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2 Katedra logiky

3 MIKLUÁŠ MRVA

4 REFLECTION PRINCIPLES AND LARGE
5 CARDINALS

6 Bakalářská práce

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8 2015

¹⁰ Prohlašuji, že jsem bakalářskou práci vypracoval samostatně a že jsem uvedl
¹¹ všechny použité prameny a literaturu.

¹² V Praze 14. dubna 2015

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Abstract

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Práce zkoumá vztah tzv. principů reflexe a velkých kardinálů. Lévy ukázal, že v ZFC platí tzv. věta o reflexi a dokonce, že věta o reflexi je ekvivalentní schématu nahrazení a axiomu nekonečna nad teorií ZFC bez axiomu nekonečna a schématu nahrazení. Tedy lze na větu o reflexi pohlížet jako na svého druhu axiom nekonečna. Práce zkoumá do jaké míry a jakým způsobem lze větu o reflexi zobecnit a jaký to má vliv na existenci tzv. velkých kardinálů. Práce definuje nedosažitelné, Mahlovy a nepopsatelné kardinály a ukáže, jak je lze zavést pomocí reflexe. Přirozenou limitou kardinálů získaných reflexí jsou kardinály nekonzistentní s L. Práce nabídne intuitivní zdůvodnění, proč tomu tak je.

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Abstract

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This thesis aims to examine relations between so called "Reflection Principles" and Large cardinals. Lévy has shown that Reflection Theorem is a sound theorem of ZFC and it is equivalent to Replacement Scheme and the Axiom of Infinity. From this point of view, Reflection theorem can be seen a specific version of an Axiom of Infinity. This paper aims to examine the Reflection Principle and its generalisations with respect to existence of Large Cardinals. This thesis will establish Inaccessible, Mahlo and Indescribable cardinals and their definition via reflection. A natural limit of Large Cardinals obtained via reflection are cardinals inconsistent with L. The thesis will offer an intuitive explanation of why this is the case.

38 **Contents**

1 Introduction

1.1 Motivation and Origin

The Universe of sets cannot be uniquely characterized (i. e. distinguished from all its initial elements) by any internal structural property of the membership relation in it, which is expressible in any logic of finite or transfinite type, including infinitary logics of any cardinal order.

— Kurt Gödel [?]

To understand why we need reflection in the first place, let's think about infinity for a moment. In the intuitive sense, infinity is an upper limit of all numbers. But for centuries, this was merely a philosophical concept, closely bound to religious and metaphysical way of thinking, considered separate from numbers used for calculations or geometry. It was a rather vague concept. In ancient Greece, Aristotle's response to famous Zeno's paradoxes introduced the distinction between actual and potential infinity. He argued, that potential infinity is (in today's words) well defined, as opposed to actual infinity, which remained a vague incoherent concept. He didn't think it's possible for infinity to inhabit a bounded place in space or time, rejecting Zeno's thought experiments as a whole. Aristotle's thoughts shaped western thinking partly due to Aquinas, who himself believed actual infinity to be more of a metaphysical concept for describing God than a mathematical property attributed to any other entity. In his *Summa Theologica*¹ he argues:

A geometrician does not need to assume a line actually infinite, but takes some actually finite line, from which he subtracts whatever he finds necessary; which line he calls infinite.

Less than hundred years later, Gregory of Rimini wrote

If God can endlessly add a cubic foot to a stone—which He can—then He can create an infinitely big stone. For He need only add one cubic foot at some time, another half an hour later, another a quarter of an hour later than that, and so on ad infinitum. He would then have before Him an infinite stone at the end of the hour.

Which is basically a Zeno's Paradox made plausible with God being the actor. In contrast to Aquinas' position, Gregory of Rimini theoretically constructs an object with actual infinite magnitude that is essentially different from God.

¹Part I, Question 7, Article 3, Reply to Objection 1

Even later, in the 17th century, pushing the property of infiniteness from the Creator to his creation, Nature, Leibniz wrote to Foucher in 1662:

I am so in favor of the actual infinite that instead of admitting that Nature abhors it, as is commonly said, I hold that Nature makes frequent use of it everywhere, in order to show more effectively the perfections of its Author. Thus I believe that there is no part of matter which is not, I do not say divisible, but actually divided; and consequently the least particle ought to be considered as a world full of an infinity of different creatures.

But even though he used potential infinity in what would become foundations of modern Calculus and argued for actual infinity in Nature, Leibniz refused the existence of an infinite, thinking that Galileo's Paradoxon² is in fact a contradiction. The so called Galileo's Paradoxon is an observation Galileo Galilei made in his final book "Discourses and Mathematical Demonstrations Relating to Two New Sciences". He states that if all numbers are either squares and non-squares, there seem to be less squares than there is all numbers. On the other hand, every number can be squared and every square has its square root. Therefore, there seem to be as many squares as there are all numbers. Galileo concludes, that the idea of comparing sizes makes sense only in the finite realm.

Salviati: So far as I see we can only infer that the totality of all numbers is infinite, that the number of squares is infinite, and that the number of their roots is infinite; neither is the number of squares less than the totality of all the numbers, nor the latter greater than the former; and finally the attributes "equal," "greater," and "less," are not applicable to infinite, but only to finite, quantities. When therefore Simplicio introduces several lines of different lengths and asks me how it is possible that the longer ones do not contain more points than the shorter, I answer him that one line does not contain more or less or just as many points as another, but that each line contains an infinite number.

Leibniz insists in part being smaller than the whole saying

Among numbers there are infinite roots, infinite squares, infinite cubes. Moreover, there are as many roots as numbers. And there are as many squares as roots. Therefore there are as many squares

²zneni galileova paradoxu

as numbers, that is to say, there are as many square numbers as there are numbers in the universe. Which is impossible. Hence it follows either that in the infinite the whole is not greater than the part, which is the opinion of Galileo and Gregory of St. Vincent, and which I cannot accept; or that infinity itself is nothing, i.e. that it is not one and not a whole.

TODO Hegel–strucne?

TODO Cantor

TODO mene teologie, vice matematiky

TODO definovat pojmy (trida etc)

TODO neni V v nejakem smyslu porad potencialni nekonecno, zatimco mnoziny vetsi nez omega jsou aktualni? nebo jsou potencialni protoze se stavaji pres indukci, od spoda.

In his work, he defined transfinite numbers to extend existing natural number structure so it contains more objects that behave like natural numbers and are based on an object (rather a meta-object) that doesn't explicitly exist in the structure, but is closely related to it. This is the first instance of reflection. This paper will focus on taking this principle a step further, extending Cantor's (or Zermelo–Fraenkel's, to be more precise) universe so it includes objects so big, they could be considered the universe itself, in a certain sense.

TODO dal asi smazat

The original idea behind reflection principles probably comes from what could be informally called “universality of the universe”. The effort to precisely describe the universe of sets was natural and could be regarded as one of the impulses for formalization of naive set theory. If we try to express the universe as a set $\{x|x = x\}$, a paradox appears, because either our set is contained in itself and therefore is contained in a set (itself again), which contradicts the intuitive notion of a universe that contains everything but is not contained itself.

TODO ???

If there is an object containing all sets, it must not be a set itself. The notion of class seems inevitable. Either directly the ways for example the Bernays–Gödel set theory, we will also discuss later in this paper, does in, or on a meta-level like the Zermelo–Fraenkel set theory, that doesn't refer to them in the axioms but often works with the notion of a universal class. duet Another obstacle of constructing a set of all sets comes from Georg Cantor, who proved that the set of all subsets of a set (let x be the set and $\mathcal{P}((x)$ its powerset) is strictly larger than x . That would turn every aspiration to

finally establish an universal set into a contradictory infinite regression.³ We will use V to denote the class of all sets. From previous thoughts we can easily argue, that it is impossible to construct a property that holds for V and no set and is neither paradoxical like $\{x|x = x\}$ nor trivial. Previous observation can be transposed to a rather naive formulation of the reflection principle:

Reflection Any property which holds in V already holds in some initial segment of V .

To avoid vagueness of the term "property", we could informally reformulate the above statement into a schema:

For every first-order formula⁴ φ holds in $V \leftrightarrow \varphi$ holds in some initial segment of V .

Interested reader should note that this is a theorem scheme rather than a single theorem.⁵

1.2 A few historical remarks on reflection

Reflection made its first in set-theoretical appearance in Gödel's proof of GCH in L (citace Kanamori ? Lévy and set theory), but it was around even earlier as a concept. Gödel himself regarded it as very close to Russel's reducibility axiom (an earlier equivalent of the axiom schema of Zermelo's separation). Richard Montague then studied reflection properties as a tool for verifying that Replacement is not finitely axiomatizable (citace?). a few years later Lévy proved in [?] the equivalence of reflection with Axiom of infinity together with Replacement in proof we shall examine closely in chapter 2.

TODO co dal? recent results?

³An intuitive analogy of this *reductio ad infinitum* is the status of ω , which was originally thought to be an unreachable absolute, only to become starting point of Cantor's hierarchy of sets growing beyond all boundaries around the end of the 19th century

⁴this also works for finite sets of formulas [?, p. 168]

⁵If there were a single theorem stating "for any formula φ that holds in V there is an initial segment of V where φ also holds", we would obtain the following contradiction with the second Gödel's theorem: In ZFC, any finite group of axioms of ZFC holds in some initial segment of the universe. If we take the largest of those initial segments it is still strictly smaller than the universe and thus we have, via compactness, constructed a model of ZFC within ZFC. That is, of course a harsh contradiction. This also leads to an elegant way to prove that ZFC is not finitely axiomatizable.

1.3 Reflection in Platonism and Structuralism

TODO cite "reflection in a structuralist setting"

TODO veci o tom, ze reflexe je ok protoze reflektuje veci ktere objektivne plati, protoze plati pro V ...

TODO souvislost s kompaktnosti, hranice formalich systemu nebo alespon ZFC

1.4 Notation and terminology

1.4.1 The Language of Set Theory

We are about to define basic set-theoretical terminology on which the rest of this thesis will be built. For Chapter 2, the underlying theory will be the *Zermelo – Fraenkel* set theory with the Axiom of Choice (ZFC), a first-order set theory in the language $\mathcal{L} = \{=, \in\}$, which will be sometimes referred to as *the language of set theory*. In Chapter 3⁶, we shall always make it clear whether we are in first-order ZFC or second-order ZFC₂, which will be precisely defined later in this chapter. When in second-order theory, we will usually denote type 1 variables, which are elements of the domain of discourse⁷ by lower-case letters, mostly $u, v, w, x, y, z, p_1, p_2, p_3, \dots$ while type 2 variables, which represent n -ary relations of the domain of discourse for any natural number n , are usually denoted by upper-case letters A, B, C, X, Y, Z . Note that those may be used both as relations and functions, see the definition of a function below.⁸

TODO uppercase M is a set!

TODO "M is a limit ordinal" je ve skutecnosti formule, nekam to sem napis!

The informal notions of *class* and *property* will be used throughout this thesis. They both represent formulas with respect to the domain of discourse. If $\varphi(x, p_1, \dots, p_n)$ is a formula in the language of set theory, we call

$$A = \{x : \varphi(x, p_1, \dots, p_n)\} \quad (1.1)$$

a class of all sets satisfying $\varphi(x, p_1, \dots, p_n)$ in a sense that

$$x \in A \leftrightarrow \varphi(x, p_1, \dots, p_n) \quad (1.2)$$

One can easily define for classes A, B the operations like $A \cap B, A \cup B, A \setminus C, \bigcup A$, but it is elementary and we won't do it here, see the first part of

⁶TODO bude jich vic? Chapter 4 taky?

⁷co je "domain of discourse"?

⁸TODO ref?

[?] for technical details. The following axioms are the tools by which decide whether particular classes are in fact sets. A class that fails to be considered a set is called a *proper class*.

