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4 REFLECTION PRINCIPLES AND LARGE
5 CARDINALS
6 Bakalářská práce

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¹⁰ Prohlašuj, že jsem bakalářskou práci vypracoval samostatně a že jsem uvedl
¹¹ všechny použité prameny a literaturu.

¹² V Praze 14. dubna 2015

Abstract

Práce zkoumá vztah tzv. principů reflexe a velkých kardinálů. Lévy ukázal, že v ZFC platí tzv. věta o reflexi a dokonce, že věta o reflexi je ekvivalentní schématu nahrazení a axiomu nekonečna nad teorií ZFC bez axiomu nekonečna a schématu nahrazení. Tedy lze na větu o reflexi pohlížet jako na svého druhu axiom nekonečna. Práce zkoumá do jaké míry a jakým způsobem lze větu o reflexi zobecnit a jaký to má vliv na existenci tzv. velkých kardinálů. Práce definuje nedosažitelné, Mahlovy a nepopsatelné kardinály a ukáže, jak je lze zavést pomocí reflexe. Přirozenou limitou kardinálů získaných reflexí jsou kardinály nekonzistentní s L. Práce nabídne intuitivní zdůvodnění, proč tomu tak je.

Abstract

This thesis aims to examine relations between so called "Reflection Principles" and Large cardinals. Lévy has shown that Reflection Theorem is a sound theorem of ZFC and it is equivalent to Replacement Scheme and the Axiom of Infinity. From this point of view, Reflection theorem can be seen a specific version of an Axiom of Infinity. This paper aims to examine the Reflection Principle and it's generalisations with respect to existence of Large Cardinals. This thesis will establish Inaccessible, Mahlo and Indescribable cardinals and their definition via reflection. A natural limit of Large Cardinals obtained via reflection are cardinals inconsistent with L. The thesis will offer an intuitive explanation of why this is the case.

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1 Introduction

1.1 Motivation and Origin

The Universe of sets cannot be uniquely characterized (i. e. distinguished from all its initial elements) by any internal structural property of the membership relation in it, which is expressible in any logic of finite or transfinite type, including infinitary logics of any cardinal order.

— Kurt Gödel [9]

To understand why we need reflection in the first place, let's think about infinity for a moment. In the intuitive sense, infinity is an upper limit of all numbers. But for centuries, this was merely a philosophical concept, closely bound to religious and metaphysical way of thinking, considered separate from numbers used for calculations or geometry. It was a rather vague concept. In ancient Greece, Aristotle's response to famous Zeno's paradoxes introduced the distinction between actual and potential infinity. He argued, that potential infinity is (in today's words) well defined, as opposed to actual infinity, which remained a vague incoherent concept. He didn't think it's possible for infinity to inhabit a bounded place in space or time, rejecting Zeno's thought experiments as a whole. Aristotle's thoughts shaped western thinking partly due to Aquinas, who himself believed actual infinity to be more of a metaphysical concept for describing God than a mathematical property attributed to any other entity. In his *Summa Theologica* ¹ he argues:

A geometrician does not need to assume a line actually infinite, but takes some actually finite line, from which he subtracts whatever he finds necessary; which line he calls infinite.

Less than hundred years later, Gregory of Rimini wrote

If God can endlessly add a cubic foot to a stone—which He can—then He can create an infinitely big stone. For He need only add one cubic foot at some time, another half an hour later, another a quarter of an hour later than that, and so on ad infinitum. He would then have before Him an infinite stone at the end of the hour.

Which is basically a Zeno's Paradox made plausible with God being the actor. In contrast to Aquinas' position, Gregory of Rimini theoretically constructs an object with actual infinite magnitude that is essentially different from God.

¹Part I, Question 7, Article 3, Reply to Objection 1

Even later, in the 17th century, pushing the property of infiniteness from the Creator to his creation, Nature, Leibniz wrote to Foucher in 1662:

I am so in favor of the actual infinite that instead of admitting that Nature abhors it, as is commonly said, I hold that Nature makes frequent use of it everywhere, in order to show more effectively the perfections of its Author. Thus I believe that there is no part of matter which is not, I do not say divisible, but actually divided; and consequently the least particle ought to be considered as a world full of an infinity of different creatures.

But even though he used potential infinity in what would become foundations of modern Calculus and argued for actual infinity in Nature, Leibniz refused the existence of an infinite, thinking that Galileo's Paradoxon² is in fact a contradiction. The so called Galileo's Paradoxon is an observation Galileo Galilei made in his final book "Discourses and Mathematical Demonstrations Relating to Two New Sciences". He states that if all numbers are either squares and non-squares, there seem to be less squares than there is all numbers. On the other hand, every number can be squared and every square has its square root. Therefore, there seem to be as many squares as there are all numbers. Galileo concludes, that the idea of comparing sizes makes sense only in the finite realm.

Salviati: So far as I see we can only infer that the totality of all numbers is infinite, that the number of squares is infinite, and that the number of their roots is infinite; neither is the number of squares less than the totality of all the numbers, nor the latter greater than the former; and finally the attributes "equal," "greater," and "less," are not applicable to infinite, but only to finite, quantities. When therefore Simplicio introduces several lines of different lengths and asks me how it is possible that the longer ones do not contain more points than the shorter, I answer him that one line does not contain more or less or just as many points as another, but that each line contains an infinite number.

Leibniz insists in part being smaller than the whole saying

Among numbers there are infinite roots, infinite squares, infinite cubes. Moreover, there are as many roots as numbers. And there are as many squares as roots. Therefore there are as many squares

²zneni galileova paradoxu

as numbers, that is to say, there are as many square numbers as there are numbers in the universe. Which is impossible. Hence it follows either that in the infinite the whole is not greater than the part, which is the opinion of Galileo and Gregory of St. Vincent, and which I cannot accept; or that infinity itself is nothing, i.e. that it is not one and not a whole.

TODO nejakej Hegel–strucne?

TODO Cantor

TODO mene teologie, vice matematiky

TODO definovat pojmy (trida etc)

TODO neni V v nejakem smyslu porad potencialni nekonecno, zatimco mnoziny vetsi nez omega jsou aktualni? nebo jsou potencialni protoze se stavaji pres indukci, od spoda.

In his work, he defined transfinite numbers to extend existing natural number structure so it contains more objects that behave like natural numbers and are based on an object (rather a meta-object) that doesn't explicitly exist in the structure, but is closely related to it. This is the first instance of reflection. This paper will focus on taking this principle a step further, extending Cantor's (or Zermelo–Fraenkel's, to be more precise) universe so it includes objects so big, they could be considered the universe itself, in a certain sense.

TODO dal asi smazat

The original idea behind reflection principles probably comes from what could be informally called “universality of the universe”. The effort to precisely describe the universe of sets was natural and could be regarded as one of the impulses for formalization of naive set theory. If we try to express the universe as a set $\{x|x = x\}$, a paradox appears, because either our set is contained in itself and therefore is contained in a set (itself again), which contradicts the intuitive notion of a universe that contains everything but is not contained itself.

TODO ???

