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4 REFLECTION PRINCIPLES AND LARGE
5 CARDINALS
6 Bakalářská práce

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¹⁰ Prohlašuji, že jsem bakalářskou práci vypracoval samostatně a že jsem uvedl
¹¹ všechny použité prameny a literaturu.

¹² V Praze 22. května 2016

Abstract

Práce zkoumá vztah tzv. principů reflexe a velkých kardinálů. Lévy ukázal, že v ZFC platí tzv. věta o reflexi a dokonce, že věta o reflexi je ekvivalentní schématu nahrazení a axiomu nekonečna nad teorií ZFC bez axiomu nekonečna a schématu nahrazení. Tedy lze na větu o reflexi pohlížet jako na svého druhu axiom nekonečna. Práce zkoumá do jaké míry a jakým způsobem lze větu o reflexi zobecnit a jaký to má vliv na existenci tzv. velkých kardinálů. Práce definuje nedosažitelné, Mahlovy a nepopsatelné kardinály a ukáže, jak je lze zavést pomocí reflexe. Přirozenou limitou kardinálů získaných reflexí jsou kardinály nekonzistentní s L. Práce nabídne intuitivní zdůvodnění, proč tomu tak je.

Abstract

This thesis aims to examine relations between the so called Reflection Principles and Large cardinals. Lévy has shown that the Reflection Theorem is a sound theorem of ZF and it is equivalent to the Replacement Scheme and the Axiom of Infinity. From this point of view, Reflection theorem can be seen a specific version of an Axiom of Infinity. This paper aims to examine the Reflection Principle and its generalisations with respect to the existence of Large Cardinals. This thesis will establish the Inaccessible, Mahlo and Indescribable cardinals and show how can those be defined via reflection. A natural limit of Large Cardinals obtained via reflection are cardinals inconsistent with L. This thesis will offer an intuitive explanation of why this holds.

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1 Introduction

1.1 Motivation and Origin

“The Universe of sets cannot be uniquely characterised (i. e. distinguished from all its initial elements) by any internal structural property of the membership relation in it, which is expressible in any logic of finite or transfinite type, including infinitary logics of any cardinal order.”

— Kurt Gödel [Wang, 1997]

1.2 Notation and Terminology

1.2.1 The Language of Set Theory

This text assumes the knowledge of basic terminology and some results from first-order predicate logic, see [Hamilton, 1988]. We won’t introduce the notions of *language*, *function symbol*, *predicate*, *term*, *model* and *interpretation* that are used in (1.42).

All proofs are based on [Jech, 2006] unless explicitly stated otherwise. Notable amount of inspiration is also drawn from [Kanamori, 2003] and [Drake, 1974].

We will now shortly review the basic notions that allow us to define the *Zermelo-Fraenkel* set theory.

When we talk about a *class*, we have the notion of a definable class in mind. If $\varphi(x, p_1, \dots, p_n)$ is a formula in the language of set theory, we call

$$A = \{x : \varphi(x, p_1, \dots, p_n)\} \quad (1.1)$$

a class of all sets satisfying $\varphi(x, p_1, \dots, p_n)$ in a sense that

$$x \in A \leftrightarrow \varphi(x, p_1, \dots, p_n) \quad (1.2)$$

for some p_1, \dots, p_n . Given classes A, B , one can easily define the elementary set operations such as $A \cap B$, $A \cup B$, $A \setminus B$, $\bigcup A$, see the first chapter of [Jech, 2006] for details. Axioms are the tools by which we can decide whether a particular class is “small enough” to be considered a set¹. A class that fails to be considered a set is called a *proper class*.

We will often write something like “ M is a limit ordinal”, it should always be clear that this can be rewritten as a formula that was introduced earlier.

¹“Small enough” means that it doesn’t lead to a paradox similar to the famous Russell’s paradox.

89 1.2.2 The Axioms

90 **Definition 1.1** (*The Existence of a Set*)

$$\exists x(x = x) \quad (1.3)$$

91 **Definition 1.2** (*Axiom of Extensionality*)

$$\forall x, y(x = y \leftrightarrow \forall z(z \in x \leftrightarrow z \in y)) \quad (1.4)$$

92 **Definition 1.3** (*Axiom Schema of Specification*)

93 *The following yields an axiom for every first-order formula $\varphi(x, p_1, \dots, p_n)$*
 94 *with no free variables other than x, p_1, \dots, p_n .*

$$\forall x, p_1, \dots, p_n \exists y \forall z(z \in y \leftrightarrow z \in x \ \& \ \varphi(z, p_1, \dots, p_n)) \quad (1.5)$$

95 We will now provide two definitions that are not axioms, but will be
 96 helpful in establishing some axioms in a more comprehensible way.

97 **Definition 1.4** ($x \subseteq y, x \subset y$)

$$x \subseteq y \leftrightarrow (\forall z \in x)z \in y \quad (1.6)$$

$$x \subset y \leftrightarrow x \subseteq y \ \& \ x \neq y \quad (1.7)$$

99 *We read $x \subseteq y$ as x is a subset of y and $x \subset y$ as x is a proper subset of y .*

100 **Definition 1.5** (*Empty Set*) *For an arbitrary set x , the empty set, repre-*
 101 *sented by the symbol " \emptyset ", is the set defined by the following formula:*

$$(\forall y \in x)(y \in \emptyset \leftrightarrow \neg(y = y)) \quad (1.8)$$

102 \emptyset is a set due to Specification, there is only one such set due to Extension-
 103 ality.

104 **Definition 1.6** (*Axiom of Pairing*)

$$\forall x, y \exists z \forall q(q \in z \leftrightarrow q = x \vee q = y) \quad (1.9)$$

105 **Definition 1.7** (*Axiom of Union*)

$$\forall x \exists y \forall z(z \in y \leftrightarrow \exists q(z \in q \ \& \ q \in x)) \quad (1.10)$$

106 **Definition 1.8** (*Axiom of Foundation*)

$$\forall x(x \neq \emptyset \rightarrow (\exists y \in x)(x \cap y = \emptyset)) \quad (1.11)$$

107 **Definition 1.9** (*Axiom of Powerset*)

$$\forall x \exists y \forall z (z \in y \leftrightarrow z \subseteq x) \quad (1.12)$$

108 **Definition 1.10** (*Axiom of Infinity*)

$$\exists x (\emptyset \in x \ \& \ (\forall y \in x)(y \cup \{y\} \in x)) \quad (1.13)$$

109 The least set satisfying (??) is denoted ω .

110 **Definition 1.11** (*Function*)

111 Given an arbitrary first-order formula $\varphi(x, y, p_1, \dots, p_n)$, we say that φ is a
112 function iff

$$\forall x, y, z, p_1, \dots, p_n (\varphi(x, y, p_1, \dots, p_n) \ \& \ \varphi(x, z, p_1, \dots, p_n) \rightarrow y = z) \quad (1.14)$$

113 When a $\varphi(x, y)$ is a function, we also write the following:

$$\varphi(x, y) \Leftrightarrow f(x) = y \quad (1.15)$$

114 Alternatively, $f = \{\langle x, y \rangle : \varphi(x, y)\}$ is a class.

115 Let us introduce a few more definitions that will make the two remaining
116 axioms more comprehensible.

117 **Definition 1.12** (*Powerset Function*)

118 Given a set x , the powerset of x , denoted $\mathcal{P}(x)$ and satisfying (1.9), is
119 defined as follows:

$$\mathcal{P}(x) \stackrel{\text{def}}{=} \{y : y \subseteq x\} \quad (1.16)$$

120 **Definition 1.13** (*Domain of a Function*)

121 Let f be a function. We call the domain of f the set of all sets for which f
122 is defined. We use “ $\text{Dom}(f)$ ” to refer to this set.

$$x \in \text{Dom}(f) \Leftrightarrow \exists y (f(x) = y) \quad (1.17)$$

123 We say “ f is a function on A ”, A being a class, if $A = \text{dom}(f)$.

124 **Definition 1.14** (*Range of a Function*)

125 Let f be a function. We call the range of f the set of all sets that are images
126 of other sets via f . We use “ $\text{Rng}(f)$ ” to refer to this set.

$$x \in \text{Rng}(f) \Leftrightarrow \exists y (f(y) = x) \quad (1.18)$$

127 We say that f is a *function into* A , A being a class, iff $\text{rng}(f) \subseteq A$. We say
 128 that f is a *function onto* A iff $\text{rng}(f) = A$. We say a function f is a *one to*
 129 *one function*, iff

$$(\forall x_1, x_2 \in \text{dom}(f))(f(x_1) = f(x_2) \rightarrow x_1 = x_2) \quad (1.19)$$

130 We say that f is a *bijection* iff it is a one to one function that is onto.

131 Note that $\text{Dom}(f)$ and $\text{Rng}(f)$ are not definitions in a strict sense, they
 132 are in fact definition schemas that yield definitions for every function f given.
 133 Also note that they can be easily modified for φ instead of f , with the only
 134 difference being the fact that it is then defined only for those φ s that are
 135 functions, which must be taken into account. This is worth noting as we will
 136 use the notions of *function* and *formula* interchangeably.

137 **Definition 1.15** (*Function Defined For All Ordinals*)

138 We say a function f is defined for all ordinals, this is sometimes written
 139 $f : \text{Ord} \rightarrow A$ for any class A , if $\text{Dom}(f) = \text{Ord}$. Alternatively,

$$(\forall \alpha \in \text{Ord})(\exists y \in A)(f(\alpha) = y) \quad (1.20)$$

140 And now for the axioms.

141 **Definition 1.16** (*Axiom Schema of Replacement*)

142 The following is an axiom for every first-order formula $\varphi(x, p_1, \dots, p_n)$ with
 143 no free variables other than x, p_1, \dots, p_n .

$$“\varphi \text{ is a function}” \rightarrow \forall x \exists y \forall z (z \in y \leftrightarrow (\exists q \in x)(\varphi(x, y, p_1, \dots, p_n))) \quad (1.21)$$

144 **Definition 1.17** (*Choice function*)

145 We say that a function f is a *choice function* on x iff

$$\text{dom}(f) = x \setminus \{\emptyset\} \ \& \ (\forall y \in \text{dom}(f))(f(y) \in y) \quad (1.22)$$

146 **Definition 1.18** (*Axiom of Choice*)

147 For every set x there is a function f that is a *choice function* on x .

148 One might be unsettled by the fact that this definition quantifies over func-
 149 tions, which are generally classes, but in this particular case, since $\text{dom}(f) =$
 150 x and x is a set, f is also a set due to *Replacement*².

151 We will refer to the axioms by their name, written in italic type, e.g.
 152 *Foundation* refers to the Axiom of Foundation. Now we need to define the
 153 set theories to be used in the article.

²If the underlying theory includes of implies *Replacement*.

154 **Definition 1.19** (S)

155 We call **S** an axiomatic theory in the language $\mathcal{L} = \{=, \in\}$ with exactly the
 156 following axioms:

- 157 (i) Existence of a set (see (1.1))
- 158 (ii) Extensionality (see (1.2))
- 159 (iii) Specification (see (1.3))
- 160 (iv) Foundation (see (1.8))
- 161 (v) Pairing (see (1.6))
- 162 (vi) Union (see (1.7))
- 163 (vii) Powerset (see (1.9))

164 **Definition 1.20** (ZF)

165 We call **ZF** an axiomatic theory in the language $\mathcal{L} = \{=, \in\}$ that contains
 166 all the axioms of **S** in addition to the following:

- 167 (i) Replacement schema (see (1.16))
- 168 (ii) Infinity (see (1.10))

169 Existence of a set is usually left out because it is a consequence of infinity.

170 **Definition 1.21** (ZFC)

171 **ZFC** is an axiomatic theory in the language $\mathcal{L} = \{=, \in\}$ that contains all the
 172 axioms of **ZF** plus Choice (1.18).

173

174 **1.2.3 The Transitive Universe**175 **Definition 1.22** (Transitive Class)

176 We say a class A is transitive iff

$$(\forall x \in A)(x \subseteq A) \quad (1.23)$$

177 **Definition 1.23** (Well Ordered Class) A class A is said to be well ordered
 178 by \in iff the following hold:

- 179 (i) $(\forall x \in A)(x \not\subseteq x)$ (Antireflexivity)
- 180 (ii) $(\forall x, y, z \in A)(x \in y \ \& \ y \in z \rightarrow x \in z)$ (Transitivity)
- 181 (iii) $(\forall x, y \in A)(x = y \vee x \in y \vee y \in x)$ (Linearity)
- 182 (iv) $(\forall x \subseteq A)(x \neq \emptyset \rightarrow (\exists y \in x)(\forall z \in x)(z = y \vee z \in y))$ (Existence of the
 183 least element)

184 **Definition 1.24** (Ordinal Number)

185 A set x is said to be an ordinal number if it is transitive and well-ordered
 186 by \in .

For the sake of brevity, we usually just say “ x is an *ordinal*”. Note that “ x is an ordinal” is a well-defined formula in the language of set theory, since 1.22 is a first-order formula and 1.23 is in fact a conjunction of four first-order formulas. Ordinals will be usually denoted by lower case greek letters, starting from the beginning of the alphabet: $\alpha, \beta, \gamma, \dots$. Given two different ordinals α, β , we will write $\alpha < \beta$ for $\alpha \in \beta$, see Lemma 2.11 in [Jech, 2006] for technical details.

Definition 1.25 (*Non-Zero Ordinal*) We say an ordinal α is non-zero iff $\alpha \neq \emptyset$.

Definition 1.26 (*Successor Ordinal*) Consider the following function defined for all ordinals. Let β be an arbitrary ordinal. We call S the successor function.

$$S(\beta) = \beta \cup \{\beta\} \quad (1.24)$$

An ordinal α is called a successor ordinal iff there is an ordinal β , such that $\alpha = S(\beta)$. We also write $\alpha = \beta + 1$.

Definition 1.27 (*Limit Ordinal*) A non-zero ordinal α is called a limit ordinal iff it is not a successor ordinal.