1.4.2 The Axioms

Definition 1.1 (*The existence of a set*)

$$\exists x(x = x) \quad (1.3)$$

The above axiom is usually not used because it can be deduced from the axiom of *Infinity* (see below), but since we will be using set theories that omit *Infinity*, this will be useful.

Definition 1.2 (*Extensionality*)

$$\forall x, y(\forall z(z \in x \leftrightarrow z \in y) \leftrightarrow x = y) \quad (1.4)$$

Definition 1.3 (*Specification*)

The following is a schema for every first-order formula $\varphi(x, p_1, \dots, p_n)$ with no free variables other than x, p_1, \dots, p_n .

$$\forall x, p_1, \dots, p_n \exists y \forall z(z \in y \leftrightarrow (z \in x \ \& \ \varphi(z, p_1, \dots, p_n))) \quad (1.5)$$

We will now provide two definitions that are not axioms, but will be helpful in establishing some of the other axioms in a more intuitive way.

Definition 1.4 ($x \subseteq y, x \subset y$)

$$x \subseteq y \leftrightarrow \forall z(z \in x \rightarrow z \in y) \quad (1.6)$$

$$x \subset y \leftrightarrow x \subseteq y \ \& \ x \neq y \quad (1.7)$$

Definition 1.5 (*Empty set*)

$$\emptyset \stackrel{\text{def}}{=} \{x : x \neq x\} \quad (1.8)$$

To make sure that \emptyset is a set, note that there exists at least one set y from ??, then consider the following alternative definition.

$$\emptyset' \stackrel{\text{def}}{=} \{x : \varphi(x) \ \& \ x \in y\} \text{ where } y \ \varphi \text{ is the formula } "x \neq x". \quad (1.9)$$

It should be clear that $\emptyset' = \emptyset$.⁹

Now we can introduce more axioms.

⁹For details, see page 8 in [?].

228 **Definition 1.6** (*Foundation*)

$$\forall x(x \neq \emptyset \rightarrow \exists(y \in x)(\forall z \neg(z \in y \ \& \ z \in x))) \quad (1.10)$$

229 **Definition 1.7** (*Pairing*)

$$\forall x, y \exists z \forall q (q \in z \leftrightarrow q \in x \vee q \in y) \quad (1.11)$$

230 **Definition 1.8** (*Union*)

$$\forall x \exists y \forall z (z \in x \leftrightarrow \exists q (z \in q \ \& \ q \in x)) \quad (1.12)$$

231 **Definition 1.9** (*Powerset*)

$$\forall x \exists y \forall z (z \subseteq x \leftrightarrow z \in y) \quad (1.13)$$

232 **Definition 1.10** (*Infinity*)

$$\exists x (\forall y (y \in x) \rightarrow (y \cup \{y\} \in x)) \quad (1.14)$$

233 Let us introduce a few more definitions that will make the two remaining
234 axioms more comprehensible.

235 **Definition 1.11** (*Function*)

236 Given arbitrary first-order formula $\varphi(x, y, p_1, \dots, p_n)$, we say that φ is a func-
237 tion iff

$$\forall x, y, z, p_1, \dots, p_n (\varphi(x, y, p_1, \dots, p_n) \ \& \ \varphi(x, z, p_1, \dots, p_n) \rightarrow y = z) \quad (1.15)$$

238 When a $\varphi(x, y)$ is a function, we also write the following:

$$f(x) = y \leftrightarrow \varphi(x, y) \quad (1.16)$$

239 Note that this f is in fact a formula

240 TODO $f = \{(x, y) : \varphi(x, y)\}$!!! f muze byt mnozina i trida! ¹⁰

241 **Definition 1.12** (*Dom(f)*)

242 Let f be a function. We read the following as " $Dom(f)$ is the domain of f ".

$$Dom(f) \stackrel{\text{def}}{=} \{x : \exists y (f(x) = y)\} \quad (1.17)$$

243 We say " f is a function on A ", A being a class, if $A = dom(f)$.

¹⁰This can also be done for φ s with more than two free variables by either setting $f(x, p_1, \dots, p_n) = y \leftrightarrow \varphi(x, y, p_1, \dots, p_n)$ or saying that φ codes more functions, determined by the various parameters, so $f(x) = y \leftrightarrow \varphi(x, y, t_1, \dots, t_n)$ for given terms t_1, \dots, t_n .

244 **Definition 1.13** (*Rng(f)*)

245 *Let f be a function. We read the following as " $Rng(f)$ is the range of f ".*

$$Rng(f) \stackrel{\text{def}}{=} \{x : \exists y(f(x) = y)\} \quad (1.18)$$

246 We say that f is a function into A , A being a class, if $rng(f) \subseteq A$.

247 Note that $Dom(f)$ and $Rng(f)$ are not definitions in a strict sense, they
 248 are in fact definition schemas that yield definitions for every function f given.
 249 Also note that they can be easily modified for φ instead of f , with the only
 250 difference that then it is defined only for those φ s that are functions.

251 **Definition 1.14** (*Powerset*)

252 *TODO*

253 And now for the axioms.

254 **Definition 1.15** (*Replacement*)

255 *The following is a schema for every first-order formula $\varphi(x, p_1, \dots, p_n)$ with*
 256 *no free variables other than x, p_1, \dots, p_n .*

$$"\varphi \text{ is a function}" \rightarrow \forall x \exists y \forall z (z \in y \leftrightarrow (\exists q \in x)(\varphi(x, y, p_1, \dots, p_n))) \quad (1.19)$$

257 **Definition 1.16** (*Choice*)

258 *This is also a schema. For every A , a family of non-empty sets¹¹, such that*
 259 *$\emptyset \notin S$, there is a function f such that for every $x \in A$*

$$f(x) \in x \quad (1.20)$$

260 We will refer the axioms by their name, written in italic type, e.g. *Founda-*
 261 *tion* refers to the Axiom of Foundation. Now we need to define some basic
 262 set theories to be used in the article. There will be others introduced in Chap-
 263 ter 3, but those will usually be defined just by appending additional axioms
 264 or schemata to one of the following.

265 **Definition 1.17** (**S**)

266 *We call **S** a set theory with the following axioms:*

- 267 (i) Existence of a set (see ??)
- 268 (ii) Extensionality (see ??)
- 269 (iii) Specification (see ??)
- 270 (iv) Foundation (see ??)
- 271 (v) Pairing (see ??)

¹¹We say a class A is a "family of non-empty sets" iff there is B such that $A \subseteq \mathcal{P}(B)$

- 272 (vi) Union (see ??)
 273 (vii) Powerset (see ??)

274 **Definition 1.18** (ZF)

275 We call ZF a set theory that contains all the axioms of the theory S^{12} in
 276 addition to the following

- 277 (i) Replacement schema (see ??)
 278 (ii) Infinity (see ??)

279 **Definition 1.19** (ZFC)

280 ZFC is a theory that contains all the axioms of ZF plus Choice (??).

281

282 **1.4.3 The transitive universe**

283 **Definition 1.20** (Transitive class)

284 We say a class A is transitive iff

$$\forall x(x \in A \rightarrow x \subseteq A) \quad (1.21)$$

285 **Definition 1.21** Well Ordered Class A class A is said to be well ordered by
 286 \in iff the following hold:

- 287 (i) $(\forall x \in A)(x \not\in x)$ (Antireflexivity)
 288 (ii) $(\forall x, y, z \in A)(x \in y \ \& \ y \in z \rightarrow x \in z)$ (Transitivity)
 289 (iii) $(\forall x, y \in A)(x = y \vee x \in y \vee y \in x)$ (Linearity)
 290 (iv) $(\forall x)(x \subseteq A \ \& \ x \neq \emptyset \rightarrow (\exists y \in x)(\forall z \in x)(z = y \vee z \in y))$

291 **Definition 1.22** (Ordinal number)

292 A set x is said to be an ordinal number, also known as an ordinal, if it is
 293 transitive and well-ordered by \in .

294 For the sake of brevity, we usually just say " x is an ordinal". Note that " x is
 295 an ordinal" is a well-defined formula, since ?? is a formula and ?? is in fact a
 296 conjunction of four formulas. Ordinals will be usually denoted by lower case
 297 greek letters, starting from the beginning: $\alpha, \beta, \gamma, \dots$. Given two different
 298 ordinals α, β , we will write $\alpha < \beta$ for $\alpha \in \beta$, see [?]Lemma 2.11 for technical
 299 details.

¹²With the exception of *Existence of a set*

300 **Definition 1.23** (*Successor Ordinal*)

301 Consider the following operation

$$\beta + 1 \stackrel{\text{def}}{=} \beta \cup \{\beta\} \quad (1.22)$$

302 An ordinal α is called a successor ordinal iff there is an ordinal β , such that
 303 $\alpha = \beta + 1$

304 **Definition 1.24** (*Limit Ordinal*)

305 A non-zero ordinal α ¹³ is called a limit ordinal iff it is not a successor ordinal.

306 **Definition 1.25** (*Ord*)

307 The class of all ordinal numbers, which we will denote Ord ¹⁴ be the following
 308 class:

$$\text{Ord} \stackrel{\text{def}}{=} \{x : x \text{ is an ordinal}\} \quad (1.23)$$

309 The following construction will be often referred to as the *Von Neumann's*
 310 *Hierarchy*, sometimes also the *Von Neumann's Universe*.

311 **Definition 1.26** (*Von Neumann's Hierarchy*)

312 The Von Neumann's Hierarchy is a collection of sets indexed by elements of
 313 Ord , defined recursively in the following way:

(i)

$$V_0 = \emptyset \quad (1.24)$$

(ii)

$$V_{\alpha+1} = \mathcal{P}(V_\alpha) \text{ for any ordinal } \alpha \quad (1.25)$$

(iii)

$$V_\lambda = \bigcup_{\beta < \lambda} V_\beta \text{ for a limit ordinal } \lambda \quad (1.26)$$

314 **Definition 1.27** (*Rank*)

315 Given a set x , we say that the rank of x (written as $\text{rank}(x)$) is the least
 316 ordinal α such that

$$x \in V_{\alpha+1} \quad (1.27)$$

317 Due to *Regularity*, every set has a rank.¹⁵

318 **Definition 1.28** (ω)

319

$$\omega \stackrel{\text{def}}{=} \bigcap \{x : x \text{ is a limit ordinal}\} \quad (1.28)$$

320

¹³ $\alpha \neq \emptyset$

¹⁴It is sometimes denoted On , but we will stick to the notation in [?]

¹⁵See chapter 6 of [?] for details.

1.4.4 Cardinal numbers

Definition 1.29 (Cardinality)

Given a set x , let the cardinality of x , written $|x|$, be defined as the smallest ordinal number such that there is an injective mapping from x to α .

For formal details as well as why every set can be well-ordered assuming Choice, see [?].

Definition 1.30 (Aleph function)

Let ω be the set defined by ???. We will recursively define the function \aleph for all ordinals.

- (i) $\aleph_0 = \omega$
- (ii) $\aleph_{\alpha+1}$ is the least cardinal larger than \aleph_α ¹⁶
- (iii) $\aleph_\lambda = \bigcup_{\beta < \lambda} \aleph_\beta$ for a limit ordinal λ

Definition 1.31 (Cardinal number)

We say a set x is a cardinal number, usually called a cardinal, if either $x \in \omega$. Cardinals will be notated by lower-case greek letters starting from $\kappa, \lambda, \mu, \dots$ ¹⁷.

Definition 1.32 (Cofinality)

Let λ be a limit ordinal. The cofinality of λ , written $cf(\lambda)$, is the least limit ordinal α such that there is an increasing α -sequence¹⁸ $\langle \lambda_\beta : \beta < \alpha \rangle$ with $\lim_{\beta \rightarrow \alpha} \lambda_\beta = \lambda$.

