Univerzita Karlova v Praze, Filozofická fakulta
Katedra logiky

Mikuláš Mrva

REFLECTION PRINCIPLES AND LARGE CARDINALS

Bakalářská práce

Vedoucí práce: Mgr. Radek Honzík, Ph.D.

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Prohlašuji, že jsem bakalářskou práci vypracoval samostatně a že jsem uvedl všechny použité prameny a literaturu.

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Mikuláš Mrva

Abstract

Práce zkoumá vztah tzv. principů reflexe a velkých kardinálů. Lévy ukázal, že v ZFC platí tzv. věta o reflexi a dokonce, že věta o reflexi je ekvivalentní schématu nahrazení a axiomu nekonečna nad teorií ZFC bez axiomu nekonečna a schématu nahrazení. Tedy lze na větu o reflexi pohlížet jako na svého druhu axiom nekonečna. Práce zkoumá do jaké míry a jakým způsobem lze větu o reflexi zobecnit a jaký to má vliv na existenci tzv. velkých kardinálů. Práce definuje nedosažitelné, Mahlovy a nepopsatelné kardinály a ukáže, jak je lze zavést pomocí reflexe. Přirozenou limitou kardinálů získaných reflexí jsou kardinály nekonzistentní s L. Práce nabídne intuitivní zdůvodněn, proč tomu tak je.

Abstract

This thesis aims to examine relations between the so called Reflection Principles and Large cardinals. Lévy has shown that the Reflection Theorem is a sound theorem of ZF and it is equivalent to the Replacement Scheme and the Axiom of Infinity. From this point of view, Reflection theorem can be seen a specific version of an Axiom of Infinity. This paper aims to examine the Reflection Principle and its generalisations with respect to the existence of Large Cardinals. This thesis will establish the Inaccessible, Mahlo and Indescribable cardinals and show how can those be defined via reflection. A natural limit of Large Cardinals obtained via reflection are cardinals inconsistent with L. This thesis will offer an intuitive explanation of why this holds.

Contents

1 Introduction

1.1 Motivation and Origin

> The Universe of sets cannot be uniquely characterised (i. e. distinguished from all its initial elements) by any internal structural property of the membership relation in it, which is expressible in any logic of finite of transfinite type, including infinitary logics of any cardinal order.
>
> — Kurt Gödel [Wang, 1997]

1.2 Notation and Terminology

1.2.1 The Language of Set Theory

This text assumes the knowledge of basic terminology and some results from first-order predicate logic. [1]

We will now shortly review the basic notions that allow us to define *Zermelo–Fraenkel* set theory.

When we talk about *class*, we have the notion of definable class in mind. If $\varphi(x, p_1, \ldots, p_n)$ is a formula in the language of set theory, we call

$$A = \{x : \varphi(x)\} \tag{1.1}$$

a class of all sets satisfying $\varphi(x)$ in a sense that

$$x \in A \leftrightarrow \varphi(x) \tag{1.2}$$

Given classes A, B, one can easily define the elementary set operations such as $A \cap B$, $A \cup B$, $A \setminus C$, $\bigcup A$, see the first part of [Jech, 2006] for details. The following axioms are the tools by which decide whether particular classes are in fact sets. A class that fails to be considered a set is called a *proper class.*

Speaking of formulas, we will often use syntax like "M is a limit ordinal". It should be clear that this statement can be rewritten as a formula that was introduced earlier in the text.

1.2.2 The Axioms

Definition 1.1 *(The Existence of a Set)*

$$\exists x (x = x) \tag{1.3}$$

[1] todo odkaz na pripadny zdroj? svejdar? neco en?

The above axiom is usually not used because it can be deduced from the axiom of *Infinity* (see below), but since we will be using set theories that omit *Infinity*, this will be useful.

Definition 1.2 *(Extensionality)*

$$\forall x, y(x = y \leftrightarrow \forall z(z \in x \leftrightarrow z \in y)) \tag{1.4}$$

Definition 1.3 *(Specification)*
The following is a schema for every first-order formula $\varphi(x, p_1, \ldots, p_n)$ with no free variables other than $x, p_1, \ldots, p_n$.

$$\forall x, p_1, \ldots, p_n \exists y \forall z(z \in y \leftrightarrow z \in x \;\&\; \varphi(z, p_1, \ldots, p_n)) \tag{1.5}$$

We will now provide two definitions that are not axioms, but will be helpful in establishing some axioms in a more comprehensible way.

Definition 1.4 *($x \subseteq y$, $x \subset y$)*

$$x \subseteq y \leftrightarrow (\forall z \in x) z \in y \tag{1.6}$$

$$x \subset y \leftrightarrow x \subseteq y \;\&\; x \neq y \tag{1.7}$$

We read $x \subseteq y$ as x is a subset of y *and $x \subset y$ as* x is a proper subset of y.

Definition 1.5 *(Empty Set) For an arbitrary set x, the empty set, written as $\emptyset$, is defined by the following formula*

$$(\forall y \in x)(y \in \emptyset \leftrightarrow \neg(y = y)) \tag{1.8}$$

$\emptyset$ is a set due to Specification*. While the empty set could also be defined by the formula $\forall y(y \in\leftrightarrow \neg(y = y))$, the former version is Δ_0, which we will find useful later on. The two definitions yield the same set for every x given thanks to* extensionality*.*

Definition 1.6 *(Pairing)*

$$\forall x, y \exists z \forall q(q \in z \leftrightarrow q = x \vee q = y) \tag{1.9}$$

Definition 1.7 *(Union)*

$$\forall x \exists y \forall z(z \in y \leftrightarrow \exists q(z \in q \;\&\; q \in x)) \tag{1.10}$$

Definition 1.8 *(Set Intersection)*

$$x \cap y = \{z : z \in x \;\&\; z \in y\} \tag{1.11}$$

Definition 1.9 *(Set Union)*

$$x \cup y = \{z : z \in x \vee z \in y\} \tag{1.12}$$

Now we can introduce more axioms.

Definition 1.10 *(Foundation)*

$$\forall x(x \neq \emptyset \rightarrow (\exists y \in x)(x \cap y = \emptyset)) \tag{1.13}$$

Definition 1.11 *(Powerset)*

$$\forall x \exists y \forall z(z \subseteq x \leftrightarrow z \in y) \tag{1.14}$$

Definition 1.12 *(Infinity)*

$$\exists x(\emptyset \in x \;\&\; (\forall y \in x)(y \cup \{y\} \in x)) \tag{1.15}$$

Let us introduce a few more definitions that will make the two remaining axioms more comprehensible.

Definition 1.13 *(Powerset function)*
Given a set x, the powerset of x, *denoted $\mathscr{P}(x)$ and satisfying 1.11, is defined as follows:*

$$\mathscr{P}(x) \stackrel{\mathsf{def}}{=} \{y : y \subseteq x\} \tag{1.16}$$

Definition 1.14 *(Function)*
Given arbitrary first-order formula $\varphi(x, y, p_1, \ldots, p_n)$, we say that φ is a function iff

$$\forall x, y, z, p_1, \ldots, p_n(\varphi(x, y, p_1, \ldots, p_n) \;\&\; \varphi(x, z, p_1, \ldots, p_n) \rightarrow y = z) \tag{1.17}$$

When a $\varphi(x, y)$ is a function, we also write the following:

$$f(x) = y \leftrightarrow \varphi(x, y) \tag{1.18}$$

[2] Alternatively, $f = \{\langle x, y\rangle : \varphi(x, y)\}$ is a class.

