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2 Katedra logiky

3 MIKLUÁŠ MRVA

4 REFLECTION PRINCIPLES AND LARGE
5 CARDINALS
6 Bakalářská práce

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8 2015

¹⁰ Prohlašuji, že jsem bakalářskou práci vypracoval samostatně a že jsem uvedl
¹¹ všechny použité prameny a literaturu.

¹² V Praze 14. dubna 2015

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Abstract

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Práce zkoumá vztah tzv. principů reflexe a velkých kardinálů. Lévy ukázal, že v ZFC platí tzv. věta o reflexi a dokonce, že věta o reflexi je ekvivalentní schématu nahrazení a axiomu nekonečna nad teorií ZFC bez axiomu nekonečna a schématu nahrazení. Tedy lze na větu o reflexi pohlížet jako na svého druhu axiom nekonečna. Práce zkoumá do jaké míry a jakým způsobem lze větu o reflexi zobecnit a jaký to má vliv na existenci tzv. velkých kardinálů. Práce definuje nedosažitelné, Mahlovy a nepopsatelné kardinály a ukáže, jak je lze zavést pomocí reflexe. Přirozenou limitou kardinálů získaných reflexí jsou kardinály nekonzistentní s L. Práce nabídne intuitivní zdůvodnění, proč tomu tak je.

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Abstract

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This thesis aims to examine relations between so called "Reflection Principles" and Large cardinals. Lévy has shown that Reflection Theorem is a sound theorem of ZFC and it is equivalent to Replacement Scheme and the Axiom of Infinity. From this point of view, Reflection theorem can be seen a specific version of an Axiom of Infinity. This paper aims to examine the Reflection Principle and its generalisations with respect to existence of Large Cardinals. This thesis will establish Inaccessible, Mahlo and Indescribable cardinals and their definition via reflection. A natural limit of Large Cardinals obtained via reflection are cardinals inconsistent with L. The thesis will offer an intuitive explanation of why this is the case.

38 **Contents**

1 Introduction

1.1 Motivation and Origin

The Universe of sets cannot be uniquely characterized (i. e. distinguished from all its initial elements) by any internal structural property of the membership relation in it, which is expressible in any logic of finite or transfinite type, including infinitary logics of any cardinal order.

— Kurt Gödel [?]

To understand why we need reflection in the first place, let's think about infinity for a moment. In the intuitive sense, infinity is an upper limit of all numbers. But for centuries, this was merely a philosophical concept, closely bound to religious and metaphysical way of thinking, considered separate from numbers used for calculations or geometry. It was a rather vague concept. In ancient Greece, Aristotle's response to famous Zeno's paradoxes introduced the distinction between actual and potential infinity. He argued, that potential infinity is (in today's words) well defined, as opposed to actual infinity, which remained a vague incoherent concept. He didn't think it's possible for infinity to inhabit a bounded place in space or time, rejecting Zeno's thought experiments as a whole. Aristotle's thoughts shaped western thinking partly due to Aquinas, who himself believed actual infinity to be more of a metaphysical concept for describing God than a mathematical property attributed to any other entity. In his *Summa Theologica*¹ he argues:

A geometrician does not need to assume a line actually infinite, but takes some actually finite line, from which he subtracts whatever he finds necessary; which line he calls infinite.

Less than hundred years later, Gregory of Rimini wrote

If God can endlessly add a cubic foot to a stone—which He can—then He can create an infinitely big stone. For He need only add one cubic foot at some time, another half an hour later, another a quarter of an hour later than that, and so on ad infinitum. He would then have before Him an infinite stone at the end of the hour.

Which is basically a Zeno's Paradox made plausible with God being the actor. In contrast to Aquinas' position, Gregory of Rimini theoretically constructs an object with actual infinite magnitude that is essentially different from God.

¹Part I, Question 7, Article 3, Reply to Objection 1

Even later, in the 17th century, pushing the property of infiniteness from the Creator to his creation, Nature, Leibniz wrote to Foucher in 1662:

I am so in favor of the actual infinite that instead of admitting that Nature abhors it, as is commonly said, I hold that Nature makes frequent use of it everywhere, in order to show more effectively the perfections of its Author. Thus I believe that there is no part of matter which is not, I do not say divisible, but actually divided; and consequently the least particle ought to be considered as a world full of an infinity of different creatures.

But even though he used potential infinity in what would become foundations of modern Calculus and argued for actual infinity in Nature, Leibniz refused the existence of an infinite, thinking that Galileo's Paradoxon² is in fact a contradiction. The so called Galileo's Paradoxon is an observation Galileo Galilei made in his final book "Discourses and Mathematical Demonstrations Relating to Two New Sciences". He states that if all numbers are either squares and non-squares, there seem to be less squares than there is all numbers. On the other hand, every number can be squared and every square has its square root. Therefore, there seem to be as many squares as there are all numbers. Galileo concludes, that the idea of comparing sizes makes sense only in the finite realm.

Salviati: So far as I see we can only infer that the totality of all numbers is infinite, that the number of squares is infinite, and that the number of their roots is infinite; neither is the number of squares less than the totality of all the numbers, nor the latter greater than the former; and finally the attributes "equal," "greater," and "less," are not applicable to infinite, but only to finite, quantities. When therefore Simplicio introduces several lines of different lengths and asks me how it is possible that the longer ones do not contain more points than the shorter, I answer him that one line does not contain more or less or just as many points as another, but that each line contains an infinite number.

Leibniz insists in part being smaller than the whole saying

Among numbers there are infinite roots, infinite squares, infinite cubes. Moreover, there are as many roots as numbers. And there are as many squares as roots. Therefore there are as many squares

²zneni galileova paradoxu

as numbers, that is to say, there are as many square numbers as there are numbers in the universe. Which is impossible. Hence it follows either that in the infinite the whole is not greater than the part, which is the opinion of Galileo and Gregory of St. Vincent, and which I cannot accept; or that infinity itself is nothing, i.e. that it is not one and not a whole.

TODO Hegel–strucne?

TODO Cantor

TODO mene teologie, vice matematiky

TODO definovat pojmy (trida etc)

TODO neni V v nejakem smyslu porad potencialni nekonecno, zatimco mnoziny vetsi nez omega jsou aktualni? nebo jsou potencialni protoze se stavaji pres indukci, od spoda.

In his work, he defined transfinite numbers to extend existing natural number structure so it contains more objects that behave like natural numbers and are based on an object (rather a meta-object) that doesn't explicitly exist in the structure, but is closely related to it. This is the first instance of reflection. This paper will focus on taking this principle a step further, extending Cantor's (or Zermelo–Fraenkel's, to be more precise) universe so it includes objects so big, they could be considered the universe itself, in a certain sense.

TODO dal asi smazat

The original idea behind reflection principles probably comes from what could be informally called “universality of the universe”. The effort to precisely describe the universe of sets was natural and could be regarded as one of the impulses for formalization of naive set theory. If we try to express the universe as a set $\{x|x = x\}$, a paradox appears, because either our set is contained in itself and therefore is contained in a set (itself again), which contradicts the intuitive notion of a universe that contains everything but is not contained itself.

TODO ???

If there is an object containing all sets, it must not be a set itself. The notion of class seems inevitable. Either directly the ways for example the Bernays–Gödel set theory, we will also discuss later in this paper, does in, or on a meta-level like the Zermelo–Fraenkel set theory, that doesn't refer to them in the axioms but often works with the notion of a universal class. duet Another obstacle of constructing a set of all sets comes from Georg Cantor, who proved that the set of all subsets of a set (let x be the set and $\mathcal{P}(x)$ its powerset) is strictly larger than x . That would turn every aspiration to

finally establish an universal set into a contradictory infinite regression.³ We will use V to denote the class of all sets. From previous thoughts we can easily argue, that it is impossible to construct a property that holds for V and no set and is neither paradoxical like $\{x|x = x\}$ nor trivial. Previous observation can be transposed to a rather naive formulation of the reflection principle:

Reflection Any property which holds in V already holds in some initial segment of V .

To avoid vagueness of the term "property", we could informally reformulate the above statement into a schema:

For every first-order formula⁴ φ holds in $V \leftrightarrow \varphi$ holds in some initial segment of V .

Interested reader should note that this is a theorem scheme rather than a single theorem.⁵

1.2 A few historical remarks on reflection

Reflection made its first in set-theoretical appearance in Gödel's proof of GCH in L (citace Kanamori ? Lévy and set theory), but it was around even earlier as a concept. Gödel himself regarded it as very close to Russel's reducibility axiom (an earlier equivalent of the axiom schema of Zermelo's separation). Richard Montague then studied reflection properties as a tool for verifying that Replacement is not finitely axiomatizable (citace?). a few years later Lévy proved in [?] the equivalence of reflection with Axiom of infinity together with Replacement in proof we shall examine closely in chapter 2.

TODO co dal? recent results?

³An intuitive analogy of this *reductio ad infinitum* is the status of ω , which was originally thought to be an unreachable absolute, only to become starting point of Cantor's hierarchy of sets growing beyond all boundaries around the end of the 19th century

⁴this also works for finite sets of formulas [?, p. 168]

⁵If there were a single theorem stating "for any formula φ that holds in V there is an initial segment of V where φ also holds", we would obtain the following contradiction with the second Gödel's theorem: In ZFC, any finite group of axioms of ZFC holds in some initial segment of the universe. If we take the largest of those initial segments it is still strictly smaller than the universe and thus we have, via compactness, constructed a model of ZFC within ZFC. That is, of course a harsh contradiction. This also leads to an elegant way to prove that ZFC is not finitely axiomatizable.

1.3 Reflection in Platonism and Structuralism

TODO cite "reflection in a structuralist setting"

TODO veci o tom, ze reflexe je ok protoze reflektuje veci ktere objektivne plati, protoze plati pro V ...

TODO souvislost s kompaktnosti, hranice formalich systemu nebo alespon ZFC

1.4 Notation and terminology

1.4.1 The Language of Set Theory

We are about to define basic set-theoretical terminology on which the rest of this thesis will be built. For Chapter 2, the underlying theory will be the *Zermelo – Fraenkel* set theory with the Axiom of Choice (ZFC), a first-order set theory in the language $\mathcal{L} = \{=, \in\}$, which will be sometimes referred to as *the language of set theory*. In Chapter 3⁶, we shall always make it clear whether we are in first-order ZFC or second-order ZFC₂, which will be precisely defined later in this chapter. When in second-order theory, we will usually denote type 1 variables, which are elements of the domain of discourse⁷ by lower-case letters, mostly $u, v, w, x, y, z, p_1, p_2, p_3, \dots$ while type 2 variables, which represent n -ary relations of the domain of discourse for any natural number n , are usually denoted by upper-case letters A, B, C, X, Y, Z . Note that those may be used both as relations and functions, see the definition of a function below.⁸

The informal notions of *class* and *property* will be used throughout this thesis. They both represent formulas with respect to the domain of discourse. If $\varphi(x, p_1, \dots, p_n)$ is a formula in the language of set theory, we call

$$A = \{x : \varphi(x, p_1, \dots, p_n)\} \quad (1.1)$$

a class of all sets satisfying $\varphi(x, p_1, \dots, p_n)$ in a sense that

$$x \in A \leftrightarrow \varphi(x, p_1, \dots, p_n) \quad (1.2)$$

One can easily define for classes A, B the operations like $A \cap B, A \cup B, A \setminus C, \bigcup A$, but it is elementary and we won't do it here, see the first part of [?] for technical details. The following axioms are the tools by which decide whether particular classes are in fact sets. A class that fails to be considered a set is called a *proper class*.

