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4 REFLECTION PRINCIPLES AND LARGE
5 CARDINALS

6 Bakalářská práce

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¹⁰ Prohlašuj, že jsem bakalářskou práci vypracoval samostatně a že jsem uvedl
¹¹ všechny použité prameny a literaturu.

¹² V Praze 14. dubna 2015

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Abstract

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Práce zkoumá vztah tzv. principů reflexe a velkých kardinálů. Lévy ukázal, že v ZFC platí tzv. věta o reflexi a dokonce, že věta o reflexi je ekvivalentní schématu nahrazení a axiomu nekonečna nad teorií ZFC bez axiomu nekonečna a schématu nahrazení. Tedy lze na větu o reflexi pohlížet jako na svého druhu axiom nekonečna. Práce zkoumá do jaké míry a jakým způsobem lze větu o reflexi zobecnit a jaký to má vliv na existenci tzv. velkých kardinálů. Práce definuje nedosažitelné, Mahlovy a nepopsatelné kardinály a ukáže, jak je lze zavést pomocí reflexe. Přirozenou limitou kardinálů získaných reflexí jsou kardinály nekonzistentní s L. Práce nabídne intuitivní zdůvodnění, proč tomu tak je.

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Abstract

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Resumé práce v anglickém jazyce.

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1 Introduction

1.1 Motivation and Origin

The Universe of sets cannot be uniquely characterized (i. e. distinguished from all its initial elements) by any internal structural property of the membership relation in it, which is expressible in any logic of finite or transfinite type, including infinitary logics of any cardinal order.

— Kurt Gödel [4]

To understand why we need reflection in the first place, let's think about infinity for a moment. In the intuitive sense, infinity is an upper limit of all numbers. But for centuries, this was merely a philosophical concept, closely bound to religious and metaphysical way of thinking, considered separate from numbers used for calculations or geometry. It was a rather vague concept. In ancient Greece, Aristotle's response to famous Zeno's paradoxes introduced the distinction between actual and potential infinity. He argued, that potential infinity is (in today's words) well defined, as opposed to actual infinity, which remained a vague incoherent concept. He didn't think it's possible for infinity to inhabit a bounded place in space or time, rejecting Zeno's thought experiments as a whole. Aristotle's thoughts shaped western thinking partly due to Aquinas, who himself believed actual infinity to be more of a metaphysical concept for describing God than a mathematical property attributed to any other entity. In his *Summa Theologica*¹ he argues:

A geometrician does not need to assume a line actually infinite, but takes some actually finite line, from which he subtracts whatever he finds necessary; which line he calls infinite.

Less than hundred years later, Gregory of Rimini wrote

If God can endlessly add a cubic foot to a stone—which He can—then He can create an infinitely big stone. For He need only add one cubic foot at some time, another half an hour later, another a quarter of an hour later than that, and so on ad infinitum. He would then have before Him an infinite stone at the end of the hour.

Which is basically a Zeno's Paradox made plausible with God being the actor. In contrast to Aquinas' position, Gregory of Rimini theoretically constructs

¹Part I, Question 7, Article 3, Reply to Objection 1

an object with actual infinite magnitude that is essentially different from God. Even later, in the 17th century, pushing the property of infiniteness from the Creator to his creation, Nature, Leibniz wrote to Foucher in 1662:

I am so in favor of the actual infinite that instead of admitting that Nature abhors it, as is commonly said, I hold that Nature makes frequent use of it everywhere, in order to show more effectively the perfections of its Author. Thus I believe that there is no part of matter which is not, I do not say divisible, but actually divided; and consequently the least particle ought to be considered as a world full of an infinity of different creatures.

But even though he used potential infinity in what would become foundations of modern Calculus and argued for actual infinity in Nature, Leibniz refused the existence of an infinite, thinking that Galileo's Paradoxon² is in fact a contradiction. The so called Galileo's Paradoxon is an observation Galileo Galilei made in his final book "Discourses and Mathematical Demonstrations Relating to Two New Sciences". He states that if all numbers are either squares and non-squares, there seem to be less squares than there is all numbers. On the other hand, every number can be squared and every square has it's square root. Therefore, there seem to be as many squares as there are all numbers. Galileo concludes, that the idea of comparing sizes makes sense only in the finite realm.

