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3 MIKLUÁŠ MRVA

4 REFLECTION PRINCIPLES AND LARGE
5 CARDINALS
6 Bakalářská práce

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¹⁰ Prohlašuji, že jsem bakalářskou práci vypracoval samostatně a že jsem uvedl
¹¹ všechny použité prameny a literaturu.

¹² V Praze 14. dubna 2015

Abstract

Práce zkoumá vztah tzv. principů reflexe a velkých kardinálů. Lévy ukázal, že v ZFC platí tzv. věta o reflexi a dokonce, že věta o reflexi je ekvivalentní schématu nahrazení a axiomu nekonečna nad teorií ZFC bez axiomu nekonečna a schématu nahrazení. Tedy lze na větu o reflexi pohlížet jako na svého druhu axiom nekonečna. Práce zkoumá do jaké míry a jakým způsobem lze větu o reflexi zobecnit a jaký to má vliv na existenci tzv. velkých kardinálů. Práce definuje nedosažitelné, Mahlovy a nepopsatelné kardinály a ukáže, jak je lze zavést pomocí reflexe. Přirozenou limitou kardinálů získaných reflexí jsou kardinály nekonzistentní s L. Práce nabídne intuitivní zdůvodnění, proč tomu tak je.

Abstract

This thesis aims to examine relations between so called "Reflection Principles" and Large cardinals. Lévy has shown that Reflection Theorem is a sound theorem of ZFC and it is equivalent to Replacement Scheme and the Axiom of Infinity. From this point of view, Reflection theorem can be seen a specific version of an Axiom of Infinity. This paper aims to examine the Reflection Principle and its generalisations with respect to existence of Large Cardinals. This thesis will establish Inaccessible, Mahlo and Indescribable cardinals and their definition via reflection. A natural limit of Large Cardinals obtained via reflection are cardinals inconsistent with L. The thesis will offer an intuitive explanation of why this is the case.

1 Introduction

1.1 Motivation and Origin

The Universe of sets cannot be uniquely characterized (i. e. distinguished from all its initial elements) by any internal structural property of the membership relation in it, which is expressible in any logic of finite or transfinite type, including infinitary logics of any cardinal order.

— Kurt Gödel [?]

To understand why we need reflection in the first place, let's think about infinity for a moment. In the intuitive sense, infinity is an upper limit of all numbers. But for centuries, this was merely a philosophical concept, closely bound to religious and metaphysical way of thinking, considered separate from numbers used for calculations or geometry. It was a rather vague concept. In ancient Greece, Aristotle's response to famous Zeno's paradoxes introduced the distinction between actual and potential infinity. He argued, that potential infinity is (in today's words) well defined, as opposed to actual infinity, which remained a vague incoherent concept. He didn't think it's possible for infinity to inhabit a bounded place in space or time, rejecting Zeno's thought experiments as a whole. Aristotle's thoughts shaped western thinking partly due to Aquinas, who himself believed actual infinity to be more of a metaphysical concept for describing God than a mathematical property attributed to any other entity. In his *Summa Theologica* ¹ he argues:

A geometrician does not need to assume a line actually infinite, but takes some actually finite line, from which he subtracts whatever he finds necessary; which line he calls infinite.

Less than hundred years later, Gregory of Rimini wrote

If God can endlessly add a cubic foot to a stone—which He can—then He can create an infinitely big stone. For He need only add one cubic foot at some time, another half an hour later, another a quarter of an hour later than that, and so on ad infinitum. He would then have before Him an infinite stone at the end of the hour.

Which is basically a Zeno's Paradox made plausible with God being the actor. In contrast to Aquinas' position, Gregory of Rimini theoretically constructs an object with actual infinite magnitude that is essentially different from God.

¹Part I, Question 7, Article 3, Reply to Objection 1

Even later, in the 17th century, pushing the property of infiniteness from the Creator to his creation, Nature, Leibniz wrote to Foucher in 1662:

I am so in favor of the actual infinite that instead of admitting that Nature abhors it, as is commonly said, I hold that Nature makes frequent use of it everywhere, in order to show more effectively the perfections of its Author. Thus I believe that there is no part of matter which is not, I do not say divisible, but actually divided; and consequently the least particle ought to be considered as a world full of an infinity of different creatures.

But even though he used potential infinity in what would become foundations of modern Calculus and argued for actual infinity in Nature, Leibniz refused the existence of an infinite, thinking that Galileo's Paradoxon² is in fact a contradiction. The so called Galileo's Paradoxon is an observation Galileo Galilei made in his final book "Discourses and Mathematical Demonstrations Relating to Two New Sciences". He states that if all numbers are either squares and non-squares, there seem to be less squares than there is all numbers. On the other hand, every number can be squared and every square has its square root. Therefore, there seem to be as many squares as there are all numbers. Galileo concludes, that the idea of comparing sizes makes sense only in the finite realm.

Salviati: So far as I see we can only infer that the totality of all numbers is infinite, that the number of squares is infinite, and that the number of their roots is infinite; neither is the number of squares less than the totality of all the numbers, nor the latter greater than the former; and finally the attributes "equal," "greater," and "less," are not applicable to infinite, but only to finite, quantities. When therefore Simplicio introduces several lines of different lengths and asks me how it is possible that the longer ones do not contain more points than the shorter, I answer him that one line does not contain more or less or just as many points as another, but that each line contains an infinite number.

Leibniz insists in part being smaller than the whole saying

Among numbers there are infinite roots, infinite squares, infinite cubes. Moreover, there are as many roots as numbers. And there are as many squares as roots. Therefore there are as many squares

²zneni galileova paradoxu

as numbers, that is to say, there are as many square numbers as there are numbers in the universe. Which is impossible. Hence it follows either that in the infinite the whole is not greater than the part, which is the opinion of Galileo and Gregory of St. Vincent, and which I cannot accept; or that infinity itself is nothing, i.e. that it is not one and not a whole.

TODO Hegel–strucne?

TODO Cantor

TODO mene teologie, vice matematiky

TODO definovat pojmy (trida etc)

TODO neni V v nejakem smyslu porad potencialni nekonecno, zatimco mnoziny vetsi nez omega jsou aktualni? nebo jsou potencialni protoze se stavaji pres indukci, od spoda.

In his work, he defined transfinite numbers to extend existing natural number structure so it contains more objects that behave like natural numbers and are based on an object (rather a meta-object) that doesn't explicitly exist in the structure, but is closely related to it. This is the first instance of reflection. This paper will focus on taking this principle a step further, extending Cantor's (or Zermelo–Fraenkel's, to be more precise) universe so it includes objects so big, they could be considered the universe itself, in a certain sense.

TODO dal asi smazat

The original idea behind reflection principles probably comes from what could be informally called “universality of the universe”. The effort to precisely describe the universe of sets was natural and could be regarded as one of the impulses for formalization of naive set theory. If we try to express the universe as a set $\{x|x = x\}$, a paradox appears, because either our set is contained in itself and therefore is contained in a set (itself again), which contradicts the intuitive notion of a universe that contains everything but is not contained itself.

TODO ???

If there is an object containing all sets, it must not be a set itself. The notion of class seems inevitable. Either directly the ways for example the Bernays–Gödel set theory, we will also discuss later in this paper, does in, or on a meta-level like the Zermelo–Fraenkel set theory, that doesn't refer to them in the axioms but often works with the notion of a universal class. duet Another obstacle of constructing a set of all sets comes from Georg Cantor, who proved that the set of all subsets of a set (let x be the set and $\mathcal{P}((x)$ its powerset) is strictly larger than x . That would turn every aspiration to

finally establish an universal set into a contradictory infinite regression.³ We will use V to denote the class of all sets. From previous thoughts we can easily argue, that it is impossible to construct a property that holds for V and no set and is neither paradoxical like $\{x|x = x\}$ nor trivial. Previous observation can be transposed to a rather naive formulation of the reflection principle:

Reflection Any property which holds in V already holds in some initial segment of V .

To avoid vagueness of the term "property", we could informally reformulate the above statement into a schema:

For every first-order formula⁴ φ holds in $V \leftrightarrow \varphi$ holds in some initial segment of V .

Interested reader should note that this is a theorem scheme rather than a single theorem.⁵

1.2 A few historical remarks on reflection

Reflection made its first in set-theoretical appearance in Gödel's proof of GCH in L (citace Kanamori ? Lévy and set theory), but it was around even earlier as a concept. Gödel himself regarded it as very close to Russel's reducibility axiom (an earlier equivalent of the axiom schema of Zermelo's separation). Richard Montague then studied reflection properties as a tool for verifying that Replacement is not finitely axiomatizable (citace?). a few years later Lévy proved in [?] the equivalence of reflection with Axiom of infinity together with Replacement in proof we shall examine closely in chapter 2.

TODO co dal? recent results?

³An intuitive analogy of this *reductio ad infinitum* is the status of ω , which was originally thought to be an unreachable absolute, only to become starting point of Cantor's hierarchy of sets growing beyond all boundaries around the end of the 19th century

⁴this also works for finite sets of formulas [?, p. 168]

⁵If there were a single theorem stating "for any formula φ that holds in V there is an initial segment of V where φ also holds", we would obtain the following contradiction with the second Gödel's theorem: In ZFC, any finite group of axioms of ZFC holds in some initial segment of the universe. If we take the largest of those initial segments it is still strictly smaller than the universe and thus we have, via compactness, constructed a model of ZFC within ZFC. That is, of course a harsh contradiction. This also leads to an elegant way to prove that ZFC is not finitely axiomatizable.

1.3 Reflection in Platonism and Structuralism

TODO cite "reflection in a structuralist setting"

TODO veci o tom, ze reflexe je ok protoze reflektuje veci ktere objektivne plati, protoze plati pro V ...