Definition 1.28 (*Ord*) The class of all ordinal numbers, which we will denote “ Ord ”³ is the proper class defined as follows.

$$x \in Ord \leftrightarrow x \text{ is an ordinal} \quad (1.25)$$

Definition 1.29 (*Von Neumann’s Hierarchy*) The Von Neumann’s Hierarchy is a collection of sets indexed by elements of Ord , defined recursively in the following way:

$$(i) \quad V_0 = \emptyset \quad (1.26)$$

$$(ii) \quad V_{\alpha+1} = \mathcal{P}(V_\alpha) \text{ for any ordinal } \alpha \quad (1.27)$$

$$(iii) \quad V_\lambda = \bigcup_{\beta < \lambda} V_\beta \text{ for a limit ordinal } \lambda \quad (1.28)$$

³Other authors use “ On ”, we will stick to the notation used in [Jech, 2006].

We will also refer to the Von Neumann's Hierarchy as Von Neumann's Universe or the Cumulative Hierarchy. This definition is only correct in a theory that contains or implies Replacement because otherwise it's not clear that the successor step is a set.

Definition 1.30 (*Rank*)

Given a set x , we say that the rank of x (written as $\text{rank}(x)$) is the least ordinal α such that $x \in V_{\alpha+1}$

Due to *Regularity*, every set has a rank.⁴ The Von Neumann's hierarchy defined above can also be defined by the fact that every V_α is a set of all set with rank less than α .

Definition 1.31 (*Order-type*)

Given an arbitrary well-ordered set x , we say that an ordinal α is the order-type of x iff x and α are isomorphic.

222

1.2.4 Cardinal Numbers

Definition 1.32 (*Cardinality*)

Given a set x , let the cardinality of x , written $|x|$, be defined as the smallest ordinal number such that there is a one to one mapping from x onto α .

Definition 1.33 (*Aleph function*)

Let ω be the set defined by ???. We will recursively define the function \aleph for all ordinals.

(i) $\aleph_0 = \omega$

(ii) $\aleph_{\alpha+1}$ is the least cardinal larger than \aleph_α ⁵

(iii) $\aleph_\lambda = \bigcup_{\beta < \lambda} \aleph_\beta$ for a limit ordinal λ

If $\kappa = \aleph_\alpha$ and α is a successor ordinal, we call κ a successor cardinal. If α is a limit ordinal, we call κ a limit cardinal.

Definition 1.34 (*Cardinal number*)

236

(i) A set x is called a finite cardinal iff $x \in \omega$.

(ii) A set is called an infinite cardinal iff there is an ordinal α such that

$\aleph_\alpha = x$

239

⁴See chapter 6 of [Jech, 2006] for details.

⁵"The least cardinal larger than \aleph_α " is sometimes notated as \aleph_α^+ .

(iii) A set is called a cardinal iff it is either a finite cardinal or an infinite cardinal.

We say κ is an uncountable cardinal iff it is an infinite ordinal and $\aleph_0 < \kappa$. Infinite cardinals will be notated by lowercase greek letters from the middle of the alphabet, e.g. κ, μ, ν, \dots with the exception of λ , which is next to κ in the greek alphabet, but is also sometimes used for limit ordinal.

For formal details as well as why every set can be well-ordered assuming *Choice*, and therefore has a cardinality, see [Jech, 2006].

Definition 1.35 (*Sequence*)

We say that a function $\varphi(x, y)$ is a sequence iff there is an ordinal α such that $\text{dom}(\varphi) = \alpha$. In other words, a function is called a sequence if it is defined exactly for every ordinal from 0 to some α . We then say it is an α -sequence. We usually write $\langle \beta_i : i \in \alpha \rangle$ or $\langle \beta_0, \beta_1, \dots \rangle$ when referring to a sequence, β_i then denotes the elements of $\text{rng}(\varphi)$ for every $i \in \text{dom}(\varphi)$.

Definition 1.36 (*Cofinal Subset*)

Given a class A of ordinals, we say that $B \subseteq A$ is cofinal in A iff

$$(\forall x \in A)(\exists y \in B)(x \in y) \quad (1.29)$$

In other words, B is cofinal in A iff it is unbounded in A .

Definition 1.37 (*Cofinality of a Limit Ordinal*)

Let λ be a limit ordinal. We say that the cofinality of λ is κ iff κ is the least ordinal, such that there is a cofinal κ -sequence $\langle \beta_\xi : \xi < \kappa \rangle$, such that

$$\sup(\{\beta_\xi : \xi < \kappa\}) = \lambda \quad (1.30)$$

We write $cf(\lambda) = \kappa$.

Note that $cf(\alpha)$ is always a cardinal⁶.

Definition 1.38 (*Regular Cardinal*)

We say an infinite cardinal κ is regular iff $cf(\kappa) = \kappa$.

Definition 1.39 (*Strong Limit Cardinal*)

We say that an ordinal κ is a strong limit cardinal if it is a limit cardinal and

$$(\forall \alpha \in \kappa)(|\mathcal{P}(\alpha)| \in \kappa). \quad (1.31)$$

⁶If $cf(\alpha)$ is not a cardinal, so $|cf(\alpha)| < cf(\alpha)$, then there is a mapping from $|cf(\alpha)|$ onto $cf(\alpha)$. But then the range of this mapping is a cofinal subset of $cf(\alpha)$ that is strictly smaller than $cf(\alpha)$.

267 **Definition 1.40** (*Generalised Continuum Hypothesis*)

268

$$(\forall \alpha \in \text{Ord}) \aleph_{\alpha+1} = |\mathcal{P}(\aleph_\alpha)| \quad (1.32)$$

269 If *GCH* holds (for example in Gödel's *L*, see chapter 3), the notions of limit
270 cardinal and strong limit cardinal are equivalent.

271

272 1.2.5 Relativisation and Absoluteness

273 **Definition 1.41** (*Relativization*)

274 Let M be a class, $R \subseteq M \times M$ and let $\varphi(p_1, \dots, p_n)$ be a first-order formula
275 with no free variables besides p_1, \dots, p_n . The relativization of φ to M and R
276 is the formula, written as $\varphi^{M,R}$, defined in the following inductive manner:

- 277 (i) $(x \in y)^{M,R} \leftrightarrow R(x, y)$
- 278 (ii) $(x = y)^{M,R} \leftrightarrow x = y$
- 279 (iii) $(\neg \varphi)^{M,R} \leftrightarrow \neg \varphi^{M,R}$
- 280 (iv) $(\varphi \ \& \ \psi)^{M,R} \leftrightarrow \varphi^{M,R} \ \& \ \psi^{M,R}$
- 281 (v) $(\varphi \vee \psi)^{M,R} \leftrightarrow \varphi^{M,R} \vee \psi^{M,R}$
- 282 (vi) $(\varphi \rightarrow \psi)^{M,R} \leftrightarrow \varphi^{M,R} \rightarrow \psi^{M,R}$
- 283 (vii) $(\exists x \varphi(x))^{M,R} \leftrightarrow (\exists x \in M) \varphi^{M,R}(x)$
- 284 (viii) $(\forall x \varphi(x))^{M,R} \leftrightarrow (\forall x \in M) \varphi^{M,R}(x)$

285 When $R = \in \cap (M \times M)$, we usually write φ^M instead of $\varphi^{M,R}$. When we
286 talk about $\varphi^M(p_1, \dots, p_n)$, it is understood that $p_1, \dots, p_n \in M$.

287 **Definition 1.42** (*Satisfaction in a Structure*)

288 Let M be a set and R a binary relation on M . Let *Terms* be the set of all
289 terms, let $e : \text{Terms} \rightarrow M$ any evaluation function. Let φ be a first-order
290 formula in the language of set theory.

291 iff any of the following hold

292 We say that φ holds in $\langle M, R \rangle$ under the evaluation e , we write $\langle M, R \rangle \models \varphi[e]$,

293 iff any of the following hold

- 294 (i) φ is the formula “ $s = t$ ”, s, t are terms and $e(s) = e(t)$.
- 295 (ii) φ is the formula “ $s \in t$ ”, s, t are terms and the pair $\langle e(s), e(t) \rangle$ is in R .
- 296 (iii) φ is the formula “ $\neg \psi$ ” and not $\langle M, R \rangle \models \psi[e]$
- 297 (iv) φ is the formula “ $\psi_1 \ \& \ \psi_2$ ” and both $\langle M, R \rangle \models \psi_1[e]$ and $\langle M, R \rangle \models \psi_2[e]$.
- 298 (v) φ is the formula “ $\psi_1 \vee \psi_2$ ” and either $\langle M, R \rangle \models \psi_1[e]$ or $\langle M, R \rangle \models \psi_2[e]$.
- 299 (vi) φ is the formula “ $\psi_1 \rightarrow \psi_2$ ” and either not $\langle M, R \rangle \models \psi_1[e]$ or
300 $\langle M, R \rangle \models \psi_2[e]$.
- 301 (vii) φ is the formula “ $\psi_1 \rightarrow \psi_2$ ” and either not $\langle M, R \rangle \models \psi_1[e]$ or
302 $\langle M, R \rangle \models \psi_2[e]$.

- 303 (viii) φ is the formula “ $\forall x_1\psi$ ” and $\langle M, R \rangle \models \psi[e']$ for every e' that differs
 304 from e only in the value of x_1 .
 305 (ix) φ is the formula “ $\forall x_1\psi$ ” and $\langle M, R \rangle \models \psi[e']$ for every e' that differs
 306 from e only in the value of x_1 .
 307 (x) φ is the formula “ $\exists x_1\psi$ ” and $\langle M, R \rangle \models \psi[e']$ for some e' that differs
 308 from e only in the value of x_1 .
 309 We also write $\langle M, R \rangle \models \varphi$, which

310 Note that we say that M is a set.

311 We will use $\langle M, R \rangle \models \varphi(p_1, \dots, p_n)$ and $\varphi^M(p_1, \dots, p_n)$ interchangeably.

312 **Definition 1.43** (Absoluteness) Given a transitive class M , we say a for-
 313 mula φ is absolute in M if for all $p_1, \dots, p_n \in M$

$$\varphi^M(p_1, \dots, p_n) \leftrightarrow \varphi(p_1, \dots, p_n) \quad (1.33)$$

314 **Definition 1.44** (Hierarchy of First-Order Formulas)

- 315
 316 (I) A first-order formula φ is Δ_0 iff it is logically equivalent to a first-order
 317 formula φ' satisfying any of the following:
 318 (i) φ' contains no quantifiers
 319 (ii) y is a set, ψ is a Δ_0 -formula, and φ' is either $(\exists x \in y)\psi(y)$ or
 320 $(\forall x \in y)\psi(y)$.
 321 (iii) ψ_1, ψ_2 are Δ_0 -formulas and φ' is any of the following: $\psi_1 \vee \psi_2$,
 322 $\psi_1 \& \psi_2$, $\psi_1 \rightarrow \psi_2$, $\neg\psi_2$,
 323 (II) If a formula is Δ_0 it is also Σ_0 and Π_0
 324 (III) A formula φ is $\Pi_n + 1$ if it is logically equivalent to a formula φ' such
 325 that $\varphi' = \forall x\psi$ where ψ is a Σ_n -formula for any $n < \omega$.
 326 (IV) A formula φ is $\Sigma_n + 1$ if it is logically equivalent to a formula φ' such
 327 that $\varphi' = \exists x\psi$ where ψ is a Π_n -formula for any $n < \omega$.

328 Note that we can use the pairing function so that for $\forall p_1, \dots, p_n \psi(p_1, \dots, p_n)$,
 329 there is a logically equivalent formula of the form $\forall x\psi'(x)$.

330 **Lemma 1.45** (Δ_0 absoluteness) Let φ be a Δ_0 -formula, then φ is absolute
 331 in any transitive class M .

332 *Proof.* This will be proven by induction over the complexity of a given Δ_0 -
 333 formula φ . Let M be an arbitrary transitive class.

334 Atomic formulas are always absolute by the definition of relativisation,
 335 see (1.41). Suppose that Δ_0 -formulas ψ_1 and ψ_2 are absolute in M . Then

from relativization, $(\psi_1 \& \psi_2)^M \leftrightarrow \psi_1^M \& \psi_2^M$, which is, from the induction hypothesis, equivalent to $\psi_1 \& \psi_2$. The same holds for \vee, \rightarrow, \neg .

Suppose that a Δ_0 -formula ψ is absolute in M . Let y be a set and let $\varphi = (\exists x \in y)\psi(x)$. From relativization, $(\exists x\psi(x))^M \leftrightarrow (\exists x \in M)\psi^M(x)$. Since the hypotheses makes it clear that $\psi^M \leftrightarrow \psi$, we get $((\exists x \in y)\psi(x))^M \leftrightarrow (\exists x \in y \cap M)\psi(x)$, which is the equivalent of $\varphi^M \leftrightarrow \varphi$. The same applies to $\varphi = (\forall x \in y)\psi(x)$. \square

Lemma 1.46 (*Downward Absoluteness*)

Let φ be a Π_1 -formula and M a transitive class. Then the following holds:

$$(\forall p_1, \dots, p_n \in M)(\varphi(p_1, \dots, p_n) \rightarrow \varphi(p_1, \dots, p_n)^M) \quad (1.34)$$

Proof. Since $\varphi(p_1, \dots, p_n)$ is Π_1 , there is a Δ_0 -formula $\psi(p_1, \dots, p_n, x)$ such that $\varphi = \forall x\psi(p_1, \dots, p_n, x)$. From relativization and lemma (1.45), $\varphi^M(p_1, \dots, p_n) \leftrightarrow (\forall x \in M)\psi(p_1, \dots, p_n, x)$.

Assume that for $p_1, \dots, p_n \in M$ fixed, that $\forall x\psi(p_1, \dots, p_n, x)$ holds, but $(\forall x \in M)\psi(p_1, \dots, p_n, x)$ does not. Therefore $\exists x\neg\psi(p_1, \dots, p_n, x)$, which contradicts $\forall x\psi(p_1, \dots, p_n, x)$. \square

Lemma 1.47 (*Upward Absoluteness*)

Let φ be a Σ_1 -formula and M a transitive class. Then the following holds:

$$(\forall p_1, \dots, p_n \in M)(\varphi^M(p_1, \dots, p_n) \rightarrow \varphi(p_1, \dots, p_n)) \quad (1.35)$$

Proof. Since $\varphi(p_1, \dots, p_n)$ is Σ_1 , there is a Δ_0 -formula $\psi(p_1, \dots, p_n, x)$ such that $\varphi = \exists x\psi(p_1, \dots, p_n, x)$. From relativization and lemma (1.45), $\varphi^M(p_1, \dots, p_n) \leftrightarrow (\exists x \in M)\psi(p_1, \dots, p_n, x)$.

