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REFLECTION PRINCIPLES AND LARGE CARDINALS

Bakalářská práce

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Abstract

Práce zkoumá vztah tzv. principů reflexe a velkých kardinálů. Lévy ukázal, že v ZFC platí tzv. věta o reflexi a dokonce, že věta o reflexi je ekvivalentní schématu nahrazení a axiomu nekonečna nad teorií ZFC bez axiomu nekonečna a schématu nahrazení. Tedy lze na větu o reflexi pohlížet jako na svého druhu axiom nekonečna. Práce zkoumá do jaké míry a jakým způsobem lze větu o reflexi zobecnit a jaký to má vliv na existenci tzv. velkých kardinálů. Práce definuje nedosažitelné, Mahlovy a nepopsatelné kardinály a ukáže, jak je lze zavést pomocí reflexe. Přirozenou limitou kardinálů získaných reflexí jsou kardinály nekonzistentní s L. Práce nabídne intuitivní zdůvodněn, proč tomu tak je.

Abstract

This thesis aims to examine relations between the so called Reflection Principles and Large cardinals. Lévy has shown that the Reflection Theorem is a sound theorem of ZF and it is equivalent to the Replacement Scheme and the Axiom of Infinity. From this point of view, Reflection theorem can be seen a specific version of an Axiom of Infinity. This paper aims to examine the Reflection Principle and its generalisations with respect to the existence of Large Cardinals. This thesis will establish the Inaccessible, Mahlo and Indescribable cardinals and show how can those be defined via reflection. A natural limit of Large Cardinals obtained via reflection are cardinals inconsistent with L. This thesis will offer an intuitive explanation of why this holds.

Contents

1 Introduction

1.1 Motivation and Origin

> "The Universe of sets cannot be uniquely characterised (i. e. distinguished from all its initial elements) by any internal structural property of the membership relation in it, which is expressible in any logic of finite of transfinite type, including infinitary logics of any cardinal order."
>
> — Kurt Gödel [Wang, 1997]

1.2 Notation and Terminology

1.2.1 The Language of Set Theory

This text assumes the knowledge of basic terminology and some results from first-order predicate logic, for example [Hamilton, 1988]. We won't introduce the notions of *language*, *function symbol*, *predicate*, *term*, *model* and *interpretation* that are used in (1.42).

All proofs are based on [Jech, 2006] unless explicitly stated otherwise. Notable amount of inspiration is also drawn from [Kanamori, 2003] and [Drake, 1974].

We will now shortly review the basic notions that allow is to define the *Zermelo-Fraenkel* set theory.

When we talk about a *class*, we have the notion of a definable class in mind. If $\varphi(x, p_1, \ldots, p_n)$ is a formula in the language of set theory, we call

$$A = \{x : \varphi(x, p_1, \ldots, p_n)\} \tag{1.1}$$

a class of all sets satisfying $\varphi(x, p_1, \ldots, p_n)$ in a sense that

$$x \in A \leftrightarrow \varphi(x, p_1, \ldots, p_n) \tag{1.2}$$

Given classes A, B, one can easily define the elementary set operations such as $A \cap B$, $A \cup B$, $A \setminus B$, $\bigcup A$, see the first part of [Jech, 2006] for details. Axioms are the tools by which we can decide whether a particular class is "small enough" to be considered a set[1]. A class that fails to be considered a set is called a *proper class*.

We will often write something like "M is a limit ordinal", it should always be clear that this can be rewritten as a formula that was introduced earlier.

[1] "Small enough" means that it doesn't introduce a paradox similar to Russell's.

1.2.2 The Axioms

Definition 1.1 *(The Existence of a Set)*

$$\exists x(x = x) \tag{1.3}$$

Definition 1.2 *(Axiom of Extensionality)*

$$\forall x, y(x = y \leftrightarrow \forall z(z \in x \leftrightarrow z \in y)) \tag{1.4}$$

Definition 1.3 *(Axiom Schema of Specification)* *The following yields an axiom for every first-order formula $\varphi(x, p_1, \ldots, p_n)$ with no free variables other than $x, p_1, \ldots, p_n$.*

$$\forall x, p_1, \ldots, p_n \exists y \forall z(z \in y \leftrightarrow z \in x \;\&\; \varphi(z, p_1, \ldots, p_n)) \tag{1.5}$$

We will now provide two definitions that are not axioms, but will be helpful in establishing some axioms in a more comprehensible way.

Definition 1.4 *($x \subseteq y$, $x \subset y$)*

$$x \subseteq y \leftrightarrow (\forall z \in x) z \in y \tag{1.6}$$

$$x \subset y \leftrightarrow x \subseteq y \;\&\; x \neq y \tag{1.7}$$

We read $x \subseteq y$ as x is a subset of y *and $x \subset y$ as* x is a proper subset of y.

Definition 1.5 *(Empty Set) For an arbitrary set x, the empty set, represented by the symbol "$\emptyset$", is the set defined by the following formula:*

$$(\forall y \in x)(y \in \emptyset \leftrightarrow \neg(y = y)) \tag{1.8}$$

$\emptyset$ is a set due to Specification*, there is only one such set due to* Extensionality.

Definition 1.6 *(Axiom of Pairing)*

$$\forall x, y \exists z \forall q(q \in z \leftrightarrow q = x \vee q = y) \tag{1.9}$$

Definition 1.7 *(Axiom of Union)*

$$\forall x \exists y \forall z(z \in y \leftrightarrow \exists q(z \in q \;\&\; q \in x)) \tag{1.10}$$

Now we can introduce more axioms.

Definition 1.8 *(Axiom of Foundation)*

$$\forall x(x \neq \emptyset \rightarrow (\exists y \in x)(x \cap y = \emptyset)) \tag{1.11}$$

Definition 1.9 *(Axiom of Powerset)*

$$\forall x \exists y \forall z(z \in y \leftrightarrow z \subseteq x) \tag{1.12}$$

Definition 1.10 *(Axiom of Infinity)*

$$\exists x(\emptyset \in x \;\&\; (\forall y \in x)(y \cup \{y\} \in x)) \tag{1.13}$$

The least set satisfying this is denoted "ω".

Definition 1.11 *(Function)*
Given arbitrary first-order formula $\varphi(x, y, p_1, \ldots, p_n)$, we say that φ is a function iff

$$\forall x, y, z, p_1, \ldots, p_n(\varphi(x, y, p_1, \ldots, p_n) \;\&\; \varphi(x, z, p_1, \ldots, p_n) \rightarrow y = z) \tag{1.14}$$

When a $\varphi(x, y)$ is a function, we also write the following:

$$f(x) = y \leftrightarrow \varphi(x, y) \tag{1.15}$$

Alternatively, $f = \{\langle x, y\rangle : \varphi(x, y)\}$ is a class.

Let us introduce a few more definitions that will make the two remaining axioms more comprehensible.

Definition 1.12 *(Powerset Function)*
Given a set x, the powerset of x, *denoted $\mathscr{P}(x)$ and satisfying (1.9), is defined as follows:*

$$\mathscr{P}(x) \stackrel{\text{def}}{=} \{y : y \subseteq x\} \tag{1.16}$$

Definition 1.13 *(Domain of a Function)*
Let f be a function. We call the domain of f *the set of all sets for which f is defined. We use "$Dom(f)$" to refer to this set.*

$$x \in Dom(f) \leftrightarrow \exists y(f(x) = y) \tag{1.17}$$

We say "f is a function on A", A being a class, if $A = dom(f)$.

Definition 1.14 *(Range of a Function)*
Let f be a function. We call the range of f *the set of all sets that are images of other sets via f. We use "$Rng(f)$" to refer to this set.*

$$x \in Rng(f) \leftrightarrow \exists y(f(y) = x) \tag{1.18}$$

We say that f is *a function into A*, A being a class, iff $rng(f) \subseteq A$. We say that f is *a function onto A* iff $rng(f) = A$. We say a function f is a *one to one function*, iff

$$(\forall x_1, x_2 \in dom(f))(f(x_1) = f(x_2) \to x_1 = x_2) \tag{1.19}$$

We say that f is a bijection iff it is a one to one function that is onto.

Note that *Dom(f)* and *Rng(f)* are not definitions in a strict sense, they are in fact definition schemas that yield definitions for every function f given. Also note that they can be easily modified for φ instead of f, with the only difference being the fact that it is then defined only for those φs that are functions, which must be taken into account. This is worth noting as we will use the notions of *function* and *formula* interchangably.

Definition 1.15 *(Function Defined For All Ordinals)*
We say a function f is defined for all ordinals, *this is sometimes written $f : Ord \to A$ for any class A, if $Dom(f) = Ord$. Alternatively,*

$$(\forall \alpha \in Ord)(\exists y \in A)(f(\alpha) = y)) \tag{1.20}$$

And now for the axioms.

Definition 1.16 *(Axiom Schema of Replacement)*
The following is an axiom for every first-order formula $\varphi(x, p_1, \ldots, p_n)$ with no free variables other than $x, p_1, \ldots, p_n$.

$$\text{“}\varphi \textit{ is a function}'' \to \forall x \exists y \forall z (z \in y \leftrightarrow (\exists q \in x)(\varphi(x, y, p_1, \ldots, p_n))) \tag{1.21}$$

Definition 1.17 *(Choice function)*
We say that a function f is a choice function on x *iff*

$$dom(f) = x \setminus \{\emptyset\}) \;\&\; (\forall y \in dom(f))(f(y) \in y) \tag{1.22}$$

Definition 1.18 *(Axiom of Choice)*
For every set x there is a function f that is a choice function on x.

One might be unsettled by the fact that this definition quantifies over functions, which are generally classes, but in this particular case, since $dom(f) = x$ and x is a set, f is also a set due to *Replacement*[2].

We will refer to the axioms by their name, written in italic type, e.g. *Foundation* refers to the Axiom of Foundation. Now we need to define the set theories to be used in the article.

[2] If the underlying theory includes of implies *Replacement*.

Definition 1.19 (S)
We call S *an axiomatic theory in the language* $\mathscr{L} = \{=, \in\}$ *with exactly the following axioms:*

(i) Existence of a set *(see (1.1))*
(ii) Extensionality *(see (1.2))*
(iii) Specification *(see (1.3))*
(iv) Foundation *(see (1.8))*
(v) Pairing *(see (1.6))*
(vi) Union *(see (1.7))*
(vii) Powerset *(see (1.9))*

Definition 1.20 (ZF)
We call ZF *an axiomatic theory in the language* $\mathscr{L} = \{=, \in\}$ *that contains all the axioms of* S *in addition to the following:*

(i) Replacement *schema (see (1.16))*
(ii) Infinity *(see (1.10))*

Existence of a set *is usually left out because it is a consequence of* infinity.

Definition 1.21 (ZFC)
ZFC *is an axiomatic theory in the language* $\mathscr{L} = \{=, \in\}$ *that contains all the axioms of* ZF *plus* Choice *(1.18).*

1.2.3 The Transitive Universe

Definition 1.22 *(Transitive Class)*
We say a class A *is* transitive *iff*

$$(\forall x \in A)(x \subseteq A) \tag{1.23}$$

Definition 1.23 *(Well Ordered Class) A class* A *is said to be* well ordered by $\in$ *iff the following hold:*

(i) $(\forall x \in A)(x \notin x)$ *(Antireflexivity)*
(ii) $(\forall x, y, z \in A)(x \in y \,\&\, y \in z \to x \in z)$ *(Transitivity)*
(iii) $(\forall x, y \in A)(x = y \vee x \in y \vee y \in x)$ *(Linearity)*
(iv) $(\forall x \subseteq A)(x \neq \emptyset \to (\exists y \in x)(\forall z \in x)(z = y \vee z \in y)))$ *(Existence of the least element)*

Definition 1.24 *(Ordinal Number)*
A set x *is said to be an* ordinal number *if it is* transitive *and* well-ordered by $\in$.

For the sake of brevity, we usually just say "x is an *ordinal*". Note that "x is an ordinal" is a well-defined formula in the language of set theory, since 1.22 is a first-order formula and 1.23 is in fact a conjunction of four first-order formulas. Ordinals will be usually denoted by lower case greek letters, starting from the beginning of the alphabet: $\alpha, \beta, \gamma, \ldots$. Given two different ordinals α, β, we will write $\alpha < \beta$ for $\alpha \in \beta$, see Lemma 2.11 in [Jech, 2006] for technical details.

Definition 1.25 *(Non-Zero Ordinal) We say an ordinal α is* non-zero *iff $\alpha \neq \emptyset$.*

Definition 1.26 *(Successor Ordinal)*
Consider the following function defined for all ordinals. Let β be an arbitrary ordinal. We call S the successor function.

$$S(\beta) = \beta \cup \{\beta\} \tag{1.24}$$

An ordinal α is called a successor ordinal *iff there is an ordinal β, such that $\alpha = S(\beta)$. We also write $\alpha = \beta + 1$.*

Definition 1.27 *(Limit Ordinal)*
A non-zero ordinal α is called a limit ordinal *iff it is not a successor ordinal.*

Definition 1.28 *(Ord)*
The class of all ordinal numbers*, which we will denote "Ord"*[3] *is the proper class defined as follows.*

$$x \in Ord \leftrightarrow x \textit{ is an ordinal} \tag{1.25}$$

Definition 1.29 *(Von Neumann's Hierarchy)*
The Von Neumann's Hierarchy *is a collection of sets indexed by elements of Ord, defined recursively in the following way:*

(i)

$$V_0 = \emptyset \tag{1.26}$$

(ii)

$$V_{\alpha+1} = \mathscr{P}(V_\alpha) \textit{ for any ordinal } \alpha \tag{1.27}$$

(iii)

$$V_\lambda = \bigcup_{\beta<\lambda} V_\beta \textit{ for a limit ordinal } \lambda \tag{1.28}$$

[3]Other authors use "*On*", we will stick to the notation used in [Jech, 2006].

