Univerzita Karlova v Praze, Filozofická fakulta
Katedra logiky

Mikluáš Mrva

REFLECTION PRINCIPLES AND LARGE CARDINALS

Bakalářská práce

Vedoucí práce: Mgr. Radek Honzík, Ph.D.

2015

Prohlašuji, že jsem bakalářkou práci vypracoval samostatně a že jsem uvedl všechny použité prameny a literaturu.

V Praze 14. dubna 2015

Mikuláš Mrva

Abstract

Práce zkoumá vztah tzv. principů reflexe a velkých kardinálů. Lévy ukázal, že v ZFC platí tzv. věta o reflexi a dokonce, že věta o reflexi je ekvivalentní schématu nahrazení a axiomu nekonečna nad teorií ZFC bez axiomu nekonečna a schématu nahrazení. Tedy lze na větu o reflexi pohlížet jako na svého druhu axiom nekonečna. Práce zkoumá do jaké míry a jakým způsobem lze větu o reflexi zobecnit a jaký to má vliv na existenci tzv. velkých kardinálů. Práce definuje nedosažitelné, Mahlovy a nepopsatelné kardinály a ukáže, jak je lze zavést pomocí reflexe. Přirozenou limitou kardinálů získaných reflexí jsou kardinály nekonzistentní s L. Práce nabídne intuitivní zdůvodněn, proč tomu tak je.

Abstract

This thesis aims to examine relations between so called "Reflection Principles" and Large cardinals. Lévy has shown that Reflection Theorem is a sound theorem of ZFC and it is equivalent to Replacement Scheme and the Axiom of Infinity. From this point of view, Reflection theorem can be seen a specific version of an Axiom of Infinity. This paper aims to examine the Reflection Principle and its generalisations with respect to existence of Large Cardinals. This thesis will establish Inaccessible, Mahlo and Indescribable cardinals and their definition via reflection. A natural limit of Large Cardinals obtained via reflection are cardinals inconsistent with L. The thesis will offer an intuitive explanation of why this is the case.

Contents

1 Introduction

1.1 Motivation and Origin

> The Universe of sets cannot be uniquely characterized (i. e. distinguished from all its initial elements) by any internal structural property of the membership relation in it, which is expressible in any logic of finite of transfinite type, including infinitary logics of any cardinal order.

— Kurt Gödel [10]

To understand why do need reflection in the first place, let's think about infinity for a moment. In the intuitive sense, infinity is an upper limit of all numbers. But for centuries, this was merely a philosophical concept, closely bound to religious and metaphysical way of thinking, considered separate from numbers used for calculations or geometry. It was a rather vague concept. In ancient Greece, Aristotle's response to famous Zeno's paradoxes introduced the distinction between actual and potential infinity. He argued, that potential infinity is (in today's words) well defined, as opposed to actual infinity, which remained a vague incoherent concept. He didn't think it's possible for infinity to inhabit a bounded place in space or time, rejecting Zeno's thought experiments as a whole. Aristotle's thoughts shaped western thinking partly due to Aquinas, who himself believed actual infinity to be more of a metaphysical concept for describing God than a mathematical property attributed to any other entity. In his Summa Theologica [1] he argues:

> A geometrician does not need to assume a line actually infinite, but takes some actually finite line, from which he subtracts whatever he finds necessary; which line he calls infinite.

Less than hundred years later, Gregory of Rimini wrote

> If God can endlessly add a cubic foot to a stone–which He can–then He can create an infinitely big stone. For He need only add one cubic foot at some time, another half an hour later, another a quarter of an hour later than that, and so on ad infinitum. He would then have before Him an infinite stone at the end of the hour.

Which is basically a Zeno's Paradox made plausible with God being the actor. In contrast to Aquinas' position, Gregory of Rimini theoretically constructs an object with actual infinite magnitude that is essentially different from God.

[1] Part I, Question 7, Article 3, Reply to Objection 1

Even later, in the 17th century, pushing the property of infinitness from the Creator to his creation, Nature, Leibniz wrote to Foucher in 1962:

> I am so in favor of the actual infinite that instead of admitting that Nature abhors it, as is commonly said, I hold that Nature makes frequent use of it everywhere, in order to show more effectively the perfections of its Author. Thus I believe that there is no part of matter which is not, I do not say divisible, but actually divided; and consequently the least particle ought to be considered as a world full of an infinity of different creatures.

But even though he used potential infinity in what would become foundations of modern Calculus and argued for actual infinity in Nature, Leibniz refused the existence of an infinite, thinking that Galileo's Paradoxon[2] is in fact a contradiction. The so called Galileo's Paradoxon is an observation Galileo Galilei made in his final book "Discourses and Mathematical Demonstrations Relating to Two New Sciences". He states that if all numbers are either squares and non–squares, there seem to be less squares than there is all numbers. On the other hand, every number can be squared and every square has it's square root. Therefore, there seem to be as many squares as there are all numbers. Galileo concludes, that the idea of comparing sizes makes sense only in the finite realm.

> Salviati: So far as I see we can only infer that the totality of all numbers is infinite, that the number of squares is infinite, and that the number of their roots is infinite; neither is the number of squares less than the totality of all the numbers, nor the latter greater than the former; and finally the attributes "equal," "greater," and "less," are not applicable to infinite, but only to finite, quantities. When therefore Simplicio introduces several lines of different lengths and asks me how it is possible that the longer ones do not contain more points than the shorter, I answer him that one line does not contain more or less or just as many points as another, but that each line contains an infinite number.

Leibniz insists in part being smaller than the whole saying

> Among numbers there are infinite roots, infinite squares, infinite cubes. Moreover, there are as many roots as numbers. And there are as many squares as roots. Therefore there are as many squares

[2]zneni galileova paradoxu

> as numbers, that is to say, there are as many square numbers as there are numbers in the universe. Which is impossible. Hence it follows either that in the infinite the whole is not greater than the part, which is the opinion of Galileo and Gregory of St. Vincent, and which I cannot accept; or that infinity itself is nothing, i.e. that it is not one and not a whole.

TODO Hegel–strucne?

TODO Cantor

TODO mene teologie, vice matematiky

TODO definovat pojmy (trida etc)

TODO neni V v nejakem smyslu porad potencialni nekonecno, zatimco mnoziny vetsi nez omega jsou aktualni? nebo jsou potencialni protoze se staveji pres indukci, od spoda.

In his work, he defined transfinite numbers to extend existing natural number structure so it contains more objects that behave like natural numbers and are based on an object (rather a meta-object) that doesn't explicitly exist in the structure, but is closely related to it. This is the first instance of reflection. This paper will focus on taking this principle a step further, extending Cantor's (or Zermelo–Fraenkel's, to be more precise) universe so it includes objects so big, they could be considered the universe itself, in a certain sense.

TODO dal asi smazat

The original idea behind reflection principles probably comes from what could be informally called "universality of the universe". The effort to precisely describe the universe of sets was natural and could be regarded as one of the impulses for formalization of naive set theory. If we try to express the universe as a set $\{x|x = x\}$, a paradox appears, because either our set is contained in itself and therefore is contained in a set (itself again), which contradicts the intuitive notion of a universe that contains everything but is not contained itself.

TODO ???

If there is an object containing all sets, it must not be a set itself. The notion of class seems inevitable. Either directly the ways for example the Bernays–Gödel set theory, we will also discuss later in this paper, does in, or on a meta–level like the Zermelo–Fraenkel set theory, that doesn't refer to them in the axioms but often works with the notion of a universal class. duet Another obstacle of constructing a set of all sets comes from Georg Cantor, who proved that the set of all subsets of a set (let x be the set and $\mathscr{P}(()x)$ its powerset) is strictly larger that x. That would turn every aspiration to finally establish an universal set into a contradictory infinite regression.[3]. We will use V to denote the class of all sets. From previous thoughts we can easily argue, that it is impossible to construct a property that holds for V and no set and is neither paradoxical like $\{x|x = x\}$ nor trivial. Previous observation can be transposed to a rather naive formulation of the reflection principle:

Reflection Any property which holds in V already holds in some initial segment of V.

To avoid vagueness of the term "property", we could informally reformulate the above statement into a schema:

For every first-order formula[4] φ holds in $V \leftrightarrow \varphi$ holds in some initial segment of V.

Interested reader should note that this is a theorem scheme rather than a single theorem. [5]

1.2 A few historical remarks on reflection

Reflection made its first in set-theoretical appearance in Gödel's proof of GCH in L (citace Kanamori ? Lévy and set theory), but it was around even earlier as a concept. Gödel himself regarded it as very close to Russel's reducibility axiom (an earlier equivalent of the axiom schema of Zermelo's separation). Richard Montague then studied reflection properties as a tool for verifying that Replacement is not finitely axiomatizable (citace?). a few years later Lévy proved in [2] the equivalence of reflection with Axiom of infinity together with Replacement in proof we shall examine closely in chaper 2.

TODO co dal? recent results?

[3] An intuitive analogy of this *reductio ad infinitum* is the status of ω, which was originally thought to be an unreachable absolute, only to become starting point of Cantor's hierarchy of sets growing beyond all boundaries around the end of the 19^{th} century

[4] this also works for finite sets of formulas [4, p. 168]

[5] If there were a single theorem stating "for any formula φ that holds in V there is an initial segment of V where φ also holds", we would obtain the following contradiction with the second Gödel's theorem: In ZFC, any finite group of axioms of ZFC holds in some initial segment of the universe. If we take the largest of those initial segments it is still strictly smaller than the universe and thus we have, via compactness, constructed a model of ZFC within ZFC. That is, of course a harsh contradiction. This also leads to an elegant way to prove that ZFC is not finitely axiomatizable.

1.3 Reflection in Platonism and Structuralism

TODO cite "reflection in a structuralist setting"

TODO veci o tom, ze reflexe je ok protoze reflektuje veci ktere objektivne plati, protoze plati pro V ...

TODO souvislost s kompaktnosti, hranice formalich systemu nebo alespon ZFC

1.4 Notation and Terminology

1.4.1 The Language of Set Theory

We are about to define basic set-theoretical terminology on which the rest of this thesis will be built. For Chapter 2, the underlying theory will be the *Zermelo –Fraenkel* set theory with the Axiom of Choice (ZFC), a first-order set theory in the language $\mathscr{L} = \{=, \in\}$, which will be sometimes referred to as *the language of set theory*. In Chapter 3[6], we shall always make it clear whether we are in first-order ZFC or second-order ZFC_2, which will be precisely defined later in this chapter. When in second-order theory, we will usually denote type 1 variables, which are elements of the domain of discourse[7] by lower-case letters, mostly $u, v, w, x, y, z, p_1, p_2, p_3, \dots$ while type 2 variables, which represent n-ary relations of the domain of discourse for any natural number n, are usually denoted by upper-case letters A, B, C, X, Y, Z. Note that those may be used both as relations and functions, see the definition of a function below.[8]

TODO uppercase M is a set!

TODO "M is a limit ordinal" je ve skutecnosti formule, nekam to sem napis!

The informal notions of *class* and *property* will be used throughout this thesis. They both represent formulas with respect to the domain of discourse. If $\varphi(x, p_1, \dots, p_n)$ is a formula in the language of set theory, we call

$$A = \{x : \varphi(x, p_1, \dots, p_n)\} \tag{1.1}$$

a class of all sets satisfying $\varphi(x, p_1, \dots, p_n)$ in a sense that

$$x \in A \leftrightarrow \varphi(x, p_1, \dots, p_n) \tag{1.2}$$

One can easily define for classes A, B the operations like $A \cap B$, $A \cup B$, $A \setminus C$, $\bigcup A$, but it is elementary and we won't do it here, see the first part of

[6] TODO bude jich vic? Chapter 4 taky?

[7] co je "domain of discourse"?

[8] TODO ref?

[4] for technical details. The following axioms are the tools by which decide whether particular classes are in fact sets. A class that fails to be considered a set is called a *proper class*.

1.4.2 The Axioms

Definition 1.1 *(The existence of a set)*

$$\exists x(x = x) \tag{1.3}$$

The above axiom is usually not used because it can be deduced from the axiom of *Infinity* (see below), but since we will be using set theories that omit *Infinity*, this will be useful.