Definition 1.15 *(Dom(f))*
Let f be a function. We read the following as "Dom(f) is the domain of f".

$$Dom(f) \stackrel{\mathsf{def}}{=} \{x : \exists y(f(x) = y)\} \tag{1.19}$$

[2] This can also be done for φs with more than two free variables by either setting $f(x, p_1, \ldots p_n) = y \leftrightarrow \varphi(x, yp_1, \ldots, p_n)$ or saying that φ codes more functions, determined by the various parameters, so given $t_1, \ldots, t_n$, $f(x) = y \leftrightarrow \varphi(x, y, t_1, \ldots, t_n)$.

We say "f is a function on A", A being a class, if $A = dom(f)$.

Definition 1.16 *(Rng(f))*
Let f be a function. We read the following as "Rng(f) is the range of f".

$$Rng(f) \stackrel{\mathsf{def}}{=} \{x : \exists y(f(x) = y)\} \tag{1.20}$$

We say that f is *a function into* A, A being a class, if $rng(f) \subseteq A$. We say that f is *a function onto* A if $rng(f) = A$, in other words,

$$(\forall y \in A)(\exists x \in dom(f))(f(x) = y) \tag{1.21}$$

We say a function f is a *one to one function*, iff

$$(\forall x_1, x_2 \in dom(f))(f(x_1) = f(x_2) \rightarrow x_1 = x_2) \tag{1.22}$$

f is a bijection iff it is a one to one function that is onto.

Note that *Dom(f)* and *Rng(f)* are not definitions in a strict sense, they are in fact definition schemas that yield definitions for every function f given. Also note that they can be easily modified for φ instead of f, with the only difference being the fact that it is then defined only for those φs that are functions, which must be taken into account. This is worth noting as we will sometimes interchange the notions of *function* and *formula*.

Definition 1.17 *(Function Defined For All Ordinals)*
We say a function f is defined for all ordinals*, this is sometimes written $f : Ord \rightarrow A$ for any class A, if $Dom(f) = Ord$. Alternatively,*

$$(\forall \alpha \in Ord)(\exists y \in A)(f(\alpha) = y)) \tag{1.23}$$

And now for the axioms.

Definition 1.18 *(Replacement)*
The following is a schema for every first-order formula $\varphi(x, p_1, \ldots, p_n)$ with no free variables other than $x, p_1, \ldots, p_n$.

$$\text{"}\varphi \textit{ is a function"} \rightarrow \forall x \exists y \forall z(z \in y \leftrightarrow (\exists q \in x)(\varphi(x, y, p_1, \ldots, p_n))) \tag{1.24}$$

Definition 1.19 *(Choice)*

$$\begin{gathered}\forall x \exists f((f \textit{ is a choice function with } dom(f) = x \setminus \{\emptyset\}) \\ \& \, \forall y((y \in y \ \& \ y \neq \emptyset) \rightarrow f(y) \in y))\end{gathered} \tag{1.25}$$

We will refer the axioms by their name, written in italic type, e.g. *Foundation* refers to the Axiom of Foundation. Now we need to define some basic set theories to be used in the article.

Definition 1.20 (S)
We call S *an axiomatic theory in the language* $\mathscr{L} = \{=, \in\}$ *with exactly the following axioms:*
(i) Existence of a set *(see 1.1)*
(ii) Extensionality *(see 1.2)*
(iii) Specification *(see 1.3)*
(iv) Foundation *(see 1.10)*
(v) Pairing *(see 1.6)*
(vi) Union *(see 1.7)*
(vii) Powerset *(see 1.11)*

Definition 1.21 (ZF)
We call ZF *an axiomatic theory in the language* $\mathscr{L} = \{=, \in\}$ *that contains all the axioms of* S *in addition to the following*
(i) Replacement *schema (see 1.18)*
(ii) Infinity *(see 1.12)*
Existence of a set *is usually left out because it is a consequence of* infinity.

Definition 1.22 (ZFC)
ZFC *is an axiomatic theory in the language* $\mathscr{L} = \{=, \in\}$ *that contains all the axioms of* ZF *plus* Choice *(1.19).*

1.2.3 The Transitive Universe

Definition 1.23 *(Transitive Class)*
We say a class A *is* transitive *iff*

$$(\forall x \in A)(x \subseteq A) \tag{1.26}$$

Definition 1.24 *(Well Ordered Class) A class* A *is said to be* well ordered by $\in$ *iff the following hold:*
(i) $(\forall x \in A)(x \notin x)$ *(Antireflexivity)*
(ii) $(\forall x, y, z \in A)(x \in y \;\&\; y \in z \rightarrow x \in z)$ *(Transitivity)*
(iii) $(\forall x, y \in A)(x = y \vee x \in y \vee y \in x)$ *(Linearity)*
(iv) $(\forall x \subseteq A)(x \neq \emptyset \rightarrow (\exists y \in x)(\forall z \in x)(z = y \vee z \in y)))$ *(Existence of the least element)*

Definition 1.25 *(Ordinal Number)*
A set x is said to be an ordinal number *if it is* transitive *and* well-ordered by $\in$.

For the sake of brevity, we usually just say "x is an *ordinal*". Note that "x is an ordinal" is a well-defined formula in the language of set theory, since 1.23 is a first-order formula and 1.24 is in fact a conjunction of four first-order formulas. Ordinals will be usually denoted by lower case greek letters, starting from the beginning: $\alpha, \beta, \gamma, \ldots$. Given two different ordinals α, β, we will write $\alpha < \beta$ for $\alpha \in \beta$, see [Jech, 2006]Lemma 2.11 for technical details.

Definition 1.26 *(Non-Zero Ordinal) We say an ordinal α is* non-zero *iff $\alpha \neq \emptyset$.*

Definition 1.27 *(Successor Ordinal)*
Consider the following operation, let β be an ordinal.

$$S(\beta) = \beta \cup \{\beta\} \tag{1.27}$$

An ordinal α is called a successor ordinal *iff there is an ordinal β, such that $\alpha = S(\beta)$. We will sometimes also write $\alpha = \beta + 1$.*

Definition 1.28 *(Limit Ordinal)*
A non-zero ordinal α is called a limit ordinal *iff it is not a successor ordinal.*

Definition 1.29 *(Ord)*
The class of all ordinal numbers*, which we will denote Ord*[3] *is the proper class defined as follows.*

$$Ord = \{x : x \textit{ is an ordinal}\} \tag{1.28}$$

Definition 1.30 *(Von Neumann's Hierarchy)*
The Von Neumann's Hierarchy *is a collection of sets indexed by elements of Ord, defined recursively in the following way:*

(i)

$$V_0 = \emptyset \tag{1.29}$$

(ii)

$$V_{\alpha+1} = \mathscr{P}(V_\alpha) \textit{ for any ordinal } \alpha \tag{1.30}$$

[3]It is sometimes denoted *On*, but we will stick to the notation used in [Jech, 2006]

(iii)

$$V_\lambda = \bigcup_{\beta<\lambda} V_\beta \text{ for a limit ordinal } \lambda \tag{1.31}$$

We will also refer to the Von Neumann's Hierarchy *as* Von Neumann's Universe *or the* Cumulative Hierarchy.