We will also refer to the Von Neumann's Hierarchy *as* Von Neumann's Universe *or the* Cumulative Hierarchy. *This definition is only correct in a theory that contains or implies* Replacement *because otherwise it's not clear that the successor step is a set.*

Definition 1.30 *(Rank)*
Given a set x, we say that the rank of x (written as $rank(x)$) is the least ordinal α such that $x \in V_{\alpha+1}$

Due to *Regularity*, every set has a rank.[4] The Von Neumann's hierarchy defined above can also be defined by the fact that every V_α is a set of all set with rank less than α.

Definition 1.31 *(Order-type)*
Given an arbitrary well-ordered set x, we say that an ordinal α is the order-type *of x iff x and α are isomorphic.*

1.2.4 Cardinal Numbers

Definition 1.32 *(Cardinality)*
Given a set x, let the cardinality of x, written $|x|$, be defined as the smallest ordinal number such that there is a one to one mapping from x onto α.

Definition 1.33 *(Aleph function)*
Let ω be the set defined by **??**. *We will recursively define the function $\aleph$ for all ordinals.*

(i) $\aleph_0 = \omega$

(ii) $\aleph_{\alpha+1}$ *is the least cardinal larger than* $\aleph_\alpha$[5]

(iii) $\aleph_\lambda = \bigcup_{\beta<\lambda} \aleph_\beta$ *for a limit ordinal* λ

If $\kappa = \aleph_\alpha$ and α is a successor ordinal, we call κ a successor cardinal. *If α is a limit ordinal, we call κ a* limit cardinal.

Definition 1.34 *(Cardinal number)*

(i) A set x is called a finite cardinal *iff $x \in \omega$.*

(ii) A set is called an infinite cardinal *iff there is an ordinal α such that $\aleph_\alpha = x$*

[4]See chapter 6 of [Jech, 2006] for details.

[5]"The least cardinal larger than $\aleph_\alpha$" is sometimes notated as $\aleph_\alpha^+$.

(iii) A set is called a cardinal *iff it is either a* finite cardinal *or an* infinite cardinal.

We say κ is an uncountable cardinal iff it is an infinite ordinal and $\aleph_0 < \kappa$. Infinite cardinals will be notated by lowercase greek letters from the middle of the alphabet, e.g. $\kappa, \mu, \nu, \ldots$ with the exception of λ, which is next to κ in the greek alphabet, but is also sometimes used for limit ordinal.

For formal details as well as why every set can be well-ordered assuming *Choice*, and therefore has a cardinality, see [Jech, 2006].

Definition 1.35 *(Sequence)*
We say that a function $\varphi(x, y)$ is a sequence *iff there is an ordinal α such that $dom(\varphi) = \alpha$. In other words, a function is called a sequence if it is defined exactly for every ordinal from 0 to some α. We then say it is an α-sequence. We usually write $\langle \beta_i : i \in \alpha \rangle$ or $\langle \beta_0, \beta_1, \ldots \rangle$ when referring to a sequence, β_i then denotes the elements of $rng(\varphi)$ for every $i \in dom(\varphi)$.*

Definition 1.36 *(Cofinal Subset)*
Given a class A of ordinals, we say that $B \subseteq A$ is cofinal in A *iff*

$$(\forall x \in A)(\exists y \in B)(x \in y) \tag{1.29}$$

In other words, B is cofinal in A iff it is unbounded in A.

Definition 1.37 *(Cofinality of a Limit Ordinal)*
Let λ be a limit ordinal. We say that the cofinality *of λ is κ iff κ is the least ordinal, such that there is a cofinal κ-sequence $\langle \beta_\xi : \xi < \kappa \rangle$, such that*

$$sup(\{\beta_\xi : \xi < \kappa\}) = \lambda \tag{1.30}$$

We write $cf(\lambda) = \kappa$.

Note that $cf(\alpha)$ is alway a cardinal[6].

Definition 1.38 *(Regular Cardinal)*
We say an infinite cardinal κ is regular *iff $cf(\kappa) = \kappa$.*

Definition 1.39 *(Strong Limit Cardinal)*
We say that an ordinal κ is a strong limit cardinal *if it is a* limit cardinal *and*

$$(\forall \alpha \in \kappa)(|\mathscr{P}(\alpha)| \in \kappa). \tag{1.31}$$

[6] If $cf(\alpha)$ is not a cardinal, so $|cf(\alpha)| < cf(\alpha)$, then there is a mapping from $|cf(\alpha)|$ onto $cf(\alpha)$. But then the range of this mapping is a cofinal subset of $cf(\alpha)$ that is strictly smaller than $cf(\alpha)$.

Definition 1.40 *(Generalised Continuum Hypothesis)*

$$(\forall \alpha \in Ord)\aleph_{\alpha+1} = |\mathscr{P}(\aleph_\alpha)| \tag{1.32}$$

If GCH holds (for example in Gödel's L, see chapter 3), the notions of limit cardinal and strong limit cardinal are equivalent.

1.2.5 Relativisation and Absoluteness

Definition 1.41 *(Relativization)*
Let M be a class, $R \subseteq M \times M$ and let $\varphi(p_1, \dots, p_n)$ be a first-order formula with no free variables besides $p_1, \dots, p_n$. The relativization of φ to M and R *is the formula, written as $\varphi^{M,R}$, defined in the following inductive manner:*

(i) $(x \in y)^{M,R} \leftrightarrow R(x, y)$
(ii) $(x = y)^{M,R} \leftrightarrow x = y$
(iii) $(\neg\varphi)^{M,R} \leftrightarrow \neg\varphi^{M,R}$
(iv) $(\varphi \,\&\, \psi)^{M,R} \leftrightarrow \varphi^{M,R} \,\&\, \psi^{M,R}$
(v) $(\varphi \vee \psi)^{M,R} \leftrightarrow \varphi^{M,R} \vee \psi^{M,R}$
(vi) $(\varphi \to \psi)^{M,R} \leftrightarrow \varphi^{M,R} \to \psi^{M,R}$
(vii) $(\exists x\varphi(x))^{M,R} \leftrightarrow (\exists x \in M)\varphi^{M,R}(x)$
(viii) $(\forall x\varphi(x))^{M,R} \leftrightarrow (\forall x \in M)\varphi^{M,R}(x)$

When $R = \in \cap (M \times M)$, we usually write φ^M instead of $\varphi^{M,R}$. When we talk about $\varphi^M(p_1, \dots, p_n)$, it is understood that $p_1, \dots, p_n \in M$.

Definition 1.42 *(Satisfaction in a Structure)*
Let M be a set and R a binary relation on M. Let $Terms$ be the set of all terms, let $e : Terms \to M$ any evaluation function. Let φ be a first-order formula in the language of set theory.

iff any of the following hold

We say that φ holds in $\langle M, R\rangle$ under the evaluation e, *we write $\langle M, R\rangle \models \varphi[e]$, iff any of the following hold*

(i) φ is the formula "$s = t$", s, t are terms and $e(s) = e(t)$.
(ii) φ is the formula "$s \in t$", s, t are terms and the pair $\rangle e(s), e(t)\langle$ is in R.
(iii) φ is the formula "$\neg\psi$" and not $\langle M, R\rangle \models \psi[e]$
(iv) φ is the formula "$\psi_1 \,\&\, \psi_2$" and both $\langle M, R\rangle \models \psi_1[e]$ and $\langle M, R\rangle \models \psi_2[e]$.
(v) φ is the formula "$\psi_1 \vee \psi_2$" and either $\langle M, R\rangle \models \psi_1[e]$ or $\langle M, R\rangle \models \psi_2[e]$.
(vi) φ is the formula "$\psi_1 \to \psi_2$" and either not $\langle M, R\rangle \models \psi_1[e]$ or $\langle M, R\rangle \models \psi_2[e]$.
(vii) φ is the formula "$\psi_1 \to \psi_2$" and either not $\langle M, R\rangle \models \psi_1[e]$ or $\langle M, R\rangle \models \psi_2[e]$.

(viii) φ *is the formula "*$\forall x_1 \psi$*" and* $\langle M, R\rangle \models \psi[e']$ *for every* e' *that differs from* e *only in the value of* x_1*.*

(ix) φ *is the formula "*$\forall x_1 \psi$*" and* $\langle M, R\rangle \models \psi[e']$ *for every* e' *that differs from* e *only in the value of* x_1*.*

(x) φ *is the formula "*$\exists x_1 \psi$*" and* $\langle M, R\rangle \models \psi[e']$ *for some* e' *that differs from* e *only in the value of* x_1*.*

We also write $\langle M, R\rangle \models \varphi$*, which*

Note that we say that M is a set.

We will use $\langle M, R\rangle \models \varphi(p_1, \ldots, p_n)$ and $\varphi^M(p_1, \ldots, p_n)$ interchangably.

Definition 1.43 *(Absoluteness) Given a transitive class* M*, we say a formula* φ *is* absolute in M *if for all* $p_1, \ldots, p_n \in M$

$$\varphi^M(p_1, \ldots, p_n) \leftrightarrow \varphi(p_1, \ldots, p_n) \tag{1.33}$$

Definition 1.44 *(Hierarchy of First-Order Formulas)*

(I) A first-order formula φ *is* Δ_0 *iff it is logically equivalent to a first-order formula* φ' *satisfying any of the following:*
  (i) φ' *contains no quantifiers*
  (ii) y *is a set,* ψ *is a* Δ_0*-formula, and* φ' *is either* $(\exists x \in y)\psi(y)$ *or* $(\forall x \in y)\psi(y)$*.*
  (iii) ψ_1, ψ_2 *are* Δ_0*-formulas and* φ' *is any of the following:* $\psi_1 \vee \psi_2$*,* $\psi_1 \;\&\; \psi_2$*,* $\psi_1 \rightarrow \psi_2$*,* $\neg\psi_2$*,*

(II) If a formula is Δ_0 *it is also* Σ_0 *and* Π_0

(III) A formula φ *is* $\Pi_n + 1$ *if it is logically equivalent to a formula* φ' *such that* $\varphi' = \forall x \psi$ *where* ψ *is a* Σ_n*-formula for any* $n < \omega$*.*

(IV) A formula φ *is* $\Sigma_n + 1$ *if it is logically equivalent to a formula* φ' *such that* $\varphi' = \forall x \psi$ *where* ψ *is a* Π_n*-formula for any* $n < \omega$*.*

Note that we can use the pairing function so that for $\forall p_1, \ldots, p_n \psi(p_1, \ldots, p_n)$, there is a logically equivalent formula of the form $\forall x \psi'(x)$.

Lemma 1.45 *(*Δ_0 *absoluteness) Let* φ *be a* Δ_0*-formula, then* φ *is absolute in any transitive class* M*.*

Proof. This will be proven by induction over the complexity of a given Δ_0-formula φ. Let M be an arbitrary transitive class.

Atomic formulas are always absolute by the definition of relativisation, see (1.41). Suppose that Δ_0-formulas ψ_1 and ψ_2 are absolute in M. Then from relativization, $(\psi_1 \,\&\, \psi_2)^M \leftrightarrow \psi_1^M \,\&\, \psi_2^M$, which is, from the induction hypothesis, equivalent to $\psi_1 \,\&\, \psi_2$. The same holds for $\vee, \rightarrow, \neg$.

Suppose that a Δ_0-formula ψ is absolute in M. Let y be a set and let $\varphi = (\exists x \in y)\psi(x)$. From relativization, $(\exists x \psi(x))^M \leftrightarrow (\exists x \in M)\psi^M(x)$. Since the hypotheses makes it clear that $\psi^M \leftrightarrow \psi$, we get $((\exists x \in y)\psi(x))^M \leftrightarrow (\exists x \in y \cap M)\psi(x)$, which is the equivalent of $\varphi^M \leftrightarrow \varphi$. The same applies to $\varphi = (\forall x \in y)\psi(x)$. □

Lemma 1.46 *(Downward Absoluteness)*
Let φ be a Π_1-formula and M a transitive class. Then the following holds:

$$(\forall p_1, \ldots, p_n \in M)(\varphi(p_1, \ldots, p_n) \rightarrow \varphi(p_1, \ldots, p_n)^M) \tag{1.34}$$

Proof. Since $\varphi(p_1, \ldots, p_n)$ is Π_1, there is a Δ_0-formula $\psi(p_1, \ldots, p_n, x)$ such that $\varphi = \forall x \psi(p_1, \ldots, p_n, x)$. From relativization and lemma (1.45), $\varphi^M(p_1, \ldots, p_n) \leftrightarrow (\forall x \in M)\psi(p_1, \ldots, p_n, x)$.

Assume that for $p_1, \ldots, p_n \in M$ fixed, that $\forall x \psi(p_1, \ldots, p_n, x)$ holds, but $(\forall x \in M)\psi(p_1, \ldots, p_n, x)$ does not. Therefore $\exists x \neg\psi(p_1, \ldots, p_n, x)$, which contradicts $\forall x \psi(p_1, \ldots, p_n, x)$. □

Lemma 1.47 *(Upward Absoluteness)*
Let φ be a Σ_1-formula and M a transitive class. Then the following holds:

$$(\forall p_1, \ldots, p_n \in M)(\varphi^M(p_1, \ldots, p_n) \rightarrow \varphi(p_1, \ldots, p_n)) \tag{1.35}$$

Proof. Since $\varphi(p_1, \ldots, p_n)$ is Σ_1, there is a Δ_0-formula $\psi(p_1, \ldots, p_n, x)$ such that $\varphi = \exists x \psi(p_1, \ldots, p_n, x)$. From relativization and lemma (1.45), $\varphi^M(p_1, \ldots, p_n) \leftrightarrow (\exists x \in M)\psi(p_1, \ldots, p_n, x)$.