⁶TODO bude jich vic? Chapter 4 taky?

⁷co je "domain of discourse"?

⁸TODO ref?

207 1.4.2 The Axioms

208 **Definition 1.1** (*The existence of a set*)

$$\exists x(x = x) \quad (1.3)$$

209 The above axiom is usually not used because it can be deduced from the
 210 axiom of *Infinity* (see below), but since we will be using set theories that
 211 omit *Infinity*, this will be useful.

212 **Definition 1.2** (*Extensionality*)

$$\forall x, y(\forall z(z \in x \leftrightarrow z \in y) \leftrightarrow x = y) \quad (1.4)$$

213 **Definition 1.3** (*Specification*)

214 *The following is a schema for every first-order formula $\varphi(x, p_1, \dots, p_n)$ with*
 215 *no free variables other than x, p_1, \dots, p_n .*

$$\forall x, p_1, \dots, p_n \exists y \forall z(z \in y \leftrightarrow (z \in x \ \& \ \varphi(z, p_1, \dots, p_n))) \quad (1.5)$$

216 We will now provide two definitions that are not axioms, but will be
 217 helpful in establishing some of the other axioms in a more intuitive way.

218 **Definition 1.4** ($x \subseteq y, x \subset y$)

$$x \subseteq y \leftrightarrow \forall z(z \in x \rightarrow z \in y) \quad (1.6)$$

219

$$x \subset y \leftrightarrow x \subseteq y \ \& \ x \neq y \quad (1.7)$$

220 **Definition 1.5** (*Empty set*)

$$\emptyset \stackrel{\text{def}}{=} \{x : x \neq x\} \quad (1.8)$$

221 To make sure that \emptyset is a set, note that there exists at least one set y from
 222 \mathcal{V} , then consider the following alternative definition.

$$\emptyset' \stackrel{\text{def}}{=} \{x : \varphi(x) \ \& \ x \in y\} \text{ where } y \ \varphi \text{ is the formula " } x \neq x \text{ ".} \quad (1.9)$$

223 It should be clear that $\emptyset' = \emptyset$.⁹

224 Now we can introduce more axioms.

⁹For details, see page 8 in [?].

225 **Definition 1.6** (*Foundation*)

$$\forall x(x \neq \emptyset \rightarrow \exists z(z \in x \ \& \ \forall u \neg(u \in z \ \& \ u \in x))) \quad (1.10)$$

226 **Definition 1.7** (*Pairing*)

$$\forall x, y \exists z \forall q (q \in z \leftrightarrow q \in x \vee q \in y) \quad (1.11)$$

227 **Definition 1.8** (*Union*)

$$\forall x \exists y \forall z (z \in x \leftrightarrow \exists q (z \in q \ \& \ q \in x)) \quad (1.12)$$

228 **Definition 1.9** (*Powerset*)

$$\forall x \exists y \forall z (z \subseteq x \leftrightarrow z \in y) \quad (1.13)$$

229 **Definition 1.10** (*Infinity*)

$$\exists x (\forall y (y \in x) \rightarrow (y \cup \{y\} \in x)) \quad (1.14)$$

230 Let us introduce a few more definitions that will make the two remaining
231 axioms more comprehensible.

232 **Definition 1.11** (*Function*)

233 *Given arbitrary first-order formula $\varphi(x, y, p_1, \dots, p_n)$, we say that φ is a func-*
234 *tion iff*

$$\forall x, y, z, p_1, \dots, p_n (\varphi(x, y, p_1, \dots, p_n) \ \& \ \varphi(x, z, p_1, \dots, p_n) \rightarrow y = z) \quad (1.15)$$

235 When a $\varphi(x, y)$ is a function, we also write the following:

$$f(x) = y \leftrightarrow \varphi(x, y) \quad (1.16)$$

236 Note that this f is in fact a formula

237 TODO $f = \{(x, y) : \varphi(x, y)\}$!!! f muze byt mnozina i trida! ¹⁰

238 **Definition 1.12** (*Dom(f)*)

239 *Let f be a function. We read the following as " $\text{Dom}(f)$ is the domain of f ".*

$$\text{Dom}(f) \stackrel{\text{def}}{=} \{x : \exists y (f(x) = y)\} \quad (1.17)$$

240 We say " f is a function on A ", A being a class, if $A = \text{dom}(f)$.

¹⁰This can also be done for φ s with more than two free variables by either setting $f(x, p_1, \dots, p_n) = y \leftrightarrow \varphi(x, y, p_1, \dots, p_n)$

241 **Definition 1.13** (*Rng(f)*)

242 *Let f be a function. We read the following as "Rng(f) is the range of f ".*

$$Rng(f) \stackrel{\text{def}}{=} \{x : \exists y(f(x) = y)\} \quad (1.18)$$

243 We say that f is a function into A , A being a class, if $rng(f) \subseteq A$.

244 Note that $Dom(f)$ and $Rng(f)$ are not definitions in a strict sense, they
 245 are in fact definition schemas that yield definitions for every function f given.
 246 Also note that they can be easily modified for φ instead of f , with the only
 247 difference that then it is defined only for those φ s that are functions.

248 **Definition 1.14** (*Powerset*)

249 *TODO*

250 And now for the axioms.

251 **Definition 1.15** (*Replacement*)

252 *The following is a schema for every first-order formula $\varphi(x, p_1, \dots, p_n)$ with*
 253 *no free variables other than x, p_1, \dots, p_n .*

$$"\varphi \text{ is a function}" \rightarrow \forall x \exists y \forall z (z \in y \leftrightarrow (\exists q \in x)(\varphi(x, y, p_1, \dots, p_n))) \quad (1.19)$$

254 **Definition 1.16** (*Choice*)

255 *This is also a schema. For every A , a family of non-empty sets¹¹, such that*
 256 *$\emptyset \notin S$, there is a function f such that for every $x \in A$*

$$f(x) \in x \quad (1.20)$$

257 We will refer the axioms by their name, written in italic type, e.g. *Founda-*
 258 *tion* refers to the Axiom of Foundation. Now we need to define some basic
 259 set theories to be used in the article. There will be others introduced in Chap-
 260 ter 3, but those will usually be defined just by appending additional axioms
 261 or schemata to one of the following.

262 **Definition 1.17** (**S**)

263 *We call **S** a set theory with the following axioms:*

- 264 (i) Existence of a set (see ??)
- 265 (ii) Extensionality (see ??)
- 266 (iii) Specification (see ??)
- 267 (iv) Foundation (see ??)
- 268 (v) Pairing (see ??)

¹¹We say a class A is a "family of non-empty sets" iff there is B such that $A \subseteq \mathcal{P}(B)$

- 269 (vi) Union (see ??)
 270 (vii) Powerset (see ??)

271 **Definition 1.18** (ZF)

272 We call ZF a set theory that contains all the axioms of the theory S^{12} in
 273 addition to the following

- 274 (i) Replacement schema (see ??)
 275 (ii) Infinity (see ??)

276 **Definition 1.19** (ZFC)

277 ZFC is a theory that contains all the axioms of ZF plus Choice (??).

278

279 **1.4.3 The transitive universe**

280 **Definition 1.20** (Transitive class)

281 We say a class A is transitive iff

$$\forall x(x \in A \rightarrow x \subseteq A) \quad (1.21)$$

282 **Definition 1.21** Well Ordered Class A class A is said to be well ordered by
 283 \in iff the following hold:

- 284 (i) $(\forall x \in A)(x \not\subseteq x)$ (Antireflexivity)
 285 (ii) $(\forall x, y, z \in A)(x \in y \ \& \ y \in z \rightarrow x \in z)$ (Transitivity)
 286 (iii) $(\forall x, y \in A)(x = y \vee x \in y \vee y \in x)$ (Linearity)
 287 (iv) $(\forall x)(x \subseteq A \ \& \ x \neq \emptyset \rightarrow (\exists y \in x)(\forall z \in x)(z = y \vee z \in y))$

288 **Definition 1.22** (Ordinal number)

289 A set x is said to be an ordinal number, also known as an ordinal, if it is
 290 transitive and well-ordered by \in .

291 For the sake of brevity, we usually just say " x is an ordinal". Note that " x is
 292 an ordinal" is a well-defined formula, since ?? is a formula and ?? is in fact a
 293 conjunction of four formulas. Ordinals will be usually denoted by lower case
 294 greek letters, starting from the beginning: $\alpha, \beta, \gamma, \dots$. Given two different
 295 ordinals α, β , we will write $\alpha < \beta$ for $\alpha \in \beta$, see [?]Lemma 2.11 for technical
 296 details.

¹²With the exception of *Existence of a set*

297 **Definition 1.23** (*Successor Ordinal*)

298 Consider the following operation

$$\beta + 1 \stackrel{\text{def}}{=} \beta \cup \{\beta\} \quad (1.22)$$

299 An ordinal α is called a successor ordinal iff there is an ordinal β , such that
300 $\alpha = \beta + 1$

301 **Definition 1.24** (*Limit Ordinal*)

302 A non-zero ordinal α ¹³ is called a limit ordinal iff it is not a successor ordinal.

303 **Definition 1.25** (*Ord*)

304 The class of all ordinal numbers, which we will denote Ord ¹⁴ be the following
305 class:

$$\text{Ord} \stackrel{\text{def}}{=} \{x : x \text{ is an ordinal}\} \quad (1.23)$$

306 The following construction will be often referred to as the *Von Neumann's*
307 *Hierarchy*, sometimes also the *Von Neumann's Universe*.

308 **Definition 1.26** (*Von Neumann's Hierarchy*)

309 The Von Neumann's Hierarchy is a collection of sets indexed by elements of
310 Ord , defined recursively in the following way:

(i)

$$V_0 = \emptyset \quad (1.24)$$

(ii)

$$V_{\alpha+1} = \mathcal{P}(V_\alpha) \text{ for any ordinal } \alpha \quad (1.25)$$

(iii)

$$V_\lambda = \bigcup_{\beta < \lambda} V_\beta \text{ for a limit ordinal } \lambda \quad (1.26)$$

311 **Definition 1.27** (*Rank*)

312 Given a set x , we say that the rank of x (written as $\text{rank}(x)$) is the least
313 ordinal α such that

$$x \in V_{\alpha+1} \quad (1.27)$$

314 Due to *Regularity*, every set has a rank.¹⁵

315 **Definition 1.28** (ω)

316

$$\omega \stackrel{\text{def}}{=} \bigcap \{x : x \text{ is a limit ordinal}\} \quad (1.28)$$

317

¹³ $\alpha \neq \emptyset$

¹⁴It is sometimes denoted On , but we will stick to the notation in [?]

¹⁵See chapter 6 of [?] for details.

