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4 REFLECTION PRINCIPLES AND LARGE
5 CARDINALS
6 Bakalářská práce

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¹⁰ Prohlašuji, že jsem bakalářskou práci vypracoval samostatně a že jsem uvedl
¹¹ všechny použité prameny a literaturu.

¹² V Praze 22. května 2016

Abstract

Práce zkoumá vztah tzv. principů reflexe a velkých kardinálů. Lévy ukázal, že v ZFC platí tzv. věta o reflexi a dokonce, že věta o reflexi je ekvivalentní schématu nahrazení a axiomu nekonečna nad teorií ZFC bez axiomu nekonečna a schématu nahrazení. Tedy lze na větu o reflexi pohlížet jako na svého druhu axiom nekonečna. Práce zkoumá do jaké míry a jakým způsobem lze větu o reflexi zobecnit a jaký to má vliv na existenci tzv. velkých kardinálů. Práce definuje nedosažitelné, Mahlovy a nepopsatelné kardinály a ukáže, jak je lze zavést pomocí reflexe. Přirozenou limitou kardinálů získaných reflexí jsou kardinály nekonzistentní s L. Práce nabídne intuitivní zdůvodnění, proč tomu tak je.

Abstract

This thesis aims to examine relations between the so called Reflection Principles and Large cardinals. Lévy has shown that the Reflection Theorem is a sound theorem of ZF and it is equivalent to the Replacement Scheme and the Axiom of Infinity. From this point of view, Reflection theorem can be seen a specific version of an Axiom of Infinity. This paper aims to examine the Reflection Principle and its generalisations with respect to the existence of Large Cardinals. This thesis will establish the Inaccessible, Mahlo and Indescribable cardinals and show how can those be defined via reflection. A natural limit of Large Cardinals obtained via reflection are cardinals inconsistent with L. This thesis will offer an intuitive explanation of why this holds.

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1 Introduction

1.1 Motivation and Origin

“The Universe of sets cannot be uniquely characterised (i. e. distinguished from all its initial elements) by any internal structural property of the membership relation in it, which is expressible in any logic of finite or transfinite type, including infinitary logics of any cardinal order.”

— Kurt Gödel [Wang, 1997]

1.2 Notation and Terminology

1.2.1 The Language of Set Theory

This text assumes the knowledge of basic terminology and some results from first-order predicate logic, for example [?]. We won’t introduce the notions of *language*, *function symbol*, *predicate*, *term*, *model* and *interpretation* that are used in (1.42).

All proofs are based on [Jech, 2006] unless explicitly stated otherwise. Notable amount of inspiration is also drawn from [Kanamori, 2003] and [Drake, 1974].

We will now shortly review the basic notions that allow us to define the *Zermelo-Fraenkel* set theory.

When we talk about a *class*, we have the notion of a definable class in mind. If $\varphi(x, p_1, \dots, p_n)$ is a formula in the language of set theory, we call

$$A = \{x : \varphi(x, p_1, \dots, p_n)\} \quad (1.1)$$

a class of all sets satisfying $\varphi(x, p_1, \dots, p_n)$ in a sense that

$$x \in A \leftrightarrow \varphi(x, p_1, \dots, p_n) \quad (1.2)$$

Given classes A, B , one can easily define the elementary set operations such as $A \cap B$, $A \cup B$, $A \setminus B$, $\bigcup A$, see the first part of [Jech, 2006] for details. Axioms are the tools by which we can decide whether a particular class is “small enough” to be considered a set¹. A class that fails to be considered a set is called a *proper class*.

We will often write something like “ M is a limit ordinal”, it should always be clear that this can be rewritten as a formula that was introduced earlier.

¹“Small enough” means that it doesn’t introduce a paradox similar to Russell’s.

87 **1.2.2 The Axioms**88 **Definition 1.1** (*The Existence of a Set*)

$$\exists x(x = x) \quad (1.3)$$

89 **Definition 1.2** (*Axiom of Extensionality*)

$$\forall x, y(x = y \leftrightarrow \forall z(z \in x \leftrightarrow z \in y)) \quad (1.4)$$

90 **Definition 1.3** (*Axiom Schema of Specification*)

91 *The following yields an axiom for every first-order formula $\varphi(x, p_1, \dots, p_n)$*
 92 *with no free variables other than x, p_1, \dots, p_n .*

$$\forall x, p_1, \dots, p_n \exists y \forall z(z \in y \leftrightarrow z \in x \ \& \ \varphi(z, p_1, \dots, p_n)) \quad (1.5)$$

93 We will now provide two definitions that are not axioms, but will be
 94 helpful in establishing some axioms in a more comprehensible way.

95 **Definition 1.4** ($x \subseteq y, x \subset y$)

$$x \subseteq y \leftrightarrow (\forall z \in x)z \in y \quad (1.6)$$

96

$$x \subset y \leftrightarrow x \subseteq y \ \& \ x \neq y \quad (1.7)$$

97 *We read $x \subseteq y$ as x is a subset of y and $x \subset y$ as x is a proper subset of y .*

98 **Definition 1.5** (*Empty Set*) *For an arbitrary set x , the empty set, repre-*
 99 *sented by the symbol " \emptyset ", is the set defined by the following formula:*

$$(\forall y \in x)(y \in \emptyset \leftrightarrow \neg(y = y)) \quad (1.8)$$

100 \emptyset is a set due to Specification, there is only one such set due to Extension-
 101 ality.

102 **Definition 1.6** (*Axiom of Pairing*)

$$\forall x, y \exists z \forall q(q \in z \leftrightarrow q = x \vee q = y) \quad (1.9)$$

103 **Definition 1.7** (*Axiom of Union*)

$$\forall x \exists y \forall z(z \in y \leftrightarrow \exists q(z \in q \ \& \ q \in x)) \quad (1.10)$$

104 Now we can introduce more axioms.

105 **Definition 1.8** (*Axiom of Foundation*)

$$\forall x(x \neq \emptyset \rightarrow (\exists y \in x)(x \cap y = \emptyset)) \quad (1.11)$$

106 **Definition 1.9** (*Axiom of Powerset*)

$$\forall x \exists y \forall z (z \in y \leftrightarrow z \subseteq x) \quad (1.12)$$

107 **Definition 1.10** (*Axiom of Infinity*)

$$\exists x(\emptyset \in x \ \& \ (\forall y \in x)(y \cup \{y\} \in x)) \quad (1.13)$$

108 The least set satisfying this is denoted “ ω ”.

109 **Definition 1.11** (*Function*)

110 Given arbitrary first-order formula $\varphi(x, y, p_1, \dots, p_n)$, we say that φ is a func-
111 tion iff

$$\forall x, y, z, p_1, \dots, p_n (\varphi(x, y, p_1, \dots, p_n) \ \& \ \varphi(x, z, p_1, \dots, p_n) \rightarrow y = z) \quad (1.14)$$

112 When a $\varphi(x, y)$ is a function, we also write the following:

$$f(x) = y \leftrightarrow \varphi(x, y) \quad (1.15)$$

113 Alternatively, $f = \{\langle x, y \rangle : \varphi(x, y)\}$ is a class.

114 Let us introduce a few more definitions that will make the two remaining
115 axioms more comprehensible.

116 **Definition 1.12** (*Powerset Function*)

117 Given a set x , the powerset of x , denoted $\mathcal{P}(x)$ and satisfying (1.9), is
118 defined as follows:

$$\mathcal{P}(x) \stackrel{\text{def}}{=} \{y : y \subseteq x\} \quad (1.16)$$

119 **Definition 1.13** (*Domain of a Function*)

120 Let f be a function. We call the domain of f the set of all sets for which f
121 is defined. We use “ $\text{Dom}(f)$ ” to refer to this set.

$$x \in \text{Dom}(f) \leftrightarrow \exists y(f(x) = y) \quad (1.17)$$

122 We say “ f is a function on A ”, A being a class, if $A = \text{dom}(f)$.

123 **Definition 1.14** (*Range of a Function*)

124 Let f be a function. We call the range of f the set of all sets that are images
125 of other sets via f . We use “ $\text{Rng}(f)$ ” to refer to this set.

$$x \in \text{Rng}(f) \leftrightarrow \exists y(f(y) = x) \quad (1.18)$$

126 We say that f is a *function into* A , A being a class, iff $\text{rng}(f) \subseteq A$. We say
 127 that f is a *function onto* A iff $\text{rng}(f) = A$. We say a function f is a *one to*
 128 *one function*, iff

$$(\forall x_1, x_2 \in \text{dom}(f))(f(x_1) = f(x_2) \rightarrow x_1 = x_2) \quad (1.19)$$

129 We say that f is a *bijection* iff it is a one to one function that is onto.

130 Note that $\text{Dom}(f)$ and $\text{Rng}(f)$ are not definitions in a strict sense, they
 131 are in fact definition schemas that yield definitions for every function f given.
 132 Also note that they can be easily modified for φ instead of f , with the only
 133 difference being the fact that it is then defined only for those φ s that are
 134 functions, which must be taken into account. This is worth noting as we will
 135 use the notions of *function* and *formula* interchangeably.

136 **Definition 1.15** (*Function Defined For All Ordinals*)

137 We say a function f is defined for all ordinals, this is sometimes written
 138 $f : \text{Ord} \rightarrow A$ for any class A , if $\text{Dom}(f) = \text{Ord}$. Alternatively,

$$(\forall \alpha \in \text{Ord})(\exists y \in A)(f(\alpha) = y) \quad (1.20)$$

139 And now for the axioms.

140 **Definition 1.16** (*Axiom Schema of Replacement*)

141 The following is an axiom for every first-order formula $\varphi(x, p_1, \dots, p_n)$ with
 142 no free variables other than x, p_1, \dots, p_n .

$$“\varphi \text{ is a function}” \rightarrow \forall x \exists y \forall z (z \in y \leftrightarrow (\exists q \in x)(\varphi(x, y, p_1, \dots, p_n))) \quad (1.21)$$

143 **Definition 1.17** (*Choice function*)

144 We say that a function f is a *choice function* on x iff

$$\text{dom}(f) = x \setminus \{\emptyset\} \ \& \ (\forall y \in \text{dom}(f))(f(y) \in y) \quad (1.22)$$

145 **Definition 1.18** (*Axiom of Choice*)

146 For every set x there is a function f that is a *choice function* on x .

147 One might be unsettled by the fact that this definition quantifies over func-
 148 tions, which are generally classes, but in this particular case, since $\text{dom}(f) =$
 149 x and x is a set, f is also a set due to *Replacement*².

150 We will refer to the axioms by their name, written in italic type, e.g.
 151 *Foundation* refers to the Axiom of Foundation. Now we need to define the
 152 set theories to be used in the article.

²If the underlying theory includes of implies *Replacement*.

