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REFLECTION PRINCIPLES AND LARGE CARDINALS

Bakalářská práce

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Prohlašuj, že jsem bakalářkou práci vypracoval samostatně a že jsem uvedl všechny použité prameny a literaturu.

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Abstract

Práce zkoumá vztah tzv. principů reflexe a velkých kardinálů. Lévy ukázal, že v ZFC platí tzv. věta o reflexi a dokonce, že věta o reflexi je ekvivalentní schématu nahrazení a axiomu nekonečna nad teorií ZFC bez axiomu nekonečna a schématu nahrazení. Tedy lze na větu o reflexi pohlížet jako na svého druhu axiom nekonečna. Práce zkoumá do jaké míry a jakým způsobem lze větu o reflexi zobecnit a jaký to má vliv na existenci tzv. velkých kardinálů. Práce definuje nedosažitelné, Mahlovy a nepopsatelné kardinály a ukáže, jak je lze zavést pomocí reflexe. Přirozenou limitou kardinálů získaných reflexí jsou kardinály nekonzistentní s L. Práce nabídne intuitivní zdůvodněn, proč tomu tak je.

Abstract

This thesis aims to examine relations between so called "Reflection Principles" and Large cardinals. Lévy has shown that Reflection Theorem is a sound theorem of ZFC and it is equivalent to Replacement Scheme and the Axiom of Infinity. From this point of view, Reflection theorem can be seen a specific version of an Axiom of Infinity. This paper aims to examine the Reflection Principle and it's generalisations with respect to existence of Large Cardinals. This thesis will establish Inaccessible, Mahlo and Indescribable cardinals and their definition via reflection. A natural limit of Large Cardinals obtained via reflection are cardinals inconsistent with L. The thesis will offer an intuitive explanation of why this is the case.

Contents

1 Introduction

1.1 Motivation and Origin

> The Universe of sets cannot be uniquely characterized (i. e. distinguished from all its initial elements) by any internal structural property of the membership relation in it, which is expressible in any logic of finite of transfinite type, including infinitary logics of any cardinal order.

— Kurt Gödel [4]

To understand why do need reflection in the first place, let's think about infinity for a moment. In the intuitive sense, infinity is an upper limit of all numbers. But for centuries, this was merely a philosophical concept, closely bound to religious and metaphysical way of thinking, considered separate from numbers used for calculations or geometry. It was a rather vague concept. In ancient Greece, Aristotle's response to famous Zeno's paradoxes introduced the distinction between actual and potential infinity. He argued, that potential infinity is (in today's words) well defined, as opposed to actual infinity, which remained a vague incoherent concept. He didn't think it's possible for infinity to inhabit a bounded place in space or time, rejecting Zeno's thought experiments as a whole. Aristotle's thoughts shaped western thinking partly due to Aquinas, who himself believed actual infinity to be more of a metaphysical concept for describing God than a mathematical property attributed to any other entity. In his Summa Theologica [1] he argues:

> A geometrician does not need to assume a line actually infinite, but takes some actually finite line, from which he subtracts whatever he finds necessary; which line he calls infinite.

Less than hundred years later, Gregory of Rimini wrote

> If God can endlessly add a cubic foot to a stone–which He can–then He can create an infinitely big stone. For He need only add one cubic foot at some time, another half an hour later, another a quarter of an hour later than that, and so on ad infinitum. He would then have before Him an infinite stone at the end of the hour.

Which is basically a Zeno's Paradox made plausible with God being the actor. In contrast to Aquinas' position, Gregory of Rimini theoretically constructs an object with actual infinite magnitude that is essentially different from

[1]Part I, Question 7, Article 3, Reply to Objection 1

God. Even later, in the 17th century, pushing the property of infinitness from the Creator to his creation, Nature, Leibniz wrote to Foucher in 1962:

> I am so in favor of the actual infinite that instead of admitting that Nature abhors it, as is commonly said, I hold that Nature makes frequent use of it everywhere, in order to show more effectively the perfections of its Author. Thus I believe that there is no part of matter which is not, I do not say divisible, but actually divided; and consequently the least particle ought to be considered as a world full of an infinity of different creatures.

But even though he used potential infinity in what would become foundations of modern Calculus and argued for actual infinity in Nature, Leibniz refused the existence of an infinite, thinking that Galileo's Paradoxon[2] is in fact a contradiction. The so called Galileo's Paradoxon is an observation Galileo Galilei made in his final book "Discourses and Mathematical Demonstrations Relating to Two New Sciences". He states that if all numbers are either squares and non–squares, there seem to be less squares than there is all numbers. On the other hand, every number can be squared and every square has it's square root. Therefore, there seem to be as many squares as there are all numbers. Galileo concludes, that the idea of comparing sizes makes sense only in the finite realm.

> Salviati: So far as I see we can only infer that the totality of all numbers is infinite, that the number of squares is infinite, and that the number of their roots is infinite; neither is the number of squares less than the totality of all the numbers, nor the latter greater than the former; and finally the attributes "equal," "greater," and "less," are not applicable to infinite, but only to finite, quantities. When therefore Simplicio introduces several lines of different lengths and asks me how it is possible that the longer ones do not contain more points than the shorter, I answer him that one line does not contain more or less or just as many points as another, but that each line contains an infinite number.