Assume that for $p_1, \dots, p_n \in M$ fixed, that $(\exists x \in M)\psi(p_1, \dots, p_n, x)$ holds, but $\exists x\psi(p_1, \dots, p_n, x)$ does not. This is an obvious contradiction. \square

1.2.6 More Functions

Definition 1.48 (*Strictly Increasing Function*)

A function $f : Ord \rightarrow Ord$ is said to be strictly increasing iff

$$\forall \alpha, \beta \in Ord(\alpha < \beta \rightarrow f(\alpha) < f(\beta)). \quad (1.36)$$

Definition 1.49 (*Continuous Function*)

A function $f : Ord \rightarrow Ord$ is said to be continuous iff

$$\lambda \text{ is limit} \rightarrow f(\lambda) = \bigcup_{\alpha < \lambda} f(\alpha). \quad (1.37)$$

363 **Definition 1.50** (*Normal Function*)

364 *A function $f : \text{Ord} \rightarrow \text{Ord}$ is said to be normal iff it is strictly increasing*
 365 *and continuous.*

366 **Definition 1.51** (*Fixed Point*)

367 *We say x is a fixed point of a function f iff $x = f(x)$.*

368 **Definition 1.52** (*Unbounded Class*)

369 *We say a class A of ordinals is unbounded iff*

$$\forall x(\exists y \in A)(x < y) \quad (1.38)$$

370 **Definition 1.53** (*Limit Point*)

371 *Given a class $x \subseteq \text{Ord}$, we say that $\alpha \neq \emptyset$ is a limit point of x iff*

$$\alpha = \bigcup (x \cap \alpha) \quad (1.39)$$

372 **Definition 1.54** (*Closed Class*)

373 *We say a class $A \subseteq \text{Ord}$ is closed iff it contains all its limit points.*

374 **Definition 1.55** (*Club set*)

375 *For a regular uncountable cardinal κ , a set $x \subset \kappa$ is a closed unbounded*
 376 *subset, abbreviated as a club set, iff x is both closed and unbounded in κ .*

377 **Definition 1.56** (*Stationary set*)

378 *For a regular uncountable cardinal κ , we say a set $A \subset \kappa$ is stationary in κ*
 379 *iff it intersects every club subset of κ .*

380 1.2.7 Structure, Substructure and Embedding

381 Structures will be denoted $\langle M, \in, R \rangle$ where M is a domain, \in stands for the
 382 standard membership relation, it is assumed to be restricted to the domain⁷,
 383 $R \subseteq M$ is a relation on the domain. When R is not needed, we can as well
 384 only write M instead of $\langle M, \in \rangle$.

385 **Definition 1.57** (*Elementary Embedding*)

386 *Given the structures $\langle M_0, \in, R \rangle$, $\langle M_1, \in, R \rangle$ and a one-to-one function $j :$
 387 $M_0 \rightarrow M_1$, we say j is an elementary embedding of M_0 into M_1 , we write
 388 $j : M_0 \prec M_1$, when the following holds for every formula $\varphi(p_1, \dots, p_n)$ and
 389 every $p_1, \dots, p_n \in M_0$:*

$$\langle M_0, \in, R \rangle \models \varphi(p_1, \dots, p_n) \leftrightarrow \langle M_1, \in, R \rangle \models \varphi(j(p_1), \dots, j(p_n)) \quad (1.40)$$

⁷To be totally explicit, we should write $\langle M, \in \cap M \times M, R \rangle$.

390 **Definition 1.58** (*Elementary Substructure*)

391 *Given the structures $\langle M_0, \in, R \rangle$, $\langle M_1, \in, R \rangle$ and a one-to-one function $j :$
 392 $M_0 \rightarrow M_1$ such that $j : M_0 \prec M_1$, we say that M_0 is an elementary sub-
 393 structure of M_1 , denoted as $M_0 \prec M_1$, iff j is an identity on M_0 . In other
 394 words*

$$\langle M_0, \in, R \rangle \models \varphi(p_1, \dots, p_n) \leftrightarrow \langle M_1, \in, R \rangle \models \varphi(p_1, \dots, p_n) \quad (1.41)$$

395 *for $p_1, \dots, p_n \in M_0$*

2 Levy's First-Order Reflection

2.1 Lévy's Original Paper

This section is based on Lévy's paper *Axiom Schemata of Strong Infinity in Axiomatic Set Theory*, [Lévy, 1960]. It presents Lévy's general reflection principle and its equivalence to *Replacement* and *Infinity* under S^8 .

First, we should point out that set theory has changed over the last 66 years and show a few notable, albeit only formal, differences. One might be confused by the fact that Lévy treats the *Subsets* axiom, which we call *Specification*, as a single axiom rather than a schema. He even takes the conjunction of all axioms of ZF and treats it like a formula. This is possible because the underlying logic calculus is different. Lévy works with set theories formulated in the *non-simple applied first order functional calculus*, see beginning of *Chapter IV* in [Church, 1996] for details. For now, we only need to know that the calculus contains a substitution rule for functional variables. This way, *Subsets* is de facto a schema even though it sometimes treated as a single formula⁹ but the logic is still first-order since one can't quantify over functional variables. We will use the usual first-order axiomatization of ZFC as seen on [Jech, 2006]. It should also be noted that the logical connectives look different. The now usual symbol for an universal quantifier does not appear, $\forall x\varphi(x)$ would be written as $(x)\varphi(x)$. The symbol for negation is " \sim ", implication is written as " \supset " and equivalence is " \equiv ". We will use standard notation with " \neg ", " \rightarrow " and " \leftrightarrow " respectively when presenting Lévy's results.

This subsection uses ZF instead of the usual ZFC as the underlying theory.

Definition 2.1 (Standard Complete Model of a Set Theory)

Let Q be an arbitrary axiomatic set theory. We say that u is a standard complete model of Q iff

- (i) $(\forall \sigma \in Q)(\langle u, \in \rangle \models \sigma)$
- (ii) $\forall y(y \in u \rightarrow y \subset u)$ (u is transitive)

We write $Scm^Q(u)$.

Definition 2.2 (Cardinals Inaccessible With Respect to Q)

Let Q be an arbitrary axiomatic set theory. We say that a cardinal κ is inaccessible with respect to theory Q iff

$$Scm^Q(V_\kappa) \quad (2.42)$$

⁸See definition (1.19).

⁹This way, the conjunction of all axioms is then in fact an axiom schema.

429 We write $In^Q(\kappa)$.¹⁰

430 **Definition 2.3** (*Inaccessible Cardinal With Respect to ZF*)

431 When a cardinal κ is inaccessible with respect to ZF, we only say that it is
432 inaccessible. We write $In(\kappa)$.

$$In(\kappa) \leftrightarrow In^{ZF}(\kappa) \quad (2.43)$$

433 The above definition of inaccessibles is used because it doesn't require *Choice*.

434 For the definition of relativization, see (1.41). The notation used by Lévy is
435 " $Rel(u, \varphi)$ ", we will stick to " φ^u ".

436 **Definition 2.4** (N)

437 The following is an axiom schema of complete reflection over ZF, denoted N .
438 For every first-order formula φ in the language of set theory with no free variables
439 except for p_1, \dots, p_n , the following is an instance of schema N .

$$\exists u(Scm^{ZF}(u) \ \& \ (\forall p_1, \dots, p_n \in u)(\varphi \leftrightarrow \varphi^u)) \quad (2.44)$$

440 **Definition 2.5** (N')

441 For any first-order formulas $\varphi_1, \dots, \varphi_m$ in the language of set theory with no
442 free variables except for p_1, \dots, p_n , the following is an instance of schema N' .

$$\exists u(z \in u \ \& \ Scm^{ZF}(u) \ \& \ (\forall p_1, \dots, p_n \in u)(\varphi_1 \leftrightarrow \varphi_1^u) \ \& \ \dots \ \& \ \varphi_m \leftrightarrow \varphi_m^u)) \quad (2.45)$$

443 **Definition 2.6** (N')

444 For any first-order formulas $\varphi_1, \dots, \varphi_m$ in the language of set theory with no
445 free variables except for p_1, \dots, p_n , the following is an instance of schema N' .

$$\exists u(Scm^{ZF}(u) \ \& \ (\forall p_1, \dots, p_n \in u)(\varphi_1 \leftrightarrow \varphi_1^u) \ \& \ \dots \ \& \ \varphi_m \leftrightarrow \varphi_m^u)) \quad (2.46)$$

446 Let S be an axiomatic set theory defined in (1.19).

447 This is *Theorem 2* in [Lévy, 1960]

448 **Lemma 2.7** ($N \leftrightarrow N'' \leftrightarrow N'$)

449 The schemas N , N' and N'' are equivalent under S .

¹⁰To be able to define V_κ , we need to work in a logic that contains the *Replacement Schema* or any of it's equivalents. It should be noted that we don't work in an arbitrary theory Q , but in ZF, which contains the *Replacement Schema*. $Scm^Q(V_\kappa)$ in fact says "ZF thinks that V_κ is a transitive model of Q ".

450 *Proof.* We will execute this proof in the theory ZF, but the reader should note
 451 that we are neither using *Replacement* nor *Infinity*, so for schemas similar to N ,
 452 N' , N'' but with " $Scm^S(u)$ " instead of " $Scm^{ZF}(u)$ ", the proof works equally
 453 well.

454 Clearly, $N' \rightarrow N'' \rightarrow N$.

455 Now, assuming N and given the formulas $\varphi_1, \dots, \varphi_n$, we will prove N'' .
 456 Consider the following formula:

$$\psi = \bigvee_{i=1}^t t = i \ \& \ \varphi_i \quad (2.47)$$

457 We will take advantage of the fact that natural numbers are defined by atomic
 458 formulas and therefore absolute in transitive structures. From N , we get such
 459 u that $Scm^{ZF}(u) \ \& \ (\forall p_1, \dots, p_n \in u)(\bigvee_{i=1}^t t = i \ \& \ \varphi_i \leftrightarrow \bigvee_{i=1}^t t = i \ \& \ \varphi_i^u)$. This
 460 already satisfies N'' .

461 In order to prove N' from N'' , let's add two more formulas. Given p_1, \dots, p_n ,
 462 we denote

$$\varphi_{m+1} = \exists u(z \in u \ \& \ Scm^{ZF}(u) \ \& \ (\forall p_1, \dots, p_n \in u)(\bigvee_{i=1}^m \varphi_i = \varphi_i^u)) \quad (2.48)$$

$$\varphi_{m+2} = \forall z \varphi_{m+1} \quad (2.49)$$

464 So, by N'' , we have a set u that satisfies $Scm^{ZF}(u)$ as well as the following:

$$(\forall p_1, \dots, p_n \in u)(\varphi_i \leftrightarrow \varphi_i^u) \text{ for } 1 \leq i \leq m \quad (2.50)$$

$$z \in u \rightarrow \varphi_{m+1} \leftrightarrow \varphi_{m+1}^u \quad (2.51)$$

$$\varphi_{m+2} \leftrightarrow \varphi_{m+2}^u \quad (2.52)$$

467 By $Scm^{ZF}(u)$ and (2.50), we get $(\forall z \in u)\varphi_{m+1}$, so together with (2.51), we get
 468 $(\forall z \in u)\varphi_{m+1}^u$, exactly φ_{m+2}^u , so by (2.52) we get φ_{m+2} . But φ_{m+2} is exactly
 469 the instance of N' we were looking for. \square

470 Definition 2.8 (N_0)

471 Axiom schema N_0 is similar to N defined above, but with S instead of ZF. For
 472 every φ , a first-order formula in the language of set theory with no free variables
 473 except p_1, \dots, p_n , the following is an instance of N_0 .

$$\exists u(Scm^S(u) \ \& \ (\forall p_1, \dots, p_n \in u)(\varphi \leftrightarrow \varphi^u)) \quad (2.53)$$

474 We will now show that in S , N_0 implies both *Replacement* and *Infinity*.

475
476 Let N_0 be defined as in (2.8), for *Infinity* see (1.10).

477 **Theorem 2.9** *In S , the axiom schema N_0 implies Infinity.*

478 *Proof.* Let $\varphi = \forall x \exists y (y = x \cup \{x\})$. This clearly holds in S because given a
479 set x , there is a set $y = x \cup \{x\}$ obtained via *Pairing* and *Union*. From N_0 ,
480 there is a set u such that φ^u holds. This u satisfies the conditions required by
481 *Infinity*. \square

482 Lévy proves this theorem in a different way. He argues that for an arbitrary
483 formula φ , N_0 gives us $\exists u Scm^S(u)$ and this u already satisfies *Infinity*. To do
484 this, we would need to prove lemma (2.15) earlier on, we will do that later in
485 this chapter.

486
487 Let S be a set theory defined in (1.19), N_0 a schema defined in (2.8) and
488 *Replacement* a schema defined in (1.16).