Definition 1.33 (Limit Cardinal)

We say that a cardinal κ is a limit cardinal if

$$(\exists \alpha \in Ord)(\kappa = \aleph_\alpha) \quad (1.29)$$

Definition 1.34 (Strong Limit Cardinal)

We say that an ordinal κ is a strong limit cardinal if it is a limit cardinal and

$$\forall \alpha (\alpha \in \kappa \rightarrow \mathcal{P}(\alpha) \in \kappa) \quad (1.30)$$

Definition 1.35 (Generalised Continuum Hypothesis)

$$\aleph_{\alpha+1} = 2^{\aleph_\alpha} \quad (1.31)$$

If *GCH* holds (for example in Gödel's *L*, see chapter 3), the notions of a limit cardinal and a strong limit cardinal are equivalent.

¹⁶"The least cardinal larger than \aleph_α " is sometimes notated as \aleph_α^+

¹⁷ λ is also sometimes used for limit ordinals, the distinction should be clear from the context.

¹⁸TODO def α -sequence

1.4.5 Relativisation

Definition 1.36 (Relativization)

Let M be a class, R a binary relation on M and let $\varphi(p_1, \dots, p_n)$ be a first-order formula with n parameters. The relativization of φ to M and R is the formula, written as $\varphi^{M,R}(p_1, \dots, p_n)$, defined in the following inductive manner:

- (i) $(x \in y)^{M,R} \leftrightarrow R(x, y)$
- (ii) $(x = y)^{M,R} \leftrightarrow x = y$
- (iii) $(\neg \varphi)^{M,R} \leftrightarrow \neg \varphi^{M,R}$
- (iv) $(\varphi \ \& \ \psi)^{M,R} \leftrightarrow \varphi^{M,R} \ \& \ \psi^{M,R}$
- (v) $(\exists x \varphi)^{M,R} \leftrightarrow (\exists x \in M) \varphi^{M,R}$

1.4.6 More functions

TODO def $f : Ord \rightarrow Ord$, asi u powersetu.

Definition 1.37 (Strictly increasing function)

A function $f : Ord \rightarrow Ord$ is said to be strictly increasing iff

$$\forall \alpha, \beta \in Ord (\alpha < \beta \rightarrow f(\alpha) < f(\beta)). \quad (1.32)$$

Definition 1.38 (Continuous function)

A function $f : Ord \rightarrow Ord$ is said to be continuous iff

$$\alpha \text{ is limit} \rightarrow f(\alpha) = \bigcup_{\alpha < \lambda} f(\alpha). \quad (1.33)$$

Definition 1.39 (Normal function)

A function $f : Ord \rightarrow Ord$ is said to be normal if it is strictly increasing and continuous.

Definition 1.40 Fixed point

We say α is a fixed point of ordinal function f when $\alpha = f(\alpha)$.

1.4.7 Higher-Order Logic

Since we will utilise some basic tools of set theories formalized in second- and occasionally higher-order logic, we need to establish the basics here. This part is heavily inspired by Preliminaries from [?].

TODO viz kanamori p. 6

379 TODO proc se neda formalizovat obecne splnovani ve vyssich radech?
380 cite?

381 While higher-order satisfaction relation for proper classes is unformal-
382 izable¹⁹, we can formalize satisfaction in a structure. For the rest of this
383 chapter, let D be a domain of such structure.

384 TODO druhoradove splnovani?
385

386 **Definition 1.41** (*Hierarchy of formulas*)

387 Let φ be a formula. $((v \text{ logice radu } n)) \Pi_n^m$ und Σ_n^m

388 **Lemma 1.42** Δ_0 formulas are absolute in transitive sets, in other words, let
389 φ be a first-order Δ_0 formula and let M be a transitive class.

$$\varphi \leftrightarrow \varphi^M \quad (1.34)$$

390 **Definition 1.43** (ZFC_2)

391 TODO ?

392 TODO nenechat do patricne kapitoly? asi jo.

393 **Definition 1.44** (Reflection_1)

$$ASD \quad (1.35)$$

395

396 2 Levy's first-order reflection

397 2.1 Lévy's Original Paper

398 This section will try to present Lévy's proof of a general reflection principle
399 being equivalent to *Replacement* and *Infinity* under ZF minus *Replacement*
400 and *Infinity* from his 1960 paper *Axiom Schemata of Strong Infinity in Ax-*
401 *iomatic Set Theory*²⁰.

402 When reading said article, one should bear in mind that it was written in
403 a period when set theory was semantically oriented, so while there are many
404 statements about a model of ZF, usually denoted u , this is equivalent to
405 today's universal class V , so it doesn't necessarily mean that there is a set u

¹⁹TODO CITE KDE? Tarski nebo tak neco?

²⁰[?]

that is a model of ZF. We will review the notion of a standard complete model used by Lévy throughout the paper in a moment. Let's first say that the set theory ZF was formulated in the "non-simple applied first order functional calculus", is

TODO viz A. Church nebo tak neco.

The axioms are equivalent to those defined in ??, except for the *Axiom of Subsets*, which is just a different name for *Specification*. Besides ZF and S, defined in ?? and yrefdef:s respectively, the set theories theories Z, and SF are used in the text. Z is ZF minus replacement, SF is ZF minus *Infinity*. Also note that universal quantifier does not appear, $\forall x\varphi(x)$ would be written as $(x)\varphi(x)$. The symbol for negation is " \sim ", implication is written as " \supset " and equivalence is " \equiv ", we will use " \neg ", " \rightarrow " and " \leftrightarrow ".

Next two definitions are not used in contemporary set theory, but they illustrate 1960's set theory mind-set and they are used heavily in Lévy's text, so we will include and explain them for clarity. Generally, in this chapter, Q stands for an arbitrary axiomatic set theory used for general definitions, u is usually a model of Q, counterpart of today's V .

This subsection uses ZF instead of the usual ZFC as the underlying theory.

Definition 2.1 (*Standard model of a set theory*)

Let Q be a axiomatic set theory in first-order logic. We say the the a class u is a standard model of Q with respect to a membership relation E , written as $Sm^Q(u)$, iff both of the following hold

(i) $(x, y) \in E \leftrightarrow y \in u \ \& \ x \in y$

(ii) $y \in u \ \& \ x \in y \rightarrow x \in u$

Definition 2.2 *Standard complete model of a set theory*

Let Q and E be like in ??. We say that that u is a standard complete model of Q with respect to a membership relation E iff both of the following hold

(i) u is a transitive set with respect to \in

(ii) $\forall E((x, y) \in E \leftrightarrow (y \in u \ \& \ x \in y) \ \& \ Sm^Q(u, E))$

this is written as $Scm^Q(u)$.

Definition 2.3 (*Inaccessible cardinal with respect to Q*)

Let Q be an axiomatic first-order set theory. We say that a cardinal κ is inaccessible with respect to Q, we write $In^Q(\kappa)$.

$$In^Q(\kappa) \stackrel{\text{def}}{=} Scm^Q(V_\kappa). \quad (2.36)$$

Definition 2.4 (*Inaccessible cardinal with respect to ZF*)

When a cardinal κ is inaccessible with respect to ZF, we only say that it is inaccessible. We write $In(\kappa)$.

$$In(\kappa) \stackrel{\text{def}}{=} In^{ZF}(\kappa) \quad (2.37)$$

2.2 $S \models (N_0 \leftrightarrow \text{Replacement} \ \& \ \text{Infinity})$ 2. Levy's first-order reflection

442 The above definition of inaccessibles is used because it doesn't require *Choice*.

443 For the definition of relativization, see ???. The syntax used by Lévy is
444 $Rel(u, \varphi)$, we will use φ^u , which is more usual these days.

445 **Definition 2.5** (N)

446 *The following is an axiom schema of complete reflection over ZF, denoted as*
447 *N .*

$$N \stackrel{\text{def}}{=} \exists u (Scm^{ZF}(u) \ \& \ \forall x_1, \dots, x_n (x_1, \dots, x_n \in u \rightarrow \varphi \leftrightarrow \varphi^u)) \quad (2.38)$$

448 *where φ is a formula which contains no free variables except for x_1, \dots, x_n .*

449 **Definition 2.6** (N_0)

450 *With S instead of ZF we obtain what will now be called N_0 .*

$$N_0 \stackrel{\text{def}}{=} \exists u (Scm^S(u) \ \& \ \forall x_1, \dots, x_n (x_1, \dots, x_n \in u \rightarrow \varphi \leftrightarrow \varphi^u)) \quad (2.39)$$

451 *where φ is a formula which contains no free variables except for x_1, \dots, x_n .*

452 **2.2** $S \models (N_0 \leftrightarrow \text{Replacement} \ \& \ \text{Infinity})$

453 Let S be a set theory defined in ???.

454 **Lemma 2.7** *The following holds for every u .*

$$"u \text{ is a limit ordinal}" \leftrightarrow Scm^S(u) \quad (2.40)$$

455 *Proof.* TODO !

456 —

457 In order to prove that it is a model of S , we would need to verify all
458 axioms of S . We have already shown that ω is closed under the powerset
459 operation. Foundation, extensionality and comprehension are clear from the
460 fact that we work in ZF^{21} , pairing is clear from the fact, that given two sets
461 x, y , they have ranks α, β , without loss of generality we can assume that
462 $\alpha \leq \beta$, which means that $x \in V_\alpha \in V_\beta$, therefore V_β is a set that satisfies the
463 pairing axiom: it contains both x and y .

464 □

465 Let N_0 be defined as in ???, for *Infinity* see ???.

466 **Theorem 2.8** *In S , the schema N_0 implies Infinity.*

²¹We only need to verify axioms that provide means of constructing larger sets from smaller to make sure they don't exceed ω . Since ω is an initial segment of ZF , the axiom scheme of specification can't be broken, the same holds for foundation and extensionality.

467 *Proof.* Lévy skips this proof because it seems too obvious to him, but let's do
 468 it here for plasticity. For an arbitrary φ , N_0 gives us $\exists u Scm^S(u)$, but from
 469 lemma ??, we know that this u is a limit ordinal. This u already satisfies
 470 *Infinity*. \square

471

472 Let N_0 be defined as in ??, for *Replacement* see ??, S is again the set
 473 theory defined in ??.

474 **Theorem 2.9** *In S , the schema N_0 implies Replacement.*

475 *Proof.* Let $\varphi(x, y, p_1, \dots, p_n)$ be a formula with no free variables except
 476 x, y, p_1, \dots, p_n for an arbitrary natural number n .

$$\begin{aligned} \chi = & \forall x, y, z (\varphi(x, y, p_1, \dots, p_n) \ \& \ \varphi(x, z, p_1, \dots, p_n) \rightarrow y = z) \\ & \rightarrow \forall x \exists y \forall z (z \in y \leftrightarrow \exists q (q \in x \ \& \ \varphi(q, z, p_1, \dots, p_n))) \end{aligned} \quad (2.41)$$

477 Let χ be an instance of *Replacement* schema for given φ . Let the follow-
 478 ing formulas be instances of the N_0 schema for formulas φ , $\exists y \varphi$, χ and
 479 $\forall x, p_1, \dots, p_n \chi$ respectively:

480 We can deduce the following from N_0 :

- 481 (i) $x, y, p_1, \dots, p_n \in u \rightarrow (\varphi \leftrightarrow \varphi^u)$
- 482 (ii) $x, p_1, \dots, p_n \in u \rightarrow (\exists y \varphi \leftrightarrow (\exists y \varphi)^u)$
- 483 (iii) $x, p_1, \dots, p_n \in u \rightarrow (\chi \leftrightarrow \chi^u)$
- 484 (iv) $\forall x, p_1, \dots, p_n (\chi \leftrightarrow (\forall x, p_1, \dots, p_n \chi)^u)$

485 From relativization, we also know that $(\exists y \varphi)^u$ is equivalent to $(\exists y \in u) \varphi^u$.
 486 Therefore (ii) is equivalent to

$$x, p_1, \dots, p_n \in u \rightarrow (\exists y \in u) \varphi^u. \quad (2.42)$$

487 If φ is a function²², then for every $x \in u$, which is also $x \subset u$ by the
 488 transitivity of $Scm^S(u)$, it maps elements of x onto u . From the axiom scheme
 489 of comprehension²³, we can find y , a set of all images of elements of x . That
 490 gives us $x, p_1, \dots, p_n \in u \rightarrow \chi$. By (iii) we get $x, p_1, \dots, p_n \in u \rightarrow \chi^u$, the
 491 universal closure of this formula is $(\forall x, p_1, \dots, p_n \chi)^u$, which together with
 492 (iv) yields $\forall x, p_1, \dots, p_n \chi$. Via universal instantiation, we end up with χ .
 493 We have inferred replacement for a given arbitrary formula. \square

494 What we have just proven is just a single theorem from the above men-
 495 tioned article by Lévy, we will introduce other interesting propositions, mostly
 496 related to the existence of large cardinals, later in their appropriate context
 497 in chapter 3.