1.4.4 Cardinal numbers

Definition 1.29 (Cardinality)

Given a set x , let the cardinality of x , written $|x|$, be defined as the smallest ordinal number such that there is an injective mapping from x to α .

For formal details as well as why every set can be well-ordered assuming Choice, see [?].

Definition 1.30 (Aleph function)

Let ω be the set defined by ???. We will recursively define the function \aleph for all ordinals.

- (i) $\aleph_0 = \omega$
- (ii) $\aleph_{\alpha+1}$ is the least cardinal larger than \aleph_α ¹⁶
- (iii) $\aleph_\lambda = \bigcup_{\beta < \lambda} \aleph_\beta$ for a limit ordinal λ

Definition 1.31 (Cardinal number)

We say a set x is a cardinal number, usually called a cardinal, if either $x \in \omega$ Cardinals will be notated by lower-case greek letters starting from $\kappa, \lambda, \mu, \dots$ ¹⁷.

Definition 1.32 (Cofinality)

Let λ be a limit ordinal. The cofinality of λ , written $cf(\lambda)$, is the least limit ordinal α such that there is an increasing α -sequence¹⁸ $\langle \lambda_\beta : \beta < \alpha \rangle$ with $\lim_{\beta \rightarrow \alpha} \lambda_\beta = \lambda$.

Definition 1.33 (Limit Cardinal)

We say that a cardinal κ is a limit cardinal if

$$(\exists \alpha \in Ord)(\kappa = \aleph_\alpha) \quad (1.29)$$

Definition 1.34 (Strong Limit Cardinal)

We say that an ordinal κ is a strong limit cardinal if it is a limit cardinal and

$$\forall \alpha (\alpha \in \kappa \rightarrow \mathcal{P}(\alpha) \in \kappa) \quad (1.30)$$

Definition 1.35 (Generalised Continuum Hypothesis)

$$\aleph_{\alpha+1} = 2^{\aleph_\alpha} \quad (1.31)$$

If *GCH* holds (for example in Gödel's *L*, see chapter 3), the notions of a limit cardinal and a strong limit cardinal are equivalent.

¹⁶"The least cardinal larger than \aleph_α " is sometimes notated as \aleph_α^+

¹⁷ λ is also sometimes used for limit ordinals, the distinction should be clear from the context.

¹⁸TODO def α -sequence

1.4.5 Relativisation

Definition 1.36 (Relativization)

Let M be a class, R a binary relation on M and let $\varphi(p_1, \dots, p_n)$ be a first-order formula with n parameters. The relativization of φ to M and R is the formula, written as $\varphi^{M,R}(p_1, \dots, p_n)$, defined in the following inductive manner:

- (i) $(x \in y)^{M,R} \leftrightarrow R(x, y)$
- (ii) $(x = y)^{M,R} \leftrightarrow x = y$
- (iii) $(\neg \varphi)^{M,R} \leftrightarrow \neg \varphi^{M,R}$
- (iv) $(\varphi \ \& \ \psi)^{M,R} \leftrightarrow \varphi^{M,R} \ \& \ \psi^{M,R}$
- (v) $(\exists x \varphi)^{M,R} \leftrightarrow (\exists x \in M) \varphi^{M,R}$

1.4.6 Higher-Order Logic

Since we will utilise some basic tools of set theories formalized in second- and occasionally higher-order logic, we need to establish the basics here. This part is heavily inspired by Preliminaries from [?].

TODO viz kanamori p. 6

TODO proc se neda formalizovat obecne splnovani ve vyssich radech? cite?

While higher-order satisfaction relation for proper classes is unformalizable¹⁹, we can formalize satisfaction in a structure. For the rest of this chapter, let D be a domain of such structure.

TODO druhoradove splnovani?

Definition 1.37 (Hierarchy of formulas)

Let φ be a formula. $((v \text{ logice radu } n)) \Pi_n^m$ und Σ_n^m

Lemma 1.38 Δ_0 formulas are absolute in transitive sets, in other words, let φ be a first-order Δ_0 formula and let M be a transitive class.

$$\varphi \leftrightarrow \varphi^M \quad (1.32)$$

Definition 1.39 (ZFC₂)

TODO ?

TODO nenechat do patricne kapitoly? asi jo.

¹⁹TODO CITE KDE? Tarski nebo tak neco?

³⁷⁹ **Definition 1.40** (Reflection_1)

³⁸⁰

$$ASD \tag{1.33}$$

³⁸¹

2 Levy's first-order reflection

2.1 Lévy's Original Paper

This section will try to present Lévy's proof of a general reflection principle being equivalent to *Replacement* and *Infinity* under ZF minus *Replacement* and *Infinity* from his 1960 paper *Axiom Schemata of Strong Infinity in Axiomatic Set Theory*²⁰. When reading said article, one should bear in mind that it was written in a period when set theory was semantically oriented, so while there are many statements about a model of ZF, usually denoted u , this is equivalent to today's universal class V , so it doesn't necessarily mean that there is a set u that is a model of ZF. We will review the notion of a standard complete model used by Lévy throughout the paper in a moment. Let's first say that the set theory ZF was formulated in the "non-simple applied first order functional calculus", is

TODO viz church nebo tak neco.

The axioms are equivalent to those defined in ??, except for the *Axiom of Subsets*, which is just a different name for *Specification*. Besides ZF, the theories Z, S, and SF are used in the text. Z is ZF minus replacement, S is ZF minus *Replacement* and *Infinity*, and finally SF is ZF minus *Infinity*. The axiom of *Subsets* is an older name for the axiom scheme of specification (and it's not a scheme since we are now working in second order logic). Also note that universal quantifier does not appear, $\forall x\varphi(x)$ would be written as $(x)\varphi(x)$, the symbol for negation is " \sim ", implication is written as " \supset " and equivalence is " \equiv ", we will use " \neg ", " \rightarrow " and " \leftrightarrow ".

Note that this chapter uses ZF instead of the usual ZFC, unless explicitly stated otherwise.

The follow

Next two definitions are not used in contemporary set theory, but they illustrate 1960's set theory mind-set and they are used heavily in Lévy's text, so we will include and explain them for clarity. Generally, in this chapter, Q stands for an arbitrary axiomatic set theory used for general definitions, u is usually a model of Q, counterpart of today's V .

Definition 2.1 (*Standard model of a set theory*)

Let Q be a axiomatic set theory in first-order logic. We say the the a class u is a standard model of Q with respect to a membership relation E , written as $Sm^Q(u)$, iff both of the following hold

(i) $(x, y) \in E \leftrightarrow y \in u \ \& \ x \in y$

(ii) $y \in u \ \& \ x \in y \rightarrow x \in u$

²⁰[?]

419 **Definition 2.2** *Standard complete model of a set theory*

420 *Let Q and E be like in ???. We say that that u is a standard complete model*
 421 *of Q with respect to a membership relation E iff both of the following hold*

422 *(i) u is a transitive set with respect to \in*

423 *(ii) $\forall E((x, y) \in E \leftrightarrow (y \in u \ \& \ x \in y) \ \& \ Sm^Q(u, E))$*

424 *this is written as $Scm^Q(u)$.*

425 **Definition 2.3** *(Inaccessible cardinal with respect to Q)*

426 *Let Q be an axiomatic first-order set theory. We say that a cardinal κ is*
 427 *inaccessible with respect to Q , we write $In^Q(\kappa)$, iff*

$$Scm^Q(V_\kappa). \quad (2.34)$$

428 **Definition 2.4** *(Inaccessible cardinal with respect to ZF)*

429 *When a cardinal κ is inaccessible with respect to ZF , we only say that it is*
 430 *inaccessible. In the abbreviated version, we just leave out the superscript.*

$$In(\kappa) \leftrightarrow In^{ZF}(\kappa) \quad (2.35)$$

431 **Definition 2.5** *(N)*

432 *The following is an axiom schema of complete reflection over ZF , denoted as*
 433 *N .*

$$N \leftrightarrow \exists u(Scm^{ZF}(u) \ \& \ \forall x_1, \dots, x_n(x_1, \dots, x_n \in u \rightarrow \varphi \leftrightarrow \varphi^u)) \quad (2.36)$$

434 *where φ is a formula which contains no free variables except for x_1, \dots, x_n .*

435 **Definition 2.6** *(N_0)*

436 *If we substitute ZF for S , which is ZF minus Replacement and Infinity, we*
 437 *obtain what will now be called N_0 .*

$$N_0 \leftrightarrow \exists u(Scm^S(u) \ \& \ \forall x_1, \dots, x_n(x_1, \dots, x_n \in u \rightarrow \varphi \leftrightarrow \varphi^u)) \quad (2.37)$$

438 *where φ is a formula which contains no free variables except for x_1, \dots, x_n .*

439 Once we have established the definitions, it's time to prove something
 440 interesting.

441 **2.2** $S \models (N_0 \leftrightarrow \text{Replacement} \ \& \ \text{Infinity})$

442 Let N_0 be defined as in ??, for *Infinity* see ??.

443 **Theorem 2.7** *In S , the schema N_0 implies Infinity.*

444 *Proof.* For any φ , N_0 gives us $\exists u Scm^S(u)$, which means that there is a set u
 445 that is identical to V_α for some alpha, so $\exists \alpha Scm^S(V_\alpha)$. We don't know the
 446 exact size of this α , but we know that $\alpha \geq \omega$, otherwise α would be finite,
 447 therefore not closed under the powerset operation, which would contradict
 448 *Powerset*. In order to prove that it is a model of S , we would need to verify
 449 all axioms of S . We have already shown that ω is closed under the powerset
 450 operation. Foundation, extensionality and comprehension are clear from the
 451 fact that we work in ZF^{21} , pairing is clear from the fact, that given two sets
 452 x, y , they have ranks α, β , without loss of generality we can assume that
 453 $\alpha \leq \beta$, which means that $x \in V_\alpha \in V_\beta$, therefore V_β is a set that satisfies the
 454 pairing axiom: it contains both x and B .

455 Note that this implies that any (strong) limit cardinal is a model of S .

456 We now want to prove that V_α leads to existence of an inductive set,
 457 which is a set that satisfies $\exists A(\emptyset \in A \& \forall x \in A ((x \cup \{x\}) \in A))$. If we can
 458 find a way to construct V_ω from any V_α satisfying $\alpha \geq \omega$, we are done. Since
 459 ω is the least limit ordinal, all we need is the following

$$\bigcap \{V_\kappa \mid \forall \lambda (\lambda < \kappa \rightarrow \exists \mu (\lambda < \mu < \kappa))\} \quad (2.38)$$

460 because V_κ is a transitive set for every κ , thus the intersection is non-empty
 461 unless empty set satisfies the property or the set of V_κ s is itself empty. \square

462

463 Let N_0 be defined as in ??, for *Replacement* see ??.

464 **Theorem 2.8** *In S , the schema N_0 implies Replacement.*

465 *Proof.* Let $\varphi(v, w, x_1, \dots, x_n)$ be a formula with no free variables except
 466 v, w, x_1, \dots, x_n where n is any natural number. Let χ be an instance of
 467 *Replacement* schema for this φ which is what we want to prove:

$$\begin{aligned} \chi = & \forall r, s, t (\varphi(r, s, x_1, \dots, x_n) \& \varphi(r, t, x_1, \dots, x_n) \rightarrow s = t) \\ & \rightarrow \forall x \exists y \forall w (w \in y \leftrightarrow \exists v (v \in x \& \varphi(v, w, x_1, \dots, x_n))) \end{aligned} \quad (2.39)$$

468 We can deduce the following from N_0 :

- 469 (i) $x_1, \dots, x_n, v, w \in u \rightarrow (\varphi \leftrightarrow \varphi^u)$
- 470 (ii) $x_1, \dots, x_n, v \in u \rightarrow (\exists w \varphi \leftrightarrow (\exists w \varphi)^u)$
- 471 (iii) $x_1, \dots, x_n, x \in u \rightarrow (\chi \leftrightarrow \chi^u)$
- 472 (iv) $\forall x_1, \dots, x_n \forall x (\chi \leftrightarrow (\forall x_1, \dots, x_n \forall x \chi)^u)$

²¹We only need to verify axioms that provide means of constructing larger sets from smaller to make sure they don't exceed ω . Since ω is an initial segment of ZF , the axiom scheme of specification can't be broken, the same holds for foundation and extensionality.