Definition 1.2 *(Extensionality)*

$$\forall x, y(\forall z(z \in x \leftrightarrow z \in y) \leftrightarrow x = y) \tag{1.4}$$

Definition 1.3 *(Specification)*
The following is a schema for every first-order formula $\varphi(x, p_1, \ldots, p_n)$ with no free variables other than $x, p_1, \ldots, p_n$.

$$\forall x, p_1, \ldots, p_n \exists y \forall z(z \in y \leftrightarrow (z \in x \,\&\, \varphi(z, p_1, \ldots, p_n))) \tag{1.5}$$

We will now provide two definitions that are not axioms, but will be helpful in establishing some of the other axioms in a more intuitive way.

Definition 1.4 *($x \subseteq y$, $x \subset y$)*

$$x \subseteq y \leftrightarrow \forall z(z \in x \rightarrow z \in y) \tag{1.6}$$

$$x \subset y \leftrightarrow x \subseteq y \,\&\, x \neq y \tag{1.7}$$

Definition 1.5 *(Empty set)*

$$\emptyset \stackrel{\mathsf{def}}{=} \{x : x \neq x\} \tag{1.8}$$

To make sure that $\emptyset$ is a set, note that there exists at least one set y from 1.1, then consider the following alternative definition.

$$\emptyset' \stackrel{\mathsf{def}}{=} \{x : \varphi(x) \,\&\, x \in y\} \text{ where } y\ \varphi \text{ is the formula "} x \neq x \text{".} \tag{1.9}$$

It should be clear that $\emptyset' = \emptyset$.[9]

Now we can introduce more axioms.

[9] For details, see page 8 in [4].

Definition 1.6 *(Foundation)*

$$\forall x(x \neq \emptyset \to \exists(y \in x)(\forall z \neg(z \in y \ \& \ z \in x))) \tag{1.10}$$

Definition 1.7 *(Pairing)*

$$\forall x, y \exists z \forall q(q \in z \leftrightarrow q \in z \vee q \in y) \tag{1.11}$$

Definition 1.8 *(Union)*

$$\forall x \exists y \forall z(z \in x \leftrightarrow \exists q(z \in q \ \& \ q \in x)) \tag{1.12}$$

Definition 1.9 *(Powerset)*

$$\forall x \exists y \forall z(z \subseteq x \leftrightarrow z \in y) \tag{1.13}$$

Definition 1.10 *(Infinity)*

$$\exists x(\forall y \in x)(y \cup \{y\} \in x) \tag{1.14}$$

Let us introduce a few more definitions that will make the two remaining axioms more comprehensible.

Definition 1.11 *(Function)*
Given arbitrary first-order formula $\varphi(x, y, p_1, \ldots, p_n)$, we say that φ is a function iff

$$\forall x, y, z, p_1, \ldots, p_n(\varphi(x, y, p_1, \ldots, p_n) \ \& \ \varphi(x, z, p_1, \ldots, p_n) \to y = z) \tag{1.15}$$

When a $\varphi(x, y)$ is a function, we also write the following:

$$f(x) = y \leftrightarrow \varphi(x, y) \tag{1.16}$$

Note that this f is in fact a formula

TODO $f = \{(x, y) : \varphi(x, y)\}$!!! f muze byt mnozina i trida! [10]

Definition 1.12 *(Dom(f))*
Let f be a function. We read the following as "Dom(f) is the domain of f".

$$Dom(f) \stackrel{\text{def}}{=} \{x : \exists y(f(x) = y)\} \tag{1.17}$$

We say "f is a function on A", A being a class, if $A = dom(f)$.

[10] This can also be done for φs with more than two free variables by either setting $f(x, p_1, \ldots p_n) = y \leftrightarrow \varphi(x, yp_1, \ldots, p_n)$ or saying that φ codes more functions, determined by the various parameters, so $f(x) = y \leftrightarrow \varphi(x, y, t_1, \ldots, t_n)$ for given terms $t_1, \ldots, t_n$.

Definition 1.13 *(Rng(f))*
Let f be a function. We read the following as "Rng(f) is the range of f".

$$Rng(f) \stackrel{\text{def}}{=} \{x : \exists y(f(x) = y)\} \tag{1.18}$$

We say that f is i function into A, A being a class, if $rng(f) \subseteq A$.

Note that *Dom(f)* and *Rng(f)* are not definitions in a strict sense, they are in fact definition schemas that yield definitions for every function f given. Also note that they can be easily modified for φ instead of f, with the only difference that then it is defined only for those φs that are functions.

Definition 1.14 *(Powerset)*
TODO

And now for the axioms.

Definition 1.15 *(Replacement)*
The following is a schema for every first-order formula $\varphi(x, p_1, \ldots, p_n)$ with no free variables other than $x, p_1, \ldots, p_n$.

$$\text{"}\varphi \textit{ is a function"} \rightarrow \forall x \exists y \forall z (z \in y \leftrightarrow (\exists q \in x)(\varphi(x, y, p_1, \ldots, p_n)) \tag{1.19}$$

Definition 1.16 *(Choice)*
This is also a schema. For every A, a family of non-empty sets[11]*, such that $\emptyset \notin S$, there is a function f such that for every $x \in A$*

$$f(x) \in x \tag{1.20}$$

We will refer the axioms by their name, written in italic type, e.g. *Foundation* refers to the Axiom of Foundation. Now we need to define some basic set theories to be used in the article. There will be others introduce in Chapter 3, but those will usually be defined just by appending additional axioms or schemata to one of the following.

Definition 1.17 (S)
We call S *a set theory with the following axioms:*
(i) Existence of a set *(see 1.1)*
(ii) Extensionality *(see 1.2)*
(iii) Specification *(see 1.3)*
(iv) Foundation *(see 1.6)*
(v) Pairing *(see 1.7)*

[11]We say a class A is a "family of non-empty sets" iff there is B such that $A \subseteq \mathscr{P}(B)$

(vi) Union *(see 1.8)*
(vii) Powerset *(see 1.9)*

Definition 1.18 (ZF)
We call ZF *a set theory that contains all the axioms of the theory* S[12] *in addition to the following*
(i) Replacement *schema (see 1.15)*
(ii) Infinity *(see 1.10)*

Definition 1.19 (ZFC)
ZFC *is a theory that contains all the axioms of* ZF *plus* Choice *(1.16).*

1.4.3 The Transitive Universe

Definition 1.20 *(Transitive class)*
We say a class A is transitive *iff*

$$\forall x(x \in A \to x \subseteq A) \tag{1.21}$$

Definition 1.21 *Well Ordered Class A class A is said to be* well ordered by $\in$ *iff the following hold:*
(i) $(\forall x \in A)(x \notin x)$ *(Antireflexivity)*
(ii) $(\forall x, y, z \in A)(x \in y \,\&\, y \in z \to x \in z)$ *(Transitivity)*
(iii) $(\forall x, y \in A)(x = y \vee x \in y \vee y \in x)$ *(Linearity)*
(iv) $(\forall x)(x \subseteq A \,\&\, x \neq \emptyset \to (\exists y \in x)(\forall z \in x)(z = y \vee z \in y)))$

Definition 1.22 *(Ordinal number)*
A set x is said to be an ordinal number*, also known as an* ordinal*, if it is* transitive *and* well-ordered by $\in$.

For the sake of brevity, we usually just say "x is an *ordinal*". Note that "x is an ordinal" is a well-defined formula, since 1.20 is a formula and 1.21 is in fact a conjunction of four formulas. Ordinals will be usually denoted by lower case greek letters, starting from the beginning: $\alpha, \beta, \gamma, \ldots$. Given two different ordinals α, β, we will write $\alpha < \beta$ for $\alpha \in \beta$, see [4]Lemma 2.11 for technical details.

[12] With the exception of *Existence of a set*

Definition 1.23 *(Successor Ordinal)*
Consider the following operation

$$\beta + 1 \stackrel{\text{def}}{=} \beta \cup \{\beta\} \tag{1.22}$$

An ordinal α is called a successor ordinal *iff there is an ordinal β, such that $\alpha = \beta + 1$*

Definition 1.24 *(Limit Ordinal)*
A non-zero ordinal α[13] *is called a* limit ordinal *iff it is not a successor ordinal.*

Definition 1.25 *(Ord)*
The class of all ordinal numbers*, which we will denote Ord*[14] *be the following class:*

$$Ord \stackrel{\text{def}}{=} \{x : x \textit{ is an ordinal}\} \tag{1.23}$$

The following construction will be often referred to as the *Von Neumann's Hierarchy*, sometimes also the *Von Neumann's Universe*.

Definition 1.26 *(Von Neumann's Hierarchy)*
The Von Neumann's Hierarchy *is a collection of sets indexed by elements of Ord, defined recursively in the following way:*

(i)

$$V_0 = \emptyset \tag{1.24}$$

(ii)

$$V_{\alpha+1} = \mathscr{P}(V_\alpha) \textit{ for any ordinal } \alpha \tag{1.25}$$

(iii)

$$V_\lambda = \bigcup_{\beta<\lambda} V_\beta \textit{ for a limit ordinal } \lambda \tag{1.26}$$

Definition 1.27 *(Rank)*
Given a set x, we say that the rank of x (written as $rank(x)$) is the least ordinal α such that

$$x \in V_{\alpha+1} \tag{1.27}$$

Due to *Regularity*, every set has a rank.[15]

Definition 1.28 *(ω)*

$$\omega \stackrel{\text{def}}{=} \bigcap \{x : xisalimitordinal)\} \tag{1.28}$$

[13] $\alpha \neq \emptyset$
[14] It is sometimes denoted On, but we will stick to the notation in [4]
[15] See chapter 6 of [4] for details.

1.4.4 Cardinal Numbers

Definition 1.29 *(Cardinality)*
Given a set x, let the cardinality of x, written $|x|$, be defined as the smallest ordinal number such that there is an injective mapping from x to α.

For formal details as well as why every set can be well-ordered assuming *Choice*, see [4].

Definition 1.30 *(Aleph function)*
Let ω be the set defined by **??**. *We will recursively define the function $\aleph$ for all ordinals.*

(i) $\aleph_0 = \omega$

(ii) $\aleph_{\alpha+1}$ *is the least cardinal larger than* $\aleph_\alpha$[16]

(iii) $\aleph_\lambda = \bigcup_{\beta<\lambda} \aleph_\beta$ *for a limit ordinal* λ

Definition 1.31 *(Cardinal number)*
We say a set x is a cardinal number, *usually called a* cardinal, *if either $x \in \omega$*

Cardinals will be notated by lower-case greek letters starting from $\kappa, \lambda, \mu, \ni$ $, \dots$[17].

Definition 1.32 *(Cofinality)*
Let λ be a limit ordinal. The cofinality *of λ, written $cf(\lambda)$, is the least limit ordinal α such that there is an increasing α-sequence*[18] $\langle \lambda_\beta : \beta < \alpha \rangle$ *with* $lim_{\beta \to \alpha} \lambda_\beta = \lambda$.

Definition 1.33 *(Limit Cardinal)*
We say that a cardinal κ is a limit cardinal *if*

$$(\exists \alpha \in Ord)(\kappa = \aleph_\alpha) \tag{1.29}$$

Definition 1.34 *(Strong Limit Cardinal)*
We say that an ordinal κ is a strong limit cardinal *if it is a* limit cardinal *and*

$$\forall \alpha (\alpha \in \kappa \to \mathscr{P}(\alpha) \in \kappa) \tag{1.30}$$

Definition 1.35 *(Generalised Continuum Hypothesis)*

$$\aleph_{\alpha+1} = 2^{\aleph_\alpha} \tag{1.31}$$

If *GCH* holds (for example in Gödel's L, see chapter 3), the notions of a limit cardinal and a strong limit cardinal are equivalent.

[16]"The least cardinal larger than $\aleph_\alpha$" is sometimes notated as $\aleph_\alpha^+$

[17]λ is also sometimes used for limit ordinals, the distinction should be clear from the context.

[18]TODO def α-sequence

1.4.5 Relativisation

Definition 1.36 *(Relativization)*
Let M be a class, R a binary relation on M and let $\varphi(p_1, \ldots, p_n)$ be a first-order formula with n parameters. The relativization of φ to M and R *is the formula, written as $\varphi^{M,R}(p_1, \ldots, p_n)$, defined in the following inductive manner:*

(i) $(x \in y)^{M,R} \leftrightarrow R(x, y)$
(ii) $(x = y)^{M,R} \leftrightarrow x = y$
(iii) $(\neg\varphi)^{M,R} \leftrightarrow \neg\varphi^{M,R}$
(iv) $(\varphi \,\&\, \psi)^{M,R} \leftrightarrow \varphi^{M,R} \,\&\, \psi^{M,R}$
(v) $(\exists x\varphi)^{M,R} \leftrightarrow (\exists x \in M)\varphi^{M,R}$

1.4.6 More functions

TODO def $f : Ord \to Ord$, asi u powersetu.

