of ZFC just like Kanamori does it in in [1, 1.2] and Drake in [3, Chapter 4].

(i) *Extensionality*:
(see 1.2)
$$V_\kappa \models \forall x, y(\forall z(z \in x \leftrightarrow z \in y) \to x = y) \tag{3.72}$$
We need to prove that, given two sets that are equal in V, they are equal in V_κ, in other words, that the *Extensionality* formula is reflected, that is
$$V_\kappa \models \forall x, y \in V_\kappa(\forall z \in V_\kappa(z \in x \leftrightarrow z \in y) \to x = y) \tag{3.73}$$
But that comes from transitivity. If x and y are in V_κ their members are also in V_κ.

(ii) *Foundation*:
(see 1.6)
$$V_\kappa \models \forall x(\exists z(z \in x) \to \exists z(z \in x \,\&\, \forall u \neg(u \in z \,\&\, u \in x))) \tag{3.74}$$
The argument for *Foundation* is almost identical to the one for *Extensionality*. For any set $x \in V_\kappa$, transitivity of V_κ makes sure that every element of x is also an element of V_κ and the same holds for the elements of elements of x et cetera. So statements about those elements are absolute between any transitive structures. V and V_κ are both transitive therefore *Foundation* holds and so does its relativisation to V_κ, $Foundation^{V_\kappa}$.

(iii) *Powerset*:
(see 1.9)
$$V_\kappa \models \forall x \exists y \forall z(z \subseteq x \to z \in y). \tag{3.75}$$
If we take x, an element of V_κ, $\mathscr{P}(x)$ has to be an element of V_κ to, because it is transitive and a strong limit cardinal.

(iv) *Pairing*:
(see 1.7)
$$V_\kappa \models \forall x, y \exists z(x \in z \wedge y \in z). \tag{3.76}$$
Pairing holds from similar argument like above: let x and y be elements of V_κ, so there are ordinals $\alpha, \beta < \kappa$ such that $x \in V_\alpha$, $y \in V_\beta$. Without any loss of generality, suppose $\alpha < \beta$, threfore $V_\alpha \subset V_\beta$ which, from transitivity of the cumulative hierarchy, means that $x \in V_\beta$, then $\{x, y\} \in V_{\beta+1}$ which is still in V_κ because it is a strong limit cardinal.

(v) *Union*:
(see 1.8)

$$V_\kappa \models \forall x\, \exists y\, \forall z\, \forall w((w \in z \wedge z \in x) \to w \in y). \tag{3.77}$$

We want to see that for every $x \in V_\kappa$, this is equivalent to

$$V_\kappa \models \forall x \in V_\kappa, \exists y \in V_\kappa \forall z \in V_\kappa \forall w \in V_\kappa((w \in z \wedge z \in x) \to w \in y). \tag{3.78}$$

Since V_κ is transitive, if $x \in V_\kappa$, all of its elements as well as their elements are in V_κ. To see that they also form a set themselves we only need to remember that V_κ is limit and therefore if α is the least ordinal such that $x \in V_\alpha$, $\bigcup x \in V_{\alpha+1}$.

(vi) *Replacement, Infinity*:
(see 1.15, 1.10)
We know that those hold from 2.14.

We will now show that if a set is a model of ZFC, it is in fact an inaccessible cardinal. So let V_κ be a model of ZFC which means that it is closed under the powerset operation, in other words:

$$\forall \lambda(\lambda < \kappa \to 2^\lambda \ < \kappa) \tag{3.79}$$

which is exactly the definition of strong limitness. κ is regular from the following argument by contradiction:
Let us suppose for a moment that κ is singular. Therefore there is an ordinal $\alpha < \kappa$ and a function $F:\ \alpha \to \kappa$ such that the range of F in unbounded in κ, in other words, $F[\alpha] \subseteq V_\kappa$ and $sup(F[\alpha]) = kappa$. In order to achieve the desired contradiction, we need to see that it is the case that $F[\alpha] \in V_\kappa$. Let $\varphi(x, y)$ be the following first-order formula:

$$F(x) \ = \ y \tag{3.80}$$

Then there is an instance of *Replacement* that states the following:

$$\begin{gathered}(\forall x, y, z(\varphi(x, y) \ \&\ \varphi(x, z) \to y \ = \ z)) \to \\ \to (\forall x \exists y \forall z(z \in y \leftrightarrow \exists w(\varphi(w, z))))\end{gathered} \tag{3.81}$$

Which in turn means that there is a set $y = F[\alpha]$ and $y \in V_\kappa$, which is the contradiction with $sup(y) = \kappa$ we are looking for. □

The same holds for ZFC_2, the proof is very similar.