489 **Theorem 2.10** *In S , the axiom schema N_0 implies Replacement.*

490 *Proof.* Let $\varphi(x, y, p_1, \dots, p_n)$ be a formula with no free variables except x, y, p_1, \dots, p_n .
491 Let a set x be given and let χ be an instance of the *Replacement* schema for
492 the φ given. We want to verify in S that given a formula φ , the instance of N_0
493 for φ implies χ .

$$\begin{aligned} \chi = \forall x', y', z (\varphi(x', y', p_1, \dots, p_n) \ \& \ \varphi(x', z, p_1, \dots, p_n) \rightarrow y' = z') \\ \rightarrow \exists y \forall z (z \in y \leftrightarrow (\exists q \in x) (\varphi(x, y, p_1, \dots, p_n))) \end{aligned} \quad (2.54)$$

494 Since it can be shown that N_0 is equivalent to N'_0 similar to N' in lemma
495 (2.7), there is a set u such that $Scm^S(u)$, $x \in u$ and all of the following hold:

- 496 (i) $\varphi \leftrightarrow \varphi^u$
497 (ii) $\exists y \varphi \leftrightarrow (\exists y \varphi)^u$

498 From relativization, $(\exists y \varphi)^u$ is equivalent to $(\exists y \in u) \varphi^u$, together with (i) and
499 (ii), we get

$$(\exists y \in u) \varphi \leftrightarrow \exists y \varphi \quad (2.55)$$

500 If φ is a function, it maps the elements of x , which are also elements of u
501 due to transitivity of u , to elements of u . From *Specification*, $y = \{z \in u\}$ we
502 can find y , a set of all images of the elements of x via φ . So we have satisfied
503 *Replacement* – given a function and a set, we have proven that image of the set
504 via given function is also a set. \square

505 What we have just proven is only a single theorem from Lévy's aforementioned
506 article, we will introduce other interesting results, dealing with inaccessible and
507 Mahlo cardinals, later in their appropriate context in chapter 3.

2.2 Contemporary Restatement

We will now introduce and prove a theorem that is called Lévy's Reflection in contemporary set theory. The only difference is that while Lévy originally reflects a formula φ from V to a set u which is a *standard complete model* of S , we say that there is a V_λ for a limit λ that reflects φ . Those two conditions are equivalent due to lemma (2.15).

Lemma 2.11 *Let $\varphi_1, \dots, \varphi_n$ be first-order formulas in the language of set theory, all with m free variables¹¹.*

- (i) *For each set M_0 there is such set M that $M_0 \subset M$ and the following holds for every i , $1 \leq i \leq n$:*

$$\exists x \varphi_i(p_1, \dots, p_{m-1}, x) \rightarrow (\exists x \in M) \varphi_i(p_1, \dots, p_{m-1}, x) \quad (2.56)$$

for every $p_1, \dots, p_{m-1} \in M$.

- (ii) *Furthermore, there is a limit ordinal λ such that $M_0 \subset V_\lambda$ and the following holds for each i , $1 \leq i \leq n$:*

$$\exists x \varphi_i(p_1, \dots, p_{m-1}, x) \rightarrow (\exists x \in V_\lambda) \varphi_i(p_1, \dots, p_{m-1}, x) \quad (2.57)$$

for every $p_1, \dots, p_{m-1} \in M$.

- (iii) *Assuming Choice, there is M , $M_0 \subset M$ such that (2.56) holds for every M , $i \leq n$ and $|M| \leq |M_0| \cdot \aleph_0$.*

Proof. We will simultaneously prove statements (i) and (ii), denoting M^T the transitive set required by part (ii). Steps in the construction of M^T that are not explicitly included are equivalent to steps for M .

Let us first define an operation $H_i(p_1, \dots, p_{m-1})$ that yields the set of x 's with minimal rank¹² satisfying $\varphi_i(p_1, \dots, p_{m-1}, x)$ for p_1, \dots, p_{m-1} and for every i , $1 \leq i \leq n$.

$$H_i(p_1, \dots, p_n) = \{x \in C_i : (\forall z \in C)(\text{rank}(x) \leq \text{rank}(z))\} \quad (2.58)$$

for each $1 \leq i \leq n$, where

$$C_i = \{x : \varphi_i(p_1, \dots, p_{m-1}, x)\} \text{ for } 1 \leq i \leq n \quad (2.59)$$

¹¹For formulas with a different number of free variables, take for m the highest number of parameters among those formulas. Add spare parameters to every formula that has less than m parameters in a way that preserves the last parameter, which we will denote x . E.g. let φ'_i be the a formula with k parameters, $k < m$. Let us set $\varphi_i(p_1, \dots, p_{m-1}, x) = \varphi'_i(p_1, \dots, p_{k-1}, x)$, notice that the parameters p_k, \dots, p_{m-1} are not used.

¹²Rank is defined in (1.30).

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Next, let's construct M from given M_0 by induction.

$$M_{i+1} = M_i \cup \bigcup_{j=0}^n \{H_j(p_1, \dots, p_{m-1}) : p_1, \dots, p_{m-1} \in M_i\} \quad (2.60)$$

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In other words, in each step we include into the construction the elements satisfying $\varphi(p_1, \dots, p_{m-1}, x)$ for p_1, \dots, p_{m-1} from the previous step. For statement (ii), this is the only part that differs from (i). To end up with a transitive M , we need to extend every step to it's transitive closure transitive closure of M_{i+1} from (i). In other words, let γ be the smallest ordinal such that

$$(M_i^T \cup \bigcup_{j=0}^n \{\bigcup \{H_j(p_1, \dots, p_{m-1}) : p_1, \dots, p_{m-1} \in M_i\}\}) \subset V_\gamma \quad (2.61)$$

538

Then the incremental step is

$$M_{i+1}^T = V_\gamma \quad (2.62)$$

539

and the final M is obtained by joining the previous steps.

$$M = \bigcup_{i=0}^{\infty} M_i, \quad M^T = \bigcup_{i=0}^{\infty} M_i^T = V_\lambda \text{ for some limit } \lambda. \quad (2.63)$$

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We have yet to finish part (iii). Let's try to construct a set M' that satisfies the same conditions like M but is kept as small as possible. Assuming the Axiom of Choice, we can modify the construction so that the cardinality of M' is at most $|M_0| \cdot \aleph_0$. Note that the size of M in the previous construction is determined by the size of M_0 and, most importantly, by the size of $H_i(p_1, \dots, p_{m-1})$ for every i , $1 \leq i \leq n$ in individual iterations of the construction. Since (i) only ensures the existence of an x that satisfies $\varphi_i(p_1, \dots, p_{m-1}, x)$ for any i , $1 \leq i \leq n$, we only need to add one x for every set of parameters but $H_i(u_1, \dots, u_{m-1})$ can be arbitrarily large. Let F be a choice function on $\mathcal{P}(M')$. Also let $h_i(p_1, \dots, p_{m-1}) = F(H_i(p_1, \dots, p_{m-1}))$ for i , where $1 \leq i \leq n$, which means that h is a function that outputs an x that satisfies $\varphi_i(p_1, \dots, p_{m-1}, x)$ for i such that $1 \leq i \leq n$ and has minimal rank among all such sets. The induction step needs to be redefined to

$$M'_{i+1} = M'_i \cup \bigcup_{j=0}^n \{h_j(p_1, \dots, p_{m-1}) : p_1, \dots, p_{m-1} \in M'_i\} \quad (2.64)$$

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This way, the amount of elements added to M'_{i+1} in each step of the construction is the same as the amount of m -tuples of parameters that yielded elements not

556 included in M'_i . It is easy to see that if M_0 is finite, M' is countable because
 557 it was constructed as a countable union of sets that are themselves at most
 558 countable. If M_0 is countable or larger, the cardinality of M' is equal to the
 559 cardinality of M_0 .¹³ Therefore $|M'| \leq |M_0| \cdot \aleph_0$ \square

560 **Theorem 2.12** (*Lévy's first-order reflection theorem*)

561 Let $\varphi(p_1, \dots, p_n)$ be a first-order formula.

562 (i) For every set M_0 there exists a set M such that $M_0 \subset M$ and the following
 563 holds:

$$\varphi^M(p_1, \dots, p_n) \leftrightarrow \varphi(p_1, \dots, p_n) \quad (2.65)$$

564 for every $p_1, \dots, p_n \in M$.

565 (ii) For every set M_0 there is a transitive set M , $M_0 \subset M$ such that the
 566 following holds:

$$\varphi^M(p_1, \dots, p_n) \leftrightarrow \varphi(p_1, \dots, p_n) \quad (2.66)$$

567 for every $p_1, \dots, p_n \in M$.

568 (iii) For every set M_0 there is a limit ordinal λ such that $M_0 \subset V_\lambda$ and the
 569 following holds:

$$\varphi^{V_\lambda}(p_1, \dots, p_n) \leftrightarrow \varphi(p_1, \dots, p_n) \quad (2.67)$$

570 for every $p_1, \dots, p_n \in M$.

571 (iv) Assuming Choice, for every set M_0 there is M such that $M_0 \subset M$ and
 572 $|M| \leq |M_0| \cdot \aleph_0$ and the following holds:

$$\varphi^M(p_1, \dots, p_n) \leftrightarrow \varphi(p_1, \dots, p_n) \quad (2.68)$$

573 for every $p_1, \dots, p_n \in M$.

574 *Proof.* Let's now prove (i) for given φ via induction by complexity. We can safely
 575 assume that φ contains no quantifiers besides " \exists " and no logical connectives
 576 other than " \neg " and "&". Let $\varphi_1, \dots, \varphi_n$ be all subformulas of φ . Then there
 577 is a set M , obtained by the means of lemma (2.11), for all of the formulas
 578 $\varphi_1, \dots, \varphi_n$.

579 Let's first consider atomic formulas in the form of either $x_1 = x_2$ or $x_1 \in x_2$.
 580 It is clear from relativisation¹⁴ that (2.65) holds for both cases, $(x_1 = x_2)^M \leftrightarrow$
 581 $(x_1 = x_2)$ and $(x_1 \in x_2)^M \leftrightarrow (x_1 \in x_2)$.
 582

¹³It can not be smaller because $|M'_{i+1}| \geq |M'_i|$ for every i . It may not be significantly larger because the maximum of elements added is the number of n -tuples in M'_i , which is of the same cardinality as M'_i .

¹⁴See (1.41.). This only holds for relativization to $M, \in \cap M \times M$, not M, R for an arbitrary R .

583 We now want to verify the inductive step. First, take $\varphi = \neg\varphi'$. From
 584 relativization, we get $(\neg\varphi')^M \leftrightarrow \neg(\varphi'^M)$. Because the induction hypothesis tells
 585 us that $\varphi'^M \leftrightarrow \varphi'$, the following holds:

$$(\neg\varphi')^M \leftrightarrow \neg(\varphi'^M) \leftrightarrow \neg\varphi' \quad (2.69)$$

586 The same holds for $\varphi = \varphi_1 \ \& \ \varphi_2$. From the induction hypothesis, we know
 587 that $\varphi_1^M \leftrightarrow \varphi_1$ and $\varphi_2^M \leftrightarrow \varphi_2$, which together with relativization for formulas in
 588 the form of $\varphi_1 \ \& \ \varphi_2$ gives us

$$(\varphi_1 \ \& \ \varphi_2)^M \leftrightarrow \varphi_1^M \ \& \ \varphi_2^M \leftrightarrow \varphi_1 \ \& \ \varphi_2 \quad (2.70)$$

589 Let's now examine the case when $\varphi = \exists x\varphi'(p_1, \dots, p_n, x)$. The induction
 590 hypothesis tells us that $\varphi'^M(p_1, \dots, p_n, x) \leftrightarrow \varphi'(p_1, \dots, p_n, x)$, so, together with
 591 above lemma (2.11), the following holds:

$$\begin{aligned} & \varphi(p_1, \dots, p_n, x) \\ & \leftrightarrow \exists x\varphi'(p_1, \dots, p_n, x) \\ & \leftrightarrow (\exists x \in M)\varphi'(p_1, \dots, p_n, x) \\ & \leftrightarrow (\exists x \in M)\varphi'^M(p_1, \dots, p_n, x) \\ & \leftrightarrow (\exists x\varphi'(p_1, \dots, p_n, x))^M \\ & \leftrightarrow \varphi^M(p_1, \dots, p_n, x) \end{aligned} \quad (2.71)$$

592 Which is what we wanted to prove for part (i).

593

594 We now need to verify that the same holds for any finite number of formulas
 595 $\varphi_1, \dots, \varphi_n$. This has in fact been already done since lemma (2.11) gives us a set
 596 M for any finite amount of formulas and given M_0 . We can therefore find a set
 597 M for the union of all of their subformulas. When we obtain such M , it should
 598 be clear that it also reflects every formula in $\varphi_1, \dots, \varphi_n$.

599

600 Since V_λ is a transitive set, by proving (iii) we also satisfy (ii). To do so, we
 601 only need to look at part (ii) of lemma (2.11). All of the above proof also holds
 602 for $M = V_{\lambda \text{ambda}}$.

603 To finish part (iv), we take M of size $\leq |M_0| \cdot \aleph_0$, which exists due to part
 604 (iii) of lemma (2.11), the rest being identical. \square

605

606 Let S be a set theory defined in (1.19), for ZFC see definition (1.21).

607 The two following lemmas are based on [Drake, 1974][Chapter 3, Theorem
 608 1.2].

609 **Lemma 2.13** *If M is a transitive set, then $\langle M, \in \rangle \models$ Extensionality.*

610 *Proof.* Given a transitive set M , we want to show that the following holds.

$$\langle M, \in \rangle \models \forall x, y (x = y \leftrightarrow \forall z (z \in x \leftrightarrow z \in y)) \quad (2.72)$$

611 Given arbitrary $x, y \in M$, we want to prove that $\langle M, \in \rangle \models (x = y \leftrightarrow \forall z (z \in$
 612 $x \leftrightarrow z \in y))$. This is equivalent to $\langle M, \in \rangle \models x = y$ iff $\langle M, \in \rangle \models \forall z (z \in$
 613 $x \leftrightarrow z \in y)$, which is the same as $x = y$ iff $\langle M, \in \rangle \models \forall z (z \in x \leftrightarrow z \in y)$.

614 So all elements of x are also elements of y in M , and vice versa. Because M is
 615 transitive, all elements of x and y are in M , so $\langle M, \in \rangle \models \forall z (z \in x \leftrightarrow z \in y)$
 616 holds iff x and y contain the same elements and are therefore equal. \square

617 **Lemma 2.14** *If M is a transitive set, then $\langle M, \in \rangle \models \text{Foundation}$.*

618 *Proof.* We want to prove the following:

$$\langle M, \in \rangle \models \forall x (x \neq \emptyset \rightarrow (\exists y \in x)(x \cap y = \emptyset)) \quad (2.73)$$

619 Given an arbitrary non-empty $x \in M$ let's show that $\langle M, \in \rangle \models (\exists y \in$
 620 $x)(x \cap y = \emptyset)$.