It is easy to see that (i), (ii), (iii) are the instances of N_0 for φ , $\exists w\varphi$ and χ respectively. From relativization we also know that $(\exists w\varphi)^u$ is equivalent to $\exists w(w \in u \& \varphi^u)$. Therefore (ii) is equivalent to

$$x_1, \dots, x_n, v \in u \rightarrow (\exists w(w \in u \& \varphi^u)). \quad (2.40)$$

If φ is a function²², then for every $x \in u$, which is also $x \subset u$ by the transitivity of $Scm^S(u)$, it maps elements of x onto u . From the axiom scheme of comprehension²³, we can find y , a set of all images of elements of x . That gives us $x_1, \dots, x_n, x \in u \rightarrow \chi$. By (iii) we get $x_1, \dots, x_n, x \in u \rightarrow \chi^u$, the universal closure of this formula is $(\forall x_1, \dots, x_n \forall x \chi)^u$, which together with (iv) yields $\forall x_1, \dots, x_n \forall x \chi$. By the means of specification we end up with χ , Q.E.D. \square

What we have just proven is just a single theorem from said article, we will introduce other interesting propositions, mostly related to the existence of large cardinals, later in their appropriate context in chapter 3.

2.3 Contemporary restatement

We will now prove what is also Lévy's reflection theorem, but a little stronger, rephrased with more up to date set theory. The main difference is, that while Lévy reflects φ from V into a set u that is a "standard complete model of S "²⁴, we say that there is a V_α that reflects φ . In other words, we don't need α to be an inaccessible cardinal like Lévy does.

We will prove the equivalence of N_0 with *Replacement* and *Infinity* in S in two parts. First, we will show that *Reflection*₁ is a theorem of ZF , then the second implication which proves *Infinity* and *Replacement* from N_0 .

The following lemma is usually done in more parts, the first being with one formula and the other with n . We will only state and prove the generalised version for n formulas, knowing that $n = 1$ is just a specific case and the proof is exactly the same.

Lemma 2.9 *Let $\varphi_1, \dots, \varphi_n$ be formulas with m parameters*²⁵.

²² $\forall r, s, t(\varphi(r, s) \& \varphi(r, t) \rightarrow r = t)$

²³Lévy's uses its equivalent, axiom of subsets

²⁴Any limit ordinal is in fact a model of S , we shall pay more attention to that in a moment.

²⁵For formulas with a different number of parameters, take for m the highest number of parameters among given formulas. Add spare parameters to the other formulas so that x remains the last parameter. That can be done in a following manner: Let φ'_i be the a formula with k parameters, $k < m$. Let us set $\varphi_i(u_1, \dots, u_{m-1}, x) = \varphi'_i(u_1, \dots, u_{k-1}, u_k, \dots, u_{m-1}, x)$, notice that u_k, \dots, u_{m-1} are the aforementioned spare variables.

500 (i) For each set M_0 there is such M that $M_0 \subset M$ and the following holds
 501 for every $i \leq n$:

$$\exists x \varphi_i(u_1, \dots, u_{m-1}, x) \rightarrow (\exists x \in M) \varphi_i(u_1, \dots, u_{m-1}, x) \quad (2.41)$$

502 for every $u_1, \dots, u_{m-1} \in M$.

503 (ii) Furthermore there is an ordinal α such that $M_0 \subset V_\alpha$ and the following
 504 holds for each $i \leq n$:

$$\exists x \varphi_i(u_1, \dots, u_{m-1}, x) \rightarrow (\exists x \in V_\alpha) \varphi_i(u_1, \dots, u_{m-1}, x) \quad (2.42)$$

505 for every $u_1, \dots, u_{m-1} \in M$.

506 (iii) Assuming Choice, there is M , $M_0 \subset M$ such that ?? holds for every
 507 M , $i \leq n$ and $|M| \leq |M_0| \cdot \aleph_0$.

508 *Proof.* We will simultaneously prove statements (i) and (ii), denoting M^T
 509 the transitive set required by part (ii). Unless explicitly stated otherwise for
 510 specific steps, it is thought to be equivalent to M .

511 Let us first define operation $H(u_1, \dots, u_{m-1})$ that gives us the set of
 512 x 's with minimal rank satisfying $\varphi_i(u_1, \dots, u_{m-1}, x)$ for given parameters
 513 u_1, \dots, u_{m-1} for every $i \leq n$.

$$H_i(u_1, \dots, u_n) = \{x \in C_i : (\forall z \in C)(\text{rank}(x) \leq \text{rank}(z))\} \quad (2.43)$$

514 for each $i \leq n$, where

$$C_i = \{x : \varphi_i(u_1, \dots, u_{m-1}, x)\} \text{ for } i \leq n \quad (2.44)$$

515

516 Next, let's construct M from given M_0 by induction.

$$M_{i+1} = M_i \cup \bigcup_{j=0}^n \{H_j(u_1, \dots, u_{m-1}) : u_1, \dots, u_{m-1} \in M_i\} \quad (2.45)$$

517 In other words, in each step we add the elements satisfying $\varphi(u_1, \dots, u_{m-1}, x)$
 518 for all parameters that were either available earlier or were added in the
 519 previous step. For statement (ii), this is the only part that differs from (i).
 520 Let us take for each step transitive closure of M_{i+1} from (i). In other words,
 521 let γ be the smallest ordinal such that

$$(M_i^T \cup \bigcup_{j=0}^n \{H_j(u_1, \dots, u_{m-1}) : u_1, \dots, u_{m-1} \in M_i\}) \subset V_\gamma \quad (2.46)$$

522 Then the incremental step is like so:

$$M_{i+1}^T = V_\gamma \quad (2.47)$$

523 The final M is obtained by joining all incremental steps together.

$$M = \bigcup_{i=0}^{\infty} M_i, \quad M^T = \bigcup_{i=0}^{\infty} M_i^T \quad (2.48)$$

524
525 We have yet to finish part (iii). Let's try to construct a set M' that
526 satisfies the same conditions like M but is kept as small as possible. Assuming
527 the Axiom of Choice, we can modify the process so that cardinality of M' is
528 at most $|M_0| \cdot \aleph_0$. Note that the size of M' is determined by the size of M_0 and,
529 most importantly, by the size of $H_i(u_1, \dots, u_{m-1})$ for any $i \leq n$ in individual
530 levels of the construction. Since the lemma only states existence of some x
531 that satisfies $\varphi_i(u_1, \dots, u_{m-1}, x)$ for any $i \leq n$, we only need to add one x for
532 every set of parameters but $H_i(u_1, \dots, u_{m-1})$ can be arbitrarily large. Since
533 Axiom of Choice ensures that there is a choice function, let F be a choice
534 function on $\mathcal{P}(M')$. Also let $h_i(u_1, \dots, u_{m-1}) = F(H_i(u_1, \dots, u_{m-1}))$ for
535 $i \leq n$, which means that h is a function that outputs an x that satisfies
536 $\varphi_i(u_1, \dots, u_{m-1}, x)$ for $i \leq n$ and has minimal rank among all such witnesses.
537 The induction step needs to be redefined to

$$M'_{i+1} = M'_i \cup \bigcup_{j=0}^n \{H_j(u_1, \dots, u_{m-1}) : u_1, \dots, u_{m-1} \in M'_i\} \quad (2.49)$$

538 In every step, the amount of elements added in M'_{i+1} is equivalent to the
539 amount of sets of parameters the yielded elements not included in M'_i . So
540 the cardinality of M'_{i+1} exceeds the cardinality of M'_i only for finite M'_i . It
541 is easy to see that if M_0 is finite, M' is countable because it was built from
542 countable union of finite sets. If M_0 is countable or larger, cardinality of M'
543 is equal to the cardinality of M_0 .²⁶ Therefore $|M'| \leq |M_0| \cdot \aleph_0$ \square

544 And now for the theorem itself

545 **Theorem 2.10** (*Lévy's first-order reflection theorem*)

546 Let $\varphi(x_1, \dots, x_n)$ be a first-order formula.

547 (i) For every set M_0 there exists M such that $M_0 \subset M$ and the following
548 holds:

$$\varphi^M(x_1, \dots, x_n) \leftrightarrow \varphi(x_1, \dots, x_n) \quad (2.50)$$

549 for every $x_1, \dots, x_n \in M$.

²⁶It can not be smaller because $|M'_{i+1}| \geq |M'_i|$ for every i . It may not be significantly larger because the maximum of elements added is the number of n -tuples in M'_i , which is of the same cardinality is M'_i .

550 (ii) For every set M_0 there is a transitive set M , $M_0 \subset M$ such that the
 551 following holds:

$$\varphi^M(x_1, \dots, x_n) \leftrightarrow \varphi(x_1, \dots, x_n) \quad (2.51)$$

552 for every $x_1, \dots, x_n \in M$.

553 (iii) For every set M_0 there is α such that $M_0 \subset V_\alpha$ and the following holds:

$$\varphi^{V_\alpha}(x_1, \dots, x_n) \leftrightarrow \varphi(x_1, \dots, x_n) \quad (2.52)$$

554 for every $x_1, \dots, x_n \in M$.

555 (iv) Assuming Choice, for every set M_0 there is M such that $M_0 \subset M$ and
 556 $|M| \leq |M_0| \cdot \aleph_0$ and the following holds:

$$\varphi^M(x_1, \dots, x_n) \leftrightarrow \varphi(x_1, \dots, x_n) \quad (2.53)$$

557 for every $x_1, \dots, x_n \in M$.