Definition 1.37 *(Strictly increasing function)*
A function $f : Ord \to Ord$ is said to be strictly increasing *iff*

$$\forall \alpha, \beta \in Ord(\alpha < \beta \to f(\alpha) < f(\beta)). \tag{1.32}$$

Definition 1.38 *(Continuous function)*
A function $f : Ord \to Ord$ is said to be continuous *iff*

$$\alpha \textit{ is limit} \to f(\lambda) = \bigcup_{\alpha<\lambda} f(\alpha). \tag{1.33}$$

Definition 1.39 *(Normal function)*
A function $f : Ord \to Ord$ is said to be normal *if it is* strictly increasing *and* continuous.

Definition 1.40 *Fixed point*
We say α is a fixed point of ordinal function f if $\alpha = f(\alpha)$.

Definition 1.41 *(Unbounded class)*
We say a class A is unbounded if

$$\forall x(\exists y \in A)(x < y) \tag{1.34}$$

Definition 1.42 *(Class Unbounded in α)*
Let α be a limit ordinal. We say that $x \subset \alpha$ is unbounded in α *iff*

$$\forall \beta \in Ord(\beta < \alpha \to \exists \gamma(\gamma \in x(\beta \leq \gamma < \alpha))) \tag{1.35}$$

Definition 1.43 *(Closed class)*
For a limit ordinal $A \subseteq \lambda$, we say that A is closed in λ iff for every non-zero ordinal $\alpha < \lambda$: if $A \cap \alpha$ is unbounded in α then $\alpha \in A$.

Definition 1.44 *(Club set)*
For a regular uncountable cardinal κ, a set $x \subset \kappa$ is a closed unbounded *subset, abbreviated as a* club set*, iff x is both closed and unbounded in κ.*

Definition 1.45 *(Stationary set)*
For a regular uncountable cardinal κ, we say a set $A \subset \kappa$ is stationary in κ iff it intersects every club subset of κ.

1.4.7 Structure, Substructure and Embedding

Structures will be denoted $\langle M, \in, R\rangle$ where M is a domain, $\in$ stands for the standard membership relation, it is assumed to be restricted to the domain[19], $R \subseteq M$ is a relation on the domain. When R is not needed, we may as well only write M instead of $\langle M, \in\rangle$.

Definition 1.46 *(Elementary Embedding)*
Given the structures $\langle M_1, \in, R\rangle$, $\langle M_2, \in, R\rangle$ and a one-to-one function $j : M_1 \to M_2$, we say j is an elementary embedding *of M_1 into M_2, we write $j : M_1 \prec M_2$, when the following holds for every formula $\varphi(p_1, \ldots, p_n)$ and every $p_1, \ldots, p_n \in M_1$:*

$$\langle M_1, \in, R\rangle \models \varphi(p_1, \ldots, p_n) \leftrightarrow \langle M_2, \in, R\rangle \models \varphi(j(p_1), \ldots, j(p_n)) \tag{1.36}$$

Definition 1.47 *(Elementary Substructure)*
Given the structures $\langle M_1, \in, R\rangle$, $\langle M_2, \in, R\rangle$ and a one-to-one function $j : M_1 \to M_2$ such that $j : M_1 \prec M_2$, we say that M_1 is an elementary substructure *of M_2, denoted as $M_1 \prec M_2$, iff j is an identity on M_1. In other words*

$$\langle M_1, \in, R\rangle \models \varphi(p_1, \ldots, p_n) \leftrightarrow \langle M_2, \in, R\rangle \models \varphi(p_1, \ldots, p_n) \tag{1.37}$$

[19] To be totally correct, we should write $\langle M, \in \cap M \times M, R\rangle$

2 Levy's first-order reflection

2.1 Lévy's Original Paper

This section will try to present Lévy's proof of a general reflection principle being equivalent to *Replacement* and *Infinity* under ZF minus *Replacement* and *Infinity* from his 1960 paper *Axiom Schemata of Strong Infinity in Axiomatic Set Theory*[20].

When reading said article, one should bear in mind that it was written in a period when set theory was semantically oriented, so while there are many statements about a model of ZF, usually denoted u, this is equivalent to today's universal class V, so it doesn't necessarily mean that there is a set u that is a model of ZF. We will review the notion of a standard complete model used by Lévy throughout the paper in a moment. Let's first say that the set theory ZF was formulated in the "non-simple applied first order functional calculus", is

TODO viz A. Church nebo tak neco.

The axioms are equivalent to those defined in 1.18, except for the *Axiom of Subsets*, which is just a different name for *Specification*. Besides ZF and S, defined in 1.18 and yrefdef:s respectively, the set theories theories Z, and SF are used in the text. Z is ZF minus replacement, SF is ZF minus *Infinity*. Also note that universal quantifier does not appear, $\forall x \varphi(x)$ would be written as $(x)\varphi(x)$. The symbol for negation is "$\sim$", implication is written as "$\supset$" and equivalence is "$\equiv$", we will use "$\neg$", "$\rightarrow$" and "$\leftrightarrow$".

Next two definitions are not used in contemporary set theory, but they illustrate 1960's set theory mind-set and they are used heavily in Lévy's text, so we will include and explain them for clarity. Generally, in this chapter, Q stands for an arbitrary axiomatic set theory used for general definitions, u is usually a model of Q, counterpart of today's V.

This subsection uses ZF instead of the usual ZFC as the underlying theory.

Definition 2.1 *(Standard model of a set theory)*
Let Q be a axiomatic set theory in first-order logic. We say the the a class u is a standard model of Q with respect to a membership relation E, written as $Sm^{\mathsf{Q}}(u)$*, iff both of the following hold*

(i) $(x, y) \in E \leftrightarrow y \in u \,\&\, x \in y$

(ii) $y \in u \,\&\, x \in y \rightarrow x \in u$

Definition 2.2 *Standard complete model of a set theory*
Let Q and E be like in 2.1. We say that that u is a standard complete model of Q with respect to a membership relation E iff both of the following hold

[20] [2]

(i) u is a transitive set with respect to $\in$
(ii) $\forall E((x, y) \in E \leftrightarrow (y \in u \,\&\, x \in y) \,\&\, Sm^{\mathsf{Q}}(u, E))$
this is written as $Scm^{\mathsf{Q}}(u)$.

Definition 2.3 *(Inaccessible cardinal with respect to Q)*
Let Q be an axiomatic first-order set theory. We say that a cardinal κ is inaccessible with respect to Q, we write $In^{\mathsf{Q}}(\kappa)$.

$$In^{\mathsf{Q}}(\kappa) \stackrel{\text{def}}{=} Scm^{\mathsf{Q}}(V_\kappa). \tag{2.38}$$

Definition 2.4 *(Inaccessible cardinal with respect to ZF)*
When a cardinal κ is inaccessible with respect to ZF, we only say that it is inaccessible. We write $In(\kappa)$.

$$In(\kappa) \stackrel{\text{def}}{=} In^{\mathsf{ZF}}(\kappa) \tag{2.39}$$

The above definition of inaccessibles is used because it doesn't require *Choice*.

For the definition of relativization, see 1.36. The syntax used by Lévy is $Rel(u, \varphi)$, we will use φ^u, which is more usual these days.

Definition 2.5 *(N)*
The following is an axiom schema of complete reflection over ZF, denoted as N.

$$N \stackrel{\text{def}}{=} \exists u(Scm^{\mathsf{ZF}}(u) \,\&\, \forall x_1, \ldots, x_n(x_1, \ldots, x_n \in u \to \varphi \leftrightarrow \varphi^u)) \tag{2.40}$$

where φ is a formula which contains no free variables except for $x_1, \ldots, x_n$.

Definition 2.6 *(N_0)*
With S instead of ZFwe obtain what will now be called N_0.

$$N_0 \stackrel{\text{def}}{=} \exists u(Scm^{\mathsf{S}}(u) \,\&\, \forall x_1, \ldots, x_n(x_1, \ldots, x_n \in u \to \varphi \leftrightarrow \varphi^u)) \tag{2.41}$$

where φ is a formula which contains no free variables except for $x_1, \ldots, x_n$.

2.2 $\mathsf{S} \models (\mathsf{N}_0 \leftrightarrow$ ***Replacement*** & ***Infinity***)

Let S be a set theory defined in 1.17.

Lemma 2.7 *The following holds for every u.*

$$\text{"}u \text{ is a limit ordinal"} \leftrightarrow Scm^{\mathsf{S}}(u) \tag{2.42}$$

Proof. TODO !

—-

In order to prove that it is a model of S, we would need to verify all axioms of S. We have already shown that ω is closed under the powerset operation. Foundation, extensionality and comprehension are clear from the fact that we work in ZF[21], pairing is clear from the fact, that given two sets x, y, they have ranks α, β, without loss of generality we can assume that $\alpha \leq \beta$, which means that $x \in V_\alpha \in V_\beta$, therefore V_β is a set that satisfies the paring axiom: it contains both x and B. □

Let N_0 be defined as in 2.6, for *Infinity* see 1.10.

Theorem 2.8 *In* S*, the schema* N_0 *implies* Infinity.

Proof. Lévy skips this proof because it seems too obvious to him, but let's do it here for plasticity. For an arbitrary φ, N_0 gives us $\exists u Scm^{\mathsf{S}}(u)$, but from lemma 2.7, we know that this u is a limit ordinal. This u already satisfies *Infinity.* □

Let N_0 be defined as in 2.6, for *Replacement* see 1.15, S is again the set theory defined in 1.17.

Theorem 2.9 *In* S*, the schema* N_0 *implies* Replacement.

Proof. Let $\varphi(x, y, p_1, \ldots, p_n)$ be a formula with no free variables except $x, y, p_1, \ldots, p_n$ for an arbitrary natural number n.

$$\begin{aligned} \chi = \forall x, y, z(\varphi(x, y, p_1, \ldots, p_n) \ \&\ \varphi(x, z, p_1, \ldots, p_n) \rightarrow y = z) \\ \rightarrow \forall x \exists y \forall z (z \in y \leftrightarrow \exists q (q \in x \ \&\ \varphi(q, z, p_1, \ldots, p_n))) \end{aligned} \tag{2.43}$$

Let χ be an instance of *Replacement* schema for given φ. Let the following formulas be instances of the N_0 schema for formulas φ, $\exists y \varphi$, χ and $\forall x, p_1, \ldots, p_n \chi$ respectively:

We can deduce the following from N_0:

(i) $x, y, p_1, \ldots, p_n \in u \rightarrow (\varphi \leftrightarrow \varphi^u)$
(ii) $x, p_1, \ldots, p_n \in u \rightarrow (\exists y \varphi \leftrightarrow (\exists y \varphi)^u)$
(iii) $x, p_1, \ldots, p_n \in u \rightarrow (\chi \leftrightarrow \chi^u)$
(iv) $\forall x, p_1, \ldots, p_n (\chi \leftrightarrow (\forall x, p_1, \ldots, p_n \chi)^u)$

[21] We only need to verify axioms that provide means of constructing larger sets from smaller to make sure they don't exceed ω. Since ω is an initial segment of ZF, the axiom scheme of specification can't be broken, the same holds for foundation and extensionality.

From relativization, we also know that $(\exists y\varphi)^u$ is equivalent to $(\exists y \in u)\varphi^u$. Therefore (**ii**) is equivalent to

$$x, p_1, \ldots, p_n \in u \to (\exists y \in u)\varphi^u. \tag{2.44}$$

If φ is a function[22], then for every $x \in u$, which is also $x \subset u$ by the transitivity of $Scm^{\mathsf{S}}(u)$, it maps elements of x onto u. From the axiom scheme of comprehension[23], we can find y, a set of all images of elements of x. That gives us $x, p_1, \ldots, p_n \in u \to \chi$. By (**iii**) we get $x, p_1, \ldots, p_n \in u \to \chi^u$, the universal closure of this formula is $(\forall x, p_1, \ldots, p_n\chi)^u$, which together with (**iv**) yields $\forall x, p_1, \ldots, p_n\chi$. Via universal instantiation, we end up with χ. We have inferred replacement for a given arbitrary formula. □

What we have just proven is just a single theorem from the above mentioned article by Lévy, we will introduce other interesting propositions, mostly related to the existence of large cardinals, later in their appropriate context in chapter 3.

