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2 Katedra logiky

3 MIKLUÁŠ MRVA

4 REFLECTION PRINCIPLES AND LARGE
5 CARDINALS
6 Bakalářská práce

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¹⁰ Prohlašuji, že jsem bakalářskou práci vypracoval samostatně a že jsem uvedl
¹¹ všechny použité prameny a literaturu.

¹² V Praze 14. dubna 2015

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Abstract

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Práce zkoumá vztah tzv. principů reflexe a velkých kardinálů. Lévy ukázal, že v ZFC platí tzv. věta o reflexi a dokonce, že věta o reflexi je ekvivalentní schématu nahrazení a axiomu nekonečna nad teorií ZFC bez axiomu nekonečna a schématu nahrazení. Tedy lze na větu o reflexi pohlížet jako na svého druhu axiom nekonečna. Práce zkoumá do jaké míry a jakým způsobem lze větu o reflexi zobecnit a jaký to má vliv na existenci tzv. velkých kardinálů. Práce definuje nedosažitelné, Mahlovy a nepopsatelné kardinály a ukáže, jak je lze zavést pomocí reflexe. Přirozenou limitou kardinálů získaných reflexí jsou kardinály nekonzistentní s L. Práce nabídne intuitivní zdůvodnění, proč tomu tak je.

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Abstract

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This thesis aims to examine relations between so called "Reflection Principles" and Large cardinals. Lévy has shown that Reflection Theorem is a sound theorem of ZFC and it is equivalent to Replacement Scheme and the Axiom of Infinity. From this point of view, Reflection theorem can be seen a specific version of an Axiom of Infinity. This paper aims to examine the Reflection Principle and its generalisations with respect to existence of Large Cardinals. This thesis will establish Inaccessible, Mahlo and Indescribable cardinals and their definition via reflection. A natural limit of Large Cardinals obtained via reflection are cardinals inconsistent with L. The thesis will offer an intuitive explanation of why this is the case.

38 **Contents**

1 Introduction

1.1 Motivation and Origin

The Universe of sets cannot be uniquely characterized (i. e. distinguished from all its initial elements) by any internal structural property of the membership relation in it, which is expressible in any logic of finite or transfinite type, including infinitary logics of any cardinal order.

— Kurt Gödel [?]

To understand why we need reflection in the first place, let's think about infinity for a moment. In the intuitive sense, infinity is an upper limit of all numbers. But for centuries, this was merely a philosophical concept, closely bound to religious and metaphysical way of thinking, considered separate from numbers used for calculations or geometry. It was a rather vague concept. In ancient Greece, Aristotle's response to famous Zeno's paradoxes introduced the distinction between actual and potential infinity. He argued, that potential infinity is (in today's words) well defined, as opposed to actual infinity, which remained a vague incoherent concept. He didn't think it's possible for infinity to inhabit a bounded place in space or time, rejecting Zeno's thought experiments as a whole. Aristotle's thoughts shaped western thinking partly due to Aquinas, who himself believed actual infinity to be more of a metaphysical concept for describing God than a mathematical property attributed to any other entity. In his *Summa Theologica*¹ he argues:

A geometrician does not need to assume a line actually infinite, but takes some actually finite line, from which he subtracts whatever he finds necessary; which line he calls infinite.

Less than hundred years later, Gregory of Rimini wrote

If God can endlessly add a cubic foot to a stone—which He can—then He can create an infinitely big stone. For He need only add one cubic foot at some time, another half an hour later, another a quarter of an hour later than that, and so on ad infinitum. He would then have before Him an infinite stone at the end of the hour.

Which is basically a Zeno's Paradox made plausible with God being the actor. In contrast to Aquinas' position, Gregory of Rimini theoretically constructs an object with actual infinite magnitude that is essentially different from God.

¹Part I, Question 7, Article 3, Reply to Objection 1

Even later, in the 17th century, pushing the property of infiniteness from the Creator to his creation, Nature, Leibniz wrote to Foucher in 1662:

I am so in favor of the actual infinite that instead of admitting that Nature abhors it, as is commonly said, I hold that Nature makes frequent use of it everywhere, in order to show more effectively the perfections of its Author. Thus I believe that there is no part of matter which is not, I do not say divisible, but actually divided; and consequently the least particle ought to be considered as a world full of an infinity of different creatures.

But even though he used potential infinity in what would become foundations of modern Calculus and argued for actual infinity in Nature, Leibniz refused the existence of an infinite, thinking that Galileo's Paradoxon² is in fact a contradiction. The so called Galileo's Paradoxon is an observation Galileo Galilei made in his final book "Discourses and Mathematical Demonstrations Relating to Two New Sciences". He states that if all numbers are either squares and non-squares, there seem to be less squares than there is all numbers. On the other hand, every number can be squared and every square has its square root. Therefore, there seem to be as many squares as there are all numbers. Galileo concludes, that the idea of comparing sizes makes sense only in the finite realm.

Salviati: So far as I see we can only infer that the totality of all numbers is infinite, that the number of squares is infinite, and that the number of their roots is infinite; neither is the number of squares less than the totality of all the numbers, nor the latter greater than the former; and finally the attributes "equal," "greater," and "less," are not applicable to infinite, but only to finite, quantities. When therefore Simplicio introduces several lines of different lengths and asks me how it is possible that the longer ones do not contain more points than the shorter, I answer him that one line does not contain more or less or just as many points as another, but that each line contains an infinite number.

Leibniz insists in part being smaller than the whole saying

Among numbers there are infinite roots, infinite squares, infinite cubes. Moreover, there are as many roots as numbers. And there are as many squares as roots. Therefore there are as many squares

²zneni galileova paradoxu

as numbers, that is to say, there are as many square numbers as there are numbers in the universe. Which is impossible. Hence it follows either that in the infinite the whole is not greater than the part, which is the opinion of Galileo and Gregory of St. Vincent, and which I cannot accept; or that infinity itself is nothing, i.e. that it is not one and not a whole.

TODO Hegel–strucne?

TODO Cantor

TODO mene teologie, vice matematiky

TODO definovat pojmy (trida etc)

TODO neni V v nejakem smyslu porad potencialni nekonecno, zatimco mnoziny vetsi nez omega jsou aktualni? nebo jsou potencialni protoze se stavaji pres indukci, od spoda.

In his work, he defined transfinite numbers to extend existing natural number structure so it contains more objects that behave like natural numbers and are based on an object (rather a meta-object) that doesn't explicitly exist in the structure, but is closely related to it. This is the first instance of reflection. This paper will focus on taking this principle a step further, extending Cantor's (or Zermelo–Fraenkel's, to be more precise) universe so it includes objects so big, they could be considered the universe itself, in a certain sense.

TODO dal asi smazat

The original idea behind reflection principles probably comes from what could be informally called “universality of the universe”. The effort to precisely describe the universe of sets was natural and could be regarded as one of the impulses for formalization of naive set theory. If we try to express the universe as a set $\{x|x = x\}$, a paradox appears, because either our set is contained in itself and therefore is contained in a set (itself again), which contradicts the intuitive notion of a universe that contains everything but is not contained itself.

TODO ???

If there is an object containing all sets, it must not be a set itself. The notion of class seems inevitable. Either directly the ways for example the Bernays–Gödel set theory, we will also discuss later in this paper, does in, or on a meta-level like the Zermelo–Fraenkel set theory, that doesn't refer to them in the axioms but often works with the notion of a universal class. duet Another obstacle of constructing a set of all sets comes from Georg Cantor, who proved that the set of all subsets of a set (let x be the set and $\mathcal{P}(x)$ its powerset) is strictly larger than x . That would turn every aspiration to

finally establish an universal set into a contradictory infinite regression.³ We will use V to denote the class of all sets. From previous thoughts we can easily argue, that it is impossible to construct a property that holds for V and no set and is neither paradoxical like $\{x|x = x\}$ nor trivial. Previous observation can be transposed to a rather naive formulation of the reflection principle:

Reflection Any property which holds in V already holds in some initial segment of V .

To avoid vagueness of the term "property", we could informally reformulate the above statement into a schema:

For every first-order formula⁴ φ holds in $V \leftrightarrow \varphi$ holds in some initial segment of V .

Interested reader should note that this is a theorem scheme rather than a single theorem.⁵

1.2 A few historical remarks on reflection

Reflection made its first in set-theoretical appearance in Gödel's proof of GCH in L (citace Kanamori ? Lévy and set theory), but it was around even earlier as a concept. Gödel himself regarded it as very close to Russel's reducibility axiom (an earlier equivalent of the axiom schema of Zermelo's separation). Richard Montague then studied reflection properties as a tool for verifying that Replacement is not finitely axiomatizable (citace?). a few years later Lévy proved in [?] the equivalence of reflection with Axiom of infinity together with Replacement in proof we shall examine closely in chapter 2.

TODO co dal? recent results?

³An intuitive analogy of this *reductio ad infinitum* is the status of ω , which was originally thought to be an unreachable absolute, only to become starting point of Cantor's hierarchy of sets growing beyond all boundaries around the end of the 19th century

⁴this also works for finite sets of formulas [?, p. 168]

⁵If there were a single theorem stating "for any formula φ that holds in V there is an initial segment of V where φ also holds", we would obtain the following contradiction with the second Gödel's theorem: In ZFC, any finite group of axioms of ZFC holds in some initial segment of the universe. If we take the largest of those initial segments it is still strictly smaller than the universe and thus we have, via compactness, constructed a model of ZFC within ZFC. That is, of course a harsh contradiction. This also leads to an elegant way to prove that ZFC is not finitely axiomatizable.

1.3 Reflection in Platonism and Structuralism

TODO cite "reflection in a structuralist setting"

TODO veci o tom, ze reflexe je ok protoze reflektuje veci ktere objektivne plati, protoze plati pro V ...

TODO souvislost s kompaktnosti, hranice formalich systemu nebo alespon ZFC

1.4 Notation and Terminology

1.4.1 The Language of Set Theory

We are about to define basic set-theoretical terminology on which the rest of this thesis will be built. For Chapter 2, the underlying theory will be the *Zermelo – Fraenkel* set theory with the Axiom of Choice (ZFC), a first-order set theory in the language $\mathcal{L} = \{=, \in\}$, which will be sometimes referred to as *the language of set theory*. In Chapter 3⁶, we shall always make it clear whether we are in first-order ZFC or second-order ZFC₂, which will be precisely defined later in this chapter. When in second-order theory, we will usually denote type 1 variables, which are elements of the domain of discourse⁷ by lower-case letters, mostly $u, v, w, x, y, z, p_1, p_2, p_3, \dots$ while type 2 variables, which represent n -ary relations of the domain of discourse for any natural number n , are usually denoted by upper-case letters A, B, C, X, Y, Z . Note that those may be used both as relations and functions, see the definition of a function below.⁸

TODO uppercase M is a set!

TODO "M is a limit ordinal" je ve skutecnosti formule, nekam to sem napis!

The informal notions of *class* and *property* will be used throughout this thesis. They both represent formulas with respect to the domain of discourse. If $\varphi(x, p_1, \dots, p_n)$ is a formula in the language of set theory, we call

$$A = \{x : \varphi(x, p_1, \dots, p_n)\} \quad (1.1)$$

a class of all sets satisfying $\varphi(x, p_1, \dots, p_n)$ in a sense that

$$x \in A \leftrightarrow \varphi(x, p_1, \dots, p_n) \quad (1.2)$$

One can easily define for classes A, B the operations like $A \cap B, A \cup B, A \setminus C, \bigcup A$, but it is elementary and we won't do it here, see the first part of

⁶TODO bude jich vic? Chapter 4 taky?

⁷co je "domain of discourse"?

⁸TODO ref?

[?] for technical details. The following axioms are the tools by which decide whether particular classes are in fact sets. A class that fails to be considered a set is called a *proper class*.

1.4.2 The Axioms

Definition 1.1 (*The existence of a set*)

$$\exists x(x = x) \quad (1.3)$$

The above axiom is usually not used because it can be deduced from the axiom of *Infinity* (see below), but since we will be using set theories that omit *Infinity*, this will be useful.

Definition 1.2 (*Extensionality*)

$$\forall x, y(\forall z(z \in x \leftrightarrow z \in y) \leftrightarrow x = y) \quad (1.4)$$

Definition 1.3 (*Specification*)

The following is a schema for every first-order formula $\varphi(x, p_1, \dots, p_n)$ with no free variables other than x, p_1, \dots, p_n .

$$\forall x, p_1, \dots, p_n \exists y \forall z(z \in y \leftrightarrow (z \in x \ \& \ \varphi(z, p_1, \dots, p_n))) \quad (1.5)$$

We will now provide two definitions that are not axioms, but will be helpful in establishing some of the other axioms in a more intuitive way.

Definition 1.4 ($x \subseteq y, x \subset y$)

$$x \subseteq y \leftrightarrow \forall z(z \in x \rightarrow z \in y) \quad (1.6)$$

$$x \subset y \leftrightarrow x \subseteq y \ \& \ x \neq y \quad (1.7)$$

Definition 1.5 (*Empty set*)

$$\emptyset \stackrel{\text{def}}{=} \{x : x \neq x\} \quad (1.8)$$

To make sure that \emptyset is a set, note that there exists at least one set y from 1.1, then consider the following alternative definition.

$$\emptyset' \stackrel{\text{def}}{=} \{x : \varphi(x) \ \& \ x \in y\} \text{ where } y \ \varphi \text{ is the formula } "x \neq x". \quad (1.9)$$

It should be clear that $\emptyset' = \emptyset$.⁹

Now we can introduce more axioms.