Salviati: So far as I see we can only infer that the totality of all numbers is infinite, that the number of squares is infinite, and that the number of their roots is infinite; neither is the number of squares less than the totality of all the numbers, nor the latter greater than the former; and finally the attributes "equal," "greater," and "less," are not applicable to infinite, but only to finite, quantities. When therefore Simplicio introduces several lines of different lengths and asks me how it is possible that the longer ones do not contain more points than the shorter, I answer him that one line does not contain more or less or just as many points as another, but that each line contains an infinite number.

Leibniz insists in part being smaller than the whole saying

Among numbers there are infinite roots, infinite squares, infinite cubes. Moreover, there are as many roots as numbers. And there are as many squares as roots. Therefore there are as many squares

²zneni galileova paradoxu

119 as numbers, that is to say, there are as many square numbers as
 120 there are numbers in the universe. Which is impossible. Hence it
 121 follows either that in the infinite the whole is not greater than the
 122 part, which is the opinion of Galileo and Gregory of St. Vincent,
 123 and which I cannot accept; or that infinity itself is nothing, i.e.
 124 that it is not one and not a whole.

125 In his work, he defined transfinite numbers to extend existing natural
 126 number structure so it contains more objects that behave like natural num-
 127 bers and are based on an object (rather a meta-object) that doesn't explicitly
 128 exist in the structure, but is closely related to it. This is the first instance
 129 of reflection. This paper will focus on taking this principle a step further,
 130 extending Cantor's (or Zermelo–Fraenkel's, to be more precise) universe so
 131 it includes objects so big, they could be considered the universe itself, in a
 132 certain sense.

133 The original idea behind reflection principles probably comes from what
 134 could be informally called “universality of the universe”. The effort to pre-
 135 cisely describe the universe of sets was natural and could be regarded as one
 136 of the impulses for formalization of naive set theory. If we try to express
 137 the universe as a set $\{x|x = x\}$, a paradox appears, because either our set
 138 is contained in itself and therefore is contained in a set (itself again), which
 139 contradicts the intuitive notion of a universe that contains everything but is
 140 not contained itself. If there is an object containing all sets, it must not be
 141 a set itself. The notion of class seems inevitable. Either directly the ways
 142 for example the Bernays–Gödel set theory, we will also discuss later in this
 143 paper, does in, or on a meta-level like the Zermelo–Fraenkel set theory, that
 144 doesn't refer to them in the axioms but often works with the notion of a
 145 universal class. duet Another obstacle of constructing a set of all sets comes
 146 from Georg Cantor, who proved that the set of all subsets of a set (let A
 147 be the set and $\mathcal{P}((A)$ its powerset) is strictly larger than A . That would
 148 turn every aspiration to finally establish an universal set into a contradictory
 149 infinite regression.³ We will use V for the class of all sets.

³An intuitive analogy of this *reductio ad infinitum* is the status of ω , which was orig-
 inally thought to be an unreachable absolute, only to become starting point of Cantor's
 hierarchy of sets growing beyond all boundaries around the end of the 19th century

From previous thoughts we can easily argue, that it is impossible to construct a property that holds for V and no set and is neither paradoxical like $\{x|x = x\}$ nor trivial. Previous observation can be transposed to a rather naive formulation of the reflection principle:

(Refl) Any property which holds in V already holds in some initial segment of V .

To avoid vagueness of the term "property", we could informally reformulate the above statement into a schema:

For every first-order formula⁴ φ holds in $V \leftrightarrow \varphi$ holds in some initial segment of V .

Interested reader should note that this is a theorem scheme rather than a single theorem.⁵

1.2 A few historical remarks on reflection

Reflection made it's first in set-theoretical appearance in Gödel's proof of GCH in L (citace Kanamori ? Levy and set theory), but it was around even earlier as a concept. Gödel himself regarded it as very close to Russel's reducibility axiom (an earlier equivalent of the axiom schema of Zermelo's separation). Richard Montague then studied reflection properties as a tool for verifying that Replacement is not finitely axiomatizable (citace?). A few years later Levy proved (citace?) equivalence of reflection with Axiom of infinity together with Replacement.