153 **Definition 1.19** (S)

154 We call **S** an axiomatic theory in the language $\mathcal{L} = \{=, \in\}$ with exactly the
 155 following axioms:

- 156 (i) Existence of a set (see (1.1))
- 157 (ii) Extensionality (see (1.2))
- 158 (iii) Specification (see (1.3))
- 159 (iv) Foundation (see (1.8))
- 160 (v) Pairing (see (1.6))
- 161 (vi) Union (see (1.7))
- 162 (vii) Powerset (see (1.9))

163 **Definition 1.20** (ZF)

164 We call **ZF** an axiomatic theory in the language $\mathcal{L} = \{=, \in\}$ that contains
 165 all the axioms of **S** in addition to the following:

- 166 (i) Replacement schema (see (1.16))
- 167 (ii) Infinity (see (1.10))

168 Existence of a set is usually left out because it is a consequence of infinity.

169 **Definition 1.21** (ZFC)

170 **ZFC** is an axiomatic theory in the language $\mathcal{L} = \{=, \in\}$ that contains all the
 171 axioms of **ZF** plus Choice (1.18).

172

173 **1.2.3 The Transitive Universe**174 **Definition 1.22** (Transitive Class)

175 We say a class A is transitive iff

$$(\forall x \in A)(x \subseteq A) \quad (1.23)$$

176 **Definition 1.23** (Well Ordered Class) A class A is said to be well ordered
 177 by \in iff the following hold:

- 178 (i) $(\forall x \in A)(x \not\subseteq x)$ (Antireflexivity)
- 179 (ii) $(\forall x, y, z \in A)(x \in y \ \& \ y \in z \rightarrow x \in z)$ (Transitivity)
- 180 (iii) $(\forall x, y \in A)(x = y \vee x \in y \vee y \in x)$ (Linearity)
- 181 (iv) $(\forall x \subseteq A)(x \neq \emptyset \rightarrow (\exists y \in x)(\forall z \in x)(z = y \vee z \in y))$ (Existence of the
 182 least element)

183 **Definition 1.24** (Ordinal Number)

184 A set x is said to be an ordinal number if it is transitive and well-ordered
 185 by \in .

For the sake of brevity, we usually just say “ x is an *ordinal*”. Note that “ x is an ordinal” is a well-defined formula in the language of set theory, since 1.22 is a first-order formula and 1.23 is in fact a conjunction of four first-order formulas. Ordinals will be usually denoted by lower case greek letters, starting from the beginning of the alphabet: $\alpha, \beta, \gamma, \dots$. Given two different ordinals α, β , we will write $\alpha < \beta$ for $\alpha \in \beta$, see Lemma 2.11 in [Jech, 2006] for technical details.

Definition 1.25 (*Non-Zero Ordinal*) We say an ordinal α is non-zero iff $\alpha \neq \emptyset$.

Definition 1.26 (*Successor Ordinal*) Consider the following function defined for all ordinals. Let β be an arbitrary ordinal. We call S the successor function.

$$S(\beta) = \beta \cup \{\beta\} \quad (1.24)$$

An ordinal α is called a successor ordinal iff there is an ordinal β , such that $\alpha = S(\beta)$. We also write $\alpha = \beta + 1$.

Definition 1.27 (*Limit Ordinal*) A non-zero ordinal α is called a limit ordinal iff it is not a successor ordinal.

Definition 1.28 (*Ord*) The class of all ordinal numbers, which we will denote “ Ord ”³ is the proper class defined as follows.

$$x \in \text{Ord} \leftrightarrow x \text{ is an ordinal} \quad (1.25)$$

Definition 1.29 (*Von Neumann’s Hierarchy*) The Von Neumann’s Hierarchy is a collection of sets indexed by elements of Ord , defined recursively in the following way:

$$(i) \quad V_0 = \emptyset \quad (1.26)$$

$$(ii) \quad V_{\alpha+1} = \mathcal{P}(V_\alpha) \text{ for any ordinal } \alpha \quad (1.27)$$

$$(iii) \quad V_\lambda = \bigcup_{\beta < \lambda} V_\beta \text{ for a limit ordinal } \lambda \quad (1.28)$$

³Other authors use “ On ”, we will stick to the notation used in [Jech, 2006].

We will also refer to the Von Neumann's Hierarchy as Von Neumann's Universe or the Cumulative Hierarchy. This definition is only correct in a theory that contains or implies Replacement because otherwise it's not clear that the successor step is a set.

Definition 1.30 (*Rank*)

Given a set x , we say that the rank of x (written as $\text{rank}(x)$) is the least ordinal α such that $x \in V_{\alpha+1}$

Due to *Regularity*, every set has a rank.⁴ The Von Neumann's hierarchy defined above can also be defined by the fact that every V_α is a set of all set with rank less than α .

Definition 1.31 (*Order-type*)

Given an arbitrary well-ordered set x , we say that an ordinal α is the order-type of x iff x and α are isomorphic.

221

1.2.4 Cardinal Numbers

Definition 1.32 (*Cardinality*)

Given a set x , let the cardinality of x , written $|x|$, be defined as the smallest ordinal number such that there is a one to one mapping from x onto α .

Definition 1.33 (*Aleph function*)

Let ω be the set defined by ???. We will recursively define the function \aleph for all ordinals.

(i) $\aleph_0 = \omega$

(ii) $\aleph_{\alpha+1}$ is the least cardinal larger than \aleph_α ⁵

(iii) $\aleph_\lambda = \bigcup_{\beta < \lambda} \aleph_\beta$ for a limit ordinal λ

If $\kappa = \aleph_\alpha$ and α is a successor ordinal, we call κ a successor cardinal. If α is a limit ordinal, we call κ a limit cardinal.

Definition 1.34 (*Cardinal number*)

235

(i) A set x is called a finite cardinal iff $x \in \omega$.

(ii) A set is called an infinite cardinal iff there is an ordinal α such that

$\aleph_\alpha = x$

238

⁴See chapter 6 of [Jech, 2006] for details.

⁵"The least cardinal larger than \aleph_α " is sometimes notated as \aleph_α^+ .

(iii) A set is called a cardinal iff it is either a finite cardinal or an infinite cardinal.

We say κ is an uncountable cardinal iff it is an infinite ordinal and $\aleph_0 < \kappa$. Infinite cardinals will be notated by lowercase greek letters from the middle of the alphabet, e.g. κ, μ, ν, \dots with the exception of λ , which is next to κ in the greek alphabet, but is also sometimes used for limit ordinal.

For formal details as well as why every set can be well-ordered assuming *Choice*, and therefore has a cardinality, see [Jech, 2006].

Definition 1.35 (*Sequence*)

We say that a function $\varphi(x, y)$ is a sequence iff there is an ordinal α such that $\text{dom}(\varphi) = \alpha$. In other words, a function is called a sequence if it is defined exactly for every ordinal from 0 to some α . We then say it is an α -sequence. We usually write $\langle \beta_i : i \in \alpha \rangle$ or $\langle \beta_0, \beta_1, \dots \rangle$ when referring to a sequence, β_i then denotes the elements of $\text{rng}(\varphi)$ for every $i \in \text{dom}(\varphi)$.

Definition 1.36 (*Cofinal Subset*)

Given a class A of ordinals, we say that $B \subseteq A$ is cofinal in A iff

$$(\forall x \in A)(\exists y \in B)(x \in y) \quad (1.29)$$

In other words, B is cofinal in A iff it is unbounded in A .

Definition 1.37 (*Cofinality of a Limit Ordinal*)

Let λ be a limit ordinal. We say that the cofinality of λ is κ iff κ is the least ordinal, such that there is a cofinal κ -sequence $\langle \beta_\xi : \xi < \kappa \rangle$, such that

$$\sup(\{\beta_\xi : \xi < \kappa\}) = \lambda \quad (1.30)$$

We write $cf(\lambda) = \kappa$.

Note that $cf(\alpha)$ is always a cardinal⁶.

Definition 1.38 (*Regular Cardinal*)

We say an infinite cardinal κ is regular iff $cf(\kappa) = \kappa$.

Definition 1.39 (*Strong Limit Cardinal*)

We say that an ordinal κ is a strong limit cardinal if it is a limit cardinal and

$$(\forall \alpha \in \kappa)(|\mathcal{P}(\alpha)| \in \kappa). \quad (1.31)$$

⁶If $cf(\alpha)$ is not a cardinal, so $|cf(\alpha)| < cf(\alpha)$, then there is a mapping from $|cf(\alpha)|$ onto $cf(\alpha)$. But then the range of this mapping is a cofinal subset of $cf(\alpha)$ that is strictly smaller than $cf(\alpha)$.

266 **Definition 1.40** (*Generalised Continuum Hypothesis*)

267

$$(\forall \alpha \in \text{Ord}) \aleph_{\alpha+1} = |\mathcal{P}(\aleph_\alpha)| \quad (1.32)$$

268 If *GCH* holds (for example in Gödel's *L*, see chapter 3), the notions of limit
269 cardinal and strong limit cardinal are equivalent.

270

271 1.2.5 Relativisation and Absoluteness

272 **Definition 1.41** (*Relativization*)

273 Let M be a class, $R \subseteq M \times M$ and let $\varphi(p_1, \dots, p_n)$ be a first-order formula
274 with no free variables besides p_1, \dots, p_n . The relativization of φ to M and R
275 is the formula, written as $\varphi^{M,R}$, defined in the following inductive manner:

- 276 (i) $(x \in y)^{M,R} \leftrightarrow R(x, y)$
- 277 (ii) $(x = y)^{M,R} \leftrightarrow x = y$
- 278 (iii) $(\neg \varphi)^{M,R} \leftrightarrow \neg \varphi^{M,R}$
- 279 (iv) $(\varphi \ \& \ \psi)^{M,R} \leftrightarrow \varphi^{M,R} \ \& \ \psi^{M,R}$
- 280 (v) $(\varphi \vee \psi)^{M,R} \leftrightarrow \varphi^{M,R} \vee \psi^{M,R}$
- 281 (vi) $(\varphi \rightarrow \psi)^{M,R} \leftrightarrow \varphi^{M,R} \rightarrow \psi^{M,R}$
- 282 (vii) $(\exists x \varphi(x))^{M,R} \leftrightarrow (\exists x \in M) \varphi^{M,R}(x)$
- 283 (viii) $(\forall x \varphi(x))^{M,R} \leftrightarrow (\forall x \in M) \varphi^{M,R}(x)$

284 When $R = \in \cap (M \times M)$, we usually write φ^M instead of $\varphi^{M,R}$. When we
285 talk about $\varphi^M(p_1, \dots, p_n)$, it is understood that $p_1, \dots, p_n \in M$.

286 **Definition 1.42** (*Satisfaction in a Structure*)

287 Let M be a set and R a binary relation on M . Let *Terms* be the set of all
288 terms, let $e : \text{Terms} \rightarrow M$ any evaluation function. Let φ be a first-order
289 formula in the language of set theory.