Definition 1.31 *(Rank)*
Given a set x, we say that the rank of x (written as rank(x)) is the least ordinal α such that

$$x \in V_{\alpha+1} \tag{1.32}$$

Due to *Regularity*, every set has a rank.[4]

Definition 1.32 *(ω)*

$$\omega \stackrel{\text{def}}{=} \bigcap\{x : \text{"}x \text{ is a limit ordinal"})\} \tag{1.33}$$

ω is non-empty if *Infinity* or any equivalent holds.

Definition 1.33 *(Lévy's Hierarchy)*
!!! pozor na konflikt s analytickou (vyres podle kanamoriho) TODO

1.2.4 Cardinal Numbers

Definition 1.34 *(Cardinality)*
Given a set x, let the cardinality of x, written $|x|$, be defined as the smallest ordinal number such that there is a one to one mapping from x to α.

Definition 1.35 *(Aleph function)*
Let ω be the set defined by 1.32. We will recursively define the function $\aleph$ for all ordinals.
(i) $\aleph_0 = \omega$
(ii) $\aleph_{\alpha+1}$ *is the least cardinal larger than* $\aleph_\alpha$[5]
(iii) $\aleph_\lambda = \bigcup_{\beta<\lambda} \aleph_\beta$ *for a limit ordinal* λ
If $\kappa = \aleph_\alpha$ and α is a succesor ordinal, we call κ a succesor cardinal. *If α is a limit ordinal, we call κ a* limit cardinal.

[4]See chapter 6 of [Jech, 2006] for details.
[5]"The least cardinal larger than $\aleph_\alpha$" is sometimes notated as $\aleph_\alpha^+$

Definition 1.36 *(Cardinal number)*

(i) A set x is called a finite cardinal *iff $x \in \omega$.*
(ii) A set is called an infinite cardinal *iff there is an ordinal α such that $\aleph_\alpha = x$*
(iii) A set is called a cardinal *iff it is either a* finite cardinal *or an* infinite cardinal.

We say κ is an uncountable cardinal if it is an infinite ordinal and $\aleph_0 > \kappa$. Infinite cardinals will be notated by lowercase greek letters from the middle if the alphabet, e.g. $\kappa, \mu, \nu, \ldots$.[6]

For formal details as well as why every set can be well-ordered assuming *Choice*, and therefore has a cardinaliy, see [Jech, 2006].

Definition 1.37 *(Cofinality of a Limit Ordinal)*
Let λ be a limit ordinal. We say that the cofinality *of λ is α iff α is the smallest limit ordinal, such that there is an α-sequence $\langle \beta_\xi : \xi < \alpha \rangle$, such that*

$$sup(\beta_\xi : \xi < \alpha) = \lambda \tag{1.34}$$

We writte $cf(\lambda) = \alpha$.

Definition 1.38 *(Regular Cardinal)*
We say a cardinal κ is regular iff $cf(\kappa) = \kappa$

Definition 1.39 *(Limit Cardinal)*
We say that a cardinal κ is a limit cardinal *if*

$$(\exists \alpha \in Ord)(\kappa = \aleph_\alpha) \tag{1.35}$$

Definition 1.40 *(Strong Limit Cardinal)*
We say that an ordinal κ is a strong limit cardinal *if it is a* limit cardinal *and*

$$(\forall \alpha \in \kappa)(\mathscr{P}(\alpha) \in \kappa) \tag{1.36}$$

Definition 1.41 *(Generalised Continuum Hypothesis)*

$$\aleph_{\alpha+1} = \mathscr{P}(\aleph_\alpha) \tag{1.37}$$

If *GCH* holds (for example in Gödel's L, see chapter 3), the notions of a limit cardinal and a strong limit cardinal are equivalent.

[6] Except λ which is preferably used for limit ordinals.

1.2.5 Relativisation and Absoluteness

Definition 1.42 *(Relativization)*
Let M be a class, $R \subseteq M \times M$ and let $\varphi(p_1, \ldots, p_n)$ be a first-order formula with no free variables besides $p_1, \ldots,_n$. The relativization of φ to M and R *is the formula, written as $\varphi^{M,R}(p_1, \ldots, p_n)$, defined in the following inductive manner:*

(i) $(x \in y)^{M,R} \leftrightarrow R(x, y)$
(ii) $(x = y)^{M,R} \leftrightarrow x = y$
(iii) $(\neg\varphi)^{M,R} \leftrightarrow \neg\varphi^{M,R}$
(iv) $(\varphi \;\&\; \psi)^{M,R} \leftrightarrow \varphi^{M,R} \;\&\; \psi^{M,R}$
(v) $(\varphi \vee \psi)^{M,R} \leftrightarrow \varphi^{M,R} \vee \psi^{M,R}$
(vi) $(\varphi \to \psi)^{M,R} \leftrightarrow \varphi^{M,R} \to \psi^{M,R}$
(vii) $(\exists x\varphi(x))^{M,R} \leftrightarrow (\exists x \in M)\varphi^{M,R}(x)$
(viii) $(\forall x\varphi(x))^{M,R} \leftrightarrow (\forall x \in M)\varphi^{M,R}(x)$

When $R = \in \cap (M \times M)$, we usually write φ^M instead of $\varphi^{M,R}$. When we talk about $\varphi^M(p_1, \ldots, p_n)$, it is understood that $p_1, \ldots, p_n \in M$. We will also use $M \models \varphi(p_1, \ldots, p_n)$ and $\varphi^M(p_1, \ldots, p_n)$ interchangably.

Definition 1.43 *(Absoluteness) Given a transitive class M, we say a formula φ is* absolute in M *if for all $p_1, \ldots, p_n \in M$*

$$\varphi^M(p_1, \ldots, p_n) \leftrightarrow \varphi(p_1, \ldots, p_n) \tag{1.38}$$

Definition 1.44 *(Hierarchy of First-Order Formulas)*

A first-order formula φ is Δ_0 iff it is logically equivalent to a first-order formula φ' satisfying any of the following:

(i) *φ' contains no quantifiers*
(ii) *y is a set, ψ is a Δ_0 formula, and φ' is either $(\exists x \in y)\psi(y)$ or $(\forall x \in y)\psi(y)$.*
(iii) *ψ_1, ψ_2 are Δ_0 formulas and φ' is any of the following: $\psi_1 \vee \psi_2$, $\psi_1 \;\&\; \psi_2$, $\psi_1 \to \psi_2$, $\neg\psi_2$,*

(I) *If a formula is Δ_0 it is also Σ_0 and Π_0*
(II) *A formula φ is $\Pi_n + 1$ if it is logically equivalent to a formula φ' such that $\varphi' = \forall x\psi$ where ψ is a Σ_n-formula for any $n < \omega$.*
(III) *A formula φ is $\Sigma_n + 1$ if it is logically equivalent to a formula φ' such that $\varphi' = \forall x\psi$ where ψ is a Π_n-formula for any $n < \omega$.*

Note that we can use the pairing function so that for $\forall p_1, \ldots, p_n\psi(p_1, \ldots, p_n)$, there a logically equivalent formula of the form $\forall x\psi'(x)$.

