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4 REFLECTION PRINCIPLES AND LARGE  
5 CARDINALS  
6 Bakalářská práce

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<sup>10</sup> Prohlašuji, že jsem bakalářskou práci vypracoval samostatně a že jsem uvedl  
<sup>11</sup> všechny použité prameny a literaturu.

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## Abstract

Práce zkoumá vztah tzv. principů reflexe a velkých kardinálů. Lévy ukázal, že v ZFC platí tzv. věta o reflexi a dokonce, že věta o reflexi je ekvivalentní schématu nahrazení a axiomu nekonečna nad teorií ZFC bez axiomu nekonečna a schématu nahrazení. Tedy lze na větu o reflexi pohlížet jako na svého druhu axiom nekonečna. Práce zkoumá do jaké míry a jakým způsobem lze větu o reflexi zobecnit a jaký to má vliv na existenci tzv. velkých kardinálů. Práce definuje nedosažitelné, Mahlovy a nepopsatelné kardinály a ukáže, jak je lze zavést pomocí reflexe. Přirozenou limitou kardinálů získaných reflexí jsou kardinály nekonzistentní s L. Práce nabídne intuitivní zdůvodnění, proč tomu tak je.

## Abstract

This thesis aims to examine relations between the so called Reflection Principles and Large cardinals. Lévy has shown that the Reflection Theorem is a sound theorem of ZF and it is equivalent to the Replacement Scheme and the Axiom of Infinity. From this point of view, Reflection theorem can be seen a specific version of an Axiom of Infinity. This paper aims to examine the Reflection Principle and its generalisations with respect to the existence of Large Cardinals. This thesis will establish the Inaccessible, Mahlo and Indescribable cardinals and show how can those be defined via reflection. A natural limit of Large Cardinals obtained via reflection are cardinals inconsistent with L. This thesis will offer an intuitive explanation of why this holds.

# Contents

39	<b>Contents</b>	
40	<b>1 Introduction</b>	<b>4</b>
41	1.1 Motivation and Origin . . . . .	4
42	1.2 Notation and Terminology . . . . .	4
43	1.2.1 The Language of Set Theory . . . . .	4
44	1.2.2 The Axioms . . . . .	5
45	1.2.3 The Transitive Universe . . . . .	8
46	1.2.4 Cardinal Numbers . . . . .	10
47	1.2.5 Relativisation and Absoluteness . . . . .	11
48	1.2.6 More Functions . . . . .	13
49	1.2.7 Structure, Substructure and Embedding . . . . .	14
50	<b>2 Levy's First-Order Reflection</b>	<b>15</b>
51	2.1 Lévy's Original Paper . . . . .	15
52	2.2 Contemporary Restatement . . . . .	18
53	<b>3 Conclusion</b>	<b>26</b>

# 1 Introduction

## 1.1 Motivation and Origin

“The Universe of sets cannot be uniquely characterised (i. e. distinguished from all its initial elements) by any internal structural property of the membership relation in it, which is expressible in any logic of finite or transfinite type, including infinitary logics of any cardinal order.”

— Kurt Gödel [Wang, 1997]

## 1.2 Notation and Terminology

### 1.2.1 The Language of Set Theory

This text assumes the knowledge of basic terminology and some results from first-order predicate logic.<sup>1</sup>

We will now shortly review the basic notions that allow us to define the *Zermelo–Fraenkel* set theory.

When we talk about *class*, we have the notion of definable class in mind. If  $\varphi(x, p_1, \dots, p_n)$  is a formula in the language of set theory, we call

$$A = \{x : \varphi(x)\} \tag{1.1}$$

a class of all sets satisfying  $\varphi(x)$  in a sense that

$$x \in A \leftrightarrow \varphi(x) \tag{1.2}$$

Given classes  $A, B$ , one can easily define the elementary set operations such as  $A \cap B$ ,  $A \cup B$ ,  $A \setminus B$ ,  $\bigcup A$ , see the first part of [Jech, 2006] for details. Axioms are the tools by which we can decide whether a particular class is “small enough” to be considered a set<sup>2</sup>. A class that fails to be considered a set is called a *proper class*.

We will often write “ $M$  is a limit ordinal”, it should always be clear that this can be rewritten as a formula that was introduced earlier.

77

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<sup>1</sup>todo odkaz na pripadny zdroj? svejdar? neco en?

<sup>2</sup>“Small enough” means that it doesn’t introduce a paradox similar to Russell’s.

78 **1.2.2 The Axioms**79 **Definition 1.1** (*The Existence of a Set*)

$$\exists x(x = x) \quad (1.3)$$

80 **Definition 1.2** (*Axiom of Extensionality*)

$$\forall x, y(x = y \leftrightarrow \forall z(z \in x \leftrightarrow z \in y)) \quad (1.4)$$

81 **Definition 1.3** (*Axiom Schema of Specification*)82 *The following yields an axiom for every first-order formula  $\varphi(x, p_1, \dots, p_n)$*   
83 *with no free variables other than  $x, p_1, \dots, p_n$ .*

$$\forall x, p_1, \dots, p_n \exists y \forall z(z \in y \leftrightarrow z \in x \ \& \ \varphi(z, p_1, \dots, p_n)) \quad (1.5)$$

84 We will now provide two definitions that are not axioms, but will be  
85 helpful in establishing some axioms in a more comprehensible way.86 **Definition 1.4** ( $x \subseteq y, x \subset y$ )

$$x \subseteq y \leftrightarrow (\forall z(z \in x \rightarrow z \in y)) \quad (1.6)$$

87

$$x \subset y \leftrightarrow x \subseteq y \ \& \ x \neq y \quad (1.7)$$

88 *We read  $x \subseteq y$  as  $x$  is a subset of  $y$  and  $x \subset y$  as  $x$  is a proper subset of  $y$ .*89 **Definition 1.5** (*Empty Set*) *For an arbitrary set  $x$ , the empty set, repre-*  
90 *sented by the symbol " $\emptyset$ ", is defined by the following formula:*

$$(\forall y(y \in x) \rightarrow (y \in \emptyset \leftrightarrow \neg(y = y))) \quad (1.8)$$

91  $\emptyset$  is a set due to Specification. While the empty set could also be defined by  
92 the formula  $\forall y(y \in \emptyset \leftrightarrow \neg(y = y))$ , the version we use is  $\Delta_0$ , which we will find  
93 useful later. The two definitions yield the same set for every  $x$  given because  
94 of Extensionality.95 **Definition 1.6** (*Axiom of Pairing*)

$$\forall x, y \exists z \forall q(q \in z \leftrightarrow q = x \vee q = y) \quad (1.9)$$

96 **Definition 1.7** (*Axiom of Union*)

$$\forall x \exists y \forall z(z \in y \leftrightarrow \exists q(z \in q \ \& \ q \in x)) \quad (1.10)$$

97 Now we can introduce more axioms.

98 **Definition 1.8** (*Axiom of Foundation*)

$$\forall x(x \neq \emptyset \rightarrow (\exists y \in x)(x \cap y = \emptyset)) \quad (1.11)$$

99 **Definition 1.9** (*Axiom of Powerset*)

$$\forall x \exists y \forall z (z \in y \leftrightarrow z \subseteq x) \quad (1.12)$$

100 **Definition 1.10** (*Axiom of Infinity*)

$$\exists x(\emptyset \in x \ \& \ (\forall y \in x)(y \cup \{y\} \in x)) \quad (1.13)$$

101 *The least set satisfying this is denoted “ $\omega$ ”.*

102 Let us introduce a few more definitions that will make the two remaining  
103 axioms more comprehensible.

104 **Definition 1.11** (*Powerset Function*)

105 *Given a set  $x$ , the powerset of  $x$ , denoted  $\mathcal{P}(x)$  and satisfying 1.9, is defined*  
106 *as follows:*

$$\mathcal{P}(x) \stackrel{\text{def}}{=} \{y : y \subseteq x\} \quad (1.14)$$

107 **Definition 1.12** (*Function*)

108 *Given arbitrary first-order formula  $\varphi(x, y, p_1, \dots, p_n)$ , we say that  $\varphi$  is a func-*  
109 *tion iff*

$$\forall x, y, z, p_1, \dots, p_n (\varphi(x, y, p_1, \dots, p_n) \ \& \ \varphi(x, z, p_1, \dots, p_n) \rightarrow y = z) \quad (1.15)$$