Assume that for $p_1, \ldots, p_n \in M$ fixed, that $(\exists x \in M)\psi(p_1, \ldots, p_n, x)$ holds, but $\exists x \psi(p_1, \ldots, p_n, x)$ does not. This is an obvious contradiction. □

1.2.6 More Functions

Definition 1.48 *(Strictly Increasing Function)*
A function $f : Ord \rightarrow Ord$ is said to be strictly increasing *iff*

$$\forall \alpha, \beta \in Ord(\alpha < \beta \rightarrow f(\alpha) < f(\beta)). \tag{1.36}$$

Definition 1.49 *(Continuous Function)*
A function $f : Ord \rightarrow Ord$ is said to be continuous *iff*

$$\lambda \textit{ is limit } \rightarrow f(\lambda) = \bigcup_{\alpha < \lambda} f(\alpha). \tag{1.37}$$

Definition 1.50 *(Normal Function)*
A function $f : Ord \to Ord$ is said to be normal *iff it is* strictly increasing *and* continuous.

Definition 1.51 *(Fixed Point)*
We say x is a fixed point of a function f iff $x = f(x)$.

Definition 1.52 *(Unbounded Class)*
We say a class A of ordinals is unbounded iff

$$\forall x(\exists y \in A)(x < y) \tag{1.38}$$

Definition 1.53 *(Limit Point)*
Given a class $x \subseteq Ord$, we say that $\alpha \neq \emptyset$ is a limit point of x iff

$$\alpha = \bigcup (x \cap \alpha) \tag{1.39}$$

Definition 1.54 *(Closed Class)*
We say a class $A \subseteq Ord$ is closed iff it contains all its limit points.

Definition 1.55 *(Club set)*
For a regular uncountable cardinal κ, a set $x \subset \kappa$ is a closed unbounded *subset, abbreviated as a* club set*, iff x is both closed and unbounded in κ.*

Definition 1.56 *(Stationary set)*
For a regular uncountable cardinal κ, we say a set $A \subset \kappa$ is stationary in κ iff it intersects every club subset of κ.

1.2.7 Structure, Substructure and Embedding

Structures will be denoted $\langle M, \in, R\rangle$ where M is a domain, $\in$ stands for the standard membership relation, it is assumed to be restricted to the domain[7], $R \subseteq M$ is a relation on the domain. When R is not needed, we can as well only write M instead of $\langle M, \in\rangle$.

Definition 1.57 *(Elementary Embedding)*
Given the structures $\langle M_0, \in, R\rangle$, $\langle M_1, \in, R\rangle$ and a one-to-one function $j : M_0 \to M_1$, we say j is an elementary embedding *of M_0 into M_1, we write $j : M_0 \prec M_1$, when the following holds for every formula $\varphi(p_1, \ldots, p_n)$ and every $p_1, \ldots, p_n \in M_0$:*

$$\langle M_0, \in, R\rangle \models \varphi(p_1, \ldots, p_n) \leftrightarrow \langle M_1, \in, R\rangle \models \varphi(j(p_1), \ldots, j(p_n)) \tag{1.40}$$

[7] To be totally explicit, we should write $\langle M, \in \cap M \times M, R\rangle$.

Definition 1.58 *(Elementary Substructure)*
Given the structures $\langle M_0, \in, R\rangle$, $\langle M_1, \in, R\rangle$ *and a one-to-one function* $j : M_0 \to M_1$ *such that* $j : M_0 \prec M_1$, *we say that* M_0 *is an* elementary substructure *of* M_1, *denoted as* $M_0 \prec M_1$, *iff* j *is an identity on* M_0. *In other words*

$$\langle M_0, \in, R\rangle \models \varphi(p_1, \ldots, p_n) \leftrightarrow \langle M_1, \in, R\rangle \models \varphi(p_1, \ldots, p_n) \tag{1.41}$$

for $p_1, \ldots, p_n \in M_0$

2 Levy's First-Order Reflection

2.1 Lévy's Original Paper

This section is based on Lévy's paper *Axiom Schemata of Strong Infinity in Axiomatic Set Theory*, [Lévy, 1960]. It presents Lévy's general reflection principle and its equivalence to *Replacement* and *Infinity* under S[8].

First, we should point out that set theory has changed over the last 66 years and show a few notable, albeit only formal, differences. One might be confused by the fact that Lévy treats the *Subsets* axiom, which we call *Specification*, as a single axiom rather than a schema. He even takes the conjunction of all axioms of ZF and treats it like a formula. This is possible because the underlying logic calculus is different. Lévy works with set theories formulated in the *non-simple applied first order functional calculus*, see beginning of *Chapter IV* in [Church, 1996] for details. For now, we only need to know that the calculus contains a substitution rule for functional variables. This way, *Subsets* is de facto a schema even though it sometimes treated as a single formula[9] but the logic is still first-order since one can't quantify over functional variables. We will use the usual first-order axiomatization of ZFC as seen on [Jech, 2006]. It should also be noted that the logical connectives look different. The now usual symbol for an universal quantifier does not appear, $\forall x\varphi(x)$ would be written as $(x)\varphi(x)$. The symbol for negation is "$\sim$", implication is written as "$\supset$" and equivalence is "$\equiv$". We will use standard notation with "$\neg$", "$\rightarrow$" and "$\leftrightarrow$" respectively when presenting Lévy's results.

This subsection uses ZF instead of the usual ZFC as the underlying theory.

Definition 2.1 *(Standard Complete Model of a Set Theory)*
Let Q *be an arbitrary axiomatic set theory. We say that* u *is a standard complete model of* Q *iff*

(i) $(\forall\sigma \in \mathsf{Q})(\langle \mathsf{u}, \in\rangle \models \sigma)$

(ii) $\forall y(y \in u \rightarrow y \subset u)$ *(u is transitive)*

We write $Scm^{\mathsf{Q}}(u)$.

Definition 2.2 *(Cardinals Inaccessible With Respect to* Q*)*
Let Q *be an arbitrary axiomatic set theory. We say that a cardinal* κ *is inaccessible with respect to theory* Q *iff*

$$Scm^{\mathsf{Q}}(V_\kappa) \tag{2.42}$$

[8] See definition (1.19).

[9] This way, the conjunction of all axioms is then in fact an axiom schema.

We write $In^{\mathsf{Q}}(\kappa)$.[10]

Definition 2.3 *(Inaccessible Cardinal With Respect to* ZF*)*
When a cardinal κ is inaccessible with respect to ZF*, we only say that it is inaccessible. We write $In(\kappa)$.*

$$In(\kappa) \leftrightarrow In^{\mathsf{ZF}}(\kappa) \tag{2.43}$$

The above definition of inaccessibles is used because it doesn't require *Choice*.

For the definition of relativization, see (1.41). The notation used by Lévy is "$Rel(u, \varphi)$", we will stick to "φ^u".

Definition 2.4 *(N)*
The following is an axiom schema of complete reflection over ZF*, denoted N. For every first-order formula φ in the language of set theory with no free variables except for $p_1, \ldots, p_n$, the following is an instance of schema N.*

$$\exists u(Scm^{\mathsf{ZF}}(u) \ \&\ (\forall p_1, \ldots, p_n \in u)(\varphi \leftrightarrow \varphi^u)) \tag{2.44}$$

Definition 2.5 *(N')*
For any first-order formulas $\varphi_1, \ldots, \varphi_m$ in the language of set theory with no free variables except for $p_1, \ldots, p_n$, the following is an instance of schema N'.

$$\exists u(z \in u \ \&\ Scm^{\mathsf{ZF}}(u) \ \&\ (\forall p_1, \ldots, p_n \in u)(\varphi_1 \leftrightarrow \varphi_1^u) \ \&\ \ldots \ \&\ \varphi_m \leftrightarrow \varphi_m^u)) \tag{2.45}$$

Definition 2.6 *(N')*
For any first-order formulas $\varphi_1, \ldots, \varphi_m$ in the language of set theory with no free variables except for $p_1, \ldots, p_n$, the following is an instance of schema N'.

$$\exists u(Scm^{\mathsf{ZF}}(u) \ \&\ (\forall p_1, \ldots, p_n \in u)(\varphi_1 \leftrightarrow \varphi_1^u) \ \&\ \ldots \ \&\ \varphi_m \leftrightarrow \varphi_m^u)) \tag{2.46}$$

Let S be an axiomatic set theory defined in (1.19).

This is *Theorem 2* in [Lévy, 1960]

Lemma 2.7 *($N \leftrightarrow N'' \leftrightarrow N'$)*
The schemas N, N' and N'' are equivalent under S.

[10] To be able to define V_κ, we need to work in a logic that contains the *Replacement Schema* or any of it's equivalents. It should be noted that we don't work in an arbitrary theory Q, but in ZF, which contains the *Replacement Schema*. $Scm^{\mathsf{Q}}(V_\kappa)$ in fact says "ZF thinks that V_κ is a transitive model of Q".

Proof. We will execute this proof in the theory ZF, but the reader should note that we are neither using *Replacement* nor *Infinity*, so for schemas similar to N, N', N'' but with "$Scm^{\mathsf{S}}(u)$" instead of "$Scm^{\mathsf{ZF}}(u)$", the proof works equally well.

Clearly, $N' \to N'' \to N$.

Now, assuming N and given the formulas $\varphi_1, \ldots, \varphi_n$, we will prove N''. Consider the following formula:

$$\psi = \bigvee_{i=1}^{t} t = i \;\&\; \varphi_i \tag{2.47}$$

We will take advantage of the fact that natural numbers are defined by atomic formulas and therefore absolute in transitive structures. From N, we get such u that $Scm^{\mathsf{ZF}}(u) \;\&\; (\forall p_1, \ldots, p_n \in u)(\bigvee_{i=1}^{t} t = i \;\&\; \varphi_i \leftrightarrow \bigvee_{i=1}^{t} t = i \;\&\; \varphi_i^u)$. This already satisfies N''.

In order to prove N' from N'', let's add two more formulas. Given $p_1, \ldots, p_n$, we denote

$$\varphi_{m+1} = \exists u(z \in u \;\&\; Scm^{\mathsf{ZF}}(u) \;\&\; (\forall p_1, \ldots, p_n \in u)(\bigvee_{i=1}^{m} \varphi_i = \varphi_i^u)) \tag{2.48}$$

$$\varphi_{m+2} = \forall z \varphi_{m+1} \tag{2.49}$$

So, by N'', we have a set u that satisfies $Scm^{\mathsf{ZF}}(u)$ as well as the following:

$$(\forall p_1, \ldots, p_n \in u)(\varphi_i \leftrightarrow \varphi_i^u) \text{ for } 1 \leq i \leq m \tag{2.50}$$

$$z \in u \to \varphi_{m+1} \;\leftrightarrow \varphi_{m+1}^u \tag{2.51}$$

$$\varphi_{m+2} \leftrightarrow \varphi_{m+2}^u \tag{2.52}$$

By $Scm^{\mathsf{ZF}}(u)$ and (2.50), we get $(\forall z \in u)\varphi_{m+1}$, so together with (2.51), we get $(\forall z \in u)\varphi_{m+1}^u$, exactly φ_{m+2}^u, so by (2.52) we get φ_{m+2}. But φ_{m+2} is exactly the instance of N' we were looking for. □

Definition 2.8 *(N_0)*
Axiom schema N_0 is similar to N defined above, but with S *instead of* ZF*. For every φ, a first-order fomula in the language of set theory with no free variables except $p_1, \ldots, p_n$, the following is an instance of N_0.*

$$\exists u(Scm^{\mathsf{S}}(u) \;\&\; (\forall p_1, \ldots, p_n \in u)(\varphi \leftrightarrow \varphi^u)) \tag{2.53}$$

We will now show that in S, N_0 implies both *Replacement* and *Infinity*.

Let N_0 be defined as in (2.8), for *Infinity* see (1.10).

Theorem 2.9 *In* S*, the axiom schema* N_0 *implies* Infinity*.*

Proof. Let $\varphi = \forall x \exists y(y = x \cup \{x\})$. This clearly holds in S because given a set x, there is a set $y = x \cup \{x\}$ obtained via *Pairing* and *Union*. From N_0, there is a set u such that φ^u holds. This u satisfies the conditions required by *Infinity*. □

Lévy proves this theorem in a different way. He argues that for an arbitrary formula φ, N_0 gives us $\exists u Scm^{\mathsf{S}}(u)$ and this u already satisfies *Infinity*. To do this, we would need to prove lemma (2.15) earlier on, we will do that later in this chapter.

Let S be a set theory defined in (1.19), N_0 a schema defined in (2.8) and *Replacement* a schema defined in (1.16).

