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4 REFLECTION PRINCIPLES AND LARGE
5 CARDINALS
6 Bakalářská práce

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¹⁰ Prohlašuj, že jsem bakalářskou práci vypracoval samostatně a že jsem uvedl
¹¹ všechny použité prameny a literaturu.

¹² V Praze 14. dubna 2015

Abstract

Práce zkoumá vztah tzv. principů reflexe a velkých kardinálů. Lévy ukázal, že v ZFC platí tzv. věta o reflexi a dokonce, že věta o reflexi je ekvivalentní schématu nahrazení a axiomu nekonečna nad teorií ZFC bez axiomu nekonečna a schématu nahrazení. Tedy lze na větu o reflexi pohlížet jako na svého druhu axiom nekonečna. Práce zkoumá do jaké míry a jakým způsobem lze větu o reflexi zobecnit a jaký to má vliv na existenci tzv. velkých kardinálů. Práce definuje nedosažitelné, Mahlovy a nepopsatelné kardinály a ukáže, jak je lze zavést pomocí reflexe. Přirozenou limitou kardinálů získaných reflexí jsou kardinály nekonzistentní s L. Práce nabídne intuitivní zdůvodnění, proč tomu tak je.

Abstract

This thesis aims to examine relations between so called "Reflection Principles" and Large cardinals. Lévy has shown that Reflection Theorem is a sound theorem of ZFC and it is equivalent to Replacement Scheme and the Axiom of Infinity. From this point of view, Reflection theorem can be seen a specific version of an Axiom of Infinity. This paper aims to examine the Reflection Principle and it's generalisations with respect to existence of Large Cardinals. This thesis will establish Inaccessible, Mahlo and Indescribable cardinals and their definition via reflection. A natural limit of Large Cardinals obtained via reflection are cardinals inconsistent with L. The thesis will offer an intuitive explanation of why this is the case.

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1 Introduction

1.1 Motivation and Origin

The Universe of sets cannot be uniquely characterized (i. e. distinguished from all its initial elements) by any internal structural property of the membership relation in it, which is expressible in any logic of finite or transfinite type, including infinitary logics of any cardinal order.

— Kurt Gödel [9]

To understand why we need reflection in the first place, let's think about infinity for a moment. In the intuitive sense, infinity is an upper limit of all numbers. But for centuries, this was merely a philosophical concept, closely bound to religious and metaphysical way of thinking, considered separate from numbers used for calculations or geometry. It was a rather vague concept. In ancient Greece, Aristotle's response to famous Zeno's paradoxes introduced the distinction between actual and potential infinity. He argued, that potential infinity is (in today's words) well defined, as opposed to actual infinity, which remained a vague incoherent concept. He didn't think it's possible for infinity to inhabit a bounded place in space or time, rejecting Zeno's thought experiments as a whole. Aristotle's thoughts shaped western thinking partly due to Aquinas, who himself believed actual infinity to be more of a metaphysical concept for describing God than a mathematical property attributed to any other entity. In his *Summa Theologica* ¹ he argues:

A geometrician does not need to assume a line actually infinite, but takes some actually finite line, from which he subtracts whatever he finds necessary; which line he calls infinite.

Less than hundred years later, Gregory of Rimini wrote

If God can endlessly add a cubic foot to a stone—which He can—then He can create an infinitely big stone. For He need only add one cubic foot at some time, another half an hour later, another a quarter of an hour later than that, and so on ad infinitum. He would then have before Him an infinite stone at the end of the hour.

Which is basically a Zeno's Paradox made plausible with God being the actor. In contrast to Aquinas' position, Gregory of Rimini theoretically constructs an object with actual infinite magnitude that is essentially different from God.

¹Part I, Question 7, Article 3, Reply to Objection 1

Even later, in the 17th century, pushing the property of infiniteness from the Creator to his creation, Nature, Leibniz wrote to Foucher in 1662:

I am so in favor of the actual infinite that instead of admitting that Nature abhors it, as is commonly said, I hold that Nature makes frequent use of it everywhere, in order to show more effectively the perfections of its Author. Thus I believe that there is no part of matter which is not, I do not say divisible, but actually divided; and consequently the least particle ought to be considered as a world full of an infinity of different creatures.

But even though he used potential infinity in what would become foundations of modern Calculus and argued for actual infinity in Nature, Leibniz refused the existence of an infinite, thinking that Galileo's Paradoxon² is in fact a contradiction. The so called Galileo's Paradoxon is an observation Galileo Galilei made in his final book "Discourses and Mathematical Demonstrations Relating to Two New Sciences". He states that if all numbers are either squares and non-squares, there seem to be less squares than there is all numbers. On the other hand, every number can be squared and every square has its square root. Therefore, there seem to be as many squares as there are all numbers. Galileo concludes, that the idea of comparing sizes makes sense only in the finite realm.

Salviati: So far as I see we can only infer that the totality of all numbers is infinite, that the number of squares is infinite, and that the number of their roots is infinite; neither is the number of squares less than the totality of all the numbers, nor the latter greater than the former; and finally the attributes "equal," "greater," and "less," are not applicable to infinite, but only to finite, quantities. When therefore Simplicio introduces several lines of different lengths and asks me how it is possible that the longer ones do not contain more points than the shorter, I answer him that one line does not contain more or less or just as many points as another, but that each line contains an infinite number.

Leibniz insists in part being smaller than the whole saying

Among numbers there are infinite roots, infinite squares, infinite cubes. Moreover, there are as many roots as numbers. And there are as many squares as roots. Therefore there are as many squares

²zneni galileova paradoxu

as numbers, that is to say, there are as many square numbers as there are numbers in the universe. Which is impossible. Hence it follows either that in the infinite the whole is not greater than the part, which is the opinion of Galileo and Gregory of St. Vincent, and which I cannot accept; or that infinity itself is nothing, i.e. that it is not one and not a whole.

TODO nejakej Hegel–strucne?

TODO Cantor

TODO mene teologie, vice matematiky

TODO definovat pojmy (trida etc)

TODO neni V v nejakem smyslu porad potencialni nekonecno, zatimco mnoziny vetsi nez omega jsou aktualni? nebo jsou potencialni protoze se stavaji pres indukci, od spoda.

In his work, he defined transfinite numbers to extend existing natural number structure so it contains more objects that behave like natural numbers and are based on an object (rather a meta-object) that doesn't explicitly exist in the structure, but is closely related to it. This is the first instance of reflection. This paper will focus on taking this principle a step further, extending Cantor's (or Zermelo–Fraenkel's, to be more precise) universe so it includes objects so big, they could be considered the universe itself, in a certain sense.

TODO dal asi smazat

The original idea behind reflection principles probably comes from what could be informally called “universality of the universe”. The effort to precisely describe the universe of sets was natural and could be regarded as one of the impulses for formalization of naive set theory. If we try to express the universe as a set $\{x|x = x\}$, a paradox appears, because either our set is contained in itself and therefore is contained in a set (itself again), which contradicts the intuitive notion of a universe that contains everything but is not contained itself.

TODO ???

If there is an object containing all sets, it must not be a set itself. The notion of class seems inevitable. Either directly the ways for example the Bernays–Gödel set theory, we will also discuss later in this paper, does in, or on a meta-level like the Zermelo–Fraenkel set theory, that doesn't refer to them in the axioms but often works with the notion of a universal class. duet Another obstacle of constructing a set of all sets comes from Georg Cantor, who proved that the set of all subsets of a set (let A be the set and $\mathcal{P}(A)$ its powerset) is strictly larger than A . That would turn every aspiration to

finally establish an universal set into a contradictory infinite regression.³ We will use V to denote the class of all sets. From previous thoughts we can easily argue, that it is impossible to construct a property that holds for V and no set and is neither paradoxical like $\{x|x = x\}$ nor trivial. Previous observation can be transposed to a rather naive formulation of the reflection principle:

Reflection Any property which holds in V already holds in some initial segment of V .