Leibniz insists in part being smaller than the whole saying

> Among numbers there are infinite roots, infinite squares, infinite cubes. Moreover, there are as many roots as numbers. And there are as many squares as roots. Therefore there are as many squares as numbers, that is to say, there are as many square numbers as

[2] zneni galileova paradoxu

> there are numbers in the universe. Which is impossible. Hence it follows either that in the infinite the whole is not greater than the part, which is the opinion of Galileo and Gregory of St. Vincent, and which I cannot accept; or that infinity itself is nothing, i.e. that it is not one and not a whole.

TODO nejakej Hegel–strucne?
TODO Cantor
TODO mene teologie, vice matematiky
TODO definovat pojmy (trida etc)

In his work, he defined transfinite numbers to extend existing natural number structure so it contains more objects that behave like natural numbers and are based on an object (rather a meta-object) that doesn't explicitly exist in the structure, but is closely related to it. This is the first instance of reflection. This paper will focus on taking this principle a step further, extending Cantor's (or Zermelo–Fraenkel's, to be more precise) universe so it includes objects so big, they could be considered the universe itself, in a certain sense.

TODO dal asi smazat

The original idea behind reflection principles probably comes from what could be informally called "universality of the universe". The effort to precisely describe the universe of sets was natural and could be regarded as one of the impulses for formalization of naive set theory. If we try to express the universe as a set $\{x|x = x\}$, a paradox appears, because either our set is contained in itself and therefore is contained in a set (itself again), which contradicts the intuitive notion of a universe that contains everything but is not contained itself.

TODO ???

If there is an object containing all sets, it must not be a set itself. The notion of class seems inevitable. Either directly the ways for example the Bernays–Gödel set theory, we will also discuss later in this paper, does in, or on a meta–level like the Zermelo–Fraenkel set theory, that doesn't refer to them in the axioms but often works with the notion of a universal class. duet Another obstacle of constructing a set of all sets comes from Georg Cantor, who proved that the set of all subsets of a set (let A be the set and $\mathscr{P}(()A)$ its powerset) is strictly larger that A. That would turn every aspiration to finally establish an universal set into a contradictory infinite regression.[3]. We will use V to denote the class of all sets. From previous thoughts we can

[3] An intuitive analogy of this *reductio ad infinitum* is the status of ω, which was originally thought to be an unreachable absolute, only to become starting point of Cantor's hierarchy of sets growing beyond all boundaries around the end of the 19^{th} century

easily argue, that it is impossible to construct a property that holds for V and no set and is neither paradoxical like $\{x|x = x\}$ nor trivial. Previous observation can be transposed to a rather naive formulation of the reflection principle:

(Refl) Any property which holds in V already holds in some initial segment of V.

To avoid vagueness of the term "property", we could informally reformulate the above statement into a schema:

For every first-order formula[4] φ holds in $V \leftrightarrow \varphi$ holds in some initial segment of V.

Interested reader should note that this is a theorem scheme rather than a single theorem. [5]

1.2 A few historical remarks on reflection

Reflection made it's first in set-theoretical appearance in Gödel's proof of GCH in L (citace Kanamori ? Lévy and set theory), but it was around even earlier as a concept. Gödel himself regarded it as very close to Russel's reducibility axiom (an earlier equivalent of the axiom schema of Zermelo's separation). Richard Montague then studied reflection properties as a tool for verifying that Replacement is not finitely axiomatizable (citace?). a few years later Lévy proved (citace? 1960a) equivalence of reflection with Axiom of infinity together with Replacement in proof we shall examine closely in chaper 2.

TODO co dal? recent results?

[4]this also works for finite sets of formulas [3, p. 168]

[5]If there were a single theorem stating "for any formula φ that holds in V there is an initial segment of V where φ also holds", we would obtain the following contradiction with the second Gödel's theorem: In ZFC, any finite group of axioms of ZFC holds in some initial segment of the universe. If we take the largest of those initial segments it is still strictly smaller than the universe and thus we have, via compactness, constructed a model of ZFC within ZFC. That is, of course a harsh contradiction. This also leads to an elegant way to prove that ZFC is not finitely axiomatizable.

2 Levy's first-order reflection

2.1 Introduction

This section will try to present Lévy's proof of a general reflection principle being equivalent to Replacement and Infinity under ZF minus Replacement and Infinity. We will first introduce a few axioms and definitions that were a different in Lévy's paper[2], but are equivalent to today's terms. We will write them in contemporary notation, our aim is the result, not history of set theory notation.

Please note that Lévy's paper was written in a period when Set theory was oriented towards semantics, which means that everything was done in a model. All proofs were theodel that of ZFC was V_α (notated as $R(\alpha)$ at the time) for some cardinal α, which means that α is a inaccessible cadinal. Please bear in mind that this is vastly different from saying that there is an inaccessible α inside the model. This V_α is also referred to as $Scm^{\mathsf{Q}}(u)$, which means that u ($u = V_\alpha$ in our case) is a standard complete model of an undisclosed axiomatic set theory Q formulated in the "non-simple applied first order functional calculus", which is second-order theory is today's terminology, we are allowed to quantify over functions and thus get rid of axiom schemes. (Note that Lévy always speaks of "the axiom of replacement"). Besides placeholder set theory Q and ZF, which the reader should be familiar with, theories Z, S, and SF are used in the text. Z is ZF minus replacement, S is ZF minus replacement and infinity, and finally SF is ZF minus infinity. "The axiom of subsets" is an older name for the axiom scheme of specification (and it's not a scheme since we are now working in second order logic). Also note that universal quantifier does not appear, $\forall x \varphi(x)$ would be written as $(x)\varphi(x)$, the symbol for negation is "$\sim$".