TODO souvislost s kompaktnosti, hranice formalich systemu nebo alespon ZFC

1.4 Notation and terminology

1. *Reflection* je obecne reflexe (jaka presne?)
2. *Reflection*₁ je reflexe prvoradovych formuli TODO presna formulace!
3. etc...

V a V_α odkazuji k Von Neumannove hierarchii (pro jistotu)

zakladni definice

Definition 1.1 (*Ord*)

TODO *Ord* je trida vsech ordinalu

Definition 1.2 (*Function*)

We say that a first-order formula $\varphi(x, y, u_1, \dots, u_n)$ with no free variable besides x, y, u_1, \dots, u_n is a function iff

$$\forall x, y, z (\varphi(x, y, u_1, \dots, u_n) \ \& \ \varphi(x, z, u_1, \dots, u_n) \rightarrow y = z) \quad (1.1)$$

TODO kofinalita

Definition 1.3 (*Cofinality*)

Let κ be a cardinal. The cofinality of κ , written as $cf(\kappa)$ is defined as follows

$$TODO \quad (1.2)$$

Definition 1.4 (*Limit ordinal*)

We say that an ordinal α is a limit ordinal iff

$$\alpha \in Ord \ \& \ \exists x (x \in \alpha) \ \& \ \forall x (x \in \alpha \rightarrow x + 1 \in \alpha) \quad (1.3)$$

TODO def \aleph_α ?

Definition 1.5 (*Limit Cardinal*)

We say that a cardinal κ is a limit cardinal iff

$$\exists \alpha (\alpha \in Ord \ \& \ \kappa = \aleph_\alpha) \quad (1.4)$$

204 **Definition 1.6** (*Strong Limit Cardinal*)

205 We say that an ordinal κ is a strong limit cardinal iff it is a limit cardinal
206 and

$$\forall \alpha (\alpha \in \kappa \rightarrow \mathcal{P}(\alpha) \in \kappa) \quad (1.5)$$

207

208 Vypsát axiomy ZFC a jake formulace používám

209 *Replacement*, *Replacement₂* a *Subsets*

210 **Definition 1.7** (*Extensionality*)

211

$$\text{Extensionality} \leftrightarrow \forall x, y (\forall z (z \in x \leftrightarrow z \in y) \rightarrow x = y) \quad (1.6)$$

212 **Definition 1.8** (*Foundation*)

213

$$\text{Foundation} \leftrightarrow \forall x (\exists z (z \in x) \rightarrow \exists z (z \in x \ \& \ \forall u \neg (u \in z \ \& \ u \in x))) \quad (1.7)$$

214 **Definition 1.9** (*Pairing*)

215

$$\text{Pairing} \leftrightarrow \forall x, y \exists z (x \in z \ \& \ y \in z) \quad (1.8)$$

216 **Definition 1.10** (*Union*)

217

$$\text{Union} \leftrightarrow \forall x \exists y \forall z \forall w ((w \in z \wedge z \in x) \rightarrow w \in y) \quad (1.9)$$

218 **Definition 1.11** (*Powerset*)

219

$$\text{Powerset} \leftrightarrow \forall x \exists y \forall z (z \subseteq x \rightarrow z \in y) \quad (1.10)$$

220 **Definition 1.12** (*Infinity*)

221

$$\text{Infinity} \leftrightarrow \exists x (\exists y (y \in x) \ \& \ \forall y (y \in x \rightarrow y + 1 \in x)) \quad (1.11)$$

222 **Definition 1.13** (*Specification*)

223 The following is a schema for every first-order formula φ .

$$\text{Specification} \leftrightarrow \forall x \exists y \forall z (z \in y \leftrightarrow (z \in x \ \& \ \varphi(z, x))) \quad (1.12)$$

224 **Definition 1.14** (*Replacement*)

225 The following is a schema for every first-order formula φ .

$$\begin{aligned} \text{Replacement} \leftrightarrow \\ \forall x, y, z (\varphi(x, y) \ \& \ \varphi(x, z) \rightarrow y = z) \rightarrow \\ \rightarrow (\forall x \exists y \forall z (z \in y \leftrightarrow \exists w \in x (\varphi(w, z)))) \end{aligned} \quad (1.13)$$

226 **Definition 1.15** (Choice)

227

$$\text{Choice} \leftrightarrow \text{TODO} \quad (1.14)$$

228

229 **Definition 1.16** (S)

230 *TODO*

231 **Definition 1.17** (ZF)

232 *TODO*

233 **Definition 1.18** (ZFC)

234 *TODO*

235 **Definition 1.19** (ZFC₂)

236 *TODO*

237

238 *TODO jak se vyrobi separation z replacementu?*

239 *TODO definice druhoradoveho splnovani (nebo to vyhod)*

240 **Lemma 1.20** δ_0 formulas are absolute in transitive sets, in other words, let

241 φ be a first-order δ_0 formula and let M be a transitive class.

$$\varphi \leftrightarrow \varphi^M \quad (1.15)$$

242 **Definition 1.21** (Analytical hierarchy of formulas)

243 *ASDF Π_n^m und Σ_n^m*

244 **Definition 1.22** (Reflection₁)

245

$$\text{ASD} \quad (1.16)$$

246

247 *Asi vsechno budeme delat v ZFC, nic bychom neziskali, pokud ne.*

248 *TODO mozna zminit Levyho "sentential reflection"? pouzivame to v*

249 *indescribable*

250 2 Levy's first-order reflection

251 2.1 Introduction

252 This section will try to present Lévy's proof of a general reflection principle
 253 being equivalent to Replacement and Infinity under ZF minus Replacement
 254 and Infinity. We will first introduce a few axioms and definitions that were
 255 a different in Lévy's paper[?], but are equivalent to today's terms. We will
 256 write them in contemporary notation, our aim is the result, not history of
 257 set theory notation.

258 Please note that Lévy's paper was written in a period when Set theory
 259 was oriented towards semantics, which means that everything was done in
 260 a model. All proofs were a model that of ZFC was V_κ (notated as $R(\kappa)$ at
 261 the time) for some cardinal κ , which means that κ is a inaccessible cardinal.
 262 Please bear in mind that this is vastly different from saying that there is
 263 an inaccessible κ inside the model. This V_κ is also referred to as $Scm^Q(u)$,
 264 which means that u is a standard complete model of an undisclosed axiomatic
 265 set theory Q formulated in the "non-simple applied first order functional
 266 calculus", which is second-order theory is today's terminology, we are allowed
 267 to quantify over functions and thus get rid of axiom schemes. (Note that Lévy
 268 always speaks of "the axiom of replacement"). Besides placeholder set theory
 269 Q , and ZF, which the reader should be familiar with, theories Z , S , and SF
 270 are used in the text. Z is ZF minus replacement, S is ZF minus replacement
 271 and infinity, and finally SF is ZF minus infinity. The axiom of *Subsets* is
 272 an older name for the axiom scheme of specification (and it's not a scheme
 273 since we are now working in second order logic). Also note that universal
 274 quantifier does not appear, $\forall x\varphi(x)$ would be written as $(x)\varphi(x)$, the symbol
 275 for negation is " \sim ", we will use " \neg " the whole time.

276 TODO nebudeme tady pouzivat ZFC, ale jenom ZF. (jenom v tehle kapi-
 277 tole)

278 2.2 Lévy's Original Paper

279 The following are a few definitions that are used in Lévy's original article. ⁶

280 **Definition 2.1** (*Relativization*)[?, Definition 12.6]

281 Let M be a class, E a binary relation on M and let $\varphi(x_1, \dots, x_n)$ be a first-
 282 order formula with n parameters. The relativization of φ to M and E is the

⁶While some of them won't be of much use in this paper, they will provide extremely helpful when reading the original article as set theory notation and terminology has evolved in the last 50 years considerably.

283 formula written as

$$\varphi^{M,E}(x_1, \dots, x_n) \quad (2.17)$$

284 Defined in the following inductive manner:

$$\begin{aligned} (x \in y)^{M,E} &\leftrightarrow xEx \\ (x = y)^{M,E} &\leftrightarrow x = y \\ (\neg\varphi)^{M,E} &\leftrightarrow \neg\varphi^{M,E} \\ (\varphi \ \& \ \psi)^{M,E} &\leftrightarrow \varphi^{M,E} \ \& \ \psi^{M,E} \\ (\exists x\varphi)^{M,E} &\leftrightarrow (\exists x \in M)\varphi^{M,E} \end{aligned} \quad (2.18)$$

285 Next two definitions are not used in contemporary set theory, but they
286 illustrate 1960's set theory mind-set and they are used heavily in Lévy's text,
287 so we will include and explain them for clarity. Generally in this chapter, \mathbf{Q}
288 stands for an undisclosed axiomatic set theory, u is usually a model, coun-
289 terpart of today's V^7 , E is a relation that serves as \in in the given model.

290 **Definition 2.2** (Standard model of a set theory)

291 Let \mathbf{Q} be a axiomatic set theory in first-order logic. We say the the a class u
292 is a standard model of \mathbf{Q} with respect to a membership relation E , written as
293 $Sm^{\mathbf{Q}}(u)$, iff both of the following hold

- 294 (i) $(x, y) \in E \leftrightarrow y \in u \ \& \ x \in y$
295 (ii) $y \in u \ \& \ x \in y \rightarrow x \in u$

296 **Definition 2.3** Standard complete model of a set theory

297 Let \mathbf{Q} and E be like in ???. We say that that u is a standard complete model
298 of \mathbf{Q} with respect to a membership relation E iff both of the following hold

- 299 (i) u is a transitive set with respect to \in
300 (ii) $\forall E((x, y) \in E \leftrightarrow (y \in u \ \& \ x \in y) \ \& \ Sm^{\mathbf{Q}}(u, E))$

301 this is written as $Scm^{\mathbf{Q}}(u)$.