Theorem 3.15

$$V_\kappa \models \mathsf{ZFC}_2 \leftrightarrow \kappa \textit{ is inaccessible} \tag{3.82}$$

Proof. κ is a strong limit cardinal because from ZFC_2 and *Powerset* we know that for every $\lambda < \kappa$, we know that $2^\lambda < \kappa$.

κ is also regular, because otherwise there would be an ordinal α and a function $F: \alpha \to \kappa$ with a range unbounded in κ. *Replacement*^{2} gives us a set $y = F[\alpha]$, so $y \in V_\kappa$, which contradicts the fact that $sup(y) = kappa$. It can not be the case that $\kappa \in V_\kappa$.

The other direction is exactly like the first part of above theorem 3.14. □

This is how the existence of an inaccessible cardinal is established in [2].

Definition 3.16 N

$$\exists u(In(\alpha) \;\&\; \forall p_1, \ldots, p_n(p_1, \ldots, p_n \in u \to (\varphi \leftrightarrow \varphi^u))) \tag{3.83}$$

It is interesting to see that the above schema yields the first inaccessible cardinal if we take for φ the conjunction of all axioms of ZF_2.

To see that inaccessible cardinal can be also obtained by a fixed-point axiom (or a scheme if were in first-order logic), see the following theorem by Lévy, we won't repeat the proof here, it is available in [2, Theorem 3],

Theorem 3.17

$$M \leftrightarrow N \tag{3.84}$$

We have transcended ZFC, but that is just a start. Naturally, we could go on and consider the next inaccessible cardinal, which is inaccessible with respect to the theory $\mathsf{ZFC} + \exists\kappa(\kappa \models \mathsf{ZFC})$. But let's try to find a faster way up, informally at first.

Since we can find an inaccessible set larger than any chosen set M_0, it is clear that there are arbitrarily large inaccessible cardinals in V, they are "unbounded"[43] in V. If V were a cardinal, we could say that there are V inaccesible cardinals less than V, but this statement of course makes no sense in set theory as is because V is not a set. But being more careful, we could find a property that can be formalized in second-order logic and reflect it to an initial segment of V. That would allow us to construct large cardinals

[43] The notion is formaly defined for sets, but the meaning should be obvious.

more efficiently than by adding inaccessibles one by one. The property we are looking for ought to look like something like this:

$$\begin{gathered}\kappa \text{ is an inaccessible cardinal and} \\ \text{there are } \kappa \text{ inaccessible cardinals } \mu \ < \ \kappa\end{gathered} \tag{3.85}$$

This is in fact a fixed-point type of statement. We shall call those cardinals hyper-inaccessible. Now consider the following definition.

Definition 3.18 0-*inaccessible cardinal*
A cardinal κ is 0*-inaccessible if it is inaccessible.*

We can define *α-weakly-inaccessible* cardinals analogously with the only difference that those are limit, not strongly limit.

Definition 3.19 *α-hyper-inaccessible cardinal*
For any ordinal α, κ is called α-inaccessible, if κ is inaccessible and for each β ¡ α, the set of β-inaccessible cardinals less than κ is unbounded in κ.

Because κ is inaccessible and therefore regular, the number of β-inaccessibles below κ is equal to κ. We have therefore successfully formalized the above vague notion of hyper-inaccessible cardinal into a hierarchy of α-inaccessibles.

Let's now consider iterating this process over again. Since, informally, V would be α-inaccessible for any α, this property of the universal class could possibly be reflected to an initial segment, the smallest of those will be the first hyper-inaccessible cardinal. Such κ is larger than any α-inaccessible since from regularity of κ, for given $\alpha \ < \ \kappa$, κ is κ-th α-hyper-inaccessible cardinal. It is in fact "inaccessible" via α-inaccessibility.