⁴this also works for finite sets of formulas [3, p. 168]

⁵If there were a single theorem stating "for any formula φ that holds in V there is an initial segment of V where φ also holds", we would obtain the following contradiction with the second Gödel's theorem: In ZFC, any finite group of axioms of ZFC holds in some initial segment of the universe. If we take the largest of those initial segments it is still strictly smaller than the universe and thus we have, via compactness, constructed a model of ZFC within ZFC. That is, of course a harsh contradiction. This also leads to an elegant way to prove that ZFC is not finitely axiomatizable.

2 Levy's Reflection

2.1 Levy's Axiom Schemata of Strong Infinity

This section will try to present Levy's proof of a general reflection principle being equivalent to Replacement and Infinity under ZF minus Replacement and Infinity. We will first introduce a few axioms and definitions that were a bit different in Levy's paper[2], but are equivalent to today's terms. We will write them in contemporary notation, our aim is the result, not history of set theory notation.

Definition 2.1 *The Axiom of Subsets*

$$\forall x \exists y \forall z (z \in y \leftrightarrow (z \in x)) \quad (2.1)$$

Definition 2.2 *Standard Complete Model of $S(Scm^S)$*

$$Scm^Q(V_\alpha) \leftrightarrow In^Q(\alpha) \quad (2.2)$$

Definition 2.3 *S is a set theory obtained from ZF by leaving out the Replacement Scheme and the Axiom of Infinity.*

Definition 2.4 N_0

$$\exists u (Scm^S(u) \& x_1, \dots, x_n \in u \rightarrow \varphi \leftrightarrow \varphi^u) \quad (2.3)$$

where φ is a formula which does not contain free variables except x_1, \dots, x_n .

observation: $Scm^{yS}(u) = In(\kappa) \leftrightarrow \exists \kappa (V_\kappa = u)$

Theorem 2.5 *In S , the schema N_0 implies the Axiom of Infinity.*

Proof. This is pretty straightforward since we already have $In(\kappa)$, which is itself an inductive set. \square

Theorem 2.6 *In S , the schema N_0 implies Replacement schema.*

Proof. Let $\varphi(v, w)$ be a formula with no free variables except v, w, x_1, \dots, x_n where n is any natural number. Let χ be an instance of replacement schema for this φ :

$$\chi = \forall r, s, t (\varphi(r, s) \& \varphi(r, t) \rightarrow s = t) \rightarrow \exists y \forall w (w \in y \leftrightarrow \exists v (v \in x \& \varphi(v, w))) \quad (2.4)$$

We can deduce the following from N_0 :

- 194 (i) $x_1, \dots, x_n, v, w \in u \rightarrow (\varphi \leftrightarrow \varphi^u)$
 195 (ii) $x_1, \dots, x_n, v \in u \rightarrow (\exists w \varphi \leftrightarrow (\exists w \varphi)^u)$
 196 (iii) $x_1, \dots, x_n, x \in u \rightarrow (\chi \leftrightarrow \chi^u)$
 197 (iv) $\forall x_1, \dots, x_n \forall x (\chi \leftrightarrow (\forall x_1, \dots, x_n \forall x \chi)^u)$

198 Note that (i), (ii), (iii) are obtained from instances of N_0 for φ , $\exists w \varphi$ and
 199 χ respectively. From relativization we also know that $(\exists w \varphi)^u$ is equivalent to
 200 $\exists w (w \in u \& \varphi^u)$. Therefore (ii) is equivalent to $x_1, \dots, x_n, v \in u \rightarrow (\exists w (w \in$
 201 $u \& \varphi^u))$.

