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REFLECTION PRINCIPLES AND LARGE CARDINALS

Bakalářská práce

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Prohlašuji, že jsem bakalářskou práci vypracoval samostatně a že jsem uvedl všechny použité prameny a literaturu.

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Abstract

Práce zkoumá vztah tzv. principů reflexe a velkých kardinálů. Lévy ukázal, že v ZFC platí tzv. věta o reflexi a dokonce, že věta o reflexi je ekvivalentní schématu nahrazení a axiomu nekonečna nad teorií ZFC bez axiomu nekonečna a schématu nahrazení. Tedy lze na větu o reflexi pohlížet jako na svého druhu axiom nekonečna. Práce zkoumá do jaké míry a jakým způsobem lze větu o reflexi zobecnit a jaký to má vliv na existenci tzv. velkých kardinálů. Práce definuje nedosažitelné, Mahlovy a nepopsatelné kardinály a ukáže, jak je lze zavést pomocí reflexe. Přirozenou limitou kardinálů získaných reflexí jsou kardinály nekonzistentní s L. Práce nabídne intuitivní zdůvodněn, proč tomu tak je.

Abstract

This thesis aims to examine relations between the so called Reflection Principles and Large cardinals. Lévy has shown that the Reflection Theorem is a sound theorem of ZF and it is equivalent to the Replacement Scheme and the Axiom of Infinity. From this point of view, Reflection theorem can be seen a specific version of an Axiom of Infinity. This paper aims to examine the Reflection Principle and its generalisations with respect to the existence of Large Cardinals. This thesis will establish the Inaccessible, Mahlo and Indescribable cardinals and show how can those be defined via reflection. A natural limit of Large Cardinals obtained via reflection are cardinals inconsistent with L. This thesis will offer an intuitive explanation of why this holds.

Contents

1 Introduction

1.1 Motivation and Origin

> "The Universe of sets cannot be uniquely characterised (i. e. distinguished from all its initial elements) by any internal structural property of the membership relation in it, which is expressible in any logic of finite of transfinite type, including infinitary logics of any cardinal order."

— Kurt Gödel [Wang, 1997]

1.2 Notation and Terminology

1.2.1 The Language of Set Theory

This text assumes the knowledge of basic terminology and some results from first-order predicate logic.[1] All proofs are based on [Jech, 2006] unless explicitly stated otherwise. Notable amount of material is also drawn from [Kanamori, 2003] and [Drake, 1974].

We will now shortly review the basic notions that allow us to define the *Zermelo–Fraenkel* set theory.

When we talk about *class*, we have the notion of definable class in mind. If $\varphi(x, p_1, \ldots, p_n)$ is a formula in the language of set theory, we call

$$A = \{x : \varphi(x)\} \tag{1.1}$$

a class of all sets satisfying $\varphi(x)$ in a sense that

$$x \in A \leftrightarrow \varphi(x) \tag{1.2}$$

Given classes A, B, one can easily define the elementary set operations such as $A \cap B$, $A \cup B$, $A \setminus B$, $\bigcup A$, see the first part of [Jech, 2006] for details. Axioms are the tools by which we can decide whether a particular class is "small enough" to be considered a set[2]. A class that fails to be considered a set is called a *proper class*.

We will often write "M is a limit ordinal", it should always be clear that this can be rewritten as a formula that was introduced earlier.

[1] todo odkaz na pripadny zdroj? svejdar? neco en?

[2] "Small enough" means that it doesn't introduce a paradox similar to Russell's.

1.2.2 The Axioms

Definition 1.1 *(The Existence of a Set)*

$$\exists x(x = x) \tag{1.3}$$

Definition 1.2 *(Axiom of Extensionality)*

$$\forall x, y(x = y \leftrightarrow \forall z(z \in x \leftrightarrow z \in y)) \tag{1.4}$$

Definition 1.3 *(Axiom Schema of Specification)*
The following yields an axiom for every first-order formula $\varphi(x, p_1, \ldots, p_n)$ with no free variables other than $x, p_1, \ldots, p_n$.

$$\forall x, p_1, \ldots, p_n \exists y \forall z(z \in y \leftrightarrow z \in x \;\&\; \varphi(z, p_1, \ldots, p_n)) \tag{1.5}$$

We will now provide two definitions that are not axioms, but will be helpful in establishing some axioms in a more comprehensible way.

Definition 1.4 *($x \subseteq y$, $x \subset y$)*

$$x \subseteq y \leftrightarrow (\forall z \in x) z \in y \tag{1.6}$$

$$x \subset y \leftrightarrow x \subseteq y \;\&\; x \neq y \tag{1.7}$$

We read $x \subseteq y$ as x is a subset of y *and $x \subset y$ as* x is a proper subset of y.

Definition 1.5 *(Empty Set) For an arbitrary set x, the empty set, represented by the symbol "$\emptyset$", is defined by the following formula:*

$$(\forall y \in x)(y \in \emptyset \leftrightarrow \neg(y = y)) \tag{1.8}$$

$\emptyset$ is a set due to Specification. *While the empty set could also be defined by the formula $\forall y(y \in\leftrightarrow \neg(y = y))$, the version we use is Δ_0, which we will find useful later. The two definitions yield the same set for every x given because of* Extensionality.

Definition 1.6 *(Axiom of Pairing)*

$$\forall x, y \exists z \forall q(q \in z \leftrightarrow q = x \vee q = y) \tag{1.9}$$

Definition 1.7 *(Axiom of Union)*

$$\forall x \exists y \forall z(z \in y \leftrightarrow \exists q(z \in q \;\&\; q \in x)) \tag{1.10}$$

Now we can introduce more axioms.

Definition 1.8 *(Axiom of Foundation)*

$$\forall x(x \neq \emptyset \to (\exists y \in x)(x \cap y = \emptyset)) \tag{1.11}$$

Definition 1.9 *(Axiom of Powerset)*

$$\forall x \exists y \forall z(z \in y \leftrightarrow z \subseteq x) \tag{1.12}$$

Definition 1.10 *(Axiom of Infinity)*

$$\exists x(\emptyset \in x \;\&\; (\forall y \in x)(y \cup \{y\} \in x)) \tag{1.13}$$

The least set satisfying this is denoted "ω".

Let us introduce a few more definitions that will make the two remaining axioms more comprehensible.

Definition 1.11 *(Powerset Function)*
Given a set x, the powerset of x, *denoted $\mathscr{P}(x)$ and satisfying 1.9, is defined as follows:*

$$\mathscr{P}(x) \overset{\text{def}}{=} \{y : y \subseteq x\} \tag{1.14}$$

Definition 1.12 *(Function)*
Given arbitrary first-order formula $\varphi(x, y, p_1, \ldots, p_n)$, we say that φ is a function iff

$$\forall x, y, z, p_1, \ldots, p_n(\varphi(x, y, p_1, \ldots, p_n) \;\&\; \varphi(x, z, p_1, \ldots, p_n) \to y = z) \tag{1.15}$$

When a $\varphi(x, y)$ is a function, we also write the following:

$$f(x) = y \leftrightarrow \varphi(x, y) \tag{1.16}$$

Alternatively, $f = \{\langle x, y\rangle : \varphi(x, y)\}$ is a class.

Definition 1.13 *(Domain of a Function)*
Let f be a function. We call the domain of f *the set of all sets for which f yields a value. We use "$Dom(f)$" to refer to this set.*

$$x \in Dom(f) \leftrightarrow \exists y(f(x) = y) \tag{1.17}$$

We say "f is a function on A", A being a class, if $A = dom(f)$.

Definition 1.14 *(Range of a Function)*
Let f be a function. We call the range of f *the set of all sets that are images of other sets via f. We use "$Rng(f)$" to refer to this set.*

$$x \in Rng(f) \leftrightarrow \exists y(f(y) = x) \tag{1.18}$$

We say that f is *a function into* A, A being a class, if $rng(f) \subseteq A$. We say that f is *a function onto* A if $rng(f) = A$We say a function f is a *one to one function*, iff

$$(\forall x_1, x_2 \in dom(f))(f(x_1) = f(x_2) \to x_1 = x_2) \tag{1.19}$$

We say that f is a bijection iff it is a one to one function that is onto.

Note that *Dom(f)* and *Rng(f)* are not definitions in a strict sense, they are in fact definition schemas that yield definitions for every function f given. Also note that they can be easily modified for φ instead of f, with the only difference being the fact that it is then defined only for those φs that are functions, which must be taken into account. This is worth noting as we will use the notions of *function* and *formula* interchangably.