Lévy then mentions Mahlo's arithmetic construction of cardinals, noting, that he will use similar strategy to build higher levels of strong axioms of infinity.

TODO porovnani Mahlovy a Lévyho konstrukce

TODO asi doplnit jak to souvisi se soucasnou definici slabe Mahlovych kardinalu pres stacionarni mnoziny?

2.2 Preliminaries

Definition 2.1 *Relativization TODO (jech:161)*

2.3 Lévy's Original Proof From 1960

Definition 2.2 $N_0(\varphi)$

$$\exists u(Scm^{\mathsf{S}}(u)\&x_1,\ldots,x_n \in u \to \varphi \leftrightarrow \varphi^u) \tag{2.1}$$

where φ is a formula which does not contain free variables except $x_1,\ldots,x_n$.

TODO muzu vyhodit

Theorem 2.3 *In* S*, the schema* N_0 *implies the Axiom of Infinity.*

Proof. For any φ, N_0 gives us $\exists u Scm^{\mathsf{S}}(u)$, which means that there is a set u that is identical to V_α for some alpha, so $\exists\alpha Scm^{\mathsf{S}}(V_\alpha)$. We don't know the exact size of this α, but we know that $\alpha \geq \omega$, otherwise α would be finite, therefore not closed under the powerset operation, which would contradict the axiom of powersets. In order to prove that it is a model of S, we would need to verify all axioms of S . We have already shown that ω is closed under the powerset operation. Foundation, extensionality and comprehension are clear from the fact that we work in ZF[6], pairing is clear from the fact, that given two sets A, B, they have ranks a, b, without loss of generality we can assume that $a \leq b$, which means that $A \in V_a \in V_b$, therefore V_b is a set that satisfies the paring axiom: it contains both A and B.

TODO vyhodit axiomy, staci vyrobit ω

We now want to prove that V_α leads to existence of an inductive set, which is a set that satisfies $\exists A(\emptyset \in A \& \forall x \in A((x \cup \{x\}) \in A))$. If we can find a way to construct V_ω from any V_α satisfying $\alpha \geq \omega$, we are done. Since ω is the least limit ordinal, all we need is the following

$$\bigcap\{V_\kappa \mid \forall\lambda(\lambda < \kappa \to \exists\mu(\lambda < \mu < \kappa))\} \tag{2.2}$$

because V_κ is a transitive set for every κ, thus the intersection is non-empty unless empty set satisfies the property or the set of V_κs is itself empty. □

Theorem 2.4 *In* S*, the schema* N_0 *implies Replacement schema.*

Proof. TODO vysvetlit! (podle contemporary verze)

[6]We only need to verify axioms that provide means of constructing larger sets from smaller to make sure they don't exceed ω. Since ω is an initial segment of ZF, the axiom scheme of specification can't be broken, the same holds for foundation and extensionality.

Let $\varphi(v, w)$ be a formula wth no free variables except $v, w, x_1, \ldots, x_n$ where n is any natural number. Let χ be an instance of replacement schema for this φ:

$$\chi = \forall r, s, t(\varphi(r, s)\&\varphi(r, t) \to s = t) \to \forall x\exists y\forall w(w \in y \leftrightarrow \exists v(v \in x\&\varphi(v, w))) \tag{2.3}$$

We can deduce the following from N_0:

(i) $x_1, \ldots, x_n, v, w \in u \to (\varphi \leftrightarrow \varphi^u)$
(ii) $x_1, \ldots, x_n, v \in u \to (\exists w\varphi \leftrightarrow (\exists w\varphi)^u)$
(iii) $x_1, \ldots, x_n, x \in u \to (\chi \leftrightarrow \chi^u)$
(iv) $\forall x_1, \ldots, x_n\forall x(\chi \leftrightarrow (\forall x_1, \ldots, x_n\forall x\chi)^u)$

Note that (i), (ii), (iii) are obtained from instances of N_0 for φ, $\exists w\varphi$ and χ respectively. From relativization we also know that $(\exists w\varphi)^u$ is equivalent to $\exists w(w \in u\&\varphi^u)$. Therefore (ii) is equivalent to $x_1, \ldots, x_n, v \in u \to (\exists w(w \in u\&\varphi^u)$.

If φ is a function $(\forall r, s, t(\varphi(r, s)\&\varphi(r, t) \to r = t))$, then for every $x \in u$, which is also $x \subset u$ by $Scm^{\mathsf{S}}(u)$, it maps elements of x onto u. From the axiom scheme of comprehension[7], we can find a set of all images of elements of x. Let's call it y. That gives us $x_1, \ldots, x_n, x \in u \to \chi$. By (iii) we get $x_1, \ldots, x_n, x \in u \to \chi^u$, closure of this formula is $(\forall x_1, \ldots, x_n\forall x\chi)^u$, which together with (iv) yields $\forall x_1, \ldots, x_n\forall x\chi$. By the means of specification we end up with χ, which is all we need for now.

TODO btw co je x? nemela by tam tam byt nejaka volna promenna? $\square$

2.4 Contemporary restatement

TODO nejaky uvod. Levy rika ze existuje $Scm^S(u)$ reflektujici varphi, coz uz nepotrebujeme. atd. Ze prvoradova reflexe je theorem ZFC, vys uz je to nezavisle tvrzeni. (mozna dk. nezavislosti?)

The following lemma is usually done in parts, the first being with one formula and the other with n. Will will only state and prove the generalised version for n formulas.