202 If φ is a function $(\forall r, s, t (\varphi(r, s) \& \varphi(r, t) \rightarrow r = t))$, then for every $x \in u$,
 203 which is also $x \subset u$ by $\text{Scm}^S(u)$, it maps elements of x onto u . From the
 204 axiom scheme of comprehension⁶, we can find a set of all images of elements
 205 of x . Let's call it y . That gives us $x_1, \dots, x_n, x \in u \rightarrow \chi$. By (iii) we get
 206 $x_1, \dots, x_n, x \in u \rightarrow \chi^u$, closure of this formula is $(\forall x_1, \dots, x_n \forall x \chi)^u$, which
 207 together with (iv) yields $\forall x_1, \dots, x_n \forall x \chi$. By the means of specification we
 208 end up with χ , which is all we need for now. \square

209 2.2 Contemporary restatement

210 **Theorem 2.7 (Lévy) ZFC:**

- 211 (i) Let $\varphi(x_1, \dots, x_n)$ be a first-order formula with free variables shown.
 212 Then for each set M_0 there exists a set $M \supset M_0$ such that

$$\varphi^M(x_1, \dots, x_n) \leftrightarrow \varphi(x_1, \dots, x_n) \quad (2.5)$$

213 (We say that M reflects φ)

- 214 (ii) There is transitive $M \supset M_0$ that reflects φ ; moreover, there is a limit
 215 ordinal α such that $M \subset V_\alpha$ and V_α reflects φ .

216 In order to prove this theorem let's first state a lemma, similarly to [3].

217 **Lemma 2.8** (i) Let $\varphi(u_1, \dots, u_n, x)$ be a formula. For each set M_0 there
 218 exists a set $M \supset M_0$ such that

$$\text{If } \exists x \varphi(u_1, \dots, u_n, x) \text{ then } (\exists x \in M) \varphi(u_1, \dots, u_n, x) \quad (2.6)$$

- 219 (ii) If $\varphi_1, \dots, \varphi_k$ are formulas, then for each M_0 there is an $M \supset M_0$ such
 220 that 2.6 holds for each $\varphi_1, \dots, \varphi_k$.

221 *Proof.* Let's first prove (i). For every u_1, \dots, u_n , let

$$H(u_1, \dots, u_n) = \hat{C} \quad (2.7)$$

⁶axiom of subsets in Levy's version

222 where \hat{C} is defined as follows:

$$\hat{C} = \{x \in C : (\forall z \in C) \text{ rank } x \leq \text{rank } z\}, \quad (2.8)$$

223

$$C = \{x : \varphi(u_1, \dots, u_n, x)\}. \quad (2.9)$$

224 Intuitively, C is a set of all witnesses of property φ with n fixed parameters.
 225 \hat{C} contains the elements of C that are minimal with respect to rank.
 226 $H(u_1, \dots, u_n)$ is in fact a set with the following property

$$\text{if } \exists x \varphi(u_1, \dots, u_n, x), \text{ then } (\exists x \in H(u_1, \dots, u_n)) \varphi(u_1, \dots, u_n, x) \quad (2.10)$$

227 In other words, if there are witnesses of φ being valid with fixed parameters
 228 u_1, \dots, u_n , at least one of them has is an element of $H(u_1, \dots, u_n)$.
 229 We can now inductively construct the set M . Note that M_0 is given to us
 230 from the very beginning.

$$M_{i+1} = M_i \cup \bigcup \{H(u_1, \dots, u_n) : u_1, \dots, u_n \in M_i\}, \quad (2.11)$$

231

$$M = \bigcup_{i=0}^{\infty} M_i \quad (2.12)$$

232 We have defined H and M in a way that if $u_1, \dots, u_n \in M$, then there is
 233 some $i \in \mathbb{N}$ such that $u_1, \dots, u_n \in M_i$ and if $\varphi(u_1, \dots, u_n, x)$ holds for some
 234 x , it then holds for some $x \in M_{i+1}$.