Definition 3.20 *Hyper-inaccessible cardinal*
κ is called the hyper-inaccessible, also 0*-hyper-inaccessible, cardinal if it is α-inaccessible for every $\alpha \ < \ \kappa$.*

Definition 3.21 *α-hyper-inaccessible cardinal*
For any ordinal α, κ is called α-hyper-inaccessible cardinal if for each ordinal $\beta \ < \ \alpha$, the set of β-hyper-inaccessible cardinals less the κ is inbounded in κ.

Obviously we could go on and iterate it ad libitum, but the nomenclature would be increasingly confusing. A smarter way to accomplish the same goal is carried out in the following section.

3.4 Mahlo Cardinals

While the previous chapter introduced us to a notion of inaccessibility and the possibility of iterating it ad libitum in new theories, there is an even faster way to travel upwards in the cumulative hierarchy, that was proposed by Paul Mahlo in his papers (see [5], [6] and [7]) at the very beginning of the 20th century, and which can be easily reformulated using $(Reflection)$. To see how Lévy's initial statement of reflection was influenced by Mahlo's work, refer to section 2.1. The aim of the following paragraphs is to give an intuitive explanation of the idea behind Mahlo's hierarchy of cardinals, all claims made here ought to be stated formally later in the very same chapter.

At the very end of section 3.3, we have tried to establish the notion of hyper-inaccessibility and iterate it to yield even larger large cardinals. In order to avoid too bulky cardinal names, let's try a different route and establish those cardinals directly via reflection.

The following two definitions come from [8] and while they are rather informal, we will find them very helpful for understanding the Mahlo cardinals.

Definition 3.22 *(Fixed-point property)*
For any first-order formula $\psi(x, p_1, \ldots, p_n)$ with no free variables other than $x, p_1, \ldots, p_n$, which is any property of ordinals, we say that a property φ is a fixed-point property *if φ has the form*

$$\begin{gathered}x \textit{ is an inaccessible cardinal and} \\ \textit{there are } x \textit{ ordinals less than } x \textit{ that have the property } \psi(x, p_1, \ldots, p_n).\end{gathered} \tag{3.86}$$

Definition 3.23 *(Fixed-point reflection)*
If φ is a fixed-point property that holds for V, it also holds for some V_α, an initial segment of V.

Obviously those are in no way rigorous definitions because we have no idea what $\psi(x, p_1, \ldots, p_n)$ looks like. Let's try to restate the same idea in a useful way. But first, let's show that the formal counterpart of the idea of containing "enough" ordinals with a property is the notion of stationary set.

Definition 3.24 *(Supremum)*
Given x a set of ordinals, the supremum of x, denoted sup(x), is the least upper bound of x.

$$sup(x) = bigcupx \tag{3.87}$$

Definition 3.25 *(Limit point)*
Given x, a set of ordinals and an ordinal α, we say that α is a limit point *of x if $sup(x \cap \alpha) = alpha$*

Definition 3.26 *(Set Unbounded in α) Let α be an ordinal. We say that $x \subset \alpha$ is* unbounded in α *iff*

$$\forall \beta \in Ord(\beta < \alpha \to \exists \gamma(\gamma \in x(\beta \leq \gamma < \alpha))) \tag{3.88}$$

In other words, κ is a mahlo cardinal if it is inaccessible and every club set in κ contains an inaccessible cardinal. This is exactly the notion of fixed-point reflection we were trying to show earlier.

[3]

Definition 3.27 *The folllowing definitions are equivalent:*
(i) κ is Mahlo
(ii) κ is weakly Mahlo and strong limit
(iii) The set $\{\lambda < \kappa : \lambda$ is inaccessible$\}$ is stationary in κ.
(iv) Every normal function on κ has an inaccessible fixed point.

Proof. (**i**) $\leftrightarrow$ (**ii**) Let κ_1 be a mahlo cardinal and let κ_2 be a strong limit weakly Mahlo cardinal. We know from the definitions that the set $\{\lambda < \kappa : \lambda$ is inaccessible$\}$ is stationary in both κ_1 and κ_2, the only difference being that κ_1 is a strongly limit cardinal, but κ_2 would be limit from weak Mahloness, wasn't it for the fact that it is also strong limit. This eliminates the only difference between them and therefore κ_1 is also strong limit weakly Mahlo cardinal and κ_2 is Mahlo.