290 iff any of the following hold

291 We say that φ holds in $\langle M, R \rangle$ under the evaluation e , we write $\langle M, R \rangle \models \varphi[e]$,

292 iff any of the following hold

- 293 (i) φ is the formula “ $s = t$ ”, s, t are terms and $e(s) = e(t)$.
- 294 (ii) φ is the formula “ $s \in t$ ”, s, t are terms and the pair $\langle e(s), e(t) \rangle$ is in R .
- 295 (iii) φ is the formula “ $\neg \psi$ ” and not $\langle M, R \rangle \models \psi[e]$
- 296 (iv) φ is the formula “ $\psi_1 \ \& \ \psi_2$ ” and both $\langle M, R \rangle \models \psi_1[e]$ and $\langle M, R \rangle \models \psi_2[e]$.
- 297 (v) φ is the formula “ $\psi_1 \vee \psi_2$ ” and either $\langle M, R \rangle \models \psi_1[e]$ or $\langle M, R \rangle \models \psi_2[e]$.
- 298 (vi) φ is the formula “ $\psi_1 \rightarrow \psi_2$ ” and either not $\langle M, R \rangle \models \psi_1[e]$ or
299 $\langle M, R \rangle \models \psi_2[e]$.
- 300 (vii) φ is the formula “ $\psi_1 \rightarrow \psi_2$ ” and either not $\langle M, R \rangle \models \psi_1[e]$ or
301 $\langle M, R \rangle \models \psi_2[e]$.

- 302 (viii) φ is the formula “ $\forall x_1 \psi$ ” and $\langle M, R \rangle \models \psi[e']$ for every e' that differs
 303 from e only in the value of x_1 .
 304 (ix) φ is the formula “ $\forall x_1 \psi$ ” and $\langle M, R \rangle \models \psi[e']$ for every e' that differs
 305 from e only in the value of x_1 .
 306 (x) φ is the formula “ $\exists x_1 \psi$ ” and $\langle M, R \rangle \models \psi[e']$ for some e' that differs
 307 from e only in the value of x_1 .
 308 We also write $\langle M, R \rangle \models \varphi$, which

309 Note that we say that M is a set.

310 We will use $\langle M, R \rangle \models \varphi(p_1, \dots, p_n)$ and $\varphi^M(p_1, \dots, p_n)$ interchangeably.

311 **Definition 1.43** (Absoluteness) Given a transitive class M , we say a for-
 312 mula φ is absolute in M if for all $p_1, \dots, p_n \in M$

$$\varphi^M(p_1, \dots, p_n) \leftrightarrow \varphi(p_1, \dots, p_n) \quad (1.33)$$

313 **Definition 1.44** (Hierarchy of First-Order Formulas)

- 314
 315 (I) A first-order formula φ is Δ_0 iff it is logically equivalent to a first-order
 316 formula φ' satisfying any of the following:
 317 (i) φ' contains no quantifiers
 318 (ii) y is a set, ψ is a Δ_0 -formula, and φ' is either $(\exists x \in y)\psi(y)$ or
 319 $(\forall x \in y)\psi(y)$.
 320 (iii) ψ_1, ψ_2 are Δ_0 -formulas and φ' is any of the following: $\psi_1 \vee \psi_2$,
 321 $\psi_1 \& \psi_2$, $\psi_1 \rightarrow \psi_2$, $\neg \psi_2$,
 322 (II) If a formula is Δ_0 it is also Σ_0 and Π_0
 323 (III) A formula φ is $\Pi_n + 1$ if it is logically equivalent to a formula φ' such
 324 that $\varphi' = \forall x \psi$ where ψ is a Σ_n -formula for any $n < \omega$.
 325 (IV) A formula φ is $\Sigma_n + 1$ if it is logically equivalent to a formula φ' such
 326 that $\varphi' = \exists x \psi$ where ψ is a Π_n -formula for any $n < \omega$.

327 Note that we can use the pairing function so that for $\forall p_1, \dots, p_n \psi(p_1, \dots, p_n)$,
 328 there is a logically equivalent formula of the form $\forall x \psi'(x)$.

329 **Lemma 1.45** (Δ_0 absoluteness) Let φ be a Δ_0 -formula, then φ is absolute
 330 in any transitive class M .

331 *Proof.* This will be proven by induction over the complexity of a given Δ_0 -
 332 formula φ . Let M be an arbitrary transitive class.

333 Atomic formulas are always absolute by the definition of relativisation,
 334 see (1.41). Suppose that Δ_0 -formulas ψ_1 and ψ_2 are absolute in M . Then

335 from relativization, $(\psi_1 \& \psi_2)^M \leftrightarrow \psi_1^M \& \psi_2^M$, which is, from the induction
 336 hypothesis, equivalent to $\psi_1 \& \psi_2$. The same holds for \vee, \rightarrow, \neg .

337 Suppose that a Δ_0 -formula ψ is absolute in M . Let y be a set and
 338 let $\varphi = (\exists x \in y)\psi(x)$. From relativization, $(\exists x\psi(x))^M \leftrightarrow (\exists x \in M)\psi^M(x)$.
 339 Since the hypotheses makes it clear that $\psi^M \leftrightarrow \psi$, we get $((\exists x \in y)\psi(x))^M \leftrightarrow$
 340 $(\exists x \in y \cap M)\psi(x)$, which is the equivalent of $\varphi^M \leftrightarrow \varphi$. The same applies to
 341 $\varphi = (\forall x \in y)\psi(x)$. \square

342 **Lemma 1.46** (*Downward Absoluteness*)

343 *Let φ be a Π_1 -formula and M a transitive class. Then the following holds:*

$$(\forall p_1, \dots, p_n \in M)(\varphi(p_1, \dots, p_n) \rightarrow \varphi(p_1, \dots, p_n)^M) \quad (1.34)$$

344 *Proof.* Since $\varphi(p_1, \dots, p_n)$ is Π_1 , there is a Δ_0 -formula $\psi(p_1, \dots, p_n, x)$ such
 345 that $\varphi = \forall x\psi(p_1, \dots, p_n, x)$. From relativization and lemma (1.45), $\varphi^M(p_1, \dots, p_n) \leftrightarrow$
 346 $(\forall x \in M)\psi(p_1, \dots, p_n, x)$.

347 Assume that for $p_1, \dots, p_n \in M$ fixed, that $\forall x\psi(p_1, \dots, p_n, x)$ holds, but
 348 $(\forall x \in M)\psi(p_1, \dots, p_n, x)$ does not. Therefore $\exists x\neg\psi(p_1, \dots, p_n, x)$, which
 349 contradicts $\forall x\psi(p_1, \dots, p_n, x)$. \square

350 **Lemma 1.47** (*Upward Absoluteness*)

351 *Let φ be a Σ_1 -formula and M a transitive class. Then the following holds:*

$$(\forall p_1, \dots, p_n \in M)(\varphi^M(p_1, \dots, p_n) \rightarrow \varphi(p_1, \dots, p_n)) \quad (1.35)$$

352 *Proof.* Since $\varphi(p_1, \dots, p_n)$ is Σ_1 , there is a Δ_0 -formula $\psi(p_1, \dots, p_n, x)$ such
 353 that $\varphi = \exists x\psi(p_1, \dots, p_n, x)$. From relativization and lemma (1.45), $\varphi^M(p_1, \dots, p_n) \leftrightarrow$
 354 $(\exists x \in M)\psi(p_1, \dots, p_n, x)$.

355 Assume that for $p_1, \dots, p_n \in M$ fixed, that $(\exists x \in M)\psi(p_1, \dots, p_n, x)$
 356 holds, but $\exists x\psi(p_1, \dots, p_n, x)$ does not. This is an obvious contradiction. \square

357 1.2.6 More Functions

358 **Definition 1.48** (*Strictly Increasing Function*)

359 *A function $f : Ord \rightarrow Ord$ is said to be strictly increasing iff*

$$\forall \alpha, \beta \in Ord(\alpha < \beta \rightarrow f(\alpha) < f(\beta)). \quad (1.36)$$

360 **Definition 1.49** (*Continuous Function*)

361 *A function $f : Ord \rightarrow Ord$ is said to be continuous iff*

$$\lambda \text{ is limit} \rightarrow f(\lambda) = \bigcup_{\alpha < \lambda} f(\alpha). \quad (1.37)$$

362 **Definition 1.50** (*Normal Function*)

363 *A function $f : \text{Ord} \rightarrow \text{Ord}$ is said to be normal iff it is strictly increasing*
 364 *and continuous.*

365 **Definition 1.51** (*Fixed Point*)

366 *We say x is a fixed point of a function f iff $x = f(x)$.*

367 **Definition 1.52** (*Unbounded Class*)

368 *We say a class A of ordinals is unbounded iff*

$$\forall x(\exists y \in A)(x < y) \quad (1.38)$$

369 **Definition 1.53** (*Limit Point*)

370 *Given a class $x \subseteq \text{Ord}$, we say that $\alpha \neq \emptyset$ is a limit point of x iff*

$$\alpha = \bigcup (x \cap \alpha) \quad (1.39)$$

371 **Definition 1.54** (*Closed Class*)

372 *We say a class $A \subseteq \text{Ord}$ is closed iff it contains all its limit points.*

373 **Definition 1.55** (*Club set*)

374 *For a regular uncountable cardinal κ , a set $x \subset \kappa$ is a closed unbounded*
 375 *subset, abbreviated as a club set, iff x is both closed and unbounded in κ .*

376 **Definition 1.56** (*Stationary set*)

377 *For a regular uncountable cardinal κ , we say a set $A \subset \kappa$ is stationary in κ*
 378 *iff it intersects every club subset of κ .*

379 1.2.7 Structure, Substructure and Embedding

380 Structures will be denoted $\langle M, \in, R \rangle$ where M is a domain, \in stands for the
 381 standard membership relation, it is assumed to be restricted to the domain⁷,
 382 $R \subseteq M$ is a relation on the domain. When R is not needed, we can as well
 383 only write M instead of $\langle M, \in \rangle$.

384 **Definition 1.57** (*Elementary Embedding*)

385 *Given the structures $\langle M_0, \in, R \rangle$, $\langle M_1, \in, R \rangle$ and a one-to-one function $j :$
 386 $M_0 \rightarrow M_1$, we say j is an elementary embedding of M_0 into M_1 , we write
 387 $j : M_0 \prec M_1$, when the following holds for every formula $\varphi(p_1, \dots, p_n)$ and
 388 every $p_1, \dots, p_n \in M_0$:*

$$\langle M_0, \in, R \rangle \models \varphi(p_1, \dots, p_n) \leftrightarrow \langle M_1, \in, R \rangle \models \varphi(j(p_1), \dots, j(p_n)) \quad (1.40)$$

⁷To be totally explicit, we should write $\langle M, \in \cap M \times M, R \rangle$.