²²See definition ??

²³Lévy uses its equivalent, axiom of subsets

2.3 Contemporary restatement

We will now prove what is also Lévy's first-order reflection theorem, but rephrased with up to date set theory terminology. The main difference is, that while Lévy reflects φ from V to a set u that is a "standard complete model of \mathbf{S} ", we say that there is a V_α for a limit α that reflects φ . We will argue that those are equivalent.²⁴

Definition 2.10 (*Reflection₁*)

Let $\varphi(p_1, \dots, p_n)$ be a first-order formula in the language of set theory. Then the following holds for any such φ .

$$\forall M_0 \exists M (M_0 \subseteq M \ \& \ (\varphi^M(p_1, \dots, p_n) \leftrightarrow \varphi(p_1, \dots, p_n))) \quad (2.43)$$

Note that this restatement on Lévy's N_0 from the previous chapter, see definition ???. We prefer to call it *Reflection₁* so it complies with how other axioms and schemata are called.²⁵ Note that the subscript 1 refers to the fact that $\varphi(p_1, \dots, p_n)$ is a first-order formula, and since we're using the work "reflection" in less strict meaning throughout this thesis, distinguishing between the two just by using italic font face for the schema might cause confusion.

We will now prove the equivalence of *Reflection₁* with *Replacement* and *Infinity* in \mathbf{S} in two parts. First, we will show that N_0 is a theorem of ZFC, then we shall show that the second implication, which proves *Infinity* and *Replacement* from N_0 , also holds.

The following lemma is usually done in more parts, the first being for one formula, the other for n formulas. We will only state and prove the more general version for n formulas, knowing that setting $n = 1$ turns it to a specific version.

Lemma 2.11 Let $\varphi_1, \dots, \varphi_n$ be formulas with m parameters²⁶.

(i) For each set M_0 there is such set M that $M_0 \subset M$ and the following holds for every i , $1 \leq i \leq n$:

$$\exists x \varphi_i(p_1, \dots, p_{m-1}, x) \rightarrow (\exists x \in M) \varphi_i(p_1, \dots, p_{m-1}, x) \quad (2.44)$$

for every $p_1, \dots, p_{m-1} \in M$.

²⁴TODO nekde na to bude lemma!

²⁵We will not use the name N_0 , because it might be confusing to work N_0 and M_0 where M_0 is a set and N_0 is an axiom schema.

²⁶For formulas with a different number of parameters, take for m the highest number of parameters among those formulas. Add spare parameters to every formula that has less than m parameters in a way that preserves the last parameter, which we will denote x . E.g. let φ'_i be the a formula with k parameters, $k < m$. Let us set $\varphi_i(p_1, \dots, p_{m-1}, x) \stackrel{\text{def}}{=} \varphi'_i(p_1, \dots, p_{k-1}, x)$, notice that the parameters p_k, \dots, p_{m-1} are not used.

525 (ii) Furthermore there is an ordinal α such that $M_0 \subset V_\alpha$ and the following
 526 holds for each i , $1 \leq i \leq n$:

$$\exists x \varphi_i(p_1, \dots, p_{m-1}, x) \rightarrow (\exists x \in V_\alpha) \varphi_i(p_1, \dots, p_{m-1}, x) \quad (2.45)$$

527 for every $p_1, \dots, p_{m-1} \in M$.

528 (iii) Assuming Choice, there is M , $M_0 \subset M$ such that ?? holds for every
 529 M , $i \leq n$ and $|M| \leq |M_0| \cdot \aleph_0$.

530 *Proof.* We will simultaneously prove statements (i) and (ii), denoting M^T
 531 the transitive set required by part (ii). Unless explicitly stated otherwise for
 532 specific steps, it is thought to be equivalent to M .

533 Let us first define operation $H(p_1, \dots, p_{m-1})$ that gives us the set of
 534 x 's with minimal rank²⁷ satisfying $\varphi_i(p_1, \dots, p_{m-1}, x)$ for given parameters
 535 p_1, \dots, p_{m-1} for every i such that $1 \leq i \leq n$.

$$H_i(p_1, \dots, p_n) = \{x \in C_i : (\forall z \in C)(\text{rank}(x) \leq \text{rank}(z))\} \quad (2.46)$$

536 for each $1 \leq i \leq n$, where

$$C_i = \{x : \varphi_i(p_1, \dots, p_{m-1}, x)\} \text{ for } 1 \leq i \leq n \quad (2.47)$$

537

538 Next, let's construct M from given M_0 by induction.

$$M_{i+1} = M_i \cup \bigcup_{j=0}^n \{H_j(p_1, \dots, p_{m-1}) : p_1, \dots, p_{m-1} \in M_i\} \quad (2.48)$$

539 In other words, in each step we add the elements satisfying $\varphi(p_1, \dots, p_{m-1}, x)$
 540 for all parameters that were either available earlier or were added in the
 541 previous step. For statement (ii), this is the only part that differs from (i).
 542 Let us take for each step transitive closure of M_{i+1} from (i). In other words,
 543 let γ be the smallest ordinal such that

$$(M_i^T \cup \bigcup_{j=0}^n \{\bigcup \{H_j(p_1, \dots, p_{m-1}) : p_1, \dots, p_{m-1} \in M_i\}\}) \subset V_\gamma \quad (2.49)$$

544 Then the incremental step is like so:

$$M_{i+1}^T = V_\gamma \quad (2.50)$$

²⁷Rank is defined in ??

545 The final M is obtained by joining all the individual steps.

$$M = \bigcup_{i=0}^{\infty} M_i, \quad M^T = \bigcup_{i=0}^{\infty} M_i^T = V_{\alpha} \quad 28 \quad (2.51)$$

546

547 We have yet to finish part (iii). Let's try to construct a set M' that
 548 satisfies the same conditions like M but is kept as small as possible. Assuming
 549 the Axiom of Choice, we can modify the process so that the cardinality of
 550 M' is at most $|M_0| \cdot \aleph_0$. Note that the size of M' is determined by the size of
 551 M_0 and, most importantly, by the size of $H_i(p_1, \dots, p_{m-1})$ for any i , $1 \leq i \leq n$
 552 in individual levels of the construction. Since the lemma only states existence
 553 of some x that satisfies $\varphi_i(p_1, \dots, p_{m-1}, x)$ for any $1 \leq i \leq n$, we only need to
 554 add one x for every set of parameters but $H_i(u_1, \dots, u_{m-1})$ can be arbitrarily
 555 large. Since Axiom of Choice ensures that there is a choice function, let F be
 556 a choice function on $\mathcal{P}(M')$. Also let $h_i(p_1, \dots, p_{m-1}) = F(H_i(p_1, \dots, p_{m-1}))$
 557 for i , where $1 \leq i \leq n$, which means that h is a function that outputs an x
 558 that satisfies $\varphi_i(p_1, \dots, p_{m-1}, x)$ for i such that $1 \leq i \leq n$ and has minimal
 559 rank among all such witnesses. The induction step needs to be redefined to

$$M'_{i+1} = M'_i \cup \bigcup_{j=0}^n \{H_j(p_1, \dots, p_{m-1}) : p_1, \dots, p_{m-1} \in M'_i\} \quad (2.52)$$

560 This way, the amount of elements added to M'_{i+1} in each step of the construc-
 561 tion is the same as the amount of sets of parameters that yielded elements not
 562 included in M'_i . It is easy to see that if M_0 is finite, M' is countable because
 563 it was constructed as a countable union of finite sets. If M_0 is countable or
 564 larger, the cardinality of M' is equal to the cardinality of M_0 .²⁹ Therefore
 565 $|M'| \leq |M_0| \cdot \aleph_0$ □

566 **Theorem 2.12** (*Lévy's first-order reflection theorem*)

567 Let $\varphi(p_1, \dots, p_n)$ be a first-order formula.

568 (i) For every set M_0 there exists M such that $M_0 \subset M$ and the following
 569 holds:

$$\varphi^M(p_1, \dots, p_n) \leftrightarrow \varphi(p_1, \dots, p_n) \quad (2.53)$$

570 for every $p_1, \dots, p_n \in M$.

²⁹It can not be smaller because $|M'_{i+1}| \geq |M'_i|$ for every i . It may not be significantly larger because the maximum of elements added is the number of n -tuples in M'_i , which is of the same cardinality is M'_i .

571 (ii) For every set M_0 there is a transitive set M , $M_0 \subset M$ such that the
 572 following holds:

$$\varphi^M(p_1, \dots, p_n) \leftrightarrow \varphi(p_1, \dots, p_n) \quad (2.54)$$

573 for every $p_1, \dots, p_n \in M$.

574 (iii) For every set M_0 there is α such that $M_0 \subset V_\alpha$ and the following holds:

$$\varphi^{V_\alpha}(p_1, \dots, p_n) \leftrightarrow \varphi(p_1, \dots, p_n) \quad (2.55)$$

575 for every $p_1, \dots, p_n \in M$.

576 (iv) Assuming Choice, for every set M_0 there is M such that $M_0 \subset M$ and
 577 $|M| \leq |M_0| \cdot \aleph_0$ and the following holds:

$$\varphi^M(p_1, \dots, p_n) \leftrightarrow \varphi(p_1, \dots, p_n) \quad (2.56)$$

578 for every $p_1, \dots, p_n \in M$.

579 *Proof.* Before we start, note that the following holds for any set M if φ is an
 580 atomic formula, as a direct consequence of relativisation to M, \in ³⁰.

$$\varphi \leftrightarrow \varphi^M \quad (2.57)$$

581 Let's now prove (i) for given φ via induction by complexity. We can safely
 582 assume that φ contains no quantifiers besides " \exists " and no logical connectives
 583 other than " \neg " and "&". Let $\varphi_1, \dots, \varphi_n$ be all subformulas of φ . Then
 584 there is a set M , obtained by the means of lemma ??, for all of the formulas
 585 $\varphi_1, \dots, \varphi_n$.

586 We know that $\psi \leftrightarrow \psi^M$ for atomic ψ , we need to verify that it won't fail
 587 in the inductive step. Let us consider $\psi = \neg\psi'$ along with the definition of
 588 relativization for those formulas in ??.

$$(\neg\psi')^M \leftrightarrow \neg(\psi'^M) \quad (2.58)$$

589 Because the induction hypothesis says that ?? holds for every subformula of
 590 ψ , we can assume that $\psi'^M \leftrightarrow \psi'$, therefore the following holds:

$$(\neg\psi')^M \leftrightarrow \neg(\psi'^M) \leftrightarrow \neg\psi' \quad (2.59)$$

591 The same holds for $\psi = \psi_1 \& \psi_2$. From the induction hypothesis, we
 592 know that $\psi_1^M \leftrightarrow \psi_1$ and $\psi_2^M \leftrightarrow \psi_2$, which together with relativization for
 593 formulas in the form of $\psi_1 \& \psi_2$ gives us

$$(\psi_1 \& \psi_2)^M \leftrightarrow \psi_1^M \& \psi_2^M \leftrightarrow \psi_1 \& \psi_2 \quad (2.60)$$

³⁰See ??. Also note that this works for relativization to M, \in , not M, E where E is an arbitrary membership relation on M .