(**i**) $\to$ (**iii**) We know that κ is uncountable, regular, strong limit and that the set $S = \{\lambda < \kappa : \lambda$ is regular$\}$ is stationary in κ. We want to prove that $S' = \{\lambda < \kappa : \lambda$ is inaccessible$\}$ is thus also stationary in κ.

Since stationary set intersects every club set in κ, let C be any such set. Let $D = \{\lambda < \kappa : \lambda$ is strong limit$\}$. D is a club set because TODO. Since intersection of less than κ club sets is a club set, $C \cap D \neq \emptyset$.

TODO proc $\lambda = S \cap C \cap D$ je inaccessible?

(iii) $\to$ **(iv)**

TODO jak to dela Lévy?

(iv) $\to$ **(i)**

TODO jak to dela Lévy?

range kazde normalni funkce je club v On. (nevadi ze On je trida?)

co treba lemma ze pevne body tvori taky club set

mozna by stacilo ze jsou unbounded, tedy kazda normalni funkce ma libovolne velke pevne body. □

TODO obdoba pro α-Mahlo kardinaly?

TODO κ is hyper-Mahlo iff κ is inaccessible and the set $\{\lambda < \kappa : \lambda \text{ is Mahlo}\}$ is stationary in κ. to je to samy jako α-Mahlo, ne?

3.5 Indescribality

α-Mahlo are the extreme of regular fixed-point axioms, they are about as high as we can get via normal functions and stationary sets.

Let's try a different strategy. Remember how we said that (Regular, Limit and) various Large cardinals are in a way all determined by being unreachable by a specific process of creating bigger cardinals from already available ones? TODO indescribable – reflecting indescribability – we can't reach V by a Σ^1_1 formula, so there's some initial segment V_α that is also unreachable (we say indescribable) by the means of a ... formula

Let's recall complete reflection theorem first, consider the following:

$$\text{For every sentence } \varphi \text{, there is a limit ordinal } \alpha \text{ such that } \varphi^{V}_{\alpha} \leftrightarrow \varphi \tag{3.89}$$

We may also require that $\alpha < \beta$, where β is an arbitrary ordinal given.

For the exact definition of Π^m_n and Σ^m_n see 1.41

Definition 3.28 *(Π^m_n-indescribable cardinal) We say that κ is Π^m_n-indescribable iff for any Π^m_n sentence φ such that $V_\kappa \models \varphi$ there is an $\alpha < \kappa$ such that $V_\alpha \models \varphi$*

Definition 3.29 *(Σ^m_n-indescribable cardinal) We say that κ is Σ^m_n-indescribable iff for any Σ^m_n sentence φ such that $V_\kappa \models \varphi$ there is an $\alpha < \kappa$ such that $V_\alpha \models \varphi$*

Lemma 3.30 *Let κ be a cardinal, the following holds for any $n \in \omega$. κ is Π^1_n-indescribable iff κ is $\Sigma^1_n + 1$-indescribable*

Proof. The forward direction is obvious, we can always add a spare quantifier over a type 2 variable to turn a Π^1_n formula φ into a $\exists P\varphi$ which is thus a $\Sigma^1_n + 1$ formula.[44]

To prove the opposite direction, suppose that $V_\kappa \models \exists X\varphi(X)$ where X is a type 2 variable and φ is a Π^1_n formula with one free variable of type 2. This means that there is a set $S \subseteq V_\kappa$ that is a witness of $\exists X\varphi(X)$, in other words, $\varphi(S)$ holds. We can replace every occurence of X in φ by a new predicate symbol S, this allows us to say that κ is Π^1_n-indescribable (with respect to $\langle V_\kappa, \in, R, S\rangle$). [45] □

The above lemma tells us that we as long as we stay in the realm of type 1 and type 2 variables, we only need to classify indescribable cardinals with respect to Π^1_n-indescribability.

Theorem 3.31 *Let κ be an ordinal. The following are equivalent.*
(i) κ is inaccessible
(ii) κ is Π^1_0-indescribable.

Note that Π^1_0 formulas are those that contain zero unbound quantifiers over type-2 variables, they are in fact first-order formulas. We have already shown in 3.14 that there is no way to reach an inaccesible cardinal via first-order formulas in ZFC. We will now prove it again in for formal clarity.