302 **Definition 2.4** (Inaccessible cardinal with respect to \mathbf{Q})

303 Let \mathbf{Q} be an axiomatic first-order set theory. We say that a cardinal κ is
304 inaccessible with respect to \mathbf{Q} , we write $In^{\mathbf{Q}}(\kappa)$, iff

$$Scm^{\mathbf{Q}}(V_\kappa). \quad (2.19)$$

⁷Which is of course not referred to as a model, but it is used in a similar fashion, in this case the term "model" was a metamathematical notion because it was not based on any underlying structure of theory. It can be easily formalized in any set theory, but it's not helpful for our case.

2.3 $S \models (N_0 \leftrightarrow \text{Replacement} \ \& \ \text{Infinity})$ 2. Levy's first-order reflection

305 **Definition 2.5** (*Inaccessible cardinal with respect to ZF*)

306 When a cardinal κ is inaccessible with respect to ZF, we only say that it is
307 inaccessible. In the abbreviated version, we just leave out the superscript.

$$In(\kappa) \leftrightarrow In^{ZF}(\kappa) \quad (2.20)$$

308 **Definition 2.6** (N)

309 The following is an axiom schema of complete reflection over ZF, denoted as
310 N .

$$N \leftrightarrow \exists u (Scm^{ZF}(u) \ \& \ \forall x_1, \dots, x_n (x_1, \dots, x_n \in u \rightarrow \varphi \leftrightarrow \varphi^u)) \quad (2.21)$$

311 where φ is a formula which contains no free variables except for x_1, \dots, x_n .

312 **Definition 2.7** (N_0)

313 If we substitute ZF for S , which is ZF minus Replacement and Infinity, we
314 obtain what will now be called N_0 .

$$N_0 \leftrightarrow \exists u (Scm^S(u) \ \& \ \forall x_1, \dots, x_n (x_1, \dots, x_n \in u \rightarrow \varphi \leftrightarrow \varphi^u)) \quad (2.22)$$

315 where φ is a formula which contains no free variables except for x_1, \dots, x_n .

316 Once we have established the definitions, it's time to prove something
317 interesting.

318 **2.3** $S \models (N_0 \leftrightarrow \text{Replacement} \ \& \ \text{Infinity})$

319 Let N_0 be defined as in ??, for *Infinity* see ??.

320 **Theorem 2.8** In S , the schema N_0 implies Infinity.

321 *Proof.* For any φ , N_0 gives us $\exists u Scm^S(u)$, which means that there is a set u
322 that is identical to V_α for some alpha, so $\exists \alpha Scm^S(V_\alpha)$. We don't know the
323 exact size of this α , but we know that $\alpha \geq \omega$, otherwise α would be finite,
324 therefore not closed under the powerset operation, which would contradict
325 *Powerset*. In order to prove that it is a model of S , we would need to verify
326 all axioms of S . We have already shown that ω is closed under the powerset
327 operation. Foundation, extensionality and comprehension are clear from the
328 fact that we work in ZF^8 , pairing is clear from the fact, that given two sets
329 x, y , they have ranks α, β , without loss of generality we can assume that

⁸We only need to verify axioms that provide means of constructing larger sets from smaller to make sure they don't exceed ω . Since ω is an initial segment of ZF , the axiom scheme of specification can't be broken, the same holds for foundation and extensionality.

$\alpha \leq \beta$, which means that $x \in V_\alpha \in V_\beta$, therefore V_β is a set that satisfies the pairing axiom: it contains both x and B .

Note that this implies that any (strong) limit cardinal is a model of \mathbf{S} .

We now want to prove that V_α leads to existence of an inductive set, which is a set that satisfies $\exists A(\emptyset \in A \ \& \ \forall x \in A ((x \cup \{x\}) \in A))$. If we can find a way to construct V_ω from any V_α satisfying $\alpha \geq \omega$, we are done. Since ω is the least limit ordinal, all we need is the following

$$\bigcap \{V_\kappa \mid \forall \lambda (\lambda < \kappa \rightarrow \exists \mu (\lambda < \mu < \kappa))\} \quad (2.23)$$

because V_κ is a transitive set for every κ , thus the intersection is non-empty unless empty set satisfies the property or the set of V_κ s is itself empty. \square

Let N_0 be defined as in ??, for *Replacement* see ??.

Theorem 2.9 *In \mathbf{S} , the schema N_0 implies Replacement.*

Proof. Let $\varphi(v, w, x_1, \dots, x_n)$ be a formula wth no free variables except v, w, x_1, \dots, x_n where n is any natural number. Let χ be an instance of replacement schema for this φ which is what we want to prove:

$$\begin{aligned} \chi = & \forall r, s, t (\varphi(r, s, x_1, \dots, x_n) \& \varphi(r, t, x_1, \dots, x_n) \rightarrow s = t) \\ & \rightarrow \forall x \exists y \forall w (w \in y \leftrightarrow \exists v (v \in x \& \varphi(v, w, x_1, \dots, x_n))) \end{aligned} \quad (2.24)$$

We can deduce the following from N_0 :

- (i) $x_1, \dots, x_n, v, w \in u \rightarrow (\varphi \leftrightarrow \varphi^u)$
- (ii) $x_1, \dots, x_n, v \in u \rightarrow (\exists w \varphi \leftrightarrow (\exists w \varphi)^u)$
- (iii) $x_1, \dots, x_n, x \in u \rightarrow (\chi \leftrightarrow \chi^u)$
- (iv) $\forall x_1, \dots, x_n \forall x (\chi \leftrightarrow (\forall x_1, \dots, x_n \forall x \chi)^u)$

It is easy to see that (i), (ii), (iii) are the instances of N_0 for φ , $\exists w \varphi$ and χ respectively. From relativization we also know that $(\exists w \varphi)^u$ is equivalent to $\exists w (w \in u \& \varphi^u)$. Therefore (ii) is equivalent to

$$x_1, \dots, x_n, v \in u \rightarrow (\exists w (w \in u \& \varphi^u)). \quad (2.25)$$

If φ is a function⁹, then for every $x \in u$, which is also $x \subset u$ by the transitivity of $\text{Scm}^S(u)$, it maps elements of x onto u . From the axiom scheme of comprehension¹⁰, we can find y , a set of all images of elements of x . That gives us $x_1, \dots, x_n, x \in u \rightarrow \chi$. By (iii) we get $x_1, \dots, x_n, x \in u \rightarrow \chi^u$, the universal closure of this formula is $(\forall x_1, \dots, x_n \forall x \chi)^u$, which together with

⁹ $\forall r, s, t (\varphi(r, s) \& \varphi(r, t) \rightarrow r = t)$

¹⁰Lévy's uses its equivalent, axiom of subsets

(iv) yields $\forall x_1, \dots, x_n \forall x \chi$. By the means of specification we end up with χ , Q.E.D. \square

What we have just proven is just a single theorem from said article, we will introduce other interesting propositions, mostly related to the existence of large cardinals, later in their appropriate context in chapter 3.

2.4 Contemporary restatement

We will now prove what is also Lévy's reflection theorem, but a little stronger, rephrased with more up to date set theory. The main difference is, that while Lévy reflects φ from V into a set u that is a "standard complete model of S "¹¹, we say that there is a V_α that reflects φ . In other words, we don't need α to be an inaccessible cardinal like Lévy does.

We will prove the equivalence of N_0 with *Replacement* and *Infinity* in S in two parts. First, we will show that *Reflection*₁ is a theorem of ZF, then the second implication which proves *Infinity* and *Replacement* from N_0 .

The following lemma is usually done in more parts, the first being with one formula and the other with n . We will only state and prove the generalised version for n formulas, knowing that $n = 1$ is just a specific case and the proof is exactly the same.

Lemma 2.10 *Let $\varphi_1, \dots, \varphi_n$ be formulas with m parameters¹².*

(i) *For each set M_0 there is such M that $M_0 \subset M$ and the following holds for every $i \leq n$:*

$$\exists x \varphi_i(u_1, \dots, u_{m-1}, x) \rightarrow (\exists x \in M) \varphi_i(u_1, \dots, u_{m-1}, x) \quad (2.26)$$

for every $u_1, \dots, u_{m-1} \in M$.

(ii) *Furthermore there is an ordinal α such that $M_0 \subset V_\alpha$ and the following holds for each $i \leq n$:*

$$\exists x \varphi_i(u_1, \dots, u_{m-1}, x) \rightarrow (\exists x \in V_\alpha) \varphi_i(u_1, \dots, u_{m-1}, x) \quad (2.27)$$

for every $u_1, \dots, u_{m-1} \in M$.

¹¹Any limit ordinal is in fact a model of S , we shall pay more attention to that in a moment.

¹²For formulas with a different number of parameters, take for m the highest number of parameters among given formulas. Add spare parameters to the other formulas so that x remains the last parameter. That can be done in a following manner: Let φ'_i be the a formula with k parameters, $k < m$. Let us set $\varphi_i(u_1, \dots, u_{m-1}, x) = \varphi'_i(u_1, \dots, u_{k-1}, u_k, \dots, u_{m-1}, x)$, notice that u_k, \dots, u_{m-1} are the aforementioned spare variables.

383 (iii) Assuming Choice, there is M , $M_0 \subset M$ such that ?? holds for every
 384 M , $i \leq n$ and $|M| \leq |M_0| \cdot \aleph_0$.

385 *Proof.* We will simultaneously prove statements (i) and (ii), denoting M^T
 386 the transitive set required by part (ii). Unless explicitly stated otherwise for
 387 specific steps, it is thought to be equivalent to M .