Lemma 2.5 *Lemma Let $\varphi_1, \ldots, \varphi_n$ be any formulas with m parameters*[8].

[7] axiom of subsets in Levy's version

[8] For formulas with different number of parameters take for m the highest number of parameters among given formulas. Add spare parameters to the other formulas so that x remains the last parameter. That can be done in a following manner: Let φ_i' be the a formula with k parameters, $k < m$. Let us set $\varphi_i(u_1, \ldots, u_{m-1}, x) = \varphi_i'(u_1, \ldots, u_{k-1}, u_k, \ldots, u_{m-1}, x)$, notice that $u_k, \ldots, u_{m-1}$ are spare variables added just for formal simplicity.

(i) For each set M_0 there is such M that $M_0 \subset M$ and the following holds for every $i \leq n$:

$$\exists x \varphi_i(u_1, \ldots, u_{m-1}, x) \to (\exists x \in M) \varphi_i(u_1, \ldots, u_{m-1}, x) \tag{2.4}$$

(ii) Furthermore there is an ordinal α such that $M_0 \subset V_\alpha$ and the following holds for each $i \leq n$:

$$\exists x \varphi_i(u_1, \ldots, u_{m-1}, x) \to (\exists x \in V_\alpha) \varphi_i(u_1, \ldots, u_{m-1}, x) \tag{2.5}$$

Proof. We will simultaneously prove statements (i) and (ii), denoting M^T the transitive model required by part (ii). Unless explicitly stated otherwise, it is equivalent to M.

Let us first define operation $H(u_1, \ldots, u_{m-1})$ that gives us the set of x's with minimal rank, satisfying $\varphi_i(u_1, ldots, u_{m-1}, x)$ for given parameteres $u_1, \ldots, u_{m-1}$ for every $i \leq n$.

$$H_i(u_1, \ldots, u_n) = \{x \in C_i : (\forall z \in C)(rank(x) \leq rank(z))\} \text{ for } i \leq n \tag{2.6}$$

where

$$C_i = \{x : \varphi_i(u_1, \ldots, u_{m-1}, x)\} \text{ for } i \leq n \tag{2.7}$$

Next, let's construct M from given M_0 by induction.

$$\tag{2.8}$$

In other words, in each step we add the elements satisfying $\varphi(u_1, \ldots, u_{m-1}, x)$ for those parameters that were either available earlier or were added in the previous step.

For statement (ii), this is the only part that differs from (i). Let us take for each step transitive closure of M_{i+1} from (i). In other words, let γ be the smallest ordinal such that

$$(M_i^T \cup \bigcup_{j=0}^{n} \{\bigcup \{H_j(u_1, \ldots, u_{m-1}) : u_1, \ldots, u_{m-1} \in M_i\}\}) \subset V_\gamma \tag{2.9}$$

Then the incremetal step is like so:

$$M_{i+1}^T = V_\gamma \tag{2.10}$$

The final M is obtained by joining all incremental steps together.

$$M = \bigcup_{i=0}^{\infty} M_i, \ M^T = \bigcup_{i=0}^{\infty} M_i^T \tag{2.11}$$

Let's try to construct a set M' that satisfies the same conditions like M but is kept as small as possible. Assuming the Axiom of Choice, we can modify the process so that cardinality of M' is at most $|M_0|\cdot\aleph_0$. Note that the size of M' is determined by the size of M_0 an, most importantly, by the size of $H_i(u_1,\ldots,u_{m-1})$ for any $i \leq n$ in individual levels of the construction. Since the lemma only states existence of some x that satisfies $\varphi_i(u_1,\ldots,u_{m-1},x)$ for any $i \leq n$, we only need to add one x for every set of parameters but $H_i(u_1,\ldots,u_{m-1})$ can be arbitrarily large. Since Axiom of Choice ensures that there is a choice function, let F be a choice function on $\mathscr{P}(()M')$. Also let $h_i(u_1,\ldots,u_{m-1}) = F(H_i(u_1,\ldots,u_{m-1}))$ for $i \leq n$, which means that h is a function that outputs an x that satisfies $\varphi_i(u_1,\ldots,u_{m-1},x)$ for $i \leq n$ and has minimal rank among all such witnesses. The induction step needs to be redefined to

$$M'_{i+1} = M'_i \cup \bigcup_j = 0^n\{h_j(u_1,\ldots,u_{m-1}) : u_1,\ldots,u_{m-1} \in M'_i\} \tag{2.12}$$

In every step, the amount of elements added in M'_{i+1} is equivalent to the amount of sets of parameters the yielded elements not included in M'_i. So the cardinality of M'_{i+1} exceeds the cardinality of M'_i only for finite M'_i. It is easy to see that if M_0 is finite, M' is countable because it was built from countable union of finite sets. If M_0 is countable or larger, cardinaly of M' is equal to the cardinality of M_0.[9] Therefore $|M'| \leq |M_0| \cdot \aleph_0$

□

TODO proc $\leq$ a ne $=$?
TODO trochu vyjasnit parametry

Theorem 2.6 *First-order Reflection $\varphi(x_1,\ldots,x_n)$ is a first-order formula.*

(i) For every set M_0 there exists M such that $M_0 \subset M$ and the following holds:

$$\varphi^M(x_1,\ldots,x_n) \leftrightarrow \varphi(x_1,\ldots,x_n) \tag{2.13}$$

for every $x_1,\ldots,x_n$.