110 When a  $\varphi(x, y)$  is a function, we also write the following:

$$f(x) = y \leftrightarrow \varphi(x, y) \quad (1.16)$$

111 Alternatively,  $f = \{\langle x, y \rangle : \varphi(x, y)\}$  is a class.

112 **Definition 1.13** (*Domain of a Function*)

113 *Let  $f$  be a function. We call the domain of  $f$  the set of all sets for which  $f$*   
114 *yields a value. We use “ $\text{Dom}(f)$ ” to refer to this set.*

$$x \in \text{Dom}(f) \leftrightarrow \exists y(f(x) = y) \quad (1.17)$$

115 We say “ $f$  is a function on  $A$ ”,  $A$  being a class, if  $A = \text{dom}(f)$ .

116 **Definition 1.14** (*Range of a Function*)

117 *Let  $f$  be a function. We call the range of  $f$  the set of all sets that are images*  
118 *of other sets via  $f$ . We use “ $\text{Rng}(f)$ ” to refer to this set.*

$$x \in \text{Rng}(f) \leftrightarrow \exists y(f(y) = x) \quad (1.18)$$

119 We say that  $f$  is a *function into*  $A$ ,  $A$  being a class, if  $\text{rng}(f) \subseteq A$ . We say  
 120 that  $f$  is a *function onto*  $A$  if  $\text{rng}(f) = A$ . We say a function  $f$  is a *one to one*  
 121 *function*, iff

$$(\forall x_1, x_2 \in \text{dom}(f))(f(x_1) = f(x_2) \rightarrow x_1 = x_2) \quad (1.19)$$

122 We say that  $f$  is a *bijection* iff it is a one to one function that is onto.

123 Note that  $\text{Dom}(f)$  and  $\text{Rng}(f)$  are not definitions in a strict sense, they  
 124 are in fact definition schemas that yield definitions for every function  $f$  given.  
 125 Also note that they can be easily modified for  $\varphi$  instead of  $f$ , with the only  
 126 difference being the fact that it is then defined only for those  $\varphi$ s that are  
 127 functions, which must be taken into account. This is worth noting as we will  
 128 use the notions of *function* and *formula* interchangeably.

129 **Definition 1.15** (*Function Defined For All Ordinals*)

130 We say a function  $f$  is defined for all ordinals, this is sometimes written  
 131  $f : \text{Ord} \rightarrow A$  for any class  $A$ , if  $\text{Dom}(f) = \text{Ord}$ . Alternatively,

$$(\forall \alpha \in \text{Ord})(\exists y \in A)(f(\alpha) = y) \quad (1.20)$$

132 And now for the axioms.

133 **Definition 1.16** (*Axiom Schema of Replacement*)

134 The following is an axiom for every first-order formula  $\varphi(x, p_1, \dots, p_n)$  with  
 135 no free variables other than  $x, p_1, \dots, p_n$ .

$$“\varphi \text{ is a function}” \rightarrow \forall x \exists y \forall z (z \in y \leftrightarrow (\exists q \in x)(\varphi(x, y, p_1, \dots, p_n))) \quad (1.21)$$

136 **Definition 1.17** (*Choice*)

137

$$\begin{aligned} &\forall x \exists f ((f \text{ is a choice function with } \text{dom}(f) = x \setminus \{\emptyset\}) \\ &\quad \& \forall y ((y \in x \& y \neq \emptyset) \rightarrow f(y) \in y)) \end{aligned} \quad (1.22)$$

138 We will refer to the axioms by their name, written in italic type, e.g.  
 139 *Foundation* refers to the Axiom of Foundation. Now we need to define the  
 140 set theories to be used in the article.

141 **Definition 1.18** (S)

142 We call  $\mathbf{S}$  an axiomatic theory in the language  $\mathcal{L} = \{=, \in\}$  with exactly the  
 143 following axioms:

- 144 (i) Existence of a set (see 1.1)
- 145 (ii) Extensionality (see 1.2)
- 146 (iii) Specification (see 1.3)



- 147 (iv) Foundation (see 1.8)
- 148 (v) Pairing (see 1.6)
- 149 (vi) Union (see 1.7)
- 150 (vii) Powerset (see 1.9)

151 **Definition 1.19** (ZF)

152 We call ZF an axiomatic theory in the language  $\mathcal{L} = \{=, \in\}$  that contains  
 153 all the axioms of S in addition to the following:

- 154 (i) Replacement schema (see 1.16)
- 155 (ii) Infinity (see 1.10)
- 156 Existence of a set is usually left out because it is a consequence of infinity.

157 **Definition 1.20** (ZFC)

158 ZFC is an axiomatic theory in the language  $\mathcal{L} = \{=, \in\}$  that contains all the  
 159 axioms of ZF plus Choice (1.17).

160

161 **1.2.3 The Transitive Universe**

162 **Definition 1.21** (Transitive Class)

163 We say a class  $A$  is transitive iff

$$(\forall x \in A)(x \subseteq A) \quad (1.23)$$

164 **Definition 1.22** (Well Ordered Class) A class  $A$  is said to be well ordered  
 165 by  $\in$  iff the following hold:

- 166 (i)  $(\forall x \in A)(x \not\subseteq x)$  (Antireflexivity)
- 167 (ii)  $(\forall x, y, z \in A)(x \in y \ \& \ y \in z \rightarrow x \in z)$  (Transitivity)
- 168 (iii)  $(\forall x, y \in A)(x = y \vee x \in y \vee y \in x)$  (Linearity)
- 169 (iv)  $(\forall x \subseteq A)(x \neq \emptyset \rightarrow (\exists y \in x)(\forall z \in x)(z = y \vee z \in y))$  (Existence of the  
 170 least element)

171 **Definition 1.23** (Ordinal Number)

172 A set  $x$  is said to be an ordinal number if it is transitive and well-ordered  
 173 by  $\in$ .

174 For the sake of brevity, we usually just say “ $x$  is an ordinal”. Note that  
 175 “ $x$  is an ordinal” is a well-defined formula in the language of set theory, since  
 176 1.21 is a first-order formula and 1.22 is in fact a conjunction of four first-  
 177 order formulas. Ordinals will be usually denoted by lower case greek letters,  
 178 starting from the beginning of the alphabet:  $\alpha, \beta, \gamma, \dots$ . Given two different  
 179 ordinals  $\alpha, \beta$ , we will write  $\alpha < \beta$  for  $\alpha \in \beta$ , see Lemma 2.11 in [Jech, 2006]  
 180 for technical details.

181 **Definition 1.24** (*Non-Zero Ordinal*) We say an ordinal  $\alpha$  is non-zero iff  
 182  $\alpha \neq \emptyset$ .

183 **Definition 1.25** (*Successor Ordinal*)  
 184 Consider the following function defined for all ordinals. Let  $\beta$  be an arbitrary  
 185 ordinal. We call  $S$  the successor function.

$$S(\beta) = \beta \cup \{\beta\} \quad (1.24)$$

186 An ordinal  $\alpha$  is called a successor ordinal iff there is an ordinal  $\beta$ , such that  
 187  $\alpha = S(\beta)$ . We also write  $\alpha = \beta + 1$ .

188 **Definition 1.26** (*Limit Ordinal*)  
 189 A non-zero ordinal  $\alpha$  is called a limit ordinal iff it is not a successor ordinal.

190 **Definition 1.27** (*Ord*)  
 191 The class of all ordinal numbers, which we will denote “ $\text{Ord}$ ”<sup>3</sup> is the proper  
 192 class defined as follows.

$$x \in \text{Ord} \leftrightarrow x \text{ is an ordinal} \quad (1.25)$$

193 **Definition 1.28** (*Von Neumann’s Hierarchy*)  
 194 The Von Neumann’s Hierarchy is a collection of sets indexed by elements of  
 195  $\text{Ord}$ , defined recursively in the following way:

$$(i) \quad V_0 = \emptyset \quad (1.26)$$

$$(ii) \quad V_{\alpha+1} = \mathcal{P}(V_\alpha) \text{ for any ordinal } \alpha \quad (1.27)$$

$$(iii) \quad V_\lambda = \bigcup_{\beta < \lambda} V_\beta \text{ for a limit ordinal } \lambda \quad (1.28)$$

196 We will also refer to the Von Neumann’s Hierarchy as Von Neumann’s Uni-  
 197 verse or the Cumulative Hierarchy.

198 **Definition 1.29** (*Rank*)  
 199 Given a set  $x$ , we say that the rank of  $x$  (written as  $\text{rank}(x)$ ) is the least  
 200 ordinal  $\alpha$  such that  $x \in V_{\alpha+1}$

201 Due to *Regularity*, every set has a rank.<sup>4</sup>

202

<sup>3</sup>Other authors use “ $\text{On}$ ”, we will stick to the notation used in [Jech, 2006]

<sup>4</sup>See chapter 6 of [Jech, 2006] for details.