594

595 Let's now examine the case when from the induction hypethesis, M re-
 596 flects $\psi'(p_1, \dots, p_n, x)$ and we are interested in $\psi = \exists x \psi'(p_1, \dots, p_n, x)$. The
 597 induction hypothesis tells us that

$$\varphi'^M(p_1, \dots, p_n, x) \leftrightarrow \psi'(p_1, \dots, p_n, x) \quad (2.61)$$

598 so, together with above lemma ??, the following holds:

$$\begin{aligned} & \psi(p_1, \dots, p_n, x) \\ & \leftrightarrow \exists x \psi'(p_1, \dots, p_n, x) \\ & \leftrightarrow (\exists x \in M) \psi'(p_1, \dots, p_n, x) \\ & \leftrightarrow (\exists x \in M) \psi'^M(p_1, \dots, p_n, x) \\ & \leftrightarrow (\exists x \psi'(p_1, \dots, p_n, x))^M \\ & \leftrightarrow \psi^M(p_1, \dots, p_n, x) \end{aligned} \quad (2.62)$$

599 Which is what we have needed to prove. ?? holds for all subformulas
 600 $\varphi_1, \dots, \varphi_n$ of a given formula φ .

601

602 So far we have proven part (i) of this theorem for one formula φ , we only
 603 need to verify that the same holds for any finite number of formulas. This has
 604 in fact been already done since lemma ?? gives us M for any (finite) amount
 605 of formulas, we can find a set M for the union of all of their subformulas.
 606 We can than use the induction above to verify that M reflects each of the
 607 formulas individually iff it reflects all of its subformulas.

608

609 Since V_α is a transitive set, by proving (iii) we also satisfy (ii). To do so,
 610 we only need to look at part (ii) of lemma ?. All of the above proof also
 611 holds for $M = V_\alpha$.

612 To finish part (iv), we take M of size $\leq |M_0| \cdot \aleph_0$, which exists due to
 613 part (iii) of lemma ?, the rest being identical. \square

614

615 Let S be a set theory defined in ??, for ZFC see ??.

616 **Lemma 2.13** *Let M be a set. Then the following holds:*

$$\text{ZFC} \models (M \models S) \leftrightarrow "M \text{ is a limit cardinal}" \quad (2.63)$$

617 *Proof.* For the left-to-right direction, we shall verify that if M is a model
 618 of S , it necessarily is a limit cardinal. From *Powerset*³¹, we know that for

³¹??.

any $x \in M$, $\mathcal{P}(x) \in M$. But that is already the definition of a strong limit cardinal³².

For the converse, we need to see that if there is a limit ordinal α , such that $V_\alpha = M$, the axioms of **S** hold M .

(i) *Existence of a set* (see ??)

There obviously is a set $x \in M$

(ii) *Extensionality* (see ??)

Since *Extensionality*^M is a Δ_0 formula, it holds in any transitive class by ??.

(iii) *Specification* (see ??)

TODO

(iv) *Foundation* (see ??)

Foundation^M is also a Δ_0 formula, so it holds by ?? since M is transitive because it is a cardinal.

(v) *Pairing* (see ??)

TODO

(vi) *Union* (see ??)

TODO

(vii) *Powerset* (see ??)

TODO

□

Let *Infinity* and *Replacement* be as defined in ?? and ?? respectively.

Theorem 2.14 *Reflection₁ is equivalent to Infinity & Replacement under S.*

Proof. Since ?? already gives us one side of the implication, we are only interested in showing the converse which we shall do in two parts:

TODO N_0 prepsat zpatky na *Reflection₁*

$N_0 \rightarrow \text{Infinity}$ From N_0 (??), we know that for any first-order formula φ and a set M_0 , there is a M such that $M_0 \subseteq M$ and $\varphi^M \leftrightarrow \varphi$. Let's pick *Powerset* for φ , then by N_0 there is a set that satisfies *Powerset*, ergo there is a strong limit cardinal, which in turn satisfies *Infinity*.

650

Reflection \rightarrow Replacement

Given a formula $\varphi(x, y, p_1, \dots, p_n)$, we can suppose that it is reflected in any M ³³ What we want to obtain is the following:

$$\forall x, y, z (\varphi(x, y, p_1, \dots, p_n) \ \& \ \varphi(x, z, p_1, \dots, p_n) \rightarrow y = z) \rightarrow \rightarrow \forall X \exists Y \forall y (y \in Y \leftrightarrow \exists x (\varphi(x, y, p_1, \dots, p_n))) \quad (2.64)$$

³²see ??

³³Which means that for $x, y, p_1, \dots, p_n \in M$, $\varphi^M(x, y, p_1, \dots, p_n) \leftrightarrow \varphi(x, y, p_1, \dots, p_n)$.

We do also know that $x, y \in M$, in other words for every $X, Y = \{y \mid \varphi(x, y, p_1, \dots, p_n)\}$ and we know that $X \subset M$ and $Y \subset M$, which, together with the comprehension schema implies that Y , the image of X over φ , is a set. \square

We have shown that *Reflection* for first-order formulas, *Reflection*₁ is a theorem of ZF, which means that it won't yield us any large cardinals. We have also shown that it can be used instead of the *Infinity* and *Replacement* scheme, but $\text{ZF} + \text{Reflection}_1$ is a conservative extension of ZF. Besides being a starting point for more general and powerful statements, it can be used to show that ZF is not finitely axiomatizable. That follows from the fact that *Reflection* gives a model to any finite number of (consistent) formulas. So if $\varphi_1, \dots, \varphi_n$ for any finite n would be the axioms of ZF, *Reflection* would always contain a model of itself, which would in turn contradict the Second Gödel's Theorem³⁴. Notice that, in a way, reflection is complementary to compactness. Compactness argues that given a set of sentences, if every finite subset yields a model, so does the whole set. Reflection, on the other hand, says that while the whole set has no model in the underlying theory, every finite subset does have one.

Also, notice how reflection can be used in ways similar to upward Löwenheim–Skolem theorem. Since Reflection extends any set M_0 into a model of given formulas $\varphi_1, \dots, \varphi_n$, we can choose the lower bound of the size of M by appropriately choosing M_0 .

In the next section, we will try to generalize *Reflection* in a way that transcends ZF and finally yields some large cardinals.

3 Reflection And Large Cardinals

In this chapter we aim to examine stronger reflection properties in order to reach cardinals unavailable in ZFC. Like we said in the first chapter, the variety of reflection principles comes from the fact that there are many way to formalize "properties of the universal class". It is not always obvious what properties hold for V because, (TODO Tarski) We have shown that reflecting properties as first-order formulas doesn't allow us to leave ZFC. We will broaden the class of admissible properties to be reflected and see whether there is a natural limit in the height or width on the reflected universe and also see that no matter how far we go, the universal class is still as elusive as it is when seen from S . That is because for every process for obtaining larger sets such as for example the powerset operation in ZFC, this process

³⁴See chapter ?? for further details.

can't reach V and thus, from reflection, there is an initial segment of V that can't be reached via said process.

To see why this is important, let's dedicate a few lines to the intuition behind the notions of limitness, regularity and inaccessibility in a manner strongly influenced by [?]. To see why limit and strongly limit cardinals are worth mentioning, note that they are "limit" not only in a sense of being a supremum of an ordinal sequence, they also show that a certain way of obtaining larger sets from smaller ones is limited. We will see that all of the alternatives offered in this thesis are in a sense limited. \aleph_λ is a limit cardinal iff there is no α such that $\aleph_{\alpha+1} = \aleph_\lambda$. Strongly limit cardinals point to the limits of the powerset operation. It has been too obvious so far, so let's look at the regular cardinals in this manner. Regular cardinals are those that cannot be³⁵, expressed as a supremum of smaller amount of smaller objects³⁶. More precisely, κ is regular if there is no way to define it as a union of less than κ ordinals, all smaller than κ . So unless there already is a set of size κ , *Replacement* is useless in determining whether κ is really a set. Note that assuming *Choice*, successor cardinals are always regular, so most³⁷ limit cardinals are singular cardinals. So if one is traversing the class of all cardinals upwards, successor steps are still sets thanks to the powerset axiom while singular limits cardinal are not proper classes because they are suprema of images of smaller sets via *Replacement*. Regular cardinals are, in a way, limits of how far can we get by taking limits of increasing sequences of ordinals obtained via *Replacement*.

TODO prepsat – regularita a replacement, proc reflexe dava silnejsi veci
That all being said, it is easy to see that no cardinals in ZFC are both strongly limit and regular because there is no way to ensure they are sets and not proper classes in ZFC. The only exception to this rule is \aleph_0 which needs *Infinity* to exist. It should now be obvious why the fact that κ is inaccessible implies that $\kappa = \aleph_\kappa$.³⁸

We will also examine the connection between reflection principles and (regular) fixed points of ordinal functions in a manner proposed by Lévy in [?]. We will also see that, like Lévy has proposed in the same paper, there is a meaningful way to extend the relation between S and ZFC into a hierarchy of stronger axiomatic set theories.

³⁵Assuming *Choice*.

³⁶Just like ω can not be expressed as a supremum of a finite set consisting solely of finite numbers.

³⁷All provable to exist in ZFC

³⁸This doesn't work backwards, the least fixed point of the \aleph function is the limit of $\{\aleph_0, \aleph_{\aleph_0}, \aleph_{\aleph_{\aleph_0}}, \dots\}$, it is singular since the sequence has countably many elements.

3.1 Regular Fixed-Point Axioms

Lévy's article mentions various schemata that are not instances of reflection per se. We will mention them because they are equivalent to *Reflection*₁³⁹.

Definition 3.1 (*Axiom M_1*)

"Every normal function defined for all ordinals has at least one inaccessible number in its range."

Lévy uses "*M*" to refer to this axiom but since we also use "*M*" for sets and models, for example in ??, we will call the above axiom "*Axiom M_1* " to avoid confusion.

Now we will express *Axiom M_1* to formula to make it clear that it is an axiom scheme and the same can be done with *Axiom M'_1* as well as *Axiom M''_1* introduced immediately afterwards. Since it is an axiom schema and we will later dive into second-order logic, we may also want to refer to *Axiom M_2* as opposed *Axiom M_1* , the former being a single second-order sentence obtained by the obvious modification of *Axiom M_1* .⁴⁰

Let $\varphi(x, y, p_1, \dots, p_n)$ be a first-order formula with no free variables besides x, y, p_1, \dots, p_n . The following is equivalent to *Axiom M_1* .

$$\begin{aligned} & \text{"}\varphi \text{ is a normal function"} \ \& \ \forall x(x \in \text{Ord} \rightarrow \exists y(\varphi(x, y, p_1, \dots, p_n))) \rightarrow \\ & \rightarrow \exists y(\exists x \varphi(x, y, p_1, \dots, p_n) \ \& \ cf(y) = y \ \& \ (\forall x \in \kappa)(\exists y \in \kappa)(x > y)) \end{aligned} \quad (3.65)$$

41

Definition 3.2 (*Axiom M'_1*)

Every normal function defined for all ordinals has at least one fixed point which is inaccessible.

Definition 3.3 (*Axiom M''_1*)

"Every normal function defined for all ordinals has arbitrarily great fixed points which are inaccessible."

The following axiom is proposed by Drake in [?].

Definition 3.4 (*Axiom F_1*)

Every normal function defined for all ordinals has a regular fixed point.

³⁹For definition, see ??

⁴⁰Second-order set theory will be introduced in the next subsection.