Definition 1.15 *(Function Defined For All Ordinals)*
We say a function f is defined for all ordinals*, this is sometimes written $f : Ord \to A$ for any class A, if $Dom(f) = Ord$. Alternatively,*

$$(\forall \alpha \in Ord)(\exists y \in A)(f(\alpha) = y)) \tag{1.20}$$

And now for the axioms.

Definition 1.16 *(Axiom Schema of Replacement)*
The following is an axiom for every first-order formula $\varphi(x, p_1, \dots, p_n)$ with no free variables other than $x, p_1, \dots, p_n$.

$$\text{“}\varphi \textit{ is a function}'' \to \forall x \exists y \forall z (z \in y \leftrightarrow (\exists q \in x)(\varphi(x, y, p_1, \dots, p_n))) \tag{1.21}$$

Definition 1.17 *(Choice)*

$$\begin{gathered} \forall x \exists f((f \textit{ is a choice function with } dom(f) = x \setminus \{\emptyset\}) \\ \& \ \forall y((y \in y \ \& \ y \neq \emptyset) \to f(y) \in y)) \end{gathered} \tag{1.22}$$

We will refer to the axioms by their name, written in italic type, e.g. *Foundation* refers to the Axiom of Foundation. Now we need to define the set theories to be used in the article.

Definition 1.18 (S)
We call S *an axiomatic theory in the language $\mathscr{L} = \{=, \in\}$ with exactly the following axioms:*
(i) Existence of a set *(see 1.1)*
(ii) Extensionality *(see 1.2)*
(iii) Specification *(see 1.3)*

(iv) Foundation *(see 1.8)*
(v) Pairing *(see 1.6)*
(vi) Union *(see 1.7)*
(vii) Powerset *(see 1.9)*

Definition 1.19 (ZF)
We call ZF *an axiomatic theory in the language* $\mathscr{L} = \{=, \in\}$ *that contains all the axioms of* S *in addition to the following:*
(i) Replacement *schema (see 1.16)*
(ii) Infinity *(see 1.10)*
Existence of a set *is usually left out because it is a consequence of* infinity*.*

Definition 1.20 (ZFC)
ZFC *is an axiomatic theory in the language* $\mathscr{L} = \{=, \in\}$ *that contains all the axioms of* ZF *plus* Choice *(1.17).*

1.2.3 The Transitive Universe

Definition 1.21 *(Transitive Class)*
We say a class A *is* transitive *iff*

$$(\forall x \in A)(x \subseteq A) \tag{1.23}$$

Definition 1.22 *(Well Ordered Class) A class* A *is said to be* well ordered by $\in$ *iff the following hold:*
(i) $(\forall x \in A)(x \notin x)$ *(Antireflexivity)*
(ii) $(\forall x, y, z \in A)(x \in y \; \& \; y \in z \to x \in z)$ *(Transitivity)*
(iii) $(\forall x, y \in A)(x = y \vee x \in y \vee y \in x)$ *(Linearity)*
(iv) $(\forall x \subseteq A)(x \neq \emptyset \to (\exists y \in x)(\forall z \in x)(z = y \vee z \in y)))$ *(Existence of the least element)*

Definition 1.23 *(Ordinal Number)*
A set x *is said to be an* ordinal number *if it is* transitive *and* well-ordered by $\in$*.*

For the sake of brevity, we usually just say "x is an *ordinal*". Note that "x is an ordinal" is a well-defined formula in the language of set theory, since 1.21 is a first-order formula and 1.22 is in fact a conjunction of four first-order formulas. Ordinals will be usually denoted by lower case greek letters, starting from the beginning of the alphabet: $\alpha, \beta, \gamma, \ldots$. Given two different ordinals α, β, we will write $\alpha < \beta$ for $\alpha \in \beta$, see Lemma 2.11 in [Jech, 2006] for technical details.

Definition 1.24 *(Non-Zero Ordinal) We say an ordinal α is* non-zero *iff $\alpha \neq \emptyset$.*

Definition 1.25 *(Successor Ordinal)*
Consider the following function defined for all ordinals. Let β be an arbitrary ordinal. We call S the successor function*.*

$$S(\beta) = \beta \cup \{\beta\} \tag{1.24}$$

An ordinal α is called a successor ordinal *iff there is an ordinal β, such that $\alpha = S(\beta)$. We also write $\alpha = \beta + 1$.*

Definition 1.26 *(Limit Ordinal)*
A non-zero ordinal α is called a limit ordinal *iff it is not a successor ordinal.*

Definition 1.27 *(Ord)*
The class of all ordinal numbers*, which we will denote "Ord"*[3] *is the proper class defined as follows.*

$$x \in Ord \leftrightarrow x \textit{ is an ordinal} \tag{1.25}$$

Definition 1.28 *(Von Neumann's Hierarchy)*
The Von Neumann's Hierarchy *is a collection of sets indexed by elements of Ord, defined recursively in the following way:*

(i)
$$V_0 = \emptyset \tag{1.26}$$

(ii)
$$V_{\alpha+1} = \mathscr{P}(V_\alpha) \textit{ for any ordinal } \alpha \tag{1.27}$$

(iii)
$$V_\lambda = \bigcup_{\beta<\lambda} V_\beta \textit{ for a limit ordinal } \lambda \tag{1.28}$$

We will also refer to the Von Neumann's Hierarchy *as* Von Neumann's Universe *or the* Cumulative Hierarchy*.*

Definition 1.29 *(Rank)*
Given a set x, we say that the rank of x (written as $rank(x)$) is the least ordinal α such that $x \in V_{\alpha+1}$

Due to *Regularity*, every set has a rank.[4]

[3] Other authors use "*On*", we will stick to the notation used in [Jech, 2006]

[4] See chapter 6 of [Jech, 2006] for details.

1.2.4 Cardinal Numbers

Definition 1.30 *(Cardinality)*
Given a set x, let the cardinality of x, written $|x|$, be defined as the smallest ordinal number such that there is a one to one mapping from x to α.

Definition 1.31 *(Aleph function)*
Let ω be the set defined by **??**. *We will recursively define the function $\aleph$ for all ordinals.*

(i) $\aleph_0 = \omega$
(ii) $\aleph_{\alpha+1}$ *is the least cardinal larger than* $\aleph_\alpha$[5]
(iii) $\aleph_\lambda = \bigcup_{\beta<\lambda} \aleph_\beta$ *for a limit ordinal* λ

If $\kappa = \aleph_\alpha$ and α is a successor ordinal, we call κ a successor cardinal. *If α is a limit ordinal, we call κ a* limit cardinal.

Definition 1.32 *(Cardinal number)*

(i) A set x is called a finite cardinal *iff $x \in \omega$.*
(ii) A set is called an infinite cardinal *iff there is an ordinal α such that $\aleph_\alpha = x$*
(iii) A set is called a cardinal *iff it is either a* finite cardinal *or an* infinite cardinal.

We say κ is an uncountable cardinal iff it is an infinite ordinal and $\aleph_0 < \kappa$. Infinite cardinals will be notated by lowercase greek letters from the middle of the alphabet, e.g. $\kappa, \mu, \nu, \ldots$.[6]

For formal details as well as why every set can be well-ordered assuming *Choice*, and therefore has a cardinality, see [Jech, 2006].

Definition 1.33 *(Sequence)*
We say that a function $\varphi(x, y)$ is a sequence *iff there is an ordinal α such that $dom(\varphi) = \alpha$. In other words, a function is called a sequence if it is defined exactly for every ordinal from 0 to some α. We then say it is an α-sequence. We usually write $\langle \beta_i : i \in \alpha \rangle$ or $\langle \beta_0, \beta_1, \ldots \rangle$ when referring to a sequence, ξ_i denote the elements of $rng(\varphi)$ for every $i \in dom(\varphi)$.*

Definition 1.34 *(Cofinal Subset)*
Given a class A, we say that $B \subseteq A$ is cofinal in A *iff*

$$(\forall x \in A)(\exists y \in B)(x \in y) \tag{1.29}$$

[5] "The least cardinal larger than $\aleph_\alpha$" is sometimes notated as $\aleph_\alpha^+$

[6] Except λ which is preferably used for limit ordinals.

Definition 1.35 *(Cofinality of a Limit Ordinal)*
Let λ be a limit ordinal. We say that the cofinality *of λ is κ iff κ is the least cardinal, such that there is a cofinal κ-sequence $\langle \beta_\xi : \xi < \kappa \rangle$, such that*

$$sup(\{\beta_\xi : \xi < \kappa\}) = \lambda \tag{1.30}$$

We write $cf(\lambda) = \kappa$.