389 **Definition 1.58** (*Elementary Substructure*)

390 *Given the structures $\langle M_0, \in, R \rangle$, $\langle M_1, \in, R \rangle$ and a one-to-one function $j :$
 391 $M_0 \rightarrow M_1$ such that $j : M_0 \prec M_1$, we say that M_0 is an elementary sub-
 392 structure of M_1 , denoted as $M_0 \prec M_1$, iff j is an identity on M_0 . In other
 393 words*

$$\langle M_0, \in, R \rangle \models \varphi(p_1, \dots, p_n) \leftrightarrow \langle M_1, \in, R \rangle \models \varphi(p_1, \dots, p_n) \quad (1.41)$$

394 *for $p_1, \dots, p_n \in M_0$*

2 Levy's First-Order Reflection

2.1 Lévy's Original Paper

This section is based on Lévy's paper *Axiom Schemata of Strong Infinity in Axiomatic Set Theory*, [Lévy, 1960]. It presents Lévy's general reflection principle and its equivalence to *Replacement* and *Infinity* under S^8 .

First, we should point out that set theory has changed over the last 66 years and show a few notable, albeit only formal, differences. One might be confused by the fact that Lévy treats the *Subsets* axiom, which we call *Specification*, as a single axiom rather than a schema. He even takes the conjunction of all axioms of ZF and treats it like a formula. This is possible because the underlying logic calculus is different. Lévy works with set theories formulated in the *non-simple applied first order functional calculus*, see beginning of *Chapter IV* in [Church, 1996] for details. For now, we only need to know that the calculus contains a substitution rule for functional variables. This way, *Subsets* is de facto a schema even though it sometimes treated as a single formula⁹ but the logic is still first-order since one can't quantify over functional variables. We will use the usual first-order axiomatization of ZFC as seen on [Jech, 2006]. It should also be noted that the logical connectives look different. The now usual symbol for an universal quantifier does not appear, $\forall x\varphi(x)$ would be written as $(x)\varphi(x)$. The symbol for negation is " \sim ", implication is written as " \supset " and equivalence is " \equiv ". We will use standard notation with " \neg ", " \rightarrow " and " \leftrightarrow " respectively when presenting Lévy's results.

This subsection uses ZF instead of the usual ZFC as the underlying theory.

Definition 2.1 (Standard Complete Model of a Set Theory)

Let Q be an arbitrary axiomatic set theory. We say that u is a standard complete model of Q iff

- (i) $(\forall \sigma \in Q)(\langle u, \in \rangle \models \sigma)$
- (ii) $\forall y(y \in u \rightarrow y \subset u)$ (u is transitive)

We write $Scm^Q(u)$.

Definition 2.2 (Cardinals Inaccessible With Respect to Q)

Let Q be an arbitrary axiomatic set theory. We say that a cardinal κ is inaccessible with respect to theory Q iff

$$Scm^Q(V_\kappa) \quad (2.42)$$

⁸See definition (1.19).

⁹This way, the conjunction of all axioms is then in fact an axiom schema.

428 We write $In^Q(\kappa)$.¹⁰

429 **Definition 2.3** (*Inaccessible Cardinal With Respect to ZF*)

430 When a cardinal κ is inaccessible with respect to ZF, we only say that it is
431 inaccessible. We write $In(\kappa)$.

$$In(\kappa) \leftrightarrow In^{ZF}(\kappa) \quad (2.43)$$

432 The above definition of inaccessibles is used because it doesn't require *Choice*.

433 For the definition of relativization, see (1.41). The notation used by Lévy is
434 " $Rel(u, \varphi)$ ", we will stick to " φ^u ".

435 **Definition 2.4** (*N*)

436 The following is an axiom schema of complete reflection over ZF, denoted *N*.
437 For every first-order formula φ in the language of set theory with no free variables
438 except for p_1, \dots, p_n , the following is an instance of schema *N*.

$$\exists u(Scm^{ZF}(u) \ \& \ (\forall p_1, \dots, p_n \in u)(\varphi \leftrightarrow \varphi^u)) \quad (2.44)$$

439 **Definition 2.5** (*N'*)

440 For any first-order formulas $\varphi_1, \dots, \varphi_m$ in the language of set theory with no
441 free variables except for p_1, \dots, p_n , the following is an instance of schema *N'*.

$$\exists u(z \in u \ \& \ Scm^{ZF}(u) \ \& \ (\forall p_1, \dots, p_n \in u)(\varphi_1 \leftrightarrow \varphi_1^u) \ \& \ \dots \ \& \ \varphi_m \leftrightarrow \varphi_m^u)) \quad (2.45)$$

442 **Definition 2.6** (*N'*)

443 For any first-order formulas $\varphi_1, \dots, \varphi_m$ in the language of set theory with no
444 free variables except for p_1, \dots, p_n , the following is an instance of schema *N'*.

$$\exists u(Scm^{ZF}(u) \ \& \ (\forall p_1, \dots, p_n \in u)(\varphi_1 \leftrightarrow \varphi_1^u) \ \& \ \dots \ \& \ \varphi_m \leftrightarrow \varphi_m^u)) \quad (2.46)$$

445 Let *S* be an axiomatic set theory defined in (1.19).

446 This is *Theorem 2* in [?]

447 **Lemma 2.7** ($N \leftrightarrow N'' \leftrightarrow N'$)

448 The schemas *N*, *N'* and *N''* are equivalent under *S*.

¹⁰To be able to define V_κ , we need to work in a logic that contains the *Replacement Schema* or any of it's equivalents. It should be noted that we don't work in an arbitrary theory *Q*, but in ZF, which contains the *Replacement Schema*. $Scm^Q(V_\kappa)$ in fact says "ZF thinks that V_κ is a transitive model of *Q*".

449 *Proof.* We will execute this proof in the theory ZF, but the reader should note
 450 that we are neither using *Replacement* nor *Infinity*, so for schemas similar to N ,
 451 N' , N'' but with " $Scm^S(u)$ " instead of " $Scm^{ZF}(u)$ ", the proof works equally
 452 well.

453 Clearly, $N' \rightarrow N'' \rightarrow N$.

454 Now, assuming N and given the formulas $\varphi_1, \dots, \varphi_n$, we will prove N'' .
 455 Consider the following formula:

$$\psi = \bigvee_{i=1}^t t = i \ \& \ \varphi_i \quad (2.47)$$

456 We will take advantage of the fact that natural numbers are defined by atomic
 457 formulas and therefore absolute in transitive structures. From N , we get such
 458 u that $Scm^{ZF}(u) \ \& \ (\forall p_1, \dots, p_n \in u)(\bigvee_{i=1}^t t = i \ \& \ \varphi_i \leftrightarrow \bigvee_{i=1}^t t = i \ \& \ \varphi_i^u)$. This
 459 already satisfies N'' .

460 In order to prove N' from N'' , let's add two more formulas. Given p_1, \dots, p_n ,
 461 we denote

$$\varphi_{m+1} = \exists u(z \in u \ \& \ Scm^{ZF}(u) \ \& \ (\forall p_1, \dots, p_n \in u)(\bigvee_{i=1}^m \varphi_i = \varphi_i^u)) \quad (2.48)$$

$$\varphi_{m+2} = \forall z \varphi_{m+1} \quad (2.49)$$

463 So, by N'' , we have a set u that satisfies $Scm^{ZF}(u)$ as well as the following:

$$(\forall p_1, \dots, p_n \in u)(\varphi_i \leftrightarrow \varphi_i^u) \text{ for } 1 \leq i \leq m \quad (2.50)$$

$$z \in u \rightarrow \varphi_{m+1} \leftrightarrow \varphi_{m+1}^u \quad (2.51)$$

$$\varphi_{m+2} \leftrightarrow \varphi_{m+2}^u \quad (2.52)$$

466 By $Scm^{ZF}(u)$ and (2.50), we get $(\forall z \in u)\varphi_{m+1}$, so together with (2.51), we get
 467 $(\forall z \in u)\varphi_{m+1}^u$, exactly φ_{m+2}^u , so by (2.52) we get φ_{m+2} . But φ_{m+2} is exactly
 468 the instance of N' we were looking for. \square

469 Definition 2.8 (N_0)

470 Axiom schema N_0 is similar to N defined above, but with S instead of ZF. For
 471 every φ , a first-order formula in the language of set theory with no free variables
 472 except p_1, \dots, p_n , the following is an instance of N_0 .

$$\exists u(Scm^S(u) \ \& \ (\forall p_1, \dots, p_n \in u)(\varphi \leftrightarrow \varphi^u)) \quad (2.53)$$

473 We will now show that in S , N_0 implies both *Replacement* and *Infinity*.

474
475 Let N_0 be defined as in (2.8), for *Infinity* see (1.10).

476 **Theorem 2.9** *In S , the axiom schema N_0 implies Infinity.*

477 *Proof.* Let $\varphi = \forall x \exists y (y = x \cup \{x\})$. This clearly holds in S because given a
478 set x , there is a set $y = x \cup \{x\}$ obtained via *Pairing* and *Union*. From N_0 ,
479 there is a set u such that φ^u holds. This u satisfies the conditions required by
480 *Infinity*. \square

481 Lévy proves this theorem in a different way. He argues that for an arbitrary
482 formula φ , N_0 gives us $\exists u Scm^S(u)$ and this u already satisfies *Infinity*. To do
483 this, we would need to prove lemma (2.15) earlier on, we will do that later in
484 this chapter.

485
486 Let S be a set theory defined in (1.19), N_0 a schema defined in (2.8) and
487 *Replacement* a schema defined in (1.16).

488 **Theorem 2.10** *In S , the axiom schema N_0 implies Replacement.*

489 *Proof.* Let $\varphi(x, y, p_1, \dots, p_n)$ be a formula with no free variables except x, y, p_1, \dots, p_n .
490 Let a set x be given and let χ be an instance of the *Replacement* schema for
491 the φ given. We want to verify in S that given a formula φ , the instance of N_0
492 for φ implies χ .

$$\begin{aligned} \chi = \forall x', y', z (\varphi(x', y', p_1, \dots, p_n) \ \& \ \varphi(x', z, p_1, \dots, p_n) \rightarrow y' = z') \\ \rightarrow \exists y \forall z (z \in y \leftrightarrow (\exists q \in x) (\varphi(x, y, p_1, \dots, p_n))) \end{aligned} \quad (2.54)$$

493 Since it can be shown that N_0 is equivalent to N'_0 similar to N' in lemma
494 (2.7), there is a set u such that $Scm^S(u)$, $x \in u$ and all of the following hold:

- 495 (i) $\varphi \leftrightarrow \varphi^u$
496 (ii) $\exists y \varphi \leftrightarrow (\exists y \varphi)^u$

497 From relativization, $(\exists y \varphi)^u$ is equivalent to $(\exists y \in u) \varphi^u$, together with (i) and
498 (ii), we get

$$(\exists y \in u) \varphi \leftrightarrow \exists y \varphi \quad (2.55)$$

499 If φ is a function, it maps the elements of x , which are also elements of u
500 due to transitivity of u , to elements of u . From *Specification*, $y = \{z \in u\}$ we
501 can find y , a set of all images of the elements of x via φ . So we have satisfied
502 *Replacement* – given a function and a set, we have proven that image of the set
503 via given function is also a set. \square

504 What we have just proven is only a single theorem from Lévy's aforementioned
505 article, we will introduce other interesting results, dealing with inaccessible and
506 Mahlo cardinals, later in their appropriate context in chapter 3.