235

236 In order to modify this proof to work also for (ii), we need to change the
 237 definition of $H(u_1, \dots, u_n) = \hat{C}$ to $H_i(u_1, \dots, u_n) = \hat{C}_i$ where \hat{C}_i uses C_i
 238 instead of C , which in turn contains φ_i in place of φ . Next, we modify the
 239 construction of M in a similar manner:

240

$$M_{i+1} = M_i \cup \bigcup \left\{ \bigcup_{j \in 1, \dots, k} \{H_j(u_1, \dots, u_n)\} : u_1, \dots, u_n \in M_i \right\}, \quad (2.13)$$

241 Last step of the construction stays the same, which means we are finished
 242 with this lemma. \square

243

244 We are now ready to prove our first version of the Reflection principle. *Proof.*
 245 Let $\varphi(x_1, \dots, x_n)$ be a formula with no universal quantifiers and $\varphi_1, \dots, \varphi_k$
 246 all sub formulas in φ . Given a set M_0 , thanks to the previous lemma we
 247 know, that there exists a set $M \supset M_0$, such that

$$\exists x \varphi_j(u, \dots, x) \rightarrow (\exists x \in M) \varphi_j(u, \dots, x), \quad j = 1, \dots, k \quad (2.14)$$

248 for all $u, \dots \in M$.

249

250 TODO (ii)

□

251 **Theorem 2.9** *(Refl) is equivalent to (Infinity) & (Replacement) under ZFC*
 252 *minus (Infinity) & (Replacement)*

253 *Proof.* Since (Refl) is a sound theorem in ZFC, we are only interested in
 254 showing the converse: (Refl) \rightarrow (Infinity)

255 This is the easy part since Infinity says that *there is an infinite set* and
 256 (Refl) is just a stronger version that says "there is an inaccessible cardinal"
 257 which is all we need.

258 (Refl) \rightarrow (Replacement)

259

□

260 **Definition 2.10** *Let $\varphi(R)$ be a Π_m^n -formula which contains only one free*
 261 *variable R which is second-order. Given $R \subseteq V_\kappa$, we say that $\varphi(R)$ reflects*
 262 *in V_κ if there is some $\alpha < \kappa$ such that:*

$$\text{If } (V_\kappa, \in, R) \models \varphi(R), \text{ then } (V_\alpha, \in, R \cap V_\alpha) \models \varphi(R \cap V_\alpha). \quad (2.15)$$

263 3 Large Cardinals

264 3.1 Preliminaries

265 To avoid confusion⁷, let's first define some basic terms.

266 **Definition 3.1** (*weak limit cardinal*) κ is a weak limit cardinal if it is
267 \aleph_α for some limit α .

268 **Definition 3.2** (*strong limit cardinal*) κ is a strong limit cardinal if for
269 every $\lambda < \kappa$, $2^\lambda < \kappa$

270 3.2 Inaccessibility

271 **Definition 3.3** (*weak inaccessibility*) κ is weakly inaccessible \leftrightarrow it is regu-
272 lar and weakly limit.

273 **Definition 3.4** (*inaccessibility*) κ is inaccessible \leftrightarrow it is regular and strongly
274 limit.

275 **Theorem 3.5** [Lévy] The following are equivalent:

- 276 (i) κ is inaccessible.
- 277 (ii) For every $R \subseteq V_\kappa$ and every first-order formula $\varphi(R)$, $\varphi(R)$ reflects in
278 V_κ .
- 279 (iii) For every $R \subseteq V_\kappa$, the set $C = \{\alpha < \kappa \mid \langle V_\alpha, \in, R \cap V_\alpha \rangle \prec \langle V_\kappa, \in, R \rangle\}$ is
280 closed unbounded.

281 *Proof.* Let's start with (i) \rightarrow (iii) in a way similar to [1].

282 The set $\{\alpha < \kappa \mid \langle V_\alpha, \in, R \cap V_\alpha \rangle \prec \langle V_\kappa, \in, R \rangle\}$ is clearly closed, it remains to
283 show that it is also unbounded. To do so, let $\alpha < \kappa$ be arbitrary. Define
284 $\alpha_n < \kappa$ for $n \in \omega$ by recursion as follows:

285 Set $\alpha_0 = \alpha$. Given $\alpha_n < \kappa$ define α_{n+1} to be the least $\beta \geq \alpha_n$ such as when-
286 ever $y_1, \dots, y_k \in V_{\alpha_n}$ and $\langle V_\kappa, \in, R \rangle \models \exists v_0 \varphi[v_0, y_1, \dots, y_k]$ for some formula
287 φ , there is an $x \in V_\beta$ such that $\langle V_\kappa, \in, R \rangle \models \varphi[x, y_1, \dots, y_k]$.