(ii) For every set M_0 there is a transitive set M, $M_0 \subset M$ such that the following holds:

$$\varphi^M(x_1,\ldots,x_n) \leftrightarrow \varphi(x_1,\ldots,x_n) \tag{2.14}$$

for every $x_1,\ldots,x_n$.

[9]It can not be smaller because $|M'_{i+1}| \geq |M'_i|$ for every i. It may not be significantly larger because the maximum of elements added is the number of n-tuples in M'_i, which is of the same cardinality is M'_i. ((proc? Ramsey?))

(iii) *For every set* M_0 *there is* α *such that* $M_0 \subset V_\alpha$ *and the following holds:*

$$\varphi^{V_\alpha}(x_1, \ldots, x_n) \leftrightarrow \varphi(x_1, \ldots, x_n) \tag{2.15}$$

for every $x_1, \ldots, x_n$.

(iv) *Assuming the Axiom of Choice, for every set* M_0 *there is* M *such that* $M_0 \subset M$ *and* $|M| \leq |M_0| \cdot \aleph_0$ *and the following holds:*

$$\varphi^M(x_1, \ldots, x_n) \leftrightarrow \varphi(x_1, \ldots, x_n) \tag{2.16}$$

for every $x_1, \ldots, x_n$.*)*

Proof. Let's prove (i) for one formula φ via induction by complexity first. We can safely assume that φ contains no quantifiers besides $\exists$ and no logical connectives other than $\neg$ and &. Assume that this M is obtained from lemma 2.5.

The fact, that atomic formulas are reflected in every M comes directly from definition of relativization and the fact that they contain no quantifiers.[10]

The same holds for formulas in the form of $\varphi = \neg\varphi'$. Let us recall the definition of relativization for those formulas in .

$$(\neg\varphi_1)^M \leftrightarrow \neg(\varphi_1^M) \tag{2.17}$$

Because we can assume from induction that $\varphi'^M \leftrightarrow \varphi'$, the following holds:

$$(\neg\varphi')^M \leftrightarrow \neg(\varphi'^M) \leftrightarrow \neg\varphi' \tag{2.18}$$

The same holds for $\varphi = \varphi_1 \ \& \ \varphi_2$. From the induction hypothesis we know that $\varphi_1^M \leftrightarrow \varphi_1$ and $\varphi_2^M \leftrightarrow \varphi_2$, which together with relativization for formulas of form $\varphi_1 \& \ \varphi_2$ gives up

$$(\varphi_1 \ \& \ \varphi_2)^M \leftrightarrow \varphi_1^M \ \& \ \varphi_2^M \leftrightarrow \varphi_1 \ \& \ \varphi_2 \tag{2.19}$$

Let's now examine the case when from the induction hypethesis, M reflects $\varphi'(u_1, \ldots, u_n, x)$ and we are interested in $\varphi = \exists x \varphi'(u_1, \ldots, u_n, x)$. The induction hypothesis tells us, that

$$\varphi'^M(u_1, \ldots, u_n, x) \leftrightarrow \varphi'(u_1, \ldots, u_n, x) \tag{2.20}$$

Together with above lemma 2.5 the following holds:

$$\exists x \varphi'(u_1, \ldots, u_n, x) \tag{2.21}$$

[10]Note that this does not hold generally for relativizations to M, E, but only for relativization to $M, \in$, which is our case.

$$\leftrightarrow (\exists x \in M)\varphi'(u_1, \dots, u_n, x) \tag{2.22}$$
$$\leftrightarrow (\exists x \in M)\varphi'^M(u_1, \dots, u_n, x) \tag{2.23}$$
$$\leftrightarrow (\exists x \varphi'(u_1, \dots, u_n, x))^M \tag{2.24}$$

Which is what we needed to prove.

So far we have proven part (i) of this theorem for one formula φ, we only need to verify that the same holds for any finite number of formulas. This has in fact been already done since lemma 2.5 gives us M for any (finite) amount of formulas. We can than use the induction above to verify that it reflects each of the formulas individually.

Now we want to verify other parts of our theorem. Since V_α is a transitive set, by proving (iii) we also satisfy (ii). To do so, we only need to look at part (ii) of lemma 2.5. All of the above proof also holds for $M = V_\alpha$. To finish part (iv)

□

Theorem 2.7 *(Refl) is equivalent to (Infinity)* & *(Replacement) under ZFC minus (Infinity)* & *(Replacement)*

Proof. Since 2.6 already gives one side of the implication, we are only interested in showing the converse:

(**Refl**) $\to$ (**Infinity**)

Let us first find a formula to be reflected that requires a set M at least as large as V_ω. Let us consider the following formula:

$$\varphi'(x) = \forall \lambda(\lambda < x \to \exists \mu(\lambda < \mu < x)) \tag{2.25}$$

Because φ says "there is a limit ordinal", if it holds for some x, the Infinity axiom is very easy to satisfy. But we know that there are limit ordinals in ZF, therefore $\varphi = \exists x \varphi'(x)$ is a valid statement. (Refl) then gives us a set M in which φ^M holds, that is, a set that contains a limit ordinal. So the set of off limit ordinals is non-empty and because ordinals are well-founded, it has a minimal element. Let's call it μ. $\mu = \bigcap\{V_\kappa \ : \ \forall \lambda(\lambda < \kappa \to \exists \mu(\lambda < \mu < \kappa))\}$ We can see that μ is the least limit ordinal and therefore it satisfies (Infinity).