If there is an object containing all sets, it must not be a set itself. The notion of class seems inevitable. Either directly the ways for example the Bernays–Gödel set theory, we will also discuss later in this paper, does in, or on a meta-level like the Zermelo–Fraenkel set theory, that doesn't refer to them in the axioms but often works with the notion of a universal class. Another obstacle of constructing a set of all sets comes from Georg Cantor, who proved that the set of all subsets of a set (let A be the set and $\mathcal{P}(A)$ its powerset) is strictly larger than A . That would turn every aspiration to

finally establish an universal set into a contradictory infinite regression.³ We will use V to denote the class of all sets. From previous thoughts we can easily argue, that it is impossible to construct a property that holds for V and no set and is neither paradoxical like $\{x|x = x\}$ nor trivial. Previous observation can be transposed to a rather naive formulation of the reflection principle:

Reflection Any property which holds in V already holds in some initial segment of V .

To avoid vagueness of the term "property", we could informally reformulate the above statement into a schema:

For every first-order formula⁴ φ holds in $V \leftrightarrow \varphi$ holds in some initial segment of V .

Interested reader should note that this is a theorem scheme rather than a single theorem.⁵

1.2 A few historical remarks on reflection

Reflection made it's first in set-theoretical appearance in Gödel's proof of GCH in L (citace Kanamori ? Lévy and set theory), but it was around even earlier as a concept. Gödel himself regarded it as very close to Russel's reducibility axiom (an earlier equivalent of the axiom schema of Zermelo's separation). Richard Montague then studied reflection properties as a tool for verifying that Replacement is not finitely axiomatizable (citace?). a few years later Lévy proved in [2] the equivalence of reflection with Axiom of infinity together with Replacement in proof we shall examine closely in chapter 2.

TODO co dal? recent results?

³An intuitive analogy of this *reductio ad infinitum* is the status of ω , which was originally thought to be an unreachable absolute, only to become starting point of Cantor's hierarchy of sets growing beyond all boundaries around the end of the 19th century

⁴this also works for finite sets of formulas [3, p. 168]

⁵If there were a single theorem stating "for any formula φ that holds in V there is an initial segment of V where φ also holds", we would obtain the following contradiction with the second Gödel's theorem: In ZFC, any finite group of axioms of ZFC holds in some initial segment of the universe. If we take the largest of those initial segments it is still strictly smaller than the universe and thus we have, via compactness, constructed a model of ZFC within ZFC. That is, of course a harsh contradiction. This also leads to an elegant way to prove that ZFC is not finitely axiomatizable.

1.3 Notation (??) TODO

1. *Reflection* je obecne reflexe (jaka presne)
2. *Reflection*₁ je reflexe prvoradovych formul
3. *Reflection*₂ je reflexe druhoradovych formul
4. etc...

204 2 Levy's first-order reflection

205 2.1 Introduction

206 This section will try to present Lévy's proof of a general reflection principle
 207 being equivalent to Replacement and Infinity under ZF minus Replacement
 208 and Infinity. We will first introduce a few axioms and definitions that were
 209 a different in Lévy's paper[2], but are equivalent to today's terms. We will
 210 write them in contemporary notation, our aim is the result, not history of
 211 set theory notation.

212 Please note that Lévy's paper was written in a period when Set theory
 213 was oriented towards semantics, which means that everything was done in
 214 a model. All proofs were theodel that of ZFC was V_α (notated as $R(\alpha)$ at
 215 the time) for some cardinal α , which means that α is a inaccessible cadinal.
 216 Please bear in mind that this is vastly different from saying that there is
 217 an inaccessible α inside the model. This V_α is also referred to as $Scm^Q(u)$,
 218 which means that u ($u = V_\alpha$ in our case) is a standard complete model of
 219 an undisclosed axiomatic set theory Q formulated in the "non-simple applied
 220 first order functional calculus", which is second-order theory is today's ter-
 221 minology, we are allowed to quantify over functions and thus get rid of axiom
 222 schemes. (Note that Lévy always speaks of "the axiom of replacement"). Be-
 223 sides placeholder set theory Q and ZF, which the reader should be familiar
 224 with, theories Z , S , and SF are used in the text. Z is ZF minus replacement,
 225 S is ZF minus replacement and infinity, and finally SF is ZF minus infinity.
 226 "The axiom of subsets" is an older name for the axiom scheme of specifica-
 227 tion (and it's not a scheme since we are now working in second order logic).
 228 Also note that universal quantifier does not appear, $\forall x\varphi(x)$ would be written
 229 as $(x)\varphi(x)$, the symbol for negation is " \sim ".

230 2.2 Lévy's Original Paper

231 The following are a few definitions that are used in Lévy's original article. ⁶

232 **Definition 2.1** *Relativization*
 233 *TODO (jech:161)*

234 Next two definitions are not used in contemporary set theory, but they
 235 illustrate 1960's set theory mind-set and they are used heavily in Lévy's text,

⁶While some of them won't be of much use in this paper, they will provide extremely helpful when reading the original article as set theory notation and terminology has evolved in the last 50 years considerably.

so we will include and explain them for clarity. Generally in this chapter, \mathbf{Q} stands for an undisclosed axiomatic set theory, u is usually a model, counterpart of today's V^7 , e is a relation that serves as \in in the given model.

TODO je to relativizovany, jak rika shepherdson?

Definition 2.2 *Standard model of a set theory*

We say that u is a standard model of \mathbf{Q} with a membership relation e , written as $Sm^{\mathbf{Q}}(u)$, if both of the following hold

- (i) $(x, y) \in e \leftrightarrow y \in u \ \& \ x \in y$
- (ii) $y \in u \ \& \ x \in y \rightarrow x \in u$

Definition 2.3 *Standard complete model of a set theory*

We say that u is a standard complete model of a set theory \mathbf{Q} with a membership relation e if:

- (i) u is a transitive set with respect to \in
 - (ii) $\forall e((x, y) \in e \leftrightarrow (y \in u \ \& \ x \in y) \ \& \ Sm^{\mathbf{Q}}(u, e))$
- this is written as $Scm^{\mathbf{Q}}(u)$.

TODO what is "simple first-order functional calculus" a "non-simple first-order functional calculus"? Levyho ucebnece?

Definition 2.4 *Cardinal inaccessible with respect to \mathbf{Q}*

$$In^{\mathbf{Q}}(\alpha) = Scm^{\mathbf{Q}}(V_{\alpha}) \quad (2.1)$$

TODO tohle je lepsi protoze nepotrebuje AC

Definition 2.5 *Strictly increasing function*

A function $F(\alpha)$ defined on the ordinal number into the ordinal numbers is said to be strictly increasing if $\forall \alpha, \beta \in On(\alpha < \beta \rightarrow F(\alpha) < F(\beta))$.

Definition 2.6 *Continuous function*

A function $F(\alpha)$ defined on the ordinal number into the ordinal numbers is said to be continuous if for any limit α , $F(\alpha) = \lim_{\beta < \alpha} F(\beta)$.