288 Since κ is inaccessible, $|V_{\alpha_n}| < \kappa$ and so $\alpha_{n+1} < \kappa$.

289 Finally, set $\alpha = \sup(\alpha_n \mid n \in \omega)$. Then $\langle V_\alpha, \in, R \cap V_\alpha \rangle \prec \langle V_\kappa, \in, R \rangle$ by the
290 usual (Tarski) criterion for elementary substructure.

291

292 The next part, proving (iii) \rightarrow (ii), should be elementary since C is closed

⁷While in most sources refer to *weak limit cardinal* as a *limit cardinal* and to *strong limit cardinal*, in some cases the distinction is *weak limit cardinal* and *limit cardinal* respectively. That's why I have decided to explicitly define those otherwise elementary terms.

unbounded, which means that it contains at least countably many elements but we need only one such α to satisfy (2.10).

Finally, we shall prove that (ii) \rightarrow (i). Since it obviously holds that $\kappa > \omega$, we have yet to prove that κ is regular and a strong limit. Let's argue by contradiction that it is regular. If it wasn't, there would be a $\beta < \kappa$ and a function $F : \beta \rightarrow \kappa$ with range unbounded in κ . Set $R = \{\beta\} \cup F$. By hypothesis there is an $\alpha < \kappa$ such that $\langle V_\alpha, \in, R \cap V_\alpha \rangle \prec \langle V_\kappa, \in, R \rangle$. Since β is the single ordinal in R , $\beta \in V_\alpha$ by elementarity. This yields the desired contradiction since the domain of $F \cap V_\alpha$ cannot be all of β .

Next, let's see whether κ is indeed a strong limit, again by contradiction. If not, there would be a $\lambda < \kappa$ such that $2^\lambda \geq \kappa$. Let $G : \mathcal{P}(\lambda) \rightarrow \kappa$ be surjective and set $R = \{\lambda + 1\} \cup G$. By hypothesis, there is an $\alpha < \kappa$ such that $\langle V_\alpha, \in, R \cap V_\alpha \rangle \prec \langle V_\kappa, \in, R \rangle$. $\lambda + 1 \in V_\alpha$ and so $\mathcal{P}(\lambda) \in V_\alpha$, but this is again a contradiction. \square

3.3 Mahlo cardinals

Definition 3.6 *Weakly Mahlo Cardinals* κ is weakly Mahlo \leftrightarrow it is a limit ordinal and the set of all regular ordinals less than κ is stationary in κ

Definition 3.7 *Mahlo cardinals* The following definitions are equivalent:

- (i) κ is Mahlo
- (ii) κ is weakly Mahlo and strong limit
- (iii) κ is inaccessible and the regular cardinals below κ form a stationary subset of κ .
- (iv) κ is regular and the stationary sets below κ form a stationary subset of κ .

Theorem 3.8 κ is Mahlo \leftrightarrow for any $R \subset V_\kappa$ there is an inaccessible cardinal $\alpha < \kappa$ such that $\langle V_\alpha, \in, R \cap V_\alpha \rangle \prec \langle V_\kappa, \in, R \rangle$.

Proof. Start with the proof of (3.5) and add the following:
 κ is Mahlo by the following contradiction. If not, there would be a C closed unbounded in κ containing no inaccessible cardinals. By the hypothesis there is an inaccessible $\alpha < \kappa$ such that $\langle V_\alpha, \in, C \cap V_\alpha \rangle \prec \langle V_\kappa, \in, C \rangle$. By elementarity $C \cap \alpha$ is unbounded in α . But then, $\alpha \in C$, which is the contradiction we need. \square

Note that Mahlo cardinals were first described in 1911, almost 50 years before Levy's reflection, which was heavily inspired by those.

3.4 Weakly Compact Cardinals

In this section, we will introduce various well-known large cardinals and establish them via reflection.