558 *Proof.* Let's prove (i) for one formula φ via induction by complexity first.
 559 We can safely assume that φ contains no quantifiers besides \exists and no logical
 560 connectives other than \neg and $\&$. Assume that this M is obtained from
 561 lemma ???. The fact, that atomic formulas are reflected in every M comes
 562 directly from definition of relativization and the fact that they contain no
 563 quantifiers.²⁷ The same holds for formulas in the form of $\varphi = \neg\varphi'$. Let us
 564 recall the definition of relativization for those formulas in ???.

$$(\neg\varphi_1)^M \leftrightarrow \neg(\varphi_1^M) \quad (2.54)$$

565 Because we can assume from induction that $\varphi'^M \leftrightarrow \varphi'$, the following holds:

$$(\neg\varphi')^M \leftrightarrow \neg(\varphi'^M) \leftrightarrow \neg\varphi' \quad (2.55)$$

566 The same holds for $\varphi = \varphi_1 \& \varphi_2$. From the induction hypothesis we know
 567 that $\varphi_1^M \leftrightarrow \varphi_1$ and $\varphi_2^M \leftrightarrow \varphi_2$, which together with relativization for formulas
 568 in the form of $\varphi_1 \& \varphi_2$ gives us

$$(\varphi_1 \& \varphi_2)^M \leftrightarrow \varphi_1^M \& \varphi_2^M \leftrightarrow \varphi_1 \& \varphi_2 \quad (2.56)$$

569

570 Let's now examine the case when from the induction hypothesis, M re-
 571 flects $\varphi'(u_1, \dots, u_n, x)$ and we are interested in $\varphi = \exists x\varphi'(u_1, \dots, u_n, x)$. The
 572 induction hypothesis tells us that

$$\varphi'^M(u_1, \dots, u_n, x) \leftrightarrow \varphi'(u_1, \dots, u_n, x) \quad (2.57)$$

²⁷Note that this does not hold generally for relativizations to M, E , but only for relativization to M, \in , which is our case.

so, together with above lemma ??, the following holds:

$$\begin{aligned}
& \varphi(u_1, \dots, u_n, x) \\
& \leftrightarrow \exists x \varphi'(u_1, \dots, u_n, x) \\
& \leftrightarrow (\exists x \in M) \varphi'(u_1, \dots, u_n, x) \\
& \leftrightarrow (\exists x \in M) \varphi'^M(u_1, \dots, u_n, x) \\
& \leftrightarrow (\exists x \varphi'(u_1, \dots, u_n, x))^M \\
& \leftrightarrow \varphi^M(u_1, \dots, u_n, x)
\end{aligned} \tag{2.58}$$

Which is what we have needed to prove:

So far we have proven part (i) of this theorem for one formula φ , we only need to verify that the same holds for any finite number of formulas. This has in fact been already done since lemma ?? gives us M for any (finite) amount of formulas. We can then use the induction above to verify that it reflects each of the formulas individually.

Now we want to verify other parts of our theorem. Since V_α is a transitive set, by proving (iii) we also satisfy (ii). To do so, we only need to look at part (ii) of lemma ?. All of the above proof also holds for $M = V_\alpha$.

To finish part (iv), we take M of size $\leq |M_0| \cdot \aleph_0$, which exists due to part (iii) of lemma ?, the rest being identical. \square

Theorem 2.11 *Reflection is equivalent to Infinity & Replacement under ZFC minus Infinity & Replacement*

Proof. Since ?? already gives one side of the implication, we are only interested in showing the converse which we shall do in two parts:

Reflection \rightarrow Infinity

Let us first find a formula to be reflected that requires a set M at least as large as V_ω . Let us consider the following formula:

$$\varphi'(x) = \forall \lambda (\lambda < x \rightarrow \exists \mu (\lambda < \mu < x)) \tag{2.59}$$

Because φ says "there is a limit ordinal", if it holds for some x , the *Infinity* axiom is very easy to satisfy. But we know that there are limit ordinals in ZF, therefore $\varphi = \exists x \varphi'(x)$ is a valid statement. *Reflection* then gives us a set M in which φ^M holds, that is, a set that contains a limit ordinal. So the set

of off limit ordinals is non-empty and because ordinals are well-founded, it has a minimal element. Let's call it μ .

$$\mu = \bigcap \{V_\kappa : \forall \lambda (\lambda < \kappa \rightarrow \exists \mu (\lambda < \mu < \kappa))\} \quad (2.60)$$

We can see that μ is the least limit ordinal and therefore it satisfies *Infinity*.

Reflection \rightarrow Replacement

Given a formula $\varphi(x, y, u_1, \dots, u_n)$, we can suppose that it is reflected in any M ²⁸ What we want to obtain is the following:

$$\forall x, y, z (\varphi(x, y, u_1, \dots, u_n) \& \varphi(x, z, u_1, \dots, u_n) \rightarrow y = z) \rightarrow \quad (2.61)$$

$$\rightarrow \forall X \exists Y \forall y (y \in Y \leftrightarrow \exists x (\varphi(x, y, u_1, \dots, u_n) \& x \in X)) \quad (2.62)$$

We do also know that $x, y \in M$, in other words for every $X, Y = \{y \mid \varphi(x, y, u_1, \dots, u_n)\}$ we know that $X \subset M$ and $Y \subset M$, which, together with the comprehension schema²⁹ implies that Y , the image of X over φ , is a set. Which is exactly the *Replacement* Schema we hoped to obtain. \square

We have shown that *Reflection* for first-order formulas, *Reflection*₁ is a theorem of ZF, which means that it won't yield us any large cardinals. We have also shown that it can be used instead of the *Infinity* and *Replacement* scheme, but ZF + *Reflection*₁ is a conservative extension of ZF. Besides being a starting point for more general and powerful statements, it can be used to show that ZF is not finitely axiomatizable. That is because *Reflection* gives a model to any finite number of (consistent) formulas. So if $\varphi_1, \dots, \varphi_n$ for any finite n would be the axioms of ZF, *Reflection* would always contain a model of itself, which would in turn contradict the Second Gödel's Theorem³⁰. Notice that, in a way, reflection is complementary to compactness. Compactness argues that given a set of sentences, if every finite subset yields a model, so does the whole set. Reflection, on the other hand, says that while the whole set has no model in the underlying theory, every finite subset does have one.

Also, notice how reflection can be used in ways similar to upward Löwenheim-Skolem theorem. Since Reflection extends any set M_0 into a model of given formulas $\varphi_1, \dots, \varphi_n$, we can choose the lower bound of the size of M by appropriately choosing M_0 .

In the next section, we will try to generalize *Reflection* in a way that transcends ZF and finally yields some large cardinals.

²⁸Which means that for $x, y, u_1, \dots, u_n \in M$, $\varphi^M(x, y, u_1, \dots, u_n) \leftrightarrow \varphi(x, y, u_1, \dots, u_n)$.

²⁹Called the axiom of subsets in Lévy's proof.

³⁰See chapter ?? for further details.

3 Reflection And Large Cardinals

In this chapter we aim to examine stronger reflection properties in order to reach cardinals unavailable in ZFC. Like we said in the first chapter, the variety of reflection principles comes from the fact that there are many way to formalize "properties of the universal class". It is not always obvious what properties hold for V because, We have shown that reflecting properties as first-order formulas doesn't allow us to leave ZFC. We will broaden the class of admissible properties to be reflected and see whether there is a natural limit in the height or width on the reflected universe and also see that no matter how far we go, the universal class is still as elusive as it is when seen from S . That is because for every process for obtaining larger sets such as for example the powerset operation in ZFC, this process can't reach V and thus, from reflection, there is an initial segment of V that can't be reached via said process.

To see why this is important, let's dedicate a few lines to the intuition behind the notions of limitness, regularity and inaccessibility in a manner strongly influenced by [?]. To see why limit and strongly limit cardinals are worth mentioning, note that they are "limit" not only in a sense of being a supremum of an ordinal sequence, they also show that a certain way of obtaining bigger objects from smaller ones limited in terms of possibilities. \aleph_λ is a limit cardinal iff there is no α such that $\aleph_{\alpha+1} = \aleph_\lambda$. Strongly limit cardinals point to the limits of the powerset operation. It has been too obvious so far, so let's look at the regular cardinals in this manner. Regular cardinals are those that cannot be³¹, expressed as a supremum of smaller amount of smaller objects³². More precisely, κ is regular if there is no way to define it as u union of less than κ ordinals, all smaller than κ . So unless there already is a set of size κ , *Replacement* is useless in determining whether κ is really a set. Note that assuming *Choice*, successor cardinals are always regular, so most³³ limit cardinals are singular cardinals. So if one is traversing the class of all cardinals upwards, successor steps are still sets thanks to the powerset axiom while singular limits cardinal are not proper classes because they are images of smaller sets via *Replacement*. Regular cardinals are, in a way, limits of how far can we get by taking limits of increasing sequences of ordinals obtained via *Replacement*.

That all being said, it is easy to see that no cardinals in ZFC are both strongly limit and regular because there is no way to ensure they are sets and

³¹Assuming *Choice*.

³²Just like ω can not be expressed as a supremum of a finite set consisting solely of finite numbers.

³³All provable to exist in ZFC

not proper classes in ZFC. The only exception to this rule is \aleph_0 which needs
Infinity to exist. It should now be obvious why the fact that κ is inaccessible
implies that $\kappa = \aleph_\kappa$.³⁴

We will also examine the connection between reflection principles and
(regular) fixed points of ordinal functions in a manner proposed by Lévy in
[?]. We will also see that, like Lévy has proposed in the same paper, there is
a meaningful way to extend the relation between S and ZFC into a hierarchy
of stronger axiomatic set theories.

3.1 Regular Fixed-Point Axioms

Lévy's article mentions various schemata that are not instances of reflection
themselves. We will mention them because they are equivalent to N_0 and
because they are fixed-point theorems, which we will find useful later in this
thesis.

Definition 3.1 (*Function*) We say that a first-order formula $\varphi(x, y, u_1, \dots, u_n)$
with no free variable besides x, y, u_1, \dots, u_n is a function iff

$$\forall x, y, z (\varphi(x, y, u_1, \dots, u_n) \ \& \ \varphi(x, z, u_1, \dots, u_n) \rightarrow y = z) \quad (3.63)$$

We will also write functions in the form of " $f(x) = y$ ". This is defined for
given $\varphi(x, y, u_1, \dots, u_n)$ and given terms t_1, \dots, t_n as follows

$$f(x) = y \leftrightarrow \varphi(x, y, t_1, \dots, t_n) \quad (3.64)$$

Ord denotes the class of all ordinal numbers.

Definition 3.2 (*Strictly increasing function*)

A function $f : Ord \rightarrow Ord$ is said to be strictly increasing iff

$$\forall \alpha, \beta \in Ord (\alpha < \beta \rightarrow f(\alpha) < f(\beta)). \quad (3.65)$$

Definition 3.3 (*Continuous function*)

A function $f : Ord \rightarrow Ord$ is said to be continuous iff

$$\alpha \text{ is limit} \rightarrow (f(\alpha) = \lim_{\beta < \alpha} f(\beta)). \quad (3.66)$$

Alternatively, a function $f : Ord \rightarrow Ord$ is continuous iff for limit λ , $f(\lambda) =$
 $\bigcup_{\alpha < \lambda} f(\alpha)$.

³⁴This doesn't work backwards, the least fixed point of the \aleph function is the limit of
 $\{\aleph_0, \aleph_{\aleph_0}, \aleph_{\aleph_{\aleph_0}}, \dots\}$, it is singular since the sequence has countably many elements.