(**Refl**) $\to$ (**Replacement**)

Given a formula $\varphi(x, y, u_1, \dots, u_n)$, we can suppose that it is reflected in any M [11] What we want to obtain is the following:

$$(\forall x, y, z(\varphi(x, y, u_1, \dots, u_n) \& \varphi(x, z, u_1, \dots, u_n) \to y = z)) \to (\forall X \exists Y (\forall y\, (y \in Y \leftrightarrow \exists x(x \in X \& \varphi(\tag{2.26}$$

[11] Which means that for $x, y, u_1, \dots, u_n \in M$, $\varphi^M(x, y, u_1, \dots, u_n) \leftrightarrow \varphi(x, y, u_1, \dots, u_n)$.

TODO co je dom?

We do also know that $x, y \in M$, in other words for every X, $Y = \{y \mid \varphi(x, y, u_1, \ldots, u_n\}$ we know that $X \subset M$ and $Y \subset M$, which, together with the comprehension schema[12] implies that Y, the image of X over φ, is a set. Which is exactly the Replacement Schema. □

TODO Plagiat – prepsat a vysvetlit TODO asi citace?

Definition 2.8 *Let $\varphi(R)$ be a Π^n_m-formula which contains only one free variable R which is second-order. Given $R \subseteq V_\kappa$, we say that $\varphi(R)$ reflects in V_κ if there is some $\alpha < \kappa$ such that:*

$$\textit{If } (V_\kappa, \in, R) \models \varphi(R), \textit{ then } (V_\alpha, \in, R \cap V_\alpha) \models \varphi(R \cap V_\alpha). \tag{2.27}$$

[12] Called the axiom of subsets in Levy's proof.

3 Large Cardinals

TODO pozor na opsane definice, prefurmulovvat!!!

3.1 Preliminaries

To avoid confusion[13], let's first define some basic terms.

Definition 3.1 *(weak limit cardinal) kappa is a* weak limit cardinal *if it is* $\aleph_\alpha$ *for some limit* α.

Definition 3.2 *(strong limit cardinal) kappa is a* strong limit cardinal *if for every* $\lambda < \kappa$, $2^\lambda < \kappa$

3.2 Inaccelssibility

Definition 3.3 *(weak inaccessibility)* κ *is* weakly inaccessible $\leftrightarrow$ *it is* regular *and* weakly limit.

Definition 3.4 *(inaccessibility)* κ *is* inaccessible $\leftrightarrow$ *it is* regular *and* strongly limit.

Theorem 3.5 *[Lévy] The following are equivalent:*

(i) κ *is inaccessible.*
(ii) For every $R \subseteq V_\kappa$ *and every first-order formula* $\varphi(R)$, $\varphi(R)$ *reflects in* V_κ.
(iii) For every $R \subseteq V_\kappa$*, the set* $C = \{\alpha < \kappa \mid \langle V_\alpha, \in, R \cap V_\alpha\rangle \prec \langle V_\kappa, \in, R\rangle\}$ *is closed unbounded.*

Proof. Let's start with (i) $\rightarrow$ (iii) in a way similar to [1].
The set $\{\alpha < \kappa \mid \langle V_\alpha, \in, R \cap V_\alpha\rangle \prec \langle V_\kappa, \in, R\rangle\}$ is clearly closed, it remains to show that it is also unbounded. To do so, let $\alpha < \kappa$ be arbitrary. Define $\alpha_n < \kappa$ for $n \in \omega$ by recursion as follows:
Set $\alpha_0 = \alpha$. Given $\alpha_n < \kappa$ define α_{n+1} to be the least $\beta \geq \alpha_n$ such as whenever $y_1, \ldots, y_k \in V_{\alpha_n}$ and $\langle V_\kappa, \in, R\rangle \models \exists v_0 \varphi[v_0, y_1, \ldots, y_k]$ for some formula φ, there is an $x \in V_\beta$ such that $\langle V_\kappa, \in, R\rangle \models \varphi[x, y_1, \ldots, y_k]$.
Since κ is inaccessible, $|V_{\alpha_n}| < \kappa$ and so $\alpha_{n+1} < \kappa$.
Finally, set $\alpha = sup(\alpha_n | n \in \omega)$. Then $\langle\, V_\alpha, \in, R \cap V_\alpha\rangle \prec \langle V_\kappa, \in, R\rangle$ by the usual (Tarski) criterion for elementary substructure.

[13] While in most sources refer to *weak limit cardinal* as a *limit cardinal* and to *strong limit cardinal*, in some cases the distinction is *weak limit cardinal* and *limit cardinal* respectively. That's why I have decided to explicitly define those otherwise elementary terms.

The next part, proving $(iii) \to (ii)$, should be elementary since C is closed unbounded, which means that it contains at least countably many elements but we need only one such α to satisfy (2.8).
Finally, we shall prove that $(ii) \to (i)$. Since it obviously holds that $\kappa > \omega$, we have yet to prove that κ is regular and a strong limit. Let's argue by contradiction that it is regular. If it wasn't, there would be a $\beta < \kappa$ and a function $F : \beta \implies \kappa$ with range unbounded in κ. Set $R = \{\beta\} \cup F$. By hypothesis there is an $\alpha < \kappa$ such that $\langle V_\alpha, \in, R \cap V_\alpha \rangle \prec \langle V_\kappa, \in, R \rangle$. Since β is the single ordinal in R, $\beta \in V_\alpha$ by elementarity. This yields the desired contradiction since the domain if $F \cap V_\alpha$ cannot be all of β.