2.2 Contemporary Restatement

We will now introduce and prove a theorem that is called Lévy's Reflection in contemporary set theory. The only difference is that while Lévy originally reflects a formula φ from V to a set u which is a *standard complete model* of S , we say that there is a V_λ for a limit λ that reflects φ . Those two conditions are equivalent due to lemma (2.15).

Lemma 2.11 *Let $\varphi_1, \dots, \varphi_n$ be first-order formulas in the language of set theory, all with m free variables¹¹.*

- (i) *For each set M_0 there is such set M that $M_0 \subset M$ and the following holds for every i , $1 \leq i \leq n$:*

$$\exists x \varphi_i(p_1, \dots, p_{m-1}, x) \rightarrow (\exists x \in M) \varphi_i(p_1, \dots, p_{m-1}, x) \quad (2.56)$$

for every $p_1, \dots, p_{m-1} \in M$.

- (ii) *Furthermore, there is a limit ordinal λ such that $M_0 \subset V_\lambda$ and the following holds for each i , $1 \leq i \leq n$:*

$$\exists x \varphi_i(p_1, \dots, p_{m-1}, x) \rightarrow (\exists x \in V_\lambda) \varphi_i(p_1, \dots, p_{m-1}, x) \quad (2.57)$$

for every $p_1, \dots, p_{m-1} \in M$.

- (iii) *Assuming Choice, there is M , $M_0 \subset M$ such that (2.56) holds for every M , $i \leq n$ and $|M| \leq |M_0| \cdot \aleph_0$.*

Proof. We will simultaneously prove statements (i) and (ii), denoting M^T the transitive set required by part (ii). Steps in the construction of M^T that are not explicitly included are equivalent to steps for M .

Let us first define an operation $H_i(p_1, \dots, p_{m-1})$ that yields the set of x 's with minimal rank¹² satisfying $\varphi_i(p_1, \dots, p_{m-1}, x)$ for p_1, \dots, p_{m-1} and for every i , $1 \leq i \leq n$.

$$H_i(p_1, \dots, p_n) = \{x \in C_i : (\forall z \in C)(\text{rank}(x) \leq \text{rank}(z))\} \quad (2.58)$$

for each $1 \leq i \leq n$, where

$$C_i = \{x : \varphi_i(p_1, \dots, p_{m-1}, x)\} \text{ for } 1 \leq i \leq n \quad (2.59)$$

¹¹For formulas with a different number of free variables, take for m the highest number of parameters among those formulas. Add spare parameters to every formula that has less than m parameters in a way that preserves the last parameter, which we will denote x . E.g. let φ'_i be the a formula with k parameters, $k < m$. Let us set $\varphi_i(p_1, \dots, p_{m-1}, x) = \varphi'_i(p_1, \dots, p_{k-1}, x)$, notice that the parameters p_k, \dots, p_{m-1} are not used.

¹²Rank is defined in (1.30).

530

531 Next, let's construct M from given M_0 by induction.

$$M_{i+1} = M_i \cup \bigcup_{j=0}^n \{H_j(p_1, \dots, p_{m-1}) : p_1, \dots, p_{m-1} \in M_i\} \quad (2.60)$$

532 In other words, in each step we include into the construction the elements sat-
 533 isfying $\varphi(p_1, \dots, p_{m-1}, x)$ for p_1, \dots, p_{m-1} from the previous step. For statement
 534 (ii), this is the only part that differs from (i). To end up with a transitive M ,
 535 we need to extend every step to it's transitive closure transitive closure of M_{i+1}
 536 from (i). In other words, let γ be the smallest ordinal such that

$$(M_i^T \cup \bigcup_{j=0}^n \{ \bigcup \{H_j(p_1, \dots, p_{m-1}) : p_1, \dots, p_{m-1} \in M_i\} \}) \subset V_\gamma \quad (2.61)$$

537 Then the incremental step is

$$M_{i+1}^T = V_\gamma \quad (2.62)$$

538 and the final M is obtained by joining the previous steps.

$$M = \bigcup_{i=0}^{\infty} M_i, \quad M^T = \bigcup_{i=0}^{\infty} M_i^T = V_\lambda \text{ for some limit } \lambda. \quad (2.63)$$

539

540 We have yet to finish part (iii). Let's try to construct a set M' that sat-
 541 isfies the same conditions like M but is kept as small as possible. Assuming
 542 the Axiom of Choice, we can modify the construction so that the cardinality
 543 of M' is at most $|M_0| \cdot \aleph_0$. Note that the size of M in the previous con-
 544 struction is determined by the size of M_0 and, most importantly, by the size of
 545 $H_i(p_1, \dots, p_{m-1})$ for every i , $1 \leq i \leq n$ in individual iterations of the construc-
 546 tion. Since (i) only ensures the existence of an x that satisfies $\varphi_i(p_1, \dots, p_{m-1}, x)$
 547 for any i , $1 \leq i \leq n$, we only need to add one x for every set of parameters but
 548 $H_i(u_1, \dots, u_{m-1})$ can be arbitrarily large. Let F be a choice function on $\mathcal{P}(M')$.
 549 Also let $h_i(p_1, \dots, p_{m-1}) = F(H_i(p_1, \dots, p_{m-1}))$ for i , where $1 \leq i \leq n$, which
 550 means that h is a function that outputs an x that satisfies $\varphi_i(p_1, \dots, p_{m-1}, x)$ for
 551 i such that $1 \leq i \leq n$ and has minimal rank among all such sets. The induction
 552 step needs to be redefined to

$$M'_{i+1} = M'_i \cup \bigcup_{j=0}^n \{h_j(p_1, \dots, p_{m-1}) : p_1, \dots, p_{m-1} \in M'_i\} \quad (2.64)$$

553 This way, the amount of elements added to M'_{i+1} in each step of the construction
 554 is the same as the amount of m -tuples of parameters that yielded elements not

555 included in M'_i . It is easy to see that if M_0 is finite, M' is countable because
 556 it was constructed as a countable union of sets that are themselves at most
 557 countable. If M_0 is countable or larger, the cardinality of M' is equal to the
 558 cardinality of M_0 .¹³ Therefore $|M'| \leq |M_0| \cdot \aleph_0$ \square

559 **Theorem 2.12** (*Lévy's first-order reflection theorem*)

560 Let $\varphi(p_1, \dots, p_n)$ be a first-order formula.

561 (i) For every set M_0 there exists a set M such that $M_0 \subset M$ and the following
 562 holds:

$$\varphi^M(p_1, \dots, p_n) \leftrightarrow \varphi(p_1, \dots, p_n) \quad (2.65)$$

563 for every $p_1, \dots, p_n \in M$.

564 (ii) For every set M_0 there is a transitive set M , $M_0 \subset M$ such that the
 565 following holds:

$$\varphi^M(p_1, \dots, p_n) \leftrightarrow \varphi(p_1, \dots, p_n) \quad (2.66)$$

566 for every $p_1, \dots, p_n \in M$.

567 (iii) For every set M_0 there is a limit ordinal λ such that $M_0 \subset V_\lambda$ and the
 568 following holds:

$$\varphi^{V_\lambda}(p_1, \dots, p_n) \leftrightarrow \varphi(p_1, \dots, p_n) \quad (2.67)$$

569 for every $p_1, \dots, p_n \in M$.

570 (iv) Assuming Choice, for every set M_0 there is M such that $M_0 \subset M$ and
 571 $|M| \leq |M_0| \cdot \aleph_0$ and the following holds:

$$\varphi^M(p_1, \dots, p_n) \leftrightarrow \varphi(p_1, \dots, p_n) \quad (2.68)$$

572 for every $p_1, \dots, p_n \in M$.

573 *Proof.* Let's now prove (i) for given φ via induction by complexity. We can safely
 574 assume that φ contains no quantifiers besides " \exists " and no logical connectives
 575 other than " \neg " and "&". Let $\varphi_1, \dots, \varphi_n$ be all subformulas of φ . Then there
 576 is a set M , obtained by the means of lemma (2.11), for all of the formulas
 577 $\varphi_1, \dots, \varphi_n$.

578 Let's first consider atomic formulas in the form of either $x_1 = x_2$ or $x_1 \in x_2$.
 579 It is clear from relativisation¹⁴ that (2.65) holds for both cases, $(x_1 = x_2)^M \leftrightarrow$
 580 $(x_1 = x_2)$ and $(x_1 \in x_2)^M \leftrightarrow (x_1 \in x_2)$.
 581

¹³It can not be smaller because $|M'_{i+1}| \geq |M'_i|$ for every i . It may not be significantly larger because the maximum of elements added is the number of n -tuples in M'_i , which is of the same cardinality as M'_i .

¹⁴See (1.41.). This only holds for relativization to $M, \in \cap M \times M$, not M, R for an arbitrary R .

582 We now want to verify the inductive step. First, take $\varphi = \neg\varphi'$. From
 583 relativization, we get $(\neg\varphi')^M \leftrightarrow \neg(\varphi'^M)$. Because the induction hypothesis tells
 584 us that $\varphi'^M \leftrightarrow \varphi'$, the following holds:

$$(\neg\varphi')^M \leftrightarrow \neg(\varphi'^M) \leftrightarrow \neg\varphi' \quad (2.69)$$

585 The same holds for $\varphi = \varphi_1 \ \& \ \varphi_2$. From the induction hypothesis, we know
 586 that $\varphi_1^M \leftrightarrow \varphi_1$ and $\varphi_2^M \leftrightarrow \varphi_2$, which together with relativization for formulas in
 587 the form of $\varphi_1 \ \& \ \varphi_2$ gives us

$$(\varphi_1 \ \& \ \varphi_2)^M \leftrightarrow \varphi_1^M \ \& \ \varphi_2^M \leftrightarrow \varphi_1 \ \& \ \varphi_2 \quad (2.70)$$

588 Let's now examine the case when $\varphi = \exists x\varphi'(p_1, \dots, p_n, x)$. The induction
 589 hypothesis tells us that $\varphi'^M(p_1, \dots, p_n, x) \leftrightarrow \varphi'(p_1, \dots, p_n, x)$, so, together with
 590 above lemma (2.11), the following holds:

$$\begin{aligned} & \varphi(p_1, \dots, p_n, x) \\ & \leftrightarrow \exists x\varphi'(p_1, \dots, p_n, x) \\ & \leftrightarrow (\exists x \in M)\varphi'(p_1, \dots, p_n, x) \\ & \leftrightarrow (\exists x \in M)\varphi'^M(p_1, \dots, p_n, x) \\ & \leftrightarrow (\exists x\varphi'(p_1, \dots, p_n, x))^M \\ & \leftrightarrow \varphi^M(p_1, \dots, p_n, x) \end{aligned} \quad (2.71)$$

591 Which is what we wanted to prove for part (i).

592
 593 We now need to verify that the same holds for any finite number of formulas
 594 $\varphi_1, \dots, \varphi_n$. This has in fact been already done since lemma (2.11) gives us a set
 595 M for any finite amount of formulas and given M_0 . We can therefore find a set
 596 M for the union of all of their subformulas. When we obtain such M , it should
 597 be clear that it also reflects every formula in $\varphi_1, \dots, \varphi_n$.