Theorem 2.10 *In* S*, the axiom schema* N_0 *implies* Replacement*.*

Proof. Let $\varphi(x, y, p_1, \ldots, p_n)$ be a formula with no free variables except $x, y, p_1, \ldots, p_n$. Let a set x be given and let χ be an instance of the *Replacement* schema for the φ given. We want to verify in S that given a formula φ, the instance of N_0 for φ implies χ.

$$\begin{aligned}\chi = \forall x', y', z(\varphi(x', y', p_1, \ldots, p_n) \mathbin{\&} \varphi(x', z, p_1, \ldots, p_n) \to y' = z') \\ \to \exists y \forall z(z \in y \leftrightarrow (\exists q \in x)(\varphi(x, y, p_1, \ldots, p_n)))\end{aligned} \tag{2.54}$$

Since it can be shown that N_0 is equivalent to N_0' similar to N' in lemma (2.7), there is a set u such that $Scm^{\mathsf{S}}(u)$, $x \in u$ and all of the following hold:
(i) $\varphi \leftrightarrow \varphi^u$
(ii) $\exists y \varphi \leftrightarrow (\exists y \varphi)^u$
From relativization, $(\exists y \varphi)^u$ is equivalent to $(\exists y \in u)\varphi^u$, together with (i) and (ii), we get

$$(\exists y \in u)\varphi \leftrightarrow \exists y \varphi \tag{2.55}$$

If φ is a function, it maps the elements of x, which are also elements of u due to transitivity of u, to elements of u. From *Specification*, $y = \{z \in u\}$ we can find y, a set of all images of the elements of x via φ. So we have satisfied *Replacement* – given a function and a set, we have proven that image of the set via given function is also a set. □

What we have just proven is only a single theorem from Lévy's aforementioned article, we will introduce other interesting results, dealing with inaccessible and Mahlo cardinals, later in their appropriate context in chapter 3.

2.2 Contemporary Restatement

We will now introduce and prove a theorem that is called Lévy's Reflection in contemporary set theory. The only difference is that while Lévy originally reflects a formula φ from V to a set u which is a *standard complete model of* S, we say that there is a V_λ for a limit λ that reflects φ. Those two conditions are equivalent due to lemma (2.15).

Lemma 2.11 *Let $\varphi_1, \ldots, \varphi_n$ be first-order formulas in the language of set theory, all with m free variables* [11]*.*

(i) For each set M_0 there is such set M that $M_0 \subset M$ and the following holds for every i, $1 \leq i \leq n$:

$$\exists x \varphi_i(p_1, \ldots, p_{m-1}, x) \rightarrow (\exists x \in M)\varphi_i(p_1, \ldots, p_{m-1}, x) \tag{2.56}$$

for every $p_1, \ldots, p_{m-1} \in M$.

(ii) Furthermore, there is a limit ordinal λ such that $M_0 \subset V_\lambda$ and the following holds for each i, $1 \leq i \leq n$:

$$\exists x \varphi_i(p_1, \ldots, p_{m-1}, x) \rightarrow (\exists x \in V_\lambda)\varphi_i(p_1, \ldots, p_{m-1}, x) \tag{2.57}$$

for every $p_1, \ldots, p_{m-1} \in M$.

(iii) Assuming Choice*, there is M, $M_0 \subset M$ such that (2.56) holds for every M, $i \leq n$ and $|M| \leq |M_0| \cdot \aleph_0$.*

Proof. We will simultaneously prove statements (i) and (ii), denoting M^T the transitive set required by part (ii). Steps in the construction of M^T that are not explicitly included are equivalent to steps for M.

Let us first define an operation $H_i(p_1, \ldots, p_{m-1})$ that yields the set of x's with minimal rank[12] satisfying $\varphi_i(p_1, \ldots, p_{m-1}, x)$ for $p_1, \ldots, p_{m-1}$ and for every i, $1 \leq i \leq n$.

$$H_i(p_1, \ldots, p_n) = \{x \in C_i : (\forall z \in C)(rank(x) \leq rank(z))\} \tag{2.58}$$

for each $1 \leq i \leq n$, where

$$C_i = \{x : \varphi_i(p_1, \ldots, p_{m-1}, x)\} \text{ for } 1 \leq i \leq n \tag{2.59}$$

[11] For formulas with a different number of free variables, take for m the highest number of parameters among those formulas. Add spare parameters to every formula that has less than m parameters in a way that preserves the last parameter, which we will denote x. E.g. let φ_i' be the a formula with k parameters, $k < m$. Let us set $\varphi_i(p_1, \ldots, p_{m-1}, x) = \varphi_i'(p_1, \ldots, p_{k-1}, x)$, notice that the parameters $p_k, \ldots, p_{m-1}$ are not used.

[12] Rank is defined in (1.30).

Next, let's construct M from given M_0 by induction.

$$M_{i+1} = M_i \cup \bigcup_{j=0}^{n} \bigcup \{H_j(p_1, \ldots, p_{m-1}) : p_1, \ldots, p_{m-1} \in M_i\} \tag{2.60}$$

In other words, in each step we include into the construction the elements satisfying $\varphi(p_1, \ldots, p_{m-1}, x)$ for $p_1, \ldots, p_{m-1}$ from the previous step. For statement (ii), this is the only part that differs from (i). To end up with a transitive M, we need to extend every step to it's transitive closure transitive closure of M_{i+1} from (i). In other words, let γ be the smallest ordinal such that

$$(M_i^T \cup \bigcup_{j=0}^{n} \{\bigcup \{H_j(p_1, \ldots, p_{m-1}) : p_1, \ldots, p_{m-1} \in M_i\}\}) \subset V_\gamma \tag{2.61}$$

Then the incremental step is

$$M_{i+1}^T = V_\gamma \tag{2.62}$$

and the final M is obtained by joining the previous steps.

$$M = \bigcup_{i=0}^{\infty} M_i, \;\; M^T = \bigcup_{i=0}^{\infty} M_i^T = V_\lambda \text{ for some limit } \lambda. \tag{2.63}$$

We have yet to finish part (iii). Let's try to construct a set M' that satisfies the same conditions like M but is kept as small as possible. Assuming the Axiom of Choice, we can modify the construction so that the cardinality of M' is at most $|M_0| \cdot \aleph_0$. Note that the size of M in the previous construction is determined by the size of M_0 and, most importantly, by the size of $H_i(p_1, \ldots, p_{m-1})$ for every i, $1 \leq i \leq n$ in individual iterations of the construction. Since (i) only ensures the existence of an x that satisfies $\varphi_i(p_1, \ldots, p_{m-1}, x)$ for any i, $1 \leq i \leq n$, we only need to add one x for every set of parameters but $H_i(u_1, \ldots, u_{m-1})$ can be arbitrarily large. Let F be a choice function on $\mathscr{P}(M')$. Also let $h_i(p_1, \ldots, p_{m-1}) = F(H_i(p_1, \ldots, p_{m-1}))$ for i, where $1 \leq i \leq n$, which means that h is a function that outputs an x that satisfies $\varphi_i(p_1, \ldots, p_{m-1}, x)$ for i such that $1 \leq i \leq n$ and has minimal rank among all such sets. The induction step needs to be redefined to

$$M'_{i+1} = M'_i \cup \bigcup_{j=0}^{n} \{h_j(p_1, \ldots, p_{m-1}) : p_1, \ldots, p_{m-1} \in M'_i\} \tag{2.64}$$

This way, the amount of elements added to M'_{i+1} in each step of the construction is the same as the amount of m-tuples of parameters that yielded elements not included in M_i'. It is easy to see that if M_0 is finite, M' is countable because it was constructed as a countable union of sets that are themselves at most countable. If M_0 is countable or larger, the cardinality of M' is equal to the cardinality of M_0.[13] Therefore $|M'| \leq |M_0| \cdot \aleph_0$ □

Theorem 2.12 *(Lévy's first-order reflection theorem)*
Let $\varphi(p_1, \dots, p_n)$ be a first-order formula.

(i) For every set M_0 there exists a set M such that $M_0 \subset M$ and the following holds:

$$\varphi^M(p_1, \dots, p_n) \leftrightarrow \varphi(p_1, \dots, p_n) \tag{2.65}$$

for every $p_1, \dots, p_n \in M$.

(ii) For every set M_0 there is a transitive set M, $M_0 \subset M$ such that the following holds:

$$\varphi^M(p_1, \dots, p_n) \leftrightarrow \varphi(p_1, \dots, p_n) \tag{2.66}$$

for every $p_1, \dots, p_n \in M$.

(iii) For every set M_0 there is a limit ordinal λ such that $M_0 \subset V_\lambda$ and the following holds:

$$\varphi^{V_\lambda}(p_1, \dots, p_n) \leftrightarrow \varphi(p_1, \dots, p_n) \tag{2.67}$$

for every $p_1, \dots, p_n \in M$.

(iv) Assuming Choice, *for every set M_0 there is M such that $M_0 \subset M$ and $|M| \leq |M_0| \cdot \aleph_0$ and the following holds:*

$$\varphi^M(p_1, \dots, p_n) \leftrightarrow \varphi(p_1, \dots, p_n) \tag{2.68}$$

for every $p_1, \dots, p_n \in M$.

Proof. Let's now prove (i) for given φ via induction by complexity. We can safely assume that φ contains no quantifiers besides "$\exists$" and no logical connectives other than "$\neg$" and "&". Let $\varphi_1, \dots, \varphi_n$ be all subformulas of φ. Then there is a set M, obtained by the means of lemma (2.11), for all of the formulas $\varphi_1, \dots, \varphi_n$.

Let's first consider atomic formulas in the form of either $x_1 = x_2$ or $x_1 \in x_2$. It is clear from relativisation[14] that (2.65) holds for both cases, $(x_1 = x_2)^M \leftrightarrow (x_1 = x_2)$ and $(x_1 \in x_2)^M \leftrightarrow (x_1 \in x_2)$.

[13] It can not be smaller because $|M_{i+1}'| \geq |M_i'|$ for every i. It may not be significantly larger because the maximum of elements added is the number of n-tuples in M_i', which is of the same cardinality as M_i'.

[14] See (1.41.). This only holds for relativization to $M, \in \cap M \times M$, not M, R for an arbitrary R.

We now want to verify the inductive step. First, take $\varphi = \neg\varphi'$. From relativization, we get $(\neg\varphi')^M \leftrightarrow \neg(\varphi'^M)$. Because the induction hypothesis tells us that $\varphi'^M \leftrightarrow \varphi'$, the following holds:

$$(\neg\varphi')^M \leftrightarrow \neg(\varphi'^M) \leftrightarrow \neg\varphi' \tag{2.69}$$

The same holds for $\varphi = \varphi_1 \ \& \ \varphi_2$. From the induction hypothesis, we know that $\varphi_1^M \leftrightarrow \varphi_1$ and $\varphi_2^M \leftrightarrow \varphi_2$, which together with relativization for formulas in the form of $\varphi_1 \ \& \ \ \varphi_2$ gives us

$$(\varphi_1 \ \& \ \varphi_2)^M \leftrightarrow \varphi_1^M \ \& \ \varphi_2^M \leftrightarrow \varphi_1 \ \& \ \varphi_2 \tag{2.70}$$

Let's now examine the case when $\varphi = \exists x \varphi'(p_1, \ldots, p_n, x)$. The induction hypothesis tells us that $\varphi'^M(p_1, \ldots, p_n, x) \leftrightarrow \varphi'(p_1, \ldots, p_n, x)$, so, together with above lemma (2.11), the following holds:

$$\begin{gathered}
\varphi(p_1, \ldots, p_n, x) \\
\leftrightarrow \exists x \varphi'(p_1, \ldots, p_n, x) \\
\leftrightarrow (\exists x \in M)\varphi'(p_1, \ldots, p_n, x) \\
\leftrightarrow (\exists x \in M)\varphi'^M(p_1, \ldots, p_n, x) \\
\leftrightarrow (\exists x \varphi'(p_1, \ldots, p_n, x))^M \\
\leftrightarrow \varphi^M(p_1, \ldots, p_n, x)
\end{gathered} \tag{2.71}$$

Which is what we wanted to prove for part (i).

We now need to verify that the same holds for any finite number of formulas $\varphi_1, \ldots, \varphi_n$. This has in fact been already done since lemma (2.11) gives us a set M for any finite amount of formulas and given M_0. We can therefore find a set M for the union of all of their subformulas. When we obtain such M, it should be clear that it also reflects every formula in $\varphi_1, \ldots, \varphi_n$.

Since V_λ is a transitive set, by proving (**iii**) we also satisfy (**ii**). To do so, we only need to look at part (**ii**) of lemma (2.11). All of the above proof also holds for $M = V_lambda$.

To finish part (**iv**), we take M of size $\leq |M_0| \cdot \aleph_0$, which exists due to part (**iii**) of lemma (2.11), the rest being identical. □

Let S be a set theory defined in (1.19), for ZFC see definition (1.21).

The two following lemmas are based on [Drake, 1974][Chapter 3, Theorem 1.2].

Lemma 2.13 *If M is a transitive set, then $\langle M, \in \rangle \models$* Extensionality*.*

Proof. Given a transitive set M, we want to show that the following holds.

$$\langle M, \in \rangle \models \forall x, y(x = y \leftrightarrow \forall z(z \in x \leftrightarrow z \in y)) \tag{2.72}$$

Given arbitrary $x, y \in M$, we want to prove that $\langle M, \in \rangle \models (x = y \leftrightarrow \forall z(z \in x \leftrightarrow z \in y))$. This is equivalent to $\langle M, \in \rangle \models x = y$ iff $\langle M, \in \rangle \models \forall z(z \in x \leftrightarrow z \in y)$, which is the same as $x = y$ iff $\langle M, \in \rangle \models \forall z(z \in x \leftrightarrow z \in y)$.