693 **Definition 3.4** (*Normal function*)

694 A function $f : Ord \rightarrow Ord$ is said to be normal if it is strictly increasing
695 and continuous.

696 **Definition 3.5** (*Normal function on a set*) Let α, δ be ordinals. A function
697 $f : \delta \rightarrow \alpha$ is called a normal function on α iff all of the following hold:

- 698 (i) f is strictly increasing on α ³⁵
699 (ii) f is continuous on α
700 (iii) the $rng(f) = \{y : \exists x(f(x) = y)\}$ is unbounded in α .

701 **Definition 3.6** *Fixed point*

702 We say α is a fixed point of ordinal function f when $\alpha = f(\alpha)$.

703 Lévy ([?]) proposes those axioms as equivalent to one on his reflection
704 principles.

705 **Definition 3.7** $M \leftrightarrow$ "Every normal function defined for all ordinals has at
706 least one inaccessible number in its range."

707 We will rewrite M as a formula to make it clear that it is an axiom scheme
708 and the same can be done with M' as well as M'' .

709 Let $\varphi(x, y, u_1, \dots, u_n)$ be a first-order formula with no free variables be-
710 sides x, y, u_1, \dots, u_n . The following is equivalent to M .

$$\varphi \text{ is a normal function } \& \forall x(x \in Ord \rightarrow \exists y(\varphi(x, y, u_1, \dots, u_n))) \rightarrow \rightarrow \exists y(\exists x \varphi(x, y, u_1, \dots, u_n) \& \dots) \quad (3.67)$$

711 **Definition 3.8** $M' \leftrightarrow$ "Every normal function defined for all ordinals has
712 at least one fixed point which is inaccessible."

713 **Definition 3.9** $M'' \leftrightarrow$ "Every normal function defined for all ordinals has
714 arbitrarily great fixed points which are inaccessible."

715 The following axiom is proposed by Drake in [?].

716 **Definition 3.10** F Every normal function for all ordinals has a regular fixed
717 point.

Theorem 3.11

$$F \leftrightarrow M \leftrightarrow M' \leftrightarrow M'' \quad (3.68)$$

718 *Proof.* One can find the proof of $M \leftrightarrow M' \leftrightarrow M''$ in [?], Theorem 1.

719 TODO podle Levyho

720

□

³⁵ x is limit $\rightarrow (f(x)) = \bigcup_{y < x} f(y)$

3.2 A Model-Theoretic Intermezzo

This is a small notational intermezzo. Reflection theorems asdasd Tarski Berkley model-theoretic methods in set theory.

This notation is used for example in [?].

TODO def $\langle V_\kappa, \in, R \rangle \models \text{asdf } Replacement$

TODO $S \rightarrow ZM \rightarrow ZM' \rightarrow ZM''$, neco jako mahlovy kardinaly, pre-sunout do dane kapitoly

3.3 Reflecting Second-order Formulas

To see that there is a way to transcend ZFC, let us briefly show how a model of ZFC can be obtained in $ZFC_2 + \text{"second-order reflection"}^{36}$. This will be more closely examined in section ??.

We know that ZFC can not be finitely axiomatized in first-order formulas, however if *Replacement* and *Comprehension* schemes can be substituted by second-order formulas, ZFC becomes ZFC_2 , which is finitely axiomatizable in second-order logic. Therefore if we take second-order reflection into consideration, we can obtain a set M that is a model of ZFC_2 . For now, we have left out the details of how exactly is first-order reflection generalised into stronger statements and how second-order axiomatization of ZFC looks like as we will examine those problems closely in the following pages.

Lower-case letters represent first-order variables and upper-case P represents a second-order variable. [?]

Definition 3.12 ($Replacement_2$)

$$\begin{aligned} \forall P(\forall x, y, z(P(x, y) \& P(x, z) \rightarrow y = z) \rightarrow \\ \rightarrow (\forall x \exists y \forall z(z \in y \leftrightarrow \exists w \in x(P(w, z)))) \end{aligned} \quad (3.69)$$

We will denote this axiom $Replacement_2$.

Definition 3.13 ($Specification_2$)

$$\forall P \forall x \exists y \forall z (z \in y \leftrightarrow (z \in x \& P(z, x))) \quad (3.70)$$

Definition 3.14 (ZFC_2)

Let ZFC_2 be a theory with all axioms identical with the axioms of ZFC with the exception of *Replacement* and *Specification* schemes, which are replaced with $Replacement_2$ and $Specification_2$ respectively.

³⁶ ZFC_2 is an axiomatization of ZFC in second-order formulas, to be more rigorously established later.

3.4 Inaccessibility

Definition 3.15 (*limit cardinal*) κ is a limit cardinal iff it is \aleph_α for some limit ordinal α .

Definition 3.16 (*strong limit cardinal*) κ is a strong limit cardinal iff it is a limit cardinal and for every $\lambda < \kappa$, $2^\lambda < \kappa$

The two above definition become equivalent when we assume *GCH*.

Definition 3.17 (*weak inaccessibility*) An uncountable cardinal κ is weakly inaccessible iff it is regular and limit.

Definition 3.18 (*inaccessibility*) An uncountable cardinal κ is inaccessible (written $In(\alpha)$) iff it is regular and strongly limit.

We will now show that the above notion is equivalent to the definition Lévy uses in [?], which is, in more contemporary notation, the following:

Theorem 3.19 *The following are equivalent:*

1. κ is inaccessible
2. $\langle V_\kappa, \in \rangle \models \text{ZFC}$

Proof. Let's first prove that if κ is inaccessible, it is a model of ZFC. We will do that by verifying the axioms of ZFC just like Kanamori does it in [?, 1.2] and Drake in [?, Chapter 4].

- (i) *Extensionality*:
(see ??)

$$V_\kappa \models \forall x, y (\forall z (z \in x \leftrightarrow z \in y) \rightarrow x = y) \quad (3.71)$$

We need to prove that, given two sets that are equal in V , they are equal in V_κ , in other words, that the *Extensionality* formula is reflected, that is

$$V_\kappa \models \forall x, y \in V_\kappa (\forall z \in V_\kappa (z \in x \leftrightarrow z \in y) \rightarrow x = y) \quad (3.72)$$

But that comes from transitivity. If x and y are in V_κ their members are also in V_κ .

- (ii) *Foundation*:
(see ??)

$$V_\kappa \models \forall x (\exists z (z \in x) \rightarrow \exists z (z \in x \ \& \ \forall u \neg (u \in z \ \& \ u \in x))) \quad (3.73)$$

The argument for *Foundation* is almost identical to the one for *Extensionality*. For any set $x \in V_\kappa$, transitivity of V_κ makes sure that every element of x is also an element of V_κ and the same holds for the elements of elements of x et cetera. So statements about those elements are absolute between any transitive structures. V and V_κ are both transitive therefore *Foundation* holds and so does its relativisation to V_κ , $Foundation^{V_\kappa}$.

(iii) *Powerset*:

(see ??)

$$V_\kappa \models \forall x \exists y \forall z (z \subseteq x \rightarrow z \in y). \quad (3.74)$$

If we take x , an element of V_κ , $\mathcal{P}(x)$ has to be an element of V_κ to, because it is transitive and a strong limit cardinal.

(iv) *Pairing*:

(see ??)

$$V_\kappa \models \forall x, y \exists z (x \in z \wedge y \in z). \quad (3.75)$$

Pairing holds from similar argument like above: let x and y be elements of V_κ , so there are ordinals $\alpha, \beta < \kappa$ such that $x \in V_\alpha$, $y \in V_\beta$. Without any loss of generality, suppose $\alpha < \beta$, therefore $V_\alpha \subset V_\beta$ which, from transitivity of the cumulative hierarchy, means that $x \in V_\beta$, then $\{x, y\} \in V_{\beta+1}$ which is still in V_κ because it is a strong limit cardinal.

(v) *Union*:

(see ??)

$$V_\kappa \models \forall x \exists y \forall z \forall w ((w \in z \wedge z \in x) \rightarrow w \in y). \quad (3.76)$$

We want to see that for every $x \in V_\kappa$, this is equivalent to

$$V_\kappa \models \forall x \in V_\kappa, \exists y \in V_\kappa \forall z \in V_\kappa \forall w \in V_\kappa ((w \in z \wedge z \in x) \rightarrow w \in y). \quad (3.77)$$

Since V_κ is transitive, if $x \in V_\kappa$, all of its elements as well as their elements are in V_κ . To see that they also form a set themselves we only need to remember that V_κ is limit and therefore if α is the least ordinal such that $x \in V_\alpha$, $\bigcup x \in V_{\alpha+1}$.

(vi) *Replacement, Infinity*:

(see ??, ??)

We know that those hold from ??.

813 We will now show that if a set is a model of ZFC, it is in fact an inaccessible
 814 cardinal. So let V_κ be a model of ZFC which means that it is closed under
 815 the powerset operation, in other words:

$$\forall \lambda (\lambda < \kappa \rightarrow 2^\lambda < \kappa) \quad (3.78)$$

816 which is exactly the definition of strong limitness. κ is regular from the
 817 following argument by contradiction:

818 Let us suppose for a moment that κ is singular. Therefore there is an ordinal
 819 $\alpha < \kappa$ and a function $F : \alpha \rightarrow \kappa$ such that the range of F is unbounded in
 820 κ , in other words, $F[\alpha] \subseteq V_\kappa$ and $\sup(F[\alpha]) = \kappa$. In order to achieve the
 821 desired contradiction, we need to see that it is the case that $F[\alpha] \in V_\kappa$. Let
 822 $\varphi(x, y)$ be the following first-order formula:

$$F(x) = y \quad (3.79)$$

823 Then there is an instance of *Replacement* that states the following:

$$\begin{aligned} &(\forall x, y, z (\varphi(x, y) \& \varphi(x, z) \rightarrow y = z)) \rightarrow \\ &\rightarrow (\forall x \exists y \forall z (z \in y \leftrightarrow \exists w (\varphi(w, z)))) \end{aligned} \quad (3.80)$$

824 Which in turn means that there is a set $y = F[\alpha]$ and $y \in V_\kappa$, which is the
 825 contradiction with $\sup(y) = \kappa$ we are looking for. \square

826

827 The same holds for ZFC₂, the proof is very similar.

Theorem 3.20

$$V_\kappa \models \text{ZFC}_2 \leftrightarrow \kappa \text{ is inaccessible} \quad (3.81)$$

828 *Proof.* κ is a strong limit cardinal because from ZFC₂ and *Powerset* we know
 829 that for every $\lambda < \kappa$, we know that $2^\lambda < \kappa$.

830 κ is also regular, because otherwise there would be an ordinal α and
 831 a function $F : \alpha \rightarrow \kappa$ with a range unbounded in κ . *Replacement*² gives us
 832 a set $y = F[\alpha]$, so $y \in V_\kappa$, which contradicts the fact that $\sup(y) = \kappa$. It
 833 can not be the case that $\kappa \in V_\kappa$.