### 1.2.4 Cardinal Numbers

#### Definition 1.30 (Cardinality)

Given a set  $x$ , let the cardinality of  $x$ , written  $|x|$ , be defined as the smallest ordinal number such that there is a one to one mapping from  $x$  to  $\alpha$ .

#### Definition 1.31 (Aleph function)

Let  $\omega$  be the set defined by  $\omega$ . We will recursively define the function  $\aleph$  for all ordinals.

(i)  $\aleph_0 = \omega$

(ii)  $\aleph_{\alpha+1}$  is the least cardinal larger than  $\aleph_\alpha$ <sup>5</sup>

(iii)  $\aleph_\lambda = \bigcup_{\beta < \lambda} \aleph_\beta$  for a limit ordinal  $\lambda$

If  $\kappa = \aleph_\alpha$  and  $\alpha$  is a successor ordinal, we call  $\kappa$  a successor cardinal. If  $\alpha$  is a limit ordinal, we call  $\kappa$  a limit cardinal.

#### Definition 1.32 (Cardinal number)

(i) A set  $x$  is called a finite cardinal iff  $x \in \omega$ .

(ii) A set is called an infinite cardinal iff there is an ordinal  $\alpha$  such that  $\aleph_\alpha = x$

(iii) A set is called a cardinal iff it is either a finite cardinal or an infinite cardinal.

We say  $\kappa$  is an uncountable cardinal iff it is an infinite ordinal and  $\aleph_0 < \kappa$ . Infinite cardinals will be notated by lowercase greek letters from the middle of the alphabet, e.g.  $\kappa, \mu, \nu, \dots$ <sup>6</sup>

For formal details as well as why every set can be well-ordered assuming *Choice*, and therefore has a cardinality, see [Jech, 2006].

#### Definition 1.33 (Sequence)

We say that a function  $\varphi(x, y)$  is a sequence iff there is an ordinal  $\alpha$  such that  $\text{dom}(\varphi) = \alpha$ . In other words, a function is called a sequence if it is defined exactly for every ordinal from 0 to some  $\alpha$ . We then say it is an  $\alpha$ -sequence. We usually write  $\langle \beta_i : i \in \alpha \rangle$  or  $\langle \beta_0, \beta_1, \dots \rangle$  when referring to a sequence,  $\xi_i$  denote the elements of  $\text{rng}(\varphi)$  for every  $i \in \text{dom}(\varphi)$ .

#### Definition 1.34 (Cofinal Subset)

Given a class  $A$ , we say that  $B \subseteq A$  is cofinal in  $A$  iff

$$(\forall x \in A)(\exists y \in B)(x \in y) \quad (1.29)$$

<sup>5</sup>“The least cardinal larger than  $\aleph_\alpha$ ” is sometimes notated as  $\aleph_\alpha^+$

<sup>6</sup>Except  $\lambda$  which is preferably used for limit ordinals.

235 **Definition 1.35** (*Cofinality of a Limit Ordinal*)

236 Let  $\lambda$  be a limit ordinal. We say that the cofinality of  $\lambda$  is  $\kappa$  iff  $\kappa$  is the least  
237 cardinal, such that there is a cofinal  $\kappa$ -sequence  $\langle \beta_\xi : \xi < \kappa \rangle$ , such that

$$\sup(\{\beta_\xi : \xi < \kappa\}) = \lambda \quad (1.30)$$

238 We write  $cf(\lambda) = \kappa$ .

239 **Definition 1.36** (*Regular Cardinal*)

240 We say a cardinal  $\kappa$  is regular iff  $cf(\kappa) = \kappa$

241 **Definition 1.37** (*Strong Limit Cardinal*)

242 We say that an ordinal  $\kappa$  is a strong limit cardinal if it is a limit cardinal  
243 and

$$(\forall \alpha \in \kappa)(\mathcal{P}(\alpha) \in \kappa) \quad (1.31)$$

244 **Definition 1.38** (*Generalised Continuum Hypothesis*)

245

$$\aleph_{\alpha+1} = \mathcal{P}(\aleph_\alpha) \quad (1.32)$$

246 If *GCH* holds (for example in Gödel's  $L$ , see chapter 3), the notions of limit  
247 cardinal and strong limit cardinal are equivalent.

248

### 249 1.2.5 Relativisation and Absoluteness

250 **Definition 1.39** (*Relativization*)

251 Let  $M$  be a class,  $R \subseteq M \times M$  and let  $\varphi(p_1, \dots, p_n)$  be a first-order formula  
252 with no free variables besides  $p_1, \dots, p_n$ . The relativization of  $\varphi$  to  $M$  and  $R$   
253 is the formula, written as  $\varphi^{M,R}(p_1, \dots, p_n)$ , defined in the following inductive  
254 manner:

- 255 (i)  $(x \in y)^{M,R} \leftrightarrow R(x, y)$
- 256 (ii)  $(x = y)^{M,R} \leftrightarrow x = y$
- 257 (iii)  $(\neg \varphi)^{M,R} \leftrightarrow \neg \varphi^{M,R}$
- 258 (iv)  $(\varphi \ \& \ \psi)^{M,R} \leftrightarrow \varphi^{M,R} \ \& \ \psi^{M,R}$
- 259 (v)  $(\varphi \vee \psi)^{M,R} \leftrightarrow \varphi^{M,R} \vee \psi^{M,R}$
- 260 (vi)  $(\varphi \rightarrow \psi)^{M,R} \leftrightarrow \varphi^{M,R} \rightarrow \psi^{M,R}$
- 261 (vii)  $(\exists x \varphi(x))^{M,R} \leftrightarrow (\exists x \in M) \varphi^{M,R}(x)$
- 262 (viii)  $(\forall x \varphi(x))^{M,R} \leftrightarrow (\forall x \in M) \varphi^{M,R}(x)$

263 When  $R = \in \cap (M \times M)$ , we usually write  $\varphi^M$  instead of  $\varphi^{M,R}$ . When we talk  
264 about  $\varphi^M(p_1, \dots, p_n)$ , it is understood that  $p_1, \dots, p_n \in M$ . We will also use  
265  $M \models \varphi(p_1, \dots, p_n)$  and  $\varphi^M(p_1, \dots, p_n)$  interchangeably.

**Definition 1.40** (*Absoluteness*) Given a transitive class  $M$ , we say a formula  $\varphi$  is absolute in  $M$  if for all  $p_1, \dots, p_n \in M$

$$\varphi^M(p_1, \dots, p_n) \leftrightarrow \varphi(p_1, \dots, p_n) \quad (1.33)$$

**Definition 1.41** (*Hierarchy of First-Order Formulas*)

- (I) A first-order formula  $\varphi$  is  $\Delta_0$  iff it is logically equivalent to a first-order formula  $\varphi'$  satisfying any of the following:
- (i)  $\varphi'$  contains no quantifiers
  - (ii)  $y$  is a set,  $\psi$  is a  $\Delta_0$  formula, and  $\varphi'$  is either  $(\exists x \in y)\psi(y)$  or  $(\forall x \in y)\psi(y)$ .
  - (iii)  $\psi_1, \psi_2$  are  $\Delta_0$  formulas and  $\varphi'$  is any of the following:  $\psi_1 \vee \psi_2$ ,  $\psi_1 \& \psi_2$ ,  $\psi_1 \rightarrow \psi_2$ ,  $\neg\psi_2$ ,
- (II) If a formula is  $\Delta_0$  it is also  $\Sigma_0$  and  $\Pi_0$
- (III) A formula  $\varphi$  is  $\Pi_n + 1$  if it is logically equivalent to a formula  $\varphi'$  such that  $\varphi' = \forall x\psi$  where  $\psi$  is a  $\Sigma_n$ -formula for any  $n < \omega$ .
- (IV) A formula  $\varphi$  is  $\Sigma_n + 1$  if it is logically equivalent to a formula  $\varphi'$  such that  $\varphi' = \forall x\psi$  where  $\psi$  is a  $\Pi_n$ -formula for any  $n < \omega$ .