Definition 1.36 *(Regular Cardinal)*
We say a cardinal κ is regular iff $cf(\kappa) = \kappa$

Definition 1.37 *(Strong Limit Cardinal)*
We say that an ordinal κ is a strong limit cardinal *if it is a* limit cardinal *and*

$$(\forall \alpha \in \kappa)(\mathscr{P}(\alpha) \in \kappa) \tag{1.31}$$

Definition 1.38 *(Generalised Continuum Hypothesis)*

$$\aleph_{\alpha+1} = \mathscr{P}(\aleph_\alpha) \tag{1.32}$$

If GCH holds (for example in Gödel's L, see chapter 3), the notions of limit cardinal and strong limit cardinal are equivalent.

1.2.5 Relativisation and Absoluteness

Definition 1.39 *(Relativization)*
Let M be a class, $R \subseteq M \times M$ and let $\varphi(p_1, \dots, p_n)$ be a first-order formula with no free variables besides $p_1, \dots, _n$. The relativization of φ to M and R *is the formula, written as $\varphi^{M,R}(p_1, \dots, p_n)$, defined in the following inductive manner:*

(i) $(x \in y)^{M,R} \leftrightarrow R(x, y)$
(ii) $(x = y)^{M,R} \leftrightarrow x = y$
(iii) $(\neg\varphi)^{M,R} \leftrightarrow \neg\varphi^{M,R}$
(iv) $(\varphi \,\&\, \psi)^{M,R} \leftrightarrow \varphi^{M,R} \,\&\, \psi^{M,R}$
(v) $(\varphi \vee \psi)^{M,R} \leftrightarrow \varphi^{M,R} \vee \psi^{M,R}$
(vi) $(\varphi \to \psi)^{M,R} \leftrightarrow \varphi^{M,R} \to \psi^{M,R}$
(vii) $(\exists x \varphi(x))^{M,R} \leftrightarrow (\exists x \in M)\varphi^{M,R}(x)$
(viii) $(\forall x \varphi(x))^{M,R} \leftrightarrow (\forall x \in M)\varphi^{M,R}(x)$

When $R = \in \cap (M \times M)$, we usually write φ^M instead of $\varphi^{M,R}$. When we talk about $\varphi^M(p_1, \dots, p_n)$, it is understood that $p_1, \dots, p_n \in M$. We will also use $M \models \varphi(p_1, \dots, p_n)$ and $\varphi^M(p_1, \dots, p_n)$ interchangably.

Definition 1.40 *(Absoluteness) Given a transitive class M, we say a formula φ is* absolute in M *if for all $p_1, \ldots, p_n \in M$*

$$\varphi^M(p_1, \ldots, p_n) \leftrightarrow \varphi(p_1, \ldots, p_n) \tag{1.33}$$

Definition 1.41 *(Hierarchy of First-Order Formulas)*

(I) A first-order formula φ is Δ_0 iff it is logically equivalent to a first-order formula φ' satisfying any of the following:
  - *(i) φ' contains no quantifiers*
  - *(ii) y is a set, ψ is a Δ_0 formula, and φ' is either $(\exists x \in y)\psi(y)$ or $(\forall x \in y)\psi(y)$.*
  - *(iii) ψ_1, ψ_2 are Δ_0 formulas and φ' is any of the following: $\psi_1 \vee \psi_2$, $\psi_1 \,\&\, \psi_2$, $\psi_1 \to \psi_2$, $\neg\psi_2$,*

(II) If a formula is Δ_0 it is also Σ_0 and Π_0

(III) A formula φ is $\Pi_n + 1$ if it is logically equivalent to a formula φ' such that $\varphi' = \forall x \psi$ where ψ is a Σ_n-formula for any $n < \omega$.

(IV) A formula φ is $\Sigma_n + 1$ if it is logically equivalent to a formula φ' such that $\varphi' = \forall x \psi$ where ψ is a Π_n-formula for any $n < \omega$.

Note that we can use the pairing function so that for $\forall p_1, \ldots, p_n \psi(p_1, \ldots, p_n)$, there is a logically equivalent formula of the form $\forall x \psi'(x)$.

Lemma 1.42 *(Δ_0 absoluteness) Let φ be a Δ_0 formula, then φ is absolute in any transitive class M.*

Proof. This will be proven by induction over the complexity of a given Δ_0 formula φ. Let M be an arbitrary transitive class.

Atomic formulas are always absolute by the definition of relativisation, see 1.39. Suppose that Δ_0 formulas ψ_1 and ψ_2 are absolute in M. Then from relativization, $(\psi_1 \,\&\, \psi_2)^M \leftrightarrow \psi_1^M \,\&\, \psi_2^M$, which is, from the induction hypothesis, equivalent to $\psi_1 \,\&\, \psi_2$. The same holds for $\vee, \to, \neg$.

Suppose that a Δ_0 formula ψ is absolute in M. Let y be a set and let $\varphi = (\exists x \in y)\psi(x)$. From relativization, $(\exists x \psi(x))^M \leftrightarrow (\exists x \in M)\psi^M(x)$. Since the hypotheses makes it clear that $\psi^M \leftrightarrow \psi$, we get $((\exists x \in y)\psi(x))^M \leftrightarrow (\exists x \in y \cap M)\psi(x)$, which is the equivalent of $\varphi^M \leftrightarrow \varphi$. The same applies to $\varphi = (\forall x \in y)\psi(x)$. □

Lemma 1.43 *(Downward Absoluteness)*
Let φ be a Π_1 formula and M a transitive class. Then the following holds:

$$(\forall p_1, \ldots, p_n \in M)(\varphi(p_1, \ldots, p_n) \to \varphi(p_1, \ldots, p_n)^M) \tag{1.34}$$

Proof. Since $\varphi(p_1, \dots, p_n)$ is Π_1, there is a Δ_0 formula $\psi(p_1, \dots, p_n, x)$ such that $\varphi = \forall x \psi(p_1, \dots, p_n, x)$. From relativization and lemma 1.42, $\varphi^M(p_1, \dots, p_n) \leftrightarrow (\forall x \in M)\psi(p_1, \dots, p_n, x)$.

Assume that for $p_1, \dots, p_n \in M$ fixed, that $\forall x \psi(p_1, \dots, p_n, x)$ holds, but $(\forall x \in M)\psi(p_1, \dots, p_n, x)$ does not. Therefore $\exists x \neg\psi(p_1, \dots, p_n, x)$, which contradicts $\forall x \psi(p_1, \dots, p_n, x)$. □

Lemma 1.44 *(Upward Absoluteness)*
Let φ be a Σ_1 formula and M a transitive class. Then the following holds:

$$(\forall p_1, \dots, p_n \in M)(\varphi^M(p_1, \dots, p_n) \to \varphi(p_1, \dots, p_n)) \tag{1.35}$$

Proof. Since $\varphi(p_1, \dots, p_n)$ is Σ_1, there is a Δ_0 formula $\psi(p_1, \dots, p_n, x)$ such that $\varphi = \exists x \psi(p_1, \dots, p_n, x)$. From relativization and lemma 1.42, $\varphi^M(p_1, \dots, p_n) \leftrightarrow (\exists x \in M)\psi(p_1, \dots, p_n, x)$.

Assume that for $p_1, \dots, p_n \in M$ fixed, that $(\exists x \in M)\psi(p_1, \dots, p_n, x)$ holds, but $\exists x \psi(p_1, \dots, p_n, x)$ does not. This is an obvious contradiction. □

1.2.6 More Functions

Definition 1.45 *(Strictly Increasing Function)*
A function $f : Ord \to Ord$ is said to be strictly increasing *iff*

$$\forall \alpha, \beta \in Ord(\alpha < \beta \to f(\alpha) < f(\beta)). \tag{1.36}$$

Definition 1.46 *(Continuous Function)*
A function $f : Ord \to Ord$ is said to be continuous *iff*

$$\lambda \textit{ is limit } \to f(\lambda) = \bigcup_{\alpha < \lambda} f(\alpha). \tag{1.37}$$

Definition 1.47 *(Normal Function)*
A function $f : Ord \to Ord$ is said to be normal *iff it is* strictly increasing *and* continuous.

Definition 1.48 *(Fixed Point)*
We say x is a fixed point of a function f iff $x = f(x)$.

Definition 1.49 *(Unbounded Class)*
We say a class A is unbounded iff

$$\forall x(\exists y \in A)(x < y) \tag{1.38}$$

Definition 1.50 *(Limit Point)*
Given a class $x \subseteq Ord$, we say that $\alpha \neq \emptyset$ is a limit point of x iff

$$\alpha = \bigcup (x \cap \alpha) \tag{1.39}$$

Definition 1.51 *(Closed Class)*
We say a class $A \subseteq Ord$ is closed iff it contains all its limit points.