Lemma 1.45 *(Δ_0 absoluteness) Let φ be a Δ_0 formula, then φ is absolute in any transitive class M.*

Proof. This will be proven by induction over the complexity of a given Δ_0 formula φ. Let M be an arbitrary transitive class. Suppose, that

Atomic formulas are always absolute by the definition of relativisation, see 1.42. Suppose that Δ_0 formulas ψ_1 and ψ_2 are absolute in M. Then from relativization, $(\psi_1 \ \& \ \psi_2)^M \leftrightarrow \psi_1^M \ \& \ \psi_2^M$, which is, from the induction hypothesis, equivalent to $\psi_1 \ \& \ \psi_2$. The same holds for $\vee, \rightarrow, \neg$.

Suppose that a Δ_0 formula ψ is absolute in M. Let y be a set and let $\varphi = (\exists x \in y)\psi(x)$. From relativization, $(\exists x \psi(x))^M \leftrightarrow (\exists x \in M)\psi^M(x)$. Since the hypotheses makes it clear that $\psi^M \leftrightarrow \psi$, we get $((\exists x \in y)\psi(x))^M \leftrightarrow (\exists x \in y \cap M)\psi(x)$, which is the equivalent of $\varphi^M \leftrightarrow \varphi$. The same applies to $\varphi = (\forall x \in y)\psi(x)$. $\square$

Lemma 1.46 *(Downward Absoluteness)*
Let φ be a Π_1 formula and M a transitive class. Then the following holds:

$$(\forall p_1, \ldots, p_n \in M)(\varphi(p_1, \ldots, p_n) \rightarrow \varphi(p_1, \ldots, p_n)^M) \tag{1.39}$$

Proof. Since $\varphi(p_1, \ldots, p_n)$ is Π_1, there is a Δ_0 formula $\psi(p_1, \ldots, p_n, x)$ such that $\varphi = \forall x \psi(p_1, \ldots, p_n, x)$. From relativization and lemma 1.45, $\varphi^M(p_1, \ldots, p_n) \leftrightarrow (\forall x \in M)\psi(p_1, \ldots, p_n, x)$.

Assume that for $p_1, \ldots, p_n \in M$ fixed, that $\forall x \psi(p_1, \ldots, p_n, x)$ holds, but $(\forall x \in M)\psi(p_1, \ldots, p_n, x)$ does not. Therefore $\exists x \neg\psi(p_1, \ldots, p_n, x)$, which contradicts $\forall x \psi(p_1, \ldots, p_n, x)$. $\square$

Lemma 1.47 *(Upward Absoluteness)*
Let φ be a Σ_1 formula and M a transitive class. Then the following holds:

$$(\forall p_1, \ldots, p_n \in M)(\varphi^M(p_1, \ldots, p_n) \rightarrow \varphi(p_1, \ldots, p_n)) \tag{1.40}$$

Proof. Since $\varphi(p_1, \ldots, p_n)$ is Σ_1, there is a Δ_0 formula $\psi(p_1, \ldots, p_n, x)$ such that $\varphi = \exists x \psi(p_1, \ldots, p_n, x)$. From relativization and lemma 1.45, $\varphi^M(p_1, \ldots, p_n) \leftrightarrow (\exists x \in M)\psi(p_1, \ldots, p_n, x)$.

Assume that for $p_1, \ldots, p_n \in M$ fixed, that $(\exists x \in M)\psi(p_1, \ldots, p_n, x)$ holds, but $\exists x \psi(p_1, \ldots, p_n, x)$ does not. This is an obvious contradiction. $\square$

1.2.6 More Functions

Definition 1.48 *(Strictly Increasing Function)*
A function $f : Ord \rightarrow Ord$ is said to be strictly increasing *iff*

$$\forall \alpha, \beta \in Ord(\alpha < \beta \rightarrow f(\alpha) < f(\beta)). \tag{1.41}$$

Definition 1.49 *(Continuous Function)*
A function $f : Ord \to Ord$ is said to be continuous *iff*

$$\alpha \textit{ is limit} \to f(\lambda) = \bigcup_{\alpha<\lambda} f(\alpha). \tag{1.42}$$

Definition 1.50 *(Normal Function)*
A function $f : Ord \to Ord$ is said to be normal *if it is* strictly increasing *and* continuous*.*

Definition 1.51 *(Fixed Point)*
We say x is a fixed point of a function f iff $x = f(x)$.

Definition 1.52 *(Unbounded Class)*
We say a class A is unbounded if

$$\forall x(\exists y \in A)(x < y) \tag{1.43}$$

Definition 1.53 *(Limit Point)*
Given a class $x \subseteq On$, we say that $\alpha \neq \emptyset$ is a limit point of x iff

$$\alpha = \bigcup(x \cap \alpha) \tag{1.44}$$

Definition 1.54 *(Closed Class)*
We say a class $A \subseteq Ord$ is closed iff it contains all of its limit points.

Definition 1.55 *(Club set)*
For a regular uncountable cardinal κ, a set $x \subset \kappa$ is a closed unbounded *subset, abbreviated as a* club set*, iff x is both closed and unbounded in κ.*

Definition 1.56 *(Stationary set)*
For a regular uncountable cardinal κ, we say a set $A \subset \kappa$ is stationary in κ iff it intersects every club subset of κ.

1.2.7 Structure, Substructure and Embedding

Structures will be denoted $\langle M, \in, R\rangle$ where M is a domain, $\in$ stands for the standard membership relation, it is assumed to be restricted to the domain[7], $R \subseteq M$ is a relation on the domain. When R is not needed, we may as well only write M instead of $\langle M, \in\rangle$.

[7] To be totally correct, we should write $\langle M, \in \cap M \times M, R\rangle$

Definition 1.57 *(Elementary Embedding)*
Given the structures $\langle M_0, \in, R\rangle$, $\langle M_1, \in, R\rangle$ *and a one-to-one function* $j : M_0 \to M_1$, *we say* j *is an* elementary embedding *of* M_0 *into* M_1, *we write* $j : M_0 \prec M_1$, *when the following holds for every formula* $\varphi(p_1, \ldots, p_n)$ *and every* $p_1, \ldots, p_n \in M_0$*:*

$$\langle M_0, \in, R\rangle \models \varphi(p_1, \ldots, p_n) \leftrightarrow \langle M_1, \in, R\rangle \models \varphi(j(p_1), \ldots, j(p_n)) \tag{1.45}$$

Definition 1.58 *(Elementary Substructure)*
Given the structures $\langle M_0, \in, R\rangle$, $\langle M_1, \in, R\rangle$ *and a one-to-one function* $j : M_0 \to M_1$ *such that* $j : M_0 \prec M_1$, *we say that* M_0 *is an* elementary substructure *of* M_1, *denoted as* $M_0 \prec M_1$, *iff* j *is an identity on* M_0. *In other words*

$$\langle M_0, \in, R\rangle \models \varphi(p_1, \ldots, p_n) \leftrightarrow \langle M_1, \in, R\rangle \models \varphi(p_1, \ldots, p_n) \tag{1.46}$$

for $p_1, \ldots, p_n \in M_0$

2 Levy's First-Order Reflection

2.1 Lévy's Original Paper

This section is based on Lévy's paper *Axiom Schemata of Strong Infinity in Axiomatic Set Theory*, [Lévy, 1960]. It presents Lévy's general reflection principle and its equivalence to *Replacement* and *Infinity* under S[8].