To avoid vagueness of the term "property", we could informally reformulate the above statement into a schema:

For every first-order formula⁴ φ holds in $V \leftrightarrow \varphi$ holds in some initial segment of V .

Interested reader should note that this is a theorem scheme rather than a single theorem.⁵

1.2 A few historical remarks on reflection

Reflection made it's first in set-theoretical appearance in Gödel's proof of GCH in L (citace Kanamori ? Lévy and set theory), but it was around even earlier as a concept. Gödel himself regarded it as very close to Russel's reducibility axiom (an earlier equivalent of the axiom schema of Zermelo's separation). Richard Montague then studied reflection properties as a tool for verifying that Replacement is not finitely axiomatizable (citace?). a few years later Lévy proved in [2] the equivalence of reflection with Axiom of infinity together with Replacement in proof we shall examine closely in chapter 2.

TODO co dal? recent results?

³An intuitive analogy of this *reductio ad infinitum* is the status of ω , which was originally thought to be an unreachable absolute, only to become starting point of Cantor's hierarchy of sets growing beyond all boundaries around the end of the 19th century

⁴this also works for finite sets of formulas [3, p. 168]

⁵If there were a single theorem stating "for any formula φ that holds in V there is an initial segment of V where φ also holds", we would obtain the following contradiction with the second Gödel's theorem: In ZFC, any finite group of axioms of ZFC holds in some initial segment of the universe. If we take the largest of those initial segments it is still strictly smaller than the universe and thus we have, via compactness, constructed a model of ZFC within ZFC. That is, of course a harsh contradiction. This also leads to an elegant way to prove that ZFC is not finitely axiomatizable.

1.3 Notation (??) TODO

1. *Reflection* je obecne reflexe (jaka presne)
 2. *Reflection*₁ je reflexe prvoradovych formul
 3. *Reflection*₂ je reflexe druhoradovych formul
 4. etc...
- V a V_α odkazuji k Von Neumannove hierarchii (pro jistotu)

2 Levy's first-order reflection

2.1 Introduction

This section will try to present Lévy's proof of a general reflection principle being equivalent to Replacement and Infinity under ZF minus Replacement and Infinity. We will first introduce a few axioms and definitions that were a different in Lévy's paper[2], but are equivalent to today's terms. We will write them in contemporary notation, our aim is the result, not history of set theory notation.

Please note that Lévy's paper was written in a period when Set theory was oriented towards semantics, which means that everything was done in a model. All proofs were theodel that of ZFC was V_α (notated as $R(\alpha)$ at the time) for some cardinal α , which means that α is a inaccessible cardinal. Please bear in mind that this is vastly different from saying that there is an inaccessible α inside the model. This V_α is also referred to as $Scm^Q(u)$, which means that u ($u = V_\alpha$ in our case) is a standard complete model of an undisclosed axiomatic set theory Q formulated in the "non-simple applied first order functional calculus", which is second-order theory is today's terminology, we are allowed to quantify over functions and thus get rid of axiom schemes. (Note that Lévy always speaks of "the axiom of replacement"). Besides placeholder set theory Q and ZF, which the reader should be familiar with, theories Z , S , and SF are used in the text. Z is ZF minus replacement, S is ZF minus replacement and infinity, and finally SF is ZF minus infinity. "The axiom of subsets" is an older name for the axiom scheme of specification (and it's not a scheme since we are now working in second order logic). Also note that universal quantifier does not appear, $\forall x\varphi(x)$ would be written as $(x)\varphi(x)$, the symbol for negation is " \sim ".

2.2 Lévy's Original Paper

The following are a few definitions that are used in Lévy's original article. ⁶

Definition 2.1 *Relativization*[3, Definition 12.6]

Let M be a class, E a binary relation on M and let $\varphi(x_1, \dots, x_n)$ be a formula.

The relativization of φ to M and E is the formula

$$\varphi^{M,E}(x_1, \dots, x_n) \tag{2.1}$$

⁶While some of them won't be of much use in this paper, they will provide extremely helpful when reading the original article as set theory notation and terminology has evolved in the last 50 years considerably.

235 Defined in the following inductive manner:

$$\begin{aligned}
 (x \in y)^{M,E} &\leftrightarrow xEx \\
 (x = y)^{M,E} &\leftrightarrow x = y \\
 (\neg\varphi)^{M,E} &\leftrightarrow \neg\varphi^{M,E} \\
 (\varphi \ \& \ \psi)^{M,E} &\leftrightarrow \varphi^{M,E} \ \& \ \psi^{M,E} \\
 (\exists x\varphi)^{M,E} &\leftrightarrow (\exists x \in M)\varphi^{M,E}
 \end{aligned} \tag{2.2}$$

236 Next two definitions are not used in contemporary set theory, but they
 237 illustrate 1960's set theory mind-set and they are used heavily in Lévy's text,
 238 so we will include and explain them for clarity. Generally in this chapter, \mathbf{Q}
 239 stands for an undisclosed axiomatic set theory, u is usually a model, coun-
 240 terpart of today's V^7 , e is a relation that serves as \in in the given model.

241 TODO je to relativizovany, jak rika shepherdson?

242 **Definition 2.2** *Standard model of a set theory*

243 We say the u is a standard model of \mathbf{Q} with a membership relation e , written
 244 as $Sm^{\mathbf{Q}}(u)$, if both of the following hold

- 245 (i) $(x, y) \in e \leftrightarrow y \in u \ \& \ x \in y$
- 246 (ii) $y \in u \ \& \ x \in y \rightarrow x \in u$

247 **Definition 2.3** *Standard complete model of a set theory*

248 We say that that u is a standard complete model of a set theory \mathbf{Q} with a
 249 membership relation e if:

- 250 (i) u is a transitive set with respect to \in
- 251 (ii) $\forall e((x, y) \in e \leftrightarrow (y \in u \ \& \ x \in y) \ \& \ Sm^{\mathbf{Q}}(u, e))$
- 252 this is written as $Scm^{\mathbf{Q}}(u)$.

253

254 TODO what is "simple first-order functional calculus" a "non-simple first-
 255 order functional calculus"? Levyho ucebnice?

256

257 **Definition 2.4** *Cardinal inaccessible with respect to \mathbf{Q}*

$$In^{\mathbf{Q}}(\alpha) = Scm^{\mathbf{Q}}(V_{\alpha}) \tag{2.3}$$

258 The following is a principle of complete reflection over ZF.

⁷Which is of course not referred to as a model, but it is used in a similar fashion, in this case the term "model" was a metamathematical notion because it was not based on any underlying structure of theory. It can be easily formalized in any set theory, but it's not helpful for our case.

2.3 $S \models \text{Reflection} \leftrightarrow (\text{Replacement} \ \& \ \text{Infinity})$ Levy's first-order reflection

259 **Definition 2.5** $N(\varphi)$

$$\exists u(Scm^{ZF}(u) \ \& \ \forall x_1, \dots, x_n(x_1, \dots, x_n \in u \rightarrow \varphi \leftrightarrow \varphi^u)) \quad (2.4)$$

260 where φ is a formula which does not contain free variables except x_1, \dots, x_n .