⁴¹" φ is a normal function" is equivalent to the following first-order formula:

752 **Lemma 3.5** (*Fixed-point lemma for normal functions*)

753 *Let f be a normal function defined for all ordinals. The all of the following*
 754 *hold*

(i)

$$\forall \lambda (" \lambda \text{ is a limit ordinal} " \rightarrow " f(\lambda) \text{ is a limit ordinal} ") \quad (3.66)$$

(ii)

$$\forall \alpha (\alpha \leq f(\alpha)) \quad (3.67)$$

(iii)

$$\forall \alpha \exists \beta (\alpha < \beta \ \& \ f(\beta) = \beta) (f \text{ has arbitrarily large fixed points.}) \quad (3.68)$$

755 *Proof.* Let f be a normal function.

756 (i) Proof of (i):

757 Suppose λ is a limit ordinal. For an arbitrary ordinal $\alpha < \lambda$, the fact
 758 that f is strictly increasing means that $f(\alpha) < f(\lambda)$ and for an or-
 759 dinal β , $\beta < \alpha$, $f(\alpha) < f(\beta)$. Because f is continuous and λ limit,
 760 $f(\lambda) = \bigcup_{\alpha < \lambda} f(\alpha)$ and since $\beta < \lambda$, $f(\beta) < f(\lambda)$. So we have found
 761 $f(\beta)$ such that $f(\alpha) < f(\beta) < f(\lambda)$, therefore $f(\lambda)$ is a limit ordinal.

762
 763 (ii) Let's prove (ii) via transfinite induction:

764 Since f is defined for all ordinals, there is an ordinal α such that $f(\emptyset) =$
 765 α and because \emptyset is the least ordinal, (ii) holds for \emptyset .

766 Suppose (ii) holds for some β form the induction hypothesis. It the
 767 holds for $\beta + 1$ because f is strictly increasing.

768 For a limit ordinal λ , suppose (ii) holds for every $\alpha < \lambda$. (i) implies that
 769 $f(\lambda)$ is also limit, so there is a strictly increasing κ -sequence $\langle \alpha_0, \alpha_1, \dots \rangle$
 770 for some κ such that $\lambda = \bigcup_{i < \kappa} \alpha_i$. Because f is stricly increasing, the
 771 κ -sequence $\langle f(\alpha_0), f(\alpha_1), \dots \rangle$ is also strictly increasing, the induction
 772 hypothesis implies that $\alpha_i \leq f(\alpha_i)$ for each $i \leq \kappa$. Thus, $\lambda \leq f(\lambda)$.

773 (iii) Proof of (iii):

774

775 For a given α , let there be a ω -sequence $\langle \alpha_0, \alpha_1, \dots \rangle$, such that $\alpha_0 = \alpha$
 776 and $\alpha_{i+1} = f(\alpha_i)$ for each $i < \omega$. This sequence is stricly increasing
 777 because so is f . Now, there's a limit ordinal $\beta = \bigcup_{i < \omega} \alpha_i$, we want to
 778 show that this is the fixed point. So $f(\beta) = f(\bigcup_{i < \omega} \alpha_i) = \bigcup_{i < \omega} f(\alpha_i)$
 779 because f is continuous. We have defined the above sequence so that
 780 $\beta, \bigcup_{i < \omega} f(\alpha_i) = \bigcup_{i < \omega} \alpha_{i+1}$, which means we are done, since $\bigcup_{i < \omega} \alpha_{i+1} =$
 781 $\bigcup_{i < \omega} \alpha_i = \beta$.

782

□

Theorem 3.6

$$\text{Axiom } M_1 \leftrightarrow \text{Axiom } M'_1 \leftrightarrow \text{Axiom } M''_1 \leftrightarrow \text{Axiom } F_1 \quad (3.69)$$

783 This is *Theorem 1* in [?]. *Proof.* It is clear that *Axiom* M''_1 is a stronger
 784 version of *Axiom* M'_1 , which is in turn a stronger version of *Axiom* M_1 , so
 785 the implication *Axiom* $M''_1 \rightarrow \text{Axiom } M'_1 \rightarrow \text{Axiom } M_1$ is satisfied.

786 We will now make sure that *Axiom* $M_1 \rightarrow \text{Axiom } M''_1$ also holds. Let
 787 f be a normal function defined for all ordinals. Let g be a normal function
 788 that counts the fixed points of f . Lemma ?? implies that there arbitrarily
 789 many fixed points of f , therefore g is defined for all ordinals. Let there be
 790 another family of functions, $h_\alpha(\beta) = g(\alpha + \beta)$, obviously h_α is defined for
 791 all ordinals for every $\alpha \in \text{Ord}$ because so is g . Given an arbitrary ordinal
 792 γ , from *Axiom* M_1 we can assume that there is an ordinal δ such that such
 793 that $h_\alpha(\delta) = \kappa$, where κ is inaccessible. But since $\kappa = g(\alpha + \delta)$, κ is a
 794 fixed point of f . To show that there are arbitrarily many fixed points of
 795 f , notice that γ is arbitrary and h_γ is a normal function, so, by lemma ??,
 796 $(\forall \alpha \in \text{Ord})(\alpha \leq f(\alpha))$, therefore $\gamma \leq \gamma + \alpha \leq \kappa$, in other words, there is κ
 797 above an arbitrary ordinal γ . \square

798 **3.2 A Model-Theoretic Intermezzo**

799 This is a small notational intermezzo. Reflection theorems asdasd Tarski
 800 Berkley model-theoretic methods in set theory.

801 This notation is used for example in [?].

802 TODO def $\langle V_\kappa, \in, R \rangle \models \text{asdf Replacement}$

803 TODO $S \rightarrow ZM \rightarrow ZM' \rightarrow ZM''$, neco jako mahlový kardinály, pre-
 804 sunout do dane kapitoly

805 **3.3 Reflecting Second-order Formulas**

806 To see that there is a way to transcend ZFC, let us briefly show how a model
 807 of ZFC can be obtained in $\text{ZFC}_2 + \text{"second-order reflection"}^{42}$. This will be
 808 more closely examined in section ??.

809 We know that ZFC can not be finitely axiomatized in first-order formulas,
 810 however if *Replacement* and *Comprehension* schemes can be substituted by
 811 second-order formulas, ZFC becomes ZFC_2 , which is finitely axiomatizable in
 812 second-order logic. Therefore if we take second-order reflection into consid-
 813 eration, we can obtain a set M that is a model of ZFC_2 . For now, we have

⁴² ZFC_2 is an axiomatization of ZFC in second-order formulas, to be more rigorously established later.

left out the details of how exactly is first-order reflection generalised into stronger statements and how second-order axiomatization of ZFC looks like as we will examine those problems closely in the following pages.

Lower-case letters represent first-order variables and upper-case P represents a second-order variable. [?]

Definition 3.7 (Replacement₂)

$$\begin{aligned} \forall P(\forall x, y, z(P(x, y) \ \& \ P(x, z) \rightarrow y = z) \rightarrow \\ \rightarrow (\forall x \exists y \forall z(z \in y \leftrightarrow \exists w \in x(P(w, z)))) \end{aligned} \quad (3.70)$$

We will denote this axiom Replacement₂.

Definition 3.8 (Specification₂)

$$\forall P \forall x \exists y \forall z (z \in y \leftrightarrow (z \in x \ \& \ P(z, x))) \quad (3.71)$$

Definition 3.9 (ZFC₂)

Let ZFC₂ be a theory with all axioms identical with the axioms of ZFC with the exception of Replacement and Specification schemes, which are replaced with Replacement₂ and Specification₂ respectively.

3.4 Inaccessibility

Definition 3.10 (limit cardinal) κ is a limit cardinal iff it is \aleph_α for some limit ordinal α .

Definition 3.11 (strong limit cardinal) κ is a strong limit cardinal iff it is a limit cardinal and for every $\lambda < \kappa$, $2^\lambda < \kappa$

The two above definition become equivalent when we assume GCH.

Definition 3.12 (weak inaccessibility) An uncountable cardinal κ is weakly inaccessible iff it is regular and limit.

Definition 3.13 (inaccessibility) An uncountable cardinal κ is inaccessible (written $In(\alpha)$) iff it is regular and strongly limit.

We will now show that the above notion is equivalent to the definition Lévy uses in [?], which is, in more contemporary notation, the following:

Theorem 3.14 The following are equivalent:

- 842 1. κ in inaccessible
 843 2. $\langle V_\kappa, \in \rangle \models \text{ZFC}$

844 *Proof.* Let's first prove that if κ is inaccessible, it is a model of ZFC. We will
 845 do that by verifying the axioms of ZFC just like Kanamori does it in in [?,
 846 1.2] and Drake in [?, Chapter 4].

- 847 (i) *Extensionality*:
 848 (see ??)

$$V_\kappa \models \forall x, y (\forall z (z \in x \leftrightarrow z \in y) \rightarrow x = y) \quad (3.72)$$

849 We need to prove that, given two sets that are equal in V , they are equal
 850 in V_κ , in other words, that the *Extensionality* formula is reflected, that
 851 is

$$V_\kappa \models \forall x, y \in V_\kappa (\forall z \in V_\kappa (z \in x \leftrightarrow z \in y) \rightarrow x = y) \quad (3.73)$$

852 But that comes from transitivity. If x and y are in V_κ their members
 853 are also in V_κ .

- 854
 855 (ii) *Foundation*:
 856 (see ??)

$$V_\kappa \models \forall x (\exists z (z \in x) \rightarrow \exists z (z \in x \ \& \ \forall u \neg (u \in z \ \& \ u \in x))) \quad (3.74)$$

857 The argument for *Foundation* is almost identical to the one for *Extensionality*.
 858 For any set $x \in V_\kappa$, transitivity of V_κ makes sure that every
 859 element of x is also an element of V_κ and the same holds for the ele-
 860 ments of elements of x et cetera. So statements about those elements
 861 are absolute between any transitive structures. V and V_κ are both tran-
 862 sitive therefore *Foundation* holds and so does its relativisation to V_κ ,
 863 *Foundation* $^{V_\kappa}$.

- 864
 865 (iii) *Powerset*:
 866 (see ??)

$$V_\kappa \models \forall x \exists y \forall z (z \subseteq x \rightarrow z \in y). \quad (3.75)$$

867 If we take x , an element of V_κ , $\mathcal{P}(x)$ has to be an element of V_κ to,
 868 because it is transitive and a strong limit cardinal.

- 869
 870 (iv) *Pairing*:
 871 (see ??)

$$V_\kappa \models \forall x, y \exists z (x \in z \wedge y \in z). \quad (3.76)$$

872 *Pairing* holds from similar argument like above: let x and y be ele-
 873 ments of V_κ , so there are ordinals $\alpha, \beta < \kappa$ such that $x \in V_\alpha$, $y \in V_\beta$.

Without any loss of generality, suppose $\alpha < \beta$, therefore $V_\alpha \subset V_\beta$ which, from transitivity of the cumulative hierarchy, means that $x \in V_\beta$, then $\{x, y\} \in V_{\beta+1}$ which is still in V_κ because it is a strong limit cardinal.

(v) *Union:*
(see ??)

$$V_\kappa \models \forall x \exists y \forall z \forall w ((w \in z \wedge z \in x) \rightarrow w \in y). \quad (3.77)$$

We want to see that for every $x \in V_\kappa$, this is equivalent to

$$V_\kappa \models \forall x \in V_\kappa, \exists y \in V_\kappa \forall z \in V_\kappa \forall w \in V_\kappa ((w \in z \wedge z \in x) \rightarrow w \in y). \quad (3.78)$$

Since V_κ is transitive, if $x \in V_\kappa$, all of its elements as well as their elements are in V_κ . To see that they also form a set themselves we only need to remember that V_κ is limit and therefore if α is the least ordinal such that $x \in V_\alpha$, $\bigcup x \in V_{\alpha+1}$.

(vi) *Replacement, Infinity:*
(see ??, ??)