Note that we can use the pairing function so that for  $\forall p_1, \dots, p_n \psi(p_1, \dots, p_n)$ , there is a logically equivalent formula of the form  $\forall x\psi'(x)$ .

**Lemma 1.42** ( $\Delta_0$  absoluteness) Let  $\varphi$  be a  $\Delta_0$  formula, then  $\varphi$  is absolute in any transitive class  $M$ .

*Proof.* This will be proven by induction over the complexity of a given  $\Delta_0$  formula  $\varphi$ . Let  $M$  be an arbitrary transitive class.

Atomic formulas are always absolute by the definition of relativisation, see 1.39. Suppose that  $\Delta_0$  formulas  $\psi_1$  and  $\psi_2$  are absolute in  $M$ . Then from relativization,  $(\psi_1 \& \psi_2)^M \leftrightarrow \psi_1^M \& \psi_2^M$ , which is, from the induction hypothesis, equivalent to  $\psi_1 \& \psi_2$ . The same holds for  $\vee, \rightarrow, \neg$ .

Suppose that a  $\Delta_0$  formula  $\psi$  is absolute in  $M$ . Let  $y$  be a set and let  $\varphi = (\exists x \in y)\psi(x)$ . From relativization,  $(\exists x\psi(x))^M \leftrightarrow (\exists x \in M)\psi^M(x)$ . Since the hypotheses makes it clear that  $\psi^M \leftrightarrow \psi$ , we get  $((\exists x \in y)\psi(x))^M \leftrightarrow (\exists x \in y \cap M)\psi(x)$ , which is the equivalent of  $\varphi^M \leftrightarrow \varphi$ . The same applies to  $\varphi = (\forall x \in y)\psi(x)$ .  $\square$

**Lemma 1.43** (*Downward Absoluteness*)

Let  $\varphi$  be a  $\Pi_1$  formula and  $M$  a transitive class. Then the following holds:

$$(\forall p_1, \dots, p_n \in M)(\varphi(p_1, \dots, p_n) \rightarrow \varphi(p_1, \dots, p_n)^M) \quad (1.34)$$

299 *Proof.* Since  $\varphi(p_1, \dots, p_n)$  is  $\Pi_1$ , there is a  $\Delta_0$  formula  $\psi(p_1, \dots, p_n, x)$  such  
 300 that  $\varphi = \forall x \psi(p_1, \dots, p_n, x)$ . From relativization and lemma 1.42,  $\varphi^M(p_1, \dots, p_n) \leftrightarrow$   
 301  $(\forall x \in M) \psi(p_1, \dots, p_n, x)$ .

302 Assume that for  $p_1, \dots, p_n \in M$  fixed, that  $\forall x \psi(p_1, \dots, p_n, x)$  holds, but  
 303  $(\forall x \in M) \psi(p_1, \dots, p_n, x)$  does not. Therefore  $\exists x \neg \psi(p_1, \dots, p_n, x)$ , which  
 304 contradicts  $\forall x \psi(p_1, \dots, p_n, x)$ .  $\square$

305 **Lemma 1.44** (*Upward Absoluteness*)

306 *Let  $\varphi$  be a  $\Sigma_1$  formula and  $M$  a transitive class. Then the following holds:*

$$(\forall p_1, \dots, p_n \in M)(\varphi^M(p_1, \dots, p_n) \rightarrow \varphi(p_1, \dots, p_n)) \quad (1.35)$$

307 *Proof.* Since  $\varphi(p_1, \dots, p_n)$  is  $\Sigma_1$ , there is a  $\Delta_0$  formula  $\psi(p_1, \dots, p_n, x)$  such  
 308 that  $\varphi = \exists x \psi(p_1, \dots, p_n, x)$ . From relativization and lemma 1.42,  $\varphi^M(p_1, \dots, p_n) \leftrightarrow$   
 309  $(\exists x \in M) \psi(p_1, \dots, p_n, x)$ .

310 Assume that for  $p_1, \dots, p_n \in M$  fixed, that  $(\exists x \in M) \psi(p_1, \dots, p_n, x)$   
 311 holds, but  $\exists x \psi(p_1, \dots, p_n, x)$  does not. This is an obvious contradiction.  $\square$

### 312 1.2.6 More Functions

313 **Definition 1.45** (*Strictly Increasing Function*)

314 *A function  $f : \text{Ord} \rightarrow \text{Ord}$  is said to be strictly increasing iff*

$$\forall \alpha, \beta \in \text{Ord} (\alpha < \beta \rightarrow f(\alpha) < f(\beta)). \quad (1.36)$$

315 **Definition 1.46** (*Continuous Function*)

316 *A function  $f : \text{Ord} \rightarrow \text{Ord}$  is said to be continuous iff*

$$\lambda \text{ is limit} \rightarrow f(\lambda) = \bigcup_{\alpha < \lambda} f(\alpha). \quad (1.37)$$

317 **Definition 1.47** (*Normal Function*)

318 *A function  $f : \text{Ord} \rightarrow \text{Ord}$  is said to be normal iff it is strictly increasing*  
 319 *and continuous.*

320 **Definition 1.48** (*Fixed Point*)

321 *We say  $x$  is a fixed point of a function  $f$  iff  $x = f(x)$ .*

322 **Definition 1.49** (*Unbounded Class*)

323 *We say a class  $A$  is unbounded iff*

$$\forall x (\exists y \in A)(x < y) \quad (1.38)$$

324 **Definition 1.50** (*Limit Point*)

325 *Given a class  $x \subseteq \text{Ord}$ , we say that  $\alpha \neq \emptyset$  is a limit point of  $x$  iff*

$$\alpha = \bigcup (x \cap \alpha) \quad (1.39)$$

326 **Definition 1.51** (*Closed Class*)

327 *We say a class  $A \subseteq \text{Ord}$  is closed iff it contains all its limit points.*

328 **Definition 1.52** (*Club set*)

329 *For a regular uncountable cardinal  $\kappa$ , a set  $x \subset \kappa$  is a closed unbounded*  
 330 *subset, abbreviated as a club set, iff  $x$  is both closed and unbounded in  $\kappa$ .*

331 **Definition 1.53** (*Stationary set*)

332 *For a regular uncountable cardinal  $\kappa$ , we say a set  $A \subset \kappa$  is stationary in  $\kappa$*   
 333 *iff it intersects every club subset of  $\kappa$ .*

### 334 1.2.7 Structure, Substructure and Embedding

335 Structures will be denoted  $\langle M, \in, R \rangle$  where  $M$  is a domain,  $\in$  stands for the  
 336 standard membership relation, it is assumed to be restricted to the domain<sup>7</sup>,  
 337  $R \subseteq M$  is a relation on the domain. When  $R$  is not needed, we can as well  
 338 only write  $M$  instead of  $\langle M, \in \rangle$ .

339 **Definition 1.54** (*Elementary Embedding*)

340 *Given the structures  $\langle M_0, \in, R \rangle$ ,  $\langle M_1, \in, R \rangle$  and a one-to-one function  $j :$   
 341  $M_0 \rightarrow M_1$ , we say  $j$  is an elementary embedding of  $M_0$  into  $M_1$ , we write  
 342  $j : M_0 \prec M_1$ , when the following holds for every formula  $\varphi(p_1, \dots, p_n)$  and  
 343 every  $p_1, \dots, p_n \in M_0$ :*

$$\langle M_0, \in, R \rangle \models \varphi(p_1, \dots, p_n) \leftrightarrow \langle M_1, \in, R \rangle \models \varphi(j(p_1), \dots, j(p_n)) \quad (1.40)$$

344 **Definition 1.55** (*Elementary Substructure*)

345 *Given the structures  $\langle M_0, \in, R \rangle$ ,  $\langle M_1, \in, R \rangle$  and a one-to-one function  $j :$   
 346  $M_0 \rightarrow M_1$  such that  $j : M_0 \prec M_1$ , we say that  $M_0$  is an elementary sub-  
 347 structure of  $M_1$ , denoted as  $M_0 \prec M_1$ , iff  $j$  is an identity on  $M_0$ . In other  
 348 words*

$$\langle M_0, \in, R \rangle \models \varphi(p_1, \dots, p_n) \leftrightarrow \langle M_1, \in, R \rangle \models \varphi(p_1, \dots, p_n) \quad (1.41)$$

349 *for  $p_1, \dots, p_n \in M_0$*

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<sup>7</sup>To be totally explicit, we should write  $\langle M, \in \cap M \times M, R \rangle$

## 2 Levy's First-Order Reflection

### 2.1 Lévy's Original Paper

This section is based on Lévy's paper *Axiom Schemata of Strong Infinity in Axiomatic Set Theory*, [Lévy, 1960]. It presents Lévy's general reflection principle and its equivalence to *Replacement* and *Infinity* under  $S^8$ .