834 The other direction is exactly like the first part of above theorem ?? \square

835

836 This is how the existence of an inaccessible cardinal is established in [?].

Definition 3.21 N

837

$$\exists u (In(\alpha) \& \forall x_1, \dots, x_n (x_1, \dots, x_n \in u \rightarrow (\varphi \leftrightarrow \varphi^u))) \quad (3.82)$$

839 It is interesting to see that the above schema yields the first inaccessible
 840 cardinal if we take for φ the conjunction of all axioms of \mathbf{ZF}_2 .

841

842 To see that inaccessible cardinal can be also obtained by a fixed-point
 843 axiom (or a scheme if were in first-order logic), see the following theorem by
 844 Lévy, we won't repeat the proof here, it is available in [?, Theorem 3],

Theorem 3.22

$$M \leftrightarrow N \quad (3.83)$$

845 We have transcended \mathbf{ZFC} , but that is just a start. Naturally, we could
 846 go on and consider the next inaccessible cardinal, which is inaccessible with
 847 respect to the theory $\mathbf{ZFC} + \exists \kappa (\kappa \models \mathbf{ZFC})$. But let's try to find a faster way
 848 up, informally at first.

849 Since we can find an inaccessible set larger than any chosen set M_0 , it
 850 is clear that there are arbitrarily large inaccessible cardinals in V , they are
 851 "unbounded"³⁷ in V . If V were a cardinal, we could say that there are V
 852 inaccessible cardinals less than V , but this statement of course makes no sense
 853 in set theory as is because V is not a set. But being more careful, we could
 854 find a property that can be formalized in second-order logic and reflect it to
 855 an initial segment of V . That would allow us to construct large cardinals
 856 more efficiently than by adding inaccessibles one by one. The property we
 857 are looking for ought to look like something like this:

$$\begin{aligned} &\kappa \text{ is an inaccessible cardinal and} \\ &\text{there are } \kappa \text{ inaccessible cardinals } \mu < \kappa \end{aligned} \quad (3.84)$$

858 This is in fact a fixed-point type of statement. We shall call those cardinals
 859 hyper-inaccessible. Now consider the following definition.

860

861 **Definition 3.23** *0-inaccessible cardinal*
 862 *A cardinal κ is 0-inaccessible if it is inaccessible.*

863 We can define α -weakly-inaccessible cardinals analogously with the only dif-
 864 ference that those are limit, not strongly limit.

865 **Definition 3.24** *α -hyper-inaccessible cardinal*
 866 *For any ordinal α , κ is called α -inaccessible, if κ is inaccessible and for each*
 867 *$\beta \prec \alpha$, the set of β -inaccessible cardinals less than κ is unbounded in κ .*

³⁷The notion is formally defined for sets, but the meaning should be obvious.

Because κ is inaccessible and therefore regular, the number of β -inaccessibles below κ is equal to κ . We have therefore successfully formalized the above vague notion of hyper-inaccessible cardinal into a hierarchy of α -inaccessibles.

Let's now consider iterating this process over again. Since, informally, V would be α -inaccessible for any α , this property of the universal class could possibly be reflected to an initial segment, the smallest of those will be the first hyper-inaccessible cardinal. Such κ is larger than any α -inaccessible since from regularity of κ , for given $\alpha < \kappa$, κ is κ -th α -hyper-inaccessible cardinal. It is in fact "inaccessible" via α -inaccessibility.

Definition 3.25 *Hyper-inaccessible cardinal*

κ is called the *hyper-inaccessible*, also *0-hyper-inaccessible*, cardinal if it is α -inaccessible for every $\alpha < \kappa$.

Definition 3.26 *α -hyper-inaccessible cardinal*

For any ordinal α , κ is called α -hyper-inaccessible cardinal if for each ordinal $\beta < \alpha$, the set of β -hyper-inaccessible cardinals less than κ is bounded in κ .

Obviously we could go on and iterate it ad libitum, but the nomenclature would be increasingly confusing. A smarter way to accomplish the same goal is carried out in the following section.

3.5 Mahlo Cardinals

While the previous chapter introduced us to a notion of inaccessibility and the possibility of iterating it ad libitum in new theories, there is an even faster way to travel upwards in the cumulative hierarchy, that was proposed by Paul Mahlo in his papers (see [?], [?] and [?]) at the very beginning of the 20th century, and which can be easily reformulated using (*Reflection*). To see how Lévy's initial statement of reflection was influenced by Mahlo's work, refer to section ???. The aim of the following paragraphs is to give an intuitive explanation of the idea behind Mahlo's hierarchy of cardinals, all claims made here ought to be stated formally later in the very same chapter.

At the very end of section ??, we have tried to establish the notion of hyper-inaccessibility and iterate it to yield even larger large cardinals. In order to avoid too bulky cardinal names, let's try a different route and establish those cardinals directly via reflection.

The following two definitions come from [?] and while they are rather informal, we will find them very helpful for understanding the Mahlo cardinals.

Definition 3.27 (*Fixed-point property*)
 For any first-order formula $\psi(x, u_1, \dots, u_n)$ with no free variables other than x, u_1, \dots, u_n , which is any property of ordinals, we say that a property φ is a fixed-point property if φ has the form

$$\begin{aligned} & x \text{ is an inaccessible cardinal and} \\ & \text{there are } x \text{ ordinals less than } x \text{ that have the property } \psi(x, u_1, \dots, u_n). \end{aligned} \quad (3.85)$$

913

Definition 3.28 (*Fixed-point reflection*)
 If φ is a fixed-point property that holds for V , it also holds for some V_α , an initial segment of V .

Obviously those are in no way rigorous definitions because we have no idea what $\psi(x, u_1, \dots, u_n)$ looks like. Let's try to restate the same idea in a useful way. But first, let's show that the formal counterpart of the idea of containing "enough" ordinals with a property is the notion of stationary set.

Definition 3.29 (*Supremum*)
 Given x a set of ordinals, the supremum of x , denoted $\sup(x)$, is the least upper bound of x .

$$\sup(x) = \bigcup x \quad (3.86)$$

Definition 3.30 (*Limit point*)
 Given x , a set of ordinals and an ordinal α , we say that α is a limit point of x if $\sup(x \cap \alpha) = \alpha$

Definition 3.31 (*Set Unbounded in α*) Let α be an ordinal. We say that $x \subset \alpha$ is unbounded in α iff

$$\forall \beta \in \text{Ord}(\beta < \alpha \rightarrow \exists \gamma(\gamma \in x(\beta \leq \gamma < \alpha))) \quad (3.87)$$

In other words, κ is a mahlo cardinal if it is inaccessible and every club set in κ contains an inaccessible cardinal. This is exactly the notion of fixed-point reflection we were trying to show earlier.

[?]

Definition 3.32 *The following definitions are equivalent:*

- (i) κ is Mahlo
- (ii) κ is weakly Mahlo and strong limit
- (iii) The set $\{\lambda < \kappa : \lambda \text{ is inaccessible}\}$ is stationary in κ .
- (iv) Every normal function on κ has an inaccessible fixed point.

Proof. (i) \leftrightarrow (ii) Let κ_1 be a mahlo cardinal and let κ_2 be a strong limit weakly Mahlo cardinal. We know from the definitions that the set $\{\lambda < \kappa : \lambda \text{ is inaccessible}\}$ is stationary in both κ_1 and κ_2 , the only difference being that κ_1 is a strongly limit cardinal, but κ_2 would be limit from weak Mahloness, wasn't it for the fact that it is also strong limit. This eliminates the only difference between them and therefore κ_1 is also strong limit weakly Mahlo cardinal and κ_2 is Mahlo.

(i) \rightarrow (iii) We know that κ is uncountable, regular, strong limit and that the set $S = \{\lambda < \kappa : \lambda \text{ is regular}\}$ is stationary in κ . We want to prove that $S' = \{\lambda < \kappa : \lambda \text{ is inaccessible}\}$ is thus also stationary in κ .

Since stationary set intersects every club set in κ , let C be any such set. Let $D = \{\lambda < \kappa : \lambda \text{ is strong limit}\}$. D is a club set because TODO. Since intersection of less than κ club sets is a club set, $C \cap D \neq \emptyset$.

TODO proc $\lambda = S \cap C \cap D$ je inaccessible?

(iii) \rightarrow (iv)

TODO jak to dela Lévy?

(iv) \rightarrow (i)

TODO jak to dela Lévy?

range kazde normalni funkce je club v On. (nevadi ze On je trida?)

co treba lemma ze pevne body tvori taky club set

mozna by stacilo ze jsou unbounded, tedy kazda normalni funkce ma libovolne velke pevne body. \square

TODO obdoba pro α -Mahlo kardinaly?

TODO κ is hyper-Mahlo iff κ is inaccessible and the set $\{\lambda < \kappa : \lambda \text{ is Mahlo}\}$ is stationary in κ . to je to samy jako α -Mahlo, ne?

3.6 Indescribability

α -Mahlo are the extreme of regular fixed-point axioms, they are about as high as we can get via normal functions and stationary sets.

Let's try a different strategy. Remember how we said that (Regular, Limit and) various Large cardinals are in a way all determined by being unreachable by a specific process of creating bigger cardinals from already available ones? TODO indescribable – reflecting indescribability – we can't reach V by a Σ_1^1 formula, so there's some initial segment V_α that is also unreachable (we say indescribable) by the means of a ... formula

Let's recall complete reflection theorem first, consider the following:

For every sentence φ , there is a limit ordinal α such that $\varphi_\alpha^V \leftrightarrow \varphi$ (3.88)

We may also require that $\alpha < \beta$, where β is an arbitrary ordinal given.

For the exact definition of Π_n^m and Σ_n^m see ??

Definition 3.33 (Π_n^m -indescribable cardinal) We say that κ is Π_n^m -indescribable iff for any Π_n^m sentence φ such that $V_\kappa \models \varphi$ there is an $\alpha < \kappa$ such that $V_\alpha \models \varphi$

Definition 3.34 (Σ_n^m -indescribable cardinal) We say that κ is Σ_n^m -indescribable iff for any Σ_n^m sentence φ such that $V_\kappa \models \varphi$ there is an $\alpha < \kappa$ such that $V_\alpha \models \varphi$

Lemma 3.35 Let κ be a cardinal, the following holds for any $n \in \omega$. κ is Π_n^1 -indescribable iff κ is $\Sigma_n^1 + 1$ -indescribable

Proof. The forward direction is obvious, we can always add a spare quantifier over a type 2 variable to turn a Π_n^1 formula φ into a $\exists P\varphi$ which is thus a $\Sigma_n^1 + 1$ formula.³⁸

To prove the opposite direction, suppose that $V_\kappa \models \exists X\varphi(X)$ where X is a type 2 variable and φ is a Π_n^1 formula with one free variable of type 2. This means that there is a set $S \subseteq V_\kappa$ that is a witness of $\exists X\varphi(X)$, in other words, $\varphi(S)$ holds. We can replace every occurrence of X in φ by a new predicate symbol S , this allows us to say that κ is Π_n^1 -indescribable (with respect to $\langle V_\kappa, \in, R, S \rangle$).³⁹ \square

³⁸Note that unlike in previous sections, φ is now a sentence so we don't have to worry whether P is free in φ .