Next, let's see whether κ is indeed a strong limit, again by contradiction. If not, there would be a $\lambda < \kappa$ such that $2^\lambda \geq \kappa$. Let $G : \mathscr{P}(\lambda) \implies \kappa$ be surjective and set $R = \{\lambda + 1\} \cup G$. By hypothesis, there is an $\alpha < \kappa$ such that $\langle V_\alpha, \in, R \cap V_\alpha \rangle \prec \langle V_\kappa, \in, R \rangle$. $\lambda + 1 \in V_\alpha$ and so $\mathscr{P}(\lambda) \in V_\alpha$, but this is again a contradiction. $\square$

3.3 Mahlo cardinals

TODO reflektuji nedosazitelnost? TODO zminit Mahlovu konstrukci?

Definition 3.6 *Weakly Mahlo Cardinals κ is* weakly Mahlo $\leftrightarrow$ *it is a limit ordinal and the set of all regular ordinals less then κ is stationary in κ*

Definition 3.7 *Mahlo cardinals The folllowing definitions are equivalent:*

(i) *κ is Mahlo*

(ii) *κ is weakly Mahlo and strong limit*

(iii) *κ is inaccessible and the regular cardinals below κ form a stationary subset of κ.*

(iv) *κ is regular and the stationary sets below κ form a stationary subset of κ.*

Theorem 3.8 *κ is Mahlo $\leftrightarrow$ for any $R \subset V_\kappa$ there is an inaccessible cardinal $\alpha < \kappa$ such that $\langle V_\alpha, \in, R \cap V_\alpha \rangle \prec \langle V_\kappa, \in, R \rangle$.*

Proof. Start with the proof of (3.5) and add the following:
κ is Mahlo by the following contradiction. If not, there would be a C closed unbounded in κ containing no inaccessible cardinals. By the hypothesis there is in inaccessible $\alpha < \kappa$ such that $\langle V_\alpha, \in, C \cap V_\alpha \rangle \prec \langle V_\kappa, \in, C \rangle$. By elementarity $C \cap \alpha$ is unbounded in α. But then, $\alpha \in C$, which is the contradiction we need. $\square$ Note that Mahlo cardinals were first described in 1911, almost 50 years before Lévy's reflection, which was heavily inspired by those.

3.4 Weakly Compact Cardinals

TODO souvislost s reflexi!

In this section, we will introduce various well-known large cardinals and establish them via reflection.

Definition 3.9 *A cardinal κ is* weakly compact *if it is uncountable and satisfies the partition property $\kappa \to (\kappa)^2$*

Lemma 3.10 *Every weakly compact cardinal is inaccessible*

Proof. Let κ b a weakly compact cardinal. To show that κ is regular, let us assume that κ i the disjoint union $\bigcup\{A_\gamma : \gamma < \lambda\}$ such that $\lambda < \kappa$ and $|A_\gamma| < \kappa$ for each $\gamma < \lambda$. We define a partition $F : [\kappa]^2 \to \{0, 1\}$ as follows: $F(\{\alpha, \beta\}) = 0$ just in cas α and β are the same size A_γ. Obviously, this partition does not have a homogenous set $H \subset \kappa$ of size κ. That κ is a strong limit cardinal follows from Lemma 9.4: (?? doplnit z jecha): If $\kappa \geq 2^\lambda$ for some $\lambda < \kappa$, the because $2^\lambda \leq (\lambda^+)^2$, we have $\kappa \leq (\lambda^+)^2$ and hence $\kappa \leq (\kappa)^2$. □

Theorem 3.11 *Let κ be a weakly compact cardinal. Then for every stationary set $S \subset \kappa$ there is an uncountable regular cardinal $\lambda < \kappa$ such that the set $S \cap \lambda$ is stationary in λ.*

Proof. TODO □

3.5 Indescribable Cardinals

Definition 3.12 (Indescribability) *For Q either Π^m_n or Σ^m_n*
A cardinal κ is $Q-$indescribable if whenever $U \subseteq V_\kappa$ and φ is a Q sentence such that $\langle V_\kappa, \in, U\rangle \models \varphi$, then for some $\alpha < \kappa$, $\langle V_\alpha, \in, U \cap V_\alpha\rangle \models \varphi$.

TODO uvod / intuice

TODO souvislost s refleu

3.6 Bernays–Gödel Set Theory

TODO Plagiat – prepsat a vysvetlit

Gödel–Bernays set theory, also known as Von Neumann–Bernays–Gödel set theory is an axiomatic set theory that explicitly talks about proper classes as well as sets, which allows it to be finitely axiomatizable, albeit our version stated below contains one schema. It is a conservative extension of Zermalo–Fraenkel set theory. Using forcing, one can prove equiconsistency of BGC and ZFC.

Bernays–Gödel set theory contains two types of objects: proper classes and sets. The notion of set, usually denoted by a lower case letter, is identical to set in ZF, whereas proper classes are usually denoted by upper case letters. The difference between the two is in a fact, that proper classes are not members of other classes, sets, on the other hand, have to be members of classes.