First, we should point out that set theory has changed over the last 66 years and show a few notable, albeit only formal, differences. Firstly, when reading Lévy's article, one should bear in mind that while the author often speaks about a model of ZF, usually denoted  $u$ , it doesn't necessarily mean that there is a set  $u$  that is a model of ZF<sup>9</sup>, we are nowadays used to using the notion of universal class  $V$  in similar sense, even though independently from a particular axiomatic set theory. The theory ZF is practically identical to the theory we have established in (1.19), the differences are only formal. One might be confused by the fact that Lévy treats the *Subsets* axiom, which we call *Specification*, as a single axiom rather than a schema. He even takes the conjunction of all axioms of ZF and treats it like a formula. This is possible because the underlying logic calculus is different. Lévy works with set theories formulated in the *non-simple applied first order functional calculus*, see Chapter IV in [Church, 1996] for details. For now, we only need to know that the calculus contains a substitution rule for functional variables. This way, *Subsets* is de facto a schema even though it sometimes treated as a single formula<sup>10</sup>. It should also be noted that the logical connectives look different. The now usual symbol for an universal quantifier does not appear,  $\forall x\varphi(x)$  would be written as  $(x)\varphi(x)$ . The symbol for negation is " $\sim$ ", implication is written as " $\supset$ " and equivalence is " $\equiv$ ". We will use standard notation with " $\neg$ ", " $\rightarrow$ " and " $\leftrightarrow$ " respectively when presenting Lévy's results.

This subsection uses ZF instead of the usual ZFC as the underlying theory.

**Definition 2.1** (*Standard Complete Model of a Set Theory*)

Let  $\mathbf{Q}$  be an arbitrary axiomatic set theory. We say that  $u$  is a standard complete model of  $\mathbf{Q}$  iff

(i)  $(\forall \sigma \in \mathbf{Q})(u \models \sigma)$

(ii)  $\forall y(y \in u \rightarrow y \subset u)$

We write  $Scm^{\mathbf{Q}}(u)$ .

<sup>8</sup>See definition (1.18).

<sup>9</sup>This is indeed impossible to prove in ZF due to Gödel's Incompleteness.

<sup>10</sup>This way, the conjunction of all axioms is then in fact an axiom schema.



385 **Definition 2.2** (*Cardinals Inaccessible With Respect to Q*)

386 *Let Q be an arbitrary axiomatic set theory. We say that a cardinal  $\kappa$  is*  
 387 *inaccessible with respect to theory Q iff*

$$Scm^Q(V_\kappa) \quad (2.42)$$

388 *We write  $In^Q(\kappa)$*

389 **Definition 2.3** (*Inaccessible Cardinal With Respect to ZF*)

390 *When a cardinal  $\kappa$  is inaccessible with respect to ZF, we only say that it is*  
 391 *inaccessible. We write  $In(\kappa)$ .*

$$In(\kappa) \leftrightarrow In^{ZF}(\kappa) \quad (2.43)$$

392 The above definition of inaccessibles is used because it doesn't require *Choice*.

393 For the definition of relativization, see (1.39). The notation used by Lévy  
 394 is " $Rel(u, \varphi)$ ", we will stick to " $\varphi^u$ ".

395 **Definition 2.4** (*N*)

396 *The following is an axiom schema of complete reflection over ZF, denoted as*  
 397 *N. For every first-order formula  $\varphi$  in the language of set theory with no free*  
 398 *variables except for  $p_1, \dots, p_n$ , the following is an instance of schema N.*

$$\exists u(Scm^{ZF}(u) \ \& \ (\forall p_1, \dots, p_n \in u)(\varphi \leftrightarrow \varphi^u)) \quad (2.44)$$

399 Let S be an axiomatic set theory defined in (1.18).

400 **Definition 2.5** (*N<sub>0</sub>*)

401 *Axiom schema N<sub>0</sub> is similar to N defined above, but with S instead of ZF.*  
 402 *For every  $\varphi$ , a first-order fomula in the language of set theory with no free*  
 403 *variables except  $p_1, \dots, p_n$ , the following is an instance of N<sub>0</sub>.*

$$\exists u(Scm^S(u) \ \& \ (\forall p_1, \dots, p_n \in u)(\varphi \leftrightarrow \varphi^u)) \quad (2.45)$$

404 We will now show that in S, N<sub>0</sub> implies both *Replacement* and *Infinity*.

405

406 Let N<sub>0</sub> be defined as in (2.5), for *Infinity* see (1.10).

407 **Theorem 2.6** *In S, the axiom schema N<sub>0</sub> implies Infinity.*

408 *Proof.* Let  $\varphi = \forall x \exists y (y = x \cup \{x\})$ . This clearly holds in S because given a  
 409 set  $x$ , there is a set  $y = x \cup \{x\}$  obtained via *Pairing* and *Union*. From N<sub>0</sub>,  
 410 there is a set  $u$  such that  $\varphi^u$  holds. This  $u$  satisfies the conditions required  
 411 by *Infinity*.  $\square$

412 Lévy proves this theorem in a different way. He argues that for an arbitrary  
 413 formula  $\varphi$ ,  $N_0$  gives us  $\exists u Scm^S(u)$  and this  $u$  already satisfies *Infinity*.  
 414 To do this, we would need to prove lemma (2.12) now, which would make  
 415 second half of this chapter quite confusing.

416

417 Let  $S$  be a set theory defined in (1.18),  $N_0$  a schema defined in (2.5) and  
 418 *Replacement* a schema defined in (1.16).

419 **Theorem 2.7** *In  $S$ , axiom the schema  $N_0$  implies Replacement.*

420 *Proof.* Let  $\varphi(x, y, p_1, \dots, p_n)$  be a formula with no free variables except  
 421  $x, y, p_1, \dots, p_n$ . Let  $\chi$  be an instance of the *Replacement* schema for the  
 422  $\varphi$  given. We want to verify that  $\chi$  holds in  $S$  with  $N_0$ .

$$\begin{aligned} \chi = \forall x, y, z (\varphi(x, y, p_1, \dots, p_n) \ \& \ \varphi(x, z, p_1, \dots, p_n) \rightarrow y = z) \\ \rightarrow \forall x \exists y \forall z (z \in y \leftrightarrow (\exists q \in x) (\varphi(x, y, p_1, \dots, p_n))) \end{aligned} \quad (2.46)$$

423 Now consider the following formulas.

- 424 (i)  $(\forall x, y, p_1, \dots, p_n \in u) (\varphi \leftrightarrow \varphi^u)$
- 425 (ii)  $(\forall x, p_1, \dots, p_n \in u) (\exists y \varphi \leftrightarrow (\exists y \varphi)^u)$
- 426 (iii)  $(\forall x, p_1, \dots, p_n \in u) (\chi \leftrightarrow \chi^u)$
- 427 (iv)  $\forall x, p_1, \dots, p_n \chi \leftrightarrow (\forall x, p_1, \dots, p_n \chi)^u$

428 The above formulas are instances of the  $N_0$  schema for  $\varphi$ ,  $\exists y \varphi$ ,  $\chi$  and the  
 429 universal closure of  $\chi$  respectively. By  $N_0$ , there exists a set  $u$  where all four  
 430 formulas hold.<sup>11</sup> From relativization,  $(\exists y \varphi)^u$  is equivalent to  $(\exists y \in u) \varphi^u$ ,  
 431 together with (i) and (ii), we get

$$(\forall x, p_1, \dots, p_n \in u) ((\exists y \in u) \varphi \leftrightarrow \exists y \varphi) \quad (2.47)$$

432 If  $\varphi$  is a function, then for every  $x \in u$ , which is also  $x \subset u$  since  $Scm^S(u)$   
 433 and therefore  $u$  is transitive, it maps elements of  $x$  into  $u$ . From the *Speci-*  
 434 *fication*, we can find  $y$ , a set of all images of the elements of  $x$ . That gives  
 435 us  $x, p_1, \dots, p_n \in u \rightarrow \chi$ . By (iii) we get that  $x, p_1, \dots, p_n \in u \rightarrow \chi^u$   
 436 holds. The universal closure of this formula is  $\forall x, p_1, \dots, p_n (x, p_1, \dots, p_n \in$   
 437  $u \rightarrow \chi^u)$  which is equivalent to  $(\forall x, p_1, \dots, p_n \in u) (\chi)^u$ , which is exactly  
 438  $(\forall x, p_1, \dots, p_n \chi)^u$ . From (iv),  $\forall x, p_1, \dots, p_n \chi$  holds.  $\square$

439 What we have just proven is only a single theorem from Lévy's afore-  
 440 mentioned article, we will introduce other interesting results, mostly related  
 441 to Mahlo and inaccessible cardinals, later in their appropriate context in  
 442 chapter 3.