³⁹A different yet interesting approach is taken by Tate in ?. He states that for $n \geq 0$, a formula of order $\leq n$ is called a Π_0^n and a Σ_0^n formula. Then a Π_{m+1}^n is a formula of form $\forall Y\psi(Y)$ where ψ is a Σ_m^n formula and Y is a variable of type n . Finally, a Σ_{m+1}^n is the negation of a Π_m^n formula. So the above holds ad definitio.

997 The above lemma tells us that we as long as we stay in the realm of type
 998 1 and type 2 variables, we only need to classify indescribable cardinals with
 999 respect to Π_n^1 -indescribability.

1000 **Theorem 3.36** *Let κ be an ordinal. The following are equivalent.*

- 1001 (i) κ is inaccessible
 1002 (ii) κ is Π_0^1 -indescribable.

1003 Note that Π_0^1 formulas are those that contain zero unbound quantifiers
 1004 over type-2 variables, they are in fact first-order formulas. We have already
 1005 shown in ?? that there is no way to reach an inaccessible cardinal via first-
 1006 order formulas in ZFC. We will now prove it again in for formal clarity.

1007 *Proof.* TODO asi pridat alternativni definici nedosazitelnosti podle kan. 6.2?
 1008 □

1009 TODO nejaka veta ze kdyz jsou Π_0^1 -indescribable, jsou i Π_n^m -indescribable
 1010 pro $m \leq 1, n \leq 0$? Drake? Obracene! Π_n^m -indescribable jsou zaroven Π_b^a -
 1011 indescribable pro $a < m, b < n$.

1012 The above theorem provides an easy way to show that every following
 1013 large cardinal is also in inaccessible cardinal⁴⁰.

1014 **Definition 3.37** (*Extension property*) *We say that a cardinal κ has the ex-*
 1015 *tension property iff for any $R \subseteq V_\kappa$ there is a transitive set $X \neq V_\kappa$ and an*
 1016 *$S \subseteq X$ such that $\langle V_\kappa, \in, R \rangle \prec \langle X, \in, S \rangle$*

1017 **Definition 3.38** (*Weakly compact cardinal*)

1018 *We say that a cardinal κ is weakly compact iff it has the extension property.*

1019 The above definitions are equivalent

1020 **Theorem 3.39** *the following are equivalent:*

- 1021
 1022 (i) κ is Weakly compact.
 1023 (ii) κ is Π_1^1 -indescribable.

1024 For a proof, see [?][Theorem 6.4]

1025 TODO asi nekde bude meritelný kardinal

1026 TODO viz Drake, Ch.9 par. 3 – tam se říká že když κ je meritelný
 1027 kardinal, pak je κ Π_1^2 -nepopsatelný kardinal

⁴⁰That is because Π_0^1 formulas are included Π_n^m formulas for $m \leq 1, n \leq 0$.

3.7 Bernays–Gödel Set Theory

TODO Jech str. 70 [?]

TODO popis

Definition 3.40 (*Gödel–Bernay set theory*)

(i) extensionality for sets

$$\forall a \forall b [\forall x (x \in a \leftrightarrow x \in b) \rightarrow a = b] \quad (3.89)$$

(ii) pairing for sets

$$\forall x \forall y \exists z \forall w [w \in z \leftrightarrow (w = x \vee w = y)] \quad (3.90)$$

(iii) union for sets

$$\forall a \exists b \forall c [c \in b \leftrightarrow \exists d (c \in d \wedge d \in a)] \quad (3.91)$$

(iv) powers for sets

$$\forall a \exists p \forall b [b \in p \leftrightarrow (c \in b \rightarrow c \in a)] \quad (3.92)$$

(v) infinity for sets

$$\text{There is an inductive set.} \quad (3.93)$$

(vi) Extensionality for classes

$$\forall x (x \in A \leftrightarrow x \in B) \rightarrow A = B \quad (3.94)$$

(vii) Foundation for classes

$$\text{Each non-empty class is disjoint from each of its elements.} \quad (3.95)$$

(viii) Limitation of size for sets

$$\text{For any class } C \text{ a set } x \text{ such that } x=C \text{ exists iff} \quad (3.96)$$

$$\text{there is no bijection between } C \text{ and the class } V \text{ of all sets} \quad (3.97)$$

(ix) Comprehension schema for classes

$$\text{For an arbitrary formula } \varphi \text{ with no quantifiers over classes, there is a class } A \text{ such that } \forall x \quad (3.98)$$

The first five axioms are identical to axioms in ZF.

Comprehension schema tells us that proper classes are basically first-order predicates. **TODO**

3.8 The Constructible Universe

The constructible universe, denoted L , is a cumulative hierarchy of sets, presented by Kurt Gödel in his 1938 paper *The Consistency of the Axiom of Choice and of the Generalised Continuum Hypothesis*. For a technical description, see below. Assertion of their equality, $V = L$, is called the *axiom of constructibility*. The axiom implies GCH and therefore also AC and contradicts the existence of some of the large cardinals, our goal is to decide whether those introduced earlier are among them.

On order to formally establish this class, we need to formalize the notion of definability first.

Definition 3.41 *We say that a set X is definable over a model $\langle M, \in \rangle$ if there is a first-order formula φ together with parameters $u_1, \dots, u_n \in M$ such that*

$$X = \{x : x \in M \ \& \ \langle M, \in \rangle \models \varphi(x, u_1, \dots, u_n)\} \quad (3.99)$$

Definition 3.42 *(Sets definable in M)*

The following is a set of all definable subsets of a given set M , denoted $\text{Def}(M)$.

$$\begin{aligned} \text{Def}(M) = \{ \{y : x \in M \wedge \langle M, \in \rangle \models \varphi(y, u_1, \dots, u_n)\} \mid \\ \varphi \text{ is a first-order formula, } u_1, \dots, u_n \in M \} \end{aligned} \quad (3.100)$$

Now we can recursively build L .

Definition 3.43 *(The Constructible universe)*

(i)

$$L_0 := \emptyset \quad (3.101)$$

(ii)

$$L_{\alpha+1} := \text{Def}(L_\alpha) \quad (3.102)$$

(iii)

$$L_\lambda = \bigcup_{\alpha < \lambda} L_\alpha \text{ If } \lambda \text{ is a limit ordinal} \quad (3.103)$$

(iv)

$$L = \bigcup_{\alpha \in \text{Ord}} L_\alpha \quad (3.104)$$

Note that while L bears very close resemblance to V , the difference is, that in every successor step of constructing V , we take every subset of V_α to be $V_{\alpha+1}$, whereas $L_{\alpha+1}$ consists only of definable subsets of L_α . Also note that L is transitive.

In order to

TODO:

1071 **Lemma 3.44** $Ord \in L$

1072 **Lemma 3.45** L is well-ordered.

1073 *TODO !!*

1074 **Theorem 3.46** Let L be as in ??.

$$L \models \text{ZFC} \quad (3.105)$$

1075 *Proof.* *TODO !!!* (strucne) vit [?][Theorem 13.3]

1076 (i) *Extensionality* (see ??):

1077 *Extensionality* holds in L because Δ_0 formulas are absolute in transitive
1078 classes by ??, *Extensionality* is Δ_0 and L is transitive.

1079 (ii) *Foundation* (see ??)

1080 Take a non-empty set X . Let $x \in X$ be a set such that $X \cap x = \emptyset$. x
1081 is therefore defined by the formula $\varphi(x, y) = (x \cap y = \emptyset)$, so $x \in L$. φ
1082 is Δ_0 and therefore holds in L by ??.

1083 (iii) *Pairing* (see ??)

1084 Since *Pairin* is also Δ_0 , it holds in L by the same argument as *Exten-*
1085 *sionality* does by ??.

1086 (iv) *Union* (see ??)

1087 *Union* is also Δ_0 , see *Extensionality* and ??.

1088 (v) *Power Set* (see ??)

1089 *Power Set* also holds by ??.

1090 (vi) *Infinity* (see ??)

1091 $\omega \in L$ by ??

1092 (vii) *Specification* (see ??)

1093 .

1094 (viii) *Replacement* (see ??)

1095 .

1096 (ix) *Choice* (see ??)

1097 .

1098

□

1099 **Definition 3.47** *Constructibility*

1100 $L = V$

1101 The following are a few interesting results that we won't prove but refer
1102 interested reader to appropriate resources instead.

1103 **Definition 3.48** (*GCH*)

1104 The following is called the *Generalised Continuum Hypothesis*, abbreviated
1105 as *GCH*. It is an independent statement in *ZFC*.

$$\text{GCH iff } \aleph_{\alpha+1} = 2^{\aleph_\alpha} \text{ for every ordinal } \alpha \quad (3.106)$$

Theorem 3.49

$$(L = V) \rightarrow GCH \quad (3.107)$$

1106 This is proven in cite{neco} Gödel? Jech? Kunen?
 1107 TODO L a velke kardinaly
 1108 TODO def Con!

Theorem 3.50

$$Con(L + \exists \kappa (\kappa \text{ " is an Inaccessible Cardinal"})) \quad (3.108)$$

Theorem 3.51

$$Con(L + \exists \kappa (\kappa \text{ " is a Mahlo Cardinal"})) \quad (3.109)$$

Theorem 3.52

$$Con(L + \exists \kappa (\kappa \text{ " is a Weakly Inaccessible Cardinal Cardinal"})) \quad (3.110)$$

Theorem 3.53

$$Con(L + \exists \kappa (\kappa \text{ " is a Measurable Cardinal"})) \quad (3.111)$$

1109 TODO co velky pismena ve jmenech kardinalu?
 1110 TODO zduvodneni
 1111
 1112 TODO kratka diskuse jestli refl implikuje transcendenci na L - polemika,
 1113 nazor - V=L a slaba kompaktnost a dalsi
 1114 TODO neco jako ze meritelny kardinal je nepopsatelny vys nez je hierar-
 1115 chie vsech tvrzeni o L?

1116 **3.9 Measurable Cardinal**

1117 TODO refaktorizovat fle:

1118 **Definition 3.54** (*Ultrafilter*)

1119 Given a set X , we say $U \subset \mathcal{P}(X)$ is an ultrafilter iff all of the following
 1120 hold:

- 1121 (i) $\emptyset \notin U$
- 1122 (ii) $\forall a, b (a \subset b \ \& \ a \in U \rightarrow b \in U)$
- 1123 (iii) $\forall a, b \in U (a \cap b) \in U$
- 1124 (iv) $\forall a (a \subset X \rightarrow (a \in U \vee (X \setminus a) \in U))$

1125 **Definition 3.55** (κ -complete ultrafilter)

1126 We say that an ultrafilter U is κ -complete iff

1127 **Definition 3.56** (non-principal ultrafilter)

1128 *TODO*

1129 **Definition 3.57** (Measurable Cardinal)

1130 Let κ be a cardinal. We say κ is a measurable cardinal iff it is an uncountable
1131 cardinal with a κ -complete, non-principal ultrafilter.

1132 **Theorem 3.58** Let κ be a cardinal. κ is a measurable cardinal iff it is a
1133 Π_1^2 -indescribable cardinal.

1134 *TODO !!!*

1135 *TODO proc?*

1136 *TODO je*

1137 **4 Conclusion**

1138 TODO na konec

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