598
 599 Since V_λ is a transitive set, by proving (iii) we also satisfy (ii). To do so, we
 600 only need to look at part (ii) of lemma (2.11). All of the above proof also holds
 601 for $M = V_{\lambda \text{ambda}}$.

602 To finish part (iv), we take M of size $\leq |M_0| \cdot \aleph_0$, which exists due to part
 603 (iii) of lemma (2.11), the rest being identical. \square

604
 605 Let S be a set theory defined in (1.19), for ZFC see definition (1.21).

606 The two following lemmas are based on [Drake, 1974][Chapter 3, Theorem
 607 1.2].

608 **Lemma 2.13** *If M is a transitive set, then $\langle M, \in \rangle \models$ Extensionality.*

609 *Proof.* Given a transitive set M , we want to show that the following holds.

$$\langle M, \in \rangle \models \forall x, y (x = y \leftrightarrow \forall z (z \in x \leftrightarrow z \in y)) \quad (2.72)$$

610 Given arbitrary $x, y \in M$, we want to prove that $\langle M, \in \rangle \models (x = y \leftrightarrow \forall z (z \in$
 611 $x \leftrightarrow z \in y))$. This is equivalent to $\langle M, \in \rangle \models x = y$ iff $\langle M, \in \rangle \models \forall z (z \in$
 612 $x \leftrightarrow z \in y)$, which is the same as $x = y$ iff $\langle M, \in \rangle \models \forall z (z \in x \leftrightarrow z \in y)$.

613 So all elements of x are also elements of y in M , and vice versa. Because M is
 614 transitive, all elements of x and y are in M , so $\langle M, \in \rangle \models \forall z (z \in x \leftrightarrow z \in y)$
 615 holds iff x and y contain the same elements and are therefore equal. \square

616 **Lemma 2.14** *If M is a transitive set, then $\langle M, \in \rangle \models \text{Foundation}$.*

617 *Proof.* We want to prove the following:

$$\langle M, \in \rangle \models \forall x (x \neq \emptyset \rightarrow (\exists y \in x)(x \cap y = \emptyset)) \quad (2.73)$$

618 Given an arbitrary non-empty $x \in M$ let's show that $\langle M, \in \rangle \models (\exists y \in$
 619 $x)(x \cap y = \emptyset)$.

620 Because M is transitive, every element of x is an element of M . Take for
 621 y the element of x with the lowest rank¹⁵. It should be clear that there is no
 622 $z \in y$ such that $z \in x$, because then $\text{rank}(z) < \text{rank}(y)$, which would be a
 623 contradiction. \square

624 Let S be a set theory as defined in (1.19).

625 **Lemma 2.15** *The following holds for every λ .*

$$"\lambda \text{ is a limit ordinal}" \rightarrow \langle V_\lambda, \in \rangle \models S \quad (2.74)$$

626 *Proof.* Given an arbitrary limit ordinal λ , we will verify the axioms of S one by
 627 one.

628 (i) *The existence of a set* comes from the fact that V_λ is a non-empty set
 629 because limit ordinal is non-zero by definition.

630 (ii) *Extensionality* holds from (2.13).

631 (iii) *Foundation* holds from (2.14).

632 (iv) *Union*:

633 Given any $x \in V_\lambda$, we want verify that $y = \bigcup x$ is also in V_λ . Note that
 634 $y = \bigcup x$ is a Δ_0 -formula.

$$y = \bigcup x \leftrightarrow (\forall z \in y)(\exists q \in x)z \in q \ \& \ (\forall z \in x)(\forall q \in z)q \in y \quad (2.75)$$

635 So by lemma (1.45)

$$y = \bigcup x \leftrightarrow \langle V_\lambda, \in \rangle \models y = \bigcup x \quad (2.76)$$

¹⁵Rank is defined in (1.30).

636 (v) *Pairing*:

637 Given two sets $x, y \in V_\lambda$, we want to show that $z = \{x, y\}$ is also an
638 element of V_λ .

$$z = \{x, y\} \leftrightarrow x \in z \ \& \ y \in z \ \& \ (\forall q \in z)(q = x \vee q = y) \quad (2.77)$$

639 So $z = \{x, y\}$ is a Δ_0 -formula, and thus by lemma (1.45) it holds that

$$z = \{x, y\} \leftrightarrow \langle V_\lambda, \in \rangle \models z = \{x, y\} \quad (2.78)$$

640 (vi) *Powerset*:

641 Given any $x \in V_\lambda$, we want to make sure that $\mathcal{P}(x) \in V_\lambda$. Let $\varphi(y)$ denote
642 the formula $y \in \mathcal{P}(x) \leftrightarrow y \subset x$. according to definition of subset (1.4),
643 $y \subset x$ is Δ_0 , so for any given $x, y \in V_\lambda$, $y = \mathcal{P}(x) \leftrightarrow \langle V_\lambda, \in \rangle \models y =$
644 $\mathcal{P}(x)$. Because λ is limit and $\text{rank}(\mathcal{P}(x)) = \text{rank}(x) + 1$, if $\mathcal{P}(x) \in V_\lambda$
645 for every $x \in V_\lambda$.

646 (vii) *Specification*:

647 Given a first-order formula φ , we want to show the following:

$$\langle V_\lambda, \in \rangle \models \forall x, p_1, \dots, p_n, \exists y \forall z (z \in y \leftrightarrow z \in x \ \& \ \varphi(z, p_1, \dots, p_n)) \quad (2.79)$$

648 Given any x along with parameters p_1, \dots, p_n in V_λ , we set

$$y = \{z \in x : \varphi^{V_\lambda}(z, p_1, \dots, p_n)\} \quad (2.80)$$

649 From transitivity of V_λ and the fact that $y \subset x$ and $x \in V_\lambda$, we know that
650 $y \in V_\lambda$, so $\langle V_\lambda, \in \rangle \models \forall z (z \in y \leftrightarrow z \in x \ \& \ \varphi(z, p_1, \dots, p_n))$.
651 □

652 **Definition 2.16** (*First-Order Reflection Schema*)

653 For every first-order formula φ , the following is an axiom:

$$\forall M_0 \exists M (M_0 \subseteq M \ \& \ (\varphi(p_1, \dots, p_n) \leftrightarrow \varphi(p_1, \dots, p_n)^M)) \quad (2.81)$$

654 We will refer to this axiom schema as First-order reflection.

655 Let *Infinity* and *Replacement* be as defined in (1.10) and (1.16) respectively.

656 **Theorem 2.17** First-order reflection is equivalent to Infinity & Replacement
657 under S.

658 *Proof.* Since (2.12) already gives us one side of the implication, we are only
659 interested in showing the converse which we shall do in two parts:

660 *First-order reflection* \rightarrow *Infinity* This is done exactly like (2.9). We pick for
 661 φ the formula $(\forall y \in x)(y \cup \{y\} \in x)$, $M_0 = \{\emptyset\}$. From (2.16), there is a set M
 662 that satisfies φ , so there is an inductive set. We have picked M_0 so that $\emptyset \in M$
 663 obviously holds and M is the witness for

$$\exists x(\emptyset \in x \ \& \ (\forall y \in x)(y \cup \{y\} \in x)) \quad (2.82)$$

664 which is exactly (1.10).

665 *First-order reflection* \rightarrow *Replacement*

666 Let's first point out that while *First-order reflection* gives us a set for one
 667 formula, we can generalize it to hold for any finite number of formulas. We will
 668 show how it is done for two formulas, which is what we will use in this proof.
 669 Given two first-order formulas φ, ψ , we can suppose that there are formulas φ'
 670 and ψ' that are equivalent to φ and ψ respectively, but their free variables are
 671 different¹⁶. Let $\xi = \varphi \ \& \ \psi$, given any M_0 , we can find a M such that $\xi \leftrightarrow \xi^M$.
 672 It is easy to see that from relativisation, the following holds:
 673

$$\varphi \ \& \ \psi \leftrightarrow \varphi' \ \& \ \psi' \leftrightarrow \xi \leftrightarrow \xi^M \leftrightarrow (\varphi' \ \& \ \psi')^M \leftrightarrow \varphi'^M \ \& \ \psi'^M \leftrightarrow \varphi^M \ \& \ \psi^M \quad (2.83)$$

674 Now given a function $\varphi(x, y)$, we know from *First-order reflection* that for
 675 every M_0 , there is a set M such that $M_0 \subseteq M$ and both

$$(\forall x, y \in M)(\varphi(x, y) \leftrightarrow \varphi^M(x, y)) \quad (2.84)$$

676 and

$$(\forall x, y \in M)(\exists y \varphi(x, y) \leftrightarrow (\exists y \varphi(x, y))^M) \quad (2.85)$$

677 hold, the latter being equivalent to

$$(\forall x, y \in M)(\exists y \varphi(x, y) \leftrightarrow (\exists y \in M) \varphi^M(x, y)) \quad (2.86)$$

678 Therefore

$$(\forall x, y \in M)(\exists y \varphi(x, y) \leftrightarrow (\exists y \in M) \varphi(x, y)) \quad (2.87)$$

679 holds too. That means that we have a set M such that for every $x \in M$, if φ is
 680 defined for x , $(\exists y \in M) \varphi(x, y)$.

681 To show that *Replacement* holds for this particular φ , we need to verify that
 682 given a set M_0 , $M'_0 = \{y : (\exists x \in M_0) \varphi(x, y)\}$ is also a set. But since $M_0 \subseteq M$

¹⁶This is plausible since we can for example substitute all free variables in φ' for x_0, x_2, x_4, \dots and use x_1, x_3, x_5, \dots for free variables in ψ' , the resulting formulas will be equivalent.

and because given any $x \in M$, there is $y \in M$ satisfying $\varphi(x, y)$, the following is a set due to *Specification*:

$$M'_0 = \{y : (\exists x \in M_0)\varphi(x, y)\} = \{y \in M : (\exists x \in M_0)\varphi(x, y)\} \quad (2.88)$$

□

We have shown that *Reflection* for first-order formulas, *First-order reflection* is a theorem of ZFC. We have also shown that it can be used instead of the *Infinity* and *Replacement* scheme, but $\text{ZFC} + \text{First-order reflection}$ is a conservative extension of ZF. Besides being a starting point for more general and powerful statements, it can be used to show that ZF is not finitely axiomatizable. This follows from the fact that *Reflection* gives a model to any consistent finite set of formulas. So if $\varphi_1, \dots, \varphi_n$ would be the axioms of ZFC, *Reflection* would prove that every model of ZFC contains a smaller model of ZFC, which would in turn contradict the Second Gödel's Theorem¹⁷.