Definition 2.7 *Normal function*

A function $F(\alpha)$ defined on the ordinal number into the ordinal numbers is said to be normal if it is strictly increasing and continuous

⁷Which is of course not referred to as a model, but it is used in a similar fashion, in this case the term "model" was a metamathematical notion because it was not based on any underlying structure of theory. It can be easily formalized in any set theory, but it's not helpful for our case.

266 TODO jak znacim usporadane dvojice?

267 TODO porovnani Mahlovy a Lévyho konstrukce, viz ref{mahlovy kardi-
268 naly}

269 TODO asi doplnit jak to souvisi se soucasnou definici slabe Mahlovych
270 kardinalu pres stacionarni mnoziny?

271

272 Lévy's article mentions various schemata that are not instances of reflec-
273 tion themselves. We will mention them because they are equivalent to N_0
274 and because they are fixed-point theorems, which we will find useful later in
275 this thesis.

276 **Definition 2.8** *M Every normal function defined for all ordinals has at least*
277 *one inaccessible number in its range.*

278 **Definition 2.9** *M' Every normal function defined for all ordinals has at*
279 *least one fixed point which is inaccessible.*

280 **Definition 2.10** *M'' Every normal function defined for all ordinals has ar-*
281 *bitrarily great fixed points which are inaccessible.*

Theorem 2.11

$$M \leftrightarrow M' \leftrightarrow M'' \quad (2.2)$$

282 We will omit this proof because it is not essential for our goal. An inter-
283 ested reader will find it in [2,].

284 TODO Pak i lambda konstrukce, zobecnene

285

286 The following is a principle of complete reflection over ZF.

287 **Definition 2.12** $N(\varphi)$

$$\exists u(Scm^{ZF}(u) \ \& \ \forall x_1, \dots, x_n(x_1, \dots, x_n \in u \rightarrow \varphi \leftrightarrow \varphi^u)) \quad (2.3)$$

288 where φ is a formula which does not contain free variables except x_1, \dots, x_n .

289 Note that this by (2.4) equivalent to $\exists u(In^{ZF}(u) \ \& \ \forall x_1, \dots, x_n(x_1, \dots, x_n \in$
290 $u \rightarrow \varphi \leftrightarrow \varphi^u))$, where $In(\alpha)$ is equivalent to the standard notion of inacces-
291 sibility.

Theorem 2.13

$$M \leftrightarrow N \quad (2.4)$$

292 *Proof.* TODO (Theorem 3)? neudalam ho spis v dalsi sekci v modernejši
293 variante? \square

294 **2.3** $S \models \text{Reflection} \leftrightarrow (\text{Replacement} \ \& \ \text{Infinity})$

295 **Definition 2.14** $N_0(\varphi)$

$$\exists u(Scm^S(u) \ \& \ \forall x_1, \dots, x_n(x_1, \dots, x_n \in u \rightarrow \varphi \leftrightarrow \varphi^u)) \quad (2.5)$$

296 where φ is a formula which does not contain free variables except x_1, \dots, x_n .

297 Note that the only difference between N and N_0 is the set theory used.

298 **Theorem 2.15** *In S , the schema N_0 implies the Axiom of Infinity.*

299 *Proof.* For any φ , N_0 gives us $\exists u Scm^S(u)$, which means that there is a set u
300 that is identical to V_α for some alpha, so $\exists \alpha Scm^S(V_\alpha)$. We don't know the
301 exact size of this α , but we know that $\alpha \geq \omega$, otherwise α would be finite,
302 therefore not closed under the powerset operation, which would contradict
303 the axiom of powersets. In order to prove that it is a model of S , we would
304 need to verify all axioms of S . We have already shown that ω is closed under
305 the powerset operation. Foundation, extensionality and comprehension are
306 clear from the fact that we work in ZF^8 , pairing is clear from the fact, that
307 given two sets A, B , they have ranks a, b , without loss of generality we can
308 assume that $a \leq b$, which means that $A \in V_a \in V_b$, therefore V_b is a set that
309 satisfies the pairing axiom: it contains both A and B .

310 TODO vyhodit axiomy, staci vyrobit ω

311 We now want to prove that V_α leads to existence of an inductive set,
312 which is a set that satisfies $\exists A(\emptyset \in A \ \& \ \forall x \in A ((x \cup \{x\}) \in A))$. If we can
313 find a way to construct V_ω from any V_α satisfying $\alpha \geq \omega$, we are done. Since
314 ω is the least limit ordinal, all we need is the following

$$\bigcap \{V_\kappa \mid \forall \lambda(\lambda < \kappa \rightarrow \exists \mu(\lambda < \mu < \kappa))\} \quad (2.6)$$

315 because V_κ is a transitive set for every κ , thus the intersection is non-empty
316 unless empty set satisfies the property or the set of V_κ s is itself empty. \square

317

318 **Theorem 2.16** *In S , the schema N_0 implies Replacement schema.*

319 *Proof.* TODO vysvetlit! (podle contemporary verze)

⁸We only need to verify axioms that provide means of constructing larger sets from smaller to make sure they don't exceed ω . Since ω is an initial segment of ZF , the axiom scheme of specification can't be broken, the same holds for foundation and extensionality.

Let $\varphi(v, w)$ be a formula wth no free variables except v, w, x_1, \dots, x_n where n is any natural number. Let χ be an instance of replacement schema for this φ :

$$\chi = \forall r, s, t(\varphi(r, s) \& \varphi(r, t) \rightarrow s = t) \rightarrow \forall x \exists y \forall w (w \in y \leftrightarrow \exists v (v \in x \& \varphi(v, w))) \quad (2.7)$$

323

We can deduce the following from N_0 :

- (i) $x_1, \dots, x_n, v, w \in u \rightarrow (\varphi \leftrightarrow \varphi^u)$
- (ii) $x_1, \dots, x_n, v \in u \rightarrow (\exists w \varphi \leftrightarrow (\exists w \varphi)^u)$
- (iii) $x_1, \dots, x_n, x \in u \rightarrow (\chi \leftrightarrow \chi^u)$
- (iv) $\forall x_1, \dots, x_n \forall x (\chi \leftrightarrow (\forall x_1, \dots, x_n \forall x \chi)^u)$

Note that (i), (ii), (iii) are obtained from instances of N_0 for φ , $\exists w \varphi$ and χ respectively. From relativization we also know that $(\exists w \varphi)^u$ is equivalent to $\exists w (w \in u \& \varphi^u)$. Therefore (ii) is equivalent to $x_1, \dots, x_n, v \in u \rightarrow (\exists w (w \in u \& \varphi^u))$.

If φ is a function $(\forall r, s, t(\varphi(r, s) \& \varphi(r, t) \rightarrow r = t))$, then for every $x \in u$, which is also $x \subset u$ by $Scm^S(u)$, it maps elements of x onto u . From the axiom scheme of comprehension⁹, we can find a set of all images of elements of x . Let's call it y . That gives us $x_1, \dots, x_n, x \in u \rightarrow \chi$. By (iii) we get $x_1, \dots, x_n, x \in u \rightarrow \chi^u$, closure of this formula is $(\forall x_1, \dots, x_n \forall x \chi)^u$, which together with (iv) yields $\forall x_1, \dots, x_n \forall x \chi$. By the means of specification we end up with χ , which is all we need for now.