621 Because M is transitive, every element of x is an element of M . Take for
 622 y the element of x with the lowest rank¹⁵. It should be clear that there is no
 623 $z \in y$ such that $z \in x$, because then $\text{rank}(z) < \text{rank}(y)$, which would be a
 624 contradiction. \square

625 Let S be a set theory as defined in (1.19).

626 **Lemma 2.15** *The following holds for every λ .*

$$"\lambda \text{ is a limit ordinal}" \rightarrow \langle V_\lambda, \in \rangle \models S \quad (2.74)$$

627 *Proof.* Given an arbitrary limit ordinal λ , we will verify the axioms of S one by
 628 one.

629 (i) *The existence of a set* comes from the fact that V_λ is a non-empty set
 630 because limit ordinal is non-zero by definition.

631 (ii) *Extensionality* holds from (2.13).

632 (iii) *Foundation* holds from (2.14).

633 (iv) *Union*:

634 Given any $x \in V_\lambda$, we want verify that $y = \bigcup x$ is also in V_λ . Note that
 635 $y = \bigcup x$ is a Δ_0 -formula.

$$y = \bigcup x \leftrightarrow (\forall z \in y)(\exists q \in x)z \in q \ \& \ (\forall z \in x)(\forall q \in z)q \in y \quad (2.75)$$

636 So by lemma (1.45)

$$y = \bigcup x \leftrightarrow \langle V_\lambda, \in \rangle \models y = \bigcup x \quad (2.76)$$

¹⁵Rank is defined in (1.30).

637 (v) *Pairing*:

638 Given two sets $x, y \in V_\lambda$, we want to show that $z = \{x, y\}$ is also an
639 element of V_λ .

$$z = \{x, y\} \leftrightarrow x \in z \ \& \ y \in z \ \& \ (\forall q \in z)(q = x \vee q = y) \quad (2.77)$$

640 So $z = \{x, y\}$ is a Δ_0 -formula, and thus by lemma (1.45) it holds that

$$z = \{x, y\} \leftrightarrow \langle V_\lambda, \in \rangle \models z = \{x, y\} \quad (2.78)$$

641 (vi) *Powerset*:

642 Given any $x \in V_\lambda$, we want to make sure that $\mathcal{P}(x) \in V_\lambda$. Let $\varphi(y)$ denote
643 the formula $y \in \mathcal{P}(x) \leftrightarrow y \subset x$. according to definition of subset (1.4),
644 $y \subset x$ is Δ_0 , so for any given $x, y \in V_\lambda$, $y = \mathcal{P}(x) \leftrightarrow \langle V_\lambda, \in \rangle \models y =$
645 $\mathcal{P}(x)$. Because λ is limit and $\text{rank}(\mathcal{P}(x)) = \text{rank}(x) + 1$, if $\mathcal{P}(x) \in V_\lambda$
646 for every $x \in V_\lambda$.

647 (vii) *Specification*:

648 Given a first-order formula φ , we want to show the following:

$$\langle V_\lambda, \in \rangle \models \forall x, p_1, \dots, p_n, \exists y \forall z (z \in y \leftrightarrow z \in x \ \& \ \varphi(z, p_1, \dots, p_n)) \quad (2.79)$$

649 Given any x along with parameters p_1, \dots, p_n in V_λ , we set

$$y = \{z \in x : \varphi^{V_\lambda}(z, p_1, \dots, p_n)\} \quad (2.80)$$

650 From transitivity of V_λ and the fact that $y \subset x$ and $x \in V_\lambda$, we know that
651 $y \in V_\lambda$, so $\langle V_\lambda, \in \rangle \models \forall z (z \in y \leftrightarrow z \in x \ \& \ \varphi(z, p_1, \dots, p_n))$.
652 □

653 **Definition 2.16** (*First-Order Reflection Schema*)

654 For every first-order formula φ , the following is an axiom:

$$\forall M_0 \exists M (M_0 \subseteq M \ \& \ (\varphi(p_1, \dots, p_n) \leftrightarrow \varphi(p_1, \dots, p_n)^M)) \quad (2.81)$$

655 We will refer to this axiom schema as First-order reflection.

656 Let *Infinity* and *Replacement* be as defined in (1.10) and (1.16) respectively.

657 **Theorem 2.17** First-order reflection is equivalent to Infinity & Replacement
658 under S.

659 *Proof.* Since (2.12) already gives us one side of the implication, we are only
660 interested in showing the converse which we shall do in two parts:

661 *First-order reflection* \rightarrow *Infinity* This is done exactly like (2.9). We pick for
 662 φ the formula $(\forall y \in x)(y \cup \{y\} \in x)$, $M_0 = \{\emptyset\}$. From (2.16), there is a set M
 663 that satisfies φ , so there is an inductive set. We have picked M_0 so that $\emptyset \in M$
 664 obviously holds and M is the witness for

$$\exists x(\emptyset \in x \ \& \ (\forall y \in x)(y \cup \{y\} \in x)) \quad (2.82)$$

665 which is exactly (1.10).

666 *First-order reflection* \rightarrow *Replacement*

667 Let's first point out that while *First-order reflection* gives us a set for one
 668 formula, we can generalize it to hold for any finite number of formulas. We will
 669 show how is it done for two formulas, which is what we will use in this proof.
 670 Given two first-order formulas φ, ψ , we can suppose that there are formulas φ'
 671 and ψ' that are equivalent to φ and ψ respectively, but their free variables are
 672 different ¹⁶. Let $\xi = \varphi \ \& \ \psi$, given any M_0 , we can find a M such that $\xi \leftrightarrow \xi^M$.
 673 It is easy to see that from relativisation, the following holds:
 674

$$\varphi \ \& \ \psi \leftrightarrow \varphi' \ \& \ \psi' \leftrightarrow \xi \leftrightarrow \xi^M \leftrightarrow (\varphi' \ \& \ \psi')^M \leftrightarrow \varphi'^M \ \& \ \psi'^M \leftrightarrow \varphi^M \ \& \ \psi^M \quad (2.83)$$

675 Now given a function $\varphi(x, y)$, we know from *First-order reflection* that for
 676 every M_0 , there is a set M such that $M_0 \subseteq M$ and both

$$(\forall x, y \in M)(\varphi(x, y) \leftrightarrow \varphi^M(x, y)) \quad (2.84)$$

677 and

$$(\forall x, y \in M)(\exists y \varphi(x, y) \leftrightarrow (\exists y \varphi(x, y))^M) \quad (2.85)$$

678 hold, the latter being equivalent to

$$(\forall x, y \in M)(\exists y \varphi(x, y) \leftrightarrow (\exists y \in M) \varphi^M(x, y)) \quad (2.86)$$

679 Therefore

$$(\forall x, y \in M)(\exists y \varphi(x, y) \leftrightarrow (\exists y \in M) \varphi(x, y)) \quad (2.87)$$

680 holds too. That means that we have a set M such that for every $x \in M$, if φ is
 681 defined for x , $(\exists y \in M) \varphi(x, y)$.

682 To show that *Replacement* holds for this particular φ , we need to verify that
 683 given a set M_0 , $M'_0 = \{y : (\exists x \in M_0) \varphi(x, y)\}$ is also a set. But since $M_0 \subseteq M$

¹⁶This is plausible since we can for example substitute all free variables in φ' for x_0, x_2, x_4, \dots and use x_1, x_3, x_5, \dots for free variables in ψ' , the resulting formulas will be equivalent.

684 and because given any $x \in M$, there is $y \in M$ satisfying $\varphi(x, y)$, the following
 685 is a set due to *Specification*:

$$M'_0 = \{y : (\exists x \in M_0)\varphi(x, y)\} = \{y \in M : (\exists x \in M_0)\varphi(x, y)\} \quad (2.88)$$

686

□

687

688 We have shown that *Reflection* for first-order formulas, *First-order reflection* is
 689 a theorem of ZFC. We have also shown that it can be used instead of the *Infinity*
 690 and *Replacement* scheme, but $\text{ZFC} + \text{First-order reflection}$ is a conservative
 691 extension of ZF. Besides being a starting point for more general and powerful
 692 statements, it can be used to show that ZF is not finitely axiomatizable. This
 693 follows from the fact that *Reflection* gives a model to any consistent finite set of
 694 formulas. So if $\varphi_1, \dots, \varphi_n$ would be the axioms of ZFC, *Reflection* would prove
 695 that every model of ZFC contains a smaller model of ZFC, which would in turn
 696 contradict the Second Gödel's Theorem¹⁷.

697

698 It is also worthwhile to note that, in a way, *Reflection* is dual to compactness.
 699 Compactness says that given a set of sentences, if every finite subset yields
 700 a model, so does the whole set. *Reflection*, on the other hand, says that while
 701 the whole set has no model in the underlying theory, every finite subset has a
 702 model.

702

703 Furthermore, *Reflection* can be used in ways similar to upward Löwenheim–Skolem
 704 theorem. Since *Reflection* extends any set M_0 into a model of given formulas
 705 $\varphi_1, \dots, \varphi_n$, we can choose the lower bound of the size of M by appropriately
 706 choosing M_0 .

706

707 In the next section, we will try to generalize *Reflection* in a way that tran-
 scends ZF and yields some large cardinals.

¹⁷See chapter ?? for further details.

3 Reflection And Large Cardinals

3.1 Regular Fixed-Point Axioms

Lévy's article mentions various schemata that are not instances of reflection per se, but deal with fixed points of normal ordinal functions. We will introduce them and show that they are equivalent to *First-Order Reflection*¹⁸.

Lemma 3.1 (*Fixed-point lemma for normal functions*)

Let f be a normal function defined for all ordinals¹⁹. Then all of the following hold:

- (i) $\forall \lambda (\text{"}\lambda \text{ is a limit ordinal"} \rightarrow \text{"}f(\lambda) \text{ is a limit ordinal"})$
- (ii) $\forall \alpha (\alpha \leq f(\alpha))$
- (iii) $\forall \alpha \exists \beta (\alpha < \beta \ \& \ f(\beta) = \beta)$
- (iv) *The fixed points of f form a closed unbounded class.*²⁰

Proof. Let f be a normal function defined for all ordinals.

(i) Suppose λ is a limit ordinal. For an arbitrary ordinal $\alpha < \lambda$, the fact that f is strictly increasing means that $f(\alpha) < f(\lambda)$ and for any ordinal β , satisfying $\alpha < \beta < \lambda$, $f(\alpha) < f(\beta) < f(\lambda)$. We know that there is such β from limitness of λ . Because f is continuous and λ is limit, $f(\lambda) = \bigcup_{\gamma < \lambda} f(\gamma)$. That means that if λ is limit, so is $f(\lambda)$.

(ii) This step will be proven using the transfinite induction. Since f is defined for all ordinals, there is an ordinal α such that $f(\emptyset) = \alpha$ and because \emptyset is the least ordinal, (ii) holds for \emptyset .

Suppose (ii) holds for some β from the induction hypothesis. It the holds for $\beta + 1$ because f is strictly increasing.

For a limit ordinal λ , suppose (ii) holds for every $\alpha < \lambda$. (i) implies that $f(\lambda)$ is also limit, so there is a strictly increasing κ -sequence $\langle \alpha_0, \alpha_1, \dots \rangle$ for some κ such that $\lambda = \bigcup_{i < \kappa} \alpha_i$. Because f is stricly increasing, the κ -sequence $\langle f(\alpha_0), f(\alpha_1), \dots \rangle$ is also strictly increasing, the induction hypothesis implies that $\alpha_i \leq f(\alpha_i)$ for each $i \leq \kappa$. Thus, $\lambda \leq f(\lambda)$.

(iii) For a given ordinal α , let there be an ω -sequence $\langle \alpha_0, \alpha_1, \dots \rangle$, such that $\alpha_0 = \alpha$ and $\alpha_{i+1} = f(\alpha_i)$ for each $i < \omega$. This sequence is stricly increasing because so is f . Now, there's a limit ordinal $\beta = \bigcup_{i < \omega} \alpha_i$, we want to show that this is the fixed point. So $f(\beta) = f(\bigcup_{i < \omega} \alpha_i) = \bigcup_{i < \omega} f(\alpha_i)$ because f is continuous. We have defined the above sequence so that $\beta, \bigcup_{i < \omega} f(\alpha_i) = \bigcup_{i < \omega} \alpha_{i+1}$, which means we are done, since $\bigcup_{i < \omega} \alpha_{i+1} = \bigcup_{i < \omega} \alpha_i = \beta$.

¹⁸For definition, see (2.16).

¹⁹For the definition of normal function, see (1.50).

²⁰See (??) for the definition of closed class, (1.52) for the definition of unboundedness.

(iv) The class of fixed points of f is obviously unbounded by (iii). It remains to show that it is closed, this is based on [Drake, 1974], chapter 4. Let Y be a non-empty set of fixed points of f such that $\bigcup Y \notin Y$. Since f is defined on ordinals, Y is a set of ordinals, so $\bigcup Y$ is an ordinal because a supremum of a set of ordinals is an ordinal. $\bigcup Y$ is a limit ordinal. If it were a successor ordinal, suppose that $\alpha + 1 = \bigcup Y$, then $\alpha \in \bigcup Y$, which means that there is some x such that $\alpha \in x \in Y$. But the least such x is $\alpha + 1$, so $\bigcup Y \in Y$.
 Note that $\alpha < \bigcup Y$ iff $\exists \xi \in Y (\alpha < \xi)$. Since f is defined for all ordinals and $\bigcup Y$ is a limit ordinal, $f(\bigcup Y) = \bigcup_{\alpha \in Y} f(\alpha)$, but because Y is a set of fixed points of f , $f(\bigcup Y) = \bigcup_{\alpha \in Y} f(\alpha) = \bigcup Y$, so $\bigcup Y$ is also a limit point of Y .