⁹For details, see page 8 in [?].

228 **Definition 1.6** (*Foundation*)

$$\forall x(x \neq \emptyset \rightarrow \exists(y \in x)(\forall z \neg(z \in y \ \& \ z \in x))) \quad (1.10)$$

229 **Definition 1.7** (*Pairing*)

$$\forall x, y \exists z \forall q (q \in z \leftrightarrow q \in x \vee q \in y) \quad (1.11)$$

230 **Definition 1.8** (*Union*)

$$\forall x \exists y \forall z (z \in x \leftrightarrow \exists q (z \in q \ \& \ q \in x)) \quad (1.12)$$

231 **Definition 1.9** (*Powerset*)

$$\forall x \exists y \forall z (z \subseteq x \leftrightarrow z \in y) \quad (1.13)$$

232 **Definition 1.10** (*Infinity*)

$$\exists x (\forall y (y \in x) \rightarrow (y \cup \{y\} \in x)) \quad (1.14)$$

233 Let us introduce a few more definitions that will make the two remaining
234 axioms more comprehensible.

235 **Definition 1.11** (*Function*)

236 Given arbitrary first-order formula $\varphi(x, y, p_1, \dots, p_n)$, we say that φ is a func-
237 tion iff

$$\forall x, y, z, p_1, \dots, p_n (\varphi(x, y, p_1, \dots, p_n) \ \& \ \varphi(x, z, p_1, \dots, p_n) \rightarrow y = z) \quad (1.15)$$

238 When a $\varphi(x, y)$ is a function, we also write the following:

$$f(x) = y \leftrightarrow \varphi(x, y) \quad (1.16)$$

239 Note that this f is in fact a formula

240 TODO $f = \{(x, y) : \varphi(x, y)\}$!!! f muze byt mnozina i trida! ¹⁰

241 **Definition 1.12** (*Dom(f)*)

242 Let f be a function. We read the following as " $Dom(f)$ is the domain of f ".

$$Dom(f) \stackrel{\text{def}}{=} \{x : \exists y (f(x) = y)\} \quad (1.17)$$

243 We say " f is a function on A ", A being a class, if $A = dom(f)$.

¹⁰This can also be done for φ s with more than two free variables by either setting $f(x, p_1, \dots, p_n) = y \leftrightarrow \varphi(x, y, p_1, \dots, p_n)$ or saying that φ codes more functions, determined by the various parameters, so $f(x) = y \leftrightarrow \varphi(x, y, t_1, \dots, t_n)$ for given terms t_1, \dots, t_n .

244 **Definition 1.13** (*Rng(f)*)

245 *Let f be a function. We read the following as " $Rng(f)$ is the range of f ".*

$$Rng(f) \stackrel{\text{def}}{=} \{x : \exists y(f(x) = y)\} \quad (1.18)$$

246 We say that f is a function into A , A being a class, if $rng(f) \subseteq A$.

247 Note that $Dom(f)$ and $Rng(f)$ are not definitions in a strict sense, they
 248 are in fact definition schemas that yield definitions for every function f given.
 249 Also note that they can be easily modified for φ instead of f , with the only
 250 difference that then it is defined only for those φ s that are functions.

251 **Definition 1.14** (*Powerset*)

252 *TODO*

253 And now for the axioms.

254 **Definition 1.15** (*Replacement*)

255 *The following is a schema for every first-order formula $\varphi(x, p_1, \dots, p_n)$ with*
 256 *no free variables other than x, p_1, \dots, p_n .*

$$"\varphi \text{ is a function}" \rightarrow \forall x \exists y \forall z (z \in y \leftrightarrow (\exists q \in x)(\varphi(x, y, p_1, \dots, p_n))) \quad (1.19)$$

257 **Definition 1.16** (*Choice*)

258 *This is also a schema. For every A , a family of non-empty sets¹¹, such that*
 259 *$\emptyset \notin S$, there is a function f such that for every $x \in A$*

$$f(x) \in x \quad (1.20)$$

260 We will refer the axioms by their name, written in italic type, e.g. *Founda-*
 261 *tion* refers to the Axiom of Foundation. Now we need to define some basic
 262 set theories to be used in the article. There will be others introduced in Chap-
 263 ter 3, but those will usually be defined just by appending additional axioms
 264 or schemata to one of the following.

265 **Definition 1.17** (**S**)

266 *We call **S** a set theory with the following axioms:*

- 267 (i) Existence of a set (see 1.1)
- 268 (ii) Extensionality (see 1.2)
- 269 (iii) Specification (see 1.3)
- 270 (iv) Foundation (see 1.6)
- 271 (v) Pairing (see 1.7)

¹¹We say a class A is a "family of non-empty sets" iff there is B such that $A \subseteq \mathcal{P}(B)$

272 (vi) Union (see 1.8)

273 (vii) Powerset (see 1.9)

274 **Definition 1.18** (ZF)

275 We call ZF a set theory that contains all the axioms of the theory S^{12} in
276 addition to the following

277 (i) Replacement schema (see 1.15)

278 (ii) Infinity (see 1.10)

279 **Definition 1.19** (ZFC)

280 ZFC is a theory that contains all the axioms of ZF plus Choice (1.16).

281

282 1.4.3 The Transitive Universe

283 **Definition 1.20** (Transitive class)

284 We say a class A is transitive iff

$$\forall x(x \in A \rightarrow x \subseteq A) \quad (1.21)$$

285 **Definition 1.21** Well Ordered Class A class A is said to be well ordered by
286 \in iff the following hold:

287 (i) $(\forall x \in A)(x \not\in x)$ (Antireflexivity)

288 (ii) $(\forall x, y, z \in A)(x \in y \ \& \ y \in z \rightarrow x \in z)$ (Transitivity)

289 (iii) $(\forall x, y \in A)(x = y \vee x \in y \vee y \in x)$ (Linearity)

290 (iv) $(\forall x)(x \subseteq A \ \& \ x \neq \emptyset \rightarrow (\exists y \in x)(\forall z \in x)(z = y \vee z \in y))$

291 **Definition 1.22** (Ordinal number)

292 A set x is said to be an ordinal number, also known as an ordinal, if it is
293 transitive and well-ordered by \in .

294 For the sake of brevity, we usually just say " x is an ordinal". Note that " x
295 is an ordinal" is a well-defined formula, since 1.20 is a formula and 1.21 is
296 in fact a conjunction of four formulas. Ordinals will be usually denoted by
297 lower case greek letters, starting from the beginning: $\alpha, \beta, \gamma, \dots$. Given two
298 different ordinals α, β , we will write $\alpha < \beta$ for $\alpha \in \beta$, see [?] Lemma 2.11 for
299 technical details.

¹²With the exception of *Existence of a set*

300 **Definition 1.23** (*Successor Ordinal*)

301 Consider the following operation

$$\beta + 1 \stackrel{\text{def}}{=} \beta \cup \{\beta\} \quad (1.22)$$

302 An ordinal α is called a successor ordinal iff there is an ordinal β , such that
 303 $\alpha = \beta + 1$

304 **Definition 1.24** (*Limit Ordinal*)

305 A non-zero ordinal α ¹³ is called a limit ordinal iff it is not a successor ordinal.

306 **Definition 1.25** (*Ord*)

307 The class of all ordinal numbers, which we will denote Ord ¹⁴ be the following
 308 class:

$$\text{Ord} \stackrel{\text{def}}{=} \{x : x \text{ is an ordinal}\} \quad (1.23)$$

309 The following construction will be often referred to as the *Von Neumann's*
 310 *Hierarchy*, sometimes also the *Von Neumann's Universe*.

311 **Definition 1.26** (*Von Neumann's Hierarchy*)

312 The Von Neumann's Hierarchy is a collection of sets indexed by elements of
 313 Ord , defined recursively in the following way:

(i)

$$V_0 = \emptyset \quad (1.24)$$

(ii)

$$V_{\alpha+1} = \mathcal{P}(V_\alpha) \text{ for any ordinal } \alpha \quad (1.25)$$

(iii)

$$V_\lambda = \bigcup_{\beta < \lambda} V_\beta \text{ for a limit ordinal } \lambda \quad (1.26)$$

314 **Definition 1.27** (*Rank*)

315 Given a set x , we say that the rank of x (written as $\text{rank}(x)$) is the least
 316 ordinal α such that

$$x \in V_{\alpha+1} \quad (1.27)$$

317 Due to *Regularity*, every set has a rank.¹⁵

318 **Definition 1.28** (ω)

319

$$\omega \stackrel{\text{def}}{=} \bigcap \{x : x \text{ is a limit ordinal}\} \quad (1.28)$$

320

¹³ $\alpha \neq \emptyset$

¹⁴It is sometimes denoted On , but we will stick to the notation in [?]

¹⁵See chapter 6 of [?] for details.

1.4.4 Cardinal Numbers

Definition 1.29 (Cardinality)

Given a set x , let the cardinality of x , written $|x|$, be defined as the smallest ordinal number such that there is an injective mapping from x to α .

For formal details as well as why every set can be well-ordered assuming Choice, see [?].

Definition 1.30 (Aleph function)

Let ω be the set defined by ???. We will recursively define the function \aleph for all ordinals.

- (i) $\aleph_0 = \omega$
- (ii) $\aleph_{\alpha+1}$ is the least cardinal larger than \aleph_α ¹⁶
- (iii) $\aleph_\lambda = \bigcup_{\beta < \lambda} \aleph_\beta$ for a limit ordinal λ

Definition 1.31 (Cardinal number)

We say a set x is a cardinal number, usually called a cardinal, if either $x \in \omega$. Cardinals will be notated by lower-case greek letters starting from $\kappa, \lambda, \mu, \dots$ ¹⁷.

Definition 1.32 (Cofinality)

Let λ be a limit ordinal. The cofinality of λ , written $cf(\lambda)$, is the least limit ordinal α such that there is an increasing α -sequence¹⁸ $\langle \lambda_\beta : \beta < \alpha \rangle$ with $\lim_{\beta \rightarrow \alpha} \lambda_\beta = \lambda$.

Definition 1.33 (Limit Cardinal)

We say that a cardinal κ is a limit cardinal if

$$(\exists \alpha \in Ord)(\kappa = \aleph_\alpha) \quad (1.29)$$

Definition 1.34 (Strong Limit Cardinal)

We say that an ordinal κ is a strong limit cardinal if it is a limit cardinal and

$$\forall \alpha (\alpha \in \kappa \rightarrow \mathcal{P}(\alpha) \in \kappa) \quad (1.30)$$

Definition 1.35 (Generalised Continuum Hypothesis)

$$\aleph_{\alpha+1} = 2^{\aleph_\alpha} \quad (1.31)$$

If *GCH* holds (for example in Gödel's *L*, see chapter 3), the notions of a limit cardinal and a strong limit cardinal are equivalent.

¹⁶"The least cardinal larger than \aleph_α " is sometimes notated as \aleph_α^+

¹⁷ λ is also sometimes used for limit ordinals, the distinction should be clear from the context.

¹⁸TODO def α -sequence

1.4.5 Relativisation and Absoluteness

Definition 1.36 (Relativization)

Let M be a class, R a binary relation on M and let $\varphi(p_1, \dots, p_n)$ be a first-order formula with n parameters. The relativization of φ to M and R is the formula, written as $\varphi^{M,R}(p_1, \dots, p_n)$, defined in the following inductive manner:

- (i) $(x \in y)^{M,R} \leftrightarrow R(x, y)$
- (ii) $(x = y)^{M,R} \leftrightarrow x = y$
- (iii) $(\neg \varphi)^{M,R} \leftrightarrow \neg \varphi^{M,R}$
- (iv) $(\varphi \ \& \ \psi)^{M,R} \leftrightarrow \varphi^{M,R} \ \& \ \psi^{M,R}$
- (v) $(\exists x \varphi)^{M,R} \leftrightarrow (\exists x \in M) \varphi^{M,R}$

Definition 1.37 (Absoluteness) Given a transitive class M , we say a formula φ is absolute in M if for all $p_1, \dots, p_n \in M$

$$\varphi^M(p_1, \dots, p_n) \leftrightarrow \varphi(p_1, \dots, p_n) \quad (1.32)$$

Definition 1.38 (Hierarchy of first-order formulas) A first-order formula φ is Δ_0 iff any of the following holds for φ

1. φ contains no quantifiers
2. y is a set, ψ is a Δ_0 formula, and φ is either $(\exists x \in y) \psi(y)$ or $(\forall x \in y) \psi(y)$.
3. ψ_1, ψ_2 are Δ_0 formulas and φ is any of the following: $\psi_1 \vee \psi_2$, $\psi_1 \ \& \ \psi_2$, $\psi_1 \rightarrow \psi_2$, $\neg \psi_2$,

1.4.6 More functions

TODO def $f : Ord \rightarrow Ord$, asi u powersetu.

Definition 1.39 (Strictly increasing function)

A function $f : Ord \rightarrow Ord$ is said to be strictly increasing iff

$$\forall \alpha, \beta \in Ord (\alpha < \beta \rightarrow f(\alpha) < f(\beta)). \quad (1.33)$$

Definition 1.40 (Continuous function)

A function $f : Ord \rightarrow Ord$ is said to be continuous iff

$$\alpha \text{ is limit} \rightarrow f(\alpha) = \bigcup_{\alpha < \lambda} f(\alpha). \quad (1.34)$$

Definition 1.41 (Normal function)

A function $f : Ord \rightarrow Ord$ is said to be normal if it is strictly increasing and continuous.