Definition 1.52 *(Club set)*
For a regular uncountable cardinal κ, a set $x \subset \kappa$ is a closed unbounded *subset, abbreviated as a* club set*, iff x is both closed and unbounded in κ.*

Definition 1.53 *(Stationary set)*
For a regular uncountable cardinal κ, we say a set $A \subset \kappa$ is stationary in κ iff it intersects every club subset of κ.

1.2.7 Structure, Substructure and Embedding

Structures will be denoted $\langle M, \in, R \rangle$ where M is a domain, $\in$ stands for the standard membership relation, it is assumed to be restricted to the domain[7], $R \subseteq M$ is a relation on the domain. When R is not needed, we can as well only write M instead of $\langle M, \in \rangle$.

Definition 1.54 *(Elementary Embedding)*
Given the structures $\langle M_0, \in, R \rangle$, $\langle M_1, \in, R \rangle$ and a one-to-one function $j : M_0 \to M_1$, we say j is an elementary embedding *of M_0 into M_1, we write $j : M_0 \prec M_1$, when the following holds for every formula $\varphi(p_1, \ldots, p_n)$ and every $p_1, \ldots, p_n \in M_0$:*

$$\langle M_0, \in, R \rangle \models \varphi(p_1, \ldots, p_n) \leftrightarrow \langle M_1, \in, R \rangle \models \varphi(j(p_1), \ldots, j(p_n)) \tag{1.40}$$

Definition 1.55 *(Elementary Substructure)*
Given the structures $\langle M_0, \in, R \rangle$, $\langle M_1, \in, R \rangle$ and a one-to-one function $j : M_0 \to M_1$ such that $j : M_0 \prec M_1$, we say that M_0 is an elementary substructure *of M_1, denoted as $M_0 \prec M_1$, iff j is an identity on M_0. In other words*

$$\langle M_0, \in, R \rangle \models \varphi(p_1, \ldots, p_n) \leftrightarrow \langle M_1, \in, R \rangle \models \varphi(p_1, \ldots, p_n) \tag{1.41}$$

for $p_1, \ldots, p_n \in M_0$

[7] To be totally explicit, we should write $\langle M, \in \cap M \times M, R \rangle$

2 Levy's First-Order Reflection

2.1 Lévy's Original Paper

This section is based on Lévy's paper *Axiom Schemata of Strong Infinity in Axiomatic Set Theory*, [Lévy, 1960]. It presents Lévy's general reflection principle and its equivalence to *Replacement* and *Infinity* under S[8].

First, we should point out that set theory has changed over the last 66 years and show a few notable, albeit only formal, differences. Firstly, when reading Lévy's article, one should bear in mind that while the author often speaks about a model of ZF, usually denoted u, it doesn't necessarily mean that there is a set u that is a model of ZF[9], we are nowadays used to using the notion of universal class V in similar sense, even though independently from a particular axiomatic set theory. The theory ZF is practically identical to the theory we have established in (1.19), the differences are only formal. One might be confused by the fact that Lévy treats the *Subsets* axiom, which we call *Specification*, as a single axiom rather than a schema. He even takes the conjunction of all axioms of ZF and treats it like a formula. This is possible because the underlying logic calculus is different. Lévy works with set theories formulated in the *non-simple applied first order functional calculus*, see Chapter IV in [Church, 1996] for details. For now, we only need to know that the calculus contains a substitution rule for functional variables. This way, *Subsets* is de facto a schema even though it sometimes treated as a single formula[10]. It should also be noted that the logical connectives look different. The now usual symbol for an universal quantifier does not appear, $\forall x \varphi(x)$ would be written as $(x)\varphi(x)$. The symbol for negation is "$\sim$", implication is written as "$\supset$" and equivalence is "$\equiv$". We will use standard notation with "$\neg$", "$\rightarrow$" and "$\leftrightarrow$" respectively when presenting Lévy's results.

This subsection uses ZF instead of the usual ZFC as the underlying theory.

Definition 2.1 *(Standard Complete Model of a Set Theory)*
Let Q be an arbitrary axiomatic set theory. We say that u is a standard complete model of Q iff

(i) $(\forall \sigma \in \mathsf{Q})(\mathsf{u} \models \sigma)$

(ii) $\forall y(y \in u \rightarrow y \subset u)$

We write $Scm^{\mathsf{Q}}(u)$.

[8] See definition (1.18).

[9] This is indeed impossible to prove in ZF due to Gödel's Incompleteness.

[10] This way, the conjunction of all axioms is then in fact an axiom schema.

Definition 2.2 *(Cardinals Inaccessible With Respect to* Q*)* *Let* Q *be an arbitrary axiomatic set theory. We say that a cardinal* κ *is inaccessible with respect to theory* Q *iff*

$$Scm^{\mathsf{Q}}(V_\kappa) \tag{2.42}$$

We write $In^{\mathsf{Q}}(\kappa)$

Definition 2.3 *(Inaccessible Cardinal With Respect to* ZF*)* *When a cardinal* κ *is inaccessible with respect to* ZF*, we only say that it is inaccessible. We write* $In(\kappa)$*.*

$$In(\kappa) \leftrightarrow In^{\mathsf{ZF}}(\kappa) \tag{2.43}$$

The above definition of inaccessibles is used because it doesn't require *Choice*.

For the definition of relativization, see (1.39). The notation used by Lévy is "$Rel(u, \varphi)$", we will stick to "φ^u".

Definition 2.4 *(N)* *The following is an axiom schema of complete reflection over* ZF*, denoted as* N*. For every first-order formula* φ *in the language of set theory with no free variables except for* $p_1, \ldots, p_n$*, the following is an instance of schema* N*.*

$$\exists u(Scm^{\mathsf{ZF}}(u) \ \& \ (\forall p_1, \ldots, p_n \in u)(\varphi \leftrightarrow \varphi^u)) \tag{2.44}$$

Let S be an axiomatic set theory defined in (1.18).

Definition 2.5 *(N_0)* *Axiom schema* N_0 *is similar to* N *defined above, but with* S *instead of* ZF*. For every* φ*, a first-order fomula in the language of set theory with no free variables except* $p_1, \ldots, p_n$*, the following is an instance of* N_0*.*

$$\exists u(Scm^{\mathsf{S}}(u) \ \& \ (\forall p_1, \ldots, p_n \in u)(\varphi \leftrightarrow \varphi^u)) \tag{2.45}$$

We will now show that in S, N_0 implies both *Replacement* and *Infinity*.

Let N_0 be defined as in (2.5), for *Infinity* see (1.10).

Theorem 2.6 *In* S*, the axiom schema* N_0 *implies* Infinity.

Proof. Let $\varphi = \forall x \exists y(y = x \cup \{x\})$. This clearly holds in S because given a set x, there is a set $y = x \cup \{x\}$ obtained via *Pairing* and *Union*. From N_0, there is a set u such that φ^u holds. This u satisfies the conditions required by *Infinity*. □

Lévy proves this theorem in a different way. He argues that for an arbitrary formula φ, N_0 gives us $\exists u Scm^{\mathsf{S}}(u)$ and this u already satisfies *Infinity*. To do this, we would need to prove lemma (2.12) now, which would make second half of this chapter quite confusing.

Let S be a set theory defined in (1.18), N_0 a schema defined in (2.5) and *Replacement* a schema defined in (1.16).

Theorem 2.7 *In* S*, axiom the schema* N_0 *implies* Replacement.

Proof. Let $\varphi(x, y, p_1, \ldots, p_n)$ be a formula with no free variables except $x, y, p_1, \ldots, p_n$. Let χ be an instance of the *Replacement* schema for the φ given. We want to verify that χ holds in S with N_0.

$$\begin{aligned} \chi = \forall x, y, z(\varphi(x, y, p_1, \ldots, p_n) \,\&\, \varphi(x, z, p_1, \ldots, p_n) \rightarrow y = z) \\ \rightarrow \forall x \exists y \forall z (z \in y \leftrightarrow (\exists q \in x)(\varphi(x, y, p_1, \ldots, p_n))) \end{aligned} \tag{2.46}$$

Now consider the following formulas.