□

Lemma 3.2 *Let α be a limit ordinal. Then the following hold:*

- (i) *If C is a club set in α , then there is an ordinal β and a normal function $f : \beta \rightarrow \alpha$ such that $\text{rng}(f) = C$. We say that f enumerates C .*
- (ii) *If β is an ordinal and f is a normal function such that $f : \beta \rightarrow \alpha$ and $\text{rng}(f)$ is unbounded in α , then $\text{rng}(f)$ is a closed unbounded set in α .*

This proof comes from (<http://euclid.colorado.edu/~monkd/m6730/gradsets09.pdf> TODO cite!) *Proof.*

- (i) Let β be the order-type²¹ of C , let f be the isomorphism from β onto C . Since $C \subseteq \alpha$, f is also an increasing function from β into α . In order to be continuous, let γ be a limit ordinal under β , let $\epsilon = \bigcup_{\delta < \gamma} f(\delta)$. We want to verify that $f(\gamma) = \epsilon$. Since ϵ is a limit ordinal, we only need to show that $C \cap \epsilon$ is unbounded in ϵ .
 Take $\zeta < \epsilon$. Then there is a $\delta < \gamma$ such that $\zeta < f(\delta)$. Since γ is limit, $\delta + 1 < \gamma$ and also $f(\delta + 1) < f(\gamma)$, we know that $f(\delta) \in C \cap \epsilon$. But that means that $C \cap \epsilon$ is unbounded in ϵ , so $\epsilon \in C$. We have also shown that ϵ is closed unbounded in the image of γ over f . Therefore, $f(\gamma) = \epsilon = \bigcup_{\delta < \gamma} f(\delta)$, so f is normal.

□ It

should be clear that while this lemma works with club subsets of an ordinal, we can formulate analogous statement for club classes, which then yields a normal function defined for all ordinals, with the only exception that there is no such β because f is a function from Ord to Ord .

Definition 3.3 (Axiom Schema M_1)

“Every normal function defined for all ordinals has at least one inaccessible number in its range.”

²¹See definition (1.31).

780 Lévy uses “ M ” to refer to this axiom but since we also use “ M ” for sets and
 781 models, for example in (2.16), we will call the above axiom “*Axiom Schema M_1* ”
 782 to avoid confusion.

783 Let $\varphi(x, y, p_1, \dots, p_n)$ be a first-order formula with no free variables besides
 784 x, y, p_1, \dots, p_n . The following is equivalent to *Axiom M_1* .

$$\begin{aligned} & \text{“}\varphi \text{ is a normal function”} \ \& \ \forall x(x \in \text{Ord} \rightarrow \exists y(\varphi(x, y, p_1, \dots, p_n))) \rightarrow \\ & \rightarrow \exists y(\exists x \varphi(x, y, p_1, \dots, p_n) \ \& \ cf(y) = y \ \& \ (\forall x \in \kappa)(\exists y \in \kappa)(x > y)) \end{aligned} \quad (3.89)$$

785 **Definition 3.4** (*Axiom Schema M_2*)

786 “Every normal function defined for all ordinals has at least one fixed point which
 787 is inaccessible.”

788 **Definition 3.5** (*Axiom Schema M_3*)

789 “Every normal function defined for all ordinals has arbitrarily great fixed points
 790 which are inaccessible.”

791 Similar axiom is proposed in [Drake, 1974].

792 **Definition 3.6** (*Axiom Schema F*)

793 “Every normal function has a regular fixed point.”

794 **Lemma 3.7** Let f be a normal function defined for all ordinals.

- 795 (i) There is a normal function g_1 defined for all ordinals that enumerates the
 796 class $\{\alpha : f(\alpha) = \alpha \ \& \ \alpha \in \text{Ord}\}$.
 797 (ii) There is a normal function g_2 defined for all ordinals that enumerates the
 798 class $\{\lambda : \text{“}f(\lambda) \text{ is a strong limit cardinal.”}\}$.

799 *Proof.* We know that (ii) holds from lemma (3.1) and lemma (3.2).

800 For (i), It should be clear that there is no largest strong limit ordinal ν ,
 801 because the limit of $\nu, \mathcal{P}(\nu), \mathcal{P}(\mathcal{P}(\nu)), \dots$ is again a limit ordinal. The class of
 802 limit ordinals is closed because a limit of strong limit ordinals is clearly always a
 803 strong limit ordinal. Let h be a function enumerating limit ordinals which exists
 804 from lemma (3.2). Then $g_1(\alpha) = f(h(\alpha))$ for every ordinal α is normal and
 805 defined for all ordinals. \square

806 The following is *Theorem 1* in [Lévy, 1960], the parts dealing with *Axiom*
 807 *Schema F* come from [Drake, 1974].

808 **Theorem 3.8** *The following are all equivalent:*

- 809 (i) *Axiom Schema M_1*
 810 (ii) *Axiom Schema M_2*

811 (iii) Axiom Schema M_3

812 (iv) Axiom Schema F

813 *Proof.* It is clear that Axiom Schema M_3 is a stronger version of Axiom Schema
814 M_2 , which is in turn a stronger version of both Axiom Schema M_1 and Axiom
815 Schema F_1 .

816 We will now prove that Axiom Schema $F \rightarrow$ Axiom Schema M_2 . Lemma
817 (3.7) tells us that given a normal function f defined for all ordinals, there is a
818 normal function g_1 defined for all ordinals that enumerates the fixed-points of f .
819 There is also a function g_2 that enumerates the strong limit ordinals in $\text{rng}(f)$.
820 By Axiom Schema F , g_2 has a regular fixed-point κ , which is also a strong limit
821 ordinal, so

$$f(\kappa) = g_2(\kappa) = \kappa \text{ and } \kappa \text{ is inaccessible.} \quad (3.90)$$

822 So every normal function d.f.a.o. has a regular fixed-point.

823 We have yet to show Axiom Schema $M_1 \rightarrow$ Axiom Schema M_3 . Again by
824 lemma (3.7), there is a normal function g defined for all ordinals that enumerates
825 the fixed points of f . Let $h_\alpha(\beta) = g(\alpha + \beta)$ for any given ordinal α , then h_α
826 is a normal function defined for all ordinals. Then, given an arbitrary α , from
827 Axiom Schema M_1 , there is a β such that $\gamma = h_\alpha(\beta)$ is inaccessible. Because
828 $\gamma = g(\alpha + \beta)$, $f(\gamma) = \gamma$. Since $\alpha \leq f'(\alpha)$ for any ordinal α and any normal
829 function f' , we know that $\alpha \leq \alpha + \gamma \leq \gamma$, so γ is inaccessible and arbitrarily
830 large, depending on the choice of α . \square

831 But how do those schemata relate to reflection? Let's introduce a stronger
832 version of *First-order reflection schema* from the previous chapter to see it more
833 clearly. But in order to do this, we must establish the inaccessible cardinal first.

834 3.2 Inaccessible Cardinal

835 **Definition 3.9** An uncountable cardinal κ is inaccessible iff it is regular and
836 strongly limit. We write $In(\kappa)$ to say that κ is an inaccessible cardinal.

837 An uncountable cardinal that is regular and limit is called a *weakly inaccessible*
838 *cardinal*, we will only use the (strongly) inaccessible cardinal, but most of the
839 results are similar for weakly inaccessible, including higher types of ordinals that
840 will be presented later in this chapter.

841 **Theorem 3.10** Let κ be an inaccessible cardinal.

$$\langle V_\kappa, \in \rangle \models \text{ZFC} \quad (3.91)$$

842 We will prove this theorem in a way similar to [Kanamori, 2003]. *Proof.* Most
 843 of this is already done in lemma (2.15), we only need to verify that *Replacement*
 844 and *Infinity* axioms hold in V_κ .

845 *Infinity* holds because κ is uncountable, so $\omega \in V_\kappa$.

846 To verify *Replacement*, let x be an element of V_κ and f a function from x to
 847 V_κ . Let $y = \{z \in V_\kappa : (\exists q \in x)f(q) = z\}$, so $y \subset V_\kappa$, it remains to show that
 848 $y \in V_\kappa$. Because f is a function, we know that $|y| \leq |x| \leq \kappa$. But since κ
 849 is regular, $\{rank(z) : z \in y\} \subseteq \alpha$ for some $\alpha < \kappa$, and so $x \in V_{\alpha+1} \subseteq V_\kappa$.
 850 Therefore $y \in V_\kappa$. \square

851 **Definition 3.11** (*Inaccessible Reflection Schema*)

852 For every first-order formula φ , the following is an axiom:

$$\forall M_0 \exists \kappa (M_0 \subseteq V_\kappa \ \& \ In(\kappa) \ \& \ (\varphi(p_1, \dots, p_n) \leftrightarrow \varphi(p_1, \dots, p_n)^{V_\kappa})) \quad (3.92)$$

853 We will refer to this axiom schema as Inaccessible reflection schema.

854 We have added the requirement that α is inaccessible, which trivially means
 855 that there is an inaccessible cardinal. By taking appropriate M_0 , it can be shown
 856 that in a theory that includes the *Inaccessible reflection schema*, there is a closed
 857 unbounded class of inaccessible cardinals. Since we know that for an inaccessible
 858 κ , V_κ is a model of ZFC, *Inaccessible reflection schema* is equivalent to

$$\forall M_0 \exists \kappa (M_0 \subseteq V_\kappa \ \& \ \langle V_\kappa, \in \rangle \models \text{ZFC} \ \& \ (\varphi(p_1, \dots, p_n) \leftrightarrow \varphi(p_1, \dots, p_n)^{V_\kappa})) \quad (3.93)$$

859 because we have proven in the last section that for an inaccessible κ , $\langle V_\kappa, \in$
 860 $\rangle \models \text{ZFC}$.

861 **Theorem 3.12** Inaccessible reflection schema is equivalent to Axiom schema
 862 F .

863 This is *Theorem 4.1* in chapter four of [Drake, 1974], also equivalent to
 864 *Theorem 3* in [Lévy, 1960]. *Proof.* Let's start by showing that *Inaccessible*
 865 *reflection schema* implies *Axiom schema F*. It should be clear that we can
 866 reflect two formulas to a single set, just form a new formula as a conjunction of
 867 universal closures of the two.

868 Given a normal function f defined for all ordinals, we want to show that it
 869 has a regular fixed point. For any ordinal α , there is an ordinal κ such that

$$\alpha < \kappa \ \& \ In(\kappa) \ \& \ (\forall \gamma, \delta \in V_\kappa) (f(\gamma) = \delta \leftrightarrow (f(\gamma) = \delta)^{V_\kappa}) \quad (3.94)$$

870 and

$$\alpha < \kappa \ \& \ In(\kappa) \ \& \ \forall \gamma \exists \delta (f(\gamma) = \delta) \leftrightarrow (\forall \gamma \exists \delta f(\gamma) = \delta)^{V_\kappa} \quad (3.95)$$

871 Since V_κ is the set of all sets of rank less than κ and since every ordinal is the
872 rank of itself, there is an inaccessible ordinal κ such that

$$\forall \gamma < \kappa \exists \delta < \kappa (f^{V_\kappa}(\gamma) = \delta) \quad (3.96)$$

873 We also know that $f(\gamma) = \delta \leftrightarrow (f(\gamma) = \delta)^{V_\kappa}$. Now since κ is a limit ordinal
874 and f is continuous we get

$$f(\kappa) = \bigcup_{\gamma < \kappa} f^{V_\kappa}(\gamma) = \bigcup_{\gamma < \kappa} f(\gamma). \quad (3.97)$$

875 From (3.96) and the fact that f is increasing, we know that $\kappa \leq \bigcup_{\gamma < \kappa} f(\gamma) \leq \kappa$.
876 Therefore κ is an inaccessible fixed point of f .

877 For the opposite direction, it suffices to show that since there is an inacces-
878 sible cardinal from *Axiom schema F*, given a first-order formula φ , there is an
879 arbitrarily large inaccessible cardinal κ for which

$$\varphi \leftrightarrow \langle V_\kappa, \in \rangle \models \varphi. \quad (3.98)$$

880 Note that the arbitrary size of κ means given an arbitrary ordinal α , there is a
881 κ satisfying (3.98). In the previous chapter, in theorem (2.12), we have shown
882 that we can easily obtain a limit ordinal satisfying (3.98). Note that since for
883 any set M_0 , there is such α that $M_0 \subseteq V_\alpha$, there is a closed unbounded class of
884 sets satisfying (3.98), which are levels in the cumulative hierarchy, so there is a
885 club sets of κ s satisfying (3.98).

886 Let f be a normal function defined for all ordinals that enumerates this club
887 class, there is such by lemma (3.2). Let g be the function that enumerates
888 strong limit ordinals in $\text{rng}(f)$. Then g has a regular fixed point κ , which is also
889 a regular fixed point of f , so (3.98) holds for κ .

890

□

891 **Definition 3.13** (ZMC)

892 We will call ZMC an axiomatic set theory that contains all axioms and schemas
893 of ZFC together with Axiom Schema M_1 .

894 We have decided to call it ZMC, because Lévy uses ZM, derived from ZF, which
895 is more intuitive, but we also need the axiom of choice, thus, ZMC.

896 As a sidenote, we should note that ZMC is extension of ZFC, which is in turn
897 an extension of S.

3.3 Mahlo Cardinals

We have shown that ZMC contains arbitrarily large inaccessible cardinals. To return to reflection-style argument, is there a set that satisfies this property? To be able to properly answer this question, we have to formulate the notion of “containing arbitrarily large cardinals” more carefully. While we have previously used club sets, this is not an option because inaccessible cardinals don’t form a club class in ZMC²², we could try to formulate stronger versions of *Axiom Schema M_1* .

Let’s shortly review what *Axiom Schema M_1* says. We have shown earlier in this chapter that there is a simple relation between normal function defined for all ordinals and closed unbounded classes. So by saying that for a class of ordinals C , a normal function f has at least one element of C in its range, we say that C is stationary. Or, as Drake puts it for C , the class of inaccessible cardinals, and a κ , in which C is stationary:

“The class of inaccessible cardinals is so rich that there are members κ of the class such that no normal function on κ can avoid this class; however we climb through κ , provided we are continuous at limits (so that we are enumerating a closed subset of κ), we shall eventually have to hit an inaccessible.”

Definition 3.14 (Mahlo Cardinal)

We say that κ is a Mahlo Cardinal iff it is an inaccessible cardinal and the set $\{\lambda < \kappa : \lambda \text{ is inaccessible}\}$ is stationary in κ .