TODO btw co je x? nemela by tam tam byt nejaka volna promenna?

TODO shrnout zbytek clanku, fixed-point vety a spol

TODO S- \mathcal{L} ZM- \mathcal{L} ZM'- \mathcal{L} ZM", neco jako mahlovy kardinaly

343

□

2.4 Contemporary restatement

TODO nejaky uvod.

TODO Levy rika ze existuje $Scm^S(u)$ reflektujici varphi, coz uz nepotrebuje. atd.

TODO Ze prvoradova reflexe je theorem ZFC, vys uz max jako axiom/schema. (spis ne)

The following lemma is usually done in more parts, the first being with one formula and the other with n . We will only state and prove the generalised version for n formulas, knowing that $n = 1$ is just a specific case and the proof is exactly the same.

⁹axiom of subsets in Levy's version

354 **Lemma 2.17** *Lemma Let $\varphi_1, \dots, \varphi_n$ be any formulas with m parameters¹⁰.*
 355 *(i) For each set M_0 there is such M that $M_0 \subset M$ and the following holds*
 356 *for every $i \leq n$:*

$$\exists x \varphi_i(u_1, \dots, u_{m-1}, x) \rightarrow (\exists x \in M) \varphi_i(u_1, \dots, u_{m-1}, x) \quad (2.8)$$

357 *for every $u_1, \dots, u_{m-1} \in M$.*
 358 *(ii) Furthermore there is an ordinal α such that $M_0 \subset V_\alpha$ and the following*
 359 *holds for each $i \leq n$:*

$$\exists x \varphi_i(u_1, \dots, u_{m-1}, x) \rightarrow (\exists x \in V_\alpha) \varphi_i(u_1, \dots, u_{m-1}, x) \quad (2.9)$$

360 *for every $u_1, \dots, u_{m-1} \in M$.*

361 *Proof.* We will simultaneously prove statements (i) and (ii), denoting M^T
 362 the transitive set required by part (ii). Unless explicitly stated otherwise for
 363 specific steps, it is thought to be equivalent to M .

364 Let us first define operation $H(u_1, \dots, u_{m-1})$ that gives us the set of
 365 x 's with minimal rank satisfying $\varphi_i(u_1, \dots, u_{m-1}, x)$ for given parameters
 366 u_1, \dots, u_{m-1} for every $i \leq n$.

$$H_i(u_1, \dots, u_n) = \{x \in C_i : (\forall z \in C)(\text{rank}(x) \leq \text{rank}(z))\} \quad (2.10)$$

367 for each $i \leq n$, where

$$C_i = \{x : \varphi_i(u_1, \dots, u_{m-1}, x)\} \text{ for } i \leq n \quad (2.11)$$

368

369 Next, let's construct M from given M_0 by induction.

$$M_{i+1} = M_i \cup \bigcup_{j=0}^n \{H_j(u_1, \dots, u_{m-1}) : u_1, \dots, u_{m-1} \in M_i\} \quad (2.12)$$

370 In other words, in each step we add the elements satisfying $\varphi(u_1, \dots, u_{m-1}, x)$
 371 for all parameters that were either available earlier or were added in the
 372 previous step. For statement (ii), this is the only part that differs from (i).

¹⁰For formulas with different number of parameters take for m the highest number of parameters among given formulas. Add spare parameters to the other formulas so that x remains the last parameter. That can be done in a following manner: Let φ'_i be the a formula with k parameters, $k < m$. Let us set $\varphi_i(u_1, \dots, u_{m-1}, x) = \varphi'_i(u_1, \dots, u_{k-1}, u_k, \dots, u_{m-1}, x)$, notice that u_k, \dots, u_{m-1} are spare variables added just for formal simplicity.

Let us take for each step transitive closure of M_{i+1} from (i). In other words, let γ be the smallest ordinal such that

$$(M_i^T \cup \bigcup_{j=0}^n \{ \bigcup \{ H_j(u_1, \dots, u_{m-1}) : u_1, \dots, u_{m-1} \in M_i \} \}) \subset V_\gamma \quad (2.13)$$

Then the incremental step is like so:

$$M_{i+1}^T = V_\gamma \quad (2.14)$$

The final M is obtained by joining all incremental steps together.

$$M = \bigcup_{i=0}^{\infty} M_i, \quad M^T = \bigcup_{i=0}^{\infty} M_i^T \quad (2.15)$$

Let's try to construct a set M' that satisfies the same conditions like M but is kept as small as possible. Assuming the Axiom of Choice, we can modify the process so that cardinality of M' is at most $|M_0| \cdot \aleph_0$. Note that the size of M' is determined by the size of M_0 and, most importantly, by the size of $H_i(u_1, \dots, u_{m-1})$ for any $i \leq n$ in individual levels of the construction. Since the lemma only states existence of some x that satisfies $\varphi_i(u_1, \dots, u_{m-1}, x)$ for any $i \leq n$, we only need to add one x for every set of parameters but $H_i(u_1, \dots, u_{m-1})$ can be arbitrarily large. Since Axiom of Choice ensures that there is a choice function, let F be a choice function on $\mathcal{P}(\bigcup M')$. Also let $h_i(u_1, \dots, u_{m-1}) = F(H_i(u_1, \dots, u_{m-1}))$ for $i \leq n$, which means that h is a function that outputs an x that satisfies $\varphi_i(u_1, \dots, u_{m-1}, x)$ for $i \leq n$ and has minimal rank among all such witnesses. The induction step needs to be redefined to

$$M'_{i+1} = M'_i \cup \bigcup_j = 0^n \{ h_j(u_1, \dots, u_{m-1}) : u_1, \dots, u_{m-1} \in M'_i \} \quad (2.16)$$

In every step, the amount of elements added in M'_{i+1} is equivalent to the amount of sets of parameters the yielded elements not included in M'_i . So the cardinality of M'_{i+1} exceeds the cardinality of M'_i only for finite M'_i . It is easy to see that if M_0 is finite, M' is countable because it was built from countable union of finite sets. If M_0 is countable or larger, cardinality of M' is equal to the cardinality of M_0 .¹¹ Therefore $|M'| \leq |M_0| \cdot \aleph_0$

□

TODO proc \leq a ne =?

¹¹It can not be smaller because $|M'_{i+1}| \geq |M'_i|$ for every i . It may not be significantly larger because the maximum of elements added is the number of n -tuples in M'_i , which is of the same cardinality is M'_i . ((proc? Ramsey?))