388 Let us first define operation $H(u_1, \dots, u_{m-1})$ that gives us the set of
 389 x 's with minimal rank satisfying $\varphi_i(u_1, \dots, u_{m-1}, x)$ for given parameters
 390 u_1, \dots, u_{m-1} for every $i \leq n$.

$$H_i(u_1, \dots, u_n) = \{x \in C_i : (\forall z \in C)(\text{rank}(x) \leq \text{rank}(z))\} \quad (2.28)$$

391 for each $i \leq n$, where

$$C_i = \{x : \varphi_i(u_1, \dots, u_{m-1}, x)\} \text{ for } i \leq n \quad (2.29)$$

392

393 Next, let's construct M from given M_0 by induction.

$$M_{i+1} = M_i \cup \bigcup_{j=0}^n \{H_j(u_1, \dots, u_{m-1}) : u_1, \dots, u_{m-1} \in M_i\} \quad (2.30)$$

394 In other words, in each step we add the elements satisfying $\varphi(u_1, \dots, u_{m-1}, x)$
 395 for all parameters that were either available earlier or were added in the
 396 previous step. For statement (ii), this is the only part that differs from (i).
 397 Let us take for each step transitive closure of M_{i+1} from (i). In other words,
 398 let γ be the smallest ordinal such that

$$(M_i^T \cup \bigcup_{j=0}^n \{H_j(u_1, \dots, u_{m-1}) : u_1, \dots, u_{m-1} \in M_i\}) \subset V_\gamma \quad (2.31)$$

399 Then the incremental step is like so:

$$M_{i+1}^T = V_\gamma \quad (2.32)$$

400 The final M is obtained by joining all incremental steps together.

$$M = \bigcup_{i=0}^{\infty} M_i, \quad M^T = \bigcup_{i=0}^{\infty} M_i^T \quad (2.33)$$

401

402 We have yet to finish part (iii). Let's try to construct a set M' that
 403 satisfies the same conditions like M but is kept as small as possible. Assuming

the Axiom of Choice, we can modify the process so that cardinality of M' is at most $|M_0| \cdot \aleph_0$. Note that the size of M' is determined by the size of M_0 and, most importantly, by the size of $H_i(u_1, \dots, u_{m-1})$ for any $i \leq n$ in individual levels of the construction. Since the lemma only states existence of some x that satisfies $\varphi_i(u_1, \dots, u_{m-1}, x)$ for any $i \leq n$, we only need to add one x for every set of parameters but $H_i(u_1, \dots, u_{m-1})$ can be arbitrarily large. Since Axiom of Choice ensures that there is a choice function, let F be a choice function on $\mathcal{P}(\bigcup_{i \leq n} H_i(u_1, \dots, u_{m-1}))$. Also let $h_i(u_1, \dots, u_{m-1}) = F(H_i(u_1, \dots, u_{m-1}))$ for $i \leq n$, which means that h is a function that outputs an x that satisfies $\varphi_i(u_1, \dots, u_{m-1}, x)$ for $i \leq n$ and has minimal rank among all such witnesses. The induction step needs to be redefined to

$$M'_{i+1} = M'_i \cup \bigcup_{j=0}^n \{H_j(u_1, \dots, u_{m-1}) : u_1, \dots, u_{m-1} \in M'_i\} \quad (2.34)$$

In every step, the amount of elements added in M'_{i+1} is equivalent to the amount of sets of parameters the yielded elements not included in M'_i . So the cardinality of M'_{i+1} exceeds the cardinality of M'_i only for finite M'_i . It is easy to see that if M_0 is finite, M' is countable because it was built from countable union of finite sets. If M_0 is countable or larger, cardinality of M' is equal to the cardinality of M_0 .¹³ Therefore $|M'| \leq |M_0| \cdot \aleph_0$ \square

And now for the theorem itself

Theorem 2.11 (*Lévy's first-order reflection theorem*)

Let $\varphi(x_1, \dots, x_n)$ be a first-order formula.

(i) For every set M_0 there exists M such that $M_0 \subset M$ and the following holds:

$$\varphi^M(x_1, \dots, x_n) \leftrightarrow \varphi(x_1, \dots, x_n) \quad (2.35)$$

for every $x_1, \dots, x_n \in M$.

(ii) For every set M_0 there is a transitive set M , $M_0 \subset M$ such that the following holds:

$$\varphi^M(x_1, \dots, x_n) \leftrightarrow \varphi(x_1, \dots, x_n) \quad (2.36)$$

for every $x_1, \dots, x_n \in M$.

(iii) For every set M_0 there is α such that $M_0 \subset V_\alpha$ and the following holds:

$$\varphi^{V_\alpha}(x_1, \dots, x_n) \leftrightarrow \varphi(x_1, \dots, x_n) \quad (2.37)$$

for every $x_1, \dots, x_n \in M$.

¹³It can not be smaller because $|M'_{i+1}| \geq |M'_i|$ for every i . It may not be significantly larger because the maximum of elements added is the number of n -tuples in M'_i , which is of the same cardinality is M'_i .

432 (iv) Assuming Choice, for every set M_0 there is M such that $M_0 \subset M$ and
 433 $|M| \leq |M_0| \cdot \aleph_0$ and the following holds:

$$\varphi^M(x_1, \dots, x_n) \leftrightarrow \varphi(x_1, \dots, x_n) \quad (2.38)$$

434 for every $x_1, \dots, x_n \in M$.

435 *Proof.* Let's prove (i) for one formula φ via induction by complexity first.
 436 We can safely assume that φ contains no quantifiers besides \exists and no logical
 437 connectives other than \neg and $\&$. Assume that this M is obtained from
 438 lemma ???. The fact, that atomic formulas are reflected in every M comes
 439 directly from definition of relativization and the fact that they contain no
 440 quantifiers.¹⁴ The same holds for formulas in the form of $\varphi = \neg\varphi'$. Let us
 441 recall the definition of relativization for those formulas in ???.

$$(\neg\varphi_1)^M \leftrightarrow \neg(\varphi_1^M) \quad (2.39)$$

442 Because we can assume from induction that $\varphi'^M \leftrightarrow \varphi'$, the following holds:

$$(\neg\varphi')^M \leftrightarrow \neg(\varphi'^M) \leftrightarrow \neg\varphi' \quad (2.40)$$

443 The same holds for $\varphi = \varphi_1 \& \varphi_2$. From the induction hypothesis we know
 444 that $\varphi_1^M \leftrightarrow \varphi_1$ and $\varphi_2^M \leftrightarrow \varphi_2$, which together with relativization for formulas
 445 in the form of $\varphi_1 \& \varphi_2$ gives us

$$(\varphi_1 \& \varphi_2)^M \leftrightarrow \varphi_1^M \& \varphi_2^M \leftrightarrow \varphi_1 \& \varphi_2 \quad (2.41)$$

446 Let's now examine the case when from the induction hypethesis, M re-
 447 flects $\varphi'(u_1, \dots, u_n, x)$ and we are interested in $\varphi = \exists x\varphi'(u_1, \dots, u_n, x)$. The
 448 induction hypothesis tells us that

$$\varphi'^M(u_1, \dots, u_n, x) \leftrightarrow \varphi'(u_1, \dots, u_n, x) \quad (2.42)$$

450 so, together with above lemma ??, the following holds:

$$\begin{aligned} & \varphi(u_1, \dots, u_n, x) \\ & \leftrightarrow \exists x\varphi'(u_1, \dots, u_n, x) \\ & \leftrightarrow (\exists x \in M)\varphi'(u_1, \dots, u_n, x) \\ & \leftrightarrow (\exists x \in M)\varphi'^M(u_1, \dots, u_n, x) \\ & \leftrightarrow (\exists x\varphi'(u_1, \dots, u_n, x))^M \\ & \leftrightarrow \varphi^M(u_1, \dots, u_n, x) \end{aligned} \quad (2.43)$$

¹⁴Note that this does not hold generally for relativizations to M, E , but only for relativization to M, \in , which is our case.

Which is what we have needed to prove:

So far we have proven part (i) of this theorem for one formula φ , we only need to verify that the same holds for any finite number of formulas. This has in fact been already done since lemma ?? gives us M for any (finite) amount of formulas. We can than use the induction above to verify that it reflects each of the formulas individually.

Now we want to verify other parts of our theorem. Since V_α is a transitive set, by proving (iii) we also satisfy (ii). To do so, we only need to look at part (ii) of lemma ?. All of the above proof also holds for $M = V_\alpha$.

To finish part (iv), we take M of size $\leq |M_0| \cdot \aleph_0$, which exists due to part (iii) of lemma ?, the rest being identical. \square

Theorem 2.12 *Reflection is equivalent to Infinity & Replacement under ZFC minus Infinity & Replacement*

Proof. Since ?? already gives one side of the implication, we are only interested in showing the converse which we shall do in two parts:

Reflection \rightarrow Infinity

Let us first find a formula to be reflected that requires a set M at least as large as V_ω . Let us consider the following formula:

$$\varphi'(x) = \forall \lambda (\lambda < x \rightarrow \exists \mu (\lambda < \mu < x)) \quad (2.44)$$

Because φ says "there is a limit ordinal", if it holds for some x , the Infinity axiom is very easy to satisfy. But we know that there are limit ordinals in ZF, therefore $\varphi = \exists x \varphi'(x)$ is a valid statement. *Reflection* then gives us a set M in which φ^M holds, that is, a set that contains a limit ordinal. So the set of off limit ordinals is non-empty and because ordinals are well-founded, it has a minimal element. Let's call it μ .

$$\mu = \bigcap \{V_\kappa : \forall \lambda (\lambda < \kappa \rightarrow \exists \mu (\lambda < \mu < \kappa))\} \quad (2.45)$$

We can see that μ is the least limit ordinal and therefore it satisfies *Infinity*.