First, we should point out that set theory has changed over the last 66 years and show a few notable, albeit only formal, differences. Firstly, when reading Lévy's article, one should bear in mind that while the author often speaks about a model of ZF, usually denoted u, it doesn't necessarily mean that there is a set u that is a model of ZF[9], we are nowadays used to using the notion of universal class V in similar sense, albeit independently from a particular axiomatic set theory. The theory ZF is practically identical to the theory we have established in (1.21), the differences are only formal. One might be confused by the fact that Lévy treats the *Subsets* axiom, which is in fact *Specification*, as a single axiom rather than a schema. He even takes the conjunction of all axioms of ZF and treats it like a formula. This is possible because the underlying logic calculus is different. Lévy works with set theories formulated in the *non-simple applied first order functional calculus*, see Chapter IV in [Church, 1996] for details. For now, we only need

[8] See definition (1.20).
[9] This is indeed impossible to prove in ZF due to Gödel's Incompleteness.

to know that the calculus contains a substitution rule for functional variables. This way, *Subsets* is de facto a schema even though it sometimes treated as a single formula[10]. It should also be noted that the logical connectives look different. The now usual symbol for an universal quantifier does not appear, $\forall x\varphi(x)$ would be written as $(x)\varphi(x)$. The symbol for negation is "$\sim$", implication is written as "$\supset$" and equivalence is "$\equiv$". We will use standard notation with "$\neg$", "$\rightarrow$" and "$\leftrightarrow$" respectively when presenting Lévy's results.

This subsection uses ZF instead of the usual ZFC as the underlying theory.

Definition 2.1 *(Standard Complete Model of a Set Theory)*
Let Q be an arbitrary axiomatic set theory. We say that u is a standard complete model of Q iff
(i) $(\forall\sigma \in \mathsf{Q})(\mathsf{u} \models \sigma)$
(ii) $\forall y(y \in u \rightarrow y \subset u)$
We write $Scm^{\mathsf{Q}}(u)$.

Definition 2.2 *(Cardinals Inaccessible With Respect to Q)*
Let Q be an arbitrary axiomatic set theory. We say that a cardinal κ is inaccessible with respect to theory Q iff

$$Scm^{\mathsf{Q}}(V_\kappa) \tag{2.47}$$

We write $In^{\mathsf{Q}}(\kappa)$

Definition 2.3 *(Inaccessible Cardinal With Respect to ZF)*
When a cardinal κ is inaccessible with respect to ZF, we only say that it is inaccessible. We write $In(\kappa)$.

$$In(\kappa) \leftrightarrow In^{\mathsf{ZF}}(\kappa) \tag{2.48}$$

The above definition of inaccessibles is used because it doesn't require *Choice*.

For the definition of relativization, see (1.42). The notation used by Lévy is "$Rel(u, \varphi)$", we will stick to "φ^u".

Definition 2.4 *(N)*
The following is an axiom schema of complete reflection over ZF, denoted as N. For every first-order formula φ in the language of set theory with no free variables except for $p_1, \ldots, p_n$, the following is an instance of schema N.

$$\exists u(Scm^{\mathsf{ZF}}(u) \;\&\; \forall p_1, \ldots, p_n(p_1, \ldots, p_n \in u \rightarrow \varphi \leftrightarrow \varphi^u)) \tag{2.49}$$

[10] This way, the conjunction of all axiom is in fact a schema.

Let S be an axiomatic set theory defined in (1.20).

Definition 2.5 *(N_0)*
Axiom schema N_0 is similar to N defined above, but with S instead of ZF. For every φ, a first-order fomula in the language of set theory with no free variables except $p_1, \ldots, p_n$, the following is an instance of N_0.

$$\exists u(Scm^{\mathsf{S}}(u) \;\&\; \forall p_1, \ldots, p_n(p_1, \ldots, p_n \in u \to \varphi \leftrightarrow \varphi^u)) \tag{2.50}$$

We will now show that in S, N_0 implies both *Replacement* and *Infinity.*

Let N_0 be defined as in (2.5), for *Infinity* see (1.12).

Theorem 2.6 *In S, the schema N_0 implies* Infinity.

Proof. Let $\varphi = \forall x \exists y(y = x \cup \{x\})$. This clearly holds in S because given a set x, there is a set $y = x \cup \{x\}$ obtained via *Pairing* and *Union.* From N_0, there is a set u such that φ^u holds. This u satisfies the conditions required by *Infinity.* $\square$ Lévy proves even more swiftly. He says that for an arbitrary formula φ, N_0 gives us $\exists u Scm^{\mathsf{S}}(u)$ and this u already satisfies *Infinity.* To do this, we would need to prove (2.12), which will happen later in this chapter, but we don't know that yet.

Let S be a set theory defined in (1.20), N_0 a schema defined in (2.5) and *Replacement* a schema defined in (1.18).

Theorem 2.7 *In S, the schema N_0 implies* Replacement.

Proof. Let $\varphi(x, y, p_1, \ldots, p_n)$ be a formula with no free variables except $x, y, p_1, \ldots, p_n$. Let χ be an instance of *Replacement* schema for the above φ.

$$\begin{aligned} \chi = \forall x, y, z(\varphi(x, y, p_1, \ldots, p_n) \;\&\; \varphi(x, z, p_1, \ldots, p_n) \to y = z) \\ \to \forall x \exists y \forall z(z \in y \leftrightarrow (\exists q \in x)(\varphi(x, y, p_1, \ldots, p_n))) \end{aligned} \tag{2.51}$$

Consider the following formulas.
(i) $x, y, p_1, \ldots, p_n \in u \to (\varphi \leftrightarrow \varphi^u)$
(ii) $x, p_1, \ldots, p_n \in u \to (\exists y \varphi \leftrightarrow (\exists y \varphi)^u)$
(iii) $x, p_1, \ldots, p_n \in u \to (\chi \leftrightarrow \chi^u)$
(iv) $\forall x, p_1, \ldots, p_n(\chi \leftrightarrow (\forall x, p_1, \ldots, p_n \chi)^u)$
The first three formulas are instances of the N_0 schema for formulas φ, $\exists y \varphi$ and χ respectively, the last is universal closure of (iii). By N_0, there exists a set u where all four formulas hold.

From relativization, $(\exists y\varphi)^u$ is equivalent to $(\exists y \in u)\varphi^u$, together with (i) and (ii), we get

$$x, p_1, \ldots, p_n \in u \to ((\exists y \in u)\varphi \leftrightarrow \exists y\varphi) \tag{2.52}$$

If φ is a function, then for every $x \in u$, which is also $x \subset u$ by the transitivity of $Scm^{\mathsf{S}}(u)$, it maps elements of x into u. From the axiom scheme of comprehensionwe can find y, a set of all images of the elements of x. That gives us $x, p_1, \ldots, p_n \in u \to \chi$. By (iii) we get $x, p_1, \ldots, p_n \in u \to \chi^u$, the universal closure of this formula is $(\forall x, p_1, \ldots, p_n\chi)^u$, which together with (iv) yields $\forall x, p_1, \ldots, p_n\chi$. Via universal instantiation, we end up with χ, which is an instance of *Replacement* for an arbitrary formula φ given. $\square$

What we have just proven is only a single theorem from Lévy's aforementioned article, we will introduce other interesting propositions, mostly related to Mahlo and inaccessible cardinals, later in their appropriate context in chapter 3.