400 **Theorem 2.18** *First-order Reflection*

401 *Let $\varphi(x_1, \dots, x_n)$ be a first-order formula.*

402 (i) *For every set M_0 there exists M such that $M_0 \subset M$ and the following*
 403 *holds:*

$$\varphi^M(x_1, \dots, x_n) \leftrightarrow \varphi(x_1, \dots, x_n) \quad (2.17)$$

404 *for every x_1, \dots, x_n .*

405 (ii) *For every set M_0 there is a transitive set M , $M_0 \subset M$ such that the*
 406 *following holds:*

$$\varphi^M(x_1, \dots, x_n) \leftrightarrow \varphi(x_1, \dots, x_n) \quad (2.18)$$

407 *for every x_1, \dots, x_n .*

408 (iii) *For every set M_0 there is α such that $M_0 \subset V_\alpha$ and the following holds:*

$$\varphi^{V_\alpha}(x_1, \dots, x_n) \leftrightarrow \varphi(x_1, \dots, x_n) \quad (2.19)$$

409 *for every x_1, \dots, x_n .*

410 (iv) *Assuming the Axiom of Choice, for every set M_0 there is M such that*
 411 *$M_0 \subset M$ and $|M| \leq |M_0| \cdot \aleph_0$ and the following holds:*

$$\varphi^M(x_1, \dots, x_n) \leftrightarrow \varphi(x_1, \dots, x_n) \quad (2.20)$$

412 *for every x_1, \dots, x_n .*

413 *Proof.* Let's prove (i) for one formula φ via induction by complexity first.
 414 We can safely assume that φ contains no quantifiers besides \exists and no logical
 415 connectives other than \neg and $\&$. Assume that this M is obtained from
 416 lemma 2.17. The fact, that atomic formulas are reflected in every M comes
 417 directly from definition of relativization and the fact that they contain no
 418 quantifiers.¹² The same holds for formulas in the form of $\varphi = \neg\varphi'$. Let us
 419 recall the definition of relativization for those formulas in 2.1.

$$(\neg\varphi_1)^M \leftrightarrow \neg(\varphi_1^M) \quad (2.21)$$

420 Because we can assume from induction that $\varphi'^M \leftrightarrow \varphi'$, the following holds:

$$(\neg\varphi')^M \leftrightarrow \neg(\varphi'^M) \leftrightarrow \neg\varphi' \quad (2.22)$$

421 The same holds for $\varphi = \varphi_1 \& \varphi_2$. From the induction hypothesis we know
 422 that $\varphi_1^M \leftrightarrow \varphi_1$ and $\varphi_2^M \leftrightarrow \varphi_2$, which together with relativization for formulas
 423 in the form of $\varphi_1 \& \varphi_2$ gives us

$$(\varphi_1 \& \varphi_2)^M \leftrightarrow \varphi_1^M \& \varphi_2^M \leftrightarrow \varphi_1 \& \varphi_2 \quad (2.23)$$

¹²Note that this does not hold generally for relativizations to M, E , but only for relativization to M, \in , which is our case.

424

425 Let's now examine the case when from the induction hypethesis, M re-
 426 flects $\varphi'(u_1, \dots, u_n, x)$ and we are interested in $\varphi = \exists x \varphi'(u_1, \dots, u_n, x)$. The
 427 induction hypothesis tells us that

$$\varphi'^M(u_1, \dots, u_n, x) \leftrightarrow \varphi'(u_1, \dots, u_n, x) \quad (2.24)$$

428 so, together with above lemma 2.17, the following holds:

$$\begin{aligned} & \varphi(u_1, \dots, u_n, x) \\ & \leftrightarrow \exists x \varphi'(u_1, \dots, u_n, x) \\ & \leftrightarrow (\exists x \in M) \varphi'(u_1, \dots, u_n, x) \\ & \leftrightarrow (\exists x \in M) \varphi'^M(u_1, \dots, u_n, x) \\ & \leftrightarrow (\exists x \varphi'(u_1, \dots, u_n, x))^M \\ & \leftrightarrow \varphi^M(u_1, \dots, u_n, x) \end{aligned} \quad (2.25)$$

429 Which is what we have needed to prove:

430

431 So far we have proven part (i) of this theorem for one formula φ , we only
 432 need to verify that the same holds for any finite number of formulas. This
 433 has in fact been already done since lemma 2.17 gives us M for any (finite)
 434 amount of formulas. We can than use the induction above to verify that it
 435 reflects each of the formulas individually.

436

437 Now we want to verify other parts of our theorem. Since V_α is a transitive
 438 set, by proving (iii) we also satisfy (ii). To do so, we only need to look at
 439 part (ii) of lemma 2.17. All of the above proof also holds for $M = V_\alpha$. To
 440 finish part (iv)

441

442 TODO spocetna varianta!!

□

443

444 **Theorem 2.19** Reflection is equivalent to Infinity & Replacement under
 445 ZFC minus Infinity & Replacement

446

447 *Proof.* Since 2.18 already gives one side of the implication, we are only
 448 interested in showing the converse which we shall do in two parts:

449 *Reflection* \rightarrow *Infinity*

450 Let us first find a formula to be reflected that requires a set M at least
 451 as large as V_ω . Let us consider the following formula:

$$\varphi'(x) = \forall \lambda (\lambda < x \rightarrow \exists \mu (\lambda < \mu < x)) \quad (2.26)$$

Because φ says "there is a limit ordinal", if it holds for some x , the Infinity axiom is very easy to satisfy. But we know that there are limit ordinals in ZF, therefore $\varphi = \exists x\varphi'(x)$ is a valid statement. *Reflection* then gives us a set M in which φ^M holds, that is, a set that contains a limit ordinal. So the set of off limit ordinals is non-empty and because ordinals are well-founded, it has a minimal element. Let's call it μ .

$$\mu = \bigcap \{V_\kappa : \forall \lambda(\lambda < \kappa \rightarrow \exists \mu(\lambda < \mu < \kappa))\} \quad (2.27)$$

We can see that μ is the least limit ordinal and therefore it satisfies *Infinity*.

Reflection \rightarrow *Replacement*

Given a formula $\varphi(x, y, u_1, \dots, u_n)$, we can suppose that it is reflected in any M ¹³ What we want to obtain is the following:

$$\forall x, y, z(\varphi(x, y, u_1, \dots, u_n) \& \varphi(x, z, u_1, \dots, u_n) \rightarrow y = z) \rightarrow \quad (2.28)$$

$$\rightarrow \forall X \exists Y \forall y (y \in Y \leftrightarrow \exists x(\varphi(x, y, u_1, \dots, u_n) \& x \in X)) \quad (2.29)$$

We do also know that $x, y \in M$, in other words for every $X, Y = \{y \mid \varphi(x, y, u_1, \dots, u_n)\}$ we know that $X \subset M$ and $Y \subset M$, which, together with the comprehension schema¹⁴ implies that Y , the image of X over φ , is a set. Which is exactly the Replacement Schema we hoped to obtain. \square

We have shown that *Reflection* for first-order formulas, *Reflection*₁ is a theorem of ZF, which means that it won't yield us any large cardinals. We have also shown that it can be used instead of the Axiom of Infinity and Replacement Scheme, but ZF + *Reflection*₁ is a conservative extension of ZF. Besides being a starting point for more general and powerful statements, it can be used to show that ZF is not finitely axiomatizable. That is because *Reflection* gives a model to any finite number of (consistent) formulas. So if $\varphi_1, \dots, \varphi_n$ for any finite n would be the axioms of ZF, *Reflection* would always contain a model of itself, which would in turn contradict the Second Gödel's Theorem¹⁵. Notice that, in a way, reflection is complementary to compactness. Compactness argues that given an infinite¹⁶ set of sentences, if every finite subset yields a model, so does the whole set. Reflection, on the other hand, says that while the whole set has no model in the underlying theory, every finite subset does have one.