261 Note that this by (2.4) equivalent to $\exists u(In^{ZF}(u) \ \& \ \forall x_1, \dots, x_n(x_1, \dots, x_n \in$
 262 $u \rightarrow \varphi \leftrightarrow \varphi^u))$, where $In(\alpha)$ is equivalent to the standard notion of inacces-
 263 sibility.

264 **2.3** $S \models \text{Reflection} \leftrightarrow (\text{Replacement} \ \& \ \text{Infinity})$

265 **Definition 2.6** $N_0(\varphi)$

$$\exists u(Scm^S(u) \ \& \ \forall x_1, \dots, x_n(x_1, \dots, x_n \in u \rightarrow \varphi \leftrightarrow \varphi^u)) \quad (2.5)$$

266 where φ is a formula which does not contain free variables except x_1, \dots, x_n .

267 Note that the only difference between N and N_0 is the set theory used.

268 **Theorem 2.7** *In S , the schema N_0 implies the Axiom of Infinity.*

269 *Proof.* For any φ , N_0 gives us $\exists u Scm^S(u)$, which means that there is a set u
 270 that is identical to V_α for some alpha, so $\exists \alpha Scm^S(V_\alpha)$. We don't know the
 271 exact size of this α , but we know that $\alpha \geq \omega$, otherwise α would be finite,
 272 therefore not closed under the powerset operation, which would contradict
 273 the axiom of powersets. In order to prove that it is a model of S , we would
 274 need to verify all axioms of S . We have already shown that ω is closed under
 275 the powerset operation. Foundation, extensionality and comprehension are
 276 clear from the fact that we work in ZF^8 , pairing is clear from the fact, that
 277 given two sets A, B , they have ranks a, b , without loss of generality we can
 278 assume that $a \leq b$, which means that $A \in V_a \in V_b$, therefore V_b is a set that
 279 satisfies the paring axiom: it contains both A and B .

280 TODO vyhodit axiomy, staci vyrobit ω

281 We now want to prove that V_α leads to existence of an inductive set,
 282 which is a set that satisfies $\exists A(\emptyset \in A \ \& \ \forall x \in A ((x \cup \{x\}) \in A))$. If we can
 283 find a way to construct V_ω from any V_α satisfying $\alpha \geq \omega$, we are done. Since
 284 ω is the least limit ordinal, all we need is the following

$$\bigcap \{V_\kappa \mid \forall \lambda(\lambda < \kappa \rightarrow \exists \mu(\lambda < \mu < \kappa))\} \quad (2.6)$$

⁸We only need to verify axioms that provide means of constructing larger sets from smaller to make sure they don't exceed ω . Since ω is an initial segment of ZF , the axiom scheme of specification can't be broken, the same holds for foundation and extensionality.

because V_κ is a transitive set for every κ , thus the intersection is non-empty unless empty set satisfies the property or the set of V_κ s is itself empty. \square

Theorem 2.8 *In S , the schema N_0 implies Replacement schema.*

Proof. TODO vysvetlit! (podle contemporary verze)

Let $\varphi(v, w)$ be a formula wth no free variables except v, w, x_1, \dots, x_n where n is any natural number. Let χ be an instance of replacement schema for this φ :

$$\chi = \forall r, s, t(\varphi(r, s) \& \varphi(r, t) \rightarrow s = t) \rightarrow \forall x \exists y \forall w (w \in y \leftrightarrow \exists v (v \in x \& \varphi(v, w))) \quad (2.7)$$

293

We can deduce the following from N_0 :

- (i) $x_1, \dots, x_n, v, w \in u \rightarrow (\varphi \leftrightarrow \varphi^u)$
- (ii) $x_1, \dots, x_n, v \in u \rightarrow (\exists w \varphi \leftrightarrow (\exists w \varphi)^u)$
- (iii) $x_1, \dots, x_n, x \in u \rightarrow (\chi \leftrightarrow \chi^u)$
- (iv) $\forall x_1, \dots, x_n \forall x (\chi \leftrightarrow (\forall x_1, \dots, x_n \forall x \chi)^u)$

Note that (i), (ii), (iii) are obtained from instances of N_0 for φ , $\exists w \varphi$ and χ respectively. From relativization we also know that $(\exists w \varphi)^u$ is equivalent to $\exists w (w \in u \& \varphi^u)$. Therefore (ii) is equivalent to $x_1, \dots, x_n, v \in u \rightarrow (\exists w (w \in u \& \varphi^u))$.

If φ is a function $(\forall r, s, t(\varphi(r, s) \& \varphi(r, t) \rightarrow r = t))$, then for every $x \in u$, which is also $x \subset u$ by $Scm^S(u)$, it maps elements of x onto u . From the axiom scheme of comprehension⁹, we can find a set of all images of elements of x . Let's call it y . That gives us $x_1, \dots, x_n, x \in u \rightarrow \chi$. By (iii) we get $x_1, \dots, x_n, x \in u \rightarrow \chi^u$, closure of this formula is $(\forall x_1, \dots, x_n \forall x \chi)^u$, which together with (iv) yields $\forall x_1, \dots, x_n \forall x \chi$. By the means of specification we end up with χ , which is all we need for now. \square

TODO zbytek clanku v par vetach, odkaz na mista ve 3: kapitole kde je budeme delat.

2.4 Contemporary restatement

We will now prove what is also Lévy's reflection theorem, but a little stronger, rephrased with more up to date set theory. The main difference is, that while Lévy reflects φ from V into a set u that is a "standard complete model of S "¹⁰, we say that there is a V_α that reflects φ .

⁹axiom of subsets in Levy's version

¹⁰Any limit ordinal is in fact a model of S , we shall pay more attention to that in a moment.

We will prove the equivalence of *Reflection*₁ with *Replacement* and *Infinity* in two parts. First, we will show that *Reflection*₁ is a theorem of ZF, then the second implication which proves *Infinity* and *Replacement* from *Reflection*₁ in S.

The following lemma is usually done in more parts, the first being with one formula and the other with n . We will only state and prove the generalised version for n formulas, knowing that $n = 1$ is just a specific case and the proof is exactly the same.

Lemma 2.9 *Lemma Let $\varphi_1, \dots, \varphi_n$ be any formulas with m parameters¹¹.*

(i) *For each set M_0 there is such M that $M_0 \subset M$ and the following holds for every $i \leq n$:*

$$\exists x \varphi_i(u_1, \dots, u_{m-1}, x) \rightarrow (\exists x \in M) \varphi_i(u_1, \dots, u_{m-1}, x) \quad (2.8)$$

for every $u_1, \dots, u_{m-1} \in M$.

(ii) *Furthermore there is an ordinal α such that $M_0 \subset V_\alpha$ and the following holds for each $i \leq n$:*

$$\exists x \varphi_i(u_1, \dots, u_{m-1}, x) \rightarrow (\exists x \in V_\alpha) \varphi_i(u_1, \dots, u_{m-1}, x) \quad (2.9)$$

for every $u_1, \dots, u_{m-1} \in M$.

(iii) *Assuming Choice, there is M , $M_0 \subset M$ such that 2.8 holds for every M , $i \leq n$ and $|M| \leq |M_0| \cdot \aleph_0$.*

Proof. We will simultaneously prove statements (i) and (ii), denoting M^T the transitive set required by part (ii). Unless explicitly stated otherwise for specific steps, it is thought to be equivalent to M .