(i) $(\forall x, y, p_1, \ldots, p_n \in u)(\varphi \leftrightarrow \varphi^u)$

(ii) $(\forall x, p_1, \ldots, p_n \in u)(\exists y \varphi \leftrightarrow (\exists y \varphi)^u)$

(iii) $(\forall x, p_1, \ldots, p_n \in u)(\chi \leftrightarrow \chi^u)$

(iv) $\forall x, p_1, \ldots, p_n \chi \leftrightarrow (\forall x, p_1, \ldots, p_n \chi)^u$

The above formulas are instances of the N_0 schema for φ, $\exists y \varphi$, χ and the universal closure of χ respectively. By N_0, there exists a set u where all four formulas hold.[11] From relativization, $(\exists y \varphi)^u$ is equivalent to $(\exists y \in u)\varphi^u$, together with (i) and (ii), we get

$$(\forall x, p_1, \ldots, p_n \in u)((\exists y \in u)\varphi \leftrightarrow \exists y \varphi) \tag{2.47}$$

If φ is a function, then for every $x \in u$, which is also $x \subset u$ since $Scm^{\mathsf{S}}(u)$ and therefore u is transitive, it maps elements of x into u. From the *Specification*, we can find y, a set of all images of the elements of x. That gives us $x, p_1, \ldots, p_n \in u \rightarrow \chi$. By (iii) we get that $x, p_1, \ldots, p_n \in u \rightarrow \chi^u$ holds. The universal closure of this formula is $\forall x, p_1, \ldots, p_n(x, p_1, \ldots, p_n \in u \rightarrow \chi^u)$ which is equivalent to $(\forall x, p_1, \ldots, p_n \in u)(\chi)^u$, which is exactly $(\forall x, p_1, \ldots, p_n \chi)^u$. From (iv), $\forall x, p_1, \ldots, p_n \chi$ holds. □

What we have just proven is only a single theorem from Lévy's aforementioned article, we will introduce other interesting results, mostly related to Mahlo and inaccessible cardinals, later in their appropriate context in chapter 3.

[11] Despite the fact that N_0 is defined for one formula, we have just used it for four at once. To make this formally possible, we can either prove that N_0 is equivalent to a more general version for any finite number of formulas or we can reflect their conjunction and argue that if $u \models \varphi_1 \,\&\, \ldots \,\&\, \varphi_n$, then $(u \models \varphi_1), \ldots, (u \models \varphi_n)$.

2.2 Contemporary Restatement

We will now introduce and prove a theorem that is called Lévy's Reflection in contemporary set theory. The only difference is that while Lévy originally reflects a formula φ from V to a set u which is a *standard complete model of* S, we say that there is a V_λ for a limit λ that reflects φ. Those two conditions are equivalent due to lemma (2.12).

Lemma 2.8 *Let $\varphi_1, \ldots, \varphi_n$ be first-order formulas in the language of set theory, all with m free variables* [12].

(i) For each set M_0 there is such set M that $M_0 \subset M$ and the following holds for every i, $1 \leq i \leq n$:

$$\exists x \varphi_i(p_1, \ldots, p_{m-1}, x) \to (\exists x \in M)\varphi_i(p_1, \ldots, p_{m-1}, x) \tag{2.48}$$

for every $p_1, \ldots, p_{m-1} \in M$.

(ii) Furthermore, there is a limit ordinal λ such that $M_0 \subset V_\lambda$ and the following holds for each i, $1 \leq i \leq n$:

$$\exists x \varphi_i(p_1, \ldots, p_{m-1}, x) \to (\exists x \in V_\lambda)\varphi_i(p_1, \ldots, p_{m-1}, x) \tag{2.49}$$

for every $p_1, \ldots, p_{m-1} \in M$.

(iii) Assuming Choice*, there is M, $M_0 \subset M$ such that (2.48) holds for every M, $i \leq n$ and $|M| \leq |M_0| \cdot \aleph_0$.*

Proof. We will simultaneously prove statements (i) and (ii), denoting M^T the transitive set required by part (ii). Steps in the construction of M^T that are not explicitly included are equivalent to steps for M.

Let us first define an operation $H_i(p_1, \ldots, p_{m-1})$ that yields the set of x's with minimal rank[13] satisfying $\varphi_i(p_1, \ldots, p_{m-1}, x)$ for $p_1, \ldots, p_{m-1}$ and for every i, $1 \leq i \leq n$.

$$H_i(p_1, \ldots, p_n) = \{x \in C_i : (\forall z \in C)(rank(x) \leq rank(z))\} \tag{2.50}$$

for each $1 \leq i \leq n$, where

$$C_i = \{x : \varphi_i(p_1, \ldots, p_{m-1}, x)\} \text{ for } 1 \leq i \leq n \tag{2.51}$$

[12] For formulas with a different number of free variables, take for m the highest number of parameters among those formulas. Add spare parameters to every formula that has less than m parameters in a way that preserves the last parameter, which we will denote x. E.g. let φ'_i be the a formula with k parameters, $k < m$. Let us set $\varphi_i(p_1, \ldots, p_{m-1}, x) = \varphi'_i(p_1, \ldots, p_{k-1}, x)$, notice that the parameters $p_k, \ldots, p_{m-1}$ are not used.

[13] Rank is defined in (1.29)

Next, let's construct M from given M_0 by induction.

$$M_{i+1} = M_i \cup \bigcup_{j=0}^{n} \bigcup \{H_j(p_1, \ldots, p_{m-1}) : p_1, \ldots, p_{m-1} \in M_i\} \tag{2.52}$$

In other words, in each step we include into the construction the elements satisfying $\varphi(p_1, \ldots, p_{m-1}, x)$ for $p_1, \ldots, p_{m-1}$ from the previous step. For statement (ii), this is the only part that differs from (i). To end up with a transitive M, we need to extend every step to it's transitive closure transitive closure of M_{i+1} from (i). In other words, let γ be the smallest ordinal such that

$$(M_i^T \cup \bigcup_{j=0}^{n} \{\bigcup \{H_j(p_1, \ldots, p_{m-1}) : p_1, \ldots, p_{m-1} \in M_i\}\}) \subset V_\gamma \tag{2.53}$$

Then the incremental step is

$$M_{i+1}^T = V_\gamma \tag{2.54}$$

and the final M is obtained by joining the previous steps.

$$M = \bigcup_{i=0}^{\infty} M_i, \;\; M^T = \bigcup_{i=0}^{\infty} M_i^T = V_\lambda \text{ for some limit } \lambda. \tag{2.55}$$

We have yet to finish part (iii). Let's try to construct a set M' that satisfies the same conditions like M but is kept as small as possible. Assuming the Axiom of Choice, we can modify the construction so that the cardinality of M' is at most $|M_0| \cdot \aleph_0$. Note that the size of M in the previous construction is determined by the size of M_0 and, most importantly, by the size of $H_i(p_1, \ldots, p_{m-1})$ for every i, $1 \leq i \leq n$ in individual iterations of the construction. Since (i) only ensures the existence of an x that satisfies $\varphi_i(p_1, \ldots, p_{m-1}, x)$ for any i, $1 \leq i \leq n$, we only need to add one x for every set of parameters but $H_i(u_1, \ldots, u_{m-1})$ can be arbitrarily large. Let F be a choice function on $\mathscr{P}(M')$. Also let $h_i(p_1, \ldots, p_{m-1}) = F(H_i(p_1, \ldots, p_{m-1}))$ for i, where $1 \leq i \leq n$, which means that h is a function that outputs an x that satisfies $\varphi_i(p_1, \ldots, p_{m-1}, x)$ for i such that $1 \leq i \leq n$ and has minimal rank among all such sets. The induction step needs to be redefined to

$$M'_{i+1} = M'_i \cup \bigcup_{j=0}^{n} \{h_j(p_1, \ldots, p_{m-1}) : p_1, \ldots, p_{m-1} \in M'_i\} \tag{2.56}$$

This way, the amount of elements added to M'_{i+1} in each step of the construction is the same as the amount of m-tuples of parameters that yielded elements not included in M'_i. It is easy to see that if M_0 is finite, M' is countable because it was constructed as a countable union of sets that are themselves at most countable. If M_0 is countable or larger, the cardinality of M' is equal to the cardinality of M_0.[14] Therefore $|M'| \leq |M_0| \cdot \aleph_0$ $\square$

Theorem 2.9 *(Lévy's first-order reflection theorem)*
Let $\varphi(p_1, \ldots, p_n)$ be a first-order formula.

(i) For every set M_0 there exists a set M such that $M_0 \subset M$ and the following holds:

$$\varphi^M(p_1, \ldots, p_n) \leftrightarrow \varphi(p_1, \ldots, p_n) \tag{2.57}$$

for every $p_1, \ldots, p_n \in M$.