Alternatively, κ is Mahlo iff $\langle V_\kappa, \in \rangle \models \text{ZMC}$ as shown above, this is also sometimes written as *Ord is Mahlo*. There are also *weakly Mahlo cardinals*, that are defined via weakly inaccessible cardinals below them, Mahlo cardinals are then also called *strongly Mahlo* to highlight the difference, but we will only use the term *Mahlo cardinal*.

Mahlo cardinals are related to reflection principles in an interesting way. Note that given a formula φ , reflection gave us a club set of ordinals α such that V_α reflects φ , all below the first inaccessible cardinal. We have then used a different reflection schema to obtain arbitrarily high inaccessible cardinals. Now we have a cardinal in which this reflection schema holds, so we are in fact reflecting reflection. Beware that this is done rather informally, because *Axiom Schema M_1* is a countable set of axioms, which is too large to be reflected via the schemas introduced so far. One way to deal with this would be to extend reflection for second- and possibly higher-order formulas, but we would have to be very careful with the notion of satisfaction. For now, let us explore where can stationary sets take us because as we have shown, their connection to reflection is quite clear.

²²Note that cofinality of the limit of the first ω inaccessibles is ω , which makes it singular.

935 What would happen if we strengthened *Axiom Schema* M_1 to say that every
936 normal function has a Mahlo cardinal in its range?

937 **Definition 3.15** (*hyper-Mahlo cardinal*)

938 We say that κ is a hyper-Mahlo cardinal iff it is inaccessible and the set $\{\lambda < \kappa : \lambda \text{ is Mahlo}\}$ is stationary in κ .
939

940 **Definition 3.16** (*hyper-hyper-Mahlo cardinal*)

941 We say that κ is a hyper-hyper-Mahlo cardinal iff it is inaccessible and the set
942 $\{\lambda < \kappa : \lambda \text{ is hyper-Mahlo}\}$ is stationary in κ .

943 It is clear that one can continue in this direction, but the nomenclature gets
944 increasingly overwhelming even if we introduce *hyper $^\alpha$ -Mahlo cardinals*. To see
945 there is a more elegant way to reach those cardinals, we will now establish an
946 operation that exhausts all such cardinals in a more unified manner.

947 **Definition 3.17** (*Mahlo Operation*)

948 Let A be a class of ordinals. Let

$$H(A) = \{\alpha \in A : A \cap \alpha \text{ is stationary in } \alpha\}. \quad (3.99)$$

949 We call H the Mahlo's operation.

950 If we pick for A the class of all inaccessible cardinals, $H(A)$ is the class of
951 Mahlo cardinals. It is easy to see that if A is the class of all α -Mahlo cardinals,
952 $H(A)$ is the class of $\alpha + 1$ -Mahlo cardinals, $H(H(A))$ is the class of $\alpha + 2$ -
953 Mahlo cardinals and so on.

954 **Definition 3.18** (*Iterated Mahlo Operation*)

955 Let A be a class of ordinals. We shall extend the Mahlo operation in the following
956 way:

- 957 (i) $H^0(A) = A$,
- 958 (ii) $H^{\alpha+1}(A) = H(H^\alpha(A))$,
- 959 (iii) $H^\lambda(A) = \bigcap_{\alpha < \lambda} H^\alpha(A)$ for limit λ .

960 Clearly if A is the class of inaccessibles, $H^\alpha(A)$ is the class of α -Mahlo
961 cardinals. To get to hyper-Mahlo cardinals, we can diagonalize the operation.

962 **Definition 3.19** (*Diagonal Mahlo Operation*)

963 Let A be a class of ordinals. Then the diagonal Mahlo operation is defined as
964 follows:

$$H^\Delta(A) = \{\alpha : \forall \beta < \alpha (\alpha \in H^\beta(A))\}. \quad (3.100)$$

965 We can further diagonalize the diagonal version and continue this process ad
966 libitum in order to reach all large cardinals accessible “from below”.

967 TODO par slov o tom co je “from below” (určité až sem?)

3.4 Indescribable Cardinals

Indescribability is another approach towards large cardinals that is based on reflection. We will briefly introduce the basic definitions and show that it yield larger objects, but most of them are not reachable from below

Most of the results presented in this subchapter are taken from [Kanamori, 2003].

Since this chapter uses higher-order logic, we need to introduce the hierarchy of formulas first.

Definition 3.20 (Higher-Order Variables)

Let M be a structure and D it's domain. In first-order logic, variables range over individuals, that is, over elements of D . We shall call those type 1 variables for the purposes of higher-order logic. Type 2 variables then range over collections, that is, the elements of $\mathcal{P}(D)$. Generally, type n variables are defined for any $n \in \omega$ such that they range over $\mathcal{P}^{n-1}(D)$.

We will use lowercase latin letters for type 1 variables for backward compatibility with first-order logic, type 2 variables will be represented by uppercase letters, mostly P, X, Y, Z . If we ever stumble upon type 3 variables in this text, they shall be represented as $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ or in a similar font.

Definition 3.21 (Full Prenex Normal Form)

We say a formula is in the prenex normal form if it is written as a block of quantifiers followed by a quantifier-free part.

We say a formula is in the full prenex normal form if it is written in prenex normal form and if there are type $n + 1$ quantifiers, they are written before type n quantifiers.

It is an elementary that every formula is equivalent to a formula in the prenex normal form.

Definition 3.22 (Hierarchy of Formulas)

Let φ be a formula in the prenex normal form.

- (i) We say φ is a Δ_0^0 -formula if it contains only bounded quantifiers.
- (ii) We say φ is a Σ_0^0 -formula or a Π_0^0 -formula if it is a Δ_0^0 -formula.
- (iii) We say φ is a Π_0^{m+1} -formula if it is a Π_n^m - or Σ_n^m -formula for any $n \in \omega$ or if it is a Π_n^m - or Σ_n^m -formula with additional free variables of type $m + 1$.
- (iv) We say φ is a Σ_0^m -formula if it is a Π_0^m -formula.
- (v) We say φ is a Σ_n^{m+1} -formula if it is of a form $\exists P_1, \dots, P_i \psi$ for any non-zero i , where ψ is a Π_n^m -formula and P_1, \dots, P_i are type $m + 1$ variables.
- (vi) We say φ is a Π_n^{m+1} -formula if it is of a form $\forall P_1, \dots, P_i \psi$ for any non-zero i , where ψ is a Σ_n^m -formula and P_1, \dots, P_i are type $m + 1$ variables.

1004 **Definition 3.23** (*Describability*)

1005 We say an ordinal α is described by a sentence φ in the language \mathcal{L} with relation
1006 symbols P_1, \dots, P_n given iff

$$\langle V_\alpha, \in, P_1, \dots, P_n \rangle \models \varphi \quad (3.101)$$

1007 but for every $\beta < \alpha$

$$\langle V_\beta, \in, P_1 \cap V_\beta, \dots, P_n \cap V_\beta \rangle \not\models \varphi \quad (3.102)$$

1008 For the definition of a Π_n^m -formula and a Σ_n^m -formula, see (3.22).

1009 **Definition 3.24** (Π_n^m -Indescribable Cardinal)

1010 We say that κ is Π_n^m -indescribable iff it is not described by any Π_n^m -formula.

1011 **Definition 3.25** (Σ_n^m -Indescribable Cardinal)

1012 We say that κ is Σ_n^m -indescribable iff it is not described by any Σ_n^m -formula.

1013 To see that this notion is based in reflection, let us recall the opening quote
1014 of this thesis by Gödel which says “The Universe of sets cannot be uniquely
1015 characterised (i. e. distinguished from all its initial elements) by any internal
1016 structural property of the membership relation in it.”. A cardinal κ is Π_n^m -
1017 indescribable²³ iff every Π_n^m -formula fails to describe V_κ and describes an initial
1018 segment instead. In a sense, V_κ reflects the “property”²⁴ of indescribability of
1019 the universal class with respect to certain classes of formulas.

1020

1021 **Lemma 3.26** Let κ be a cardinal, then the following holds for any $n \in \omega$. κ is
1022 Π_n^1 -indescribable iff κ is Σ_{n+1}^1 -indescribable

1023 *Proof.* The forward direction is obvious, we can always add a spare quantifier
1024 over a type 2 variable to turn a Π_n^1 formula φ into a $\exists P\varphi$ which is obviously a
1025 Σ_{n+1}^1 -formula.²⁵

1026 To prove the opposite direction, suppose that $\langle V_\kappa, \in \rangle \models \exists X\varphi(X)$ where
1027 X is a type 2 variable and φ is a Π_n^1 formula with one free variable of type
1028 2. This means that there is a set $S \subseteq V_\kappa$ that is a witness of $\exists X\varphi(X)$, in
1029 other words, $\varphi(S)$ holds. We can replace every occurrence of X in φ by a new

²³This holds for Σ_n^m -formulas alike.

²⁴In this case, we are not using the word to refer to a definable class, but on a meta level to refer to a property expressible in the natural language, hence the quotation marks.

²⁵Note that unlike in previous sections, it is worth noting that φ is now a sentence so we don't have to worry whether P is free in φ .

1030 predicate symbol S , this allows us to say that κ is Π_n^1 -indescribable (with respect
 1031 to $\langle V_\kappa, \in, R, S \rangle$).²⁶ \square

1032 The above lemma makes it clear that we can suppose that all formulas with
 1033 no higher than type 2 variables are Π_n^1 -formulas, $n \in \omega$, without the loss of
 1034 generality.

1035 **Lemma 3.27** *If κ is an inaccessible cardinal and given $R \subseteq V_\kappa$, then the*
 1036 *following is a club set in κ :*

$$\{\alpha : \alpha < \kappa \ \& \ \langle V_\alpha, \in, R \cap V_\alpha \rangle \prec \langle V_\kappa, \in, R \rangle\} \quad (3.103)$$

1037 *Proof.* To see that (3.103) is closed, let us recall that a $A \subseteq \kappa$ is closed iff for
 1038 every ordinal $\emptyset < \alpha < \lambda$: if $A \cap \alpha$ is unbounded in α then $\alpha \in A$. Since κ is an
 1039 inaccessible cardinal, thus strong limit, it is closed under limits of sequences of
 1040 ordinals lesser than κ .

1041 We want to verify that it is unbounded, we will use a recursively defined
 1042 sequence $\alpha_0, \alpha_1, \dots$ to build an elementary substructure of $\langle V_\kappa, \in, R \rangle$ that is
 1043 built above an arbitrary $\alpha_0 < \kappa$. Let us fix an arbitrary $\alpha_0 < \kappa$. Given α_n ,
 1044 α_{n+1} is defined as the least β , $\alpha_n \leq \beta$ that satisfies the following for any formula
 1045 φ , $p_1, \dots, p_m \in V_{\alpha_n}$, $m \in \omega$:

$$\text{If } \langle V_\kappa, \in, R \rangle \models \exists x \varphi(p_1, \dots, p_n), \text{ then } \langle V_\kappa, \in, R \rangle \models \varphi(x, p_1, \dots, p_n) \quad (3.104)$$

1046 Let $\alpha = \bigcup_{n < \omega} \alpha_n$.

1047 Then $\langle V_\alpha, \in, R \cap V_\alpha \rangle \prec \langle V_\kappa, \in, R \rangle$, in other words, for any φ with given
 1048 arbitrary parameters $p_1, \dots, p_n \in V_\alpha$, it holds that

$$\langle V_\alpha, \in, R \cap V_\alpha \rangle \models \varphi(p_1, \dots, p_n) \leftrightarrow \langle V_\kappa, \in, R \rangle \models \varphi(p_1, \dots, p_n) \quad (3.105)$$

1049 Which should be clear from the construction of α \square

1050 **Theorem 3.28** *Let κ be an ordinal. The following are equivalent.*

- 1051 (i) κ is inaccessible
 1052 (ii) κ is Π_0^1 -indescribable.

1053 *Proof.* Since Π_0^1 -sentences are first-order sentences, we want to prove that κ is
 1054 an inaccessible cardinal iff whenever a first-order tries to describe κ in the sense
 1055 of definition (3.23), the formula fails to do so and describes a initial segment

²⁶A different yet interesting approach is taken by Tate in [Tait, 2005]. He states that for $n \geq 0$, a formula of order $\leq n$ is called a Π_0^n and a Σ_0^n formula. Then a Π_{m+1}^n is a formula of form $\forall Y \psi(Y)$ where ψ is a Σ_m^n formula and Y is a variable of type n . Finally, a Σ_{m+1}^n is the negation of a Π_m^n formula. So the above holds ad definitio.

thereof instead. We have already shown in (??) that there is no way to reach an inaccessible cardinal via first-order formulas in ZFC. We will now prove it again in for formal clarity.

For (i)→(ii), suppose that κ is inaccessible.

Then there is, by lemma (3.27) a club set of ordinals α such that V_α is an elementary substructures of V_κ . For κ to be Π_0^1 -indescribable, we need to make sure that given an arbitrary first-order sentence φ satisfied in the structure $\langle V_\kappa, \in, R \rangle$, there is an ordinal $\alpha < \kappa$, such that $\langle V_\alpha, \in, R \cap V_\alpha \rangle \models \varphi$. But this follows from the definition of elementary substructure.

For (ii)→(i), suppose κ is not inaccessible, so it is either singular, or there is a cardinal $\nu < \kappa$ such that $\kappa \leq \mathcal{P}(\nu)$ or $\kappa = \omega$.

Suppose κ is singular. Then there is a cardinal $\nu < \kappa$ and a function $f : \nu \rightarrow \kappa$ such that $\text{rng}(f)$ is cofinal in κ . Since $f \subseteq V_\kappa$, we can add f as a relation to the language. We can do the same with $\{\nu\}$. That means $\langle V_\kappa, \in, P_1, P_1 \text{ with } P_1 = f, P_2 = \{\nu\} \rangle$ is a structure, let $\varphi = P_1 \neq \emptyset \ \& \ \text{rng}(P_1) = P_2$ ²⁷. Since for every $\alpha < \nu$, $P_1 \cap V_\alpha = \emptyset$, φ is false and therefore describes κ . That contradicts the fact that κ was supposed to be Π_0^1 -indescribable, but φ is a first-order formula.