Reflection \rightarrow Replacement

Given a formula $\varphi(x, y, u_1, \dots, u_n)$, we can suppose that it is reflected in any M ¹⁵ What we want to obtain is the following:

$$\forall x, y, z (\varphi(x, y, u_1, \dots, u_n) \& \varphi(x, z, u_1, \dots, u_n) \rightarrow y = z) \rightarrow \quad (2.46)$$

¹⁵Which means that for $x, y, u_1, \dots, u_n \in M$, $\varphi^M(x, y, u_1, \dots, u_n) \leftrightarrow \varphi(x, y, u_1, \dots, u_n)$.

$$\rightarrow \forall X \exists Y \forall y (y \in Y \leftrightarrow \exists x (\varphi(x, y, u_1, \dots, u_n) \ \& \ x \in X)) \quad (2.47)$$

We do also know that $x, y \in M$, in other words for every $X, Y = \{y \mid \varphi(x, y, u_1, \dots, u_n)\}$ we know that $X \subset M$ and $Y \subset M$, which, together with the comprehension schema¹⁶ implies that Y , the image of X over φ , is a set. Which is exactly the Replacement Schema we hoped to obtain. \square

We have shown that *Reflection* for first-order formulas, *Reflection*₁ is a theorem of ZF, which means that it won't yield us any large cardinals. We have also shown that it can be used instead of the Axiom of Infinity and Replacement Scheme, but ZF + *Reflection*₁ is a conservative extension of ZF. Besides being a starting point for more general and powerful statements, it can be used to show that ZF is not finitely axiomatizable. That is because *Reflection* gives a model to any finite number of (consistent) formulas. So if $\varphi_1, \dots, \varphi_n$ for any finite n would be the axioms of ZF, *Reflection* would always contain a model of itself, which would in turn contradict the Second Gödel's Theorem¹⁷. Notice that, in a way, reflection is complementary to compactness. Compactness argues that given a set of sentences, if every finite subset yields a model, so does the whole set. Reflection, on the other hand, says that while the whole set has no model in the underlying theory, every finite subset does have one.

Also, notice how reflection can be used in ways similar to upward Löwenheim–Skolem theorem. Since Reflection extends any set M_0 into a model of given formulas $\varphi_1, \dots, \varphi_n$, we can choose the lower bound of the size of M by appropriately choosing M_0 .

In the next section, we will try to generalize *Reflection* in a way that transcends ZF and finally yields some large cardinals.

¹⁶Called the axiom of subsets in Lévy's proof.

¹⁷See chapter ?? for further details.

3 Reflection And Large Cardinals

In this chapter we aim to examine stronger reflection properties in order to reach cardinals unavailable in ZFC. Like we said in the first chapter, the variety of reflection principles comes from the fact that there are many way to formalize "properties of the universal class". It is not always obvious what properties hold for V because, We have shown that reflecting properties as first-order formulas doesn't allow us to leave ZFC. We will broaden the class of admissible properties to be reflected and see whether there is a natural limit in the height or width on the reflected universe and also see that no matter how far we go, the universal class is still as elusive as it is when seen from S . That is because for every process for obtaining larger sets such as for example the powerset operation in ZFC, this process can't reach V and thus, from reflection, there is an initial segment of V that can't be reached via said process.

To see why this is important, let's dedicate a few lines to the intuition behind the notions of limitness, regularity and inaccessibility in a manner strongly influenced by [?]. To see why limit and strongly limit cardinals are worth mentioning, note that they are "limit" not only in a sense of being a supremum of an ordinal sequence, they also show that a certain way of obtaining bigger objects from smaller ones limited in terms of possibilities. \aleph_λ is a limit cardinal iff there is no α such that $\aleph_{\alpha+1} = \aleph_\lambda$. Strongly limit cardinals point to the limits of the powerset operation. It has been too obvious so far, so let's look at the regular cardinals in this manner. Regular cardinals are those that cannot be¹⁸, expressed as a supremum of smaller amount of smaller objects¹⁹. More precisely, κ is regular if there is no way to define it as a union of less than κ ordinals, all smaller than κ . So unless there already is a set of size κ , *Replacement* is useless in determining whether κ is really a set. Note that assuming *Choice*, successor cardinals are always regular, so most²⁰ limit cardinals are singular cardinals. So if one is traversing the class of all cardinals upwards, successor steps are still sets thanks to the powerset axiom while singular limit cardinal are not proper classes because they are images of smaller sets via *Replacement*. Regular cardinals are, in a way, limits of how far can we get by taking limits of increasing sequences of ordinals obtained via *Replacement*.

That all being said, it is easy to see that no cardinals in ZFC are both strongly limit and regular because there is no way to ensure they are sets and

¹⁸Assuming *Choice*.

¹⁹Just like ω can not be expressed as a supremum of a finite set consisting solely of finite numbers.

²⁰All provable to exist in ZFC

not proper classes in ZFC. The only exception to this rule is \aleph_0 which needs
Infinity to exist. It should now be obvious why the fact that κ is inaccessible
implies that $\kappa = \aleph_\kappa$.²¹

We will also examine the connection between reflection principles and
(regular) fixed points of ordinal functions in a manner proposed by Lévy in
[?]. We will also see that, like Lévy has proposed in the same paper, there is
a meaningful way to extend the relation between S and ZFC into a hierarchy
of stronger axiomatic set theories.

3.1 Regular Fixed-Point Axioms

Lévy's article mentions various schemata that are not instances of reflection
themselves. We will mention them because they are equivalent to N_0 and
because they are fixed-point theorems, which we will find useful later in this
thesis.

Definition 3.1 (*Function*) We say that a first-order formula $\varphi(x, y, u_1, \dots, u_n)$
with no free variable besides x, y, u_1, \dots, u_n is a function iff

$$\forall x, y, z (\varphi(x, y, u_1, \dots, u_n) \ \& \ \varphi(x, z, u_1, \dots, u_n) \rightarrow y = z) \quad (3.48)$$

We will also write functions in the form of " $f(x) = y$ ". This is defined for
given $\varphi(x, y, u_1, \dots, u_n)$ and given terms t_1, \dots, t_n as follows

$$f(x) = y \leftrightarrow \varphi(x, y, t_1, \dots, t_n) \quad (3.49)$$

Ord denotes the class of all ordinal numbers.

Definition 3.2 (*Strictly increasing function*)

A function $f : Ord \rightarrow Ord$ is said to be strictly increasing iff

$$\forall \alpha, \beta \in Ord (\alpha < \beta \rightarrow f(\alpha) < f(\beta)). \quad (3.50)$$

Definition 3.3 (*Continuous function*)

A function $f : Ord \rightarrow Ord$ is said to be continuous iff

$$\alpha \text{ is limit} \rightarrow (f(\alpha) = \lim_{\beta < \alpha} f(\beta)). \quad (3.51)$$

Alternatively, a function $f : Ord \rightarrow Ord$ is continuous iff for limit λ , $f(\lambda) =$
 $\bigcup_{\alpha < \lambda} f(\alpha)$.

²¹This doesn't work backwards, the least fixed point of the \aleph function is the limit of
 $\{\aleph_0, \aleph_{\aleph_0}, \aleph_{\aleph_{\aleph_0}}, \dots\}$, it is singular since the sequence has countably many elements.

570 **Definition 3.4** (*Normal function*)

571 *A function $f : Ord \rightarrow Ord$ is said to be normal if it is strictly increasing*
 572 *and continuous.*

573 **Definition 3.5** (*Normal function on a set*) *Let α, δ be ordinals. A function*
 574 *$f : \delta \rightarrow \alpha$ is called a normal function on α iff all of the following hold:*

- 575 (i) *f is strictly increasing on α^{22}*
 576 (ii) *f is continuous on α*
 577 (iii) *the $rng(f) = \{y : \exists x(f(x) = y)\}$ is unbounded in α .*

578 **Definition 3.6** *Fixed point*

579 *We say α is a fixed point of ordinal function f when $\alpha = f(\alpha)$.*

580 Lévy ([?]) proposes those axioms as equivalent to one on his reflection
 581 principles.

582 **Definition 3.7** $M \leftrightarrow$ "Every normal function defined for all ordinals has at
 583 least one inaccessible number in its range."

584 We will rewrite M as a formula to make it clear that it is an axiom scheme
 585 and the same can be done with M' as well as M'' .

586 Let $\varphi(x, y, u_1, \dots, u_n)$ be a first-order formula with no free variables be-
 587 sides x, y, u_1, \dots, u_n . The following is equivalent to M .

$$\varphi \text{ is a normal function } \& \forall x(x \in Ord \rightarrow \exists y(\varphi(x, y, u_1, \dots, u_n))) \rightarrow \rightarrow \exists y(\exists x \varphi(x, y, u_1, \dots, u_n) \& \dots) \quad (3.52)$$

588 **Definition 3.8** $M' \leftrightarrow$ "Every normal function defined for all ordinals has
 589 at least one fixed point which is inaccessible."

590 **Definition 3.9** $M'' \leftrightarrow$ "Every normal function defined for all ordinals has
 591 arbitrarily great fixed points which are inaccessible."

592 The following axiom is proposed by Drake in [?].

593 **Definition 3.10** F *Every normal function for all ordinals has a regular fixed*
 594 *point.*

Theorem 3.11

$$F \leftrightarrow M \leftrightarrow M' \leftrightarrow M'' \quad (3.53)$$

595 *Proof.* One can find the proof of $M \leftrightarrow M' \leftrightarrow M''$ in [?], *Theorem 1.*

596 TODO podle Levyho

597

□

²² x is limit $\rightarrow (f(x)) = \bigcup_{y < x} f(y)$

3.2 A Model-Theoretic Intermezzo

This is a small notational intermezzo. Reflection theorems asdasd Tarski Berkley model-theoretic methods in set theory.

This notation is used for example in [?].