(ii) For every set M_0 there is a transitive set M, $M_0 \subset M$ such that the following holds:

$$\varphi^M(p_1, \ldots, p_n) \leftrightarrow \varphi(p_1, \ldots, p_n) \tag{2.58}$$

for every $p_1, \ldots, p_n \in M$.

(iii) For every set M_0 there is a limit ordinal λ such that $M_0 \subset V_\lambda$ and the following holds:

$$\varphi^{V_\lambda}(p_1, \ldots, p_n) \leftrightarrow \varphi(p_1, \ldots, p_n) \tag{2.59}$$

for every $p_1, \ldots, p_n \in M$.

(iv) Assuming Choice*, for every set M_0 there is M such that $M_0 \subset M$ and $|M| \leq |M_0| \cdot \aleph_0$ and the following holds:*

$$\varphi^M(p_1, \ldots, p_n) \leftrightarrow \varphi(p_1, \ldots, p_n) \tag{2.60}$$

for every $p_1, \ldots, p_n \in M$.

Proof. Let's now prove (i) for given φ via induction by complexity. We can safely assume that φ contains no quantifiers besides "$\exists$" and no logical connectives other than "$\neg$" and "&". Let $\varphi_1, \ldots, \varphi_n$ be all subformulas of φ. Then there is a set M, obtained by the means of lemma (2.8), for all of the formulas $\varphi_1, \ldots, \varphi_n$.

[14] It can not be smaller because $|M'_{i+1}| \geq |M'_i|$ for every i. It may not be significantly larger because the maximum of elements added is the number of n-tuples in M'_i, which is of the same cardinality as M'_i.

Let's first consider atomic formulas in the form of either $x_1 = x_2$ or $x_1 \in x_2$. It is clear from relativisation[15] that (2.57) holds for both cases, $(x_1 = x_2)^M \leftrightarrow (x_1 = x_2)$ and $(x_1 \in x_2)^M \leftrightarrow (x_1 \in x_2)$.

We now want to verify the inductive step. First, take $\varphi = \neg\varphi'$. From relativization, we get $(\neg\varphi')^M \leftrightarrow \neg(\varphi'^M)$. Because the induction hypothesis tells us that $\varphi'^M \leftrightarrow \varphi'$, the following holds:

$$(\neg\varphi')^M \leftrightarrow \neg(\varphi'^M) \leftrightarrow \neg\varphi' \tag{2.61}$$

The same holds for $\varphi = \varphi_1 \;\&\; \varphi_2$. From the induction hypothesis, we know that $\varphi_1^M \leftrightarrow \varphi_1$ and $\varphi_2^M \leftrightarrow \varphi_2$, which together with relativization for formulas in the form of $\varphi_1 \;\&\; \varphi_2$ gives us

$$(\varphi_1 \;\&\; \varphi_2)^M \leftrightarrow \varphi_1^M \;\&\; \varphi_2^M \leftrightarrow \varphi_1 \;\&\; \varphi_2 \tag{2.62}$$

Let's now examine the case when $\varphi = \exists x \varphi'(p_1, \dots, p_n, x)$. The induction hypothesis tells us that $\varphi'^M(p_1, \dots, p_n, x) \leftrightarrow \varphi'(p_1, \dots, p_n, x)$, so, together with above lemma (2.8), the following holds:

$$\begin{gathered} \varphi(p_1, \dots, p_n, x) \\ \leftrightarrow \exists x \varphi'(p_1, \dots, p_n, x) \\ \leftrightarrow (\exists x \in M)\varphi'(p_1, \dots, p_n, x) \\ \leftrightarrow (\exists x \in M)\varphi'^M(p_1, \dots, p_n, x) \\ \leftrightarrow (\exists x \varphi'(p_1, \dots, p_n, x))^M \\ \leftrightarrow \varphi^M(p_1, \dots, p_n, x) \end{gathered} \tag{2.63}$$

Which is what we wanted to prove for part (i).

We now need to verify that the same holds for any finite number of formulas $\varphi_1, \dots, \varphi_n$. This has in fact been already done since lemma (2.8) gives us a set M for any finite amount of formulas and given M_0. We can therefore find a set M for the union of all of their subformulas. When we obtain such M, it should be clear that it also reflects every formula in $\varphi_1, \dots, \varphi_n$.

Since V_λ is a transitive set, by proving (**iii**) we also satisfy (**ii**). To do so, we only need to look at part (**ii**) of lemma (2.8). All of the above proof also holds for $M = V_lambda$.

[15] See (1.39). This only holds for relativization to $M, \in \cap M \times M$, not M, R for an arbitrary R.

To finish part (**iv**), we take M of size $\leq |M_0| \cdot \aleph_0$, which exists due to part (**iii**) of lemma (2.8), the rest being identical. □

Let S be a set theory defined in (1.18), for ZFC see definition (1.20).

The two following lemmas are based on [Drake, 1974][Chapter 3, Theorem 1.2].

Lemma 2.10 *If M is a transitive set, then $M \models$ Extensionality.*

Proof. Given a transitive set M, we want to show that the following holds.

$$M \models \forall x, y(x = y \leftrightarrow \forall z(z \in x \leftrightarrow z \in y)) \tag{2.64}$$

Given arbitrary $x, y \in M$, we want to prove that $M \models (x = y \leftrightarrow \forall z(z \in x \leftrightarrow z \in y))$. This is equivalent to $M \models x = y$ iff $M \models \forall z(z \in x \leftrightarrow z \in y)$, which is the same as $x = y$ iff $M \models \forall z(z \in x \leftrightarrow z \in y)$.

So all elements of x are also elements of y in M, and vice versa. Because M is transitive, all elements of x and y are in M, so $M \models \forall z(z \in x \leftrightarrow z \in y)$ holds iff x and y contain the same elements and are therefore equal. □

Lemma 2.11 *If M is a transitive set, then $M \models$ Foundation.*

Proof. We want to prove the following:

$$M \models \forall x(x \neq \emptyset \rightarrow (\exists y \in x)(x \cap y = \emptyset)) \tag{2.65}$$

Given an arbitrary non-empty $x \in M$ let's show that $M \models (\exists y \in x)(x \cap y = \emptyset)$.

Because M is transitive, every element of x is an element of M. Take for y the element of x with the lowest rank[16]. It should be clear that there is no $z \in y$ such that $z \in x$, because then $rank(z) < rank(y)$, which would be a contradiction. □

Let S be a set theory as defined in (1.18).

Lemma 2.12 *The following holds for every λ.*

$$\text{“}\lambda \textit{ is a limit ordinal”} \rightarrow V_\lambda \models \mathsf{S} \tag{2.66}$$

Proof. Given an arbitrary limit ordinal λ, we will verify the axioms of S one by one.

(i) *The existence of a set* comes from the fact that V_λ is a non-empty set because limit ordinal is non-zero by definition.

[16] Rank is defined in (1.29).

(ii) *Extensionality* holds from (2.10).
(iii) *Foundation* holds from (2.11).
(iv) *Union*:
Given any $x \in V_\lambda$, we want verify that $y = \bigcup x$ is also in V_λ. Note that $y = \bigcup x$ is a Δ_0-formula.

$$y = \bigcup x \leftrightarrow (\forall z \in y)(\exists q \in x) z \in q \;\&\; (\forall z \in x)(\forall q \in z) q \in y \tag{2.67}$$

So by lemma (1.42)

$$y = \bigcup x \leftrightarrow V_\lambda \models y = \bigcup x \tag{2.68}$$

(v) *Pairing*:
Given two sets $x, y \in V_\lambda$, we want to show that $z = \{x, y\}$ is also an element of V_λ.

$$z = \{x, y\} \leftrightarrow x \in z \;\&\; y \in z \;\&\; (\forall q \in z)(q = x \vee q = y) \tag{2.69}$$

So $z = \{x, y\}$ is a Δ_0-formula, and thus by lemma (1.42) it holds that

$$z = \{x, y\} \leftrightarrow V_\lambda \models z = \{x, y\} \tag{2.70}$$

(vi) *Powerset*:
Given any $x \in V_\lambda$, we want to make sure that $\mathscr{P}(x) \in V_\lambda$. Let $\varphi(y)$ denote the formula $y \in \mathscr{P}(x) \leftrightarrow y \subset x$. according to definition of subset (1.4), $y \subset x$ is Δ_0, so for any given $x, y \in V_\lambda$, $y = \mathscr{P}(x) \leftrightarrow V_\lambda \models y = \mathscr{P}(x)$. Because λ is limit and $rank(\mathscr{P}(x)) = rank(x)+1$, if $\mathscr{P}(x) \in V_\lambda$ for every $x \in V_\lambda$.
(vii) *Specification*:
Given a first-order formula φ, we want to show the following:

$$V_\lambda \models \forall x, p_1, \ldots, p_n, \exists y \forall z (z \in y \leftrightarrow z \in x \;\&\; \varphi(z, p_1, \ldots, p_n)) \tag{2.71}$$

Given any x along with parameters $p_1, \ldots, p_n$ in V_λ, we set

$$y = \{z \in x : \varphi^{V_\lambda}(z, p_1, \ldots, p_n)\} \tag{2.72}$$

From transitivity of V_λ and the fact that $y \subset x$ and $x \in V_\lambda$, we know that $y \in V_\lambda$, so $V_\lambda \models \forall z (z \in y \leftrightarrow z \in x \;\&\; \varphi(z, p_1, \ldots, p_n))$.