2.3 Contemporary restatement

We will now prove what is also Lévy's first-order reflection theorem, but rephrased with up to date set theory terminology. The main difference is, that while Lévy reflects φ from V to a set u that is a "standard complete model of S", we say that there is a V_α for a limit α that reflects φ. We will argue that those are equivalent.[24]

Definition 2.10 *(Reflection$_1$)*
Let $\varphi(p_1, \ldots, p_n)$ be a first-order formula in the language of set theory. Than the following holds for any such φ.

$$\forall M_0 \exists M(M_0 \subseteq M \;\&\; (\varphi^M(p_1, \ldots, p_n) \leftrightarrow \varphi(p_1, \ldots, p_n))) \tag{2.45}$$

Note that this is a restatement of both Lévy's N and N_0 from the previous chapter, see definitions **??**, **??**. We prefer to call it *Reflection$_1$* so it complies with how other axioms and schemata are called. [25] Note that the subscript 1 refers to the fact that $\varphi(p_1, \ldots, p_n)$ is a first-order formula, and since we're using the work "reflection" in less strict meaning throughout this thesis,

[22] See definition 1.11

[23] Lévy uses its equivalent, axiom of subsets

[24] TODO nekde na to bude lemma!

[25] We will not use the name N_0, because it might be confusing to work N_0 and M_0 where M_0 is a set and N_0 is an axiom schema.

distinguishing between the two just by using italic font face for the schema might cause confusion.

We will now prove the equivalence of *Reflection₁* with *Replacement* and *Infinity* in S in two parts. First, we will show that N_0 is a theorem of ZFC, then we shall show that the second implication, which proves *Infinity* and *Replacement* from N_0, also holds.

The following lemma is usually done in more parts, the first being for one formula, the other for n formulas. We will only state and prove the more general version for n formulas, knowing that setting $n = 1$ turns it to a specific version.

Lemma 2.11 *Let $\varphi_1, \ldots, \varphi_n$ be formulas with m parameters*[26]*.*

(i) For each set M_0 there is such set M that $M_0 \subset M$ and the following holds for every i, $1 \leq i \leq n$:

$$\exists x \varphi_i(p_1, \ldots, p_{m-1}, x) \rightarrow (\exists x \in M)\varphi_i(p_1, \ldots, p_{m-1}, x) \tag{2.46}$$

for every $p_1, \ldots, p_{m-1} \in M$.

(ii) Furthermore there is an ordinal α such that $M_0 \subset V_\alpha$ and the following holds for each i, $1 \leq i \leq n$:

$$\exists x \varphi_i(p_1, \ldots, p_{m-1}, x) \rightarrow (\exists x \in V_\alpha)\varphi_i(p_1, \ldots, p_{m-1}, x) \tag{2.47}$$

for every $p_1, \ldots, p_{m-1} \in M$.

(iii) Assuming Choice*, there is M, $M_0 \subset M$ such that 2.46 holds for every M, $i \leq n$ and $|M| \leq |M_0| \cdot \aleph_0$.*

Proof. We will simultaneously prove statements (**i**) and (**ii**), denoting M^T the transitive set required by part (**ii**). Unless explicitly stated otherwise for specific steps, it is thought to be equivalent to M.

Let us first define operation $H(p_1, \ldots, p_{m-1})$ that gives us the set of x's with minimal rank[27] satisfying $\varphi_i(p_1, \ldots, p_{m-1}, x)$ for given parameters $p_1, \ldots, p_{m-1}$ for every i such that $1 \leq i \leq n$.

$$H_i(p_1, \ldots, p_n) = \{x \in C_i : (\forall z \in C)(rank(x) \leq rank(z))\} \tag{2.48}$$

[26] For formulas with a different number of parameters, take for m the highest number of parameters among those formulas. Add spare parameters to every formula that has less than m parameters in a way that preserves the last parameter, which we will denote x. E.g. let φ'_i be the a formula with k parameters, $k < m$. Let us set $\varphi_i(p_1, \ldots, p_{m-1}, x) \stackrel{\mathsf{def}}{=} \varphi'_i(p_1, \ldots, p_{k-1}, x)$, notice that the parameters $p_k, \ldots, p_{m-1}$ are not used.

[27] Rank is defined in 1.27

for each $1 \leq i \leq n$, where

$$C_i = \{x : \varphi_i(p_1, \dots, p_{m-1}, x)\} \text{ for } 1 \leq i \leq n \tag{2.49}$$

Next, let's construct M from given M_0 by induction.

$$M_{i+1} = M_i \cup \bigcup_{j=0}^{n} \bigcup \{H_j(p_1, \dots, p_{m-1}) : p_1, \dots, p_{m-1} \in M_i\} \tag{2.50}$$

In other words, in each step we add the elements satisfying $\varphi(p_1, \dots, p_{m-1}, x)$ for all parameters that were either available earlier or were added in the previous step. For statement (**ii**), this is the only part that differs from (**i**). Let us take for each step transitive closure of M_{i+1} from (**i**). In other words, let γ be the smallest ordinal such that

$$(M_i^T \cup \bigcup_{j=0}^{n} \{\bigcup \{H_j(p_1, \dots, p_{m-1}) : p_1, \dots, p_{m-1} \in M_i\}\}) \subset V_\gamma \tag{2.51}$$

Then the incremetal step is like so:

$$M_{i+1}^T = V_\gamma \tag{2.52}$$

The final M is obtained by joining all the individual steps.

$$M = \bigcup_{i=0}^{\infty} M_i, \ M^T = \bigcup_{i=0}^{\infty} M_i^T = V_\alpha \text{ [28]} \tag{2.53}$$

We have yet to finish part (**iii**). Let's try to construct a set M' that satisfies the same conditions like M but is kept as small as possible. Assuming the Axiom of Choice, we can modify the process so that the cardinality of M' is at most $|M_0| \cdot \aleph_0$. Note that the size of M' is determined by the size of M_0 and, most importantly, by the size of $H_i(p_1, \dots, p_{m-1})$ for any i, $1 \leq i \leq n$ in individual levels of the construction. Since the lemma only states existence of some x that satisfies $\varphi_i(p_1, \dots, p_{m-1}, x)$ for any $1 \leq i \leq n$, we only need to add one x for every set of parameters but $H_i(u_1, \dots, u_{m-1})$ can be arbitrarily large. Since Axiom of Choice ensures that there is a choice function, let F be a choice function on $\mathscr{P}(M')$. Also let $h_i(p_1, \dots, p_{m-1}) = F(H_i(p_1, \dots, p_{m-1}))$ for i, where $1 \leq i \leq n$, which means that h is a function that outputs an x that satisfies $\varphi_i(p_1, \dots, p_{m-1}, x)$ for i such that $1 \leq i \leq n$ and has minimal rank among all such witnesses. The induction step needs to be redefined to

$$M'_{i+1} = M'_i \cup \bigcup_{j=0}^{n} \{H_j(p_1, \dots, p_{m-1}) : p_1, \dots, p_{m-1} \in M'_i\} \tag{2.54}$$

This way, the amount of elements added to M'_{i+1} in each step of the construction is the same as the amount of sets of parameters that yielded elements not included in M'_i. It is easy to see that if M_0 is finite, M' is countable because it was constructed as a countable union of finite sets. If M_0 is countable or larger, the cardinality of M' is equal to the cardinality of M_0.[29] Therefore $|M'| \leq |M_0| \cdot \aleph_0$ $\square$

Theorem 2.12 *(Lévy's first-order reflection theorem)*
Let $\varphi(p_1, \ldots, p_n)$ be a first-order formula.

(i) For every set M_0 there exists M such that $M_0 \subset M$ and the following holds:

$$\varphi^M(p_1, \ldots, p_n) \leftrightarrow \varphi(p_1, \ldots, p_n) \tag{2.55}$$

for every $p_1, \ldots, p_n \in M$.

(ii) For every set M_0 there is a transitive set M, $M_0 \subset M$ such that the following holds:

$$\varphi^M(p_1, \ldots, p_n) \leftrightarrow \varphi(p_1, \ldots, p_n) \tag{2.56}$$

for every $p_1, \ldots, p_n \in M$.

(iii) For every set M_0 there is α such that $M_0 \subset V_\alpha$ and the following holds:

$$\varphi^{V_\alpha}(p_1, \ldots, p_n) \leftrightarrow \varphi(p_1, \ldots, p_n) \tag{2.57}$$

for every $p_1, \ldots, p_n \in M$.

(iv) Assuming Choice, *for every set M_0 there is M such that $M_0 \subset M$ and $|M| \leq |M_0| \cdot \aleph_0$ and the following holds:*

$$\varphi^M(p_1, \ldots, p_n) \leftrightarrow \varphi(p_1, \ldots, p_n) \tag{2.58}$$

for every $p_1, \ldots, p_n \in M$.

Proof. Before we start, note that the following holds for any set M if φ is an atomic formula, as a direct consequence of relativisation to $M, \in$[30].

$$\varphi \leftrightarrow \varphi^M \tag{2.59}$$

Let's now prove (**i**) for given φ via induction by complexity. We can safely assume that φ contains no quantifiers besides "$\exists$" and no logical connectives

[29] It can not be smaller because $|M'_{i+1}| \geq |M'_i|$ for every i. It may not be significantly larger because the maximum of elements added is the number of n-tuples in M'_i, which is of the same cardinality is M'_i.

[30] See **??**. Also note that this works for relativization to $M, \in$, not M, E where E is an arbitrary membership relation on M.

other than "$\neg$" and "&". Let $\varphi_1, \dots, \varphi_n$ be all subformulas of φ. Then there is a set M, obtained by the means of lemma 2.11, for all of the formulas $\varphi_1, \dots, \varphi_n$.

We know that $\psi \leftrightarrow \psi^M$ for atomic ψ, we need to verify that it won't fail in the inductive step. Let us consider $\psi = \neg\psi'$ along with the definition of relativization for those formulas in 1.36.

$$(\neg\psi')^M \leftrightarrow \neg(\psi'^M) \tag{2.60}$$

Because the induction hypothesis says that 2.55 holds for every subformula of ψ, we can assume that $\psi'^M \leftrightarrow \psi'$, therefore the following holds:

$$(\neg\psi')^M \leftrightarrow \neg(\psi'^M) \leftrightarrow \neg\psi' \tag{2.61}$$

The same holds for $\psi = \psi_1 \,\&\, \psi_2$. From the induction hypothesis, we know that $\psi_1^M \leftrightarrow \psi_1$ and $\psi_2^M \leftrightarrow \psi_2$, which together with relativization for formulas in the form of $\psi_1 \,\&\, \psi_2$ gives us

$$(\psi_1 \,\&\, \psi_2)^M \leftrightarrow \psi_1^M \,\&\, \psi_2^M \leftrightarrow \psi_1 \,\&\, \psi_2 \tag{2.62}$$

Let's now examine the case when from the induction hypethesis, M reflects $\psi'(p_1, \dots, p_n, x)$ and we are interested in $\psi = \exists x \psi'(p_1, \dots, p_n, x)$. The induction hypothesis tells us that

$$\varphi'^M(p_1, \dots, p_n, x) \leftrightarrow \psi'(p_1, \dots, p_n, x) \tag{2.63}$$

so, together with above lemma 2.11, the following holds:

$$\begin{gathered}
\psi(p_1, \dots, p_n, x) \\
\leftrightarrow \exists x \psi'(p_1, \dots, p_n, x) \\
\leftrightarrow (\exists x \in M)\psi'(p_1, \dots, p_n, x) \\
\leftrightarrow (\exists x \in M)\psi'^M(p_1, \dots, p_n, x) \\
\leftrightarrow (\exists x \psi'(p_1, \dots, p_n, x))^M \\
\leftrightarrow \psi^M(p_1, \dots, p_n, x)
\end{gathered} \tag{2.64}$$

Which is what we have needed to prove. 2.55 holds for all subformulas $\varphi_1, \dots, \varphi_n$ of a given formula φ.

So far we have proven part (**i**) of this theorem for one formula φ, we only need to verify that the same holds for any finite number of formulas. This has in fact been already done since lemma 2.11 gives us M for any (finite) amount of formulas, we can find a set M for the union of all of their subformulas. We can than use the induction above to verify that M reflects each of the formulas individually iff it reflects all of its subformulas.

Since V_α is a transitive set, by proving (**iii**) we also satisfy (**ii**). To do so, we only need to look at part (**ii**) of lemma 2.11. All of the above proof also holds for $M = V_\alpha$.