Let us first define operation $H(u_1, \dots, u_{m-1})$ that gives us the set of x 's with minimal rank satisfying $\varphi_i(u_1, \dots, u_{m-1}, x)$ for given parameters u_1, \dots, u_{m-1} for every $i \leq n$.

$$H_i(u_1, \dots, u_n) = \{x \in C_i : (\forall z \in C)(\text{rank}(x) \leq \text{rank}(z))\} \quad (2.10)$$

for each $i \leq n$, where

$$C_i = \{x : \varphi_i(u_1, \dots, u_{m-1}, x)\} \text{ for } i \leq n \quad (2.11)$$

¹¹For formulas with different number of parameters take for m the highest number of parameters among given formulas. Add spare parameters to the other formulas so that x remains the last parameter. That can be done in a following manner: Let φ'_i be the a formula with k parameters, $k < m$. Let us set $\varphi_i(u_1, \dots, u_{m-1}, x) = \varphi'_i(u_1, \dots, u_{k-1}, u_k, \dots, u_{m-1}, x)$, notice that u_k, \dots, u_{m-1} are spare variables added just for formal simplicity.

341

342 Next, let's construct M from given M_0 by induction.

$$M_{i+1} = M_i \cup \bigcup_{j=0}^n \{H_j(u_1, \dots, u_{m-1}) : u_1, \dots, u_{m-1} \in M_i\} \quad (2.12)$$

343 In other words, in each step we add the elements satisfying $\varphi(u_1, \dots, u_{m-1}, x)$
 344 for all parameters that were either available earlier or were added in the
 345 previous step. For statement (ii), this is the only part that differs from (i).
 346 Let us take for each step transitive closure of M_{i+1} from (i). In other words,
 347 let γ be the smallest ordinal such that

$$(M_i^T \cup \bigcup_{j=0}^n \{H_j(u_1, \dots, u_{m-1}) : u_1, \dots, u_{m-1} \in M_i\}) \subset V_\gamma \quad (2.13)$$

348 Then the incremental step is like so:

$$M_{i+1}^T = V_\gamma \quad (2.14)$$

349 The final M is obtained by joining all incremental steps together.

$$M = \bigcup_{i=0}^{\infty} M_i, \quad M^T = \bigcup_{i=0}^{\infty} M_i^T \quad (2.15)$$

350

351 We have yet to finish part (iii). Let's try to construct a set M' that
 352 satisfies the same conditions like M but is kept as small as possible. Assuming
 353 the Axiom of Choice, we can modify the process so that cardinality of M' is
 354 at most $|M_0| \cdot \aleph_0$. Note that the size of M' is determined by the size of M_0 an,
 355 most importantly, by the size of $H_i(u_1, \dots, u_{m-1})$ for any $i \leq n$ in individual
 356 levels of the construction. Since the lemma only states existence of some x
 357 that satisfies $\varphi_i(u_1, \dots, u_{m-1}, x)$ for any $i \leq n$, we only need to add one x for
 358 every set of parameters but $H_i(u_1, \dots, u_{m-1})$ can be arbitrarily large. Since
 359 Axiom of Choice ensures that there is a choice function, let F be a choice
 360 function on $\mathcal{P}(\bigcup_{i=0}^n M_i)$. Also let $h_i(u_1, \dots, u_{m-1}) = F(H_i(u_1, \dots, u_{m-1}))$ for
 361 $i \leq n$, which means that h is a function that outputs an x that satisfies
 362 $\varphi_i(u_1, \dots, u_{m-1}, x)$ for $i \leq n$ and has minimal rank among all such witnesses.
 363 The induction step needs to be redefined to

$$M'_{i+1} = M'_i \cup \bigcup_{j=0}^n \{H_j(u_1, \dots, u_{m-1}) : u_1, \dots, u_{m-1} \in M'_i\} \quad (2.16)$$

In every step, the amount of elements added in M'_{i+1} is equivalent to the amount of sets of parameters the yielded elements not included in M'_i . So the cardinality of M'_{i+1} exceeds the cardinality of M'_i only for finite M'_i . It is easy to see that if M_0 is finite, M' is countable because it was built from countable union of finite sets. If M_0 is countable or larger, cardinality of M' is equal to the cardinality of M_0 .¹² Therefore $|M'| \leq |M_0| \cdot \aleph_0$ \square

And now for the theorem itself

Theorem 2.10 *First-order Reflection*

Let $\varphi(x_1, \dots, x_n)$ be a first-order formula.

(i) For every set M_0 there exists M such that $M_0 \subset M$ and the following holds:

$$\varphi^M(x_1, \dots, x_n) \leftrightarrow \varphi(x_1, \dots, x_n) \quad (2.17)$$

for every x_1, \dots, x_n .

(ii) For every set M_0 there is a transitive set M , $M_0 \subset M$ such that the following holds:

$$\varphi^M(x_1, \dots, x_n) \leftrightarrow \varphi(x_1, \dots, x_n) \quad (2.18)$$

for every x_1, \dots, x_n .

(iii) For every set M_0 there is α such that $M_0 \subset V_\alpha$ and the following holds:

$$\varphi^{V_\alpha}(x_1, \dots, x_n) \leftrightarrow \varphi(x_1, \dots, x_n) \quad (2.19)$$

for every x_1, \dots, x_n .

(iv) Assuming the Axiom of Choice, for every set M_0 there is M such that $M_0 \subset M$ and $|M| \leq |M_0| \cdot \aleph_0$ and the following holds:

$$\varphi^M(x_1, \dots, x_n) \leftrightarrow \varphi(x_1, \dots, x_n) \quad (2.20)$$

for every x_1, \dots, x_n .

Proof. Let's prove (i) for one formula φ via induction by complexity first. We can safely assume that φ contains no quantifiers besides \exists and no logical connectives other than \neg and $\&$. Assume that this M is obtained from lemma 2.9. The fact, that atomic formulas are reflected in every M comes directly from definition of relativization and the fact that they contain no

¹²It can not be smaller because $|M'_{i+1}| \geq |M'_i|$ for every i . It may not be significantly larger because the maximum of elements added is the number of n -tuples in M'_i , which is of the same cardinality is M'_i .

quantifiers.¹³ The same holds for formulas in the form of $\varphi = \neg\varphi'$. Let us recall the definition of relativization for those formulas in 2.1.

$$(\neg\varphi_1)^M \leftrightarrow \neg(\varphi_1^M) \quad (2.21)$$

Because we can assume from induction that $\varphi'^M \leftrightarrow \varphi'$, the following holds:

$$(\neg\varphi')^M \leftrightarrow \neg(\varphi'^M) \leftrightarrow \neg\varphi' \quad (2.22)$$

The same holds for $\varphi = \varphi_1 \& \varphi_2$. From the induction hypothesis we know that $\varphi_1^M \leftrightarrow \varphi_1$ and $\varphi_2^M \leftrightarrow \varphi_2$, which together with relativization for formulas in the form of $\varphi_1 \& \varphi_2$ gives us

$$(\varphi_1 \& \varphi_2)^M \leftrightarrow \varphi_1^M \& \varphi_2^M \leftrightarrow \varphi_1 \& \varphi_2 \quad (2.23)$$

Let's now examine the case when from the induction hypothesis, M reflects $\varphi'(u_1, \dots, u_n, x)$ and we are interested in $\varphi = \exists x\varphi'(u_1, \dots, u_n, x)$. The induction hypothesis tells us that

$$\varphi'^M(u_1, \dots, u_n, x) \leftrightarrow \varphi'(u_1, \dots, u_n, x) \quad (2.24)$$

so, together with above lemma 2.9, the following holds:

$$\begin{aligned} & \varphi(u_1, \dots, u_n, x) \\ & \leftrightarrow \exists x\varphi'(u_1, \dots, u_n, x) \\ & \leftrightarrow (\exists x \in M)\varphi'(u_1, \dots, u_n, x) \\ & \leftrightarrow (\exists x \in M)\varphi'^M(u_1, \dots, u_n, x) \\ & \leftrightarrow (\exists x\varphi'(u_1, \dots, u_n, x))^M \\ & \leftrightarrow \varphi^M(u_1, \dots, u_n, x) \end{aligned} \quad (2.25)$$

Which is what we have needed to prove:

So far we have proven part (i) of this theorem for one formula φ , we only need to verify that the same holds for any finite number of formulas. This has in fact been already done since lemma 2.9 gives us M for any (finite) amount of formulas. We can then use the induction above to verify that it reflects each of the formulas individually.