2.2 Contemporary Restatement

We will now introduce and prove a theorem that is called Lévy's Reflection in contemporary set theory. The only difference is that while Lévy originally reflects a formula φ from V to a set u which is a *standard complete model of* S, we say that there is a V_α for a limit α that reflects φ. Those two conditions are equivalent due to lemma (2.12).

Lemma 2.8 *Let $\varphi_1, \ldots, \varphi_n$ be first-order formulas in the language of set theory, all with m free variables*[11].

(i) For each set M_0 there is such set M that $M_0 \subset M$ and the following holds for every i, $1 \leq i \leq n$:

$$\exists x\varphi_i(p_1, \ldots, p_{m-1}, x) \to (\exists x \in M)\varphi_i(p_1, \ldots, p_{m-1}, x) \tag{2.53}$$

for every $p_1, \ldots, p_{m-1} \in M$.

(ii) Furthermore, there is an ordinal α such that $M_0 \subset V_\alpha$ and the following holds for each i, $1 \leq i \leq n$:

$$\exists x\varphi_i(p_1, \ldots, p_{m-1}, x) \to (\exists x \in V_\alpha)\varphi_i(p_1, \ldots, p_{m-1}, x) \tag{2.54}$$

for every $p_1, \ldots, p_{m-1} \in M$.

[11] For formulas with a different number of free variables, take for m the highest number of parameters among those formulas. Add spare parameters to every formula that has less than m parameters in a way that preserves the last parameter, which we will denote x. E.g. let φ_i' be the a formula with k parameters, $k < m$. Let us set $\varphi_i(p_1, \ldots, p_{m-1}, x) \overset{\mathsf{def}}{=} \varphi_i'(p_1, \ldots, p_{k-1}, x)$, notice that the parameters $p_k, \ldots, p_{m-1}$ are not used.

(iii) Assuming Choice*, there is* M*,* $M_0 \subset M$ *such that (2.53) holds for every* M*,* $i \leq n$ *and* $|M| \leq |M_0| \cdot \aleph_0$.

Proof. We will simultaneously prove statements *(i)* and *(ii)*, denoting M^T the transitive set required by part *(ii)*. Steps in the construction of M^T that are not explicitly included are equivalent to steps for M.

Let us first define an operation $H_i(p_1, \dots, p_{m-1})$ that yields the set of x's with minimal rank[12] satisfying $\varphi_i(p_1, \dots, p_{m-1}, x)$ for $p_1, \dots, p_{m-1}$ and for every i, $1 \leq i \leq n$.

$$H_i(p_1, \dots, p_n) = \{x \in C_i : (\forall z \in C)(rank(x) \leq rank(z))\} \tag{2.55}$$

for each $1 \leq i \leq n$, where

$$C_i = \{x : \varphi_i(p_1, \dots, p_{m-1}, x)\} \text{ for } 1 \leq i \leq n \tag{2.56}$$

Next, let's construct M from given M_0 by induction.

$$M_{i+1} = M_i \cup \bigcup_{j=0}^{n} \bigcup \{H_j(p_1, \dots, p_{m-1}) : p_1, \dots, p_{m-1} \in M_i\} \tag{2.57}$$

In other words, in each step we include into the construction the elements satisfying $\varphi(p_1, \dots, p_{m-1}, x)$ for $p_1, \dots, p_{m-1}$ from the previous step. For statement *(ii)*, this is the only part that differs from *(i)*. To end up with a transitive M, we need to extend every step to it's transitive closure transitive closure of M_{i+1} from *(i)*. In other words, let γ be the smallest ordinal such that

$$(M_i^T \cup \bigcup_{j=0}^{n} \{\bigcup \{H_j(p_1, \dots, p_{m-1}) : p_1, \dots, p_{m-1} \in M_i\}\}) \subset V_\gamma \tag{2.58}$$

Then the incremental step is

$$M_{i+1}^T = V_\gamma \tag{2.59}$$

and the final M is obtained by joining the previous steps.

$$M = \bigcup_{i=0}^{\infty} M_i, \;\; M^T = \bigcup_{i=0}^{\infty} M_i^T = V_\alpha \tag{2.60}$$

[12]Rank is defined in (1.31)

We have yet to finish part *(iii)*. Let's try to construct a set M' that satisfies the same conditions like M but is kept as small as possible. Assuming the Axiom of Choice, we can modify the process so that the cardinality of M' is at most $|M_0| \cdot \aleph_0$. Note that the size of M in the previous construction is determined by the size of M_0 and, most importantly, by the size of $H_i(p_1, \ldots, p_{m-1})$ for every i, $1 \leq i \leq n$ in individual iterations of the construction. Since *(i)* only ensures the existence of an x that satisfies $\varphi_i(p_1, \ldots, p_{m-1}, x)$ for any i, $1 \leq i \leq n$, we only need to add one x for every set of parameters but $H_i(u_1, \ldots, u_{m-1})$ can be arbitrarily large. Let F be a choice function on $\mathscr{P}(M')$. Also let $h_i(p_1, \ldots, p_{m-1}) = F(H_i(p_1, \ldots, p_{m-1}))$ for i, where $1 \leq i \leq n$, which means that h is a function that outputs an x that satisfies $\varphi_i(p_1, \ldots, p_{m-1}, x)$ for i such that $1 \leq i \leq n$ and has minimal rank among all such sets. The induction step needs to be redefined to

$$M'_{i+1} = M'_i \cup \bigcup_{j=0}^{n} \{H_j(p_1, \ldots, p_{m-1}) : p_1, \ldots, p_{m-1} \in M'_i\} \tag{2.61}$$

This way, the amount of elements added to M'_{i+1} in each step of the construction is the same as the amount of m-tuples of parameters that yielded elements not included in M'_i. It is easy to see that if M_0 is finite, M' is countable because it was constructed as a countable union of at most countable sets. If M_0 is countable or larger, the cardinality of M' is equal to the cardinality of M_0.[13] Therefore $|M'| \leq |M_0| \cdot \aleph_0$ □

Theorem 2.9 *(Lévy's first-order reflection theorem)*
Let $\varphi(p_1, \ldots, p_n)$ be a first-order formula.

(i) For every set M_0 there exists M such that $M_0 \subset M$ and the following holds:

$$\varphi^M(p_1, \ldots, p_n) \leftrightarrow \varphi(p_1, \ldots, p_n) \tag{2.62}$$

for every $p_1, \ldots, p_n \in M$.

(ii) For every set M_0 there is a transitive set M, $M_0 \subset M$ such that the following holds:

$$\varphi^M(p_1, \ldots, p_n) \leftrightarrow \varphi(p_1, \ldots, p_n) \tag{2.63}$$

for every $p_1, \ldots, p_n \in M$.

[13] It can not be smaller because $|M'_{i+1}| \geq |M'_i|$ for every i. It may not be significantly larger because the maximum of elements added is the number of n-tuples in M'_i, which is of the same cardinality is M'_i.