So all elements of x are also elements of y in M, and vice versa. Because M is transitive, all elements of x and y are in M, so $\langle M, \in \rangle \models \forall z(z \in x \leftrightarrow z \in y)$ holds iff x and y contain the same elements and are therefore equal. □

Lemma 2.14 *If M is a transitive set, then $\langle M, \in \rangle \models$ Foundation.*

Proof. We want to prove the following:

$$\langle M, \in \rangle \models \forall x(x \neq \emptyset \rightarrow (\exists y \in x)(x \cap y = \emptyset)) \tag{2.73}$$

Given an arbitrary non-empty $x \in M$ let's show that $\langle M, \in \rangle \models (\exists y \in x)(x \cap y = \emptyset)$.

Because M is transitive, every element of x is an element of M. Take for y the element of x with the lowest rank[15]. It should be clear that there is no $z \in y$ such that $z \in x$, because then $rank(z) < rank(y)$, which would be a contradiction. □

Let S be a set theory as defined in (1.19).

Lemma 2.15 *The following holds for every λ.*

$$\text{“}\lambda \textit{ is a limit ordinal”} \rightarrow \langle V_\lambda, \in \rangle \models \mathsf{S} \tag{2.74}$$

Proof. Given an arbitrary limit ordinal λ, we will verify the axioms of S one by one.

(i) *The existence of a set* comes from the fact that V_λ is a non-empty set because limit ordinal is non-zero by definition.
(ii) *Extensionality* holds from (2.13).
(iii) *Foundation* holds from (2.14).
(iv) *Union*:
Given any $x \in V_\lambda$, we want verify that $y = \bigcup x$ is also in V_λ. Note that $y = \bigcup x$ is a Δ_0-formula.

$$y = \bigcup x \leftrightarrow (\forall z \in y)(\exists q \in x) z \in q \ \& \ (\forall z \in x)(\forall q \in z) q \in y \tag{2.75}$$

So by lemma (1.45)

$$y = \bigcup x \leftrightarrow \langle V_\lambda, \in \rangle \models y = \bigcup x \tag{2.76}$$

[15] Rank is defined in (1.30).

(v) *Pairing*:
Given two sets $x, y \in V_\lambda$, we want to show that $z = \{x, y\}$ is also an element of V_λ.

$$z = \{x, y\} \leftrightarrow x \in z \;\&\; y \in z \;\&\; (\forall q \in z)(q = x \vee q = y) \tag{2.77}$$

So $z = \{x, y\}$ is a Δ_0-formula, and thus by lemma (1.45) it holds that

$$z = \{x, y\} \leftrightarrow \langle V_\lambda, \in\rangle \models z = \{x, y\} \tag{2.78}$$

(vi) *Powerset*:
Given any $x \in V_\lambda$, we want to make sure that $\mathscr{P}(x) \in V_\lambda$. Let $\varphi(y)$ denote the formula $y \in \mathscr{P}(x) \leftrightarrow y \subset x$. according to definition of subset (1.4), $y \subset x$ is Δ_0, so for any given $x, y \in V_\lambda$, $y = \mathscr{P}(x) \leftrightarrow \langle V_\lambda, \in\rangle \models y = \mathscr{P}(x)$. Because λ is limit and $rank(\mathscr{P}(x)) = rank(x) + 1$, if $\mathscr{P}(x) \in V_\lambda$ for every $x \in V_\lambda$.

(vii) *Specification*:
Given a first-order formula φ, we want to show the following:

$$\langle V_\lambda, \in\rangle \models \forall x, p_1, \ldots, p_n, \exists y \forall z(z \in y \leftrightarrow z \in x \;\&\; \varphi(z, p_1, \ldots, p_n)) \tag{2.79}$$

Given any x along with parameters $p_1, \ldots, p_n$ in V_λ, we set

$$y = \{z \in x : \varphi^{V_\lambda}(z, p_1, \ldots, p_n)\} \tag{2.80}$$

From transitivity of V_λ and the fact that $y \subset x$ and $x \in V_\lambda$, we know that $y \in V_\lambda$, so $\langle V_\lambda, \in\rangle \models \forall z(z \in y \leftrightarrow z \in x \;\&\; \varphi(z, p_1, \ldots, p_n))$.

□

Definition 2.16 *(First-Order Reflection Schema)*
For every first-order formula φ, the following is an axiom:

$$\forall M_0 \exists M(M_0 \subseteq M \;\&\; (\varphi(p_1, \ldots, p_n) \leftrightarrow \varphi(p_1, \ldots, p_n)^M)) \tag{2.81}$$

We will refer to this axiom schema as First-order reflection.

Let *Infinity* and *Replacement* be as defined in (1.10) and (1.16) respectively.

Theorem 2.17 First-order reflection *is equivalent to* Infinity $\&$ Replacement *under* S.

Proof. Since (2.12) already gives us one side of the implication, we are only interested in showing the converse which we shall do in two parts:

First-order reflection $\rightarrow$ *Infinity* This is done exactly like (2.9). We pick for φ the formula $(\forall y \in x)(y \cup \{y\} \in x)$, $M_0 = \{\emptyset\}$. From (2.16), there is a set M that satisfies φ, so there is an inductive set. We have picked M_0 so that $\emptyset \in M$ obviously holds and M is the witness for

$$\exists x(\emptyset \in x \ \& \ (\forall y \in x)(y \cup \{y\} \in x)) \tag{2.82}$$

which is exactly (1.10).

First-order reflection $\rightarrow$ *Replacement*

Let's first point out that while *First-order reflection* gives us a set for one formula, we can generalize it to hold for any finite number of formulas. We will show how is it done for two formulas, which is what we will use in this proof. Given two first-order formulas φ, ψ, we can suppose that there are formulas φ' and ψ' that are equivalent to φ and ψ respectively, but their free variables are different [16]. Let $\xi = \varphi \ \& \ \psi$, given any M_0, we can find a M such that $\xi \leftrightarrow \xi^M$. It is easy to see that from relativisation, the following holds:

$$\varphi \ \& \ \psi \leftrightarrow \varphi' \ \& \ \psi' \leftrightarrow \xi \leftrightarrow \xi^M \leftrightarrow (\varphi' \ \& \ \psi')^M \leftrightarrow \varphi'^M \ \& \ \psi'^M \leftrightarrow \varphi^M \ \& \ \psi^M \tag{2.83}$$

Now given a function $\varphi(x, y)$, we know from *First-order reflection* that for every M_0, there is a set M such that $M_0 \subseteq M$ and both

$$(\forall x, y \in M)(\varphi(x, y) \leftrightarrow \varphi^M(x, y)) \tag{2.84}$$

and

$$(\forall x, y \in M)(\exists y \varphi(x, y) \leftrightarrow (\exists y \varphi(x, y))^M) \tag{2.85}$$

hold, the latter being equivalent to

$$(\forall x, y \in M)(\exists y \varphi(x, y) \leftrightarrow (\exists y \in M)\varphi^M(x, y)) \tag{2.86}$$

Therefore

$$(\forall x, y \in M)(\exists y \varphi(x, y) \leftrightarrow (\exists y \in M)\varphi(x, y)) \tag{2.87}$$

holds too. That means that we have a set M such that for every $x \in M$, if φ is defined for x, $(\exists y \in M)\varphi(x, y)$.

To show that *Replacement* holds for this particular φ, we need to verify that given a set M_0, $M_0' = \{y : (\exists x \in M_0)\varphi(x, y)\}$ is also a set. But since $M_0 \subseteq M$

[16] This is plausible since we can for example substitute all free variables in φ' for $x_0, x_2, x_4, \ldots$ and use $x_1, x_3, x_5, \ldots$ for free variables in ψ', the resulting formulas will be equivalent.

and because given any $x \in M$, there is $y \in M$ satisfying $\varphi(x, y)$, the following is a set due to *Specification*:

$$M_0' = \{y : (\exists x \in M_0)\varphi(x, y)\} = \{y \in M : (\exists x \in M_0)\varphi(x, y)\} \tag{2.88}$$

$\square$

We have shown that *Reflection* for first-order formulas, *First-order reflection* is a theorem of ZFC.We have also shown that it can be used instead of the *Infinity* and *Replacement* scheme, but ZFC $+$ *First-order reflection* is a conservative extension of ZF. Besides being a starting point for more general and powerful statements, it can be used to show that ZF is not finitely axiomatizable. This follows from the fact that *Reflection* gives a model to any consistent finite set of formulas. So if $\varphi_1, \ldots, \varphi_n$ would be the axioms of ZFC, *Reflection* would prove that every model of ZFC contains a smaller model of ZFC, which would in turn contradict the Second Gödel's Theorem[17].

It is also worthwhile to note that, in a way, Reflection is dual to compactness. Compactness says that given a set of sentences, if every finite subset yields a model, so does the whole set. Reflection, on the other hand, says that while the whole set has no model in the underlying theory, every finite subset has a model.

Furthemore, *Reflection* can be used in ways similar to upward Löwenheim–Skolem theorem. Since Reflection extends any set M_0 into a model of given formulas $\varphi_1, \ldots, \varphi_n$, we can choose the lower bound of the size of M by appropriately choosing M_0.

In the next section, we will try to generalize *Reflection* in a way that transcends ZF and yields some large cardinals.

[17]See chapter ?? for further details.

3 Reflection And Large Cardinals

3.1 Regular Fixed-Point Axioms

Lévy's article mentions various schemata that are not instances of reflection per se, but deal with fixed points of normal ordinal functions. We will introduce them and show that they are equivalent to *First-Order Reflection*[18].

Lemma 3.1 *(Fixed-point lemma for normal functions)*
Let f be a normal function defined for all ordinals[19]*. Then all of the following hold:*

(i) $\forall\lambda($*"λ is a limit ordinal"* $\rightarrow$ *"f(λ) is a limit ordinal"*$)$

(ii) $\forall\alpha(\alpha \leq f(\alpha))$

(iii) $\forall\alpha\exists\beta(\alpha < \beta \;\&\; f(\beta) = \beta)$

(iv) The fixed points of f form a closed unbounded class.[20]

Proof. Let f be a normal function defined for all ordinals.

(i) Suppose λ is a limit ordinal. For an arbitrary ordinal $\alpha < \lambda$, the fact that f is strictly increasing means that $f(\alpha) < f(\lambda)$ and for any ordinal β, satisfying $\alpha < \beta < \lambda$, $f(\alpha) < f(\beta) < f(\lambda)$. We know that there is such β from limitness of λ. Because f is continuous and λ is limit, $f(\lambda) = \bigcup_{\gamma<\lambda} f(\gamma)$.That means that if λ is limit, so is $f(\lambda)$.

(ii) This step will be proven using the transfinite induction. Since f is defined for all ordinals, there is an ordinal α such that $f(\emptyset) = \alpha$ and because $\emptyset$ is the least ordinal, (ii) holds for $\emptyset$.
Suppose (ii) holds for some β form the induction hypothesis. It the holds for $\beta + 1$ because f is strictly increasing.
For a limit ordinal λ, suppose (ii) holds for every $\alpha < \lambda$. (i) implies that $f(\lambda)$ is also limit, so there is a strictly increasing κ-sequence $\langle\alpha_0, \alpha_1, \ldots\rangle$ for some κ such that $\lambda = \bigcup_{i<\kappa} \alpha_i$. Because f is stricly increasing, the κ-sequence $\langle f(\alpha_0), f(\alpha_1), \ldots$ is also strictly increasing, the induction hypothesis implies that $\alpha_i \leq f(\alpha_i)$ for each $i \leq \kappa$. Thus, $\lambda \leq f(\lambda)$.

(iii) For a given ordinal α, let there be an ω-sequence $\langle\alpha_0, \alpha_1, \ldots\rangle$, such that $\alpha_0 = \alpha$ and $\alpha_{i+1} = f(\alpha_i)$ for each $i < \omega$. This sequence is stricly increasing because so is f. Now, there's a limit ordinal $\beta = \bigcup_{i<\omega} \alpha_i$, we want to show that this is the fixed point. So $f(\beta) = f(\bigcup_{i<\omega} \alpha_i) = \bigcup_{i<\omega} f(\alpha)$ because f is continuous. We have defined the above sequence so that β, $\bigcup_{i<\omega} f(\alpha) = \bigcup_{i<\omega} \alpha_{i+1}$, which means we are done, since $\bigcup_{i<\omega} \alpha_{i+1} = \bigcup_{i<\omega} \alpha_i = \beta$.

[18] For definition, see (2.16).
[19] For the definition of normal function, see (1.50).
[20] See (**??**.) for the definition of closed class, (1.52) for the definition of unboundedness.

(iv) The class of fixed points of f is obviously unbounded by (iii). It remains to show that it is closed, this is based on [Drake, 1974], chapter 4. Let Y be a non-empty set of fixed points of f such that $\bigcup Y \notin Y$. Since f is defined on ordinals, Y is a set of ordinals, so $\bigcup Y$ is an ordinal because a supremum of a set of ordinals is an ordinal. $\bigcup Y$ is a limit ordinal. If it were a successor ordinal, suppose that $\alpha + 1 = \bigcup Y$, then $\alpha \in \bigcup Y$, which means that there is some x such that $\alpha \in x \in Y$. But the least such x is $\alpha + 1$, so $\bigcup Y \in Y$.
Note that $\alpha < \bigcup Y iff \exists \xi \in Y(\alpha < \xi)$. Since f is defined for all ordinals and $\bigcup Y$ is a limit ordinal, $f(\bigcup Y) = \bigcup_\alpha \in Y f(\alpha)$, but because Y is a set of fixed points of f, $f(\bigcup Y) = \bigcup_\alpha \in Y f(\alpha) = \bigcup Y$, so $\bigcup Y$ is also a limit point of Y.