¹³Note that this does not hold generally for relativizations to M, E , but only for relativization to M, \in , which is our case.

Now we want to verify other parts of our theorem. Since V_α is a transitive set, by proving (iii) we also satisfy (ii). To do so, we only need to look at part (ii) of lemma 2.9. All of the above proof also holds for $M = V_\alpha$.

To finish part (iv), we take M of size $\leq |M_0| \cdot \aleph_0$, which exists due to part (iii) of lemma 2.9, the rest being identical. \square

Theorem 2.11 *Reflection is equivalent to Infinity & Replacement under ZFC minus Infinity & Replacement*

Proof. Since 2.10 already gives one side of the implication, we are only interested in showing the converse which we shall do in two parts:

Reflection \rightarrow Infinity

Let us first find a formula to be reflected that requires a set M at least as large as V_ω . Let us consider the following formula:

$$\varphi'(x) = \forall \lambda (\lambda < x \rightarrow \exists \mu (\lambda < \mu < x)) \quad (2.26)$$

Because φ says "there is a limit ordinal", if it holds for some x , the Infinity axiom is very easy to satisfy. But we know that there are limit ordinals in ZF, therefore $\varphi = \exists x \varphi'(x)$ is a valid statement. *Reflection* then gives us a set M in which φ^M holds, that is, a set that contains a limit ordinal. So the set of off limit ordinals is non-empty and because ordinals are well-founded, it has a minimal element. Let's call it μ .

$$\mu = \bigcap \{V_\kappa : \forall \lambda (\lambda < \kappa \rightarrow \exists \mu (\lambda < \mu < \kappa))\} \quad (2.27)$$

We can see that μ is the least limit ordinal and therefore it satisfies *Infinity*.

Reflection \rightarrow Replacement

Given a formula $\varphi(x, y, u_1, \dots, u_n)$, we can suppose that it is reflected in any M ¹⁴ What we want to obtain is the following:

$$\forall x, y, z (\varphi(x, y, u_1, \dots, u_n) \& \varphi(x, z, u_1, \dots, u_n) \rightarrow y = z) \rightarrow \quad (2.28)$$

$$\rightarrow \forall X \exists Y \forall y (y \in Y \leftrightarrow \exists x (\varphi(x, y, u_1, \dots, u_n) \& x \in X)) \quad (2.29)$$

We do also know that $x, y \in M$, in other words for every $X, Y = \{y \mid \varphi(x, y, u_1, \dots, u_n)\}$ we know that $X \subset M$ and $Y \subset M$, which, together

¹⁴Which means that for $x, y, u_1, \dots, u_n \in M$, $\varphi^M(x, y, u_1, \dots, u_n) \leftrightarrow \varphi(x, y, u_1, \dots, u_n)$.

436 with the comprehension schema¹⁵ implies that Y , the image of X over φ , is
 437 a set. Which is exactly the Replacement Schema we hoped to obtain. \square

438

439 We have shown that *Reflection* for first-order formulas, *Reflection*₁ is
 440 a theorem of ZF, which means that it won't yield us any large cardinals.
 441 We have also shown that it can be used instead of the Axiom of Infinity and
 442 Replacement Scheme, but ZF + *Reflection*₁ is a conservative extension of
 443 ZF. Besides being a starting point for more general and powerful statements,
 444 it can be used to show that ZF is not finitely axiomatizable. That is because
 445 *Reflection* gives a model to any finite number of (consistent) formulas. So
 446 if $\varphi_1, \dots, \varphi_n$ for any finite n would be the axioms of ZF, *Reflection* would
 447 always contain a model of itself, which would in turn contradict the Second
 448 Gödel's Theorem¹⁶. Notice that, in a way, reflection is complementary to
 449 compactness. Compactness argues that given an infinite¹⁷ set of sentences,
 450 if every finite subset yields a model, so does the whole set. Reflection, on
 451 the other hand, says that while the whole set has no model in the underlying
 452 theory, every finite subset does have one.

453 Also, notice how reflection can be used in ways similar to upward Löwenheim–Skolem
 454 theorem. Since Reflection extends any set M_0 into a model of given formulas
 455 $\varphi_1, \dots, \varphi_n$, we can choose M_0 such that the final M is at least as big as we
 456 need it to be.¹⁸

457 In the next section, we will try to generalize Reflection in a way that
 458 transcends ZF and finally yields some large cardinals.

¹⁵Called the axiom of subsets in Levy's proof.

¹⁶See chapter 3.3 for further details.

¹⁷Countable?

¹⁸Too vague?

3 Reflecting Large Cardinals

TODO nekam dat M a Levyho theorem 3

TODO "v nasledujici kapitole budeme reflektovat vlastnosti univerzalni tridy a budeme se divat jake initial segmenty/kardinaly dostavame podle toho, jak se meni presny vyznam pojmu 'vlastnost' atd..."

TODO neco jako: for every process for obtaining larger sets, this process can't reach V and, from reflection, there is an initial segment of V that is strictly smaller than V but still can't be reached via said process.

In this chapter we aim to explore the possible generalisations of *Reflection* for second-order formulas and use those to establish the existence of various large cardinals. We will also argue whether there is a limit to the size of large cardinals obtainable via generalised form of *Reflection*.

3.1 Reflecting Second-order Formulas

TODO ze "uplne totalni" reflexe se zacykli a rozbije

To see that there is a way to transcend ZF , let us briefly show how a model of ZF can be obtained in $ZF_2 + \text{"second-order reflection"}^{19}$. This will be more closely examined in section 3.3.

We know that ZF can not be finitely axiomatized in first-order formulas, however if Replacement and Comprehension schemes can be substituted by second-order formulas, ZF becomes ZF_2 , which is finitely axiomatizable in second-order logic. Therefore if we take second-order reflection into consideration, we can obtain a set M that is a model of ZF_2 . For now, we have left out the details of how exactly is first-order reflection generalised into stronger statements and how second-order axiomatization of ZF looks like as we will examine those problems closely in the following pages.

We will now define reflection for second-order formulas.

Definition 3.1 *Second-order reflection*

TODO

TODO see Hanf-Scott [kanamori:61]?

TODO full reflection, partial reflection? viz Levy60, ten druhy clanek

TODO diskuse, co vede ve vysledku k nepopsatelnym kardinalum

levyho clanek – zahrnout na patricna mista tehle kapitoly

¹⁹ ZF_2 is an axiomatization of ZF in second-order formulas, to be more rigorously established later.