□

Definition 2.13 *(First-Order Reflection Schema)*
For every first-order formula φ, the following is an axiom:

$$\forall M_0 \exists M (M_0 \subseteq M \;\&\; (\varphi(p_1, \ldots, p_n) \leftrightarrow \varphi(p_1, \ldots, p_n)^M)) \tag{2.73}$$

We will refer to this axiom schema as First-order reflection.

Let *Infinity* and *Replacement* be as defined in (1.10) and (1.16) respectively.

Theorem 2.14 First-order reflection *is equivalent to* Infinity & Replacement *under* S.

Proof. Since (2.9) already gives us one side of the implication, we are only interested in showing the converse which we shall do in two parts:

First-order reflection $\to$ *Infinity* This is done exactly like (2.6). We pick for φ the formula $(\forall y \in x)(y \cup \{y\} \in x)$, $M_0 = \{\emptyset\}$. From (2.13), there is a set M that satisfies φ, so there is an inductive set. We have picked M_0 so that $\emptyset \in M$ obviously holds and M is the witness for

$$\exists x(\emptyset \in x \ \& \ (\forall y \in x)(y \cup \{y\} \in x)) \tag{2.74}$$

which is exactly (1.10).

First-order reflection $\to$ *Replacement*

Let's first point out that while *First-order reflection* gives us a set for one formula, we can generalize it to hold for any finite number of formulas. We will show how is it done for two formulas, which is what we will use in this proof. Given two first-order formulas φ, ψ, we can suppose that there are formulas φ' and ψ' that are equivalent to φ and ψ respectively, but their free variables are different [17]. Let $\xi = \varphi \ \& \ \psi$, given any M_0, we can find a M such that $\xi \leftrightarrow \xi^M$. It is easy to see that from relativisation, the following holds:

$$\varphi \ \& \ \psi \leftrightarrow \varphi' \ \& \ \psi' \leftrightarrow \xi \leftrightarrow \xi^M \leftrightarrow (\varphi' \ \& \ \psi')^M \leftrightarrow \varphi'^M \ \& \ \psi'^M \leftrightarrow \varphi^M \ \& \ \psi^M \tag{2.75}$$

Now given a function $\varphi(x, y)$, we know from *First-order reflection* that for every M_0, there is a set M such that $M_0 \subseteq M$ and both

$$(\forall x, y \in M)(\varphi(x, y) \leftrightarrow \varphi^M(x, y)) \tag{2.76}$$

and

$$(\forall x, y \in M)(\exists y \varphi(x, y) \leftrightarrow (\exists y \varphi(x, y))^M) \tag{2.77}$$

hold, the latter being equivalent to

$$(\forall x, y \in M)(\exists y \varphi(x, y) \leftrightarrow (\exists y \in M)\varphi^M(x, y)) \tag{2.78}$$

[17] This is plausible since we can for example substitute all free variables in φ' for $x_0, x_2, x_4, \ldots$ and use $x_1, x_3, x_5, \ldots$ for free variables in ψ', the resulting formulas will be equivalent.

Therefore

$$(\forall x, y \in M)(\exists y\varphi(x, y) \leftrightarrow (\exists y \in M)\varphi(x, y)) \tag{2.79}$$

holds too. That means that we have a set M such that for every $x \in M$, if φ is defined for x, $(\exists y \in M)\varphi(x, y)$.

To show that *Replacement* holds for this particular φ, we need to verify that given a set M_0, $M_0' = \{y : (\exists x \in M_0)\varphi(x, y)\}$ is also a set. But since $M_0 \subseteq M$ and because given any $x \in M$, there is $y \in M$ satisfying $\varphi(x, y)$, the following is a set due to *Specification*:

$$M_0' = \{y : (\exists x \in M_0)\varphi(x, y)\} = \{y \in M : (\exists x \in M_0)\varphi(x, y)\} \tag{2.80}$$

□

We have shown that *Reflection* for first-order formulas, *First-order reflection* is a theorem of ZFC.We have also shown that it can be used instead of the *Infinity* and *Replacement* scheme, but ZFC + *First-order reflection* is a conservative extension of ZF. Besides being a starting point for more general and powerful statements, it can be used to show that ZF is not finitely axiomatizable. This follows from the fact that *Reflection* gives a model to any consistent finite set of formulas. So if $\varphi_1, \dots, \varphi_n$ would be the axioms of ZFC, *Reflection* would prove that every model of ZFC contains a smaller model of ZFC, which would in turn contradict the Second Gödel's Theorem[18].

It is also worthwhile to note that, in a way, Reflection is dual to compactness. Compactness says that given a set of sentences, if every finite subset yields a model, so does the whole set. Reflection, on the other hand, says that while the whole set has no model in the underlying theory, every finite subset has a model.

Furthemore, *Reflection* can be used in ways similar to upward Löwenheim–Skolem theorem. Since Reflection extends any set M_0 into a model of given formulas $\varphi_1, \dots, \varphi_n$, we can choose the lower bound of the size of M by appropriately choosing M_0.

In the next section, we will try to generalize *Reflection* in a way that transcends ZF and yields some large cardinals.

[18]See chapter **??** for further details.

3 Reflection And Large Cardinals

3.1 Regular Fixed-Point Axioms

Lévy's article mentions various schemata that are not instances of reflection per se, but deal with fixed points of normal ordinal functions. We will introduce them and show that they are equivalent to *First-Order Reflection*[19].

Lemma 3.1 *(Fixed-point lemma for normal functions)*
Let f be a normal function defined for all ordinals[20]*. Then all of the following hold:*

(i) $\forall\lambda($*"λ is a limit ordinal"* $\rightarrow$ *"f(λ) is a limit ordinal"*$)$
(ii) $\forall\alpha(\alpha \leq f(\alpha))$
(iii) $\forall\alpha\exists\beta(\alpha < \beta \;\&\; f(\beta) = \beta)$
(iv) The fixed points of f form a closed unbounded class.[21]

Proof. Let f be a normal function defined for all ordinals.

(i) Suppose λ is a limit ordinal. For an arbitrary ordinal $\alpha < \lambda$, the fact that f is strictly increasing means that $f(\alpha) < f(\lambda)$ and for any ordinal β, satisfying $\alpha < \beta < \lambda$, $f(\alpha) < f(\beta) < f(\lambda)$. We know that there is such β from limitness of λ. Because f is continuous and λ is limit, $f(\lambda) = \bigcup_{\gamma<\lambda} f(\gamma)$.That means that if λ is limit, so is $f(\lambda)$.

(ii) This step will be proven using the transfinite induction. Since f is defined for all ordinals, there is an ordinal α such that $f(\emptyset) = \alpha$ and because $\emptyset$ is the least ordinal, (ii) holds for $\emptyset$.
Suppose (ii) holds for some β form the induction hypothesis. It the holds for $\beta + 1$ because f is strictly increasing.
For a limit ordinal λ, suppose (ii) holds for every $\alpha < \lambda$. (i) implies that $f(\lambda)$ is also limit, so there is a strictly increasing κ-sequence $\langle\alpha_0, \alpha_1, \ldots\rangle$ for some κ such that $\lambda = \bigcup_{i<\kappa} \alpha_i$. Because f is stricly increasing, the κ-sequence $\langle f(\alpha_0), f(\alpha_1), \ldots$ is also strictly increasing, the induction hypothesis implies that $\alpha_i \leq f(\alpha_i)$ for each $i \leq \kappa$. Thus, $\lambda \leq f(\lambda)$.