TODO def $\langle V_\kappa, \in, R \rangle \models$ asdf

TODO $S \rightarrow ZM \rightarrow ZM' \rightarrow ZM''$, neco jako mahlovy kardinaly, pre-sunout do dane kapitoly

3.3 Reflecting Second-order Formulas

To see that there is a way to transcend ZFC, let us briefly show how a model of ZFC can be obtained in $ZFC_2 + \text{"second-order reflection"}^{23}$. This will be more closely examined in section ??.

We know that ZFC can not be finitely axiomatized in first-order formulas, however if Replacement and Comprehension schemes can be substituted by second-order formulas, ZFC becomes ZFC_2 , which is finitely axiomatizable in second-order logic. Therefore if we take second-order reflection into consideration, we can obtain a set M that is a model of ZFC_2 . For now, we have left out the details of how exactly is first-order reflection generalised into stronger statements and how second-order axiomatization of ZFC looks like as we will examine those problems closely in the following pages.

Lower-case letters represent first-order variables and upper-case P represents a second-order variable. [?]

Definition 3.12 (Replacement_2)

$$\begin{aligned} \forall P(\forall x, y, z(P(x, y) \& P(x, z) \rightarrow y = z) \rightarrow \\ \rightarrow (\forall x \exists y \forall z(z \in y \leftrightarrow \exists w \in x(P(w, z)))) \end{aligned} \quad (3.54)$$

We will denote this axiom Replacement_2 .

Definition 3.13 (Specification_2)

$$\forall P \forall x \exists y \forall z (z \in y \leftrightarrow (z \in x \& P(z, x))) \quad (3.55)$$

Definition 3.14 (ZFC_2)

Let ZFC_2 be a theory with all axioms identical with the axioms of ZFC with the exception of Replacement and Specification schemes, which are replaced with Replacement_2 and Specification_2 respectively.

²³ ZFC_2 is an axiomatization of ZFC in second-order formulas, to be more rigorously established later.

3.4 Inaccessibility

Definition 3.15 (*limit cardinal*) κ is a limit cardinal iff it is \aleph_α for some limit ordinal α .

Definition 3.16 (*strong limit cardinal*) κ is a strong limit cardinal iff it is a limit cardinal and for every $\lambda < \kappa$, $2^\lambda < \kappa$

The two above definition become equivalent when we assume *GCH*.

Definition 3.17 (*weak inaccessibility*) An uncountable cardinal κ is weakly inaccessible iff it is regular and limit.

Definition 3.18 (*inaccessibility*) An uncountable cardinal κ is inaccessible (written $In(\alpha)$) iff it is regular and strongly limit.

We will now show that the above notion is equivalent to the definition Lévy uses in [?], which is, in more contemporary notation, the following:

Theorem 3.19 *The following are equivalent:*

1. κ is inaccessible
2. $\langle V_\kappa, \in \rangle \models \text{ZFC}$

Proof. Let's first prove that if κ is inaccessible, it is a model of ZFC. We will do that by verifying the axioms of ZFC just like Kanamori does it in [?, 1.2] and Drake in [?, Chapter 4].

- (i) *Extensionality*:
(see ??)

$$V_\kappa \models \forall x, y (\forall z (z \in x \leftrightarrow z \in y) \rightarrow x = y) \quad (3.56)$$

We need to prove that, given two sets that are equal in V , they are equal in V_κ , in other words, that the *Extensionality* formula is reflected, that is

$$V_\kappa \models \forall x, y \in V_\kappa (\forall z \in V_\kappa (z \in x \leftrightarrow z \in y) \rightarrow x = y) \quad (3.57)$$

But that comes from transitivity. If x and y are in V_κ their members are also in V_κ .

- (ii) *Foundation*:
(see ??)

$$V_\kappa \models \forall x (\exists z (z \in x) \rightarrow \exists z (z \in x \ \& \ \forall u \neg (u \in z \ \& \ u \in x))) \quad (3.58)$$

The argument for *Foundation* is almost identical to the one for *Extensionality*. For any set $x \in V_\kappa$, transitivity of V_κ makes sure that every element of x is also an element of V_κ and the same holds for the elements of elements of x et cetera. So statements about those elements are absolute between any transitive structures. V and V_κ are both transitive therefore *Foundation* holds and so does its relativisation to V_κ , $Foundation^{V_\kappa}$.

(iii) *Powerset*:

(see ??)

$$V_\kappa \models \forall x \exists y \forall z (z \subseteq x \rightarrow z \in y). \quad (3.59)$$

If we take x , an element of V_κ , $\mathcal{P}(\cdot)(x)$ has to be an element of V_κ to, because it is transitive and a strong limit cardinal.

(iv) *Pairing*:

(see ??)

$$V_\kappa \models \forall x, y \exists z (x \in z \wedge y \in z). \quad (3.60)$$

Pairing holds from similar argument like above: let x and y be elements of V_κ , so there are ordinals $\alpha, \beta < \kappa$ such that $x \in V_\alpha$, $y \in V_\beta$. Without any loss of generality, suppose $\alpha < \beta$, therefore $V_\alpha \subset V_\beta$ which, from transitivity of the cumulative hierarchy, means that $x \in V_\beta$, then $\{x, y\} \in V_{\beta+1}$ which is still in V_κ because it is a strong limit cardinal.

(v) *Union*:

(see ??)

$$V_\kappa \models \forall x \exists y \forall z \forall w ((w \in z \wedge z \in x) \rightarrow w \in y). \quad (3.61)$$

We want to see that for every $x \in V_\kappa$, this is equivalent to

$$V_\kappa \models \forall x \in V_\kappa, \exists y \in V_\kappa \forall z \in V_\kappa \forall w \in V_\kappa ((w \in z \wedge z \in x) \rightarrow w \in y). \quad (3.62)$$

Since V_κ is transitive, if $x \in V_\kappa$, all of its elements as well as their elements are in V_κ . To see that they also form a set themselves we only need to remember that V_κ is limit and therefore if α is the least ordinal such that $x \in V_\alpha$, $\bigcup x \in V_{\alpha+1}$.

(vi) *Replacement, Infinity*:

(see ??, ??)

We know that those hold from ??.

We will now show that if a set is a model of ZFC, it is in fact an inaccessible cardinal. So let V_κ be a model of ZFC which means that it is closed under the powerset operation, in other words:

$$\forall \lambda (\lambda < \kappa \rightarrow 2^\lambda < \kappa) \quad (3.63)$$

which is exactly the definition of strong limitness. κ is regular from the following argument by contradiction:

Let us suppose for a moment that κ is singular. Therefore there is an ordinal $\alpha < \kappa$ and a function $F : \alpha \rightarrow \kappa$ such that the range of F is unbounded in κ , in other words, $F[\alpha] \subseteq V_\kappa$ and $\sup(F[\alpha]) = \kappa$. In order to achieve the desired contradiction, we need to see that it is the case that $F[\alpha] \in V_\kappa$. Let $\varphi(x, y)$ be the following first-order formula:

$$F(x) = y \quad (3.64)$$

Then there is an instance of Axiom Schema of Replacement that states the following:

$$\begin{aligned} &(\forall x, y, z (\varphi(x, y) \& \varphi(x, z) \rightarrow y = z)) \rightarrow \\ &\rightarrow (\forall x \exists y \forall z (z \in y \leftrightarrow \exists w (\varphi(w, z)))) \end{aligned} \quad (3.65)$$

Which in turn means that there is a set $y = F[\alpha]$ and $y \in V_\kappa$, which is the contradiction with $\sup(y) = \kappa$ we are looking for. \square

The same holds for ZFC_2 , the proof is very similar.

Theorem 3.20

$$V_\kappa \models \text{ZFC}_2 \leftrightarrow \kappa \text{ is inaccessible} \quad (3.66)$$

Proof. κ is a strong limit cardinal because from ZFC_2 and *Powerset* we know that for every $\lambda < \kappa$, we know that $2^\lambda < \kappa$.

κ is also regular, because otherwise there would be an ordinal α and a function $F : \alpha \rightarrow \kappa$ with a range unbounded in κ . *Replacement*² gives us a set $y = F[\alpha]$, so $y \in V_\kappa$, which contradicts the fact that $\sup(y) = \kappa$. It can not be the case that $\kappa \in V_\kappa$.

The other direction is exactly like the first part of above theorem ?? \square

This is how the existence of an inaccessible cardinal is established in [?].

Definition 3.21 N

$$\exists u (In(\alpha) \& \forall x_1, \dots, x_n (x_1, \dots, x_n \in u \rightarrow (\varphi \leftrightarrow \varphi^u))) \quad (3.67)$$

717 It is interesting to see that the above schema yields the first inaccessible
 718 cardinal if we take for φ the conjunction of all axioms of \mathbf{ZF}_2 .

719

720 To see that inaccessible cardinal can be also obtained by a fixed-point
 721 axiom (or a scheme if were in first-order logic), see the following theorem by
 722 Lévy, we won't repeat the proof here, it is available in [?, Theorem 3],

Theorem 3.22

$$M \leftrightarrow N \quad (3.68)$$

723 We have transcended \mathbf{ZFC} , but that is just a start. Naturally, we could
 724 go on and consider the next inaccessible cardinal, which is inaccessible with
 725 respect to the theory $\mathbf{ZFC} + \exists \kappa (\kappa \models \mathbf{ZFC})$. But let's try to find a faster way
 726 up, informally at first.

727 Since we can find an inaccessible set larger than any chosen set M_0 , it
 728 is clear that there are arbitrarily large inaccessible cardinals in V , they are
 729 "unbounded"²⁴ in V . If V were a cardinal, we could say that there are V
 730 inaccessible cardinals less than V , but this statement of course makes no sense
 731 in set theory as is because V is not a set. But being more careful, we could
 732 find a property that can be formalized in second-order logic and reflect it to
 733 an initial segment of V . That would allow us to construct large cardinals
 734 more efficiently than by adding inaccessibles one by one. The property we
 735 are looking for ought to look like something like this:

$$\begin{aligned} &\kappa \text{ is an inaccessible cardinal and} \\ &\text{there are } \kappa \text{ inaccessible cardinals } \mu < \kappa \end{aligned} \quad (3.69)$$

736 This is in fact a fixed-point type of statement. We shall call those cardinals
 737 hyper-inaccessible. Now consider the following definition.