□

Lemma 3.2 *Let α be a limit ordinal. Then the following hold:*
(i) If C is a club set in α, then there is an ordinal β and a normal function $f : \beta \to \alpha$ such that $rng(f) = C$. We say that f enumrates *C.*
(ii) If β is an ordinal and f is a normal function such that $f : \beta \to \alpha$ and $rng(f)$ is unbounded in α, then $rng(f)$ is a closed unbouded set in α.

This proof comes from (http://euclid.colorado.edu/ monkd/m6730/gradsets09.pdf TODO cite!) *Proof.*
(i) Let β be the order-type[21] of C, let f be the isomorphism from β onto C. Since $C \subseteq \alpha$, f is also an increasing function from β into α. In order to be continuous, let γ be a limit ordinal under β, let $\epsilon = \bigcup_{\delta<\gamma} f(\delta)$. We want to verify that $f(\gamma) = \epsilon$. Since ϵ is a limit ordinal, we only need to show that $C \cap \epsilon$ is inbounded in ϵ.
Take $\zeta < \epsilon$. Then there is a $\delta < \ \gamma$ such that $\zeta < f(\delta)$. Since γ is limit, $\delta + 1 < \ \gamma$ and also $f(\delta + 1) < \ f(\gamma)$, we know that $f(\delta) \in C \cap \epsilon$. But that means that $C \cap \epsilon$ is unbounded in ϵ, so $\epsilon \in C$. We have also shown that ϵ is closed unbounded in the image of γ over f. Therefore, $f(\gamma) = \epsilon = \bigcup_{\delta< \ \gamma} f(\delta)$, so f is normal.

□ It should be clear that while this lemma works with club subsets of an ordinal, we can formulate analogous statement for club classes, which then yields a normal function defined for all ordinals, with the only exception that there is no such β because f is a function from Ord to Ord.

Definition 3.3 *(*Axiom Schema M_1*)*
"Every normal function defined for all ordinals has at least one inaccessible number in its range."

[21] See definition (1.31).

Lévy uses "M" to refer to this axiom but since we also use "M" for sets and models, for example in (2.16), we will call the above axiom "*Axiom Schema M_1*" to avoid confusion.

Let $\varphi(x, y, p_1, \ldots, p_n)$ be a first-order formula with no free variables besides $x, y, p_1, \ldots, p_n$. The following is equivalent to *Axiom M_1*.

$$\begin{aligned} &\text{“}\varphi \text{ is a normal function”} \ \& \ \forall x(x \in Ord \to \exists y(\varphi(x, y, p_1, \ldots, p_n))) \to \\ &\to \exists y(\exists x \varphi(x, y, p_1, \ldots, p_n) \ \& \ cf(y) = y \ \& \ (\forall x \in \kappa)(\exists y \in \kappa)(x > y)) \end{aligned} \tag{3.89}$$

Definition 3.4 *(Axiom Schema M_2)*
"Every normal function defined for all ordinals has at least one fixed point which is inaccessible."

Definition 3.5 *(Axiom Schema M_3)*
"Every normal function defined for all ordinals has arbitrarily great fixed points which are inaccessible."

Similar axiom is proposed in [Drake, 1974].

Definition 3.6 *(Axiom Schema F)*
"Every normal function has a regular fixed point."

Lemma 3.7 *Let f be a normal function defined for all ordinals.*
(i) There is a is normal function g_1 defined for all ordinals that enumerates the class $\{\alpha : f(\alpha) = \alpha \ \& \ \alpha \in Ord\}$.
(ii) There is a is normal function g_2 defined for all ordinals that enumerates the class $\{\lambda :$ "$f(\lambda)$ is a strong limit cardinal."$\}$.

Proof. We know that (ii) holds from lemma (3.1) and lemma (3.2).

For (i), It should be clear that there is no largest strong limit ordinal ν, because the limit of $\nu, \mathscr{P}(\nu), \mathscr{P}(\mathscr{P}(\nu)), \ldots$ is again a limit ordinal. The class of limit ordinals is closed because a limit of strong limit ordinals of is clearly always a strong limit ordinal. Let h be a function enumerating limit ordinals which exists from lemma (3.2). Then $g_1(\alpha) = f(h(\alpha))$ for every ordinal α is normal and defined for all ordinals. $\square$

The following is *Theorem 1* in [Lévy, 1960], the parts dealing with *Axiom Schema F* come from [Drake, 1974].

Theorem 3.8 *The following are all equivalent:*
(i) Axiom Schema M_1
(ii) Axiom Schema M_2

(iii) Axiom Schema M_3
(iv) Axiom Schema F

Proof. It is clear that *Axiom Schema M_3* is a stronger version of *Axiom Schema M_2*, which is in turn a stronger version of both *Axiom Schema M_1* and *Axiom Schema F_1*.

We will now prove that *Axiom Schema F* $\rightarrow$ *Axiom Schema M_2*. Lemma (3.7) tells us that given a normal function f defined for all ordinals, there is a normal function g_1 defined for all ordinals that enumerates the fixed-points of f. There is also a function g_2 that enumerates the strong limit ordinals in $rng(f)$. By *Axiom Schema F*, g_2 has a regular fixed-point κ, which is also a strong limit ordinal, so

$$f(\kappa) = g_2(\kappa) = \kappa \text{ and } \kappa \text{ is inaccessible.} \tag{3.90}$$

So every normal function d.f.a.o. has a regular fixed-point.

We have yet to show *Axiom Schema M_1* $\rightarrow$ *Axiom Schema M_3*. Again by lemma (3.7), there is a normal function g defined for all ordinals that enumerates the fixed points of f. Let $h_\alpha(\beta) = g(\alpha + \beta)$ for any given ordinal α, then h_α is a normal function defined for all ordinals. Then, given an arbitrary α, from *Axiom Schema M_1*, there is a β such that $\gamma = h_\alpha(\beta)$ is inaccessible. Because $\gamma = g(\alpha + \beta)$, $f(\gamma) = \gamma$. Since $\alpha \leq f'(\alpha)$ for any ordinal α and any normal function f', we know that $\alpha \leq \alpha + \gamma \leq \gamma$, so γ is inaccessible and arbitrarily large, depending on the choice of α. □

But how do those schemata relate to reflection? Let's introduce a stronger version of *First-order reflection schema* from the previous chapter to see it more clearly. But in order to do this, we must establish the inaccessible cardinal first.

3.2 Inaccessible Cardinal

Definition 3.9 *An uncountable cardinal κ is* inaccessible *iff it is* regular *and* strongly limit. *We write $In(\kappa)$ to say that κ is an inaccessible cardinal.*

An uncountable cardinal that is regular and limit is called a *weakly inaccessible cardinal*, we will only use the (strongly) inaccessible cardinal, but most of the results are similar for weakly inaccessibles, including higher types of ordinals that will be presented later in this chapter.

Theorem 3.10 *Let κ be an inaccessible cardinal.*

$$\langle V_\kappa, \in \rangle \models \text{ZFC} \tag{3.91}$$

We will prove this theorem in a way similar to [Kanamori, 2003]. *Proof.* Most of this is already done in lemma (2.15), we only need to verify that *Replacement* and *Infinity* axioms hold in V_κ.

Infinity holds because κ is uncountable, so $\omega \in V_\kappa$.

To verify *Replacement*, let x be an element of V_κ and f a function from x to V_κ. Let $y = \{z \in V_\kappa : (\exists q \in x) f(q) = z\}$, so $y \subset V_\kappa$, it remains to show that $y \in V_\kappa$. Because f is a function, we know that $|y| \leq |x| \leq \kappa$. But since κ is regular, $\{rank(z) : z \in y\} \subseteq \alpha$ for some $\alpha < \kappa$, and so $x \in V_{\alpha+1} \subseteq V_\kappa$. Therefore $y \in V_\kappa$. □

Definition 3.11 *(Inaccessible Reflection Schema)*
For every first-order formula φ, the following is an axiom:

$$\forall M_0 \exists \kappa (M_0 \subseteq V_\kappa \;\&\; In(\kappa) \;\&\; (\varphi(p_1, \ldots, p_n) \leftrightarrow \varphi(p_1, \ldots, p_n)^{V_\kappa})) \tag{3.92}$$

We will refer to this axiom schema as Inaccessible reflection schema.

We have added the requirement that α is inaccessible, which trivially means that there is an inaccessible cardinal. By taking appropriate M_0, it can be shown that in a theory that includes the *Inaccessible reflection schema*, there is a closed unbounded class of inaccessible cardinals. Since we know that for an inaccessible κ, V_κ is a model of ZFC, *Inaccessible reflection schema* is equivalent to

$$\forall M_0 \exists \kappa (M_0 \subseteq V_\kappa \;\&\; \langle V_\kappa, \in \rangle \models \text{ZFC} \;\&\; (\varphi(\mathsf{p}_1, \ldots, \mathsf{p}_n) \leftrightarrow \varphi(\mathsf{p}_1, \ldots, \mathsf{p}_n)^{\mathsf{V}_\kappa})) \tag{3.93}$$

because we have proven in the last section that for an inaccessible κ, $\langle V_\kappa, \in \rangle \models$ ZFC.

Theorem 3.12 Inaccessible reflection schema *is equivalent to* Axiom schema F.

This is *Theorem 4.1* in chapter four of [Drake, 1974], also equivalent to *Theorerem 3* in [Lévy, 1960]. *Proof.* Let's start by showing that *Inaccessible reflection schema* implies *Axiom schema F*. It should be clear that we can reflect two formulas to a single set, just form a new formula as a conjunction of universal closures of the two.

Given a normal function f defined for all ordinals, we want to show that it has a regular fixed point. For any ordinal α, there is an ordinal κ such that

$$\alpha < \kappa \;\&\; In(\kappa) \;\&\; (\forall \gamma, \delta \in V_\kappa)(f(\gamma) = \delta \leftrightarrow (f(\gamma) = \delta)^{V_\kappa}) \tag{3.94}$$

and

$$\alpha < \kappa \;\&\; In(\kappa) \;\&\; \forall \gamma \exists \delta (f(\gamma) = \delta) \leftrightarrow (\forall \gamma \exists \delta f(\gamma) = \delta)^{V_\kappa} \tag{3.95}$$

Since V_κ is the set of all sets of rank less than κ and since every ordinal is the rank of itself, there is an inaccessible ordinal κ such that

$$\forall\gamma < \kappa\exists\delta < \kappa(f^{V_\kappa}(\gamma) = \delta) \tag{3.96}$$

We also know that $f(\gamma) = \delta \leftrightarrow (f(\gamma) = \delta)^{V_\kappa}$. Now since κ is a limit ordinal and f is continuous we get

$$f(\kappa) = \bigcup_{\gamma<\kappa} f^{V_\kappa}(\gamma) = \bigcup_{\gamma<\kappa} f(\gamma). \tag{3.97}$$

From (3.96) and the fact that f is increasing, we know that $\kappa \leq \bigcup_{\gamma<\kappa} f(\gamma) \leq \kappa$. Therefore κ is an inaccessible fixed point of f.

For the opposite direction, it suffices to show that since there is an inaccessible cardinal from *Axiom schema F*, given a first-order formula φ, there is an arbitrarily large inaccessible cardinal κ for which

$$\varphi \leftrightarrow \langle V_\kappa, \in\rangle \models \varphi. \tag{3.98}$$

Note that the arbitrary size of κ means given an arbitrary ordinal α, there is a κ satisfying (3.98). In the previous chapter, in theorem (2.12), we have shown that we can easily obtain a limit ordinal satisfying (3.98). Note that since for any set M_0, there is such α that $M_0 \subseteq V_\alpha$, there is a closed unbounded class of sets satisfying (3.98), which are levels in the cumulative hierarchy, so there is a club sets of κs satisfying (3.98).

Let f be a normal function defined for all ordinals that enumerates this club class, there is such by lemma (3.2). Let g be the function that enumerates strong limit ordinals in $rng(f)$. Then g has a regular fixed point κ, which is also a regular fixed point of f, so (3.98) holds for κ.

□

Definition 3.13 *(*ZMC)
We will call ZMC *an axiomatic set theory that contains all axioms and schemas of* ZFC *together with* Axiom Schema M_1.

We have decided to call it ZMC, because Lévy uses ZM, derived from ZF, which is more intuitive, but we also need the axiom of choice, thus, ZMC.

As a sidenote, we should note that ZMC is extension of ZFC, which is in turn an extension of S.

3.3 Mahlo Cardinals

We have shown that ZMC contains arbitrarily large inaccessible cardinals. To return to reflection-style argument, is there a set that satisfies this property? To be able to properly answer this question, we have to formulate the notion of "containing arbitrarily large cardinals" more carefully. While we have previously used club sets, this is not an option because inaccessibles don't form a club class in ZMC[22], we could try to formulate stronger versions of *Axiom Schame M_1*.

Let's shortly review what *Axiom Schema M_1* says. We have shown earlier in this chapter that there is a simple relation between normal function defined for all ordinals and closed unbounded classes. So by saying that for a class of ordinals C, a normal function f has at least one element of C in its range, we say that C is stationary. Or, as Drake puts it for C, the class of inaccessible cardinals, and a κ, in which C is stationary:

> " The class of inaccessible cardinals is so rich that there are members κ of the class such that no normal function on κ can avoid this class; however we climb though κ, provided we are continuous at limits (so that we are enumerating a closed subset of κ), we shall eventually have to hit an inaccessible."