It is also worthwhile to note that, in a way, *Reflection* is dual to compactness. Compactness says that given a set of sentences, if every finite subset yields a model, so does the whole set. *Reflection*, on the other hand, says that while the whole set has no model in the underlying theory, every finite subset has a model.

Furthermore, *Reflection* can be used in ways similar to upward Löwenheim–Skolem theorem. Since *Reflection* extends any set M_0 into a model of given formulas $\varphi_1, \dots, \varphi_n$, we can choose the lower bound of the size of M by appropriately choosing M_0 .

In the next section, we will try to generalize *Reflection* in a way that transcends ZF and yields some large cardinals.

¹⁷See chapter ?? for further details.

3 Reflection And Large Cardinals

3.1 Regular Fixed-Point Axioms

Lévy's article mentions various schemata that are not instances of reflection per se, but deal with fixed points of normal ordinal functions. We will introduce them and show that they are equivalent to *First-Order Reflection*¹⁸.

Lemma 3.1 (*Fixed-point lemma for normal functions*)

Let f be a normal function defined for all ordinals¹⁹. Then all of the following hold:

- (i) $\forall \lambda (\text{"}\lambda \text{ is a limit ordinal"} \rightarrow \text{"}f(\lambda) \text{ is a limit ordinal"})$
- (ii) $\forall \alpha (\alpha \leq f(\alpha))$
- (iii) $\forall \alpha \exists \beta (\alpha < \beta \ \& \ f(\beta) = \beta)$
- (iv) *The fixed points of f form a closed unbounded class.*²⁰

Proof. Let f be a normal function defined for all ordinals.

(i) Suppose λ is a limit ordinal. For an arbitrary ordinal $\alpha < \lambda$, the fact that f is strictly increasing means that $f(\alpha) < f(\lambda)$ and for any ordinal β , satisfying $\alpha < \beta < \lambda$, $f(\alpha) < f(\beta) < f(\lambda)$. We know that there is such β from limitness of λ . Because f is continuous and λ is limit, $f(\lambda) = \bigcup_{\gamma < \lambda} f(\gamma)$. That means that if λ is limit, so is $f(\lambda)$.

(ii) This step will be proven using the transfinite induction. Since f is defined for all ordinals, there is an ordinal α such that $f(\emptyset) = \alpha$ and because \emptyset is the least ordinal, (ii) holds for \emptyset .

Suppose (ii) holds for some β from the induction hypothesis. It the holds for $\beta + 1$ because f is strictly increasing.

For a limit ordinal λ , suppose (ii) holds for every $\alpha < \lambda$. (i) implies that $f(\lambda)$ is also limit, so there is a strictly increasing κ -sequence $\langle \alpha_0, \alpha_1, \dots \rangle$ for some κ such that $\lambda = \bigcup_{i < \kappa} \alpha_i$. Because f is stricly increasing, the κ -sequence $\langle f(\alpha_0), f(\alpha_1), \dots \rangle$ is also strictly increasing, the induction hypothesis implies that $\alpha_i \leq f(\alpha_i)$ for each $i \leq \kappa$. Thus, $\lambda \leq f(\lambda)$.

(iii) For a given ordinal α , let there be an ω -sequence $\langle \alpha_0, \alpha_1, \dots \rangle$, such that $\alpha_0 = \alpha$ and $\alpha_{i+1} = f(\alpha_i)$ for each $i < \omega$. This sequence is stricly increasing because so is f . Now, there's a limit ordinal $\beta = \bigcup_{i < \omega} \alpha_i$, we want to show that this is the fixed point. So $f(\beta) = f(\bigcup_{i < \omega} \alpha_i) = \bigcup_{i < \omega} f(\alpha_i)$ because f is continuous. We have defined the above sequence so that $\beta, \bigcup_{i < \omega} f(\alpha_i) = \bigcup_{i < \omega} \alpha_{i+1}$, which means we are done, since $\bigcup_{i < \omega} \alpha_{i+1} = \bigcup_{i < \omega} \alpha_i = \beta$.

¹⁸For definition, see (2.16).

¹⁹For the definition of normal function, see (1.50).

²⁰See (??.) for the definition of closed class, (1.52) for the definition of unboundedness.

(iv) The class of fixed points of f is obviously unbounded by (iii). It remains to show that it is closed, this is based on [Drake, 1974], chapter 4. Let Y be a non-empty set of fixed points of f such that $\bigcup Y \notin Y$. Since f is defined on ordinals, Y is a set of ordinals, so $\bigcup Y$ is an ordinal because a supremum of a set of ordinals is an ordinal. $\bigcup Y$ is a limit ordinal. If it were a successor ordinal, suppose that $\alpha + 1 = \bigcup Y$, then $\alpha \in \bigcup Y$, which means that there is some x such that $\alpha \in x \in Y$. But the least such x is $\alpha + 1$, so $\bigcup Y \in Y$.
 Note that $\alpha < \bigcup Y$ iff $\exists \xi \in Y (\alpha < \xi)$. Since f is defined for all ordinals and $\bigcup Y$ is a limit ordinal, $f(\bigcup Y) = \bigcup_{\alpha \in Y} f(\alpha)$, but because Y is a set of fixed points of f , $f(\bigcup Y) = \bigcup_{\alpha \in Y} f(\alpha) = \bigcup Y$, so $\bigcup Y$ is also a limit point of Y .

□

Lemma 3.2 *Let α be a limit ordinal. Then the following hold:*

- (i) *If C is a club set in α , then there is an ordinal β and a normal function $f : \beta \rightarrow \alpha$ such that $\text{rng}(f) = C$. We say that f enumerates C .*
- (ii) *If β is an ordinal and f is a normal function such that $f : \beta \rightarrow \alpha$ and $\text{rng}(f)$ is unbounded in α , then $\text{rng}(f)$ is a closed unbounded set in α .*

This proof comes from (<http://euclid.colorado.edu/~monkd/m6730/gradsets09.pdf> TODO cite!) *Proof.*

- (i) Let β be the order-type²¹ of C , let f be the isomorphism from β onto C . Since $C \subseteq \alpha$, f is also an increasing function from β into α . In order to be continuous, let γ be a limit ordinal under β , let $\epsilon = \bigcup_{\delta < \gamma} f(\delta)$. We want to verify that $f(\gamma) = \epsilon$. Since ϵ is a limit ordinal, we only need to show that $C \cap \epsilon$ is unbounded in ϵ .
 Take $\zeta < \epsilon$. Then there is a $\delta < \gamma$ such that $\zeta < f(\delta)$. Since γ is limit, $\delta + 1 < \gamma$ and also $f(\delta + 1) < f(\gamma)$, we know that $f(\delta) \in C \cap \epsilon$. But that means that $C \cap \epsilon$ is unbounded in ϵ , so $\epsilon \in C$. We have also shown that ϵ is closed unbounded in the image of γ over f . Therefore, $f(\gamma) = \epsilon = \bigcup_{\delta < \gamma} f(\delta)$, so f is normal.

- (ii) TODO (potrebuj to?)

□ It

should be clear that while this lemma works with club subsets of an ordinal, we can formulate analogous statement for club classes, which then yields a normal function defined for all ordinals, with the only exception that there is no such β because f is a function from Ord to Ord .

²¹See definition (1.31).

777 **Definition 3.3** (Axiom Schema M_1)

778 “Every normal function defined for all ordinals has at least one inaccessible num-
779 ber in its range.”

780 Lévy uses “ M ” to refer to this axiom but since we also use “ M ” for sets and
781 models, for example in (2.16), we will call the above axiom “Axiom Schema M_1 ”
782 to avoid confusion.

783 Let $\varphi(x, y, p_1, \dots, p_n)$ be a first-order formula with no free variables besides
784 x, y, p_1, \dots, p_n . The following is equivalent to Axiom M_1 .

$$\begin{aligned} & \text{“}\varphi \text{ is a normal function”} \ \& \ \forall x(x \in \text{Ord} \rightarrow \exists y(\varphi(x, y, p_1, \dots, p_n))) \rightarrow \\ & \rightarrow \exists y(\exists x \varphi(x, y, p_1, \dots, p_n) \ \& \ cf(y) = y \ \& \ (\forall x \in \kappa)(\exists y \in \kappa)(x > y)) \end{aligned} \quad (3.89)$$

785 **Definition 3.4** (Axiom Schema M_2)

786 “Every normal function defined for all ordinals has at least one fixed point which
787 is inaccessible.”

788 **Definition 3.5** (Axiom Schema M_3)

789 “Every normal function defined for all ordinals has arbitrarily great fixed points
790 which are inaccessible.”

791 Similar axiom is proposed in [Drake, 1974].

792 **Definition 3.6** (Axiom Schema F)

793 “Every normal function has a regular fixed point.”

794 **Lemma 3.7** Let f be a normal function defined for all ordinals.

- 795 (i) There is a normal function g_1 defined for all ordinals that enumerates the
796 class $\{\alpha : f(\alpha) = \alpha \ \& \ \alpha \in \text{Ord}\}$.
797 (ii) There is a normal function g_2 defined for all ordinals that enumerates the
798 class $\{\lambda : "f(\lambda) \text{ is a strong limit cardinal."}\}$.

799 *Proof.* We know that (ii) holds from lemma (3.1) and lemma (3.2).

800 For (i), It should be clear that there is no largest strong limit ordinal ν ,
801 because the limit of $\nu, \mathcal{P}(\nu), \mathcal{P}(\mathcal{P}(\nu)), \dots$ is again a limit ordinal. The class of
802 limit ordinals is closed because a limit of strong limit ordinals is clearly always a
803 strong limit ordinal. Let h be a function enumerating limit ordinals which exists
804 from lemma (3.2). Then $g_1(\alpha) = f(h(\alpha))$ for every ordinal α is normal and
805 defined for all ordinals. \square

806 The following is *Theorem 1* in [Lévy, 1960], the parts dealing with Axiom
807 Schema F come from [Drake, 1974].

808 **Theorem 3.8** *The following are all equivalent:*

- 809 (i) Axiom Schema M_1
- 810 (ii) Axiom Schema M_2
- 811 (iii) Axiom Schema M_3
- 812 (iv) Axiom Schema F

813 *Proof.* It is clear that *Axiom Schema M_3* is a stronger version of *Axiom Schema*
 814 *M_2* , which is in turn a stronger version of both *Axiom Schema M_1* and *Axiom*
 815 *Schema F_1* .