380 **Definition 1.42** *Fixed point*

381 We say α is a fixed point of ordinal function f if $\alpha = f(\alpha)$.

382 **Definition 1.43** *(Unbounded class)*

383 We say a class A is unbounded if

$$\forall x(\exists y \in A)(x < y) \quad (1.35)$$

384 **Definition 1.44** *(Class Unbounded in α)*

385 Let α be a limit ordinal. We say that $x \subset \alpha$ is unbounded in α iff

$$\forall \beta \in \text{Ord}(\beta < \alpha \rightarrow \exists \gamma(\gamma \in x(\beta \leq \gamma < \alpha))) \quad (1.36)$$

386 **Definition 1.45** *(Closed class)*

387 For a limit ordinal $A \subseteq \lambda$, we say that A is closed in λ iff for every non-zero
388 ordinal $\alpha < \lambda$: if $A \cap \alpha$ is unbounded in α then $\alpha \in A$.

389 **Definition 1.46** *(Club set)*

390 For a regular uncountable cardinal κ , a set $x \subset \kappa$ is a closed unbounded
391 subset, abbreviated as a club set, iff x is both closed and unbounded in κ .

392 **Definition 1.47** *(Stationary set)*

393 For a regular uncountable cardinal κ , we say a set $A \subset \kappa$ is stationary in
394 κ iff it intersects every club subset of κ .

395 1.4.7 Structure, Substructure and Embedding

396 Structures will be denoted $\langle M, \in, R \rangle$ where M is a domain, \in stands for the
397 standard membership relation, it is assumed to be restricted to the domain¹⁹,
398 $R \subseteq M$ is a relation on the domain. When R is not needed, we may as well
399 only write M instead of $\langle M, \in \rangle$.

400 **Definition 1.48** *(Elementary Embedding)*

401 Given the structures $\langle M_1, \in, R \rangle$, $\langle M_2, \in, R \rangle$ and a one-to-one function $j :$
402 $M_1 \rightarrow M_2$, we say j is an elementary embedding of M_1 into M_2 , we write
403 $j : M_1 \prec M_2$, when the following holds for every formula $\varphi(p_1, \dots, p_n)$ and
404 every $p_1, \dots, p_n \in M_1$:

$$\langle M_1, \in, R \rangle \models \varphi(p_1, \dots, p_n) \leftrightarrow \langle M_2, \in, R \rangle \models \varphi(j(p_1), \dots, j(p_n)) \quad (1.37)$$

¹⁹To be totally correct, we should write $\langle M, \in \cap M \times M, R \rangle$

405 **Definition 1.49** (*Elementary Substructure*)
 406 Given the structures $\langle M_1, \in, R \rangle$, $\langle M_2, \in, R \rangle$ and a one-to-one function $j : M_1 \rightarrow M_2$ such that $j : M_1 \prec M_2$, we say that M_1 is an elementary sub-
 407 structure of M_2 , denoted as $M_1 \prec M_2$, iff j is an identity on M_1 . In other
 408 words
 409

$$\langle M_1, \in, R \rangle \models \varphi(p_1, \dots, p_n) \leftrightarrow \langle M_2, \in, R \rangle \models \varphi(p_1, \dots, p_n) \quad (1.38)$$

2 Levy's first-order reflection

2.1 Lévy's Original Paper

This section will try to present Lévy's proof of a general reflection principle being equivalent to *Replacement* and *Infinity* under ZF minus *Replacement* and *Infinity* from his 1960 paper *Axiom Schemata of Strong Infinity in Axiomatic Set Theory*²⁰.

When reading said article, one should bear in mind that it was written in a period when set theory was semantically oriented, so while there are many statements about a model of ZF, usually denoted u , this is equivalent to today's universal class V , so it doesn't necessarily mean that there is a set u that is a model of ZF. We will review the notion of a standard complete model used by Lévy throughout the paper in a moment. Let's first say that the set theory ZF was formulated in the "non-simple applied first order functional calculus", is

TODO viz A. Church nebo tak neco.

The axioms are equivalent to those defined in 1.18, except for the *Axiom of Subsets*, which is just a different name for *Specification*. Besides ZF and S, defined in 1.18 and yrefdef:s respectively, the set theories theories Z, and SF are used in the text. Z is ZF minus replacement, SF is ZF minus *Infinity*. Also note that universal quantifier does not appear, $\forall x\varphi(x)$ would be written as $(x)\varphi(x)$. The symbol for negation is " \sim ", implication is written as " \supset " and equivalence is " \equiv ", we will use " \neg ", " \rightarrow " and " \leftrightarrow ".

Next two definitions are not used in contemporary set theory, but they illustrate 1960's set theory mind-set and they are used heavily in Lévy's text, so we will include and explain them for clarity. Generally, in this chapter, Q stands for an arbitrary axiomatic set theory used for general definitions, u is usually a model of Q, counterpart of today's V .

This subsection uses ZF instead of the usual ZFC as the underlying theory.

Definition 2.1 (*Standard model of a set theory*)

Let Q be a axiomatic set theory in first-order logic. We say the the a class u is a standard model of Q with respect to a membership relation E , written as $Sm^Q(u)$, iff both of the following hold

- (i) $(x, y) \in E \leftrightarrow y \in u \ \& \ x \in y$
- (ii) $y \in u \ \& \ x \in y \rightarrow x \in u$

Definition 2.2 *Standard complete model of a set theory*

Let Q and E be like in 2.1. We say that that u is a standard complete model of Q with respect to a membership relation E iff both of the following hold

²⁰[?]

- 447 (i) u is a transitive set with respect to \in
 448 (ii) $\forall E((x, y) \in E \leftrightarrow (y \in u \& x \in y) \& Sm^Q(u, E))$
 449 this is written as $Scm^Q(u)$.

450 **Definition 2.3** (*Inaccessible cardinal with respect to Q*)
 451 Let Q be an axiomatic first-order set theory. We say that a cardinal κ is
 452 inaccessible with respect to Q , we write $In^Q(\kappa)$.

$$In^Q(\kappa) \stackrel{\text{def}}{=} Scm^Q(V_\kappa). \quad (2.39)$$

453 **Definition 2.4** (*Inaccessible cardinal with respect to ZF*)
 454 When a cardinal κ is inaccessible with respect to ZF , we only say that it is
 455 inaccessible. We write $In(\kappa)$.

$$In(\kappa) \stackrel{\text{def}}{=} In^{ZF}(\kappa) \quad (2.40)$$

456 The above definition of inaccessibles is used because it doesn't require *Choice*.
 457 For the definition of relativization, see 1.36. The syntax used by Lévy is
 458 $Rel(u, \varphi)$, we will use φ^u , which is more usual these days.

459 **Definition 2.5** (N)
 460 The following is an axiom schema of complete reflection over ZF , denoted as
 461 N .

$$N \stackrel{\text{def}}{=} \exists u(Scm^{ZF}(u) \& \forall x_1, \dots, x_n(x_1, \dots, x_n \in u \rightarrow \varphi \leftrightarrow \varphi^u)) \quad (2.41)$$

462 where φ is a formula which contains no free variables except for x_1, \dots, x_n .

463 **Definition 2.6** (N_0)
 464 With S instead of ZF we obtain what will now be called N_0 .

$$N_0 \stackrel{\text{def}}{=} \exists u(Scm^S(u) \& \forall x_1, \dots, x_n(x_1, \dots, x_n \in u \rightarrow \varphi \leftrightarrow \varphi^u)) \quad (2.42)$$

465 where φ is a formula which contains no free variables except for x_1, \dots, x_n .

466 **2.2** $S \models (N_0 \leftrightarrow \text{Replacement \& Infinity})$

467 Let S be a set theory defined in 1.17.

468 **Lemma 2.7** *The following holds for every u .*

$$"u \text{ is a limit ordinal}" \leftrightarrow Scm^S(u) \quad (2.43)$$

469 *Proof.* TODO !

470 —

471 In order to prove that it is a model of \mathbf{S} , we would need to verify all
 472 axioms of \mathbf{S} . We have already shown that ω is closed under the powerset
 473 operation. Foundation, extensionality and comprehension are clear from the
 474 fact that we work in \mathbf{ZF}^{21} , pairing is clear from the fact, that given two sets
 475 x, y , they have ranks α, β , without loss of generality we can assume that
 476 $\alpha \leq \beta$, which means that $x \in V_\alpha \in V_\beta$, therefore V_β is a set that satisfies the
 477 pairing axiom: it contains both x and B .

478 □

479 Let N_0 be defined as in 2.6, for *Infinity* see 1.10.

480 **Theorem 2.8** *In \mathbf{S} , the schema N_0 implies Infinity.*

481 *Proof.* Lévy skips this proof because it seems too obvious to him, but let's do
 482 it here for plasticity. For an arbitrary φ , N_0 gives us $\exists u \text{Scm}^{\mathbf{S}}(u)$, but from
 483 lemma 2.7, we know that this u is a limit ordinal. This u already satisfies
 484 *Infinity*. □

485

486 Let N_0 be defined as in 2.6, for *Replacement* see 1.15, \mathbf{S} is again the set
 487 theory defined in 1.17.

488 **Theorem 2.9** *In \mathbf{S} , the schema N_0 implies Replacement.*

489 *Proof.* Let $\varphi(x, y, p_1, \dots, p_n)$ be a formula with no free variables except
 490 x, y, p_1, \dots, p_n for an arbitrary natural number n .

$$\begin{aligned} \chi &= \forall x, y, z (\varphi(x, y, p_1, \dots, p_n) \ \& \ \varphi(x, z, p_1, \dots, p_n) \rightarrow y = z) \\ &\rightarrow \forall x \exists y \forall z (z \in y \leftrightarrow \exists q (q \in x \ \& \ \varphi(q, z, p_1, \dots, p_n))) \end{aligned} \quad (2.44)$$

491 Let χ be an instance of *Replacement* schema for given φ . Let the follow-
 492 ing formulas be instances of the N_0 schema for formulas $\varphi, \exists y \varphi, \chi$ and
 493 $\forall x, p_1, \dots, p_n \chi$ respectively:

494 We can deduce the following from N_0 :

- 495 (i) $x, y, p_1, \dots, p_n \in u \rightarrow (\varphi \leftrightarrow \varphi^u)$
- 496 (ii) $x, p_1, \dots, p_n \in u \rightarrow (\exists y \varphi \leftrightarrow (\exists y \varphi)^u)$
- 497 (iii) $x, p_1, \dots, p_n \in u \rightarrow (\chi \leftrightarrow \chi^u)$
- 498 (iv) $\forall x, p_1, \dots, p_n (\chi \leftrightarrow (\forall x, p_1, \dots, p_n \chi)^u)$

²¹We only need to verify axioms that provide means of constructing larger sets from smaller to make sure they don't exceed ω . Since ω is an initial segment of \mathbf{ZF} , the axiom scheme of specification can't be broken, the same holds for foundation and extensionality.

From relativization, we also know that $(\exists y\varphi)^u$ is equivalent to $(\exists y \in u)\varphi^u$.
Therefore (ii) is equivalent to

$$x, p_1, \dots, p_n \in u \rightarrow (\exists y \in u)\varphi^u. \quad (2.45)$$

If φ is a function²², then for every $x \in u$, which is also $x \subset u$ by the transitivity of $Scm^S(u)$, it maps elements of x onto u . From the axiom scheme of comprehension²³, we can find y , a set of all images of elements of x . That gives us $x, p_1, \dots, p_n \in u \rightarrow \chi$. By (iii) we get $x, p_1, \dots, p_n \in u \rightarrow \chi^u$, the universal closure of this formula is $(\forall x, p_1, \dots, p_n \chi)^u$, which together with (iv) yields $\forall x, p_1, \dots, p_n \chi$. Via universal instantiation, we end up with χ . We have inferred replacement for a given arbitrary formula. \square

What we have just proven is just a single theorem from the above mentioned article by Lévy, we will introduce other interesting propositions, mostly related to the existence of large cardinals, later in their appropriate context in chapter 3.

2.3 Contemporary restatement

We will now prove what is also Lévy's first-order reflection theorem, but rephrased with up to date set theory terminology. The main difference is, that while Lévy reflects φ from V to a set u that is a "standard complete model of S ", we say that there is a V_α for a limit α that reflects φ . We will argue that those are equivalent.²⁴

Definition 2.10 (*Reflection₁*)

Let $\varphi(p_1, \dots, p_n)$ be a first-order formula in the language of set theory. Then the following holds for any such φ .

$$\forall M_0 \exists M (M_0 \subseteq M \ \& \ (\varphi^M(p_1, \dots, p_n) \leftrightarrow \varphi(p_1, \dots, p_n))) \quad (2.46)$$

Note that this is a restatement of both Lévy's N and N_0 from the previous chapter, see definitions ??, ??. We prefer to call it *Reflection₁* so it complies with how other axioms and schemata are called.²⁵ Note that the subscript 1 refers to the fact that $\varphi(p_1, \dots, p_n)$ is a first-order formula, and since we're using the work "reflection" in less strict meaning throughout this thesis,

²²See definition 1.11

²³Lévy uses its equivalent, axiom of subsets

²⁴TODO nekde na to bude lemma!

²⁵We will not use the name N_0 , because it might be confusing to work N_0 and M_0 where M_0 is a set and N_0 is an axiom schema.

distinguishing between the two just by using italic font face for the schema might cause confusion.