738

739 **Definition 3.23** *0-inaccessible cardinal*
 740 *A cardinal κ is 0-inaccessible if it is inaccessible.*

741 We can define α -weakly-inaccessible cardinals analogously with the only dif-
 742 ference that those are limit, not strongly limit.

743 **Definition 3.24** *α -hyper-inaccessible cardinal*
 744 *For any ordinal α , κ is called α -inaccessible, if κ is inaccessible and for each*
 745 *$\beta \prec \alpha$, the set of β -inaccessible cardinals less than κ is unbounded in κ .*

²⁴The notion is formally defined for sets, but the meaning should be obvious.

Because κ is inaccessible and therefore regular, the number of β -inaccessibles below κ is equal to κ . We have therefore successfully formalized the above vague notion of hyper-inaccessible cardinal into a hierarchy of α -inaccessibles.

Let's now consider iterating this process over again. Since, informally, V would be α -inaccessible for any α , this property of the universal class could possibly be reflected to an initial segment, the smallest of those will be the first hyper-inaccessible cardinal. Such κ is larger than any α -inaccessible since from regularity of κ , for given $\alpha < \kappa$, κ is κ -th α -hyper-inaccessible cardinal. It is in fact "inaccessible" via α -inaccessibility.

Definition 3.25 *Hyper-inaccessible cardinal*

κ is called the *hyper-inaccessible*, also *0-hyper-inaccessible*, cardinal if it is α -inaccessible for every $\alpha < \kappa$.

Definition 3.26 *α -hyper-inaccessible cardinal*

For any ordinal α , κ is called α -hyper-inaccessible cardinal if for each ordinal $\beta < \alpha$, the set of β -hyper-inaccessible cardinals less than κ is bounded in κ .

Obviously we could go on and iterate it ad libitum, but the nomenclature would be increasingly confusing. A smarter way to accomplish the same goal is carried out in the following section.

3.5 Mahlo Cardinals

While the previous chapter introduced us to a notion of inaccessibility and the possibility of iterating it ad libitum in new theories, there is an even faster way to travel upwards in the cumulative hierarchy, that was proposed by Paul Mahlo in his papers (see [?], [?] and [?]) at the very beginning of the 20th century, and which can be easily reformulated using (*Reflection*). To see how Lévy's initial statement of reflection was influenced by Mahlo's work, refer to section ???. The aim of the following paragraphs is to give an intuitive explanation of the idea behind Mahlo's hierarchy of cardinals, all claims made here ought to be stated formally later in the very same chapter.

At the very end of section ??, we have tried to establish the notion of hyper-inaccessibility and iterate it to yield even larger large cardinals. In order to avoid too bulky cardinal names, let's try a different route and establish those cardinals directly via reflection.

The following two definitions come from [?] and while they are rather informal, we will find them very helpful for understanding the Mahlo cardinals.

Definition 3.27 (*Fixed-point property*)

For any first-order formula $\psi(x, u_1, \dots, u_n)$ with no free variables other than x, u_1, \dots, u_n , which is any property of ordinals, we say that a property φ is a fixed-point property if φ has the form

$$\begin{aligned} & x \text{ is an inaccessible cardinal and} \\ & \text{there are } x \text{ ordinals less than } x \text{ that have the property } \psi(x, u_1, \dots, u_n). \end{aligned} \quad (3.70)$$

Definition 3.28 (*Fixed-point reflection*)

If φ is a fixed-point property that holds for V , it also holds for some V_α , an initial segment of V .

Obviously those are in no way rigorous definitions because we have no idea what $\psi(x, u_1, \dots, u_n)$ looks like. Let's try to restate the same idea in a useful way. But first, let's show that the formal counterpart of the idea of containing "enough" ordinals with a property is the notion of stationary set.

Definition 3.29 (*Supremum*)

Given x a set of ordinals, the supremum of x , denoted $\sup(x)$, is the least upper bound of x .

$$\sup(x) = \bigcup x \quad (3.71)$$

Definition 3.30 (*Limit point*)

Given x , a set of ordinals and an ordinal α , we say that α is a limit point of x if $\sup(x \cap \alpha) = \alpha$

Definition 3.31 (*Set Unbounded in α*) Let α be an ordinal. We say that $x \subset \alpha$ is unbounded in α iff

$$\forall \beta \in \text{Ord}(\beta < \alpha \rightarrow \exists \gamma(\gamma \in x(\beta \leq \gamma < \alpha))) \quad (3.72)$$

In other words, κ is a mahlo cardinal if it is inaccessible and every club set in κ contains an inaccessible cardinal. This is exactly the notion of fixed-point reflection we were trying to show earlier.

[?]

Definition 3.32 *The following definitions are equivalent:*

- (i) κ is Mahlo
- (ii) κ is weakly Mahlo and strong limit
- (iii) The set $\{\lambda < \kappa : \lambda \text{ is inaccessible}\}$ is stationary in κ .
- (iv) Every normal function on κ has an inaccessible fixed point.

Proof. (i) \leftrightarrow (ii) Let κ_1 be a mahlo cardinal and let κ_2 be a strong limit weakly Mahlo cardinal. We know from the definitions that the set $\{\lambda < \kappa : \lambda \text{ is inaccessible}\}$ is stationary in both κ_1 and κ_2 , the only difference being that κ_1 is a strongly limit cardinal, but κ_2 would be limit from weak Mahloness, wasn't it for the fact that it is also strong limit. This eliminates the only difference between them and therefore κ_1 is also strong limit weakly Mahlo cardinal and κ_2 is Mahlo.

(i) \rightarrow (iii) We know that κ is uncountable, regular, strong limit and that the set $S = \{\lambda < \kappa : \lambda \text{ is regular}\}$ is stationary in κ . We want to prove that $S' = \{\lambda < \kappa : \lambda \text{ is inaccessible}\}$ is thus also stationary in κ .

Since stationary set intersects every club set in κ , let C be any such set. Let $D = \{\lambda < \kappa : \lambda \text{ is strong limit}\}$. D is a club set because TODO. Since intersection of less than κ club sets is a club set, $C \cap D \neq \emptyset$.

TODO proc $\lambda = S \cap C \cap D$ je inaccessible?

(iii) \rightarrow (iv)

TODO jak to dela Lévy?

(iv) \rightarrow (i)

TODO jak to dela Lévy?

range kazde normalni funkce je club v On. (nevadi ze On je trida?)

co treba lemma ze pevne body tvori taky club set

mozna by stacilo ze jsou unbounded, tedy kazda normalni funkce ma libovolne velke pevne body. \square

TODO obdoba pro α -Mahlo kardinaly?

TODO κ is hyper-Mahlo iff κ is inaccessible and the set $\{\lambda < \kappa : \lambda \text{ is Mahlo}\}$ is stationary in κ . to je to samy jako α -Mahlo, ne?

3.6 Indescribability

α -Mahlo are the extreme of regular fixed-point axioms, they are about as high as we can get via normal functions and stationary sets.

Let's try a different strategy. Remember how we said that (Regular, Limit and) various Large cardinals are in a way all determined by being unreachable by a specific process of creating bigger cardinals from already available ones? TODO indescribable – reflecting indescribability – we can't reach V by a Σ_1^1 formula, so there's some initial segment V_α that is also unreachable (we say indescribable) by the means of a ... formula

Let's recall complete reflection theorem first, consider the following:

For every sentence φ , there is a limit ordinal α such that $\varphi_\alpha^V \leftrightarrow \varphi$ (3.73)

We may also require that $\alpha < \beta$, where β is an arbitrary ordinal given.

For the exact definition of Π_n^m and Σ_n^m see ??

Definition 3.33 (Π_n^m -indescribable cardinal) We say that κ is Π_n^m -indescribable iff for any Π_n^m sentence φ such that $V_\kappa \models \varphi$ there is an $\alpha < \kappa$ such that $V_\alpha \models \varphi$

Definition 3.34 (Σ_n^m -indescribable cardinal) We say that κ is Σ_n^m -indescribable iff for any Σ_n^m sentence φ such that $V_\kappa \models \varphi$ there is an $\alpha < \kappa$ such that $V_\alpha \models \varphi$

Lemma 3.35 Let κ be a cardinal, the following holds for any $n \in \omega$. κ is Π_n^1 -indescribable iff κ is $\Sigma_n^1 + 1$ -indescribable

Proof. The forward direction is obvious, we can always add a spare quantifier over a type 2 variable to turn a Π_n^1 formula φ into a $\exists P\varphi$ which is thus a $\Sigma_n^1 + 1$ formula.²⁵

To prove the opposite direction, suppose that $V_\kappa \models \exists X\varphi(X)$ where X is a type 2 variable and φ is a Π_n^1 formula with one free variable of type 2. This means that there is a set $S \subseteq V_\kappa$ that is a witness of $\exists X\varphi(X)$, in other words, $\varphi(S)$ holds. We can replace every occurrence of X in φ by a new predicate symbol S , this allows us to say that κ is Π_n^1 -indescribable (with respect to $\langle V_\kappa, \in, R, S \rangle$).²⁶ \square

²⁵Note that unlike in previous sections, φ is now a sentence so we don't have to worry whether P is free in φ .

²⁶A different yet interesting approach is taken by Tate in ?. He states that for $n \geq 0$, a formula of order $\leq n$ is called a Π_0^n and a Σ_0^n formula. Then a Π_{m+1}^n is a formula of form $\forall Y\psi(Y)$ where ψ is a Σ_m^n formula and Y is a variable of type n . Finally, a Σ_{m+1}^n is the negation of a Π_m^n formula. So the above holds ad definitio.