<sup>11</sup>Despite the fact that  $N_0$  is defined for one formula, we have just used it for four at once. To make this formally possible, we can either prove that  $N_0$  is equivalent to a more general version for any finite number of formulas or we can reflect their conjunction and argue that if  $u \models \varphi_1 \ \& \ \dots \ \& \ \varphi_n$ , then  $(u \models \varphi_1), \dots, (u \models \varphi_n)$ .

## 2.2 Contemporary Restatement

We will now introduce and prove a theorem that is called Lévy's Reflection in contemporary set theory. The only difference is that while Lévy originally reflects a formula  $\varphi$  from  $V$  to a set  $u$  which is a *standard complete model of*  $S$ , we say that there is a  $V_\lambda$  for a limit  $\lambda$  that reflects  $\varphi$ . Those two conditions are equivalent due to lemma (2.12).

**Lemma 2.8** *Let  $\varphi_1, \dots, \varphi_n$  be first-order formulas in the language of set theory, all with  $m$  free variables<sup>12</sup>.*

(i) *For each set  $M_0$  there is such set  $M$  that  $M_0 \subset M$  and the following holds for every  $i$ ,  $1 \leq i \leq n$ :*

$$\exists x \varphi_i(p_1, \dots, p_{m-1}, x) \rightarrow (\exists x \in M) \varphi_i(p_1, \dots, p_{m-1}, x) \quad (2.48)$$

*for every  $p_1, \dots, p_{m-1} \in M$ .*

(ii) *Furthermore, there is a limit ordinal  $\lambda$  such that  $M_0 \subset V_\lambda$  and the following holds for each  $i$ ,  $1 \leq i \leq n$ :*

$$\exists x \varphi_i(p_1, \dots, p_{m-1}, x) \rightarrow (\exists x \in V_\lambda) \varphi_i(p_1, \dots, p_{m-1}, x) \quad (2.49)$$

*for every  $p_1, \dots, p_{m-1} \in M$ .*

(iii) *Assuming Choice, there is  $M$ ,  $M_0 \subset M$  such that (2.48) holds for every  $M$ ,  $i \leq n$  and  $|M| \leq |M_0| \cdot \aleph_0$ .*

*Proof.* We will simultaneously prove statements (i) and (ii), denoting  $M^T$  the transitive set required by part (ii). Steps in the construction of  $M^T$  that are not explicitly included are equivalent to steps for  $M$ .

Let us first define an operation  $H_i(p_1, \dots, p_{m-1})$  that yields the set of  $x$ 's with minimal rank<sup>13</sup> satisfying  $\varphi_i(p_1, \dots, p_{m-1}, x)$  for  $p_1, \dots, p_{m-1}$  and for every  $i$ ,  $1 \leq i \leq n$ .

$$H_i(p_1, \dots, p_n) = \{x \in C_i : (\forall z \in C)(\text{rank}(x) \leq \text{rank}(z))\} \quad (2.50)$$

for each  $1 \leq i \leq n$ , where

$$C_i = \{x : \varphi_i(p_1, \dots, p_{m-1}, x)\} \text{ for } 1 \leq i \leq n \quad (2.51)$$

<sup>12</sup>For formulas with a different number of free variables, take for  $m$  the highest number of parameters among those formulas. Add spare parameters to every formula that has less than  $m$  parameters in a way that preserves the last parameter, which we will denote  $x$ . E.g. let  $\varphi'_i$  be the a formula with  $k$  parameters,  $k < m$ . Let us set  $\varphi_i(p_1, \dots, p_{m-1}, x) = \varphi'_i(p_1, \dots, p_{k-1}, x)$ , notice that the parameters  $p_k, \dots, p_{m-1}$  are not used.

<sup>13</sup>Rank is defined in (1.29)

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Next, let's construct  $M$  from given  $M_0$  by induction.

$$M_{i+1} = M_i \cup \bigcup_{j=0}^n \{H_j(p_1, \dots, p_{m-1}) : p_1, \dots, p_{m-1} \in M_i\} \quad (2.52)$$

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In other words, in each step we include into the construction the elements satisfying  $\varphi(p_1, \dots, p_{m-1}, x)$  for  $p_1, \dots, p_{m-1}$  from the previous step. For statement (ii), this is the only part that differs from (i). To end up with a transitive  $M$ , we need to extend every step to its transitive closure transitive closure of  $M_{i+1}$  from (i). In other words, let  $\gamma$  be the smallest ordinal such that

$$(M_i^T \cup \bigcup_{j=0}^n \{\bigcup \{H_j(p_1, \dots, p_{m-1}) : p_1, \dots, p_{m-1} \in M_i\}\}) \subset V_\gamma \quad (2.53)$$

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Then the incremental step is

$$M_{i+1}^T = V_\gamma \quad (2.54)$$

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and the final  $M$  is obtained by joining the previous steps.

$$M = \bigcup_{i=0}^{\infty} M_i, \quad M^T = \bigcup_{i=0}^{\infty} M_i^T = V_\lambda \text{ for some limit } \lambda. \quad (2.55)$$

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We have yet to finish part (iii). Let's try to construct a set  $M'$  that satisfies the same conditions like  $M$  but is kept as small as possible. Assuming the Axiom of Choice, we can modify the construction so that the cardinality of  $M'$  is at most  $|M_0| \cdot \aleph_0$ . Note that the size of  $M$  in the previous construction is determined by the size of  $M_0$  and, most importantly, by the size of  $H_i(p_1, \dots, p_{m-1})$  for every  $i$ ,  $1 \leq i \leq n$  in individual iterations of the construction. Since (i) only ensures the existence of an  $x$  that satisfies  $\varphi_i(p_1, \dots, p_{m-1}, x)$  for any  $i$ ,  $1 \leq i \leq n$ , we only need to add one  $x$  for every set of parameters but  $H_i(u_1, \dots, u_{m-1})$  can be arbitrarily large. Let  $F$  be a choice function on  $\mathcal{P}(M')$ . Also let  $h_i(p_1, \dots, p_{m-1}) = F(H_i(p_1, \dots, p_{m-1}))$  for  $i$ , where  $1 \leq i \leq n$ , which means that  $h$  is a function that outputs an  $x$  that satisfies  $\varphi_i(p_1, \dots, p_{m-1}, x)$  for  $i$  such that  $1 \leq i \leq n$  and has minimal rank among all such sets. The induction step needs to be redefined to

$$M'_{i+1} = M'_i \cup \bigcup_{j=0}^n \{h_j(p_1, \dots, p_{m-1}) : p_1, \dots, p_{m-1} \in M'_i\} \quad (2.56)$$

490 This way, the amount of elements added to  $M'_{i+1}$  in each step of the con-  
 491 struction is the same as the amount of  $m$ -tuples of parameters that yielded  
 492 elements not included in  $M'_i$ . It is easy to see that if  $M_0$  is finite,  $M'$  is  
 493 countable because it was constructed as a countable union of sets that are  
 494 themselves at most countable. If  $M_0$  is countable or larger, the cardinality  
 495 of  $M'$  is equal to the cardinality of  $M_0$ .<sup>14</sup> Therefore  $|M'| \leq |M_0| \cdot \aleph_0$   $\square$

496 **Theorem 2.9** (*Lévy's first-order reflection theorem*)

497 Let  $\varphi(p_1, \dots, p_n)$  be a first-order formula.

498 (i) For every set  $M_0$  there exists a set  $M$  such that  $M_0 \subset M$  and the  
 499 following holds:

$$\varphi^M(p_1, \dots, p_n) \leftrightarrow \varphi(p_1, \dots, p_n) \quad (2.57)$$

500 for every  $p_1, \dots, p_n \in M$ .

501 (ii) For every set  $M_0$  there is a transitive set  $M$ ,  $M_0 \subset M$  such that the  
 502 following holds:

$$\varphi^M(p_1, \dots, p_n) \leftrightarrow \varphi(p_1, \dots, p_n) \quad (2.58)$$

503 for every  $p_1, \dots, p_n \in M$ .