¹³Which means that for $x, y, u_1, \dots, u_n \in M$, $\varphi^M(x, y, u_1, \dots, u_n) \leftrightarrow \varphi(x, y, u_1, \dots, u_n)$.

¹⁴Called the axiom of subsets in Levy's proof.

¹⁵See chapter 3.3 for further details.

¹⁶Countable?

483 Also, notice how reflection can be used in ways similar to upward Löwenheim–Skolem
484 theorem. Since Reflection extends any set M_0 into a model of given formulas
485 $\varphi_1, \dots, \varphi_n$, we can choose M_0 such that the final M is at least as big as we
486 need it to be.¹⁷

487 TODO znacit *Reflection* asi jako *Reflection*₁ pokud mluvíme o prvo-
488 radových formulích (definice je nahore v posledni subsection section 1)

489 In the next section, we will try to generalize Reflection in a way that
490 transcends ZF and finally yields some large cardinals.

¹⁷Too vague?

3 Reflecting Large Cardinals

TODO "v nasledujici kapitole budeme reflektovat vlastnosti univerzalni tridy a budeme se divat jake initial segmenty/kardinaly dostavame podle toho, jak se meni presny vyznam pojmu 'vlastnost' atd..."

TODO napsat nekam

TODO neco jako: for every process for obtaining larger sets, this process can't reach V and, from reflection, there is an initial segment of V that is strictly smaller than V but still can't be reached via said process.

In this chapter we aim to explore the possible generalisations of *Reflection* for second-order formulas and use those to establish the existence of various large cardinals. We will also argue whether there is a limit to the size of large cardinals obtainable via generalised form of *Reflection*.

3.1 Reflecting Second-order Formulas

TODO ze "uplne totalni" reflexe se zacykli a rozbije

To see that there is a way to transcend ZF, let us briefly show how a model of ZF can be obtained in $ZF_2 + \text{"second-order reflection"}$ ¹⁸. This will be more closely examined in section 3.3.

We know that ZF can not be finitely axiomatized in first-order formulas, however if Replacement and Comprehension schemes can be substituted by second-order formulas, ZF becomes ZF_2 , which is finitely axiomatizable in second-order logic. Therefore if we take second-order reflection into consideration, we can obtain a set M that is a model of ZF_2 . For now, we have left out the details of how exactly is first-order reflection generalised into stronger statements and how second-order axiomatization of ZF looks like as we will examine those problems closely in the following pages.

We will now define reflection for second-order formulas.

Definition 3.1 *Second-order reflection*

TODO

TODO see Hanf-Scott [kanamori:61]?

TODO full reflection, partial reflection? viz Levy60, ten druhý clanek

3.2 Preliminaries

But first, let's establish some elementary terms that will allow us to define the relevant large cardinals.

¹⁸ ZF_2 is an axiomatization of ZF in second-order formulas, to be more rigorously established later.

524 **Definition 3.2** (*limit cardinal*) *kappa* is a limit cardinal if it is \aleph_α for some
 525 limit ordinal α .

526 **Definition 3.3** (*strong limit cardinal*) *kappa* is a strong limit cardinal if for
 527 every $\lambda < \kappa$, $2^\lambda < \kappa$

528 TODO splyvaji kdyz plati GCH

529 We also need to rigorously define \mathbf{ZF}_2 , the second-order axiomatization
 530 of \mathbf{ZF} we have already used in the previous section. Let's take advantage of
 531 second-order variables and replace Replacement and Specification schemes
 532 with a single Replacement and a Specification axiom respectively. Lower-case
 533 letters represent first-order variables and upper-case P represents a second-
 534 order variable. [8]

535 **Definition 3.4** Replacement²

$$\begin{aligned} \forall P(\forall x, y, z(P(x, y) \& P(x, z) \rightarrow y = z) \rightarrow \\ \rightarrow (\forall x \exists y \forall z(z \in y \leftrightarrow \exists w \in x(P(w, z)))) \end{aligned} \quad (3.30)$$

537 We will denote this axiom Replacement₂.

538 **Definition 3.5** Specification₂

$$\forall P \forall x \exists y \forall z (z \in y \leftrightarrow [z \in x \& P(z, x)]) \quad (3.31)$$

540 **Definition 3.6** \mathbf{ZF}_2

541 Let \mathbf{ZF}_2 be a theory with all axioms identical with the axioms of \mathbf{ZF} with the
 542 exception of Replacement and Specification schemes, which are replaced with
 543 Replacement₂ and Specification₂ respectively.

544 TODO vsechny jmena axiomu emph?

545 TODO sjednotit Replacement₂ s Replacement²

546

547 TODO budeme potrebovat club sety, stacionarni mnoziny? treba aby-
 548 chom dokazali ekvivalence?

549 3.3 Inaccessibility

550 The inaccessible cardinal is the smallest of large cardinals¹⁹

551 **Definition 3.7** (*weak inaccessibility*) An uncountable cardinal κ is weakly
 552 inaccessible \leftrightarrow it is regular and limit.

¹⁹citation needed.

Definition 3.8 (*inaccessibility*) An uncountable cardinal κ is inaccessible \leftrightarrow it is regular and strongly limit.

To see why this is important, let's dedicate a few lines to the intuition behind the notions of limitness, regularity and inaccessibility in a manner strongly influenced by [7]. To see why limit and strongly limit cardinals are worth mentioning, note that they are "limit" not only in a sense of being a supremum of an ordinal sequence, they also show that a certain way of obtaining bigger objects from smaller ones²⁰ limited in terms of possibilities. \aleph_λ is a limit cardinal iff there is no α such that $\aleph_{\alpha+1} = \aleph_\lambda$. Strongly limit cardinals point to the limits of the powerset operation. It has been too obvious so far, so let's look at the regular cardinals in this manner. Regular cardinals are those that cannot be²¹, expressed as a supremum of smaller amount of smaller objects²². More precisely, κ is regular if there is no way to define it as a union of less than κ ordinals, all smaller than κ . It is interesting to point out, that many of these sets wouldn't be considered sets without *Replacement*, therefore regular cardinals are, in a way, "limits" of how far can we get by taking limits of increasing sequences of ordinals obtained via *Replacement*.

That all being said, it is easy to see that no cardinals in ZF are both strongly limit and regular, the only exception being \aleph_0 . This makes it clear why the definition 3.8 explicitly calls for $\kappa > \aleph_0$. It should be also obvious why the fact that κ is inaccessible implies that $\kappa = \aleph_\kappa$.²³

The above should make a clear picture of why *Infinity* is a specific case of *Reflection*.

TODO proc je Refl zaroven zobecneny replacement?

TODO nize budeme zkoumat ktere mnoziny jsou "nedosazitelne" reflexi ruznych skupin "properties of V".

We will now show that the above notion is equivalent to the definition Levy uses in [2], which is, in more contemporary notation, the following:

Theorem 3.9 *The following are equivalent.*²⁴

1. κ in inaccessible
2. $\langle V_\kappa, \in \rangle \models \text{ZFC}$

²⁰TODO elegantnejsi formulace?