875 The above lemma tells us that we as long as we stay in the realm of type
 876 1 and type 2 variables, we only need to classify indescribable cardinals with
 877 respect to Π_n^1 -indescribability.

878 **Theorem 3.36** *Let κ be an ordinal. The following are equivalent.*

- 879 (i) κ is inaccessible
 880 (ii) κ is Π_0^1 -indescribable.

881 Note that Π_0^1 formulas are those that contain zero unbound quantifiers
 882 over type-2 variables, they are in fact first-order formulas. We have already
 883 shown in ?? that there is no way to reach an inaccessible cardinal via first-
 884 order formulas in ZFC. We will now prove it again in for formal clarity.

885 *Proof.* TODO asi pridat alternativni definici nedosazitelnosti podle kan. 6.2?
 886 □

887 TODO nejaka veta ze kdyz jsou Π_0^1 -indescribable, jsou i Π_n^m -indescribable
 888 pro $m \leq 1, n \leq 0$? Drake? Obracene! Π_n^m -indescribable jsou zaroven Π_b^a -
 889 indescribable pro $a < m, b < n$.

890 The above theorem provides an easy way to show that every following
 891 large cardinal is also in inaccessible cardinal²⁷.

892 **Definition 3.37** (*Extension property*) *We say that a cardinal κ has the ex-*
 893 *tension property iff for any $R \subseteq V_\kappa$ there is a transitive set $X \neq V_\kappa$ and an*
 894 *$S \subseteq X$ such that $\langle V_\kappa, \in, R \rangle \prec \langle X, \in, S \rangle$*

895 **Definition 3.38** (*Weakly compact cardinal*)

896 *We say that a cardinal κ is weakly compact iff it has the extension property.*

897 The above definitions are equivalent

898 **Theorem 3.39** *the following are equivalent:*

- 899
 900 (i) κ is Weakly compact.
 901 (ii) κ is Π_1^1 -indescribable.

902 For a proof, see [?, ?]

903 TODO asi nekde bude meritelny kardinal

904 TODO viz Drake, Ch.9 par. 3 – tam se rika ze kdyz κ je meritelny
 905 kardinal, pak je κ Π_1^2 -nepopsatelný kardinal

²⁷That is because Π_0^1 formulas are included Π_n^m formulas for $m \leq 1, n \leq 0$.

3.7 Bernays–Gödel Set Theory

TODO Jech str. 70 [?]

TODO popis

Definition 3.40 (*Gödel–Bernay set theory*)

(i) extensionality for sets

$$\forall a \forall b [\forall x (x \in a \leftrightarrow x \in b) \rightarrow a = b] \quad (3.74)$$

(ii) pairing for sets

$$\forall x \forall y \exists z \forall w [w \in z \leftrightarrow (w = x \vee w = y)] \quad (3.75)$$

(iii) union for sets

$$\forall a \exists b \forall c [c \in b \leftrightarrow \exists d (c \in d \wedge d \in a)] \quad (3.76)$$

(iv) powers for sets

$$\forall a \exists p \forall b [b \in p \leftrightarrow (c \in b \rightarrow c \in a)] \quad (3.77)$$

(v) infinity for sets

$$\text{There is an inductive set.} \quad (3.78)$$

(vi) Extensionality for classes

$$\forall x (x \in A \leftrightarrow x \in B) \rightarrow A = B \quad (3.79)$$

(vii) Foundation for classes

$$\text{Each non-empty class is disjoint from each of its elements.} \quad (3.80)$$

(viii) Limitation of size for sets

$$\text{For any class } C \text{ a set } x \text{ such that } x=C \text{ exists iff} \quad (3.81)$$

$$\text{there is no bijection between } C \text{ and the class } V \text{ of all sets} \quad (3.82)$$

(ix) Comprehension schema for classes

$$\text{For an arbitrary formula } \varphi \text{ with no quantifiers over classes, there is a class } A \text{ such that } \forall x \varphi(x) \quad (3.83)$$

The first five axioms are identical to axioms in ZF.

Comprehension schema tells us that proper classes are basically first-order predicates. **TODO**

3.8 The Constructible Universe

The constructible universe, denoted L , is a cumulative hierarchy of sets, presented by Kurt Gödel in his 1938 paper *The Consistency of the Axiom of Choice and of the Generalised Continuum Hypothesis*. For a technical description, see below. Assertion of their equality, $V = L$, is called the *axiom of constructibility*. The axiom implies GCH and therefore also AC and contradicts the existence of some of the large cardinals, our goal is to decide whether those introduced earlier are among them.

On order to formally establish this class, we need to formalize the notion of definability first.

Definition 3.41 We say that a set X is definable over a model $\langle M, \in \rangle$ if there is a first-order formula φ together with parameters $u_1, \dots, u_n \in M$ such that

$$X = \{x : x \in M \ \& \ \langle M, \in \rangle \models \varphi(x, u_1, \dots, u_n)\} \quad (3.84)$$

Definition 3.42 (Sets definable in M)

The following is a set of all definable subsets of a given set M , denoted $Def(M)$.

$$Def(M) = \{\{y : x \in M \wedge \langle M, \in \rangle \models \varphi(y, u_1, \dots, u_n)\} \mid \varphi \text{ is a first-order formula, } u_1, \dots, u_n \in M\} \quad (3.85)$$

Now we can recursively build L .

Definition 3.43 (The Constructible universe)

(i)

$$L_0 := \emptyset \quad (3.86)$$

(ii)

$$L_{\alpha+1} := Def(L_\alpha) \quad (3.87)$$

(iii)

$$L_\lambda = \bigcup_{\alpha < \lambda} L_\alpha \text{ If } \lambda \text{ is a limit ordinal} \quad (3.88)$$

(iv)

$$L = \bigcup_{\alpha \in Ord} L_\alpha \quad (3.89)$$

Note that while L bears very close resemblance to V , the difference is, that in every successor step of constructing V , we take every subset of V_α to be $V_{\alpha+1}$, whereas $L_{\alpha+1}$ consists only of definable subsets of L_α . Also not that L is transitive.

TODO:

948 **Lemma 3.44** *Ord* $\in L$

949 **Lemma 3.45** *L* is well-ordered.

950 *TODO !!*

951 **Theorem 3.46** Let *L* be as in ??.

$$L \models \text{ZFC} \quad (3.90)$$

952 *Proof.* *TODO !!!* (strucne) vit [?, ?]

953 (i) *Extensionality* (see ??):

954 *Extensionality* holds in *L* because Δ_0 formulas are absolute in transitive
955 classes by ??, *Extensionality* is Δ_0 and *L* is transitive.

956 (ii) *Foundation* (see ??)

957 Take a non-empty set *X*. Let *x* $\in X$ be a set such that $X \cap x = \emptyset$. *x*
958 is therefore defined by the formula $\varphi(x, y) = (x \cap y = \emptyset)$, so $x \in L$. φ
959 is Δ_0 and therefore holds in *L* by ??.

960 (iii) *Pairing* (see ??)

961 Since *Pairin* is also Δ_0 , it holds in *L* by the same argument as *Exten-*
962 *sionality* does by ??.

963 (iv) *Union* (see ??)

964 *Union* is also Δ_0 , see *Extensionality* and ??.

965 (v) *Power Set* (see ??)

966 *Power Set* also holds by ??.

967 (vi) *Infinity* (see ??)

968 $\omega \in L$ by ??

969 (vii) *Specification* (see ??)

970 .

971 (viii) *Replacement* (see ??)

972 .

973 (ix) *Choice* (see ??)

974 .

975

□

976 **Definition 3.47** *Constructibility*

977 $L = V$

978 The following are a few interesting results that we won't prove but refer
979 interested reader to appropriate resources instead.

980 **Definition 3.48** (*GCH*)

981 The following is called the *Generalised Continuum Hypothesis*, abbreviated
982 as *GCH*. It is an independent statement in *ZFC*.

$$\text{GCH iff } \aleph_{\alpha+1} = 2^{\aleph_\alpha} \text{ for every ordinal } \alpha \quad (3.91)$$

Theorem 3.49

$$(L = V) \rightarrow GCH \quad (3.92)$$

983 This is proven in cite{neco}

984 TODO zduvodneni

985

986 TODO kratka diskuse jestli refl implikuje transcenci na L - polemika,
987 nazor - V=L a slaba kompaktnost a dalsi

988 TODO neco jako ze meritelny kardinal je nepopsatelny vys nez je hierar-
989 chie vseh tvrzeni o L?

990 3.9 Measurable Cardinal

991 **Definition 3.50** (*Ultrafilter*)

992 Given a set X , we say $U \subset \mathcal{P}(X)$ is an ultrafilter iff all of the following
993 hold:

994 (i) $\emptyset \notin U$

995 (ii) $\forall a, b \subset X \text{ \& } a \subset b \text{ \& } a \in U \rightarrow b \in U$

996 (iii) $\forall a, b \in U (a \cap b) \in U$

997 (iv) $\forall a (a \subset X \rightarrow (a \in U \vee (X \setminus a) \in U))$

998 TODO refaktorizovat fle If A is a subset of X , then either A or $X \setminus A$ is
999 an element of U . (Note: axioms 1 and 3 imply that A and $X \setminus A$ cannot
1000 both be elements of U .)

1001 **Definition 3.51** (κ -complete ultrafilter)

1002 TODO

1003 **Definition 3.52** (*non-principal ultrafilter*)

1004 TODO

1005 **Definition 3.53** (*Measurable Cardinal*)

1006 Let κ be a cardinal. We say κ is a measurable cardinal iff it is an uncountable
1007 cardinal with a κ -complete, non-principal ultrafilter.

1008 **4 Conclusion**

1009 TODO na konec

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