To finish part (**iv**), we take M of size $\leq |M_0| \cdot \aleph_0$, which exists due to part (**iii**) of lemma 2.11, the rest being identical. □

Let S be a set theory defined in 1.17, for ZFC see 1.19.

Lemma 2.13 *Let M be a set. Then the following holds:*

$$\mathsf{ZFC} \models (\mathsf{M} \models \mathsf{S}) \leftrightarrow \text{"}M \textit{ is a limit cardinal"} \tag{2.65}$$

Proof. For the left-to-right direction, we shall verify that if M is a model of S, it necessarily is a limit cardinal. From *Powerset*[31], we know that for any $x \in M$, $\mathscr{P}(x) \in M$. But that is already the definition of a strong limit cardinal[32].

For the converse, we need to see that if there is a limit ordinal α, such that $V_\alpha = M$, the axioms of S hold M.

(i) *Existence of a set* (see 1.1)
There obviously is a set $x \in M$

(ii) *Extensionality* (see 1.2)
Since $Extensionality^M$ is a Δ_0 formula, it holds in any transitive class by **??**.

(iii) *Specification* (see 1.3)
TODO

(iv) *Foundation* (see 1.6)
$Foundation^M$ is also a Δ_0 formula, so it holds by **??** since M is transitive because it is a cardinal.

(v) *Pairing* (see 1.7)
TODO

(vi) *Union* (see 1.8
TODO

(vii) *Powerset* (see 1.9)
TODO

[31] 1.9.
[32] see **??**

□

Let *Infinity* and *Replacement* be as defined in 1.10 and 1.15 respectively.

Theorem 2.14 Reflection$_1$ *is equivalent to* Infinity & Replacement *under* S.

Proof. Since 2.12 already gives us one side of the implication, we are only interested in showing the converse which we shall do in two parts:

TODO N_0 prepsat zpatky na *Reflection*$_1$

$\mathbf{N_0} \to$ *Infinity* From N_0 (**??**), we know that for any first-order formula φ and a set M_0, there is a M such that $M_0 \subseteq M$ and $\varphi^M \leftrightarrow \varphi$. Let's pick *Powerset* for φ, then by N_0 there is a set that satisfies *Powerset*, ergo there is a strong limit cardinal, which in turn satisfies *Infinity*.

Reflection $\to$ *Replacement*

Given a formula $\varphi(x, y, p_1, \ldots, p_n)$, we can suppose that it is reflected in any M [33] What we want to obtain is the following:

$$\forall x, y, z(\varphi(x, y, p_1, \ldots, p_n) \;\&\; \varphi(x, z, p_1, \ldots, p_n) \to y = z) \to\to \forall X \exists Y \forall y\ (y \in Y \leftrightarrow \exists x(\varphi(x, y, p_1, \ldots \tag{2.66}$$

We do also know that $x, y \in M$, in other words for every X, $Y = \{y \mid \varphi(x, y, p_1, \ldots, p_n)\}$ and we know that $X \subset M$ and $Y \subset M$, which, together with the comprehension schema implies that Y, the image of X over φ, is a set. □

We have shown that *Reflection* for first-order formulas, *Reflection*$_1$ is a theorem of ZF, which means that it won't yield us any large cardinals. We have also shown that it can be used instead of the *Infinity* and *Replacement* scheme, but ZF + *Reflection*$_1$ is a conservative extension of ZF. Besides being a starting point for more general and powerful statements, it can be used to show that ZF is not finitely axiomatizable. That follows from the fact that *Reflection* gives a model to any finite number of (consistent) formulas. So if $\varphi_1, \ldots, \varphi_n$ for any finite n would be the axioms of ZF, *Reflection* would always contain a model of itself, which would in turn contradict the Second Gödel's Theorem[34]. Notice that, in a way, reflection is complementary to compactness. Compactness argues that given a set of sentences, if every finite subset yields a model, so does the whole set. Reflection, on the other hand, says that while the whole set has no model in the underlying theory, every finite subset does have one.

[33] Which means that for $x, y, p_1, \ldots, p_n \in M$, $\varphi^M(x, y, p_1, \ldots, p_n) \leftrightarrow \varphi(x, y, p_1, \ldots, p_n)$.

[34] See chapter 3.2 for further details.

Also, notice how reflection can be used in ways similar to upward Löwenheim–Skolem theorem. Since Reflection extends any set M_0 into a model of given formulas $\varphi_1, \ldots, \varphi_n$, we can choose the lower bound of the size of M by appropriately choosing M_0.

In the next section, we will try to generalize *Reflection* in a way that transcends ZF and finally yields some large cardinals.

3 Reflection And Large Cardinals

In this chapter we aim to examine stronger reflection properties in order to reach cardinals unavailable in ZFC. Like we said in the first chapter, the variety of reflection principles comes from the fact that there are many way to formalize "properties of the universal class". It is not always obvious what properties hold for V because, (TODO Tarski) We have shown that reflecting properties as first-order formulas doesn't allow us to leave ZFC. We will broaden the class of admissible properties to be reflected and see whether there is a natural limit in the height or width on the reflected universe and also see that no matter how far we go, the universal class is still as elusive as it is when seen from S. That is because for every process for obtaining larger sets such as for example the powerset operation in ZFC, this process can't reach V and thus, from reflection, there is an initial segment of V that can't be reached via said process.

To see why this is important, let's dedicate a few lines to the intuition behind the notions of limitness, regularity and inaccessibility in a manner strongly influenced by [8]. To see why limit and strongly limit cardinals are worth mentioning, note that they are "limit" not only in a sense of being a supremum of an ordinal sequence, they also show that a certain way of obtaining larger sets from smaller ones is limited. We will see that all of the alternatives offered in this thesis are in a sense limited. $\aleph_\lambda$ is a limit cardinal iff there is no α such that $\aleph_{\alpha+1} = \aleph_\lambda$. Strongly limit cardinals point to the limits of the powerset operation. It has been too obvious so far, so let's look at the regular cardinals in this manner. Regular cardinals are those that cannot be[35], expressed as a supremum of smaller amount of smaller objects[36]. More precisely, κ is regular if there is no way to define it as a union of less than κ ordinals, all smaller than κ. So unless there already is a set of size κ, *Replacement* is useless in determining whether κ is really a set. Note that assuming *Choice*, successor cardinals are always regular, so most[37] limit cardinals are singular cardinals. So if one is traversing the class of all cardinals upwards, successor steps are still sets thanks to the powerset axiom while singular limits cardinal are not proper classes because they are suprema of images of smaller sets via *Replacement*. Regular cardinals are, in a way, limits of how far can we get by taking limits of increasing sequences of ordinals obtained via *Replacement*.

TODO prepsat – regularita a replacement, proc reflexe dava silnejsi veci

[35] Assuming *Choice*.

[36] Just like ω can not be expressed as a supremum of a finite set consisting solely of finite numbers.

[37] All provable to exist in ZFC

That all being said, it is easy to see that no cardinals in ZFC are both strongly limit and regular because there is no way to ensure they are sets and not proper classes in ZFC. The only exception to this rule is $\aleph_0$ which needs *Infinity* to exist. It should now be obvious why the fact that κ is inaccessible implies that $\kappa = aleph_\kappa$.[38]

We will also examine the connection between reflection principles and (regular) fixed points of ordinal functions in a manner proposed by Lévy in [2]. We will also see that, like Lévy has proposed in the same paper, there is a meaningful way to extend the relation between S and ZFC into a hierarchy of stronger axiomatic set theories.

3.1 Regular Fixed-Point Axioms

Lévy's article mentions various schemata that are not instances of reflection per se. We will mention them because they are equivalent to *Reflection₁*[39].

Definition 3.1 *(*Axiom M_1*)*
"Every normal function defined for all ordinals has at least one inaccessible number in its range."

Lévy uses "M" to refer to this axiom but since we also use "M" for sets and models, for example in 2.10, we will call the above axiom "*Axiom M_1*" to avoid confusion.

Now we will express *Axiom M_1* to formula to make it clear that it is an axiom scheme and the same can be done with *Axiom M'_1* as well as *Axiom M''_1* introduced immediately afterwards. Since it is an axiom schema and we will later dive into second-order logic, we may also want to refer to *Axiom M_2* as opposed *Axiom M_1*, the former being a single second-order sentence obtained by the obvious modification of *Axiom M_1*.[40]

Let $\varphi(x, y, p_1, \ldots, p_n)$ be a first-order formula with no free variables besides $x, y, p_1, \ldots, p_n$. The following is equivalent to *Axiom M_1*.

$$\begin{aligned} &\text{"}\varphi \text{ is a normal function" } \& \ \forall x(x \in Ord \to \exists y(\varphi(x, y, p_1, \ldots, p_n))) \to \\ &\to \exists y(\exists x \varphi(x, y, p_1, \ldots, p_n) \ \& \ cf(y) = y \ \& \ (\forall x \in \kappa)(\exists y \in \kappa)(x > y)) \end{aligned} \tag{3.67}$$

[41]

[38] This doesn't work backwards, the least fixed point of the $\aleph$ function is the limit of $\{\aleph_0,\ \aleph_{\aleph_0},\ \aleph_{\aleph_{\aleph_0}},\ \ldots\}$, it is singular since the sequence has countably many elements.

[39] For definition, see 2.10

[40] Second-order set theory will be introduced in the next subsection.

[41] "φ is a normal function" is equivalent to the following first-order formula:

Definition 3.2 *(Axiom M'_1)*
Every normal function defined for all ordinals has at least one fixed point which is inaccessible.

Definition 3.3 *(Axiom M''_1)*
"Every normal function defined for all ordinals has arbitrarily great fixed points which are inaccessible."

The following axiom is proposed by Drake in [3].

Definition 3.4 *(Axiom F_1)*
Every normal function defined for all ordinals has a regular fixed point.

Lemma 3.5 *(Fixed-point lemma for normal functions)*
Let f be a normal function defined for all ordinals. The all of the following hold

(i) $\forall\lambda$(*"λ is a limit ordinal" $\rightarrow$ "$f(\lambda)$ is a limit ordinal"*)
(ii) $\forall\alpha(\alpha \leq f(\alpha))$
(iii) $\forall\alpha\exists\beta(\alpha < \beta \ \&\ f(\beta) = \beta)$*($f$ has arbitrarily large fixed points.)*
(iv) The fixed points of f form a closed unbounded class.[42]

Proof. Let f be a normal function.

(i) Proof of (**i**):
Suppose λ is a limit ordinal. For an arbitrary ordinal $\alpha < \lambda$, the fact that f is strictly increasing means that $f(\alpha) < f(\lambda)$ and for an ordinal β, $\beta < \alpha$, $f(\alpha) < f(\beta)$. Because f is continuous and λ limit, $f(\lambda) = \bigcup_{\alpha<\lambda} f(\alpha)$ and since $\beta < \lambda$, $f(\beta) < f(\lambda)$. So we have found $f(\beta)$ such that $f(\alpha) < f(\beta) < f(\lambda)$, therefore $f(\lambda)$ is a limit ordinal.

(ii) This step will be proven using the transfinite induction. Since f is defined for all ordinals, there is an ordinal α such that $f(\emptyset) = \alpha$ and because $\emptyset$ is the least ordinal, (**ii**) holds for $\emptyset$.
Suppose (**ii**) holds for some β form the induction hypothesis. It the holds for $\beta + 1$ because f is strictly increasing.
For a limit ordinal λ, suppose (**ii**) holds for every $\alpha < \lambda$. (**i**) implies that $f(\lambda)$ is also limit, so there is a strictly increasing κ-sequence $\langle\alpha_0, \alpha_1, \ldots\rangle$ for some κ such that $\lambda = \bigcup_{i<\kappa} \alpha_i$. Because f is stricly increasing, the κ-sequence $\langle f(\alpha_0), f(\alpha_1), \ldots$ is also strictly increasing, the induction hypothesis implies that $\alpha_i \leq f(\alpha_i)$ for each $i \leq \kappa$. Thus, $\lambda \leq f(\lambda)\rangle$.

[42] See 1.43 for the definition of closed set, **??**

(iii) For a given α, let there be a ω-sequence $\langle \alpha_0, \alpha_1, \ldots \rangle$, such that $\alpha_0 = \alpha$ and $\alpha_{i+1} = f(\alpha_i)$ for each $i < \omega$. This sequence is stricly increasing because so is f. Now, there's a limit ordinal $\beta = \bigcup_{i<\omega} \alpha_i$, we want to show that this is the fixed point. So $f(\beta) = f(\bigcup_{i<\omega} \alpha_i) = \bigcup_{i<\omega} f(\alpha)$ because f is continuous. We have defined the above sequence so that β, $\bigcup_{i<\omega} f(\alpha) = \bigcup_{i<\omega} \alpha_{i+1}$, which means we are done, since $\bigcup_{i<\omega} \alpha_{i+1} = \bigcup_{i<\omega} \alpha_i = \beta$.