504 (iii) For every set  $M_0$  there is a limit ordinal  $\lambda$  such that  $M_0 \subset V_\lambda$  and the  
 505 following holds:

$$\varphi^{V_\lambda}(p_1, \dots, p_n) \leftrightarrow \varphi(p_1, \dots, p_n) \quad (2.59)$$

506 for every  $p_1, \dots, p_n \in M$ .

507 (iv) Assuming Choice, for every set  $M_0$  there is  $M$  such that  $M_0 \subset M$  and  
 508  $|M| \leq |M_0| \cdot \aleph_0$  and the following holds:

$$\varphi^M(p_1, \dots, p_n) \leftrightarrow \varphi(p_1, \dots, p_n) \quad (2.60)$$

509 for every  $p_1, \dots, p_n \in M$ .

510 *Proof.* Let's now prove (i) for given  $\varphi$  via induction by complexity. We  
 511 can safely assume that  $\varphi$  contains no quantifiers besides " $\exists$ " and no logical  
 512 connectives other than " $\neg$ " and "&". Let  $\varphi_1, \dots, \varphi_n$  be all subformulas of  $\varphi$ .  
 513 Then there is a set  $M$ , obtained by the means of lemma (2.8), for all of the  
 514 formulas  $\varphi_1, \dots, \varphi_n$ .

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<sup>14</sup>It can not be smaller because  $|M'_{i+1}| \geq |M'_i|$  for every  $i$ . It may not be significantly larger because the maximum of elements added is the number of  $n$ -tuples in  $M'_i$ , which is of the same cardinality as  $M'_i$ .

Let's first consider atomic formulas in the form of either  $x_1 = x_2$  or  $x_1 \in x_2$ . It is clear from relativisation<sup>15</sup> that (2.57) holds for both cases,  $(x_1 = x_2)^M \leftrightarrow (x_1 = x_2)$  and  $(x_1 \in x_2)^M \leftrightarrow (x_1 \in x_2)$ .

We now want to verify the inductive step. First, take  $\varphi = \neg\varphi'$ . From relativization, we get  $(\neg\varphi')^M \leftrightarrow \neg(\varphi'^M)$ . Because the induction hypothesis tells us that  $\varphi'^M \leftrightarrow \varphi'$ , the following holds:

$$(\neg\varphi')^M \leftrightarrow \neg(\varphi'^M) \leftrightarrow \neg\varphi' \quad (2.61)$$

The same holds for  $\varphi = \varphi_1 \& \varphi_2$ . From the induction hypothesis, we know that  $\varphi_1^M \leftrightarrow \varphi_1$  and  $\varphi_2^M \leftrightarrow \varphi_2$ , which together with relativization for formulas in the form of  $\varphi_1 \& \varphi_2$  gives us

$$(\varphi_1 \& \varphi_2)^M \leftrightarrow \varphi_1^M \& \varphi_2^M \leftrightarrow \varphi_1 \& \varphi_2 \quad (2.62)$$

Let's now examine the case when  $\varphi = \exists x\varphi'(p_1, \dots, p_n, x)$ . The induction hypothesis tells us that  $\varphi'^M(p_1, \dots, p_n, x) \leftrightarrow \varphi'(p_1, \dots, p_n, x)$ , so, together with above lemma (2.8), the following holds:

$$\begin{aligned} & \varphi(p_1, \dots, p_n, x) \\ & \leftrightarrow \exists x\varphi'(p_1, \dots, p_n, x) \\ & \leftrightarrow (\exists x \in M)\varphi'(p_1, \dots, p_n, x) \\ & \leftrightarrow (\exists x \in M)\varphi'^M(p_1, \dots, p_n, x) \\ & \leftrightarrow (\exists x\varphi'(p_1, \dots, p_n, x))^M \\ & \leftrightarrow \varphi^M(p_1, \dots, p_n, x) \end{aligned} \quad (2.63)$$

Which is what we wanted to prove for part (i).

We now need to verify that the same holds for any finite number of formulas  $\varphi_1, \dots, \varphi_n$ . This has in fact been already done since lemma (2.8) gives us a set  $M$  for any finite amount of formulas and given  $M_0$ . We can therefore find a set  $M$  for the union of all of their subformulas. When we obtain such  $M$ , it should be clear that it also reflects every formula in  $\varphi_1, \dots, \varphi_n$ .

Since  $V_\lambda$  is a transitive set, by proving (iii) we also satisfy (ii). To do so, we only need to look at part (ii) of lemma (2.8). All of the above proof also holds for  $M = V_{\text{lambda}}$ .

<sup>15</sup>See (1.39). This only holds for relativization to  $M, \in \cap M \times M$ , not  $M, R$  for an arbitrary  $R$ .

539 To finish part (iv), we take  $M$  of size  $\leq |M_0| \cdot \aleph_0$ , which exists due to  
 540 part (iii) of lemma (2.8), the rest being identical.  $\square$

541 Let  $\mathbf{S}$  be a set theory defined in (1.18), for ZFC see definition (1.20).

542 The two following lemmas are based on [Drake, 1974][Chapter 3, Theorem  
 543 1.2].  
 544

545 **Lemma 2.10** *If  $M$  is a transitive set, then  $M \models \text{Extensionality}$ .*

546 *Proof.* Given a transitive set  $M$ , we want to show that the following holds.

$$M \models \forall x, y (x = y \leftrightarrow \forall z (z \in x \leftrightarrow z \in y)) \quad (2.64)$$

547 Given arbitrary  $x, y \in M$ , we want to prove that  $M \models (x = y \leftrightarrow \forall z (z \in$   
 548  $x \leftrightarrow z \in y))$ . This is equivalent to  $M \models x = y$  iff  $M \models \forall z (z \in x \leftrightarrow z \in y)$ ,  
 549 which is the same as  $x = y$  iff  $M \models \forall z (z \in x \leftrightarrow z \in y)$ .

550 So all elements of  $x$  are also elements of  $y$  in  $M$ , and vice versa. Because  
 551  $M$  is transitive, all elements of  $x$  and  $y$  are in  $M$ , so  $M \models \forall z (z \in x \leftrightarrow z \in y)$   
 552 holds iff  $x$  and  $y$  contain the same elements and are therefore equal.  $\square$

553 **Lemma 2.11** *If  $M$  is a transitive set, then  $M \models \text{Foundation}$ .*

554 *Proof.* We want to prove the following:

$$M \models \forall x (x \neq \emptyset \rightarrow (\exists y \in x) (x \cap y = \emptyset)) \quad (2.65)$$

555 Given an arbitrary non-empty  $x \in M$  let's show that  $M \models (\exists y \in x) (x \cap$   
 556  $y = \emptyset)$ .

557 Because  $M$  is transitive, every element of  $x$  is an element of  $M$ . Take for  
 558  $y$  the element of  $x$  with the lowest rank<sup>16</sup>. It should be clear that there is no  
 559  $z \in y$  such that  $z \in x$ , because then  $\text{rank}(z) < \text{rank}(y)$ , which would be a  
 560 contradiction.  $\square$

561 Let  $\mathbf{S}$  be a set theory as defined in (1.18).

562 **Lemma 2.12** *The following holds for every  $\lambda$ .*

$$“\lambda \text{ is a limit ordinal}” \rightarrow V_\lambda \models \mathbf{S} \quad (2.66)$$

563 *Proof.* Given an arbitrary limit ordinal  $\lambda$ , we will verify the axioms of  $\mathbf{S}$  one  
 564 by one.

565 (i) *The existence of a set* comes from the fact that  $V_\lambda$  is a non-empty set  
 566 because limit ordinal is non-zero by definition.

---

<sup>16</sup>Rank is defined in (1.29).

567 (ii) *Extensionality* holds from (2.10).

568 (iii) *Foundation* holds from (2.11).