(iii) *For every set M_0 there is α such that $M_0 \subset V_\alpha$ and the following holds:*

$$\varphi^{V_\alpha}(p_1, \ldots, p_n) \leftrightarrow \varphi(p_1, \ldots, p_n) \tag{2.64}$$

for every $p_1, \ldots, p_n \in M$.

(iv) *Assuming* Choice, *for every set M_0 there is M such that $M_0 \subset M$ and $|M| \leq |M_0| \cdot \aleph_0$ and the following holds:*

$$\varphi^{M}(p_1, \ldots, p_n) \leftrightarrow \varphi(p_1, \ldots, p_n) \tag{2.65}$$

for every $p_1, \ldots, p_n \in M$.

Proof.

Let's now prove *(i)* for given φ via induction by complexity. We can safely assume that φ contains no quantifiers besides "$\exists$" and no logical connectives other than "$\neg$" and "&". Let $\varphi_1, \ldots, \varphi_n$ be all subformulas of φ. Then there is a set M, obtained by the means of lemma 2.8, for all of the formulas $\varphi_1, \ldots, \varphi_n$.

Let's first consider atomic formulas in the form of either $x_1 = x_2$ or $x_1 \in x_2$. It is clear from relativisation[14] that 2.62 holds for both cases.

(i) $(x_1 = x_2)^M \leftrightarrow (x_1 = x_2)$

(ii) $(x_1 \in x_2)^M \leftrightarrow (x_1 \in x_2)$

We now want to verify the inductive step. First, take $\varphi = \neg\varphi'$. From the relativization, we get

$$(\neg\varphi')^M \leftrightarrow \neg(\varphi'^M) \tag{2.66}$$

Because the induction hypothesis tell us that $\varphi'^M \leftrightarrow \varphi'$, the following holds:

$$(\neg\varphi')^M \leftrightarrow \neg(\varphi'^M) \leftrightarrow \neg\varphi' \tag{2.67}$$

The same holds for $\varphi = \varphi_1 \ \& \ \varphi_2$. From the induction hypothesis, we know that $\varphi_1^M \leftrightarrow \varphi_1$ and $\varphi_2^M \leftrightarrow \varphi_2$, which together with relativization for formulas in the form of $\varphi_1 \ \& \ \varphi_2$ gives us

$$(\varphi_1 \ \& \ \varphi_2)^M \leftrightarrow \varphi_1^M \ \& \ \varphi_2^M \leftrightarrow \varphi_1 \ \& \ \varphi_2 \tag{2.68}$$

Let's now examine the case when, from the induction hypethesis, M reflects $\varphi'(p_1, \ldots, p_n, x)$ and we are interested in $\varphi = \exists x \varphi'(p_1, \ldots, p_n, x)$. The induction hypothesis tells us that

$$\varphi'^M(p_1, \ldots, p_n, x) \leftrightarrow \varphi'(p_1, \ldots, p_n, x) \tag{2.69}$$

[14] See 1.42. This only holds for relativization to $M, \in$, not M, E for an arbitrary relation E.

so, together with above lemma 2.8, the following holds:

$$\begin{aligned}
&\varphi(p_1, \ldots, p_n, x) \\
&\leftrightarrow \exists x \varphi'(p_1, \ldots, p_n, x) \\
&\leftrightarrow (\exists x \in M)\varphi'(p_1, \ldots, p_n, x) \\
&\leftrightarrow (\exists x \in M)\varphi'^M(p_1, \ldots, p_n, x) \\
&\leftrightarrow (\exists x \varphi'(p_1, \ldots, p_n, x))^M \\
&\leftrightarrow \varphi^M(p_1, \ldots, p_n, x)
\end{aligned} \tag{2.70}$$

Which is what we wanted to prove for part (i).

We now need to verify that the same holds for any finite number of formulas $\varphi_1, \ldots, \varphi_n$. This has in fact been already done since lemma 2.8 gives us a set M for any (finite) amount of formulas and given M_0. We can therefore find a set M for the union of all of their subformulas. When we obtain such M, it should be clear that it also reflects every formula in $\varphi_1, \ldots, \varphi_n$.

Since V_α is a transitive set, by proving (**iii**) we also satisfy (**ii**). To do so, we only need to look at part (**ii**) of lemma 2.8. All of the above proof also holds for $M = V_\alpha$.

To finish part (**iv**), we take M of size $\leq |M_0| \cdot \aleph_0$, which exists due to part (**iii**) of lemma 2.8, the rest being identical. □

Let S be a set theory defined in 1.20, for ZFC see 1.22.

The two following lemmas are based on [Drake, 1974][Chapter 3, Theorem 1.2].

Lemma 2.10 *Iff M is a transitive set, then $M \models$ Extensionality.*

Proof. Given a transitive set M, we want to show that the following holds.

$$M \models \forall x, y(x = y \leftrightarrow \forall z(z \in x \leftrightarrow z \in y)) \tag{2.71}$$

Given arbitrary sets $x, y \in M$, we want to prove

$$M \models (x = y \leftrightarrow \forall z(z \in x \leftrightarrow z \in y)) \tag{2.72}$$

This is equivalent to

$$M \models x = y \text{ iff } M \models \forall z(z \in x \leftrightarrow z \in y) \tag{2.73}$$

Which is the same as

$$x = y \text{ iff } M \models \forall z(z \in x \leftrightarrow z \in y) \tag{2.74}$$

So all elements of x are also elements of y in M, and vice versa. Because M is transitive, all elements of x and y are in M, so $M \models \forall z(z \in x \leftrightarrow z \in y)$ holds iff x and y contain the same elements and are therefore equal. □

Lemma 2.11 *If M is a transitive set, then $M \models$* Foundation.

Proof. We want to prove

$$M \models \forall x(x \neq \emptyset \rightarrow (\exists y \in x)(x \cap y = \emptyset)) \tag{2.75}$$

Given an arbitrary $x \in M$ such that $x \neq \emptyset$, we want to prove

$$M \models (\exists y \in x)(x \cap y = \emptyset) \tag{2.76}$$

Because M is transitive, every element of x is an element of M. Take for y the element of x with the lowest rank[15]. It should be clear that there is no $z \in y$ such that $z \in x$, because then $rank(z) < rank(y)$, which is a contradiction. □

Let S be a set theory as defined in (1.20).

Lemma 2.12 *The following holds for every λ.*

$$\text{"}\lambda \textit{ is a limit ordinal"} \rightarrow V_\lambda \models \mathsf{S} \tag{2.77}$$

Proof. Given an arbitrary limit ordinal λ, we will verify the axioms of S one by one.

(i) *The existence of a set* comes from the fact that V_λ is a non-empty set because limit ordinal is non-zero.

(ii) *Extensionality* holds from 2.10

(iii) *Foundation* holds from 2.11

(iv) *Union*:

$$\forall x \exists y \forall z(z \in y \leftrightarrow \exists q(z \in q \,\&\, q \in x)) \tag{2.78}$$

Given any $x \in V_\lambda$, we want verify that $y = \bigcup x$ is also in V_λ. Note that $y = \bigcup x$ is a Δ_0-formula.

$$y = \bigcup x \leftrightarrow (\forall z \in y)(\exists q \in x) z \in q \,\&\, (\forall z \in x)(\forall q \in z) q \in y \tag{2.79}$$

So by lemma (1.45)

$$y = \bigcup x \leftrightarrow V_\lambda \models y = \bigcup x \tag{2.80}$$

[15] Rank is defined in (1.31).