(iv) The class of fixed points of f is obviously unbounded by (**iii**). It remains to show that it is closed. TODO def closed?

$\square$

Theorem 3.6

$$\text{Axiom } M_1 \leftrightarrow \text{Axiom } M'_1 \leftrightarrow \text{Axiom } M''_1 \leftrightarrow \text{Axiom } F_1 \tag{3.68}$$

This is *Theorem 1* in [2]. *Proof.* It is clear that *Axiom M''_1* is a stronger version of *Axiom M'_1*, which is in turn a stronger version of both *Axiom M_1* and *Axiom F_1*, so the implication *Axiom M''_1* $\to$ *Axiom M'_1* $\to$ *Axiom M_1* is satisfied and *Axiom M'_1* $\to$ *Axiom F_1* holds too.

We will now make sure that *Axiom M_1* $\to$ *Axiom M''_1* also holds. Let f be a normal function defined for all ordinals. Let g be a normal function that counts the fixed points of f. Lemma 3.5 implies that there arbitrarily many fixed points of f, therefore g is defined for all ordinals. Let there be another family of functions, $h_\alpha(\beta) = g(\alpha + \beta)$, obviously h_α is defined for all ordinals for every $\alpha \in Ord$ because so is g. Given an arbitrary ordinal γ, from *Axiom M_1* we can assume that there is an ordinal δ such that such that $h_\alpha(\delta) = \kappa$, where κ is inaccessible. But since $\kappa = g(\alpha + \delta)$, κ is a fixed point of f. To show that there are arbitrarily many fixed points of f, notice that γ is arbitrary and h_γ is a normal function, so, by lemma 3.5, $(\forall \alpha \in Ord)(\alpha \leq f(\alpha)$, therefore $\gamma \leq \gamma + \alpha \leq \kappa$, in other words, there is κ above an arbitrary ordinal γ.

Now we need to show that *Axiom F_1* implies any of the remaining axioms. TODO nevyhodime F? $\square$

Definition 3.7 ZMC
We will call ZMC *a set theory that contains all axioms and schemas of* ZFC *together with the schema* Axiom M_1.

We have decided to call it ZMC, because Lévy uses ZM, derived from ZF, which is more intuitive, but we also need the axiom of choice, thus, ZMC.

Let's now prove that in ZFC, the above *Axiom M* is equivalent to *Reflection$_1$* as defined in 2.10. This is proven in [2] as *Theorem 3.*

Theorem 3.8

$$\mathsf{ZFC} \models \text{Axiom M} \leftrightarrow \text{Reflection}_1 \tag{3.69}$$

TODO nedosazitelne kardinaly – reflektuj presne formule, schemata

3.2 Inaccessibility

Definition 3.9 *(limit cardinal) κ is a* limit cardinal *iff it is $\aleph_\alpha$ for some limit ordinal α.*

Definition 3.10 *(strong limit cardinal) κ is a* strong limit cardinal *iff it is a limit cardinal and for every $\lambda < \kappa$, $2^\lambda < \kappa$*

The two above definition become equivalent if we assume GCH.

Definition 3.11 *(weak inaccessibility) An uncountable cardinal κ is* weakly inaccessible *iff it is* regular *and* limit.

Definition 3.12 *(inaccessibility) An uncountable cardinal κ is* inaccessible *iff it is* regular *and* strongly limit.

TODO neni tohle cely hotovy v Contemporary restatement??? porovnat ktera je lepsi a sjednotit!!!

We will now show that the above notion is equivalent to the definition Lévy uses in [2], which is, in more contemporary notation, the following:

Theorem 3.13 *The following are equivalent:*

1. *κ in inaccessible*
2. *$\langle V_\kappa, \in\rangle \models \mathsf{ZFC}$*

Proof. Let's first prove that if κ is inaccessible, it is a model of ZFC. We will do that by verifying the axioms of ZFC just like Kanamori does it in in [1, 1.2] and Drake in [3, Chapter 4].

(i) *Extensionality*:
(see 1.2)

$$V_\kappa \models \forall x, y(\forall z(z \in x \leftrightarrow z \in y) \rightarrow x = y) \tag{3.70}$$

We need to prove that, given two sets that are equal in V, they are equal in V_κ, in other words, that the *Extensionality* formula is reflected, that is

$$V_\kappa \models \forall x, y \in V_\kappa(\forall z \in V_\kappa(z \in x \leftrightarrow z \in y) \rightarrow x = y) \tag{3.71}$$

But that comes from transitivity. If x and y are in V_κ their members are also in V_κ.

(ii) *Foundation*:
(see 1.6)

$$V_\kappa \models \forall x(\exists z(z \in x) \to \exists z(z \in x \mathbin{\&} \forall u \neg(u \in z \mathbin{\&} u \in x))) \tag{3.72}$$

The argument for *Foundation* is almost identical to the one for *Extensionality*. For any set $x \in V_\kappa$, transitivity of V_κ makes sure that every element of x is also an element of V_κ and the same holds for the elements of elements of x et cetera. So statements about those elements are absolute between any transitive structures. V and V_κ are both transitive therefore *Foundation* holds and so does its relativisation to V_κ, $Foundation^{V_\kappa}$.

(iii) *Powerset*:
(see 1.9)

$$V_\kappa \models \forall x \exists y \forall z(z \subseteq x \to z \in y). \tag{3.73}$$

If we take x, an element of V_κ, $\mathscr{P}(x)$ has to be an element of V_κ to, because it is transitive and a strong limit cardinal.

(iv) *Pairing*:
(see 1.7)

$$V_\kappa \models \forall x, y \exists z(x \in z \wedge y \in z). \tag{3.74}$$

Pairing holds from similar argument like above: let x and y be elements of V_κ, so there are ordinals $\alpha, \beta < \kappa$ such that $x \in V_\alpha$, $y \in V_\beta$. Without any loss of generality, suppose $\alpha < \beta$, threfore $V_\alpha \subset V_\beta$ which, from transitivity of the cumulative hierarchy, means that $x \in V_\beta$, then $\{x, y\} \in V_{\beta+1}$ which is still in V_κ because it is a strong limit cardinal.

(v) *Union*:
(see 1.8)

$$V_\kappa \models \forall x \, \exists y \, \forall z \, \forall w((w \in z \wedge z \in x) \to w \in y). \tag{3.75}$$

We want to see that for every $x \in V_\kappa$, this is equivalent to

$$V_\kappa \models \forall x \in V_\kappa, \exists y \in V_\kappa \forall z \in V_\kappa \forall w \in V_\kappa((w \in z \wedge z \in x) \to w \in y). \tag{3.76}$$

Since V_κ is transitive, if $x \in V_\kappa$, all of its elements as well as their elements are in V_κ. To see that they also form a set themselves we only need to remember that V_κ is limit and therefore if α is the least ordinal such that $x \in V_\alpha$, $\bigcup x \in V_{\alpha+1}$.

(vi) *Replacement, Infinity*:
(see 1.15, 1.10)
TODO !!!!
to spis ty pred tim zname z dukazu v S, viz contemporary restatement.
udelat z toho lemma?
co ten replacement?? druha implikace Levyho vety?

We will now show that if a set is a model of ZFC, it is in fact an inaccessible cardinal. So let V_κ be a model of ZFC which means that it is closed under the powerset operation, in other words:

$$\forall\lambda(\lambda < \kappa \rightarrow 2^\lambda \ < \kappa) \tag{3.77}$$

which is exactly the definition of strong limitness. κ is regular from the following argument by contradiction:
Let us suppose for a moment that κ is singular. Therefore there is an ordinal $\alpha < \kappa$ and a function $F : \ \alpha \rightarrow \kappa$ such that the range of F in unbounded in κ, in other words, $F[\alpha] \subseteq V_\kappa$ and $sup(F[\alpha]) = kappa$. In order to achieve the desired contradiction, we need to see that it is the case that $F[\alpha] \in V_\kappa$. Let $\varphi(x, y)$ be the following first-order formula:

$$F(x) \ = \ y \tag{3.78}$$

Then there is an instance of *Replacement* that states the following:

$$\begin{gathered}(\forall x, y, z(\varphi(x, y) \ \& \ \varphi(x, z) \rightarrow y \ = \ z)) \rightarrow \\ \rightarrow (\forall x \exists y \forall z(z \in y \leftrightarrow \exists w(\varphi(w, z))))\end{gathered} \tag{3.79}$$

Which in turn means that there is a set $y = F[\alpha]$ and $y \in V_\kappa$, which is the contradiction with $sup(y) = \kappa$ we are looking for. $\square$ TODO vyhodit sup, pouzivat radis $\bigcup$

We have transcended ZFC, but that is just a start. Naturally, we could go on and consider the next inaccessible cardinal, which is inaccessible with respect to the theory $\mathsf{ZFC} + \exists\kappa(\kappa \models \mathsf{ZFC})$. But let's try to find a faster way up, informally at first.

Since we can find an inaccessible set larger than any chosen set M_0, it is clear that there are arbitrarily large inaccessible cardinals in V, they are "unbounded"[43] in V. If V were a cardinal, we could say that there are V inaccesible cardinals less than V, but this statement of course makes no sense in set theory as is because V is not a set. But being more careful, we could find a property that can be formalized in second-order logic and reflect it to

[43] The notion is formaly defined for sets, but the meaning should be obvious.

an initial segment of V. That would allow us to construct large cardinals more efficiently than by adding inaccessibles one by one. The property we are looking for ought to look like something like this:

$$\begin{gathered}\kappa \text{ is an inaccessible cardinal and} \\ \text{there are } \kappa \text{ inaccessible cardinals } \mu \ < \ \kappa\end{gathered} \tag{3.80}$$

This is in fact a fixed-point type of statement. We shall call those cardinals hyper-inaccessible. Now consider the following definition.

Definition 3.14 0-*inaccessible cardinal*
A cardinal κ is 0*-inaccessible if it is inaccessible.*

We can define α-*weakly-inaccessible* cardinals analogously with the only difference that those are limit, not strongly limit.

Definition 3.15 α-*hyper-inaccessible cardinal*
For any ordinal α, κ is called α-inaccessible, if κ is inaccessible and for each β ¡ α, the set of β-inaccessible cardinals less than κ is unbounded in κ.

Because κ is inaccessible and therefore regular, the number of β-inaccessibles below κ is equal to κ. We have therefore successfully formalized the above vague notion of hyper-inaccessible cardinal into a hierarchy of α-inaccessibles.

Let's now consider iterating this process over again. Since, informally, V would be α-inaccessible for any α, this property of the universal class could possibly be reflected to an initial segment, the smallest of those will be the first hyper-inaccessible cardinal. Such κ is larger than any α-inaccessible since from regularity of κ, for given $\alpha \ < \ \kappa$, κ is κ-th α-hyper-inaccessible cardinal. It is in fact "inaccessible" via α-inaccessibility.

Definition 3.16 *Hyper-inaccessible cardinal*
κ is called the hyper-inaccessible, also 0*-hyper-inaccessible, cardinal if it is α-inaccessible for every $\alpha \ < \ \kappa$.*

Definition 3.17 α-*hyper-inaccessible cardinal*
For any ordinal α, κ is called α-hyper-inaccessible cardinal if for each ordinal $\beta \ < \ \alpha$, the set of β-hyper-inaccessible cardinals less the κ is inbounded in κ.

Obviously we could go on and iterate it ad libitum, yielding α-hyper-...-hyper-inaccessibles, but the nomenclature would be increasingly confusing. A smarter way to accomplish the same goal is carried out in the following section.

3.3 Mahlo Cardinals

TODO axiomy?

While the previous chapter introduced us to a notion of inaccessibility and the possibility of iterating it ad libitum in new theories, there is an even faster way to travel upwards in the cumulative hierarchy, that was proposed by Paul Mahlo in his articles (see [5], [6] and [7]) at the very beginning of the 20th century, and which can be easily reformulated using reflection.