Suppose there a cardinal ν satisfying $\kappa \leq \mathcal{P}(\nu)$. Let there be a function $f : \mathcal{P}(\nu) \rightarrow \kappa$ that is onto. Then, like in the previous paragraph, we can obtain a structure $\langle V_\kappa, \in, P_1, P_2 \rangle$, where $P_1 = f$ like before, but this time $P_2 = \mathcal{P}(\nu)$. Again, $\varphi = P_1 \neq \emptyset \ \& \ \text{rng}(P_1) = P_2$ describes κ .

Finally, suppose $\kappa = \omega$, then the sentence $\varphi = \forall x \exists y (x \in y)$ describes κ , there is obviously no $\alpha < \omega$ such that $\langle V_\alpha, \in \rangle \models \varphi$.

□

Generally, it should be clear that if a cardinal κ is Π_n^m -indescribable, it is also $\Pi_{n'}^{m'}$ -indescribable for every $m' < m, n' < n$. By the same line of thought, if a cardinal κ satisfies the property implied by Π_n^m -indescribability, it satisfies all properties implied by $\Pi_{n'}^{m'}$ -indescribability for $m' < m, n' < n$. For example, if κ is Π_n^m -indescribable for $m \geq 1$ then it is also an inaccessible cardinal.

Theorem 3.29 *If a cardinal κ is Π_1^1 -indescribable, then it is a Mahlo cardinal.*

Proof. Assuming that κ is Π_1^1 -indescribable, we want to prove that every club set of in κ contains an inaccessible cardinal.

Consider the following Π_1^1 -sentence φ :

$$\begin{aligned} \varphi = \forall P (& \text{“} P \text{ is a function”} \rightarrow \forall x \exists y \forall z (z \in y \leftrightarrow (\exists q \in x) (P(x, y, p_1, \dots, p_n)))) \\ & \& \forall x \exists y \forall z (z \in y \leftrightarrow z \subseteq x) \end{aligned} \quad (3.106)$$

²⁷ $\text{rng}(x) = y$ is a first-order formula, see (1.14).

where P is a type 2 variable and the rest are type 1 variables, “ P is a function” is a first-order formula defined in (1.11). As has been shown earlier in this chapter, given a cardinal μ , the following holds if and only if μ is inaccessible:

$$\langle V_\mu, \in \rangle \models \varphi \quad (3.107)$$

Now fix an arbitrary $C \subset \kappa$, a club set in κ . We want to show that it contains an inaccessible cardinal. Since C is a subset of κ , let's add it to the structure $\langle V_\kappa, \in \rangle$, turning it into $\langle V_\kappa, \in, C \rangle$. Then the following holds:

$$\langle V_\kappa, \in, C \rangle \models \text{Inac} \ \& \ \text{“}C \text{ is unbounded”}^{28} \quad (3.108)$$

Note that this holds because κ is Π_1^1 -indescribable, and therefore also Π_0^1 -indescribable.

So κ is itself inaccessible and therefore $\langle V_\kappa, \in, C \rangle \models \varphi$.

Since κ is Π_1^1 -indescribable and $\varphi \ \& \ \text{“}C \text{ is unbounded”}$ is equivalent to a Π_1^1 -formula, there must be an ordinal α that satisfies

$$\langle V_\alpha, \in, C \cap V_\alpha \rangle \models \text{Inac} \ \& \ \text{“}C \text{ is unbounded”}, \quad (3.109)$$

which implies that α is inaccessible; it is regular because it reflects *Replacement* and it is limit because if α were a successor ordinal, it couldn't contain an unbounded class of ordinals.

We only need to verify that $\alpha \in C$, which is clear from the fact that C is a club set in κ and it is unbounded in α . \square

There is an even stronger large cardinal property implied by Π_1^1 -indescribability that is based on reflection.

Definition 3.30 (*Extension Property*)

We say a cardinal κ has the extension property iff for all $U \subset V_\kappa$ there exists a transitive set X such that $\kappa \in X$, and a set $S \subset X$, such that (V_κ, \in, U) is an elementary substructure of (X, \in, S) .

Definition 3.31 (*Weakly Compact Cardinal*)

We say that a cardinal κ is weakly compact iff it has the extension property.

Theorem 3.32 A cardinal κ is Π_1^1 -indescribable iff it is weakly compact.

For the proof, see [Kanamori, 2003].

TODO slabe kompaktni a reflexe?

Definition 3.33 (*Ultrafilter*)

Given a set x , we say $U \subset \mathcal{P}(x)$ is an ultrafilter over x iff all of the following hold:

²⁸“ C is unbounded” is a first-order formula, see (1.52).

- 1119 (i) $\emptyset \notin U$
 1120 (ii) $\forall y, z (\subset x \ \& \ y \subset z \ \& \ y \in U \rightarrow z \in U)$
 1121 (iii) $\forall y, z \in U (y \cap z) \in U$
 1122 (iv) $\forall y (y \subset x \rightarrow (y \in U \vee (x \setminus y) \in U))$

1123 **Definition 3.34** (κ -Complete Ultrafilter)

1124 We say that an ultrafilter U is κ -complete iff

1125 **Definition 3.35** (Measurable Cardinal)

1126 Let κ be a cardinal. We say κ is a measurable cardinal iff there is a κ -complete
 1127 ultrafilter over κ .

1128 **Theorem 3.36** Let κ be a cardinal. If κ is a measurable cardinal then the
 1129 following hold:

- 1130 (i) κ is Π_1^2 -indescribable.
 1131 (ii) Given U , a normal ultrafilter over κ , a relation $R \subseteq V_\kappa$ and a Π_1^2 -formula
 1132 φ such that $\langle V_\kappa, \in, R \rangle \models \varphi$, then

$$\{\alpha < \kappa : \langle V_\alpha, \in, R \cap V_\alpha \rangle \models \varphi\} \in U \quad (3.110)$$

1133 For a proof, see Proposition 6.5 in [Kanamori, 2003].

1134 **Theorem 3.37** If κ is a measurable cardinal and U is a normal ultrafilter over
 1135 κ , the following holds:

$$\{\alpha < \kappa : "\alpha \text{ is totally indescribable}"\} \in U \quad (3.111)$$

1136 For a proof, see Proposition 6.6 in [Kanamori, 2003].

1137 3.5 The Constructible Universe

1138 The constructible universe, denoted L , is a cumulative hierarchy of sets, presented
 1139 by Kurt Gödel in his paper [Gödel and Brown, 1940]. Assertion of their equality,
 1140 $V = L$, is called the *Axiom of Constructibility*. The axiom implies GCH and
 1141 therefore also AC and contradicts the existence of some large cardinals, our goal
 1142 is to decide whether those introduced earlier are among them.

1143 On order to formally establish this class, we need to formalize the notion of
 1144 definability first.

1145 **Definition 3.38** (Definability)

1146 We say that a set X is definable over a model $\langle M, \in \rangle$ if there is a formula φ
 1147 together with parameters $p_1, \dots, p_n \in M$ such that

$$X = \{x : x \in M \ \& \ \langle M, \in \rangle \models \varphi(x, p_1, \dots, p_n)\} \quad (3.112)$$

1148 **Definition 3.39** (*The Set of Definable Subsets*)

1149 *The following is a set of all definable subsets of a given set M , denoted $Def(M)$.*

$$Def(M) = \{\{y : x \in M \wedge \langle M, \in \rangle \models \varphi(y, u_1, \dots, i_n)\} \mid \varphi \text{ is a first-order formula, } p_1, \dots, p_n \in M\} \quad (3.113)$$

1150 We will use $Def(M)$ in the following construction in the way the powerset
1151 operation is used when constructing the usual Von Neumann's hierarchy of sets²⁹.

1152 Now we can recursively build L .

1153 **Definition 3.40** (*The Constructible Universe*)

1154

$$(i) \quad L_0 = \emptyset \quad (3.114)$$

$$(ii) \quad L_{\alpha+1} = Def(L_\alpha) \text{ for any ordinal } \alpha \quad (3.115)$$

$$(iii) \quad L_\lambda = \bigcup_{\alpha < \lambda} L_\alpha \text{ For a limit ordinal } \lambda \quad (3.116)$$

$$(iv) \quad L = \bigcup_{\alpha \in Ord} L_\alpha \quad (3.117)$$

1155 Note that while L bears very close resemblance to V , the difference is, that
1156 in every successor step of constructing V , we take every subset of V_α to be
1157 $V_{\alpha+1}$, whereas $L_{\alpha+1}$ consists only of definable subsets of L_α . Also note that L
1158 is transitive.

1159 In order to

1160 **Theorem 3.41** *Let L be as in (3.40).*

$$L \text{ is a model of ZFC} \quad (3.118)$$

1161 For details, refer to Theorem 13.3 in [Jech, 2006].

1162 **Definition 3.42** (*Constructibility*)

1163 *The axiom of constructibility say that every set is constructible. It is usually*
1164 *denoted as $L = V$.*

²⁹For that reason, some authors use $\mathcal{P}^*(M)$ instead of $Def(M)$, see section 11 of [Pinter, 2014] for one such example.

Without providing a proof, we will introduce two important results established by Gödel in his aforementioned article. ZF stands for Zermelo–Fraenkel set theory as introduced in (1.20).

Theorem 3.43 (*Constructibility \rightarrow Choice*)

$$\text{ZF} \vdash \text{Constructibility} \rightarrow \text{Choice} \quad (3.119)$$

The *GCH* refers to the *Generalised Continuum Hypothesis*, see (1.40).

Theorem 3.44 (*Constructibility \rightarrow Generalised Continuum Hypothesis*)

$$\text{ZF} \vdash \text{Constructibility} \rightarrow \text{GCH} \quad (3.120)$$

It is worth mentioning that Gödel's proof of *Constructibility \rightarrow GCH* featured the first formal use of a reflection principle. For the actual proofs, see for example [Kunen, 1983],

Since *GCH* implies that κ is a limit cardinal iff κ is a strong limit cardinal for every κ , the distinctions between inaccessible and weakly inaccessible cardinals as well as between Mahlo and weakly Mahlo cardinals vanish.

Theorem 3.45 (*Inaccessibility in L*)

Let κ be an inaccessible cardinal. Then “ κ is inaccessible” ^{L} .

Proof. We want to show that the following are all true for an inaccessible cardinal κ :

- (i) “ κ is a cardinal” ^{L}
- (ii) $(\omega < \kappa)^L$
- (iii) “ κ is regular” ^{L}
- (iv) “ κ is limit” ^{L} .³⁰

Suppose “ κ is not a cardinal” ^{L} holds, then there is a cardinal μ , $\mu < \kappa$ and a function $f : \mu \rightarrow \kappa$, $f \in L$, such that “ $f : \mu \rightarrow \kappa$ is onto” ^{L} . But since “ f is onto” is a Δ_0 formula and Δ_0 formulas are absolute in transitive structures³¹ and L is a transitive class, “ f is onto” ^{L} \leftrightarrow “ f is onto”, this contradicts the fact that κ is a cardinal. $(\omega < \kappa)^L$ holds because $\omega \in \kappa$ and because ordinals remain ordinals in L , so $(\omega \in \kappa)^L$.

In order to see that “ κ is regular” ^{L} , we can repeat the argument by contradiction used to show that κ is a cardinal in L . If κ was singular, there is a $\mu < \kappa$ together with a function $f : \mu \rightarrow \kappa$ that is onto, but since “ f is onto” implies

³⁰While inaccessible cardinals are strong limit cardinals, since *GCH* holds in L , “ κ is limit” ^{L} implies “ κ is strong limit” ^{L} .

³¹See lemma (1.45).

1194 “ f is onto” ^{L} , we have reached a contradiction with the fact that κ is regular,
 1195 but singular in L .

1196 It now suffices to show that “ κ is a limit cardinal” ^{L} . That means, that for
 1197 any given $\lambda < \kappa$, we need to find an ordinal μ such that $\lambda < \mu < \kappa$ that is also
 1198 a cardinal in L . But since cardinals remain cardinals in L by an argument with
 1199 surjective functions just like above, it holds. \square

1200 **Theorem 3.46** (*Mahloness in L*)

1201 *Let κ be a Mahlo cardinal. Then “ κ is Mahlo” ^{L} .*

1202 *Proof.* Let κ be a Mahlo cardinal. From the definition of Mahloness in (3.14),
 1203 it should be clear that we want prove that κ is inaccessible in L and

$$\text{“The set } \{\alpha : \alpha \in \kappa \ \& \ \alpha \text{ is inaccessible}\} \text{ is stationary in } \kappa^L \text{”} \quad (3.121)$$

1204 Since we have shown that inaccessible cardinals remain inaccessible in L
 1205 in the previous theorem, L “ κ is inaccessible” ^{L} holds.

1206 Now consider the two following sets:

(i)

$$S \stackrel{\text{def}}{=} \{\alpha : \alpha \in \kappa \ \& \ \alpha \text{ is inaccessible}\} \quad (3.122)$$

(ii)

$$T \stackrel{\text{def}}{=} \{\alpha : \alpha \in \kappa \ \& \ \alpha \text{ is inaccessible}\}^L \quad (3.123)$$

1207 Since inaccessible cardinals are inaccessible in L from theorem (3.45), $S \subseteq T$.
 1208 So if T is stationary in κ , we are done. Suppose for contradiction that it is not
 1209 the case. Therefore there is a $C \subset \kappa$ satisfying “ C is a club set in κ^L ”, but it is
 1210 the case that $T \cap C = \emptyset$. But because “ C is a club set in κ ” is equivalent to a
 1211 Δ_0 formula, “ C is a club set in κ^M ” \leftrightarrow “ C is a club set in κ ”, ergo C is a club
 1212 set in κ . But since it has o intersection with T , it can’t have an intersection with
 1213 a subset thereof, which contradicts the fact that S is stationary in κ .

1214 κ remains Mahlo in L . \square

1215 It should be clear that the above process can be iterated over again. Since
 1216 Mahlo cardinals are absolute in L , the same argument using stationary sets can
 1217 be carried out for hyper-Mahlo cardinals and so on. It is clear that since a regular
 1218 and an inaccessible cardinal in consistent with *Constructibility*, so should be the
 1219 higher properties acquired from assuring the existence of regular, inaccessible and
 1220 Mahlo fixed points of normal functions.

1221 **4 Conclusion**

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