²¹Assuming *Choice*.

²²Just like ω can not be expressed as a supremum of a finite set consisting solely of finite numbers.

²³This doesn't work backwards, the first fixed point of the \aleph function is the limit of $\{\aleph_0, \aleph_{\aleph_0}, \aleph_{\aleph_{\aleph_0}}, \dots\}$ is singular since the sequence has countably many elements.

²⁴TODO skutecne plati na obe strany?

Proof. Let's first prove that if κ is inaccessible, it is a model of ZFC. We will do that by verifying the axioms of ZFC just like Kanamori does it in 1.2 in [1]. Because κ is a limit ordinal, there's no need for us to verify the Powerset, Foundation, Extensionality, Subset, Pairing and Union axioms and the Specification scheme. Thus we only have the Replacement Scheme to verify.

Given an arbitrary set $x \in V_\kappa$ and a function $F : x \rightarrow V_\kappa$, we need to verify that $y = F[x]$ is indeed a set and that it is an element of V_κ . The fact that F is a function implies that $|y| \leq |x|$. It follows from Specification that $y \subset V_\kappa$, which is still not exactly what we want. Let $\alpha < \kappa$ be the least ordinal such that $y \in V_\alpha$ ²⁵, since $V_{\alpha+1} = \mathcal{P}(V_\alpha)$, $y \in V_{\alpha+1}$, together with $\alpha + 1 < \kappa$ this means that $y \in V_\kappa$.

We will now show that if a set is a model of ZFC, it is in fact an inaccessible cardinal. So let V_κ be a model of ZFC which means that it is closed under the powerset operation, in other words:

$$\forall \lambda (\lambda < \kappa \rightarrow 2^\lambda < \kappa) \quad (3.32)$$

which is exactly the definition of strong limitness. κ is regular from the following argument by contradiction:

Let us suppose for a moment that κ is singular. Therefore there is an ordinal $\alpha < \kappa$ and a function $F : \alpha \rightarrow \kappa$ such that the range of F is unbounded in κ , in other words, $F[\alpha] \subseteq V_\kappa$ and $\sup(F[\alpha]) = \kappa$. In order to achieve the desired contradiction, we need to see that it is the case that $F[\alpha] \in V_\kappa$. Let $\varphi(x, y)$ be the following first-order formula:

$$F(x) = y \quad (3.33)$$

Then there is an instance of Axiom Schema of Replacement that states the following:

$$\begin{aligned} &(\forall x, y, z (\varphi(x, y) \& \varphi(x, z) \rightarrow y = z)) \rightarrow \\ &\rightarrow (\forall x \exists y \forall z (z \in y \leftrightarrow \exists w (\varphi(w, z)))) \end{aligned} \quad (3.34)$$

Which in turn means that there is a set $y = F[\alpha]$ and $y \in V_\kappa$, which is the contradiction with $\sup(y) = \kappa$ we are looking for. \square

The same holds for ZF_2 , the proof is very similar.

Theorem 3.10

$$V_\kappa \models \text{ZF}_2 \leftrightarrow \kappa \text{ is inaccessible} \quad (3.35)$$

²⁵TODO pozor – jak vime ze takove alpha existuje?

615 *Proof.* κ is a strong limit cardinal because from \mathbf{ZF}_2 and the Powerset Axiom
 616 we know that for every $\lambda < \kappa$, we know that $2^\lambda < \kappa$.

617 κ is also regular, because otherwise there would be an ordinal α and
 618 a function $F : \alpha \rightarrow \kappa$ with a range unbounded in κ . *Replacement*² gives us
 619 a set $y = F[\alpha]$, so $y \in V_\kappa$, which contradicts the fact that $\sup(y) = \kappa$. It
 620 can not be the case that $\kappa \in V_\kappa$.

621

622 The other direction is exactly like the first part of above theorem 3.9. \square

623

624 We have transcended \mathbf{ZF} , but that is just a start. Naturally, we could
 625 go on and consider the next inaccessible cardinal, which is inaccessible with
 626 respect to the theory $\mathbf{ZF} + \exists \kappa (\kappa \models \mathbf{ZF})$. But let's try to find a faster way up,
 627 informally at first.

628 TODO muzu rict "inaccessible cardinals are unbounded in V "?

629 Since we can find an inaccessible set larger than any chosen set M_0 , it is
 630 clear that inaccessible cardinals are unbounded in V . If V were a cardinal,
 631 we could say that there are V inaccessible cardinals less than V , but this
 632 statement of course makes no sense in a set theory as is because V is not a
 633 set. But being more careful, we could find a property that can be formalized
 634 in first- or second-order logic and reflect it to an initial segment of V . That
 635 would allow us to construct large cardinals more efficiently than by adding
 636 inaccessibles one by one. The property we are looking for ought to look like
 637 something like this:

$$\begin{aligned} &\kappa \text{ is an inaccessible cardinal and} \\ &\text{there are } \kappa \text{ inaccessible cardinals } \mu < \kappa \end{aligned} \tag{3.36}$$

638 This is in fact a fixed-point type of statement. We shall call those cardinals
 639 hyper-inaccessible. Now consider the following definition.

640

641 **Definition 3.11** *0-hyper-inaccessible cardinal*

642 *A cardinal κ is 0-hyper-inaccessible if it is inaccessible.*

643

644 **Definition 3.12** *α -hyper-inaccessible cardinal*

645 *For any ordinal α , κ is called α -hyper-inaccessible, if κ is inaccessible and for*
 646 *each $\beta \prec \alpha$, the set of β -hyper-inaccessible cardinals less than κ is unbounded*
 647 *in κ .*

648 TODO nepotrebuju treba vzit zvlast v uvahu limitni α ?

649
650 Because κ is inaccessible and therefore regular, the number of β -inaccessibles
651 below κ is equal to κ . We have therefore successfully formalized the above
652 vague notion of hyper-inaccessible cardinal into a hierarchy of α -hyper-inaccessibles.

653
654 Let's now consider iterating this process over again. Since, informally,
655 V would be α -hyper-inaccessible for any α , this property of the universal
656 class could possibly be reflected to an initial segment, the smallest of those
657 will be the first hyper-hyper-inaccessible cardinal. Such κ is larger than
658 any α -hyper-inaccessible since from regularity of κ , for given $\alpha < \kappa$, κ is
659 κ -th α -hyper-inaccessible cardinal. It is in fact "inaccessible" via α -hyper-
660 inaccessibility.

661
662 **Definition 3.13** *Hyper-hyper-inaccessible cardinal*
663 κ is called the hyper-hyper-inaccessible, also 0-hyper-hyper-inaccessible, car-
664 dinal if it is α -inaccessible for every $\alpha < \kappa$.

665
666 **Definition 3.14** *α -hyper-hyper-inaccessible cardinal*
667 For any ordinal α , κ is called α -hyper-hyper-inaccessible cardinal if for each
668 ordinal $\beta < \alpha$, the set of β -hyper-hyper-inaccessible cardinals less the κ is
669 inbounded in κ .