We will now prove the equivalence of *Reflection₁* with *Replacement* and *Infinity* in **S** in two parts. First, we will show that N_0 is a theorem of **ZFC**, then we shall show that the second implication, which proves *Infinity* and *Replacement* from N_0 , also holds.

The following lemma is usually done in more parts, the first being for one formula, the other for n formulas. We will only state and prove the more general version for n formulas, knowing that setting $n = 1$ turns it to a specific version.

Lemma 2.11 *Let $\varphi_1, \dots, \varphi_n$ be formulas with m parameters²⁶.*

(i) *For each set M_0 there is such set M that $M_0 \subset M$ and the following holds for every i , $1 \leq i \leq n$:*

$$\exists x \varphi_i(p_1, \dots, p_{m-1}, x) \rightarrow (\exists x \in M) \varphi_i(p_1, \dots, p_{m-1}, x) \quad (2.47)$$

for every $p_1, \dots, p_{m-1} \in M$.

(ii) *Furthermore there is an ordinal α such that $M_0 \subset V_\alpha$ and the following holds for each i , $1 \leq i \leq n$:*

$$\exists x \varphi_i(p_1, \dots, p_{m-1}, x) \rightarrow (\exists x \in V_\alpha) \varphi_i(p_1, \dots, p_{m-1}, x) \quad (2.48)$$

for every $p_1, \dots, p_{m-1} \in M$.

(iii) *Assuming Choice, there is M , $M_0 \subset M$ such that 2.47 holds for every M , $i \leq n$ and $|M| \leq |M_0| \cdot \aleph_0$.*

Proof. We will simultaneously prove statements (i) and (ii), denoting M^T the transitive set required by part (ii). Unless explicitly stated otherwise for specific steps, it is thought to be equivalent to M .

Let us first define operation $H(p_1, \dots, p_{m-1})$ that gives us the set of x 's with minimal rank²⁷ satisfying $\varphi_i(p_1, \dots, p_{m-1}, x)$ for given parameters p_1, \dots, p_{m-1} for every i such that $1 \leq i \leq n$.

$$H_i(p_1, \dots, p_n) = \{x \in C_i : (\forall z \in C)(\text{rank}(x) \leq \text{rank}(z))\} \quad (2.49)$$

²⁶For formulas with a different number of parameters, take for m the highest number of parameters among those formulas. Add spare parameters to every formula that has less than m parameters in a way that preserves the last parameter, which we will denote x . E.g. let φ'_i be the a formula with k parameters, $k < m$. Let us set $\varphi_i(p_1, \dots, p_{m-1}, x) \stackrel{\text{def}}{=} \varphi'_i(p_1, \dots, p_{k-1}, x)$, notice that the parameters p_k, \dots, p_{m-1} are not used.

²⁷Rank is defined in 1.27

551 for each $1 \leq i \leq n$, where

$$C_i = \{x : \varphi_i(p_1, \dots, p_{m-1}, x)\} \text{ for } 1 \leq i \leq n \quad (2.50)$$

552

553 Next, let's construct M from given M_0 by induction.

$$M_{i+1} = M_i \cup \bigcup_{j=0}^n \{H_j(p_1, \dots, p_{m-1}) : p_1, \dots, p_{m-1} \in M_i\} \quad (2.51)$$

554 In other words, in each step we add the elements satisfying $\varphi(p_1, \dots, p_{m-1}, x)$
 555 for all parameters that were either available earlier or were added in the
 556 previous step. For statement (ii), this is the only part that differs from (i).
 557 Let us take for each step transitive closure of M_{i+1} from (i). In other words,
 558 let γ be the smallest ordinal such that

$$(M_i^T \cup \bigcup_{j=0}^n \{\bigcup \{H_j(p_1, \dots, p_{m-1}) : p_1, \dots, p_{m-1} \in M_i\}\}) \subset V_\gamma \quad (2.52)$$

559 Then the incremental step is like so:

$$M_{i+1}^T = V_\gamma \quad (2.53)$$

560 The final M is obtained by joining all the individual steps.

$$M = \bigcup_{i=0}^{\infty} M_i, \quad M^T = \bigcup_{i=0}^{\infty} M_i^T = V_\alpha \quad (2.54)$$

561

562 We have yet to finish part (iii). Let's try to construct a set M' that
 563 satisfies the same conditions like M but is kept as small as possible. Assuming
 564 the Axiom of Choice, we can modify the process so that the cardinality of
 565 M' is at most $|M_0| \cdot \aleph_0$. Note that the size of M' is determined by the size of
 566 M_0 and, most importantly, by the size of $H_i(p_1, \dots, p_{m-1})$ for any i , $1 \leq i \leq n$
 567 in individual levels of the construction. Since the lemma only states existence
 568 of some x that satisfies $\varphi_i(p_1, \dots, p_{m-1}, x)$ for any $1 \leq i \leq n$, we only need to
 569 add one x for every set of parameters but $H_i(u_1, \dots, u_{m-1})$ can be arbitrarily
 570 large. Since Axiom of Choice ensures that there is a choice function, let F be
 571 a choice function on $\mathcal{P}(M')$. Also let $h_i(p_1, \dots, p_{m-1}) = F(H_i(p_1, \dots, p_{m-1}))$
 572 for i , where $1 \leq i \leq n$, which means that h is a function that outputs an x
 573 that satisfies $\varphi_i(p_1, \dots, p_{m-1}, x)$ for i such that $1 \leq i \leq n$ and has minimal
 574 rank among all such witnesses. The induction step needs to be redefined to

$$M'_{i+1} = M'_i \cup \bigcup_{j=0}^n \{H_j(p_1, \dots, p_{m-1}) : p_1, \dots, p_{m-1} \in M'_i\} \quad (2.55)$$

575 This way, the amount of elements added to M'_{i+1} in each step of the construc-
 576 tion is the same as the amount of sets of parameters that yielded elements not
 577 included in M'_i . It is easy to see that if M_0 is finite, M' is countable because
 578 it was constructed as a countable union of finite sets. If M_0 is countable or
 579 larger, the cardinality of M' is equal to the cardinality of M_0 .²⁹ Therefore
 580 $|M'| \leq |M_0| \cdot \aleph_0$ \square

581 **Theorem 2.12** (*Lévy's first-order reflection theorem*)

582 Let $\varphi(p_1, \dots, p_n)$ be a first-order formula.

583 (i) For every set M_0 there exists M such that $M_0 \subset M$ and the following
 584 holds:

$$\varphi^M(p_1, \dots, p_n) \leftrightarrow \varphi(p_1, \dots, p_n) \quad (2.56)$$

585 for every $p_1, \dots, p_n \in M$.

586 (ii) For every set M_0 there is a transitive set M , $M_0 \subset M$ such that the
 587 following holds:

$$\varphi^M(p_1, \dots, p_n) \leftrightarrow \varphi(p_1, \dots, p_n) \quad (2.57)$$

588 for every $p_1, \dots, p_n \in M$.

589 (iii) For every set M_0 there is α such that $M_0 \subset V_\alpha$ and the following holds:

$$\varphi^{V_\alpha}(p_1, \dots, p_n) \leftrightarrow \varphi(p_1, \dots, p_n) \quad (2.58)$$

590 for every $p_1, \dots, p_n \in M$.

591 (iv) Assuming Choice, for every set M_0 there is M such that $M_0 \subset M$ and
 592 $|M| \leq |M_0| \cdot \aleph_0$ and the following holds:

$$\varphi^M(p_1, \dots, p_n) \leftrightarrow \varphi(p_1, \dots, p_n) \quad (2.59)$$

593 for every $p_1, \dots, p_n \in M$.

594 *Proof.* Before we start, note that the following holds for any set M if φ is an
 595 atomic formula, as a direct consequence of relativisation to M, \in .³⁰

$$\varphi \leftrightarrow \varphi^M \quad (2.60)$$

596 Let's now prove (i) for given φ via induction by complexity. We can safely
 597 assume that φ contains no quantifiers besides " \exists " and no logical connectives

²⁹It can not be smaller because $|M'_{i+1}| \geq |M'_i|$ for every i . It may not be significantly larger because the maximum of elements added is the number of n -tuples in M'_i , which is of the same cardinality is M'_i .

³⁰See ???. Also note that this works for relativization to M, \in , not M, E where E is an arbitrary membership relation on M .

other than " \neg " and " $\&$ ". Let $\varphi_1, \dots, \varphi_n$ be all subformulas of φ . Then there is a set M , obtained by the means of lemma 2.11, for all of the formulas $\varphi_1, \dots, \varphi_n$.

We know that $\psi \leftrightarrow \psi^M$ for atomic ψ , we need to verify that it won't fail in the inductive step. Let us consider $\psi = \neg\psi'$ along with the definition of relativization for those formulas in 1.36.

$$(\neg\psi')^M \leftrightarrow \neg(\psi'^M) \quad (2.61)$$

Because the induction hypothesis says that ?? holds for every subformula of ψ , we can assume that $\psi'^M \leftrightarrow \psi'$, therefore the following holds:

$$(\neg\psi')^M \leftrightarrow \neg(\psi'^M) \leftrightarrow \neg\psi' \quad (2.62)$$

The same holds for $\psi = \psi_1 \& \psi_2$. From the induction hypothesis, we know that $\psi_1^M \leftrightarrow \psi_1$ and $\psi_2^M \leftrightarrow \psi_2$, which together with relativization for formulas in the form of $\psi_1 \& \psi_2$ gives us

$$(\psi_1 \& \psi_2)^M \leftrightarrow \psi_1^M \& \psi_2^M \leftrightarrow \psi_1 \& \psi_2 \quad (2.63)$$

Let's now examine the case when from the induction hypothesis, M reflects $\psi'(p_1, \dots, p_n, x)$ and we are interested in $\psi = \exists x\psi'(p_1, \dots, p_n, x)$. The induction hypothesis tells us that

$$\varphi'^M(p_1, \dots, p_n, x) \leftrightarrow \psi'(p_1, \dots, p_n, x) \quad (2.64)$$

so, together with above lemma 2.11, the following holds:

$$\begin{aligned} & \psi(p_1, \dots, p_n, x) \\ & \leftrightarrow \exists x\psi'(p_1, \dots, p_n, x) \\ & \leftrightarrow (\exists x \in M)\psi'(p_1, \dots, p_n, x) \\ & \leftrightarrow (\exists x \in M)\psi'^M(p_1, \dots, p_n, x) \\ & \leftrightarrow (\exists x\psi'(p_1, \dots, p_n, x))^M \\ & \leftrightarrow \psi^M(p_1, \dots, p_n, x) \end{aligned} \quad (2.65)$$

Which is what we have needed to prove. ?? holds for all subformulas $\varphi_1, \dots, \varphi_n$ of a given formula φ .

So far we have proven part (i) of this theorem for one formula φ , we only need to verify that the same holds for any finite number of formulas. This has in fact been already done since lemma 2.11 gives us M for any

(finite) amount of formulas, we can find a set M for the union of all of their subformulas. We can then use the induction above to verify that M reflects each of the formulas individually iff it reflects all of its subformulas.

Since V_α is a transitive set, by proving (iii) we also satisfy (ii). To do so, we only need to look at part (ii) of lemma 2.11. All of the above proof also holds for $M = V_\alpha$.

To finish part (iv), we take M of size $\leq |M_0| \cdot \aleph_0$, which exists due to part (iii) of lemma 2.11, the rest being identical. \square

Let \mathbf{S} be a set theory defined in 1.17, for ZFC see 1.19.

Lemma 2.13 *Let M be a set. Then the following holds:*

$$\text{ZFC} \models (M \models \mathbf{S}) \leftrightarrow "M \text{ is a limit cardinal}" \quad (2.66)$$

Proof. For the left-to-right direction, we shall verify that if M is a model of \mathbf{S} , it necessarily is a limit cardinal. From *Powerset*³¹, we know that for any $x \in M$, $\mathcal{P}(x) \in M$. But that is already the definition of a strong limit cardinal³².

For the converse, we need to see that if there is a limit ordinal α , such that $V_\alpha = M$, the axioms of \mathbf{S} hold in M .

(i) *Existence of a set* (see 1.1)

There obviously is a set $x \in M$

(ii) *Extensionality* (see 1.2)

Since *Extensionality*^M is a Δ_0 formula, it holds in any transitive class by ??.

(iii) *Specification* (see 1.3)

TODO

(iv) *Foundation* (see 1.6)

Foundation^M is also a Δ_0 formula, so it holds by ?? since M is transitive because it is a cardinal.

(v) *Pairing* (see 1.7)

TODO

(vi) *Union* (see 1.8)

TODO

(vii) *Powerset* (see 1.9)

TODO

³¹1.9.

³²see ??

654

□

655

Let *Infinity* and *Replacement* be as defined in 1.10 and 1.15 respectively.

656

657

Theorem 2.14 *Reflection₁ is equivalent to Infinity & Replacement under S.*

658

Proof. Since 2.12 already gives us one side of the implication, we are only interested in showing the converse which we shall do in two parts:

660

TODO N_0 prepsat zpatky na *Reflection₁*

661

$\mathbf{N}_0 \rightarrow \text{Infinity}$ From N_0 (??), we know that for any first-order formula φ and a set M_0 , there is a M such that $M_0 \subseteq M$ and $\varphi^M \leftrightarrow \varphi$. Let's pick *Powerset* for φ , then by N_0 there is a set that satisfies *Powerset*, ergo there is a strong limit cardinal, which in turn satisfies *Infinity*.