Definition 3.13 *(Gödel–Bernay set theory)*

(i) extensionality for sets

$$\forall a \forall b[\forall x(x \in a \ \leftrightarrow x \in b) \to a = b] \tag{3.28}$$

(ii) pairing for sets

$$\forall x \forall y \exists z \forall w[w \in z \leftrightarrow (w = x \vee w = y)] \tag{3.29}$$

(iii) union for sets

$$\forall a \exists b \forall c[c \in b \leftrightarrow \exists d(c \in d \wedge d \in a)] \tag{3.30}$$

(iv) powers for sets

$$\forall a \exists p \forall b[b \in p \leftrightarrow (c \in b \to c \in a)] \tag{3.31}$$

(v) infinity for sets

$$\textit{There is an inductive set.} \tag{3.32}$$

(vi) Extensionality for classes

$$\forall x(x \in A \leftrightarrow x \in B) \to A = B \tag{3.33}$$

(vii) Foundation for classes

$$\textit{Each nonempty class is disjoint from each of its elements.} \tag{3.34}$$

(viii) Limitation of size for sets

$$\textit{For any class } C \textit{ a set } x \textit{ such that } x{=}C \textit{ exists iff} \tag{3.35}$$

$$\textit{there is no bijection between } C \textit{ and the class } V \textit{ of all sets} \tag{3.36}$$

(ix) Comprehension schema for classes

$$\textit{For any formula } \varphi \textit{ with no quantifiers over classes, there is a class } A \textit{ such that } \forall x(x \in A \leftarrow \tag{3.37}$$

The first five axioms are identical to axioms in ZF.
Comprehension schema tells us, that proper classes are basically first-order predicates. ... TODO Plagiat – prepsat a vysvetlit

Definition 3.14 *We say that $\varphi(R)$ with a class parameter R reflects if there is α such that*

$$\varphi(R) \to (V_\alpha, V_{\alpha+1}) \models \varphi(R \cap V_\alpha). \tag{3.38}$$

Theorem 3.15 *There is a second-order sentence φ which is provable in* GB *such that if φ reflects at α, i.e. if*

$$\varphi \to (V_\alpha, V_{\alpha+1}) \models \varphi, \tag{3.39}$$

then α is an inaccessible cardinal.

Proof. Take φ to say "there is no function from $\gamma \in \mathrm{ORD}$ cofinal in ORD and for every $\gamma \in \mathrm{ORD}$, $2^\gamma \in \mathrm{ORD}$". Clearly, if φ reflects at some α, then α is inaccessible (here we use that the second-order variable range over $\mathscr{P}(V_\alpha) = V_{\alpha+1}$). □

As a corollary we obtain:

Corollary 3.16 *Second-order reflection in* GB *implies the existence of an inaccessible cardinal.*

3.7 Morse–Kelley Set Theory

Axioms not

(i) *Extensionality*

$$\forall X \forall Y (\forall z (z \in X \leftrightarrow z \in Y) \to X = Y). \tag{3.40}$$

(ii) *Pairing*

$$asdfg \tag{3.41}$$

(iii) *Foundation For Classes*

$$asdf \tag{3.42}$$

(iv) *Class Comprehension*

$$\forall W_1, \ldots, W_n \exists Y \forall x (x \in Y \leftrightarrow (\phi(x, W_1, \ldots, W_n) \& set(x))). \tag{3.43}$$

Where $set(x)$ is monadic predicate stating that class x is a set.

(v) *Limitation Of Size For Classes*

$$asdf \tag{3.44}$$

(vi) *Pairing*

$$asdf \tag{3.45}$$

(vii) *Pairing*

$$asdf \tag{3.46}$$

TODO

3.8 Reflection and the constructible universe

TODO reflektovat muzeme jenom kardinaly konzistentni s V=L, proc?

TODO Plagiat – prepsat a vysvetlit

L was introduced by Kurt Gödel in 1938 in his paper *The Consistency of the Axiom of Choice and of the Generalised Continuum Hypothesis* and denotes a class of sets built recursively in terms of simpler sets, somewhat similar to Von Neumann universe V. Assertion of their equality, $V = L$, is called the *axiom of constructibility.* The axiom implies GCH and therefore also AC and contradicts the existence of some of the large cardinals, our goal is to decide whether those introduced earlier are among them.

On order to formally establish this class, we need to formalize the notion of definability first:

Definition 3.17 (Definable sets)

$$Def(X) := \{\{y | x \in X \wedge \langle X, \in \rangle \models \varphi(y, z_1, \ldots, z_n)\} | \ \varphi \textit{ is a first-order formula, } z_1, \ldots, z_n \in X\} \tag{3.47}$$

Now we can recursively build L.

Definition 3.18 (The Constructible universe) *(i)*

$$L_0 := \emptyset \tag{3.48}$$

(ii)

$$L_{\alpha+1} := Def(L_\alpha) \tag{3.49}$$

(iii)

$$L_\lambda = \bigcup_{\alpha<\lambda} L_\alpha \textit{ If } \lambda \textit{ is a limit ordinal} \tag{3.50}$$

(iv)

$$L = \bigcup_{\alpha \in Ord} L_\alpha \tag{3.51}$$

TODO Plagiat – prepsat a vysvetlit

Fact 3.19 *The reflection – constructed as explained in the previous paragraph (!!! preformulovat !!!) – with second-order parameters for higher-order formulas (even of transfinite type) does not yield transcendence over L.*

TODO zduvodneni

TODO kratka diskuse jestli refl implikuje transcendenci na L - polemika, nazor - V=L a slaba kompaktnost a dalsi

4 Higher-order reflection

TODO rict ze to je zobecneni a nejaky dalsi uvodni veci

4.1 Sharp

TODO je tohle higher-order vec?

4.2 Welek: Global Reflection Principles

TODO

5 Conclusion

TODO na konec

References

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