816 We will now prove that *Axiom Schema $F \rightarrow$ Axiom Schema M_2* . Lemma
 817 (3.7) tells us that given a normal function f defined for all ordinals, there is a
 818 normal function g_1 defined for all ordinals that enumerates the fixed-points of f .
 819 There is also a function g_2 that enumerates the strong limit ordinals in $\text{rng}(f)$.
 820 By *Axiom Schema F* , g_2 has a regular fixed-point κ , which is also a strong limit
 821 ordinal, so

$$f(\kappa) = g_2(\kappa) = \kappa \text{ and } \kappa \text{ is inaccessible.} \quad (3.90)$$

822 So every normal function d.f.a.o. has a regular fixed-point.

823 We have yet to show *Axiom Schema $M_1 \rightarrow$ Axiom Schema M_3* . Again by
 824 lemma (3.7), there is a normal function g defined for all ordinals that enumerates
 825 the fixed points of f . Let $h_\alpha(\beta) = g(\alpha + \beta)$ for any given ordinal α , then h_α
 826 is a normal function defined for all ordinals. Then, given an arbitrary α , from
 827 *Axiom Schema M_1* , there is a β such that $\gamma = h_\alpha(\beta)$ is inaccessible. Because
 828 $\gamma = g(\alpha + \beta)$, $f(\gamma) = \gamma$. Since $\alpha \leq f'(\alpha)$ for any ordinal α and any normal
 829 function f' , we know that $\alpha \leq \alpha + \gamma \leq \gamma$, so γ is inaccessible and arbitrarily
 830 large, depending on the choice of α . \square

831 But how do those schemata relate to reflection? Let's introduce a stronger
 832 version of *First-order reflection schema* from the previous chapter to see it more
 833 clearly. But in order to do this, we must establish the inaccessible cardinal first.

834 3.2 Inaccessible Cardinal

835 **Definition 3.9** *An uncountable cardinal κ is inaccessible iff it is regular and*
 836 *strongly limit. We write $In(\kappa)$ to say that κ is an inaccessible cardinal.*

837 An uncountable cardinal that is regular and limit is called a *weakly inaccessible*
 838 *cardinal*, we will only use the (strongly) inaccessible cardinal, but most of the
 839 results are similar for weakly inaccessible, including higher types of ordinals that
 840 will be presented later in this chapter.

841 **Theorem 3.10** *Let κ be an inaccessible cardinal.*

$$\langle V_\kappa, \in \rangle \models \text{ZFC} \quad (3.91)$$

842 We will prove this theorem in a way similar to [Kanamori, 2003]. *Proof.* Most
 843 of this is already done in lemma (2.15), we only need to verify that *Replacement*
 844 and *Infinity* axioms hold in V_κ .

845 *Infinity* holds because κ is uncountable, so $\omega \in V_\kappa$.

846 To verify *Replacement*, let x be an element of V_κ and f a function from x to
 847 V_κ . Let $y = \{z \in V_\kappa : (\exists q \in x)f(q) = z\}$, so $y \subset V_\kappa$, it remains to show that
 848 $y \in V_\kappa$. Because f is a function, we know that $|y| \leq |x| \leq \kappa$. But since κ
 849 is regular, $\{rank(z) : z \in y\} \subseteq \alpha$ for some $\alpha < \kappa$, and so $x \in V_{\alpha+1} \subseteq V_\kappa$.
 850 Therefore $y \in V_\kappa$. \square

851 **Definition 3.11** (*Inaccessible Reflection Schema*)

852 For every first-order formula φ , the following is an axiom:

$$\forall M_0 \exists \kappa (M_0 \subseteq V_\kappa \ \& \ In(\kappa) \ \& \ (\varphi(p_1, \dots, p_n) \leftrightarrow \varphi(p_1, \dots, p_n)^{V_\kappa})) \quad (3.92)$$

853 We will refer to this axiom schema as Inaccessible reflection schema.

854 We have added the requirement that α is inaccessible, which trivially means
 855 that there is an inaccessible cardinal. By taking appropriate M_0 , it can be shown
 856 that in a theory that includes the *Inaccessible reflection schema*, there is a closed
 857 unbounded class of inaccessible cardinals. Since we know that for an inaccessible
 858 κ , V_κ is a model of ZFC, *Inaccessible reflection schema* is equivalent to

$$\forall M_0 \exists \kappa (M_0 \subseteq V_\kappa \ \& \ \langle V_\kappa, \in \rangle \models \text{ZFC} \ \& \ (\varphi(p_1, \dots, p_n) \leftrightarrow \varphi(p_1, \dots, p_n)^{V_\kappa})) \quad (3.93)$$

859 because we have proven in the last section that for an inaccessible κ , $\langle V_\kappa, \in$
 860 $\rangle \models \text{ZFC}$.

861 **Theorem 3.12** Inaccessible reflection schema is equivalent to Axiom schema
 862 F .

863 This is *Theorem 4.1* in chapter four of [Drake, 1974], also equivalent to
 864 *Theorem 3* in [?]. *Proof.* Let's start by showing that *Inaccessible reflection*
 865 *schema* implies *Axiom schema F*. It should be clear that we can reflect two
 866 formulas to a single set, just form a new formula as a conjunction of universal
 867 closures of the two.

868 Given a normal function f defined for all ordinals, we want to show that it
 869 has a regular fixed point. For any ordinal α , there is an ordinal κ such that

$$\alpha < \kappa \ \& \ In(\kappa) \ \& \ (\forall \gamma, \delta \in V_\kappa)(f(\gamma) = \delta \leftrightarrow (f(\gamma) = \delta)^{V_\kappa}) \quad (3.94)$$

870 and

$$\alpha < \kappa \ \& \ In(\kappa) \ \& \ \forall \gamma \exists \delta (f(\gamma) = \delta) \leftrightarrow (\forall \gamma \exists \delta f(\gamma) = \delta)^{V_\kappa} \quad (3.95)$$

871 Since V_κ is the set of all sets of rank less than κ and since every ordinal is the
872 rank of itself, there is an inaccessible ordinal κ such that

$$\forall \gamma < \kappa \exists \delta < \kappa (f^{V_\kappa}(\gamma) = \delta) \quad (3.96)$$

873 We also know that $f(\gamma) = \delta \leftrightarrow (f(\gamma) = \delta)^{V_\kappa}$. Now since κ is a limit ordinal
874 and f is continuous we get

$$f(\kappa) = \bigcup_{\gamma < \kappa} f^{V_\kappa}(\gamma) = \bigcup_{\gamma < \kappa} f(\gamma). \quad (3.97)$$

875 From (3.96) and the fact that f is increasing, we know that $\kappa \leq \bigcup_{\gamma < \kappa} f(\gamma) \leq \kappa$.
876 Therefore κ is an inaccessible fixed point of f .

877 For the opposite direction, it suffices to show that since there is an inacces-
878 sible cardinal from *Axiom schema F*, given a first-order formula φ , there is an
879 arbitrarily large inaccessible cardinal κ for which

$$\varphi \leftrightarrow \langle V_\kappa, \in \rangle \models \varphi. \quad (3.98)$$

880 Note that the arbitrary size of κ means given an arbitrary ordinal α , there is a
881 κ satisfying (3.98). In the previous chapter, in theorem (2.12), we have shown
882 that we can easily obtain a limit ordinal satisfying (3.98). Note that since for
883 any set M_0 , there is such α that $M_0 \subseteq V_\alpha$, there is a closed unbounded class of
884 sets satisfying (3.98), which are levels in the cumulative hierarchy, so there is a
885 club sets of κ s satisfying (3.98).

886 Let f be a normal function defined for all ordinals that enumerates this club
887 class, there is such by lemma (3.2). Let g be the function that enumerates
888 strong limit ordinals in $\text{rng}(f)$. Then g has a regular fixed point κ , which is also
889 a regular fixed point of f , so (3.98) holds for κ .

890 □

891 Definition 3.13 (ZMC)

892 We will call ZMC an axiomatic set theory that contains all axioms and schemas
893 of ZFC together with Axiom Schema M_1 .

894 We have decided to call it ZMC, because Lévy uses ZM, derived from ZF, which
895 is more intuitive, but we also need the axiom of choice, thus, ZMC.

896 3.3 Mahlo Cardinals

897 We have shown that ZMC contains arbitrarily large inaccessible cardinals. To
898 return to reflection-style argument, is there a set that satisfies this property? To
899 be able to properly answer this question, we have to formulate the notion of

"containing arbitrarily large cardinals" more carefully. While we have previously used club sets, this is not an option because inaccessible don't form a club class in ZMC^{22} , we could try to formulate stronger versions of *Axiom Schema M_1* .

Let's shortly review what *Axiom Schema M_1* says. We have shown earlier in this chapter that there is a simple relation between normal function defined for all ordinals and closed unbounded classes. So by saying that for a class of ordinals C , a normal function f has at least one element of C in its range, we say that C is stationary. Or, as Drake puts it for C , the class of inaccessible cardinals, and a κ , in which C is stationary:

"The class of inaccessible cardinals is so rich that there are members κ of the class such that no normal function on κ can avoid this class; however we climb through κ , provided we are continuous at limits (so that we are enumerating a closed subset of κ), we shall eventually have to hit an inaccessible."

Definition 3.14 (*Mahlo Cardinal*)

We say that κ is a Mahlo Cardinal iff it is an inaccessible cardinal and the set $\{\lambda < \kappa : \lambda \text{ is inaccessible}\}$ is stationary in κ .

Alternatively, κ is Mahlo iff $\langle V_\kappa, \in \rangle \models ZMC$ as shown above, this is also sometimes written as *Ord is Mahlo*. There are also *weakly Mahlo cardinals*, that are defined via weakly inaccessible cardinal below them, Mahlo cardinals are then also called *strongly Mahlo* to highlight the difference, but we will only use the term *Mahlo cardinal*.

Mahlo cardinals are related to reflection principles in an interesting way. Note that given a formula φ , reflection gave us a club set of ordinals α such that V_α reflects φ , all below the first inaccessible cardinal. We have then used a different reflection schema to obtain arbitrarily high inaccessible cardinals. Now we have a cardinal in which this reflection schema holds, so we're in fact reflecting reflection. Beware that this is done rather informally, because *Axiom Schema M_1* is a countable set of axioms, which is too large to be reflected via the schemas introduced so far. One way to deal with this would be to extend reflection for second- and possibly higher-order formulas, but we would have to be very careful with the notion of satisfaction. For now, explore where can stationary sets take us because as we have shown, their connection to reflection is quite clear.

What would happen if we strengthened *Axiom Schema M_1* to say that every normal function has a Mahlo cardinal in its range?

Definition 3.15 (*hyper-Mahlo cardinal*)

We say that κ is a hyper-Mahlo cardinal iff it is inaccessible and the set $\{\lambda < \kappa : \lambda \text{ is Mahlo}\}$ is stationary in κ .

²²Note that cofinality of the limit of the first ω inaccessibles is ω , which makes it singular.

938 **Definition 3.16** (*hyper-hyper-Mahlo cardinal*)

939 We say that κ is a hyper-hyper-Mahlo cardinal iff it is inaccessible and the set
 940 $\{\lambda < \kappa : \lambda \text{ is hyper-Mahlo}\}$ is stationary in κ .

941 It is clear that one can continue in this direction, but the nomenclature gets
 942 increasingly overwhelming even if we introduce *hyper ^{α} -Mahlo cardinals*.

943 TODO Mahlo operation

⁹⁴⁴ **4 Conclusion**

References

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