Definition 3.14 *(Mahlo Cardinal)*
We say that κ is a Mahlo Cardinal *iff it is an inaccessible cardinal and the set $\{\lambda < \kappa : \lambda$ is inaccessible$\}$ is stationary in κ.*

Alternatively, κ is Mahlo iff $\langle V_\kappa, \in\rangle \models$ ZMC as shown above, this is also sometimes written as *Ord is Mahlo*. There are also *weakly Mahlo cardinals*, that are defined via weakly inaccessible cardinals below them, Mahlo cardinals are then also called *strongly Mahlo* to highlight the difference, but we will only use the term *Mahlo cardinal*.

Mahlo cardinals are related to reflection principles in an interesting way. Note that given a formula φ, reflection gave us a club set of ordinals α such that V_α reflects φ, all below the first inaccessible cardinal. We have then used a different reflection schema to obtain arbitrarily high inaccessible cardinals. Now we have a cardinal in which this reflection schema holds, so we are in fact reflecting reflection. Beware that this is done rather informally, because *Axiom Schema M_1* is a countable set of axioms, which is too large to be reflected via the schemas introduced so far. One way to deal with this would be to extend reflection for second- and possibly higher-order formulas, but we would have to be very careful with the notion of satisfaction. For now, let us explore where can stationary sets take us because as we have shown, their connection to reflection is quite clear.

[22] Note that cofinality of the limit of the first ω inaccessibles is ω, which makes is singular.

What would happen if we strengthened *Axiom Schema M_1* to say that every normal function has a Mahlo cardinal in its range?

Definition 3.15 *(hyper-Mahlo cardinal)*
We say that κ is a hyper-Mahlo cardinal *iff it is inaccessible and the set* $\{\lambda < \kappa : \lambda \text{ is Mahlo}\}$ *is stationary in κ.*

Definition 3.16 *(hyper-hyper-Mahlo cardinal)*
We say that κ is a hyper-hyper-Mahlo cardinal *iff it is inaccessible and the set* $\{\lambda < \kappa : \lambda \text{ is hyper-Mahlo}\}$ *is stationary in κ.*

It is clear that one can continue in this direction, but the nomenclature gets increasingly overwhelming even if we introduce *hyper$^\alpha$-Mahlo cardinals*. To see there is a more elegant way to reach those cardinals, we will now establish an operation that exhausts all such cardinals in a more unified manner.

Definition 3.17 *(Mahlo Operation)*
Let A be a class of ordinals. Let

$$H(A) = \{\alpha \in A : A \cap \alpha \text{ is stationary in } \alpha\}. \tag{3.99}$$

We call H the Mahlo's operation.

If we pick for A the class of all inaccessible cardinals, $H(A)$ is the class of Mahlo cardinals. It is easy to see that is A is the class of all α-Mahlo cardinals, $H(A)$ is the class of $alpha + 1$-Mahlo cardinals, $H(H(A))$ is the class of $\alpha + 2$-Mahlo cardinals and so on.

Definition 3.18 *(Iterated Mahlo Operation)*
Let A be a class of ordinals. We shall extend the Mahlo operation in the following way:
(i) $H^0(A) = A$,
(ii) $H^{\alpha+1}(A) = H(H^\alpha(A))$,
(iii) $H^\lambda(A) = \bigcap_{\alpha< \lambda} H^\alpha(X)$ *for limit λ.*

Clearly it A is the class of inaccessibles, $H^\alpha(A)$ is the class of α-Mahlo cardinals. To get to hyper-Mahlo cardinals, we can diagonalize the operation.

Definition 3.19 *(Diagonal Mahlo Operation)*
Let A be a class of ordinals. Then the diagonal Mahlo operation *is defined as follows:*

$$H^\Delta(A) = \{\alpha : \forall\beta < \alpha(\alpha \in H^\beta(X))\}. \tag{3.100}$$

We can further diagonalize the diagonal version and continue this process ad libitum in order to reach all large cardinals accessible "from below".

TODO par slov o tom co je "from below" (určitě až sem?)

3.4 Indescribable Cardinals

Indescriability is another approach towards large cardinals that is based on reflection. We will briefly introduce the basic definitions and show that it yield larger objects, but most of them are not reachable from below

Most of the results presented in this subchapter are taken from [Kanamori, 2003].

Since this chapter uses higher-order logic, we need to introduce the hierarchy of formulas first.

Definition 3.20 *(Higher-Order Variables)*
Let M be a structure and D it's domain. In first-order logic, variables range over individuals, that is, over elements of D. We shall call those type 1 variables *for the purposes of higher-order logic. Type 2 variables then range over collections, that is, the elements of $\mathscr{P}(D)$. Generally, type n variables are defined for any $n \in \omega$ such that they range over $\mathscr{P}^{n-1}(D)$.*

We will use lowercase latin letters for type 1 variables for backward compatibility with first-order logic, type 2 variables will be represented by uppercase letters, mostly P, X, Y, Z. If we ever stumble upon type 3 variables in this text, they shall be represented as $\mathscr{X}, \mathscr{Y}, \mathscr{Z}$ or in a similar font.

Definition 3.21 *(Full Prenex Normal Form)*
We say a formula is in the prenex normal form *if it is written as a block of quantifiers followed by a quantifier-free part.*
We say a formula is in the full prenex normal form *if it is written in* prenex normal form *and if there are type $n+1$ quantifiers, they are written before type n quantifiers.*

It is an elementary that every formula is equivalent to a formula in the prenex normal form.

Definition 3.22 *(Hierarchy of Formulas)*
Let φ be a formula in the prenex formal form.

(i) We say φ is a Δ^0_0-formula if it contains only bounded quantifiers.
(ii) We say φ is a Σ^0_0-formula or a Π^0_0-formula if it is a Δ^0_0-formula.
(iii) We say φ is a Π^{m+1}_0-formula if it is a Π^m_n- or Σ^m_n-formula for any $n \in \omega$ or if it is a Π^m_n- or Σ^m_n-formula with additional free variables of type $m+1$.
(iv) We say φ is a Σ^m_0-formula if it is a Π^m_0-formula.
(v) We say φ is a Σ^m_n+1-formula if it is of a form $\exists P_1, \ldots, P_i \psi$ for any non-zero i, where ψ is a Π^m_n-formula and $P_1, \ldots, P_i$ are type $m+1$ variables.
(vi) We say φ is a Π^m_n+1-formula if it is of a form $\forall P_1, \ldots, P_i \psi$ for any non-zero i, where ψ is a Σ^m_n-formula and $P_1, \ldots, P_i$ are type $m+1$ variables.

Definition 3.23 *(Describability)*
We say an ordinal α is described by a sentence φ in the language $\mathscr{L}$ with relation symbols $P_1, \ldots, P_n$ given iff

$$\langle V_\alpha, \in, P_1, \ldots, P_n \rangle \ \models \ \varphi \tag{3.101}$$

but for every $\beta < \ \alpha$

$$\langle V_\beta, \in, P_1 \cap V_\beta, \ldots, P_n \cap V_\beta \rangle \not\models \varphi \tag{3.102}$$

For the definition of a Π^m_n-formula and a Σ^m_n-formula, see (??).

Definition 3.24 *(Π^m_n-Indescribable Cardinal)*
We say that κ is Π^m_n-indescribable iff it is not described by any Π^m_n-formula.

Definition 3.25 *(Σ^m_n-Indescribable Cardinal)*
We say that κ is Σ^m_n-indescribable iff it is not described by any Σ^m_n-formula.

To see that this notion is based in reflection, let us recall the opening quote of this thesis by Gödel which says *"The Universe of sets cannot be uniquely characterised (i. e. distinguished from all its initial elements) by any internal structural property of the membership relation in it."*. A cardinal κ is Π^m_n-indescribable[23] iff every Π^m_n-formula fails to describe V_κ and describes an initial segment instead. In a sense, V_κ reflects the "property"[24] of indescribability of the universal class with respect to certain classes of formulas.

Lemma 3.26 *Let κ be a cardinal, then the following holds for any $n \in \omega$. κ is Π^1_n-indescribable iff κ is Σ^1_{n+1}-indescribable*

Proof. The forward direction is obvious, we can always add a spare quantifier over a type 2 variable to turn a Π^1_n formula φ into a $\exists P \varphi$ which is obviously a Σ^1_{n+1}-formula.[25]

To prove the opposite direction, suppose that $\langle V_\kappa, \in \rangle \ \models \ \exists X \varphi(X)$ where X is a type 2 variable and φ is a Π^1_n formula with one free variable of type 2. This means that there is a set $S \subseteq V_\kappa$ that is a witness of $\exists X \varphi(X)$, in other words, $\varphi(S)$ holds. We can replace every occurence of X in φ by a new

[23] This holds for Σ^m_n-formulas alike.

[24] In this case, we are not using the word to refer to a definable class, but on a meta level to refer to a property expressible in the natural language, hence the quotation marks.

[25] Note that unlike in previous sections, it is worth noting that φ is now a sentence so we don't have to worry whether P is free in φ.

predicate symbol S, this allows us to say that κ is Π^1_n-indescribable (with respect to $\langle V_\kappa, \in, R, S\rangle$).[26] □

The above lemma makes it clear that we can suppose that all formulas with no higher than type 2 variables are Π^1_n-formulas, $n \in \omega$, without the loss of generality.

Lemma 3.27 *If κ is an inaccessible cardinal and given $R \subseteq V_\kappa$, then the following is a club set in κ:*

$$\{\alpha : \alpha < \kappa \ \&\ \langle V_\alpha, \in, R \cap V_\alpha\rangle \prec \langle V_\kappa, \in, R\rangle\} \tag{3.103}$$

Proof. To see that (3.103) is closed, let us recall that a $A \subseteq \kappa$ is closed iff for every ordinal $\emptyset < \alpha < \lambda$: if $A \cap \alpha$ is unbounded in α then $\alpha \in A$. Since κ is an inaccessible cardinal, thus strong limit, it is closed under limits of sequences of ordinals lesser than κ.

We want to verify that it is unbounded, we will use a recursively defined sequence $\alpha_0, \alpha_1, \ldots$ to build an elementary substructure of $\langle V_\kappa, \in, R\rangle$ that is built above an arbitrary $\alpha_0 < \kappa$. Let us fix an arbitrary $\alpha_0 < \kappa$. Given α_n, α_{n+1} is defined as the least β, $\alpha_n \leq \beta$ that satisfies the following for any formula φ, $p_1, \ldots, p_m \in V_{\alpha_n}, m \in \omega$:

$$\text{If } \langle V_\kappa, \in, R\rangle \models \exists x \varphi(p_1, \ldots, p_n), \text{ then } \langle V_\kappa, \in, R\rangle \models \varphi(x, p_1, \ldots, p_n) \tag{3.104}$$

Let $\alpha = \bigcup_{n<\omega} \alpha_n$.

Then $\langle V_\alpha, \in, R \cap V_\alpha\rangle \prec \langle V_\kappa, \in, R\rangle$, in other words, for any φ with given arbitrary parameters $p_1, \ldots, p_n \in V_\alpha$, it holds that

$$\langle V_\alpha, \in, R \cap V_\alpha\rangle \models \varphi(p_1, \ldots, p_n) \leftrightarrow \langle V_\kappa, \in, R\rangle \models \varphi(p_1, \ldots, p_n) \tag{3.105}$$

Which should be clear from the construction of α □

Theorem 3.28 *Let κ be an ordinal. The following are equivalent.*

(i) κ is inaccessible

(ii) κ is Π^1_0-indescribable.

Proof. Since Π^1_0-sentences are first-order sentences, we want to prove that κ is an inaccessible cardinal iff whenever a first-order tries to describe κ in the sense of definition (3.20), the formula fails to do so and describes a initial segment

[26] A different yet interesting approach is taken by Tate in [Tait, 2005]. He states that for $n \geq 0$, a formula of order $\leq n$ is called a Π^n_0 and a Σ^n_0 formula. Then a Π^n_{m+1} is a formula of form $\forall Y \psi(Y)$ where ψ is a Σ^n_m formula and Y is a variable of type n. Finally, a Σ^n_{m+1} is the negation of a Π^n_m formula. So the above holds ad definitio.

thereof instead. We have already shown in (??) that there is no way to reach an inaccesible cardinal via first-order formulas in ZFC. We will now prove it again in for formal clarity.

For (i)$\rightarrow$(ii), suppose that κ is inaccessible.

Then there is, by lemma (3.24) a club set of ordinals α such that V_α is an elementary substructures of V_κ. For κ to be Π^1_0-inderscribable, we need to make sure that given an arbitrary first-order sentence φ satisfied in the structure $\langle V_\kappa, \in, R\rangle$, there is an ordinal $\alpha < \kappa$, such that $\langle V_\alpha, \in, R \cap V_\alpha\rangle \models \varphi$. But this follows from the definition of elementary substructure.

For (ii)$\rightarrow$(i), suppose κ is not inaccessible, so it is either singular, or there is a cardinal $\nu < \kappa$ such that $\kappa \leq \mathscr{P}(\nu)$ or $\kappa = \omega$.

Suppose κ is singular. Then there is a cardinal $\nu < \kappa$ and a function $f : \nu \rightarrow \kappa$ such that $rng(f)$ is cofinal in κ. Since $f \subseteq V_\kappa$, we can add f as a relation to the language. We can do the same with $\{\nu\}$. That means $\langle V_\kappa, \in, P_1, P_1$ with $P_1 = f, P_2 = \{\nu\}$ is a structure, let $\varphi = P_1 \neq \emptyset \;\&\; rng(P_1) = P_2$[27]. Since for every $\alpha < \nu$, $P_1 \cap V_\alpha = \emptyset$, φ is false and therefore describes κ. That contradicts the fact that κ was supposed to be Π^1_0-indescribable, but φ is a first-order formula.