We know that those hold from ??.

We will now show that if a set is a model of ZFC, it is in fact an inaccessible cardinal. So let V_κ be a model of ZFC which means that it is closed under the powerset operation, in other words:

$$\forall \lambda (\lambda < \kappa \rightarrow 2^\lambda < \kappa) \quad (3.79)$$

which is exactly the definition of strong limitness. κ is regular from the following argument by contradiction:

Let us suppose for a moment that κ is singular. Therefore there is an ordinal $\alpha < \kappa$ and a function $F : \alpha \rightarrow \kappa$ such that the range of F is unbounded in κ , in other words, $F[\alpha] \subseteq V_\kappa$ and $\sup(F[\alpha]) = \kappa$. In order to achieve the desired contradiction, we need to see that it is the case that $F[\alpha] \in V_\kappa$. Let $\varphi(x, y)$ be the following first-order formula:

$$F(x) = y \quad (3.80)$$

Then there is an instance of *Replacement* that states the following:

$$\begin{aligned} &(\forall x, y, z (\varphi(x, y) \ \& \ \varphi(x, z) \rightarrow y = z)) \rightarrow \\ &\rightarrow (\forall x \exists y \forall z (z \in y \leftrightarrow \exists w (\varphi(w, z)))) \end{aligned} \quad (3.81)$$

Which in turn means that there is a set $y = F[\alpha]$ and $y \in V_\kappa$, which is the contradiction with $\sup(y) = \kappa$ we are looking for. \square

903

904 The same holds for ZFC_2 , the proof is very similar.**Theorem 3.15**

$$V_\kappa \models \text{ZFC}_2 \leftrightarrow \kappa \text{ is inaccessible} \quad (3.82)$$

905 *Proof.* κ is a strong limit cardinal because from ZFC_2 and *Powerset* we know
 906 that for every $\lambda < \kappa$, we know that $2^\lambda < \kappa$.

907 κ is also regular, because otherwise there would be an ordinal α and
 908 a function $F : \alpha \rightarrow \kappa$ with a range unbounded in κ . *Replacement*² gives us
 909 a set $y = F[\alpha]$, so $y \in V_\kappa$, which contradicts the fact that $\sup(y) = \kappa$.
 910 It can not be the case that $\kappa \in V_\kappa$.

911 The other direction is exactly like the first part of above theorem ?? \square

912

913 This is how the existence of an inaccessible cardinal is established in [?].

Definition 3.16 N

915

$$\exists u (In(\alpha) \ \& \ \forall p_1, \dots, p_n (p_1, \dots, p_n \in u \rightarrow (\varphi \leftrightarrow \varphi^u))) \quad (3.83)$$

916 It is interesting to see that the above schema yields the first inaccessible
 917 cardinal if we take for φ the conjunction of all axioms of ZF_2 .

918

919 To see that inaccessible cardinal can be also obtained by a fixed-point
 920 axiom (or a scheme if were in first-order logic), see the following theorem by
 921 Lévy, we won't repeat the proof here, it is available in [?, Theorem 3],

Theorem 3.17

$$M \leftrightarrow N \quad (3.84)$$

922 We have transcended ZFC , but that is just a start. Naturally, we could
 923 go on and consider the next inaccessible cardinal, which is inaccessible with
 924 respect to the theory $\text{ZFC} + \exists \kappa (\kappa \models \text{ZFC})$. But let's try to find a faster way
 925 up, informally at first.

926 Since we can find an inaccessible set larger than any chosen set M_0 , it
 927 is clear that there are arbitrarily large inaccessible cardinals in V , they are
 928 "unbounded"⁴³ in V . If V were a cardinal, we could say that there are V
 929 inaccessible cardinals less than V , but this statement of course makes no sense
 930 in set theory as is because V is not a set. But being more careful, we could
 931 find a property that can be formalized in second-order logic and reflect it to
 932 an initial segment of V . That would allow us to construct large cardinals

⁴³The notion is formally defined for sets, but the meaning should be obvious.

more efficiently than by adding inaccessible cardinals one by one. The property we are looking for ought to look like something like this:

$$\begin{aligned} &\kappa \text{ is an inaccessible cardinal and} \\ &\text{there are } \kappa \text{ inaccessible cardinals } \mu < \kappa \end{aligned} \tag{3.85}$$

This is in fact a fixed-point type of statement. We shall call those cardinals hyper-inaccessible. Now consider the following definition.

Definition 3.18 *0-inaccessible cardinal*
A cardinal κ is 0-inaccessible if it is inaccessible.

We can define α -weakly-inaccessible cardinals analogously with the only difference that those are limit, not strongly limit.

Definition 3.19 *α -hyper-inaccessible cardinal*
For any ordinal α , κ is called α -inaccessible, if κ is inaccessible and for each $\beta \prec \alpha$, the set of β -inaccessible cardinals less than κ is unbounded in κ .

Because κ is inaccessible and therefore regular, the number of β -inaccessibles below κ is equal to κ . We have therefore successfully formalized the above vague notion of hyper-inaccessible cardinal into a hierarchy of α -inaccessibles.

Let's now consider iterating this process over again. Since, informally, V would be α -inaccessible for any α , this property of the universal class could possibly be reflected to an initial segment, the smallest of those will be the first hyper-inaccessible cardinal. Such κ is larger than any α -inaccessible since from regularity of κ , for given $\alpha < \kappa$, κ is κ -th α -hyper-inaccessible cardinal. It is in fact "inaccessible" via α -inaccessibility.

Definition 3.20 *Hyper-inaccessible cardinal*
 κ is called the hyper-inaccessible, also 0-hyper-inaccessible, cardinal if it is α -inaccessible for every $\alpha < \kappa$.

Definition 3.21 *α -hyper-inaccessible cardinal*
For any ordinal α , κ is called α -hyper-inaccessible cardinal if for each ordinal $\beta < \alpha$, the set of β -hyper-inaccessible cardinals less than κ is unbounded in κ .

Obviously we could go on and iterate it ad libitum, but the nomenclature would be increasingly confusing. A smarter way to accomplish the same goal is carried out in the following section.

3.5 Mahlo Cardinals

While the previous chapter introduced us to a notion of inaccessibility and the possibility of iterating it ad libitum in new theories, there is an even faster way to travel upwards in the cumulative hierarchy, that was proposed by Paul Mahlo in his papers (see [?], [?] and [?]) at the very beginning of the 20th century, and which can be easily reformulated using (*Reflection*). To see how Lévy's initial statement of reflection was influenced by Mahlo's work, refer to section ???. The aim of the following paragraphs is to give an intuitive explanation of the idea behind Mahlo's hierarchy of cardinals, all claims made here ought to be stated formally later in the very same chapter.

At the very end of section ??, we have tried to establish the notion of hyper-inaccessibility and iterate it to yield even larger large cardinals. In order to avoid too bulky cardinal names, let's try a different route and establish those cardinals directly via reflection.

The following two definitions come from [?] and while they are rather informal, we will find them very helpful for understanding the Mahlo cardinals.

Definition 3.22 (*Fixed-point property*)

For any first-order formula $\psi(x, p_1, \dots, p_n)$ with no free variables other than x, p_1, \dots, p_n , which is any property of ordinals, we say that a property φ is a fixed-point property if φ has the form

$$\begin{aligned} &x \text{ is an inaccessible cardinal and} \\ &\text{there are } x \text{ ordinals less than } x \text{ that have the property } \psi(x, p_1, \dots, p_n). \end{aligned} \tag{3.86}$$

Definition 3.23 (*Fixed-point reflection*)

If φ is a fixed-point property that holds for V , it also holds for some V_α , an initial segment of V .

Obviously those are in no way rigorous definitions because we have no idea what $\psi(x, p_1, \dots, p_n)$ looks like. Let's try to restate the same idea in a useful way. But first, let's show that the formal counterpart of the idea of containing "enough" ordinals with a property is the notion of stationary set.

998 **Definition 3.24** (*Supremum*)

999 *Given x a set of ordinals, the supremum of x , denoted $\sup(x)$, is the least*
 1000 *upper bound of x .*

$$\sup(x) = \text{bigcup}x \quad (3.87)$$

1001 **Definition 3.25** (*Limit point*)

1002 *Given x , a set of ordinals and an ordinal α , we say that α is a limit point*
 1003 *of x if $\sup(x \cap \alpha) = \alpha$*

1004 **Definition 3.26** (*Set Unbounded in α*) *Let α be an ordinal. We say that*
 1005 *$x \subset \alpha$ is unbounded in α iff*

$$\forall \beta \in \text{Ord}(\beta < \alpha \rightarrow \exists \gamma(\gamma \in x(\beta \leq \gamma < \alpha))) \quad (3.88)$$

1006

1007 In other words, κ is a mahlo cardinal if it is inaccessible and every club
 1008 set in κ contains an inaccessible cardinal. This is exactly the notion of fixed-
 1009 point reflection we were trying to show earlier.

1010

1011 [?]

1012 **Definition 3.27** *The following definitions are equivalent:*

1013 (i) κ is Mahlo

1014 (ii) κ is weakly Mahlo and strong limit

1015 (iii) The set $\{\lambda < \kappa : \lambda \text{ is inaccessible}\}$ is stationary in κ .

1016 (iv) Every normal function on κ has an inaccessible fixed point.

1017 *Proof.* (i) \leftrightarrow (ii) Let κ_1 be a mahlo cardinal and let κ_2 be a strong limit
 1018 weakly Mahlo cardinal. We know from the definitions that the set $\{\lambda <$
 1019 $\kappa : \lambda \text{ is inaccessible}\}$ is stationary in both κ_1 and κ_2 , the only difference
 1020 being that κ_1 is a strongly limit cardinal, but κ_2 would be limit from weak
 1021 Mahloness, wasn't it for the fact that it is also strong limit. This eliminates
 1022 the only difference between them and therefore κ_1 is also strong limit weakly
 1023 Mahlo cardinal and κ_2 is Mahlo.

1024

1025 (i) \rightarrow (iii) We know that κ is uncountable, regular, strong limit and that
 1026 the set $S = \{\lambda < \kappa : \lambda \text{ is regular}\}$ is stationary in κ . We want to prove
 1027 that $S' = \{\lambda < \kappa : \lambda \text{ is inaccessible}\}$ is thus also stationary in κ .

1028 Since stationary set intersects every club set in κ , let C be any such set.
 1029 Let $D = \{\lambda < \kappa : \lambda \text{ is strong limit}\}$. D is a club set because TODO. Since
 1030 intersection of less than κ club sets is a club set, $C \cap D \neq \emptyset$.

1031 TODO proc $\lambda = S \cap C \cap D$ je inaccessible?

1032 (iii) \rightarrow (iv)
 1033 TODO jak to dela Lévy?
 1034 (iv) \rightarrow (i)
 1035 TODO jak to dela Lévy?
 1036 range kazde normalni funkce je club v On. (nevadi ze On je trida?)
 1037 co treba lemma ze pevne body tvori taky club set
 1038 mozna by stacilo ze jsou unbounded, tedy kazda normalni funkce ma
 1039 libovolne velke pevne body. \square
 1040
 1041 TODO obdoba pro α -Mahlo kardinaly?
 1042 TODO κ is hyper-Mahlo iff κ is inaccessible and the set $\{\lambda < \kappa :$
 1043 λ is Mahlo $\}$ is stationary in κ . to je to samy jako α -Mahlo, ne?