493 **Definition 3.2** *Strictly increasing function*

494 *A function $F(\alpha)$ defined on the ordinal number into the ordinal numbers is*
 495 *said to be strictly increasing if $\forall \alpha, \beta \in On(\alpha < \beta \rightarrow F(\alpha) < F(\beta))$.*

496 **Definition 3.3** *Continuous function*

497 *A function $F(\alpha)$ defined on the ordinal number into the ordinal numbers is*
 498 *said to be continuous if for any limit α , $F(\alpha) = \lim_{\beta < \alpha} F(\beta)$.*

499 **Definition 3.4** *Normal function*

500 *A function $F(\alpha)$ defined on the ordinal number into the ordinal numbers is*
 501 *said to be normal if it is strictly increasing and continuous*

502 TODO vse nize presunout do patricne sekce ve velkych kardinalech?

503
 504 TODO jak znamim usporadane dvojice?

505
 506 TODO porovnani Mahlovy a Lévyho konstrukce, viz ref{mahlovy kardi-
 507 naly}

508
 509 TODO asi doplnit jak to souvisi se soucasnou definici slabe Mahlovych
 510 kardinalu pres stacionarni mnoziny?

511
 512 Lévy's article mentions various schemata that are not instances of reflec-
 513 tion themselves. We will mention them because they are equivalent to N_0
 514 and because they are fixed-point theorems, which we will find useful later in
 515 this thesis.

516 **Definition 3.5** *M Every normal function defined for all ordinals has at least*
 517 *one inaccessible number in its range.*

518 **Definition 3.6** *M' Every normal function defined for all ordinals has at*
 519 *least one fixed point which is inaccessible.*

520 **Definition 3.7** *M'' Every normal function defined for all ordinals has arbi-*
 521 *trarily great fixed points which are inaccessible.*

522

Theorem 3.8

$$M \leftrightarrow M' \leftrightarrow M'' \quad (3.30)$$

523 An interested reader will find the proof in [2,].

524
525 TODO Pak i lambda konstrukce, zobecnene

Theorem 3.9

$$M \leftrightarrow N \quad (3.31)$$

526 *Proof.* TODO (Theorem 3)? neudelam ho spis v dalsi sekci v modernejsi
527 variante? \square

528
529 TODO shrnout zbytek clanku, fixed-point vety a spol, rict ze to udelame
530 v sekci "velke kardinaly"

531 TODO $S- > ZM- > ZM'- > ZM''$, neco jako mahlovy kardinaly,
532 presunout do dane kapitoly

533 TODO shrnout v par vetach.

534

535

3.2 Preliminaries

536
537 But first, let's establish some elementary terms that will allow us to define
538 the relevant large cardinals.

539 **Definition 3.10** (*limit cardinal*) κ is a limit cardinal if it is \aleph_α for
540 some limit ordinal α .

541 **Definition 3.11** (*strong limit cardinal*) κ is a strong limit cardinal if
542 for every $\lambda < \kappa$, $2^\lambda < \kappa$

543 TODO splyvaji kdyz plati GCH

544 We also need to rigorously define ZF_2 , the second-order axiomatization
545 of ZF we have already used in the previous section. Let's take advantage of
546 second-order variables and replace Replacement and Specification schemes
547 with a single Replacement and a Specification axiom respectively. Lower-case
548 letters represent first-order variables and upper-case P represents a second-
549 order variable. [8]

550 **Definition 3.12** Replacement²

551

$$\begin{aligned} &\forall P(\forall x, y, z(P(x, y) \& P(x, z) \rightarrow y = z) \rightarrow \\ &\rightarrow (\forall x \exists y \forall z(z \in y \leftrightarrow \exists w \in x(P(w, z)))) \end{aligned} \quad (3.32)$$

552 We will denote this axiom Replacement₂.

553 **Definition 3.13** Specification_2

554

$$\forall P \forall x \exists y \forall z (z \in y \leftrightarrow [z \in x \& P(z, x)]) \quad (3.33)$$

555 **Definition 3.14** ZF_2

556 *Let ZF_2 be a theory with all axioms identical with the axioms of ZF with the*
 557 *exception of Replacement and Specification schemes, which are replaced with*
 558 *Replacement₂ and Specification₂ respectively.*

559 TODO vsechny jmena axiomu emph?

560 TODO sjednotit *Replacement₂* s *Replacement*²

561

562 TODO budeme potrebovat club sety, stacionarni mnoziny? treba aby-
 563 chom dokazali ekvivalence?

564 3.3 Inaccessibility

565 The inaccessible cardinal is the smallest of large cardinals²⁰

566 **Definition 3.15** (*weak inaccessibility*) *An uncountable cardinal κ is weakly*
 567 *inaccessible \leftrightarrow it is regular and limit.*

568 **Definition 3.16** (*inaccessibility*) *An uncountable cardinal κ is inaccessible*
 569 *\leftrightarrow it is regular and strongly limit.*

570 To see why this is important, let's dedicate a few lines to the intuition
 571 behind the notions of limitness, regularity and inaccessibility in a manner
 572 strongly influenced by [7]. To see why limit and strongly limit cardinals are
 573 worth mentioning, note that they are "limit" not only in a sense of being
 574 a supremum of an ordinal sequence, they also show that a certain way of
 575 obtaining bigger objects from smaller ones²¹ limited in terms of possibilities.
 576 \aleph_λ is a limit cardinal iff there is no α such that $\aleph_{\alpha+1} = \aleph_\lambda$. Strongly
 577 limit cardinals point to the limits of the powerset operation. It has been too
 578 obvious so far, so let's look at the regular cardinals in this manner. Regular
 579 cardinals are those that cannot be²², expressed as a supremum of smaller
 580 amount of smaller objects²³. More precisely, κ is regular if there is no way to
 581 define it as a union of less than κ ordinals, all smaller than κ . It is interesting

²⁰citation needed.

²¹TODO elegantnejsi formulace?

²²Assuming *Choice*.

²³Just like ω can not be expressed as a supremum of a finite set consisting solely of finite numbers.

to point out, that many of these sets wouldn't be considered sets without *Replacement*, therefore regular cardinals are, in a way, "limits" of how far can we get by taking limits of increasing sequences of ordinals obtained via *Replacement*.

That all being said, it is easy to see that no cardinals in ZF are both strongly limit and regular, the only exception being \aleph_0 . This makes it clear why the definition 3.16 explicitly calls for $\kappa > \aleph_0$. It should be also obvious why the fact that κ is inaccessible implies that $\kappa = \aleph_\kappa$.²⁴

The above should make a clear picture of why *Infinity* is a specific case of *Reflection*.

TODO proc je Refl zaroven zobecneny replacement?

TODO nize budeme zkoumat ktere mnoziny jsou "nedosazitelne" reflexi ruznych skupin "properties of V".

We will now show that the above notion is equivalent to the definition Levy uses in [2], which is, in more contemporary notation, the following:

Theorem 3.17 *The following are equivalent:*

1. κ is inaccessible
2. $\langle V_\kappa, \in \rangle \models \text{ZFC}$

Proof. Let's first prove that if κ is inaccessible, it is a model of ZFC. We will do that by verifying the axioms of ZFC just like Kanamori does it in 1.2 in [1]. Because κ is a limit ordinal, there's no need for us to verify the Powerset, Foundation, Extensionality, Subset, Pairing and Union axioms and the Specification scheme. Thus we only have the Replacement Scheme to verify.

Given an arbitrary set $x \in V_\kappa$ and a function $F : x \rightarrow V_\kappa$, we need to verify that $y = F[x]$ is indeed a set and that it is an element of V_κ . The fact that F is a function implies that $|y| \leq |x|$. It follows from Specification that $y \subset V_\kappa$, which is still not exactly what we want. Let $\alpha < \kappa$ be the least ordinal such that $y \in V_\alpha$ ²⁵, since $V_{\alpha+1} = \mathcal{P}(V_\alpha)$, $y \in V_{\alpha+1}$, together with $\alpha + 1 < \kappa$ this means that $y \in V_\kappa$.