665

666

Reflection \rightarrow *Replacement*

667

Given a formula $\varphi(x, y, p_1, \dots, p_n)$, we can suppose that it is reflected in any M ³³ What we want to obtain is the following:

$$\forall x, y, z (\varphi(x, y, p_1, \dots, p_n) \ \& \ \varphi(x, z, p_1, \dots, p_n) \rightarrow y = z) \rightarrow \rightarrow \forall X \exists Y \forall y (y \in Y \leftrightarrow \exists x (\varphi(x, y, p_1, \dots, p_n) \ \& \ x \in X)) \quad (2.67)$$

669

We do also know that $x, y \in M$, in other words for every $X, Y = \{y \mid \varphi(x, y, p_1, \dots, p_n)\}$ and we know that $X \subset M$ and $Y \subset M$, which, together with the comprehension schema implies that Y , the image of X over φ , is a set. □

673

674

We have shown that *Reflection* for first-order formulas, *Reflection₁* is a theorem of ZF, which means that it won't yield us any large cardinals. We have also shown that it can be used instead of the *Infinity* and *Replacement* scheme, but $\text{ZF} + \text{Reflection}_1$ is a conservative extension of ZF. Besides being a starting point for more general and powerful statements, it can be used to show that ZF is not finitely axiomatizable. That follows from the fact that *Reflection* gives a model to any finite number of (consistent) formulas. So if $\varphi_1, \dots, \varphi_n$ for any finite n would be the axioms of ZF, *Reflection* would always contain a model of itself, which would in turn contradict the Second Gödel's Theorem³⁴. Notice that, in a way, reflection is complementary to compactness. Compactness argues that given a set of sentences, if every finite subset yields a model, so does the whole set. Reflection, on the other hand, says that while the whole set has no model in the underlying theory, every finite subset does have one.

687

³³Which means that for $x, y, p_1, \dots, p_n \in M$, $\varphi^M(x, y, p_1, \dots, p_n) \leftrightarrow \varphi(x, y, p_1, \dots, p_n)$.

³⁴See chapter ?? for further details.

688 Also, notice how reflection can be used in ways similar to upward Löwenheim–Skolem
689 theorem. Since Reflection extends any set M_0 into a model of given formulas
690 $\varphi_1, \dots, \varphi_n$, we can choose the lower bound of the size of M by appropriately
691 choosing M_0 .

692 In the next section, we will try to generalize *Reflection* in a way that
693 transcends ZF and finally yields some large cardinals.

3 Reflection And Large Cardinals

In this chapter we aim to examine stronger reflection properties in order to reach cardinals unavailable in ZFC. Like we said in the first chapter, the variety of reflection principles comes from the fact that there are many way to formalize "properties of the universal class". It is not always obvious what properties hold for V because, (TODO Tarski) We have shown that reflecting properties as first-order formulas doesn't allow us to leave ZFC. We will broaden the class of admissible properties to be reflected and see whether there is a natural limit in the height or width on the reflected universe and also see that no matter how far we go, the universal class is still as elusive as it is when seen from S . That is because for every process for obtaining larger sets such as for example the powerset operation in ZFC, this process can't reach V and thus, from reflection, there is an initial segment of V that can't be reached via said process.

To see why this is important, let's dedicate a few lines to the intuition behind the notions of limitness, regularity and inaccessibility in a manner strongly influenced by [?]. To see why limit and strongly limit cardinals are worth mentioning, note that they are "limit" not only in a sense of being a supremum of an ordinal sequence, they also show that a certain way of obtaining larger sets from smaller ones is limited. We will see that all of the alternatives offered in this thesis are in a sense limited. \aleph_λ is a limit cardinal iff there is no α such that $\aleph_{\alpha+1} = \aleph_\lambda$. Strongly limit cardinals point to the limits of the powerset operation. It has been too obvious so far, so let's look at the regular cardinals in this manner. Regular cardinals are those that cannot be³⁵, expressed as a supremum of smaller amount of smaller objects³⁶. More precisely, κ is regular if there is no way to define it as a union of less than κ ordinals, all smaller than κ . So unless there already is a set of size κ , *Replacement* is useless in determining whether κ is really a set. Note that assuming *Choice*, successor cardinals are always regular, so most³⁷ limit cardinals are singular cardinals. So if one is traversing the class of all cardinals upwards, successor steps are still sets thanks to the powerset axiom while singular limits cardinal are not proper classes because they are suprema of images of smaller sets via *Replacement*. Regular cardinals are, in a way, limits of how far can we get by taking limits of increasing sequences of ordinals obtained via *Replacement*.

TODO prepsat – regularita a replacement, proc reflexe dava silnejsi veci

³⁵Assuming *Choice*.

³⁶Just like ω can not be expressed as a supremum of a finite set consisting solely of finite numbers.

³⁷All provable to exist in ZFC

730 That all being said, it is easy to see that no cardinals in ZFC are both
 731 strongly limit and regular because there is no way to ensure they are sets and
 732 not proper classes in ZFC. The only exception to this rule is \aleph_0 which needs
 733 *Infinity* to exist. It should now be obvious why the fact that κ is inaccessible
 734 implies that $\kappa = \aleph_\kappa$.³⁸

735 We will also examine the connection between reflection principles and
 736 (regular) fixed points of ordinal functions in a manner proposed by Lévy in
 737 [?]. We will also see that, like Lévy has proposed in the same paper, there is
 738 a meaningful way to extend the relation between S and ZFC into a hierarchy
 739 of stronger axiomatic set theories.

740 3.1 Regular Fixed-Point Axioms

741 Lévy's article mentions various schemata that are not instances of reflection
 742 per se. We will mention them because they are equivalent to *Reflection*₁³⁹.

743 **Definition 3.1** (Axiom M_1)

744 "Every normal function defined for all ordinals has at least one inaccessible
 745 number in its range."

746 Lévy uses " M " to refer to this axiom but since we also use " M " for sets and
 747 models, for example in 2.10, we will call the above axiom "*Axiom M_1* " to
 748 avoid confusion.

749 Now we will express *Axiom M_1* to formula to make it clear that it is an
 750 axiom scheme and the same can be done with *Axiom M'_1* as well as *Axiom*
 751 *M''_1* introduced immediately afterwards. Since it is an axiom schema and we
 752 will later dive into second-order logic, we may also want to refer to *Axiom*
 753 *M_2* as opposed *Axiom M_1* , the former being a single second-order sentence
 754 obtained by the obvious modification of *Axiom M_1* .⁴⁰

755 Let $\varphi(x, y, p_1, \dots, p_n)$ be a first-order formula with no free variables be-
 756 sides x, y, p_1, \dots, p_n . The following is equivalent to *Axiom M_1* .

$$\begin{aligned} & \text{"}\varphi \text{ is a normal function"} \ \& \ \forall x(x \in \text{Ord} \rightarrow \exists y(\varphi(x, y, p_1, \dots, p_n))) \rightarrow \\ & \rightarrow \exists y(\exists x(\varphi(x, y, p_1, \dots, p_n) \ \& \ cf(y) = y \ \& \ (\forall x \in \kappa)(\exists y \in \kappa)(x > y)) \\ & \hspace{15em} (3.68) \end{aligned}$$

757 41

³⁸This doesn't work backwards, the least fixed point of the \aleph function is the limit of $\{\aleph_0, \aleph_{\aleph_0}, \aleph_{\aleph_{\aleph_0}}, \dots\}$, it is singular since the sequence has countably many elements.

³⁹For definition, see 2.10

⁴⁰Second-order set theory will be introduced in the next subsection.

⁴¹" φ is a normal function" is equivalent to the following first-order formula:

758 **Definition 3.2** (*Axiom M'_1*)

759 *Every normal function defined for all ordinals has at least one fixed point*
 760 *which is inaccessible.*

761 **Definition 3.3** (*Axiom M''_1*)

762 *"Every normal function defined for all ordinals has arbitrarily great fixed*
 763 *points which are inaccessible."*

764 The following axiom is proposed by Drake in [?].

765 **Definition 3.4** (*Axiom F_1*)

766 *Every normal function defined for all ordinals has a regular fixed point.*

767 **Lemma 3.5** (*Fixed-point lemma for normal functions*)

768 *Let f be a normal function defined for all ordinals. The all of the following*
 769 *hold*

- 770 (i) $\forall \lambda$ (" λ is a limit ordinal" \rightarrow " $f(\lambda)$ is a limit ordinal")
- 771 (ii) $\forall \alpha (\alpha \leq f(\alpha))$
- 772 (iii) $\forall \alpha \exists \beta (\alpha < \beta \ \& \ f(\beta) = \alpha)$ (*f has arbitrarily large fixed points.*)
- 773 (iv) *The fixed points of f form a closed unbounded class.*⁴²

774 *Proof.* Let f be a normal function.

775 (i) Proof of (i):

776 Suppose λ is a limit ordinal. For an arbitrary ordinal $\alpha < \lambda$, the fact
 777 that f is strictly increasing means that $f(\alpha) < f(\lambda)$ and for an ordi-
 778 nal β , $\beta < \alpha$, $f(\alpha) < f(\beta)$. Because f is continuous and λ limit,
 779 $f(\lambda) = \bigcup_{\alpha < \lambda} f(\alpha)$ and since $\beta < \lambda$, $f(\beta) < f(\lambda)$. So we have found
 780 $f(\beta)$ such that $f(\alpha) < f(\beta) < f(\lambda)$, therefore $f(\lambda)$ is a limit ordinal.

781
 782 (ii) This step will be proven using the transfinite induction. Since f is
 783 defined for all ordinals, there is an ordinal α such that $f(\emptyset) = \alpha$ and
 784 because \emptyset is the least ordinal, (ii) holds for \emptyset .

785 Suppose (ii) holds for some β from the induction hypothesis. It the
 786 holds for $\beta + 1$ because f is strictly increasing.

787 For a limit ordinal λ , suppose (ii) holds for every $\alpha < \lambda$. (i) implies that
 788 $f(\lambda)$ is also limit, so there is a strictly increasing κ -sequence $\langle \alpha_0, \alpha_1, \dots \rangle$
 789 for some κ such that $\lambda = \bigcup_{i < \kappa} \alpha_i$. Because f is strictly increasing, the
 790 κ -sequence $\langle f(\alpha_0), f(\alpha_1), \dots \rangle$ is also strictly increasing, the induction
 791 hypothesis implies that $\alpha_i \leq f(\alpha_i)$ for each $i \leq \kappa$. Thus, $\lambda \leq f(\lambda)$.

⁴²See 1.45 for the definition of closed set, ??

- (iii) For a given α , let there be a ω -sequence $\langle \alpha_0, \alpha_1, \dots \rangle$, such that $\alpha_0 = \alpha$ and $\alpha_{i+1} = f(\alpha_i)$ for each $i < \omega$. This sequence is strictly increasing because so is f . Now, there's a limit ordinal $\beta = \bigcup_{i < \omega} \alpha_i$, we want to show that this is the fixed point. So $f(\beta) = f(\bigcup_{i < \omega} \alpha_i) = \bigcup_{i < \omega} f(\alpha_i)$ because f is continuous. We have defined the above sequence so that $\beta, \bigcup_{i < \omega} f(\alpha_i) = \bigcup_{i < \omega} \alpha_{i+1}$, which means we are done, since $\bigcup_{i < \omega} \alpha_{i+1} = \bigcup_{i < \omega} \alpha_i = \beta$.
- (iv) The class of fixed points of f is obviously unbounded by (iii). It remains to show that it is closed. TODO def closed?

□

Theorem 3.6

$$\text{Axiom } M_1 \leftrightarrow \text{Axiom } M'_1 \leftrightarrow \text{Axiom } M''_1 \leftrightarrow \text{Axiom } F_1 \quad (3.69)$$

This is *Theorem 1* in [?]. *Proof.* It is clear that *Axiom* M''_1 is a stronger version of *Axiom* M'_1 , which is in turn a stronger version of both *Axiom* M_1 and *Axiom* F_1 , so the implication *Axiom* $M''_1 \rightarrow \text{Axiom } M'_1 \rightarrow \text{Axiom } M_1$ is satisfied and *Axiom* $M'_1 \rightarrow \text{Axiom } F_1$ holds too.

We will now make sure that *Axiom* $M_1 \rightarrow \text{Axiom } M''_1$ also holds. Let f be a normal function defined for all ordinals. Let g be a normal function that counts the fixed points of f . Lemma ?? implies that there are arbitrarily many fixed points of f , therefore g is defined for all ordinals. Let there be another family of functions, $h_\alpha(\beta) = g(\alpha + \beta)$, obviously h_α is defined for all ordinals for every $\alpha \in \text{Ord}$ because so is g . Given an arbitrary ordinal γ , from *Axiom* M_1 we can assume that there is an ordinal δ such that such that $h_\alpha(\delta) = \kappa$, where κ is inaccessible. But since $\kappa = g(\alpha + \delta)$, κ is a fixed point of f . To show that there are arbitrarily many fixed points of f , notice that γ is arbitrary and h_γ is a normal function, so, by lemma ??, $(\forall \alpha \in \text{Ord})(\alpha \leq f(\alpha))$, therefore $\gamma \leq \gamma + \alpha \leq \kappa$, in other words, there is κ above an arbitrary ordinal γ .