Definition 3.9 *A cardinal κ is weakly compact if it is uncountable and satisfies the partition property $\kappa \rightarrow (\kappa)^2$*

Lemma 3.10 *Every weakly compact cardinal is inaccessible*

Proof. Let κ be a weakly compact cardinal. To show that κ is regular, let us assume that κ is the disjoint union $\bigcup \{A_\gamma : \gamma < \lambda\}$ such that $\lambda < \kappa$ and $|A_\gamma| < \kappa$ for each $\gamma < \lambda$. We define a partition $F : [\kappa]^2 \rightarrow \{0, 1\}$ as follows: $F(\{\alpha, \beta\}) = 0$ just in case α and β are the same size A_γ . Obviously, this partition does not have a homogenous set $H \subset \kappa$ of size κ . That κ is a strong limit cardinal follows from Lemma 9.4: (?? doplnit z jecha): If $\kappa \geq 2^\lambda$ for some $\lambda < \kappa$, then because $2^\lambda \leq (\lambda^+)^2$, we have $\kappa \leq (\lambda^+)^2$ and hence $\kappa \leq (\kappa)^2$. \square

Theorem 3.11 *Let κ be a weakly compact cardinal. Then for every stationary set $S \subset \kappa$ there is an uncountable regular cardinal $\lambda < \kappa$ such that the set $S \cap \lambda$ is stationary in λ .*

Proof. TODO \square

3.5 Indescribable Cardinals

Definition 3.12 (Indescribability) *For Q either Π_n^m or Σ_n^m a cardinal κ is Q -indescribable if whenever $U \subseteq V_\kappa$ and φ is a Q sentence such that $\langle V_\kappa, \in, U \rangle \models \varphi$, then for some $\alpha < \kappa$, $\langle V_\alpha, \in, U \cap V_\alpha \rangle \models \varphi$.*

3.6 Measurable Cardinals

Definition 3.13 *An uncountable cardinal κ is measurable if there exists a κ -complete nonprincipal ultrafilter U on κ .*

Every measurable cardinal is inaccessible⁸

3.7 Supercompact cardinals

TODO

⁸See Jech[?] for a proof.

356 **3.8 Reflecting cardinals**

357 TODO

358 **3.9 Strongly compact cardinals**

359 TODO

3.10 Bernays–Gödel Set Theory

Gödel–Bernays set theory, also known as Von Neumann–Bernays–Gödel set theory is an axiomatic set theory that explicitly talks about proper classes as well as sets, which allows it to be finitely axiomatizable, albeit our version stated below contains one schema. It is a conservative extension of Zermalo–Fraenkel set theory. Using forcing, one can prove equiconsistency of BGC and ZFC.

Bernays–Gödel set theory contains two types of objects: proper classes and sets. The notion of set, usually denoted by a lower case letter, is identical to set in ZF, whereas proper classes are usually denoted by upper case letters. The difference between the two is in a fact, that proper classes are not members of other classes, sets, on the other hand, have to be members of classes.

Definition 3.14 (*Gödel–Bernay set theory*)

(i) extensionality for sets

$$\forall a \forall b [\forall x (x \in a \leftrightarrow x \in b) \rightarrow a = b] \quad (3.16)$$

(ii) pairing for sets

$$\forall x \forall y \exists z \forall w [w \in z \leftrightarrow (w = x \vee w = y)] \quad (3.17)$$

(iii) union for sets

$$\forall a \exists b \forall c [c \in b \leftrightarrow \exists d (c \in d \wedge d \in a)] \quad (3.18)$$

(iv) powers for sets

$$\forall a \exists p \forall b [b \in p \leftrightarrow (c \in b \rightarrow c \in a)] \quad (3.19)$$

(v) infinity for sets

$$\text{There is an inductive set.} \quad (3.20)$$

(vi) Extensionality for classes

$$\forall x (x \in A \leftrightarrow x \in B) \rightarrow A = B \quad (3.21)$$

(vii) Foundation for classes

$$\text{Each nonempty class is disjoint from each of its elements.} \quad (3.22)$$

(viii) Limitation of size for sets

$$\text{For any class } C \text{ a set } x \text{ such that } x=C \text{ exists iff} \quad (3.23)$$

$$\text{there is no bijection between } C \text{ and the class } V \text{ of all sets} \quad (3.24)$$