Proof. TODO asi pridat alternativni definici nedosazitelnosti podle kan. 6.2?
□

TODO nejaka veta ze kdyz jsou Π^1_0-indescribable, jsou i Π^m_n-indescribable pro $m \leq 1, n \leq 0$? Drake? Obracene! Π^m_n-indescribable jsou zaroven Π^a_b-indescribable pro $a < m, b < n$.

The above theorem provides an easy way to show that every following large cardinal is also in inaccessible cardinal[46].

Definition 3.32 *(Extension property) We say that a cardinal κ has the* extension property *iff for any $R \subseteq V_\kappa$ there is a transitive set $X \neq V_\kappa$ and an $S \subseteq X$ such that $\langle V_\kappa, \in, R\rangle \prec \langle X, \in, S\rangle$*

Definition 3.33 *(Weakly compact cardinal)*
We say that a cardinal κ is weakly compact *iff it has the extension property.*

[44]Note that unlike in previous sections, φ is now a sentence so we don't have to worry whether P is free in φ.

[45]A different yet interesting approach is taken by Tate in ??. He states that for $n \geq 0$, a formula of order $\leq n$ is called a Π^n_0 and a Σ^n_0 formula. Then a Π^n_{m+1} is a formula of form $\forall Y\psi(Y)$ where ψ is a Σ^n_m formula and Y is a variable of type n. Finally, a Σ^n_{m+1} is the negation of a Π^n_m formula. So the above holds ad definitio.

[46]That is because Π^1_0 formulas are included Π^m_n formulas for $m \leq 1, n \leq 0$.

The above definitions are equivalent

Theorem 3.34 *the following are equivalent:*

(i) κ *is* Weakly compact.

(ii) κ *is* Π^1_1*-indescribable.*

For a proof, see [1][Theorem 6.4]

TODO def totalne nepopsatelny kardinal

TODO viz Drake, Ch.9 par. 3 – tam se rika ze kdyz κ je meritelny kardinal, pak je κ Π^2_1-nepopsatelny kardinal (kanamori to rika taky)

3.6 Measurable Cardinal

TODO refaktorizovat fle:

Definition 3.35 *(Ultrafilter)*
Given a set X, we say $U \subset \mathscr{P}(X)$ is an ultrafilter *iff all of the following hold:*

(i) $\emptyset \notin U$

(ii) $\forall a, b(\subset X \;\&\; a \subset b \;\&\; a \in U \to b \in U)$

(iii) $\forall a, b \in U(a \cap b) \in U$

(iv) $\forall a(a \subset X \to (a \in U \vee (X \setminus a) \in U))$

Definition 3.36 *(κ-complete ultrafilter)*
We say that an ultrafilter U is κ-complete iff

Definition 3.37 *(non-principal ultrafilter)*
TODO

Definition 3.38 *(Measurable Cardinal)*
Let κ be a caridnal. We say is a measurable cardinal *iff it is an uncountable cardinal with a κ-complete, non-principal ultrafilter.*

Theorem 3.39 *Let κ be a cardinal. If κ is a measurable cardinal then it is Π^2_1-indescribable.*

Theorem 3.40 *Pod kazdym meritelnym kardinalem existuje ultrafiltr totalne nepopsatelnych, ktere tim padem nejsou sestrojitelne. VIZ VETA Z KANAMORIHO.*

asi nedokazovat?

3.7 The Constructible Universe

The constructible universe, denoted L, is a cumulative hierarchy of sets, presented by Kurt Gödel in his 1938 paper *The Consistency of the Axiom of Choice and of the Generalised Continuum Hypothesis*. For a technical description, see below. Assertion of their equality, $V = L$, is called the *axiom of constructibility*. The axiom implies GCH and therefore also AC and contradicts the existence of some of the large cardinals, our goal is to decide whether those introduced earlier are among them.

On order to formally establish this class, we need to formalize the notion of definability first.