(iii) For a given ordinal α, let there be an ω-sequence $\langle\alpha_0, \alpha_1, \ldots\rangle$, such that $\alpha_0 = \alpha$ and $\alpha_{i+1} = f(\alpha_i)$ for each $i < \omega$. This sequence is stricly increasing because so is f. Now, there's a limit ordinal $\beta = \bigcup_{i<\omega} \alpha_i$, we want to show that this is the fixed point. So $f(\beta) = f(\bigcup_{i<\omega} \alpha_i) = \bigcup_{i<\omega} f(\alpha)$ because f is continuous. We have defined the above sequence so that β, $\bigcup_{i<\omega} f(\alpha) = \bigcup_{i<\omega} \alpha_{i+1}$, which means we are done, since $\bigcup_{i<\omega} \alpha_{i+1} = \bigcup_{i<\omega} \alpha_i = \beta$.

[19] For definition, see (2.13).
[20] For the definition of normal function, see (1.47).
[21] See (1.51.) for the definition of closed class, (1.49) for the definition of unboundedness.

(iv) The class of fixed points of f is obviously unbounded by (iii). It remains to show that it is closed, this is based on [Drake, 1974], chapter 4. Let Y be a non-empty set of fixed points of f such that $\bigcup Y \notin Y$. Since f is defined on ordinals, Y is a set of ordinals, so $\bigcup Y$ is an ordinal because a supremum of a set of ordinals is an ordinal. $\bigcup Y$ is a limit ordinal. If it were a successor ordinal, suppose that $\alpha + 1 = \bigcup Y$, then $\alpha \in \bigcup Y$, which means that there is some x such that $\alpha \in x \in Y$. But the least such x is $\alpha + 1$, so $\bigcup Y \in Y$.
Note that $\alpha < \bigcup Y iff \exists \xi \in Y(\alpha < \xi)$. Since f is defined for all ordinals and $\bigcup Y$ is a limit ordinal, $f(\bigcup Y) = \bigcup_\alpha \in Y f(\alpha)$, but because Y is a set of fixed points of f, $f(\bigcup Y) = \bigcup_\alpha \in Y f(\alpha) = \bigcup Y$, so $\bigcup Y$ is also a limit point of Y.

□

Definition 3.2 *(*Axiom Schema M_1*)*
"Every normal function defined for all ordinals has at least one inaccessible number in its range."

Lévy uses "M" to refer to this axiom but since we also use "M" for sets and models, for example in (2.13), we will call the above axiom "*Axiom Schema M_1*" to avoid confusion.

Let $\varphi(x, y, p_1, \ldots, p_n)$ be a first-order formula with no free variables besides $x, y, p_1, \ldots, p_n$. The following is equivalent to *Axiom M_1*.

$$\begin{gathered}\text{"}\varphi \text{ is a normal function" } \& \ \forall x(x \in Ord \rightarrow \exists y(\varphi(x, y, p_1, \ldots, p_n))) \rightarrow \\ \rightarrow \exists y(\exists x \varphi(x, y, p_1, \ldots, p_n) \ \& \ cf(y) = y \ \& \ (\forall x \in \kappa)(\exists y \in \kappa)(x > y))\end{gathered} \tag{3.81}$$

Definition 3.3 *(Axiom Schema M_2)*
"Every normal function defined for all ordinals has at least one fixed point which is inaccessible."

Definition 3.4 *(Axiom Schema M_3)*
"Every normal function defined for all ordinals has arbitrarily great fixed points which are inaccessible."

Similar axiom is proposed in [Drake, 1974].

Definition 3.5 *(Axiom Schema F)*
"Every normal function has a regular fixed point."

Theorem 3.6

$$\text{Axiom } M_1 \leftrightarrow \text{Axiom } M_2 \leftrightarrow \text{Axiom } M_3 \leftrightarrow \text{Axiom Schema } F \tag{3.82}$$

This is *Theorem 1* in [Lévy, 1960].

Proof. ∎

But how do those schemata relate to reflection? Let's introduce a stronger version of *First-order reflection schema* from the previous chapter to see it more clearly. But in order to do this, we must first establish the inaccessible cardinal.

3.2 Inaccessible Cardinal

Definition 3.7 *An uncountable cardinal κ is* inaccessible *iff it is* regular *and* strongly limit. *We write $In(\kappa)$ to say that κ is an inaccessible cardinal.*

An uncountable cardinal that is regular and limit is called a *weakly limit cardinal*, we will only use the (strongly) inaccessible cardinal, but most of the results are similar, including higher types of ordinals that will be presented later in this chapter.

Theorem 3.8 *Let κ be an inaccessible cardinal.*

$$V_\kappa \models \mathsf{ZFC} \tag{3.83}$$

We will prove this theorem in a way similar to [Kanamori, 2003]. *Proof.* Most of this is already done in lemma (2.12), we only need to verify that *Replacement* and *Infinity* axioms hold in V_κ.

Infinity holds because κ is uncountable, so $\omega \in V_\kappa$.

To verify *Replacement*, let x be an element of V_κ and f a function from x to V_κ. Let $y = \{z \in V_\kappa : (\exists q \in x) f(q) = z\}$, so $y \subset V_\kappa$, it remains to show that $y \in V_\kappa$. Because f is a function, we know that $|y| \leq |x| \leq \kappa$. But since κ is regular, $\{rank(z) : z \in y\} \subseteq \alpha$ for some $\alpha < \kappa$, and so $x \in V_{\alpha+1} \subseteq V_\kappa$. Therefore $y \in V_\kappa$. ∎

Definition 3.9 *(Inaccessible Reflection Schema)*
For every first-order formula φ, the following is an axiom:

$$\forall M_0 \exists \kappa (M_0 \subseteq V_\kappa \;\&\; In(\kappa) \;\&\; (\varphi(p_1, \ldots, p_n) \leftrightarrow \varphi(p_1, \ldots, p_n)^{V_\kappa})) \tag{3.84}$$

We will refer to this axiom schema as Inaccessible reflection schema.

We have added the requirement that α is inaccessible, which trivially means that there is an inaccessible cardinal. By taking appropriate M_0, it can be shown that in a theory that includes the *Inaccessible reflection* *schema*, there is a closed unbounded class of inaccessible cardinals. Since we know that for an inaccessible κ, V_κ is a model of ZFC, *Inaccessible reflection schema* is equivalent to

$$\forall M_0 \exists \kappa (M_0 \subseteq V_\kappa \;\&\; V_\kappa \models \mathsf{ZFC} \;\&\; (\varphi(\mathsf{p}_1, \ldots, \mathsf{p}_n) \leftrightarrow \varphi(\mathsf{p}_1, \ldots, \mathsf{p}_n)^{V_\kappa})) \quad (3.85)$$

because we have proven in the last section that for an inaccessible κ, $V_\kappa \models$ ZFC.

Theorem 3.10 Inaccessible reflection schema *is equivalent to* Axiom schema *F.*

This is Theorem 4.1 in chapter four of [Drake, 1974]. *Proof.* Let's start by showing that *Inaccessible reflection schema* implies *Axiom schema F*. It should be clear that we can reflect two formulas to a single set, just form a new formula as a conjunction of universal closures of the two.

Given a normal function f defined for all ordinals, we want to show that it has a regular fixed point. For any ordinal α, there is an ordinal κ such that

$$\alpha < \kappa \;\&\; In(\kappa) \;\&\; (\forall \gamma, \delta \in V_\kappa)(f(\gamma) = \delta \leftrightarrow (f(\gamma) = \delta)^{V_\kappa}) \quad (3.86)$$

and

$$\alpha < \kappa \;\&\; In(\kappa) \;\&\; \forall \gamma \exists \delta (f(\gamma) = \delta) \leftrightarrow (\forall \gamma \exists \delta f(\gamma) = \delta)^{V_\kappa} \quad (3.87)$$

Since V_κ is the set of all sets of rank less than κ and since every ordinal is the rank of itself, there is an inaccessible ordinal κ such that

$$\forall \gamma < \kappa \exists \delta < \kappa (f^{V_\kappa}(\gamma) = \delta) \quad (3.88)$$

We also know that $f(\gamma) = \delta \leftrightarrow (f(\gamma) = \delta)^{V_\kappa}$. Now since κ is a limit ordinal and f is continuous we get

$$f(\kappa) = \bigcup_{\gamma < \kappa} f^{V_\kappa}(\gamma) = \bigcup_{\gamma < \kappa} f(\gamma). \quad (3.89)$$

From (3.88) and the fact that f is increasing, we know that $\kappa \leq \bigcup_{\gamma<\kappa} f(\gamma) \leq \kappa$. Therefore κ is an inaccessible fixed point of f.

For the opposite direction, it suffices to show that since there is an inaccessible cardinal from *Axiom schema F*, given a first-order formula φ, there is an inaccessible κ for which $\varphi \leftrightarrow V_\kappa \models \varphi$. □

4 Conclusion

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