384 (ix) Comprehension schema for classes

For any formula φ with no quantifiers over classes, there is a class A such that $\forall x(x \in A \leftrightarrow \varphi(x))$.
(3.25)

385 The first five axioms are identical to axioms in ZF.

386 Comprehension schema tells us, that proper classes are basically first-order
387 predicates. ...

388 **Definition 3.15** We say that $\varphi(R)$ with a class parameter R reflects if there
389 is α such that

$$\varphi(R) \rightarrow (V_\alpha, V_{\alpha+1}) \models \varphi(R \cap V_\alpha). \quad (3.26)$$

390 **Theorem 3.16** There is a second-order sentence φ which is provable in GB
391 such that if φ reflects at α , i.e. if

$$\varphi \rightarrow (V_\alpha, V_{\alpha+1}) \models \varphi, \quad (3.27)$$

392 then α is an inaccessible cardinal.

393 *Proof.* Take φ to say “there is no function from $\gamma \in \text{ORD}$ cofinal in ORD
394 and for every $\gamma \in \text{ORD}$, $2^\gamma \in \text{ORD}$ ”. Clearly, if φ reflects at some α ,
395 then α is inaccessible (here we use that the second-order variable range over
396 $\mathcal{P}(V_\alpha) = V_{\alpha+1}$). \square

397 As a corollary we obtain:

398 **Corollary 3.17** Second-order reflection in GB implies the existence of an
399 inaccessible cardinal.

3.11 Morse–Kelley Set Theory

Axioms not

(i) *Extensionality*

$$\forall X \forall Y (\forall z (z \in X \leftrightarrow z \in Y) \rightarrow X = Y). \quad (3.28)$$

(ii) *Pairing*

$$asdfg \quad (3.29)$$

(iii) *Foundation For Classes*

$$asdf \quad (3.30)$$

(iv) *Class Comprehension*

$$\forall W_1, \dots, W_n \exists Y \forall x (x \in Y \leftrightarrow (\phi(x, W_1, \dots, W_n) \& set(x))). \quad (3.31)$$

Where $set(x)$ is monadic predicate stating that class x is a set.

(v) *Limitation Of Size For Classes*

$$asdf \quad (3.32)$$

(vi) *Pairing*

$$asdf \quad (3.33)$$

(vii) *Pairing*

$$asdf \quad (3.34)$$

TODO

3.12 Reflection and the constructible universe

L was introduced by Kurt Gödel in 1938 in his paper *The Consistency of the Axiom of Choice and of the Generalised Continuum Hypothesis* and denotes a class of sets built recursively in terms of simpler sets, somewhat similar to Von Neumann universe V . Assertion of their equality, $V = L$, is called the *axiom of constructibility*. The axiom implies GCH and therefore also AC and contradicts the existence of some of the large cardinals, our goal is to decide whether those introduced earlier are among them.

On order to formally establish this class, we need to formalize the notion of definability first:

Definition 3.18 (Definable sets)

$$Def(X) := \{\{y | x \in X \wedge \langle X, \in \rangle \models \varphi(y, z_1, \dots, z_n)\} | \varphi \text{ is a first-order formula, } z_1, \dots, z_n \in X\}$$
(3.35)

Now we can recursively build L .

Definition 3.19 (The Constructible universe) (i)

$$L_0 := \emptyset$$
(3.36)

(ii)

$$L_{\alpha+1} := Def(L_\alpha)$$
(3.37)

(iii)

$$L_\lambda = \bigcup_{\alpha < \lambda} L_\alpha \text{ If } \lambda \text{ is a limit ordinal}$$
(3.38)

(iv)

$$L = \bigcup_{\alpha \in Ord} L_\alpha$$
(3.39)

Fact 3.20 *The reflection – constructed as explained in the previous paragraph (!!! preformulovat !!!) – with second-order parameters for higher-order formulas (even of transfinite type) does not yield transcendence over L .*

TODO zduvodneni

TODO kratka diskuse jestli refl implikuje transcendenci na L - polemika, nazor - $V=L$ a slaba kompaktnost a dalsi

4 Further reflection

4.1 Sharp

TODO

4.2 Welek: Global Reflection Principles

TODO

References

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