We will now show that if a set is a model of ZFC, it is in fact an inaccessible cardinal. So let V_κ be a model of ZFC which means that it is closed under the powerset operation, in other words:

$$\forall \lambda (\lambda < \kappa \rightarrow 2^\lambda < \kappa) \quad (3.34)$$

²⁴This doesn't work backwards, the first fixed point of the \aleph function is the limit of $\{\aleph_0, \aleph_{\aleph_0}, \aleph_{\aleph_{\aleph_0}}, \dots\}$ is singular since the sequence has countably many elements.

²⁵TODO pozor – jak vime ze takove alpha existuje?

617 which is exactly the definition of strong limitness. κ is regular from the
 618 following argument by contradiction:

619 Let us suppose for a moment that κ is singular. Therefore there is an ordinal
 620 $\alpha < \kappa$ and a function $F : \alpha \rightarrow \kappa$ such that the range of F is unbounded in
 621 κ , in other words, $F[\alpha] \subseteq V_\kappa$ and $\sup(F[\alpha]) = \kappa$. In order to achieve the
 622 desired contradiction, we need to see that it is the case that $F[\alpha] \in V_\kappa$. Let
 623 $\varphi(x, y)$ be the following first-order formula:

$$F(x) = y \quad (3.35)$$

624 Then there is an instance of Axiom Schema of Replacement that states the
 625 following:

$$\begin{aligned} &(\forall x, y, z(\varphi(x, y) \wedge \varphi(x, z) \rightarrow y = z)) \rightarrow \\ &\rightarrow (\forall x \exists y \forall z (z \in y \leftrightarrow \exists w(\varphi(w, z)))) \end{aligned} \quad (3.36)$$

626 Which in turn means that there is a set $y = F[\alpha]$ and $y \in V_\kappa$, which is the
 627 contradiction with $\sup(y) = \kappa$ we are looking for. \square

628

629 The same holds for \mathbf{ZF}_2 , the proof is very similar.

Theorem 3.18

$$V_\kappa \models \mathbf{ZF}_2 \leftrightarrow \kappa \text{ is inaccessible} \quad (3.37)$$

630 *Proof.* κ is a strong limit cardinal because from \mathbf{ZF}_2 and the Powerset Axiom
 631 we know that for every $\lambda < \kappa$, we know that $2^\lambda < \kappa$.

632 κ is also regular, because otherwise there would be an ordinal α and
 633 a function $F : \alpha \rightarrow \kappa$ with a range unbounded in κ . *Replacement*² gives us
 634 a set $y = F[\alpha]$, so $y \in V_\kappa$, which contradicts the fact that $\sup(y) = \kappa$. It
 635 can not be the case that $\kappa \in V_\kappa$.

636

637 The other direction is exactly like the first part of above theorem 3.17.

638

639

640 We have transcended \mathbf{ZF} , but that is just a start. Naturally, we could
 641 go on and consider the next inaccessible cardinal, which is inaccessible with
 642 respect to the theory $\mathbf{ZF} + \exists \kappa (\kappa \models \mathbf{ZF})$. But let's try to find a faster way up,
 643 informally at first.

644 TODO muzu rict "inaccessible cardinals are unbounded in V "?

645 Since we can find an inaccessible set larger than any chosen set M_0 , it is
 646 clear that inaccessible cardinals are unbounded in V . If V were a cardinal,
 647 we could say that there are V inaccessible cardinals less than V , but this

statement of course makes no sense in a set theory as is because V is not a set. But being more careful, we could find a property that can be formalized in first- or second-order logic and reflect it to an initial segment of V . That would allow us to construct large cardinals more efficiently than by adding inaccessibles one by one. The property we are looking for ought to look like something like this:

$$\begin{aligned} &\kappa \text{ is an inaccessible cardinal and} \\ &\text{there are } \kappa \text{ inaccessible cardinals } \mu < \kappa \end{aligned} \tag{3.38}$$

This is in fact a fixed-point type of statement. We shall call those cardinals hyper-inaccessible. Now consider the following definition.

Definition 3.19 *0-inaccessible cardinal*
A cardinal κ is 0-inaccessible if it is inaccessible.

We can define α -weakly-inaccessible cardinals analogously with the only difference that those are limit, not strongly limit.

Definition 3.20 *α -hyper-inaccessible cardinal*
For any ordinal α , κ is called α -inaccessible, if κ is inaccessible and for each $\beta \prec \alpha$, the set of β -inaccessible cardinals less than κ is unbounded in κ .

Because κ is inaccessible and therefore regular, the number of β -inaccessibles below κ is equal to κ . We have therefore successfully formalized the above vague notion of hyper-inaccessible cardinal into a hierarchy of α -inaccessibles.

Let's now consider iterating this process over again. Since, informally, V would be α -inaccessible for any α , this property of the universal class could possibly be reflected to an initial segment, the smallest of those will be the first hyper-inaccessible cardinal. Such κ is larger than any α -inaccessible since from regularity of κ , for given $\alpha < \kappa$, κ is κ -th α -hyper-inaccessible cardinal. It is in fact "inaccessible" via α -inaccessibility.

Definition 3.21 *Hyper-inaccessible cardinal*
 κ is called the hyper-inaccessible, also 0-hyper-inaccessible, cardinal if it is α -inaccessible for every $\alpha < \kappa$.

Definition 3.22 *α -hyper-inaccessible cardinal*

For any ordinal α , κ is called α -hyper-inaccessible cardinal if for each ordinal $\beta < \alpha$, the set of β -hyper-inaccessible cardinals less than κ is bounded in κ .

Obviously we could go on and iterate it ad libitum, but the nomenclature would be increasingly confusing. A smarter way to accomplish the same goal is carried out in the following section.

TODO typografie – mezery kolem vseh = a asi i vyrokovych ostatnich spojek

3.4 Mahlo Cardinals

While the previous chapter introduced us to a notion of inaccessibility and the possibility of iterating it ad libitum in new theories, there is an even faster way to travel upwards in the cumulative hierarchy, that was proposed by Paul Mahlo in his papers (see [4], [5] and [6]) at the very beginning of the 20th century, and which can be easily reformulated using (*Reflection*). To see how Lévy's initial statement of reflection was influenced by Mahlo's work, refer to section 2.2. The aim of the following paragraphs is to give an intuitive explanation of the idea behind Mahlo's hierarchy of cardinals, all claims made here ought to be stated formally later in the very same chapter.

At the very end of section 3.3, we have tried to establish the notion of hyper-inaccessibility and iterate it to yield even larger large cardinals. In order to avoid too bulky cardinal names, let's try a different route and establish those cardinals directly via reflection.

The following two definitions come from [7] and while they are rather informal, we will find them very helpful for understanding the Mahlo cardinals.

Definition 3.23 *Fixed-point property*

For any $\psi(x, u_1, \dots, u_n)$ which is any property of ordinals, we say that a property φ is a fixed-point property if φ has the form

$$x \text{ is an inaccessible cardinal and there are } x \text{ ordinals less than } x \text{ that have the property } \psi(x, u_1, \dots, u_n) \quad (3.39)$$

TODO this is fixed-point property because ... (/fixed-point ceho?)

Definition 3.24 *Fixed-point reflection*

If φ is a fixed-point property that holds for V , it also holds for some V_α , an initial segment of V .