670
671 Obviously we could go on and iterate it ad libitum, but the nomenclature
672 would be increasingly confusing. A smarter way to accomplish the same goal
673 is carried out in the following section.

674 TODO typografie – mezery kolem vseh = a asi i vyrokovych ostatnich
675 spojek

676 3.4 Mahlo Cardinals

677 While the previous chapter introduced us to a notion of inaccessibility and
678 the possibility of iterating it ad libitum in new theories, there is an even
679 faster way to travel upwards in the cumulative hierarchy, that was proposed
680 by Paul Mahlo in his papers (see [4], [5] and [6]) at the very beginning of
681 the 20th century, and which can be easily reformulated using (*Reflection*).
682 To see how Lévy's initial statement of reflection was influenced by Mahlo's

work, refer to section 2.2. The aim of the following paragraphs is to give an intuitive explanation of the idea behind Mahlo's hierarchy of cardinals, all claims made here ought to be stated formally later in the very same chapter.

At the very end of section 3.3, we have tried to establish the notion of hyper-inaccessibility and iterate it to yield even larger large cardinals. In order to avoid too bulky cardinal names, let's try a different route and establish those cardinals directly via reflection.

The following two definitions come from [7] and while they are rather informal, we will find them very helpful for understanding the Mahlo cardinals.

Definition 3.15 *Fixed-point property*

For any $\psi(x, u_1, \dots, u_n)$ which is any property of ordinals, we say that a property φ is a fixed-point property if φ has the form

$$x \text{ is an inaccessible cardinal and there are } x \text{ ordinals less than } x \text{ that have the property } \psi(x, u_1, \dots, u_n) \quad (3.37)$$

TODO this is fixed-point property because ... (/fixed-point ceho?)

Definition 3.16 *Fixed-point reflection*

If φ is a fixed-point property that holds for V , it also holds for some V_α , an initial segment of V .

Obviously those are in on way rigorous definitions because we have no idea what $\psi(x, u_1, \dots, u_n)$ looks like. Let's try to restate the same idea in a useful way. But first, let's show that the formal counterpart of the idea of containing "enough" ordinals with a property is the notion of stationary set.

Definition 3.17 *Supremum*

Given A a set of ordinals, the supremum of A , denoted $\sup(A)$, is the least upper bound of A .

$$\sup(A) = \bigcup A \quad (3.38)$$

where α is an ordinal.

Definition 3.18 *Limit point*

Given A , a set of ordinals and an ordinal α , we say that α is a limit point of A if $\sup(A \cap \alpha) = \alpha$

713 **Definition 3.19** *Club set*

714 *For a regular uncountable κ , a set $A \subset \kappa$ is a closed unbounded subset*
 715 *(often abbreviated as a club set) iff A is both closed, which means it contains*
 716 *all its limit points, and unbounded, which means that for every $\beta < \kappa$ there*
 717 *is a $\beta' \in A$ such that $\beta < \beta' < \kappa$.*

718

719 **Definition 3.20** *Stationary set*

720 *For a regular uncountable κ , a set $A \subset \kappa$ is stationary if it intersects every*
 721 *club subset of κ .*

722

723 **Definition 3.21** *Weakly Mahlo Cardinal*

724 *κ is weakly Mahlo \leftrightarrow it is a limit ordinal and the set of all regular ordinals*
 725 *less than κ is stationary in κ*

726 **Definition 3.22** *Mahlo Cardinal*

727 *κ is a Mahlo Cardinal iff it is an inaccessible cardinal and the set of all*
 728 *inaccessible ordinals less than κ is stationary in κ .*

729 Analogously,

730 **Definition 3.23** *α -Mahlo Cardinal*

731 *κ is a α -Mahlo Cardinal iff it is an α -inaccessible cardinal and the set of all*
 732 *α -inaccessible ordinals less than κ is stationary in κ .*

733

734 In other words, κ is a mahlo cardinal if it is inaccessible and every club
 735 set in κ contains an inaccessible cardinal. TODO napsat co to znamena, proc
 736 stacionarni?

737

738 TODO jak to tam dela Levy?

739

740 Thus a Mahlo cardinal κ is not only inaccessible, but also has κ inacces-
 741 sibles below it.

742

743 **Definition 3.24** *Mahlo cardinals*

744 *The following definitions are equivalent:*

745 (i) κ is Mahlo

746 (ii) κ is weakly Mahlo and strong limit

3.5 Indescribability and Weakly Compact Cardinals Reflecting Large Cardinals

- (iii) κ is inaccessible and the regular cardinals below κ form a stationary subset of κ .
(iv) κ is regular and the stationary sets below κ form a stationary subset of κ .
(v) $\{\lambda < \kappa : \lambda \text{ is inaccessible}\}$ is stationary in κ .

TODO κ is hyper-Mahlo iff κ is inaccessible and the set $\{\lambda < \kappa : \lambda \text{ is Mahlo}\}$ is stationary in κ .
” We also state the appropriate generalization for greatly Mahlo cardinals.”

3.5 Indescribability and Weakly Compact Cardinals

TODO indescribable – reflecting indescribability – we can’t reach V by a Σ_1^1 formula, so there’s some initial segment V_α that is also unreachable (we say indescribable) by the means of a ... formula
TODO co je ”partition property”?
TODO pak dk. ekvivalenci
TODO Kanamori 6.3

Definition 3.25 A cardinal κ is weakly compact if it is uncountable and satisfies the partition property $\kappa \rightarrow (\kappa)^2$

opsano z jecha!

3.6 Indescribable Cardinals

TODO uvod / intuice
TODO souvislost s reflexi

3.7 Bernays–Gödel Set Theory

TODO Plagiat – prepsat a vysvetlit
TODO

775 **3.8 Reflection and the constructible universe**

776 TODO reflektovat muzeme jenom kardinaly konzistentni s $V=L$, proc?

777 TODO Plagiat – prepsat a vysvetlit

778 L was introduced by Kurt Gödel in 1938 in his paper *The Consistency*
779 *of the Axiom of Choice and of the Generalised Continuum Hypothesis* and
780 denotes a class of sets built recursively in terms of simpler sets, somewhat
781 similar to Von Neumann universe V . Assertion of their equality, $V = L$, is
782 called the *axiom of constructibility*. The axiom implies GCH and therefore
783 also AC and contradicts the existence of some of the large cardinals, our goal
784 is to decide whether those introduced earlier are among them.

785 On order to formally establish this class, we need to formalize the notion
786 of definability first:

787 TODO zduvodneni

788
789 TODO kratka diskuse jestli refl implikuje transcendenci na L - polemika,
790 nazor - $V=L$ a slaba kompaktnost a dalsi

791

792 TODO asi nekde bude meritelny kardinal

793 4 Higher-order reflection

794 TODO rict ze to je zobecneni a nejaky dalsi uvodni veci

795 4.1 Sharp

796 TODO je tohle higher-order vec?

797 4.2 Welek: Global Reflection Principles

798 TODO ma to vubec cenu?

799 **5 Conclusion**

800 TODO na konec

References

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