Now we need to show that *Axiom* F_1 implies any of the remaining axioms. TODO nevyhodime F? □

Definition 3.7 ZMC

We will call **ZMC** a set theory that contains all axioms and schemas of **ZFC** together with the schema *Axiom* M_1 .

We have decided to call it **ZMC**, because Lévy uses **ZM**, derived from **ZF**, which is more intuitive, but we also need the axiom of choice, thus, **ZMC**.

Let's now prove that in **ZFC**, the above *Axiom* M is equivalent to *Reflection*₁ as defined in 2.10. This is proven in [?] as *Theorem 3*.

Theorem 3.8

$$\text{ZFC} \models \text{Axiom M} \leftrightarrow \text{Reflection}_1 \quad (3.70)$$

827 TODO nedosazitelne kardinaly – reflektuj presne formule, schemata

828 **3.2 Inaccessibility**

829 **Definition 3.9** (*limit cardinal*) κ is a limit cardinal iff it is \aleph_α for some
830 limit ordinal α .

831 **Definition 3.10** (*strong limit cardinal*) κ is a strong limit cardinal iff it is
832 a limit cardinal and for every $\lambda < \kappa$, $2^\lambda < \kappa$

833 The two above definition become equivalent if we assume *GCH*.

834 **Definition 3.11** (*weak inaccessibility*) An uncountable cardinal κ is weakly
835 inaccessible iff it is regular and limit.

836 **Definition 3.12** (*inaccessibility*) An uncountable cardinal κ is inaccessible
837 iff it is regular and strongly limit.

838

839 TODO neni tohle cely hotovy v Contemporary restatement??? porovnat
840 ktera je lepsi a sjednotit!!!

841 We will now show that the above notion is equivalent to the definition
842 Lévy uses in [?], which is, in more contemporary notation, the following:

843 **Theorem 3.13** *The following are equivalent:*

- 844 1. κ is inaccessible
845 2. $\langle V_\kappa, \in \rangle \models \text{ZFC}$

846 *Proof.* Let's first prove that if κ is inaccessible, it is a model of ZFC. We will
847 do that by verifying the axioms of ZFC just like Kanamori does it in in [?,
848 1.2] and Drake in [?, Chapter 4].

849 (i) *Extensionality*:
850 (see 1.2)

$$V_\kappa \models \forall x, y (\forall z (z \in x \leftrightarrow z \in y) \rightarrow x = y) \quad (3.71)$$

851 We need to prove that, given two sets that are equal in V , they are equal
852 in V_κ , in other words, that the *Extensionality* formula is reflected, that
853 is

$$V_\kappa \models \forall x, y \in V_\kappa (\forall z \in V_\kappa (z \in x \leftrightarrow z \in y) \rightarrow x = y) \quad (3.72)$$

854 But that comes from transitivity. If x and y are in V_κ their members
855 are also in V_κ .

856

- 857 (ii) *Foundation*:
858 (see 1.6)

$$V_\kappa \models \forall x(\exists z(z \in x) \rightarrow \exists z(z \in x \ \& \ \forall u \neg(u \in z \ \& \ u \in x))) \quad (3.73)$$

859 The argument for *Foundation* is almost identical to the one for *Extensionality*. For any set $x \in V_\kappa$, transitivity of V_κ makes sure that every
860 element of x is also an element of V_κ and the same holds for the elements of elements of x et cetera. So statements about those elements
861 are absolute between any transitive structures. V and V_κ are both transitive therefore *Foundation* holds and so does its relativisation to V_κ ,
862 *Foundation* $^{V_\kappa}$.
863
864
865
866

- 867 (iii) *Powerset*:
868 (see 1.9)

$$V_\kappa \models \forall x \exists y \forall z (z \subseteq x \rightarrow z \in y). \quad (3.74)$$

869 If we take x , an element of V_κ , $\mathcal{P}(x)$ has to be an element of V_κ to,
870 because it is transitive and a strong limit cardinal.
871

- 872 (iv) *Pairing*:
873 (see 1.7)

$$V_\kappa \models \forall x, y \exists z (x \in z \wedge y \in z). \quad (3.75)$$

874 *Pairing* holds from similar argument like above: let x and y be elements of V_κ , so there are ordinals $\alpha, \beta < \kappa$ such that $x \in V_\alpha$, $y \in V_\beta$.
875 Without any loss of generality, suppose $\alpha < \beta$, therefore $V_\alpha \subset V_\beta$ which,
876 from transitivity of the cumulative hierarchy, means that $x \in V_\beta$, then
877 $\{x, y\} \in V_{\beta+1}$ which is still in V_κ because it is a strong limit cardinal.
878
879

- 880 (v) *Union*:
881 (see 1.8)

$$V_\kappa \models \forall x \exists y \forall z \forall w ((w \in z \wedge z \in x) \rightarrow w \in y). \quad (3.76)$$

882 We want to see that for every $x \in V_\kappa$, this is equivalent to

$$V_\kappa \models \forall x \in V_\kappa, \exists y \in V_\kappa \forall z \in V_\kappa \forall w \in V_\kappa ((w \in z \wedge z \in x) \rightarrow w \in y). \quad (3.77)$$

883 Since V_κ is transitive, if $x \in V_\kappa$, all of its elements as well as their
884 elements are in V_κ . To see that they also form a set themselves we only
885 need to remember that V_κ is limit and therefore if α is the least ordinal
886 such that $x \in V_\alpha$, $\bigcup x \in V_{\alpha+1}$.
887

888 (vi) *Replacement, Infinity:*
 889 (see 1.15, 1.10)
 890 TODO !!!!
 891 to spis ty pred tim zname z dukazu v S, viz contemporary restatement.
 892 udelat z toho lemma?
 893 co ten replacement?? druha implikace Levyho vety?

894
 895 We will now show that if a set is a model of ZFC, it is in fact an inaccessible
 896 cardinal. So let V_κ be a model of ZFC which means that it is closed under
 897 the powerset operation, in other words:

$$\forall \lambda (\lambda < \kappa \rightarrow 2^\lambda < \kappa) \quad (3.78)$$

898 which is exactly the definition of strong limitness. κ is regular from the
 899 following argument by contradiction:

900 Let us suppose for a moment that κ is singular. Therefore there is an ordinal
 901 $\alpha < \kappa$ and a function $F : \alpha \rightarrow \kappa$ such that the range of F is unbounded
 902 in κ , in other words, $F[\alpha] \subseteq V_\kappa$ and $\sup(F[\alpha]) = \kappa$. In order to achieve
 903 the desired contradiction, we need to see that it is the case that $F[\alpha] \in V_\kappa$.
 904 Let $\varphi(x, y)$ be the following first-order formula:

$$F(x) = y \quad (3.79)$$

905 Then there is an instance of *Replacement* that states the following:

$$\begin{aligned} &(\forall x, y, z (\varphi(x, y) \ \& \ \varphi(x, z) \rightarrow y = z)) \rightarrow \\ &\rightarrow (\forall x \exists y \forall z (z \in y \leftrightarrow \exists w (\varphi(w, z)))) \end{aligned} \quad (3.80)$$

906 Which in turn means that there is a set $y = F[\alpha]$ and $y \in V_\kappa$, which is the
 907 contradiction with $\sup(y) = \kappa$ we are looking for. \square TODO vyhodit sup,
 908 pouzivat radis \bigcup

909 We have transcended ZFC, but that is just a start. Naturally, we could
 910 go on and consider the next inaccessible cardinal, which is inaccessible with
 911 respect to the theory $\text{ZFC} + \exists \kappa (\kappa \models \text{ZFC})$. But let's try to find a faster way
 912 up, informally at first.

913 Since we can find an inaccessible set larger than any chosen set M_0 , it
 914 is clear that there are arbitrarily large inaccessible cardinals in V , they are
 915 "unbounded"⁴³ in V . If V were a cardinal, we could say that there are V
 916 inaccessible cardinals less than V , but this statement of course makes no sense
 917 in set theory as is because V is not a set. But being more careful, we could
 918 find a property that can be formalized in second-order logic and reflect it to

⁴³The notion is formally defined for sets, but the meaning should be obvious.

an initial segment of V . That would allow us to construct large cardinals more efficiently than by adding inaccessibles one by one. The property we are looking for ought to look like something like this:

$$\begin{aligned} &\kappa \text{ is an inaccessible cardinal and} \\ &\text{there are } \kappa \text{ inaccessible cardinals } \mu < \kappa \end{aligned} \tag{3.81}$$

This is in fact a fixed-point type of statement. We shall call those cardinals hyper-inaccessible. Now consider the following definition.

Definition 3.14 *0-inaccessible cardinal*
A cardinal κ is 0-inaccessible if it is inaccessible.

We can define α -weakly-inaccessible cardinals analogously with the only difference that those are limit, not strongly limit.

Definition 3.15 *α -hyper-inaccessible cardinal*
For any ordinal α , κ is called α -inaccessible, if κ is inaccessible and for each $\beta \uparrow \alpha$, the set of β -inaccessible cardinals less than κ is unbounded in κ .

Because κ is inaccessible and therefore regular, the number of β -inaccessibles below κ is equal to κ . We have therefore successfully formalized the above vague notion of hyper-inaccessible cardinal into a hierarchy of α -inaccessibles.

Let's now consider iterating this process over again. Since, informally, V would be α -inaccessible for any α , this property of the universal class could possibly be reflected to an initial segment, the smallest of those will be the first hyper-inaccessible cardinal. Such κ is larger than any α -inaccessible since from regularity of κ , for given $\alpha < \kappa$, κ is κ -th α -hyper-inaccessible cardinal. It is in fact "inaccessible" via α -inaccessibility.

Definition 3.16 *Hyper-inaccessible cardinal*
 κ is called the hyper-inaccessible, also 0-hyper-inaccessible, cardinal if it is α -inaccessible for every $\alpha < \kappa$.

Definition 3.17 *α -hyper-inaccessible cardinal*
For any ordinal α , κ is called α -hyper-inaccessible cardinal if for each ordinal $\beta < \alpha$, the set of β -hyper-inaccessible cardinals less than κ is unbounded in κ .

Obviously we could go on and iterate it ad libitum, yielding α -hyper-...-hyper-inaccessibles, but the nomenclature would be increasingly confusing. A smarter way to accomplish the same goal is carried out in the following section.

3.3 Mahlo Cardinals

TODO axiomy?

While the previous chapter introduced us to a notion of inaccessibility and the possibility of iterating it ad libitum in new theories, there is an even faster way to travel upwards in the cumulative hierarchy, that was proposed by Paul Mahlo in his articles (see [?], [?] and [?]) at the very beginning of the 20th century, and which can be easily reformulated using reflection.

Theorem 3.18 *Let κ be a regular uncountable cardinal. The intersection of fewer than κ club subsets of κ is a club set.*

For the proof, see [?, Theorem 8.3]

Definition 3.19 *Weakly Mahlo Cardinal*

κ is weakly Mahlo \leftrightarrow it is a weakly-inaccessible ordinal and the set of all regular ordinals less than κ is stationary in κ

Definition 3.20 *Mahlo Cardinal*

κ is a Mahlo Cardinal iff it is an inaccessible cardinal and the set of all inaccessible ordinals less than κ is stationary in κ .

Analogously,

Definition 3.21 *α -Mahlo Cardinal*

κ is a α -Mahlo Cardinal iff it is an α -inaccessible cardinal and the set of all α -inaccessible ordinals less than κ is stationary in κ .

In other words, κ is a (weakly-)Mahlo cardinal if it is (weakly-)inaccessible and every club set in κ contains an (weakly-)inaccessible cardinal. Alternatively, a cardinal is (weakly-)Mahlo if it is (weakly-)inaccessible and there are κ (weakly-)inaccessibles below κ .

In a fashion similar to hyper-inaccessible cardinals, hyper-Mahlo cardinals can be defined as well.

TODO Lévy tady nekde? posloupnost modelu?

TODO co s nima edla Jech?

TODO proc se vys nedostaneme pevnyma bodama?

TODO explicitni reflexe? reflektuji reflexi nedosazitelnosti?

TODO Drake p.121!!

3.4 Second-order Reflection

Let's try a different approach in formalizing reflection. We have seen that reflecting individual first-order formulas doesn't even transcend ZFC, we have examined what can be done with axiom schemas. The aim of this chapter is to examine second-order formulas as possible axioms. Note that second-order variables (which will be established as type 2 variables later in the text) are subcollections of the universal class, but so are functions and relations. So first-order axiom schemata can also be interpreted as formulas with free second-order variables, which quantify over first-order variables only, we only need to customize the underlying theory accordingly. For example, the satisfaction relation was so far defined for first-order formulas only, but we will deal with that in a moment. Also note that by rewriting *replacement* and *comprehension* to single axioms, ZFC becomes finitely axiomatizable, which in turn means that the reflection theorem as stated in section does not hold for higher-order theories because of Gödel's second incompleteness theorem. We will explore stronger axioms of reflection instead.

TODO nemam nekam napsat ze u vsech velkych karidnalu je to "existujou pokud .."?

Let us establish a formal background first. We will now introduce higher-order formulas.

Definition 3.22 (Higher-order variables)

Let M be a structure and D its domain. In first-order logic, variables range over individuals, that is, over elements of D . We shall call those type 1 variables for the purposes of higher-order logic. Type 2 variables then range over collections, that is, the elements of $\mathcal{P}(D)$. Generally, type n variables are defined for any $n \in \omega$ such that they range over $\mathcal{P}^{n-1}(D)$.