1044 3.6 Indescribability

1045 α -Mahlo are the extreme of regular fixed-point axioms, they are about as
 1046 high as we can get via normal functions and stationary sets.
 1047 Let's try a different strategy. Remember how we said that (Regular, Limit
 1048 and) various Large cardinals are in a way all determined by being unreachable
 1049 by a specific process of creating bigger cardinals from already available ones?
 1050 TODO indescribable – reflecting indescribability – we can't reach V by a Σ_1^1
 1051 formula, so there's some initial segment V_α that is also unreachable (we say
 1052 indescribable) by the means of a ... formula
 1053 Let's recall complete reflection theorem first, consider the following:

For every sentence φ , there is a limit ordinal α such that $\varphi_\alpha^V \leftrightarrow \varphi$ (3.89)

1054 We may also require that $\alpha < \beta$, where β is an arbitrary ordinal given.

1055

1056 For the exact definition of Π_n^m and Σ_n^m see ??

1057 **Definition 3.28** (Π_n^m -indescribable cardinal) We say that κ is Π_n^m -indescribable
 1058 iff for any Π_n^m sentence φ such that $V_\kappa \models \varphi$ there is an $\alpha < \kappa$ such that
 1059 $V_\alpha \models \varphi$

1060 **Definition 3.29** (Σ_n^m -indescribable cardinal) We say that κ is Σ_n^m -indescribable
 1061 iff for any Σ_n^m sentence φ such that $V_\kappa \models \varphi$ there is an $\alpha < \kappa$ such that
 1062 $V_\alpha \models \varphi$

1063 **Lemma 3.30** Let κ be a cardinal, the following holds for any $n \in \omega$. κ is
 1064 Π_n^1 -indescribable iff κ is $\Sigma_n^1 + 1$ -indescribable

1065 *Proof.* The forward direction is obvious, we can always add a spare quantifier
 1066 over a type 2 variable to turn a Π_n^1 formula φ into a $\exists P\varphi$ which is thus a
 1067 $\Sigma_n^1 + 1$ formula.⁴⁴

1068 To prove the opposite direction, suppose that $V_\kappa \models \exists X\varphi(X)$ where X is
 1069 a type 2 variable and φ is a Π_n^1 formula with one free variable of type 2. This
 1070 means that there is a set $S \subseteq V_\kappa$ that is a witness of $\exists X\varphi(X)$, in other words,
 1071 $\varphi(S)$ holds. We can replace every occurrence of X in φ by a new predicate
 1072 symbol S , this allows us to say that κ is Π_n^1 -indescribable (with respect to
 1073 $\langle V_\kappa, \in, R, S \rangle$).⁴⁵ \square

1074 The above lemma tells us that we as long as we stay in the realm of type
 1075 1 and type 2 variables, we only need to classify indescribable cardinals with
 1076 respect to Π_n^1 -indescribability.

1077 **Theorem 3.31** *Let κ be an ordinal. The following are equivalent.*

- 1078 (i) κ is inaccessible
 1079 (ii) κ is Π_0^1 -indescribable.

1080 Note that Π_0^1 formulas are those that contain zero unbound quantifiers
 1081 over type-2 variables, they are in fact first-order formulas. We have already
 1082 shown in ?? that there is no way to reach an inaccessible cardinal via first-
 1083 order formulas in ZFC. We will now prove it again in for formal clarity.

1084 *Proof.* TODO asi pridat alternativni definici nedosazitelnosti podle kan. 6.2?
 1085 \square

1086 TODO nejaka veta ze kdyz jsou Π_0^1 -indescribable, jsou i Π_n^m -indescribable
 1087 pro $m \leq 1, n \leq 0$? Drake? Obracene! Π_n^m -indescribable jsou zaroven Π_b^a -
 1088 indescribable pro $a < m, b < n$.

1089 The above theorem provides an easy way to show that every following
 1090 large cardinal is also an inaccessible cardinal⁴⁶.

1091 **Definition 3.32** (*Extension property*) *We say that a cardinal κ has the ex-*
 1092 *tension property iff for any $R \subseteq V_\kappa$ there is a transitive set $X \neq V_\kappa$ and an*
 1093 *$S \subseteq X$ such that $\langle V_\kappa, \in, R \rangle \prec \langle X, \in, S \rangle$*

1094 **Definition 3.33** (*Weakly compact cardinal*)

1095 *We say that a cardinal κ is weakly compact iff it has the extension property.*

⁴⁴Note that unlike in previous sections, φ is now a sentence so we don't have to worry whether P is free in φ .

⁴⁵A different yet interesting approach is taken by Tate in ?. He states that for $n \geq 0$, a formula of order $\leq n$ is called a Π_0^n and a Σ_0^n formula. Then a Π_{m+1}^n is a formula of form $\forall Y\psi(Y)$ where ψ is a Σ_m^n formula and Y is a variable of type n . Finally, a Σ_{m+1}^n is the negation of a Π_m^n formula. So the above holds ad definitio.

⁴⁶That is because Π_0^1 formulas are included Π_n^m formulas for $m \leq 1, n \leq 0$.

1096 The above definitions are equivalent

1097 **Theorem 3.34** *the following are equivalent:*

1098

1099 (i) κ is Weakly compact.

1100 (ii) κ is Π_1^1 -indescribable.

1101 For a proof, see [?][Theorem 6.4]

1102 TODO def totalne nepopsatelny kardinal

1103 TODO viz Drake, Ch.9 par. 3 – tam se rika ze kdyz κ je meritelny
1104 kardinal, pak je κ Π_1^2 -nepopsatelny kardinal (kanamori to rika taky)

1105 3.7 Measurable Cardinal

1106 TODO refaktorizovat fle:

1107 **Definition 3.35** (*Ultrafilter*)

1108 Given a set X , we say $U \subset \mathcal{P}(X)$ is an ultrafilter iff all of the following
1109 hold:

1110 (i) $\emptyset \notin U$

1111 (ii) $\forall a, b \subset X \ \& \ a \subset b \ \& \ a \in U \rightarrow b \in U$

1112 (iii) $\forall a, b \in U \ (a \cap b) \in U$

1113 (iv) $\forall a \ (a \subset X \rightarrow (a \in U \vee (X \setminus a) \in U))$

1114 **Definition 3.36** (κ -complete ultrafilter)

1115 We say that an ultrafilter U is κ -complete iff

1116 **Definition 3.37** (*non-principal ultrafilter*)

1117 TODO

1118 **Definition 3.38** (*Measurable Cardinal*)

1119 Let κ be a cardinal. We say κ is a measurable cardinal iff it is an uncountable
1120 cardinal with a κ -complete, non-principal ultrafilter.

1121 **Theorem 3.39** Let κ be a cardinal. If κ is a measurable cardinal then it is
1122 Π_1^2 -indescribable.

1123 **Theorem 3.40** Pod kazdym meritelnym kardinalem existuje ultrafiltr to-
1124 talne nepopsatelnych, ktere tim padem nejsou sestrojitelne. VIZ VETA Z
1125 KANAMORIHO.

1126 asi nedokazovat?

3.8 The Constructible Universe

The constructible universe, denoted L , is a cumulative hierarchy of sets, presented by Kurt Gödel in his 1938 paper *The Consistency of the Axiom of Choice and of the Generalised Continuum Hypothesis*. For a technical description, see below. Assertion of their equality, $V = L$, is called the *axiom of constructibility*. The axiom implies GCH and therefore also AC and contradicts the existence of some of the large cardinals, our goal is to decide whether those introduced earlier are among them.

On order to formally establish this class, we need to formalize the notion of definability first.

Definition 3.41 We say that a set X is definable over a model $\langle M, \in \rangle$ if there is a first-order formula φ together with parameters $p_1, \dots, p_n \in M$ such that

$$X = \{x : x \in M \ \& \ \langle M, \in \rangle \models \varphi(x, p_1, \dots, p_n)\} \quad (3.90)$$

Definition 3.42 (Sets definable in M)

The following is a set of all definable subsets of a given set M , denoted $\text{Def}(M)$.

$$\begin{aligned} \text{Def}(M) = \{ \{y : x \in M \wedge \langle M, \in \rangle \models \varphi(y, u_1, \dots, u_n)\} \mid \\ \varphi \text{ is a first-order formula, } p_1, \dots, p_n \in M \} \end{aligned} \quad (3.91)$$

Now we can recursively build L .

Definition 3.43 (The Constructible universe)

(i)

$$L_0 := \emptyset \quad (3.92)$$

(ii)

$$L_{\alpha+1} := \text{Def}(L_\alpha) \quad (3.93)$$

(iii)

$$L_\lambda = \bigcup_{\alpha < \lambda} L_\alpha \text{ If } \lambda \text{ is a limit ordinal} \quad (3.94)$$

(iv)

$$L = \bigcup_{\alpha \in \text{Ord}} L_\alpha \quad (3.95)$$

Note that while L bears very close resemblance to V , the difference is, that in every successor step of constructing V , we take every subset of V_α to be $V_{\alpha+1}$, whereas $L_{\alpha+1}$ consists only of definable subsets of L_α . Also note that L is transitive.

In order to

TODO:

1151 **Lemma 3.44** $Ord \in L$

1152 **Lemma 3.45** L is well-ordered.

1153 *TODO !!*

1154 **Theorem 3.46** Let L be as in ??.

$$L \models \text{ZFC} \quad (3.96)$$

1155 *Proof.* *TODO !!!* (strucne) vit [?][Theorem 13.3]

1156 (i) *Extensionality* (see ??):

1157 *Extensionality* holds in L because Δ_0 formulas are absolute in transitive
1158 classes by ??, *Extensionality* is Δ_0 and L is transitive.

1159 (ii) *Foundation* (see ??)

1160 Take a non-empty set X . Let $x \in X$ be a set such that $X \cap x = \emptyset$. x
1161 is therefore defined by the formula $\varphi(x, y) = (x \cap y = \emptyset)$, so $x \in L$. φ
1162 is Δ_0 and therefore holds in L by ??.

1163 (iii) *Pairing* (see ??)

1164 Since *Pairin* is also Δ_0 , it holds in L by the same argument as *Exten-*
1165 *sionality* does by ??.

1166 (iv) *Union* (see ??)

1167 *Union* is also Δ_0 , see *Extensionality* and ??.

1168 (v) *Power Set* (see ??)

1169 *Power Set* also holds by ??.

1170 (vi) *Infinity* (see ??)

1171 $\omega \in L$ by ??

1172 (vii) *Specification* (see ??)

1173 .

1174 (viii) *Replacement* (see ??)

1175 .

1176 (ix) *Choice* (see ??)

1177 .

1178 □

1179 **Definition 3.47** *Constructibility*

1180 $L = V$

1181 The following are a few interesting results that we won't prove but refer
1182 interested reader to appropriate resources instead.

1183 **Definition 3.48** (*GCH*)

1184 The following is called the *Generalised Continuum Hypothesis*, abbreviated
1185 as *GCH*. It is an independent statement in *ZFC*.

$$\text{GCH iff } \aleph_{\alpha+1} = 2^{\aleph_\alpha} \text{ for every ordinal } \alpha \quad (3.97)$$

Theorem 3.49

$$(L = V) \rightarrow GCH \quad (3.98)$$

1186 This is proven in cite{neco} Gödel? Jech? Kunnen?
 1187 TODO L a velke kardinaly
 1188 TODO def Con!

Theorem 3.50

$$Con(L + \exists \kappa (\kappa \text{'' is an Inaccessible Cardinal})) \quad (3.99)$$

Theorem 3.51

$$Con(L + \exists \kappa (\kappa \text{'' is a Mahlo Cardinal})) \quad (3.100)$$

Theorem 3.52

$$Con(L + \exists \kappa (\kappa \text{'' is a Weakly Inaccessible Cardinal Cardinal})) \quad (3.101)$$

Theorem 3.53

$$Con(L + \exists \kappa (\kappa \text{'' is a Measurable Cardinal})) \quad (3.102)$$

1189 TODO co velky pismena ve jmenech kardinalu?
 1190 TODO zduvodneni
 1191
 1192 TODO kratka diskuse jestli refl implikuje transcendenci na L - polemika,
 1193 nazor - V=L a slaba kompaktnost a dalsi
 1194

1195 4 Conclusion

1196 TODO na konec

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