Definition 3.41 *We say that a set X is* definable *over a model $\langle M, \in\rangle$ if there is a first-order formula φ together with parameters $p_1, \dots, p_n \in M$ such that*

$$X = \{x : x \in M \;\&\; \langle M, \in\rangle \models \varphi(x, p_1, \dots, p_n)\} \tag{3.90}$$

Definition 3.42 *(Sets definable in M)*
The following is a set of all definable subsets of a given set M, denoted Def(M).

$$\begin{gathered} Def(M) = \{\{y : x \in M \wedge \langle M, \in\rangle \models \varphi(y, u_1, \dots, i_n)\}| \\ \varphi \textit{ is a first-order formula, } p_1, \dots, p_n \in M\} \end{gathered} \tag{3.91}$$

Now we can recursively build L.

Definition 3.43 *(The Constructible universe)*

(i)

$$L_0 := \emptyset \tag{3.92}$$

(ii)

$$L_{\alpha+1} := Def(L_\alpha) \tag{3.93}$$

(iii)

$$L_\lambda = \bigcup_{\alpha<\lambda} L_\alpha \textit{ If } \lambda \textit{ is a limit ordinal} \tag{3.94}$$

(iv)

$$L = \bigcup_{\alpha \in Ord} L_\alpha \tag{3.95}$$

Note that while L bears very close resemblance to V, the difference is, that in every successor step of constructing V, we take every subset of V_α to be $V_{\alpha+1}$, whereas $L_{\alpha+1}$ consists only of definable subsets of L_α. Also note that L is transitive.

In order to

TODO:

Lemma 3.44 *$Ord \in L$*

Lemma 3.45 *L is well-ordered.*
TODO !!

Theorem 3.46 *Let L be as in 3.43.*

$$L \models \mathsf{ZFC} \tag{3.96}$$

Proof. TODO !!! (strucne) vit [4][Theorem 13.3]

(i) *Extensionality* (see 1.2):
Extensionality holds in L because Δ_0 formulas are absolute in transitive classes by 1.42, *Extensionality* is Δ_0 and L is transitive.

(ii) *Foundation* (see 1.6)
Take a non-empty set X. Let $x \in X$ be a set such that $X \cap x = \emptyset$. x is therefore defined by the formula $\varphi(x, y) = (x \cap y = \emptyset)$, so $x \in L$. φ is Δ_0 and therefore holds in L by 1.42.

(iii) *Pairing* (see 1.7)
Since *Pairin* is also Δ_0, it holds in L by the same argument as *Extensionality* does by 1.42.

(iv) *Union* (see 1.8)
Union is also Δ_0, see *Extensionality* and 1.42.

(v) *Power Set* (see 1.9)
Power Set also holds by 1.42.

(vi) *Infinity* (see 1.10)
$\omega \in L$ by 3.44

(vii) *Specification* (see 1.3)
.

(viii) *Replacement* (see 1.15)
.

(ix) *Choice* (see 1.15)
.

□

Definition 3.47 *Constructibility*
$L = V$

The following are a few interesting results that we won't prove but refer interested reader to appropriate resources instead.

Definition 3.48 *(GCH)*
The following is called the Generalised Continuum Hypothesis, abbreviated as GCH*. It is an independent statement in* ZFC*.*

$$\text{GCH } \textit{iff } \aleph_{\alpha+1} = 2^{\aleph_\alpha} \textit{ for every ordinal } \alpha \tag{3.97}$$

Theorem 3.49

$$(L = V) \rightarrow GCH \tag{3.98}$$

This is proven in cite{neco} Gödel? Jech? Kunnen?

TODO L a velke kardinaly

TODO def Con!

Theorem 3.50

$$Con(L + \exists\kappa(\kappa\textit{” is an Inaccessible Cardinal”})) \tag{3.99}$$

Theorem 3.51

$$Con(L + \exists\kappa(\kappa\textit{” is a Mahlo Cardinal”})) \tag{3.100}$$

Theorem 3.52

$$Con(L + \exists\kappa(\kappa\textit{” is a Weakly Inaccessible Cardinal Cardinal”})) \tag{3.101}$$

Theorem 3.53

$$Con(L + \exists\kappa(\kappa\textit{” is a Measurable Cardinal”})) \tag{3.102}$$

TODO co velky pismena ve jmenech kardinalu?

TODO zduvodneni

TODO kratka diskuse jestli refl implikuje transcendenci na L - polemika, nazor - V=L a slaba kompaktnost a dalsi

4 Conclusion

TODO na konec

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