Theorem 3.18 *Let κ be a regular uncountable cardinal. The intersection of fewer than κ club subsets of κ is a club set.*

For the proof, see [4, Theorem 8.3]

Definition 3.19 *Weakly Mahlo Cardinal*
κ is weakly Mahlo $\leftrightarrow$ *it is a weakly-inaccessible ordinal and the set of all regular ordinals less then κ is stationary in κ*

Definition 3.20 *Mahlo Cardinal*
κ is a Mahlo Cardinal *iff it is an inaccessible cardinal and the set of all inaccessible ordinals less then κ is stationary in κ.*

Analogously,

Definition 3.21 *α-Mahlo Cardinal*
*κ is a α-*Mahlo Cardinal *iff it is an α-inaccessible cardinal and the set of all α-inaccessible ordinals less then κ is stationary in κ.*

In other words, κ is a (weakly-)Mahlo cardinal if it is (weakly-)inaccessible and every club set in κ contains an (weakly-)inaccessible cardinal. Alternatively, a cardinal is (weakly-)Mahlo if it is (weakly-)inaccesible and there are κ (weakly-)inaccessibles below κ.

In a fashion similar to hyper-inaccessible cardinals, hyper-Mahlo cardinals can be defined as well.

TODO Lévy tady nekde? posloupnost modelu?

TODO co s nima edla Jech?

TODO proc se vys nedostaneme pevnyma bodama?

TODO explicitni reflexe? reflektuji reflexi nedosazitelnosti?

3.4 Second-order Reflection

Let's try a different approach in formalizing reflection. We have seen that reflecting individual first-order formulas doesn't even transcend ZFC, we have examined what can be done with axiom schemas. The aim of this chapter is to examine second-order formulas as possible axioms. Note that second-order variables (which will be established as type 2 variables later in the text) are subcollections of the universal class, but so are functions and relations. So first-order axiom schemata can also be interpreted as formulas with free second-order variables, which quantify over first-order variables only, we only need to customize the underlying theory accordingly. For example, the satisfaction relation was so far defined for first-order formulas only, but we will deal with that in a moment. Also note that by rewriting *replacement* and *comprehension* to single axioms, ZFC becomes finitely axiomatizable, which in turn means that the reflection theorem as stated in section does not hold for higher-order theories because of Gödel's second incompleteness theorem. We will explore stronger axioms of reflection instead.

TODO nemam nekam napsat ze u vsech velkych karidnalu je to "existujou pokud .. "?

Let us establish a formal background first. We will now introduce higher-order formulas.

Definition 3.22 *(Higher-order variables)*
Let M be a structure and D it's domain. In first-order logic, variables range over individuals, that is, over elements of D. We shall call those type 1 variables *for the purposes of higher-order logic. Type 2 variables then range over collections, that is, the elements of $\mathscr{P}(D)$. Generally, type n variables are defined for any $n \in \omega$ such that they range over $\mathscr{P}^{n-1}(D)$.*

We will use lowercase latin letters for type 1 variables for backwards compatibility with first-order logic, type 2 variables will be represented by upper-case letters, mostly P, X, Y, Z. If we ever stumble upon type 3 variables in this text, they shall be represented as $\mathscr{X}, \mathscr{Y}, \mathscr{Z}$ or in a similar font.

Definition 3.23 *(Full prenex normal form)*
We say a formula is in the prenex normal form *if it is written as a block of quantifiers followed by a quantifier-free part.*
We say a formula is in the Full prenex normal form *if it is written in* prenex normal form *and if there are type $n+1$ quantifiers, they are written before type n quantifiers.*

It is an elementary that every formula is equivalent to a formula in the prenex normal form.

Definition 3.24 *(Hierarchy of formulas)*
Let φ be a formula in the prenex formal form.

(i) We say φ is a Δ^0_0-formula if it contains only bounded quantifiers.
(ii) We say φ is a Σ^0_0-formula or a Π^0_0-formula if it is a Δ^0_0-formula.
(iii) We say φ is a Π^{m+1}_0-formula if it is a Π^m_n- or Σ^m_n-formula for any $n \in \omega$ or if it is a Π^m_n- or Σ^m_n-formula with additional free variables of type $m+1$.
(iv) We say φ is a Σ^m_0-formula if it is a Π^m_0-formula.
(v) We say φ is a $\Sigma^m_n + 1$-formula if it is of a form $\exists P_1, \ldots, P_i \psi$ for any non-zero i, where ψ is a Π^m_n-formula and $P_1, \ldots, P_i$ are type $m+1$ variables.
(vi) We say φ is a $\Pi^m_n + 1$-formula if it is of a form $\forall P_1, \ldots, P_i \psi$ for any non-zero i, where ψ is a Σ^m_n-formula and $P_1, \ldots, P_i$ are type $m+1$ variables.

Definition 3.25 *(Reflection)*
Let $\varphi(P)$ be a Π^n_m-formula with one free variable of type type 2 denoted P. We say $\varphi(P)$ reflects in V_κ if for every $P \subseteq V_\kappa$ there is an ordinal $\alpha < \kappa$ such that the following holds:

$$\begin{gathered} \text{If } (V_\kappa, \in, R) \models \varphi(R), \\ \text{then } (V_\alpha, \in, R \cap V_\alpha) \models \varphi(R \cap V_\alpha). \end{gathered} \tag{3.81}$$

3.5 Indescribality

Since this section talks about indescribability, this is how an ordinal is described according to Drake [3, Chapter 9].

Definition 3.26 *We say an ordinal α is described by a formula $\varphi(P_1, \ldots, P_n)$ with type 2 parameters $P_1, \ldots, P_n$ given iff*

$$\langle V_\alpha, \in \rangle \models \langle \varphi(P_1, \ldots, P_n) \tag{3.82}$$

but for every $\beta < \alpha$

$$\langle V_\beta, \in \rangle \not\models \varphi(P_1 \cap V_\beta, \ldots, P_n \cap V_\beta) \tag{3.83}$$

Drake then notes that the same notion can be established for sentences if the corresponding type 2 parameters are added to the language. Since the this approach is used by Kanamori in [1], we will stick to that too.[44]

[44]The first definition is included because the author of this thesis finds it more intuitive.

Definition 3.27 *Describability*
We say an ordinal α is described by a sentence φ in the language $\mathscr{L}$ with relation symbols $P_1, \dots, P_n$ given iff

$$\langle V_\alpha, \in, P_1, \dots, P_n \rangle \models \varphi \tag{3.84}$$

but for every $\beta < \alpha$

$$\langle V_\beta, \in, P_1 \cap V_\beta, \dots, P_n \cap V_\beta \rangle \not\models \varphi \tag{3.85}$$

Definition 3.28 *(Π^m_n-indescribable cardinal) We say that κ is Π^m_n-indescribable iff it is not described by any Π^m_n-formula.*

Definition 3.29 *(Σ^m_n-indescribable cardinal) We say that κ is Σ^m_n-indescribable iff it is not described by any Σ^m_n-formula.*

Lemma 3.30 *Let κ be a cardinal, the following holds for any $n \in \omega$. κ is Π^1_n-indescribable iff κ is $\Sigma^1_n + 1$-indescribable*

Proof. The forward direction is obvious, we can always add a spare quantifier over a type 2 variable to turn a Π^1_n formula φ into a $\exists P \varphi$ which is obviously a $\Sigma^1_n + 1$ formula.[45]

To prove the opposite direction, suppose that $V_\kappa \models \exists X \varphi(X)$ where X is a type 2 variable and φ is a Π^1_n formula with one free variable of type 2. This means that there is a set $S \subseteq V_\kappa$ that is a witness of $\exists X \varphi(X)$, in other words, $\varphi(S)$ holds. We can replace every occurence of X in φ by a new predicate symbol S, this allows us to say that κ is Π^1_n-indescribable (with respect to $\langle V_\kappa, \in, R, S \rangle$). [46] $\square$

The above lemma makes it clear that we can suppose that all formulas with no higher than type 2 variables are Π^1_n-formulas, $n \in \omega$, without the loss of generality.

Lemma 3.31 *If κ is an inaccessible cardinal and given $R \subseteq V_\kappa$, then the following is a club set in κ:*

$$\{\alpha : \alpha < \kappa \ \& \ \langle V_\alpha, \in, R \cap V_\alpha \rangle \prec \langle V_\kappa, \in, R \rangle\} \tag{3.86}$$

[45] Note that unlike in previous sections, it is worth noting that φ is now a sentence so we don't have to worry whether P is free in φ.

[46] A different yet interesting approach is taken by Tate in ??. He states that for $n \geq 0$, a formula of order $\leq n$ is called a Π^n_0 and a Σ^n_0 formula. Then a Π^n_{m+1} is a formula of form $\forall Y \psi(Y)$ where ψ is a Σ^n_m formula and Y is a variable of type n. Finally, a Σ^n_{m+1} is the negation of a Π^n_m formula. So the above holds ad definitio.

Proof. To see that 3.86 is closed, let us recall that a $A \subseteq \kappa$ is closed iff for every ordinal $\alpha < \lambda$, $\alpha \neq \emptyset$: if $A \cap \alpha$ is unbounded in α then $\alpha \in A$. Since κ is an inaccessible cardinal, thus strong limit, it is closed under limits of sequences of ordinals lesser than κ.

TODO neco s V_κ, ze je tranzitivni a tak jso vsechny V_α pro $\alpha < \kappa$ $V_\alpha \in V_\kappa$

We want to verify that it is unbounded, we will use a recursively defined sequence $\alpha_0, \alpha_1, \ldots$ to build an elementary substructure of $\langle V_\kappa, \in, R\rangle$ that is built above an arbitrary $\alpha_0 < \kappa$. Let us fix an arbitrary $\alpha_0 < \kappa$. Given α_n, $\alpha_n + 1$ is defined as the least β, $\alpha_n \leq \beta$ that satisfies the following for any formula φ, $p_1, \ldots, p_m \in V_{\alpha_n}, m \in \omega$:

$$\text{If } \langle V_\kappa, \in, R\rangle \models \exists x \varphi(p_1, \ldots, p_n), \text{ then } \langle V_\kappa, \in, R\rangle \models \varphi(x, p_1, \ldots, p_n) \tag{3.87}$$

Let $\alpha = \bigcup_{n<\omega} \alpha_n$.

Then $\langle V_\alpha, \in, R \cap V_\alpha\rangle \prec \langle V_\kappa, \in, R\rangle$, in other words, for any φ with given arbitrary parameters $p_1, \ldots, p_n \in V_\alpha$, it holds that

$$\langle V_\alpha, \in, R \cap V_\alpha\rangle \models \varphi(p_1, \ldots, p_n) \leftrightarrow \langle V_\kappa, \in, R\rangle \models \varphi(p_1, \ldots, p_n) \tag{3.88}$$

Which should be clear from the construction of α □

Theorem 3.32 *Let κ be an ordinal. The following are equivalent.*

(i) κ is inaccessible

(ii) κ is Π^1_0-indescribable.

Proof. Since Π^1_0-sentences are first-order sentences, we want to prove that κ is an inaccessible cardinal iff whenever a first-order tries to describe κ in the sense of definition 3.27, the formula fails to do so and describes a initial segment thereof instead. We have already shown in 3.13 that there is no way to reach an inaccesible cardinal via first-order formulas in ZFC. We will now prove it again in for formal clarity.

For (**i**) $\rightarrow$ (**ii**), suppose that κ is inaccessible.

Then there is, by lemma 3.31 a club set of ordinals α such that V_α is an elementary substructures of V_κ. For κ to be Π^1_0inderscribable, we need to make sure that given an arbitrary first-order sentence φ satisfied in the structure $\langle V_\kappa, \in, R\rangle$, there is an ordinal $\alpha < \kappa$, such that $\langle V_\alpha, \in, R \cap V_\alpha\rangle \models \varphi$. But this follows from the definition of elementary substructure.

For (**ii**) $\rightarrow$ (**i**), suppose κ is not inaccessible, so it is either singular, or there is a cardinal $\nu < \kappa$ such that $\kappa \leq \mathscr{P}(\nu)$ or $\kappa = \omega$.

Suppose κ is singular. Then there is a cardinal $\nu < \kappa$ and a function $f : \nu \rightarrow \kappa$ such that $rng(f)$ is cofinal in κ. Since $f \subseteq V_\kappa$, we can add f as a relation to the language. We can do the same with $\{\nu\}$. That means $\langle V_\kappa, \in$

$, P_1, P_1$ with $P_1 = f, P_2 = \{\nu\}$ is a structure, let $\varphi = P_1 \neq \emptyset$ & $rng(P_1) = P_2$[47]. Since for every $\alpha < \nu$, $P_1 \cap V_\alpha = \emptyset$, φ is false and therefore describes κ. That contradicts the fact that κ was supposed to be Π^1_0-indescribable, but φ is a first-order formula.