Obviously those are in on way rigorous definitions because we have no idea what $\psi(x, u_1, \dots, u_n)$ looks like. Let's try to restate the same idea in a useful way. But first, let's show that the formal counterpart of the idea of containing "enough" ordinals with a property is the notion of stationary set.

Definition 3.25 *Supremum*

Given A a set of ordinals, the supremum of A , denoted $\sup(A)$, is the least upper bound of A .

$$\sup(A) = \bigcup A \quad (3.40)$$

where α is an ordinal.

Definition 3.26 *Limit point*

Given A , a set of ordinals and an ordinal α , we say that α is a limit point of A if $\sup(A \cap \alpha) = \alpha$

Definition 3.27 *Club set*

For a regular uncountable κ , a set $A \subset \kappa$ is a closed unbounded subset (often abbreviated as a club set) iff A is both closed, which means it contains all it's limit points, and unbounded, which means that for every $\beta \prec \kappa$ there is a $\beta' \in A$ such that $\beta < \beta' < \kappa$.

Definition 3.28 *Stationary set*

For a regular uncountable κ , a set $A \subset \kappa$ is stationary if it intersects every club subset of κ .

Theorem 3.29 *The intersection of fewer than κ^{26} club subsets of κ is a club set.*

For proof, see [3, Theorem 8.3]

Definition 3.30 *Normal function: see definition 3.4.*

²⁶ κ is again a regular uncountable cardinal and it will always be when we will be talking about club sets.

Definition 3.31 *Fixed point*

We say α is a fixed point of ordinal function f when $\alpha = f(\alpha)$.

Definition 3.32 *Weakly Mahlo Cardinal*

κ is weakly Mahlo \leftrightarrow it is a regular limit ordinal and the set of all regular ordinals less than κ is stationary in κ

Definition 3.33 *Mahlo Cardinal*

κ is a Mahlo Cardinal iff it is an inaccessible cardinal and the set of all inaccessible ordinals less than κ is stationary in κ .

It is interesting to note, that weakly-Mahlo cardinals are fixed points of α -weakly inaccessible cardinals, so if κ is weakly mahlo, .. viz Kanamori Proposition 1.1 Analogously,

Definition 3.34 α -Mahlo Cardinal

κ is a α -Mahlo Cardinal iff it is an α -inaccessible cardinal and the set of all α -inaccessible ordinals less than κ is stationary in κ .

In other words, κ is a mahlo cardinal if it is inaccessible and every club set in κ contains an inaccessible cardinal. This is exactly the notion of fixed-point reflection we were trying to show earlier.

The following are two formulations of the same principle, the former is a scheme and the latter a second-order formula, hence the indexes. TODO cite drake?

Definition 3.35 *Axiom F_1*

If F is a normal function, F has a regular fixed point.

Definition 3.36 *Axiom F_2*

$\forall F("F \text{ is a normal function}" \rightarrow \exists \alpha(" \alpha \text{ is a regular fixed point of } F"))$

27 28

TODO cite drake?

Definition 3.37 *The following definitions are equivalent:*

- (i) κ is Mahlo
- (ii) κ is weakly Mahlo and strong limit

²⁷" F is a normal function" is a first-order formula, see definition 3.4

²⁸" α is a regular fixed point of F " is in fact the following first-order formula: $\alpha = f(\alpha) = cf(\alpha)$.

(iii) The set $\{\lambda < \kappa : \lambda \text{ is inaccessible}\}$ is stationary in κ .

(iv) Every normal function on κ has an inaccessible fixed point.

Proof. (i) \leftrightarrow (ii) Let κ_1 be a mahlo cardinal and let κ_2 be a strong limit weakly Mahlo cardinal. We know from the definitions that the set $\{\lambda < \kappa : \lambda \text{ is inaccessible}\}$ is stationary in both κ_1 and κ_2 , the only difference being that κ_1 is a strongly limit cardinal, but κ_2 would be limit from weak Mahloness, wasn't it for the fact that it is also strong limit. This eliminates the only difference between them and therefore κ_1 is also strong limit weakly Mahlo cardinal and κ_2 is Mahlo.

(i) \rightarrow (iii) We know that κ is uncountable, regular, strong limit and that the set $S = \{\lambda < \kappa : \lambda \text{ is regular}\}$ is stationary in κ . We want to prove that $S' = \{\lambda < \kappa : \lambda \text{ is inaccessible}\}$ is thus also stationary in κ .

Since stationary set intersects every club set in κ , let C be any such set. Let $D = \{\lambda < \kappa : \lambda \text{ is strong limit}\}$. D is a club set because TODO. Since intersection of less than κ club sets is a club set, $C \cap D \neq \emptyset$.

TODO proc $\lambda = S \cap C \cap D$ je inaccessible?

(iii) \rightarrow (iv)

TODO jak to tam dela Levy?

(iv) \rightarrow (i)

TODO jak to tam dela Levy?

range kazde normalni funkce je club v On. (nevadi ze On je trida?)

co treba lemma ze pevne body tvori taky club set

mozna by stacilo ze jsou unbounded, tedy kazda normalni funkce ma libovolne velke pevne body. \square

TODO obdoba pro α -Mahlo kardinaly

TODO κ is hyper-Mahlo iff κ is inaccessible and the set $\{\lambda < \kappa : \lambda \text{ is Mahlo}\}$ is stationary in κ .

" We also state the appropriate generalization for greatly Mahlo cardinals."

TODO veta na zaver, shrnuti

3.5 Indescribability and Weakly Compact Cardinals

TODO indescribable – reflecting indescribability – we can't reach V by a Σ_1^1 formula, so there's some initial segment V_α that is also unreachable (we say indescribable) by the means of a ... formula

811 TODO co je "partition property"?

812 TODO pak dk. ekvivalenci

813 TODO Kanamori 6.3

814 **Definition 3.38** *A cardinal κ is weakly compact if it is uncountable and*
 815 *satisfies the partition property $\kappa \rightarrow (\kappa)^2$*

816 opsano z jecha!

817 3.6 Bernays–Gödel Set Theory

818

819 TODO Plagiat – prepsat a vysvetlit

820 TODO

821 3.7 Reflection and the constructible universe

822 TODO reflektovat muzeme jenom kardinaly konzistentni s $V=L$, proc?

823 TODO Plagiat – prepsat a vysvetlit

824 L was introduced by Kurt Gödel in 1938 in his paper *The Consistency*
 825 *of the Axiom of Choice and of the Generalised Continuum Hypothesis* and
 826 denotes a class of sets built recursively in terms of simpler sets, somewhat
 827 similar to Von Neumann universe V . Assertion of their equality, $V = L$, is
 828 called the *axiom of constructibility*. The axiom implies GCH and therefore
 829 also AC and contradicts the existence of some of the large cardinals, our goal
 830 is to decide whether those introduced earlier are among them.

831 On order to formally establish this class, we need to formalize the notion
 832 of definability first:

833 TODO zduvodneni

834

835 TODO kratka diskuse jestli refl implikuje transcendenci na L - polemika,
 836 nazor - $V=L$ a slaba kompaktnost a dalsi

837

838 TODO asi nekde bude meritelny kardinal

839 **4 Higher-order reflection**

840 TODO rict ze to je zobecneni a nejaky dalsi uvodni veci

841 **4.1 Sharp**

842 TODO je tohle higher-order vec?

843 **4.2 Welek: Global Reflection Principles**

844 TODO ma to vubec cenu?

845 **5 Conclusion**

846 TODO na konec

References

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