Suppose there a cardinal ν satisfying $\kappa \leq \mathscr{P}(\nu)$. Let there be a function $f : \mathscr{P}(\nu) \rightarrow \kappa$ that is onto. Then, like in the previous paragraph, we can obtain a structure $\langle V_\kappa, \in, P_1, P_2\rangle$, where $P_1 = f$ like before, but this time $P_2 = \mathscr{P}(\nu)$. Again, $\varphi = P_1 \neq \emptyset \;\&\; rng(P_1) = P_2$ describes κ.

Finally, suppose $\kappa = \omega$, then the sentence $\varphi = \forall x \exists y (x \in y)$ describes κ, there is obviously no $\alpha < \omega$ such that $\langle V_\alpha, \in\rangle \models \varphi$.

□

Generally, it should be clear that it a cardinal κ is Π^m_n-indescribable, it is also $\Pi^{m'}_{n'}$-indescribable for every $m' < m, n' < n$. By the same line of thought, if a cardinal κ satisfies the property implied by Π^m_n-indescribability, it satisfies all properties implied by $\Pi^{m'}_{n'}$-indescribability for $m' < m, n' < n$. For example, if κ is Π^m_n-indescribable for $m \geq 1$ then it is also an inaccessible cardinal.

Theorem 3.29 *If a cardinal κ is Π^1_1-indescribable, then it is a Mahlo cardinal.*

Proof. Assuming that κ is Π^1_1-indescribable, we want to prove that every club set of in κ contains an inaccessible cardinal.

Consider the following Π^1_1-sentence φ:

$$\begin{aligned}\varphi = \forall P(\text{“P is a function”} \rightarrow \forall x \exists y \forall z (z \in y \leftrightarrow (\exists q \in x)(P(x, y, p_1, \ldots, p_n)))) \\ \& \; \forall x \exists y \forall z (z \in y \leftrightarrow z \subseteq x)\end{aligned} \tag{3.106}$$

[27] $rng(x) = y$ is a first-order formula, see (1.14).

where P is a type 2 variable and the rest are type 1 variables, "P is a function" is a first-order formula defined in (1.11). As has been shown earlier in this chapter, given a cardinal μ, the following holds if and only if μ is inaccessible:

$$\langle V_\mu, \in\rangle \models \varphi \tag{3.107}$$

Now fix an arbitrary $C \subset \kappa$, a club set in κ. We want to show that it contains an inaccessible cardinal. Since C is a subset of κ, let's add it to the structure $\langle V_\kappa, \in\rangle$, turning it into $\langle V_\kappa, \in, C\rangle$. Then the following holds:

$$\langle V_\kappa, \in, C\rangle \models Inac \;\&\; \text{“}C \text{ is unbounded”}^{28} \tag{3.108}$$

Note that this holds because κ is Π^1_1-indescribable, and therefore also Π^1_0-indescribable. So κ is itself inaccessible and therefore $\langle V_\kappa, \in, C\rangle \models \varphi$.

Since κ is Π^1_1-indescribable and φ & "C is unbounded" is equivalent to a Π^1_1-formula, there must be an ordinal α that satisfies

$$\langle V_\alpha, \in, C \cap V_\alpha\rangle \models Inac \;\&\; \text{“}C \text{ is unbounded”}, \tag{3.109}$$

which implies that α is inaccessible; it is regular because it reflects *Replacement* and it is limit because if α were a successor ordinal, it couldn't contain an unbounded class of ordinals.

We only need to verify that $\alpha \in C$, which is clear from the fact that C is a club set in κ and it is unbounded in α. □

There is an even stronger large cardinal property implied by Π^1_1-indescribability that is based on reflection.

Definition 3.30 *(Extension Property)*
We say a cardinal κ has the extension property *iff for all $U \subset V_\kappa$ there exists a transitive set X such that $\kappa \in X$, and a set $S \subset X$, such that $(V_\kappa, \in, U)$ is an elementary substructure of $(X, \in, S)$.*

Definition 3.31 *(Weakly Compact Cardinal)*
We say that a cardinal κ is weakly compact *iff it has the extension property.*

Theorem 3.32 *A cardinal κ is Π^1_1-indescribable iff it is weakly compact.*

For the proof, see [Kanamori, 2003].

TODO slabe kompaktni a reflexe?

Definition 3.33 *(Ultrafilter)*
Given a set x, we say $U \subset \mathscr{P}(x)$ is an ultrafilter *over x iff all of the following hold:*

[28] "C is unbounded" is a first-order formula, see (1.52).

(i) $\emptyset \notin U$
(ii) $\forall y, z(\subset x \;\&\; y \subset z \;\&\; y \in U \rightarrow z \in U)$
(iii) $\forall y, z \in U(y \cap z) \in U$
(iv) $\forall y(y \subset x \rightarrow (y \in U \vee (x \setminus y) \in U))$

Definition 3.34 *(κ-Complete Ultrafilter)*
We say that an ultrafilter U is κ-complete iff

Definition 3.35 *(Measurable Cardinal)*
Let κ be a caridnal. We say κ is a measurable cardinal *iff there is a κ-complete ultrafilter over κ.*

Theorem 3.36 *Let κ be a cardinal. If κ is a measurable cardinal then the following hold:*
(i) κ is Π^2_1-indescribable.
(ii) Given U, a normal ultrafilter over κ, a relation $R \subseteq V_\kappa$ and a Π^2_1-formula φ such that $\langle V_\kappa, \in, R\rangle \models \varphi$, then

$$\{\alpha < \kappa : \langle V_\alpha, \in, R \cap V_\alpha\rangle \models \varphi\} \in U \tag{3.110}$$

For a proof, see Proposition 6.5 in [Kanamori, 2003].

Theorem 3.37 *If κ is a measurable cardinal and U is a normal ultrafilter over κ, the following holds:*

$$\{\alpha < \kappa : \text{"}\alpha \textit{ is totally indescribable"}\} \in U \tag{3.111}$$

For a proof, see Proposition 6.6 in [Kanamori, 2003].

3.5 The Constructible Universe

The constructible universe, denoted L, is a cumulative hierarchy of sets, presented by Kurt Gödel in his paper [Gödel and Brown, 1940]. Assertion of their equality, $V = L$, is called the *Axiom of Constructibility*. The axiom implies GCH and therefore also AC and contradicts the existence of some large cardinals, our goal is to decide whether those introduced earlier are among them.

On order to formally establish this class, we need to formalize the notion of definability first.

Definition 3.38 *(Definability)*
We say that a set X is definable *over a model $\langle M, \in\rangle$ if there is a formula φ together with parameters $p_1, \ldots, p_n \in M$ such that*

$$X = \{x : x \in M \;\&\; \langle M, \in\rangle \models \varphi(x, p_1, \ldots, p_n)\} \tag{3.112}$$

Definition 3.39 *(The Set of Definable Subsets)*
The following is a set of all definable subsets of a given set M, denoted Def(M).

$$\begin{gathered} Def(M) = \{\{y : x \in M \wedge \langle M, \in\rangle \models \varphi(y, u_1, \ldots, i_n)\}| \\ \varphi \textit{ is a first-order formula, } p_1, \ldots, p_n \in M\} \end{gathered} \tag{3.113}$$

We will use $Def(M)$ in the following construction in the way the powerset operation is used when constructing the usual Von Neumann's hierarchy of sets[29].

Now we can recursively build L.

Definition 3.40 *(The Constructible Universe)*

(i)

$$L_0 = \emptyset \tag{3.114}$$

(ii)

$$L_{\alpha+1} = Def(L_\alpha) \textit{ for any ordinal } \alpha \tag{3.115}$$

(iii)

$$L_\lambda = \bigcup_{\alpha<\lambda} L_\alpha \textit{ For a limit ordinal } \lambda \tag{3.116}$$

(iv)

$$L = \bigcup_{\alpha \in Ord} L_\alpha \tag{3.117}$$

Note that while L bears very close resemblance to V, the difference is, that in every successor step of constructing V, we take every subset of V_α to be $V_{\alpha+1}$, whereas $L_{\alpha+1}$ consists only of definable subsets of L_α. Also note that L is transitive.

In order to

Theorem 3.41 *Let L be as in (3.32).*

$$L \textit{ is a model of } \mathsf{ZFC} \tag{3.118}$$

For details, refer to Theorem 13.3 in [Jech, 2006].

Definition 3.42 *(Constructibility)*
The axiom of constructibility say that every set is constructible. It is usually denoted as $L = V$.

[29]For that reason, some authors use $\mathscr{P}^*(M)$ instead of $Def(M)$, see section 11 of [Pinter, 2014] for one such example.

Without providing a proof, we will introduce two important results established by Gödel in his aforementioned article. ZF stands for Zermelo–Fraenkel set theory as introduced in (1.20).

Theorem 3.43 *(Constructibility $\rightarrow$ Choice)*

$$\mathsf{ZF} \vdash \text{Constructibility} \rightarrow \text{Choice} \tag{3.119}$$

The GCH refers to the *Generalised Continuum Hypothesis*, see (1.40).

Theorem 3.44 *(Constructibility $\rightarrow$ Generalised Continuum Hypothesis)*

$$\mathsf{ZF} \vdash \text{Constructibility} \rightarrow \text{GCH} \tag{3.120}$$

It is worth mentioning that Gödel's proof of *Construcibility $\rightarrow$ GCH* featured the first formal use of a reflection principle. For the actual proofs, see for example [Kunen, 1983],

Since *GCH* implies that κ is a limit cardinal iff κ is a strong limit cardinal for every κ, the distinctions between inaccessible and weakly inaccessible cardinals as well as between Mahlo and weakly Mahlo cardinals vanish.

Theorem 3.45 *(Inaccessibility in L)*
Let κ be an inaccessible cardinal. Then "κ is inaccessible"^{L}.

Proof. We want to show that the following are all true for an inaccessible cardinal κ:
(i) "κ is a cardinal"^{L}
(ii) $(\omega < \kappa)^L$
(iii) "κ is regular"^{L}
(iv) "κ is limit"^{L}.[30]

Suppose "κ is not a cardinal"^{L} holds, then there is a cardinal μ, $\mu < \kappa$ and a function $f : \mu \rightarrow \kappa$, $f \in L$, such that "$f : \mu \rightarrow \kappa$ is onto"^{L}. But since "f is onto" is a Δ_0 formula and Δ_0 formulas are are absolute in transitive structures[31] and L is a transitive class, "f is onto"$^L \leftrightarrow$ "f is onto", this contradicts the fact that κ is a cardinal. $(\omega < \kappa)^L$ holds because $\omega \in \kappa$ and because ordinals remain ordinals in L, so $(\omega \in \kappa)^L$.

In order to see that "κ is regular"^{L}, we can repeat the argument by contradiction used to show that κ is a cardinal in L. If κ was singular, there is a $\mu < \kappa$ together with a function $f : \mu \rightarrow \kappa$ that is onto, but since "f is onto" implies

[30] While inaccessible cardinals are strong limit cardinals, since GCH holds in L, "κ is limit"^{L} implies "κ is strong limit"^{L}.

[31] See lemma (1.45).

"f is onto"^{L}, we have reached a contradiction with the fact that κ is regular, but singular in L.

It now suffices to show that "κ is a limit cardinal"^{L}. That means, that for any given $\lambda < \kappa$, we need to find an ordinal μ such that $\lambda < \mu < \kappa$ that is also a cardinal in L. But since cardinals remain cardinals in L by an argument with surjective functions just like above, it holds. □

Theorem 3.46 *(Mahloness in L)*
Let κ be a Mahlo cardinal. Then "κ is Mahlo"^{L}.

Proof. Let κ be a Mahlo cardinal. From the definition of Mahloness in (3.14), it should be clear that we want prove that κ is inaccessible in L and

$$\text{"The set } \{\alpha : \alpha \in \kappa \;\&\; \text{'}\alpha \text{ is inaccessible'}\} \text{ is stationary in } \kappa\text{"}^L \tag{3.121}$$

Since we have shown that an inaccessible cardinals remain inaccessible in L in the previous theorem, L"κ is inaccessible"^{L} holds.

Now consider the two following sets:

(i)
$$S \stackrel{\text{def}}{=} \{\alpha : \alpha \in \kappa \;\&\; \text{"}\alpha \text{ is inaccessible"}\} \tag{3.122}$$

(ii)
$$T \stackrel{\text{def}}{=} \{\alpha : \alpha \in \kappa \;\&\; \text{"}\alpha \text{ is inaccessible"}^L\} \tag{3.123}$$

Since inaccessible cardinals are inaccessible in L from theorem (3.37), $S \subseteq T$. So if T is stationary in κ, we are done. Suppose for contradiction that it is not the case. Therefore there is a $C \subset \kappa$ satisfying "C is a club set in κ"^{L}, but it is the case that $T \cap C = \emptyset$. But because "C is a club set in κ" is equivalent to a Δ_0 formula, "C is a club set in κ"$^M \leftrightarrow$ "C is a club set in κ", ergo C is a club set in κ. But since it has o intersection with T, it can't have an intersection with a subset thereof, which contradicts the fact that S is stationary in κ.

κ remains Mahlo in L. □

It should be clear that the above process can be iterated over again. Since Mahlo cardinals are absolute in L, the same argument using stationary sets can be carried out for hyper-Mahlo cardinals and so on. It is clear that since a regular and an inaccessible cardinal in consistent with *Constructibility*, so should be the higher properties acquired from assuring the existence of regular, inaccessible and Mahlo fixed points of normal functions.

4 Conclusion

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