Suppose there a cardinal ν satisfying $\kappa \leq \mathscr{P}(\nu)$. Let there be a function $f : \mathscr{P}(\nu) \to \kappa$ that is onto. Then, like in the previous paragraph, we can obtain a structure $\langle V_\kappa, \in, P_1, P_2\rangle$, where $P_1 = f$ like before, but this time $P_2 = \mathscr{P}(\nu)$. Again, $\varphi = P_1 \neq \emptyset$ & $rng(P_1) = P_2$ describes κ.

Finally, suppose $\kappa = \omega$, then the sentence $\varphi = \forall x \exists y (x \in y)$ describes κ, there is obviously no $\alpha < \ \omega$ such that $\langle V_\alpha, \in\rangle \models \varphi$.

□

Generally, it should be clear that it a cardinal κ is Π^m_n-indescribable, it is also $\Pi^{m'}_{n'}$-indescribable for every $m' < m, n' < n$. By the same line of thought, if a cardinal κ satisfies property implied by Π^m_n-indescribability, it satisfies all properties implied by $\Pi^{m'}_{n'}$-indescribability for $m' < m, n' < n$, for example κ is Π^m_n-indescribable for $m \geq 1, n \geq 0$, it is also an inaccessible cardinal.

Theorem 3.33 *If a cardinal κ is Π^1_1-indescribable, then it is a Mahlo cardinal.*

Proof. Assuming that κ is Π^1_1-indescribable, we want to prove that every club set in κ contains an inaccessible cardinal.

Consider the following Π^1_1-sentence:

$$\forall P(\text{"}P \text{ is a function"} \ \& \ \exists x(x = dom(P) \vee \mathscr{P}(x) = dom(P)) \to \to \exists y(y = rng(P))) \tag{3.89}$$

where P is a type 2 variable and x, y are type 1 variables, $rng(P)$ is defined in 1.13, $dom(P)$ in 1.12 and "P is a function" is a first-order formula defined in 1.11. We will call this sentence *Inac*, as in "inaccessible", because, given a cardinal μ, the following holds if and only if μ is inaccessible:

$$\langle V_\mu, \in\rangle \models Inac \tag{3.90}$$

So let's fix an arbitrary $C \subset \kappa$, club set in κ. We want to show that it contains an inaccessible cardinal. Since C is a subset of V_κ, let's add it to the structure $\langle V_\kappa, \in\rangle$, turning it into $\langle V_\kappa, \in, C\rangle$. Then the following holds:

$$\langle V_\kappa, \in, C\rangle \models Inac \ \& \ \text{"}C \text{ in unbounded"} \tag{3.91}$$

[47] $rng(x) = y$ is a first-order formula, see 1.13.

Note that this is correct, because, as we have noted just before introducing the statement now being proven, if κ is Π^1_1-indescribable, it is also Π^1_0-indescribable. So κ is itself inaccessible and therefore $\langle V_\kappa, \in, C\rangle \models Inac$. C is obviously picked so that it is unbounded in κ[48].

Now because we have assumed that κ is Π^1_1-indescribable and $Inac$ is a Π^1_1-formula, so $Inac$ & "C in unbounded" is equivalent to a Π^1_1-formula, there must be an ordinal α that satisfies

$$\langle V_\alpha, \in, C \cap V_\alpha\rangle \models Inac \,\&\, \text{"}C \text{ in unbounded"} \tag{3.92}$$

which implies that α is inaccessible.

To be finished, we need to verify that $\alpha \in C$. Since $\kappa = V_\kappa$ for inaccessible κ[49], $C \cap V_\alpha = C \cap \alpha$, from unboundedness of $C \cap \alpha$ in α, $\bigcup(C \cap \alpha) = \alpha$, which, together with the fact that C is a club set in κ and therefore closed in κ, yields that $\alpha \in C$. $\square$

TODO asi jako Drake, pozn ze to jde i pro hyper-Mahlovy?

Definition 3.34 *(Extension property) We say that a cardinal κ has the* extension property *iff for any $R \subseteq V_\kappa$ there is a transitive set $X \neq V_\kappa$ and an $S \subseteq X$ such that $\langle V_\kappa, \in, R\rangle \prec \langle X, \in, S\rangle$*

Definition 3.35 *(Weakly compact cardinal)*
We say that a cardinal κ is weakly compact *iff it has the extension property.*

The above definitions are equivalent

Theorem 3.36 *the following are equivalent:*

(i) κ *is* Weakly compact.
(ii) κ *is Π^1_1-indescribable.*

For a proof, see [1][Theorem 6.4]

3.6 Measurable Cardinal

TODO refaktorizovat fle:

Definition 3.37 *(Ultrafilter)*
Given a set X, we say $U \subset \mathscr{P}(X)$ is an ultrafilter *iff all of the following hold:*

[48] "C in unbounded" is a first-order formula defined in 1.41
[49] TODO link ——— ?

(i) $\emptyset \notin U$
(ii) $\forall a, b (\subset X \;\&\; a \subset b \;\&\; a \in U \to b \in U)$
(iii) $\forall a, b \in U (a \cap b) \in U$
(iv) $\forall a (a \subset X \to (a \in U \vee (X \setminus a) \in U))$

Definition 3.38 *(κ-complete ultrafilter)* *We say that an ultrafilter U is κ-complete iff*

Definition 3.39 *(non-principal ultrafilter)* *TODO*

Definition 3.40 *(Measurable Cardinal)* *Let κ be a caridnal. We say is a* measurable cardinal *iff it is an uncountable cardinal with a κ-complete, non-principal ultrafilter.*

Theorem 3.41 *Let κ be a cardinal. If κ is a measurable cardinal then it is Π^2_1-indescribable.*

Theorem 3.42 *Pod kazdym meritelnym kardinalem existuje ultrafiltr totalne nepopsatelnych, ktere tim padem nejsou sestrojitelne. VIZ VETA Z KANAMORIHO.*

asi nedokazovat?

3.7 The Constructible Universe

The constructible universe, denoted L, is a cumulative hierarchy of sets, presented by Kurt Gödel in his 1938 paper *The Consistency of the Axiom of Choice and of the Generalised Continuum Hypothesis*. For a technical description, see below. Assertion of their equality, $V = L$, is called the *axiom of constructibility*. The axiom implies GCH and therefore also AC and contradicts the existence of some of the large cardinals, our goal is to decide whether those introduced earlier are among them.

On order to formally establish this class, we need to formalize the notion of definability first.

Definition 3.43 *We say that a set X is* definable *over a model $\langle M, \in \rangle$ if there is a first-order formula φ together with parameters $p_1, \ldots, p_n \in M$ such that*

$$X = \{x : x \in M \;\&\; \langle M, \in \rangle \models \varphi(x, p_1, \ldots, p_n)\} \tag{3.93}$$

Definition 3.44 *(Sets definable in M)* *The following is a set of all definable subsets of a given set M, denoted Def(M).*

$$\begin{aligned} Def(M) = \{\{y : x \in M \wedge \langle M, \in\rangle \models \varphi(y, u_1, \ldots, i_n)\}| \\ \varphi \textit{ is a first-order formula, } p_1, \ldots, p_n \in M\} \end{aligned} \tag{3.94}$$

Now we can recursively build L.

Definition 3.45 *(The Constructible universe)*

(i)

$$L_0 := \emptyset \tag{3.95}$$

(ii)

$$L_{\alpha+1} := Def(L_\alpha) \tag{3.96}$$

(iii)

$$L_\lambda = \bigcup_{\alpha<\lambda} L_\alpha \textit{ If } \lambda \textit{ is a limit ordinal} \tag{3.97}$$

(iv)

$$L = \bigcup_{\alpha \in Ord} L_\alpha \tag{3.98}$$

Note that while L bears very close resemblance to V, the difference is, that in every successor step of constructing V, we take every subset of V_α to be $V_{\alpha+1}$, whereas $L_{\alpha+1}$ consists only of definable subsets of L_α. Also note that L is transitive.

In order to

TODO:

Lemma 3.46 $Ord \in L$

Lemma 3.47 *L is well-ordered.*
TODO !!

Theorem 3.48 *Let L be as in 3.45.*

$$L \models \mathsf{ZFC} \tag{3.99}$$

Proof. TODO !!! (strucne) vit [4][Theorem 13.3]

(i) *Extensionality* (see 1.2):
Extensionality holds in L because Δ_0 formulas are absolute in transitive classes by **??**, *Extensionality* is Δ_0 and L is transitive.

(ii) *Foundation* (see 1.6)
Take a non-empty set X. Let $x \in X$ be a set such that $X \cap x = \emptyset$. x is therefore defined by the formula $\varphi(x, y) = (x \cap y = \emptyset)$, so $x \in L$. φ is Δ_0 and therefore holds in L by **??**.

(iii) *Pairing* (see 1.7)
Since *Pairin* is also Δ_0, it holds in L by the same argument as *Extensionality* does by **??**.

(iv) *Union* (see 1.8)
Union is also Δ_0, see *Extensionality* and **??**.

(v) *Power Set* (see 1.9)
Power Set also holds by **??**.

(vi) *Infinity* (see 1.10)
$\omega \in L$ by 3.46

(vii) *Specification* (see 1.3)
.

(viii) *Replacement* (see 1.15)
.

(ix) *Choice* (see 1.15)
.

□

Definition 3.49 *Constructibility*
$L = V$

The following are a few interesting results that we won't prove but refer interested reader to appropriate resources instead.

Definition 3.50 *(GCH)*
The following is called the Generalised Continuum Hypothesis, abbreviated as GCH*. It is an independent statement in* ZFC*.*

$$\text{GCH } \textit{iff } \aleph_{\alpha+1} = 2^{\aleph_\alpha} \textit{ for every ordinal } \alpha \tag{3.100}$$

Theorem 3.51

$$(L = V) \rightarrow GCH \tag{3.101}$$

This is proven in cite{neco} Gödel? Jech? Kunnen?

TODO L a velke kardinaly

TODO def Con!

Theorem 3.52 *The existence of the inaccessible cardinal is compatible with*

Theorem 3.53

$$Con(L + \exists\kappa(\kappa\text{" }\textit{is a Mahlo Cardinal"})) \tag{3.102}$$

Theorem 3.54

$$Con(L + \exists\kappa(\kappa\text{" }\textit{is a Weakly Inaccessible Cardinal Cardinal"})) \tag{3.103}$$

Theorem 3.55

$$Con(L + \exists\kappa(\kappa\text{" }\textit{is a Measurable Cardinal"})) \tag{3.104}$$

TODO vyska / sirka univerza
TODO co velky pismena ve jmenech kardinalu?
TODO zduvodneni

TODO kratka diskuse jestli refl implikuje transcendenci na L - polemika, nazor - V=L a slaba kompaktnost a dalsi

4 Conclusion

TODO na konec

References

[1] Akihiro Kanamori (auth.). *The higher infinite: Large cardinals in set theory from their beginnings.* Springer Monographs in Mathematics. Springer-Verlag Berlin Heidelberg, 2 edition, 2003.

[2] Lévy Azriel. Axiom schemata of strong infinity in axiomatic set theory. *Pacific Journal of Mathematics*, 10, 1960.

[3] Drake F. *Set theory. An introduction to large cardinals.* Studies in Logic and the Foundations of Mathematics, Volume 76. NH, 1974.

[4] Thomas Jech. *Set theory.* Springer monographs in mathematics. Springer, the 3rd millennium ed., rev. and expanded edition, 2006.

[5] P. Mahlo. Über lineare transfinite Mengen. Leipz. Ber. 63, 187-225 (1911)., 1911.

[6] P. Mahlo. Über lineare transfinite Mengen. Leipz. Ber. 63, 187-225 (1911)., 1911.

[7] P. Mahlo. Zur Theorie und Anwendung der ϱ_v-Zahlen. II. Leipz. Ber. 65, 268-282 (1913)., 1913.

[8] Rudy von Bitter Rucker. *Infinity and the mind : the science and philosophy of the infinite.* Princeton science library. Princeton University Press, 2005 ed edition, 2005.

[9] Stewart Shapiro. Principles of reflection and second-order logic. *Journal of Philosophical Logic*, 16, 1987.

[10] Hao Wang. *"A Logical Journey: From Gödel to Philosophy"*. A Bradford Book, 1997.