(v) *Pairing*:

$$\forall x, y \exists z \forall q (q \in z \leftrightarrow q = x \vee q = y) \tag{2.81}$$

Given two sets $x, y \in V_\lambda$, we want to show that z, defined as $z = \{x, y\}$, is also an element of V_λ.

$$z = \{x, y\} \leftrightarrow x \in z \; \& \; y \in z \; \& \; (\forall q \in z)(q = x \vee q = y) \tag{2.82}$$

So $z = \{x, y\}$ is a Δ_0-formula, and thus by lemma (1.45) it holds that

$$z = \{x, y\} \leftrightarrow V_\lambda \models z = \{x, y\} \tag{2.83}$$

(vi) *Powerset*:

$$\forall x \exists y \forall z (z \subseteq x \leftrightarrow z \in y) \tag{2.84}$$

Given any $x \in V_\lambda$, we want to make sure that $\mathscr{P}(x) \in V_\lambda$. Let $\varphi(y)$ denote the formula $y \in \mathscr{P}(x) \leftrightarrow y \subset x$. according to definition of subset (1.4), $y \subset x$ is Δ_0, so for any given $x, y \in V_\lambda$, $y = \mathscr{P}(x) \leftrightarrow V_\lambda \models y = \mathscr{P}(x)$. Because λ is limit and $rank(\mathscr{P}(x)) = rank(x) + 1$, if $\mathscr{P}(x) \in V_\lambda$ for every $x \in V_\lambda$.

(vii) *Specification*:Given a first-order formula φ, we want to show the following

$$V_\lambda \models \forall x, p_1, \ldots, p_n, \exists y \forall z (z \in y \leftrightarrow z \in x \; \& \; \varphi(z, p_1, \ldots, p_n)) \tag{2.85}$$

Given any x along with parameters $p_1, \ldots, p_n$ in V_λ , we set $y = \{z \in x : \varphi^{V_\lambda}(z, p_1, \ldots, p_n)\}$. From transitivity of V_λ and the fact that $y \subset x$ and $x \in V_\lambda$, we know that $y \in V_\lambda$, so $V_\lambda \models \forall z(z \in y \leftrightarrow z \in x \; \& \; \varphi(z, p_1, \ldots, p_n))$.

□

Definition 2.13 *(First-Order Reflection)*
Let φ be a first-order formula in the language of set theory. For every set M_0 there is such set M that $M_0 \subseteq M$ and the following holds for every $p_1, \ldots, p_n \in M$:

$$\varphi(p_1, \ldots, p_n) \rightarrow M \models \varphi(p_1, \ldots, p_n) \tag{2.86}$$

We will refer to this axiom schema as First-order reflection

Let *Infinity* and *Replacement* be as defined in 1.12 and 1.18 respectively.

Theorem 2.14 First-order reflection *is equivalent to* Infinity & Replacement *under* S.

Proof. Since 2.9 already gives us one side of the implication, we are only interested in showing the converse which we shall do in two parts:

First-order reflection $\to$ *Infinity* This is done exactly like (2.6). We pick for φ the formula $(\forall y \in x)(y \cup \{y\} \in x)$, $M_0 = \{\emptyset\}$. From (2.13), there is a set M that satisfies φ, so there is an inductive set. We have picked M_0 so that $\emptyset \in M$ obviously holds and M is the witness for

$$\exists x(\emptyset \in x \ \&\ (\forall y \in x)(y \cup \{y\} \in x)) \tag{2.87}$$

which is exactly (1.12).

From *First-order reflection*, we know that for any first-order formula φ and a set M_0, there is a M such that $M_0 \subseteq M$ and $\varphi^M \leftrightarrow \varphi$. Let's pick *Powerset* for φ, then by *First-order reflection* there is a set that satisfies *Powerset*, ergo there is a strong limit cardinal, which in turn satisfies *Infinity.*

Reflection $\to$ *Replacement*

Given a formula $\varphi(x, y, p_1, \dots, p_n)$, we can suppose that given if it holds for given $x, y, p_1, \dots, p_n$, it is reflected in a set M [16] What we want to obtain is the following:

$$\begin{aligned} &\forall x, y, z(\varphi(x, y, p_1, \dots, p_n) \ \&\ \varphi(x, z, p_1, \dots, p_n) \to y = z) \to \\ &\quad \to \forall X \exists Y \forall y\ (y \in Y \leftrightarrow \exists x(\varphi(x, y, p_1, \dots, p_n) \ \&\ x \in X)) \end{aligned} \tag{2.88}$$

We do also know that $x, y \in M$, in other words for every X, $Y = \{y \mid \varphi(x, y, p_1, \dots, p_n)\}$ and we know that $X \subset M$ and $Y \subset M$, which, together with the specification schema implies that Y, the image of X over φ, is a set. □

We have shown that *Reflection* for first-order formulas, *First-order reflection* is a theorem of ZF.We have also shown that it can be used instead of the *Infinity* and *Replacement* scheme, but ZF + *First-order reflection* is a conservative extension of ZF. Besides being a starting point for more general and powerful statements, it can be used to show that ZF is not finitely axiomatizable. That follows from the fact that *Reflection* gives a model to any consistent finite set of formulas. So if $\varphi_1, \dots, \varphi_n$ for any finite n would be the axioms of ZF, *Reflection* would always contain a model of itself, which would in turn contradict the Second Gödel's Theorem[17]. Notice that, in a way, Reflection is dual to compactness. Compactness says that given a set of sentences, if every finite subset yields a model, so does the whole set. Reflection, on the other hand, says that while the whole set has no model in the underlying theory, every finite subset has a model.

[16] Which means that for $x, y, p_1, \dots, p_n \in M$, $\varphi^M(x, y, p_1, \dots, p_n) \leftrightarrow \varphi(x, y, p_1, \dots, p_n)$.

[17] See chapter **??** for further details.

Furthemore, *Reflection* can be used in ways similar to upward Löwenheim–Skolem theorem. Since Reflection extends any set M_0 into a model of given formulas $\varphi_1, \ldots, \varphi_n$, we can choose the lower bound of the size of M by appropriately choosing M_0.

In the next section, we will try to generalize *Reflection* in a way that transcends ZF and finally yields some large cardinals.

3 Conclusion

References

[Church, 1996] Church, A. (1996). *Introduction to Mathematical Logic*. Annals of Mathematics Studies. Princeton University Press.

[Drake, 1974] Drake, F. (1974). *Set theory. An introduction to large cardinals*. Studies in Logic and the Foundations of Mathematics, Volume 76. NH.

[Jech, 2006] Jech, T. (2006). *Set theory*. Springer monographs in mathematics. Springer, the 3rd millennium ed., rev. and expanded edition.

[Lévy, 1960] Lévy, A. (1960). Axiom schemata of strong infinity in axiomatic set theory. *Pacific Journal of Mathematics*, 10.

[Wang, 1997] Wang, H. (1997). *"A Logical Journey: From Gödel to Philosophy"*. A Bradford Book.