We will use lowercase latin letters for type 1 variables for backwards compatibility with first-order logic, type 2 variables will be represented by upper-case letters, mostly P, X, Y, Z . If we ever stumble upon type 3 variables in this text, they shall be represented as $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ or in a similar font.

Definition 3.23 (Full prenex normal form)

We say a formula is in the prenex normal form if it is written as a block of quantifiers followed by a quantifier-free part.

We say a formula is in the Full prenex normal form if it is written in prenex normal form and if there are type $n + 1$ quantifiers, they are written before type n quantifiers.

It is an elementary that every formula is equivalent to a formula in the prenex normal form.

1026 **Definition 3.24** (*Hierarchy of formulas*)

1027 Let φ be a formula in the prenex formal form.

- 1028 (i) We say φ is a Δ_0^0 -formula if it contains only bounded quantifiers.
- 1029 (ii) We say φ is a Σ_0^0 -formula or a Π_0^0 -formula if it is a Δ_0^0 -formula.
- 1030 (iii) We say φ is a Π_0^{m+1} -formula if it is a Π_n^m - or Σ_n^m -formula for any $n \in \omega$
 1031 or if it is a Π_n^m - or Σ_n^m -formula with additional free variables of type
 1032 $m + 1$.
- 1033 (iv) We say φ is a Σ_0^m -formula if it is a Π_0^m -formula.
- 1034 (v) We say φ is a $\Sigma_n^m + 1$ -formula if it is of a form $\exists P_1, \dots, P_i \psi$ for any
 1035 non-zero i , where ψ is a Π_n^m -formula and P_1, \dots, P_i are type $m + 1$
 1036 variables.
- 1037 (vi) We say φ is a $\Pi_n^m + 1$ -formula if it is of a form $\forall P_1, \dots, P_i \psi$ for any
 1038 non-zero i , where ψ is a Σ_n^m -formula and P_1, \dots, P_i are type $m + 1$
 1039 variables.

1040 Now that we have introduced higher types of quantifiers, we will use it
 1041 to formulate reflection. But first, let's make it clear how relativization works
 1042 for higher-order quantifiers and type 2 parameters. Let α, κ be ordinals such
 1043 that $\alpha < \kappa$, $R \subseteq V_\kappa$.

$$R^{V_\alpha} \stackrel{\text{def}}{=} R \cap V_\alpha \quad (3.82)$$

1044 And let \exists^m be a quantifier that ranges over type m variables, let P represent
 1045 a type m variable, let φ be a type m formula with the only free variable P .

$$(\exists P \varphi(P))^{V_\alpha} \stackrel{\text{def}}{=} (\exists \mathcal{P}^{(m-1)} V_\alpha) \varphi^{V_\alpha}(P) \quad (3.83)$$

1046 **Definition 3.25** (*Reflection*)

1047 Let $\varphi(R)$ be a Π_n^m -formula with one free variable of type 2 denoted P .
 1048 We say $\varphi(R)$ reflects in V_κ if for every $R \subseteq V_\kappa$ there is an ordinal $\alpha < \kappa$
 1049 such that the following holds:

$$\begin{aligned} & \text{If } (V_\kappa, \in, R) \models \varphi(R), \\ & \text{then } (V_\alpha, \in, R \cap V_\alpha) \models \varphi(R \cap V_\alpha). \end{aligned} \quad (3.84)$$

1050 This formalization of the notion of reflection allows us to describe Inac-
 1051 cessible and Mahlo cardinals more easily, which we will do in the following
 1052 section.

1053 It is important to see, that while we can now reflect Π_n^m -formulas for arbi-
 1054 trary $m, n \in \omega$, they can only have type 2 free variables. This formalization
 1055 of reflection can not be extended to higher-order parameters as is. This will
 1056 be briefly reviewed in the next paragraph.

1057 In order to extend reflection as stated above in ??, we need to make sure
 1058 that given the domain of the structure, V_κ , we know what relativization to

1059 V_α , $\alpha < \kappa$, means. Since a type 3 parameters are collections of subcollections
 1060 of V_κ and we can already relativize subcollections of V_κ , this seems to be a
 1061 reasonable way to extend relativization to type 3 parameters:

$$\mathcal{R}^{V_\alpha} = \{R^{V_\alpha} : R \in \mathcal{R}\} \quad (3.85)$$

1062 Where R^{V_α} is type 2 relativization, which is $R \cap V_\alpha$.

1063 3.5 Indescribability

1064 Since this section talks about indescribability, this is how an ordinal is de-
 1065 scribed according to Drake [?, Chapter 9].

1066 **Definition 3.26** *We say an ordinal α is described by a formula $\varphi(P_1, \dots, P_n)$
 1067 with type 2 parameters P_1, \dots, P_n given iff*

$$\langle V_\alpha, \in \rangle \models \langle \varphi(P_1, \dots, P_n) \quad (3.86)$$

1068 but for every $\beta < \alpha$

$$\langle V_\beta, \in \rangle \not\models \varphi(P_1 \cap V_\beta, \dots, P_n \cap V_\beta) \quad (3.87)$$

1069 Drake then notes that the same notion can be established for sentences
 1070 if the corresponding type 2 parameters are added to the language. Since the
 1071 this approach is used by Kanamori in [?], we will stick to that too.⁴⁴

1072 **Definition 3.27** *Describability*

1073 *We say an ordinal α is described by a sentence φ in the language \mathcal{L} with
 1074 relation symbols P_1, \dots, P_n given iff*

$$\langle V_\alpha, \in, P_1, \dots, P_n \rangle \models \varphi \quad (3.88)$$

1075 but for every $\beta < \alpha$

$$\langle V_\beta, \in, P_1 \cap V_\beta, \dots, P_n \cap V_\beta \rangle \not\models \varphi \quad (3.89)$$

1076 **Definition 3.28** (Π_n^m -indescribable cardinal) *We say that κ is Π_n^m -indescribable
 1077 iff it is not described by any Π_n^m -formula.*

1078 **Definition 3.29** (Σ_n^m -indescribable cardinal) *We say that κ is Σ_n^m -indescribable
 1079 iff it is not described by any Σ_n^m -formula.*

⁴⁴The first definition is included because the author of this thesis finds it more intuitive.

1080 To see that this notion is based in reflection, note that for Π_n^m -formulas⁴⁵,
 1081 a cardinal κ is Π_n^m -indescribable iff every Π_n^m -formula reflects in κ in the sense
 1082 of definition ??.

1083 **Lemma 3.30** *Let κ be a cardinal, the following holds for any $n \in \omega$. κ is*
 1084 *Π_n^1 -indescribable iff κ is $\Sigma_n^1 + 1$ -indescribable*

1085 *Proof.* The forward direction is obvious, we can always add a spare quantifier
 1086 over a type 2 variable to turn a Π_n^1 formula φ into a $\exists P\varphi$ which is obviously
 1087 a $\Sigma_n^1 + 1$ formula.⁴⁶

1088 To prove the opposite direction, suppose that $V_\kappa \models \exists X\varphi(X)$ where X is
 1089 a type 2 variable and φ is a Π_n^1 formula with one free variable of type 2. This
 1090 means that there is a set $S \subseteq V_\kappa$ that is a witness of $\exists X\varphi(X)$, in other words,
 1091 $\varphi(S)$ holds. We can replace every occurrence of X in φ by a new predicate
 1092 symbol S , this allows us to say that κ is Π_n^1 -indescribable (with respect to
 1093 $\langle V_\kappa, \in, R, S \rangle$).⁴⁷ \square

1094 The above lemma makes it clear that we can suppose that all formulas
 1095 with no higher than type 2 variables are Π_n^1 -formulas, $n \in \omega$, without the
 1096 loss of generality.

1097 **Lemma 3.31** *If κ is an inaccessible cardinal and given $R \subseteq V_\kappa$, then the*
 1098 *following is a club set in κ :*

$$\{\alpha : \alpha < \kappa \text{ \& } \langle V_\alpha, \in, R \cap V_\alpha \rangle \prec \langle V_\kappa, \in, R \rangle\} \quad (3.90)$$

1099 *Proof.* To see that ?? is closed, let us recall that a $A \subseteq \kappa$ is closed iff for
 1100 every ordinal $\alpha < \lambda$, $\alpha \neq \emptyset$: if $A \cap \alpha$ is unbounded in α then $\alpha \in A$. Since
 1101 κ is an inaccessible cardinal, thus strong limit, it is closed under limits of
 1102 sequences of ordinals lesser than κ .

1103 TODO neco s V_κ , ze je tranzitivni a tak jso vsechny V_α pro $\alpha < \kappa$ $V_\alpha \in V_\kappa$

1104 We want to verify that it is unbounded, we will use a recursively defined
 1105 sequence $\alpha_0, \alpha_1, \dots$ to build an elementary substructure of $\langle V_\kappa, \in, R \rangle$ that is
 1106 built above an arbitrary $\alpha_0 < \kappa$. Let us fix an arbitrary $\alpha_0 < \kappa$. Given α_n ,
 1107 $\alpha_n + 1$ is defined as the least β , $\alpha_n \leq \beta$ that satisfies the following for any
 1108 formula φ , $p_1, \dots, p_m \in V_{\alpha_n}$, $m \in \omega$:

$$\text{If } \langle V_\kappa, \in, R \rangle \models \exists x\varphi(p_1, \dots, p_n), \text{ then } \langle V_\kappa, \in, R \rangle \models \varphi(x, p_1, \dots, p_n) \quad (3.91)$$

⁴⁵This holds for Σ_n^m -formulas alike.

⁴⁶Note that unlike in previous sections, it is worth noting that φ is now a sentence so we don't have to worry whether P is free in φ .

⁴⁷A different yet interesting approach is taken by Tate in ??. He states that for $n \geq 0$, a formula of order $\leq n$ is called a Π_0^n and a Σ_0^n formula. Then a Π_{m+1}^n is a formula of form $\forall Y\psi(Y)$ where ψ is a Σ_m^n formula and Y is a variable of type n . Finally, a Σ_{m+1}^n is the negation of a Π_m^n formula. So the above holds ad definitio.

1109 Let $\alpha = \bigcup_{n < \omega} \alpha_n$.

1110 Then $\langle V_\alpha, \in, R \cap V_\alpha \rangle \prec \langle V_\kappa, \in, R \rangle$, in other words, for any φ with given
 1111 arbitrary parameters $p_1, \dots, p_n \in V_\alpha$, it holds that

$$\langle V_\alpha, \in, R \cap V_\alpha \rangle \models \varphi(p_1, \dots, p_n) \leftrightarrow \langle V_\kappa, \in, R \rangle \models \varphi(p_1, \dots, p_n) \quad (3.92)$$

1112 Which should be clear from the construction of α □

1113 **Theorem 3.32** *Let κ be an ordinal. The following are equivalent.*

- 1114 (i) κ is inaccessible
 1115 (ii) κ is Π_0^1 -indescribable.

1116 *Proof.* Since Π_0^1 -sentences are first-order sentences, we want to prove that
 1117 κ is an inaccessible cardinal iff whenever a first-order tries to describe κ in
 1118 the sense of definition ??, the formula fails to do so and describes a initial
 1119 segment thereof instead. We have already shown in ?? that there is no way
 1120 to reach an inaccessible cardinal via first-order formulas in ZFC. We will now
 1121 prove it again in for formal clarity.

1122 For (i) \rightarrow (ii), suppose that κ is inaccessible.

1123 Then there is, by lemma ?? a club set of ordinals α such that V_α is
 1124 an elementary substructures of V_κ . For κ to be Π_0^1 -indescribable, we need
 1125 to make sure that given an arbitrary first-order sentence φ satisfied in the
 1126 structure $\langle V_\kappa, \in, R \rangle$, there is an ordinal $\alpha < \kappa$, such that $\langle V_\alpha, \in, R \cap V_\alpha \rangle \models \varphi$.
 1127 But this follows from the definition of elementary substructure.

1128 For (ii) \rightarrow (i), suppose κ is not inaccessible, so it is either singular, or
 1129 there is a cardinal $\nu < \kappa$ such that $\kappa \leq \mathcal{P}(\nu)$ or $\kappa = \omega$.

1130 Suppose κ is singular. Then there is a cardinal $\nu < \kappa$ and a function
 1131 $f : \nu \rightarrow \kappa$ such that $\text{rng}(f)$ is cofinal in κ . Since $f \subseteq V_\kappa$, we can add f as a
 1132 relation to the language. We can do the same with $\{\nu\}$. That means $\langle V_\kappa, \in$
 1133 $, P_1, P_1$ with $P_1 = f, P_2 = \{\nu\}$ is a structure, let $\varphi = P_1 \neq \emptyset \ \& \ \text{rng}(P_1) =$
 1134 P_2 ⁴⁸. Since for every $\alpha < \nu$, $P_1 \cap V_\alpha = \emptyset$, φ is false and therefore describes κ .
 1135 That contradicts the fact that κ was supposed to be Π_0^1 -indescribable, but φ
 1136 is a first-order formula.

1137 Suppose there a cardinal ν satisfying $\kappa \leq \mathcal{P}(\nu)$. Let there be a function
 1138 $f : \mathcal{P}(\nu) \rightarrow \kappa$ that is onto. Then, like in the previous paragraph, we can
 1139 obtain a structure $\langle V_\kappa, \in, P_1, P_2 \rangle$, where $P_1 = f$ like before, but this time
 1140 $P_2 = \mathcal{P}(\nu)$. Again, $\varphi = P_1 \neq \emptyset \ \& \ \text{rng}(P_1) = P_2$ describes κ .