569 (iv) *Union*:

570 Given any  $x \in V_\lambda$ , we want verify that  $y = \bigcup x$  is also in  $V_\lambda$ . Note that  
 571  $y = \bigcup x$  is a  $\Delta_0$ -formula.

$$y = \bigcup x \leftrightarrow (\forall z \in y)(\exists q \in x)z \in q \ \& \ (\forall z \in x)(\forall q \in z)q \in y \quad (2.67)$$

572 So by lemma (1.42)

$$y = \bigcup x \leftrightarrow V_\lambda \models y = \bigcup x \quad (2.68)$$

573 (v) *Pairing*:

574 Given two sets  $x, y \in V_\lambda$ , we want to show that  $z = \{x, y\}$  is also an  
 575 element of  $V_\lambda$ .

$$z = \{x, y\} \leftrightarrow x \in z \ \& \ y \in z \ \& \ (\forall q \in z)(q = x \vee q = y) \quad (2.69)$$

576 So  $z = \{x, y\}$  is a  $\Delta_0$ -formula, and thus by lemma (1.42) it holds that

$$z = \{x, y\} \leftrightarrow V_\lambda \models z = \{x, y\} \quad (2.70)$$

577 (vi) *Powerset*:

578 Given any  $x \in V_\lambda$ , we want to make sure that  $\mathcal{P}(x) \in V_\lambda$ . Let  $\varphi(y)$  denote  
 579 the formula  $y \in \mathcal{P}(x) \leftrightarrow y \subset x$ . according to definition of subset (1.4),  
 580  $y \subset x$  is  $\Delta_0$ , so for any given  $x, y \in V_\lambda$ ,  $y = \mathcal{P}(x) \leftrightarrow V_\lambda \models y = \mathcal{P}(x)$ .  
 581 Because  $\lambda$  is limit and  $\text{rank}(\mathcal{P}(x)) = \text{rank}(x) + 1$ , if  $\mathcal{P}(x) \in V_\lambda$  for every  
 582  $x \in V_\lambda$ .

583 (vii) *Specification*:

584 Given a first-order formula  $\varphi$ , we want to show the following:

$$V_\lambda \models \forall x, p_1, \dots, p_n, \exists y \forall z (z \in y \leftrightarrow z \in x \ \& \ \varphi(z, p_1, \dots, p_n)) \quad (2.71)$$

585 Given any  $x$  along with parameters  $p_1, \dots, p_n$  in  $V_\lambda$ , we set

$$y = \{z \in x : \varphi^{V_\lambda}(z, p_1, \dots, p_n)\} \quad (2.72)$$

586 From transitivity of  $V_\lambda$  and the fact that  $y \subset x$  and  $x \in V_\lambda$ , we know that  
 587  $y \in V_\lambda$ , so  $V_\lambda \models \forall z (z \in y \leftrightarrow z \in x \ \& \ \varphi(z, p_1, \dots, p_n))$ .  
 588 □

589 **Definition 2.13** (*First-Order Reflection Schema*)

590 For every first-order formula  $\varphi$ , the following is an axiom:

$$\forall M_0 \exists M (M_0 \subseteq M \ \& \ (\varphi(p_1, \dots, p_n) \leftrightarrow \varphi(p_1, \dots, p_n)^M)) \quad (2.73)$$

591 We will refer to this axiom schema as First-order reflection.



592 Let *Infinity* and *Replacement* be as defined in (1.10) and (1.16) respec-  
593 tively.

594 **Theorem 2.14** First-order reflection *is equivalent to* Infinity & Replace-  
595 ment *under* S.

596 *Proof.* Since (2.9) already gives us one side of the implication, we are only  
597 interested in showing the converse which we shall do in two parts:

598 *First-order reflection  $\rightarrow$  Infinity* This is done exactly like (2.6). We pick  
599 for  $\varphi$  the formula  $(\forall y \in x)(y \cup \{y\} \in x)$ ,  $M_0 = \{\emptyset\}$ . From (2.13), there is a  
600 set  $M$  that satisfies  $\varphi$ , so there is an inductive set. We have picked  $M_0$  so  
601 that  $\emptyset \in M$  obviously holds and  $M$  is the witness for

$$\exists x(\emptyset \in x \ \& \ (\forall y \in x)(y \cup \{y\} \in x)) \quad (2.74)$$

602 which is exactly (1.10).

603

604 *First-order reflection  $\rightarrow$  Replacement*

605 Let's first point out that while *First-order reflection* gives us a set for  
606 one formula, we can generalize it to hold for any finite number of formulas.  
607 We will show how is it done for two formulas, which is what we will use in  
608 this proof. Given two first-order formulas  $\varphi, \psi$ , we can suppose that there  
609 are formulas  $\varphi'$  and  $\psi'$  that are equivalent to  $\varphi$  and  $\psi$  respectively, but their  
610 free variables are different <sup>17</sup>. Let  $\xi = \varphi \ \& \ \psi$ , given any  $M_0$ , we can find a  
611  $M$  such that  $\xi \leftrightarrow \xi^M$ . It is easy to see that from relativisation, the following  
612 holds:

$$\varphi \ \& \ \psi \leftrightarrow \varphi' \ \& \ \psi' \leftrightarrow \xi \leftrightarrow \xi^M \leftrightarrow (\varphi' \ \& \ \psi')^M \leftrightarrow \varphi'^M \ \& \ \psi'^M \leftrightarrow \varphi^M \ \& \ \psi^M \quad (2.75)$$

613 Now given a function  $\varphi(x, y)$ , we know from *First-order reflection* that  
614 for every  $M_0$ , there is a set  $M$  such that  $M_0 \subseteq M$  and both

$$(\forall x, y \in M)(\varphi(x, y) \leftrightarrow \varphi^M(x, y)) \quad (2.76)$$

615 and

$$(\forall x, y \in M)(\exists y \varphi(x, y) \leftrightarrow (\exists y \varphi(x, y))^M) \quad (2.77)$$

616 hold, the latter being equivalent to

$$(\forall x, y \in M)(\exists y \varphi(x, y) \leftrightarrow (\exists y \in M) \varphi^M(x, y)) \quad (2.78)$$

---

<sup>17</sup>This is plausible since we can for example substitute all free variables in  $\varphi'$  for  $x_0, x_2, x_4, \dots$  and use  $x_1, x_3, x_5, \dots$  for free variables in  $\psi'$ , the resulting formulas will be equivalent.

617 Therefore

$$(\forall x, y \in M)(\exists y \varphi(x, y) \leftrightarrow (\exists y \in M)\varphi(x, y)) \quad (2.79)$$

618 holds too. That means that we have a set  $M$  such that for every  $x \in M$ , if  
619  $\varphi$  is defined for  $x$ ,  $(\exists y \in M)\varphi(x, y)$ .

620 To show that *Replacement* holds for this particular  $\varphi$ , we need to verify  
621 that given a set  $M_0$ ,  $M'_0 = \{y : (\exists x \in M_0)\varphi(x, y)\}$  is also a set. But since  
622  $M_0 \subseteq M$  and because given any  $x \in M$ , there is  $y \in M$  satisfying  $\varphi(x, y)$ ,  
623 the following is a set due to *Specification*:

$$M'_0 = \{y : (\exists x \in M_0)\varphi(x, y)\} = \{y \in M : (\exists x \in M_0)\varphi(x, y)\} \quad (2.80)$$

624

□

625

626 We have shown that *Reflection* for first-order formulas, *First-order reflec-*  
627 *tion* is a theorem of ZFC. We have also shown that it can be used instead of  
628 the *Infinity* and *Replacement* scheme, but ZFC + *First-order reflection* is  
629 a conservative extension of ZF. Besides being a starting point for more gen-  
630 eral and powerful statements, it can be used to show that ZF is not finitely  
631 axiomatizable. This follows from the fact that *Reflection* gives a model to  
632 any consistent finite set of formulas. So if  $\varphi_1, \dots, \varphi_n$  would be the axioms  
633 of ZFC, *Reflection* would prove that every model of ZFC contains a smaller  
634 model of ZFC, which would in turn contradict the Second Gödel's Theorem<sup>18</sup>.

635 It is also worthwhile to note that, in a way, Reflection is dual to compact-  
636 ness. Compactness says that given a set of sentences, if every finite subset  
637 yields a model, so does the whole set. Reflection, on the other hand, says  
638 that while the whole set has no model in the underlying theory, every finite  
639 subset has a model.

640 Furthermore, *Reflection* can be used in ways similar to upward Löwenheim–Skolem  
641 theorem. Since Reflection extends any set  $M_0$  into a model of given formulas  
642  $\varphi_1, \dots, \varphi_n$ , we can choose the lower bound of the size of  $M$  by appropriately  
643 choosing  $M_0$ .

644 In the next section, we will try to generalize *Reflection* in a way that  
645 transcends ZF and finally yields some large cardinals.

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<sup>18</sup>See chapter ?? for further details.

## <sup>646</sup> 3 Conclusion

## References

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