1141 Finally, suppose $\kappa = \omega$, then the sentence $\varphi = \forall x \exists y (x \in y)$ describes κ ,
 1142 there is obviously no $\alpha < \omega$ such that $\langle V_\alpha, \in \rangle \models \varphi$. □

1143

⁴⁸ $\text{rng}(x) = y$ is a first-order formula, see 1.13.

1144 Generally, it should be clear that if a cardinal κ is Π_n^m -indescribable, it
 1145 is also $\Pi_{n'}^{m'}$ -indescribable for every $m' < m, n' < n$. By the same line of
 1146 thought, if a cardinal κ satisfies property implied by Π_n^m -indescribability, it
 1147 satisfies all properties implied by $\Pi_{n'}^{m'}$ -indescribability for $m' < m, n' < n$,
 1148 for example κ is Π_n^m -indescribable for $m \geq 1, n \geq 0$, it is also an inaccessible
 1149 cardinal.

1150 **Theorem 3.33** *If a cardinal κ is Π_1^1 -indescribable, then it is a Mahlo car-*
 1151 *dinal.*

1152 *Proof.* Assuming that κ is Π_1^1 -indescribable, we want to prove that every
 1153 club set in κ contains an inaccessible cardinal.

1154 Consider the following Π_1^1 -sentence:

$$\forall P(\text{"}P \text{ is a function"} \ \& \ \exists x(x = \text{dom}(P) \vee \mathcal{P}(x) = \text{dom}(P)) \rightarrow \rightarrow \exists y(y = \text{rng}(P))) \quad (3.93)$$

1155 where P is a type 2 variable and x, y are type 1 variables, $\text{rng}(P)$ is defined
 1156 in 1.13, $\text{dom}(P)$ in 1.12 and " P is a function" is a first-order formula defined
 1157 in 1.11. We will call this sentence *Inac*, as in "inaccessible", because, given
 1158 a cardinal μ , the following holds if and only if μ is inaccessible:

$$\langle V_\mu, \in \rangle \models \text{Inac} \quad (3.94)$$

1159 So let's fix an arbitrary $C \subset \kappa$, club set in κ . We want to show that it
 1160 contains an inaccessible cardinal. Since C is a subset of V_κ , let's add it to
 1161 the structure $\langle V_\kappa, \in \rangle$, turning it into $\langle V_\kappa, \in, C \rangle$. Then the following holds:

$$\langle V_\kappa, \in, C \rangle \models \text{Inac} \ \& \ \text{"}C \text{ in unbounded"} \quad (3.95)$$

1162 Note that this is correct, because, as we have noted just before introduc-
 1163 ing the statement now being proven, if κ is Π_1^1 -indescribable, it is also Π_0^1 -
 1164 indescribable. So κ is itself inaccessible and therefore $\langle V_\kappa, \in, C \rangle \models \text{Inac}$. C
 1165 is obviously picked so that it is unbounded in κ ⁴⁹.

1166 Now because we have assumed that κ is Π_1^1 -indescribable and *Inac* is
 1167 a Π_1^1 -formula, so *Inac* & " C in unbounded" is equivalent to a Π_1^1 -formula,
 1168 there must be an ordinal α that satisfies

$$\langle V_\alpha, \in, C \cap V_\alpha \rangle \models \text{Inac} \ \& \ \text{"}C \text{ in unbounded"} \quad (3.96)$$

1169 which implies that α is inaccessible.

⁴⁹" C in unbounded" is a first-order formula defined in 1.43

To be finished, we need to verify that $\alpha \in C$. Since $\kappa = V_\kappa$ for inaccessible κ^{50} , $C \cap V_\alpha = C \cap \alpha$, from unboundedness of $C \cap \alpha$ in α , $\bigcup(C \cap \alpha) = \alpha$, which, together with the fact that C is a club set in κ and therefore closed in κ , yields that $\alpha \in C$. \square

TODO asi jako Drake, pozn ze to jde i pro hyper-Mahlovy?

Definition 3.34 (*Extension property*) We say that a cardinal κ has the extension property iff for any $R \subseteq V_\kappa$ there is a transitive set $X \neq V_\kappa$ and an $S \subseteq X$ such that $\langle V_\kappa, \in, R \rangle \prec \langle X, \in, S \rangle$

Definition 3.35 (*Weakly compact cardinal*)

We say that a cardinal κ is weakly compact iff it has the extension property.

The above definitions are equivalent

Theorem 3.36 the following are equivalent:

(i) κ is Weakly compact.

(ii) κ is Π_1^1 -indescribable.

For a proof, see [?][Theorem 6.4]

3.6 Measurable Cardinal

TODO refaktorizovat fle:

Definition 3.37 (*Ultrafilter*)

Given a set X , we say $U \subset \mathcal{P}(X)$ is an ultrafilter iff all of the following hold:

(i) $\emptyset \notin U$

(ii) $\forall x, y (x \subset y \ \& \ x \in U \rightarrow y \in U)$

(iii) $\forall x, y \in U (x \cap y \in U)$

(iv) $\forall x (x \subset X \rightarrow (x \in U \vee (X \setminus x) \in U))$

Definition 3.38 (κ -complete ultrafilter)

We say that an ultrafilter U is κ -complete iff

Definition 3.39 (*non-principal ultrafilter*)

TODO

⁵⁰TODO link — ?

Definition 3.40 (*Measurable Cardinal*)

Let κ be a cardinal. We say κ is a measurable cardinal iff it is an uncountable cardinal with a κ -complete, non-principal ultrafilter.

Theorem 3.41 Let κ be a cardinal. If κ is a measurable cardinal then it is Π_1^2 -indescribable.

Theorem 3.42 *TODO Pod kazdym meritelnym kardinalem existuje ultra-filtr totalne nepopsatelnych, ktore tim padem nejsou sestrojitelne. VIZ VETA Z KANAMORIHO.*

3.7 The Constructible Universe

The constructible universe, denoted L , is a cumulative hierarchy of sets, presented by Kurt Gödel in his 1938 paper *The Consistency of the Axiom of Choice and of the Generalised Continuum Hypothesis*. For a technical description, see below. Assertion of their equality, $V = L$, is called the *axiom of constructibility*. The axiom implies GCH and therefore also AC and contradicts the existence of some of the large cardinals, our goal is to decide whether those introduced earlier are among them.

On order to formally establish this class, we need to formalize the notion of definability first.

Definition 3.43 We say that a set X is definable over a model $\langle M, \in \rangle$ if there is a first-order formula φ together with parameters $p_1, \dots, p_n \in M$ such that

$$X = \{x : x \in M \ \& \ \langle M, \in \rangle \models \varphi(x, p_1, \dots, p_n)\} \quad (3.97)$$

Definition 3.44 (*The set of definable subsets*)

The following is a set of all definable subsets of a given set M , denoted $Def(M)$.

$$Def(M) = \{\{y : x \in M \wedge \langle M, \in \rangle \models \varphi(y, u_1, \dots, u_n)\} \mid \varphi \text{ is a first-order formula, } p_1, \dots, p_n \in M\} \quad (3.98)$$

We will use $Def(M)$ in the following construction in the way the powerset operation is used when constructing the usual Von Neumann's hierarchy of sets⁵¹

Now we can recursively build L .

⁵¹For that reason, some authors use $\mathcal{P}^{(\cdot)}M$ instead of $Def(M)$, see section 11 of [?] for one such example.

1227 **Definition 3.45** (*The Constructible universe*)

1228

(i)

$$L_0 \stackrel{\text{def}}{=} \emptyset \quad (3.99)$$

(ii)

$$L_{\alpha+1} \stackrel{\text{def}}{=} \text{Def}(L_\alpha) \quad (3.100)$$

(iii)

$$L_\lambda = \bigcup_{\alpha < \lambda} L_\alpha \text{ If } \lambda \text{ is a limit ordinal} \quad (3.101)$$

(iv)

$$L = \bigcup_{\alpha \in \text{Ord}} L_\alpha \quad (3.102)$$

1229 Note that while L bears very close resemblance to V , the difference is,
 1230 that in every successor step of constructing V , we take every subset of V_α
 1231 to be $V_{\alpha+1}$, whereas $L_{\alpha+1}$ consists only of definable subsets of L_α . Also note
 1232 that L is transitive.

1233 In order to

1234 **Theorem 3.46** *Let L be as in ??.*

$$L \models \text{ZFC} \quad (3.103)$$

1235 For details, refer to Jech: [?][Theorem 13.3].

1236 **Definition 3.47** (*Constructibility*)

1237 *The axiom of constructibility say that every set is constructible. It is usually*
 1238 *denoted as $L = V$.*

1239 Without providing a proof, we will introduce two important results es-
 1240 tablished by Gödel in TODO citace!

1241 **Theorem 3.48** (*Constructibility \rightarrow Choice*)

$$\text{ZF} \models \text{Constructibility} \rightarrow \text{Choice} \quad (3.104)$$

1242 **Definition 3.49** (*GCH*)

1243 Generalized Continuum Hypothesis, usually denoted GCH for brevity, refers
 1244 to the following statement:

$$\aleph_{n+1} = \mathcal{P}(\aleph_n) \quad (3.105)$$

1245 **Theorem 3.50** (*Constructibility \rightarrow Continuum Hypothesis*)

$$\text{ZF} \models \text{Constructibility} \rightarrow \text{GCH} \quad (3.106)$$

1246 It is worth mentioning that Gödel's proof of *Constructibility \rightarrow GCH* featured
1247 the first formal use of a reflection principle. For the actual proofs, see for
1248 example TODO citace!! Kunen?

1249 Since *GCH* implies that κ is a limit cardinal iff κ is a strong limit cardinal
1250 for every κ , the distinctions between inaccessible and weakly inaccessible
1251 cardinals as well as between Mahlo and weakly Mahlo cardinals vanish.

1252 TODO bez Con?

1253 Pokud κ je nějaký kardinál ve V , je takový i v L ?

1254 TODO zachovávaní $\epsilon = \delta = \pi_0$ formulí v tranzitivních struk-
1255 turách!!!!!!

1256 TODO lemma: ord jsou ord v L (Kunen?)

1257 TODO lemma: card jsou card v L

1258 **Theorem 3.51** (*Inaccessibility in L*)

1259 *Let κ be an inaccessible cardinal. Then $L \models "$ κ is inaccessible".*

1260 *Proof.* TODO prepsat: !!!!!

1261 Indeed, if $L \models "$ κ is not a cardinal", there is some $\mu < \kappa$ and some $f \in L$
1262 such that $L \models f : \mu \rightarrow \kappa$ is surjective.

1263 However, this is a Σ_0 property and hence, in V , $f : \mu \rightarrow \kappa$ is surjective.

1264 Contradiction. Since $\kappa > \omega$ (as an ordinal), this also yields that $L \models$
1265 " κ is uncountable".

1266 Repeating this argument with cofinal $f : \mu \rightarrow \kappa$ yields that $L \models "$ κ is regular".

1267 Since $L \models \text{GCH}$ by ref TODO, it now suffices to prove that $L \models$
1268 " κ is a limit cardinal". This is trivial, because for any ordinal $\gamma < \kappa$, we
1269 have that $(\gamma^+)^V < \kappa$ and since cardinals in V are cardinals in L , this proves

$$L \models \forall \gamma < \kappa \exists \gamma < \mu < \kappa : "\mu \text{ is a cardinal}." \quad (3.107)$$

1270 □

1271 **Theorem 3.52** (*Mahloness in L*)

1272 *Let κ be a Mahlo cardinal. Then $L \models "$ κ is Mahlo".*

1273 *Proof.* Assume that there is a Mahlo cardinal and let's denote it κ . From
1274 the definition of Mahloness in ??, the set

$$\{\alpha : \alpha \in \kappa \ \& \ "\alpha \text{ is inaccessible}"\} \quad (3.108)$$

1275 is stationary in κ and κ is also inaccessible.

1276 Now let $T \stackrel{\text{def}}{=} \{\alpha \in \kappa : L \models \text{"}\alpha \text{ is regular"}\}$. By the argument given
 1277 above, any α that is regular in V is regular in L and hence $S \subseteq T$. Suppose
 1278 that $L \models \text{"}\kappa \text{ is not Mahlo"}\text{"}$. Then there is some $C \subseteq \kappa$ such that $L \models C$ is
 1279 club in κ and $C \cap T = \emptyset$. $L \models C$ is club in κ and $C \cap T = \emptyset$. Being club is
 1280 a Σ_0 property and hence $C \subseteq \kappa$ is club in κ . Since $S \subseteq T$, we have that
 1281 $C \cap S = \emptyset$ and hence S is not stationary in V . This is a contradiction and
 1282 we therefore must have that $L \models T$ is stationary.

1283 Thus, κ remains Mahlo in L .

1284

□

Theorem 3.53

$$\text{Con}(L) - > \text{Con}(L + \exists \kappa (\kappa \text{ is a Measurable Cardinal})) \text{ then } \text{Con}(L) \quad (3.109)$$

1285 Ktera vera?

1286 TODO vyska / sirka univerza

1287 TODO co velky pismena ve jmenech kardinalu?

1288 TODO zdvodneni

1289

1290 TODO kratka diskuse jestli refl implikuje transcendenci na L - polemika,
 1291 nazor - $V=L$ a slaba kompaktnost a dalsi

1292

1293 **4 Conclusion**

1294 TODO na konec

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