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4 REFLECTION PRINCIPLES AND LARGE  
5 CARDINALS  
6 Bakalářská práce

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<sup>10</sup> Prohlašuj, že jsem bakalářskou práci vypracoval samostatně a že jsem uvedl  
<sup>11</sup> všechny použité prameny a literaturu.

<sup>12</sup> V Praze 14. dubna 2015

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## Abstract

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Práce zkoumá vztah tzv. principů reflexe a velkých kardinálů. Lévy ukázal, že v ZFC platí tzv. věta o reflexi a dokonce, že věta o reflexi je ekvivalentní schématu nahrazení a axiomu nekonečna nad teorií ZFC bez axiomu nekonečna a schématu nahrazení. Tedy lze na větu o reflexi pohlížet jako na svého druhu axiom nekonečna. Práce zkoumá do jaké míry a jakým způsobem lze větu o reflexi zobecnit a jaký to má vliv na existenci tzv. velkých kardinálů. Práce definuje nedosažitelné, Mahlovy a nepopsatelné kardinály a ukáže, jak je lze zavést pomocí reflexe. Přirozenou limitou kardinálů získaných reflexí jsou kardinály nekonzistentní s L. Práce nabídne intuitivní zdůvodnění, proč tomu tak je.

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## Abstract

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TODO Resumé práce v anglickém jazyce.

28	<b>Contents</b>	
29	<b>1 Introduction</b>	<b>4</b>
30	1.1 Motivation and Origin . . . . .	4
31	1.2 A few historical remarks on reflection . . . . .	8
32	<b>2 Lévy's Original Proof</b>	<b>9</b>
33	2.1 Introduction . . . . .	9
34	2.2 Contemporary restatement . . . . .	11
35	<b>3 Large Cardinals</b>	<b>14</b>
36	3.1 Preliminaries . . . . .	14
37	3.2 Inaccessibility . . . . .	14
38	3.3 Mahlo cardinals . . . . .	15
39	3.4 Weakly Compact Cardinals . . . . .	16
40	3.5 Indescribable Cardinals . . . . .	16
41	3.6 Bernays–Gödel Set Theory . . . . .	17
42	3.7 Morse–Kelley Set Theory . . . . .	19
43	3.8 Reflection and the constructible universe . . . . .	20
44	<b>4 Higher-order reflection</b>	<b>21</b>
45	4.1 Sharp . . . . .	21
46	4.2 Welele: Global Reflection Principles . . . . .	21
47	<b>5 Conclusion</b>	<b>21</b>

# 1 Introduction

## 1.1 Motivation and Origin

The Universe of sets cannot be uniquely characterized (i. e. distinguished from all its initial elements) by any internal structural property of the membership relation in it, which is expressible in any logic of finite or transfinite type, including infinitary logics of any cardinal order.

— Kurt Gödel [4]

To understand why we need reflection in the first place, let's think about infinity for a moment. In the intuitive sense, infinity is an upper limit of all numbers. But for centuries, this was merely a philosophical concept, closely bound to religious and metaphysical way of thinking, considered separate from numbers used for calculations or geometry. It was a rather vague concept. In ancient Greece, Aristotle's response to famous Zeno's paradoxes introduced the distinction between actual and potential infinity. He argued, that potential infinity is (in today's words) well defined, as opposed to actual infinity, which remained a vague incoherent concept. He didn't think it's possible for infinity to inhabit a bounded place in space or time, rejecting Zeno's thought experiments as a whole. Aristotle's thoughts shaped western thinking partly due to Aquinas, who himself believed actual infinity to be more of a metaphysical concept for describing God than a mathematical property attributed to any other entity. In his *Summa Theologica*<sup>1</sup> he argues:

A geometrician does not need to assume a line actually infinite, but takes some actually finite line, from which he subtracts whatever he finds necessary; which line he calls infinite.

Less than hundred years later, Gregory of Rimini wrote

If God can endlessly add a cubic foot to a stone—which He can—then He can create an infinitely big stone. For He need only add one cubic foot at some time, another half an hour later, another a quarter of an hour later than that, and so on ad infinitum. He would then have before Him an infinite stone at the end of the hour.

Which is basically a Zeno's Paradox made plausible with God being the actor. In contrast to Aquinas' position, Gregory of Rimini theoretically constructs

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<sup>1</sup>Part I, Question 7, Article 3, Reply to Objection 1

an object with actual infinite magnitude that is essentially different from God. Even later, in the 17th century, pushing the property of infiniteness from the Creator to his creation, Nature, Leibniz wrote to Foucher in 1662:

I am so in favor of the actual infinite that instead of admitting that Nature abhors it, as is commonly said, I hold that Nature makes frequent use of it everywhere, in order to show more effectively the perfections of its Author. Thus I believe that there is no part of matter which is not, I do not say divisible, but actually divided; and consequently the least particle ought to be considered as a world full of an infinity of different creatures.

But even though he used potential infinity in what would become foundations of modern Calculus and argued for actual infinity in Nature, Leibniz refused the existence of an infinite, thinking that Galileo's Paradoxon<sup>2</sup> is in fact a contradiction. The so called Galileo's Paradoxon is an observation Galileo Galilei made in his final book "Discourses and Mathematical Demonstrations Relating to Two New Sciences". He states that if all numbers are either squares and non-squares, there seem to be less squares than there is all numbers. On the other hand, every number can be squared and every square has it's square root. Therefore, there seem to be as many squares as there are all numbers. Galileo concludes, that the idea of comparing sizes makes sense only in the finite realm.

Salviati: So far as I see we can only infer that the totality of all numbers is infinite, that the number of squares is infinite, and that the number of their roots is infinite; neither is the number of squares less than the totality of all the numbers, nor the latter greater than the former; and finally the attributes "equal," "greater," and "less," are not applicable to infinite, but only to finite, quantities. When therefore Simplicio introduces several lines of different lengths and asks me how it is possible that the longer ones do not contain more points than the shorter, I answer him that one line does not contain more or less or just as many points as another, but that each line contains an infinite number.

Leibniz insists in part being smaller than the whole saying

Among numbers there are infinite roots, infinite squares, infinite cubes. Moreover, there are as many roots as numbers. And there are as many squares as roots. Therefore there are as many squares

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<sup>2</sup>zneni galileova paradoxu

as numbers, that is to say, there are as many square numbers as there are numbers in the universe. Which is impossible. Hence it follows either that in the infinite the whole is not greater than the part, which is the opinion of Galileo and Gregory of St. Vincent, and which I cannot accept; or that infinity itself is nothing, i.e. that it is not one and not a whole.

TODO nejakej Hegel-strucne?

TODO Cantor

TODO mene teologie, vice matematiky

TODO definovat pojmy (trida etc)

In his work, he defined transfinite numbers to extend existing natural number structure so it contains more objects that behave like natural numbers and are based on an object (rather a meta-object) that doesn't explicitly exist in the structure, but is closely related to it. This is the first instance of reflection. This paper will focus on taking this principle a step further, extending Cantor's (or Zermelo–Fraenkel's, to be more precise) universe so it includes objects so big, they could be considered the universe itself, in a certain sense.

TODO dal asi smazat

The original idea behind reflection principles probably comes from what could be informally called “universality of the universe”. The effort to precisely describe the universe of sets was natural and could be regarded as one of the impulses for formalization of naive set theory. If we try to express the universe as a set  $\{x|x = x\}$ , a paradox appears, because either our set is contained in itself and therefore is contained in a set (itself again), which contradicts the intuitive notion of a universe that contains everything but is not contained itself.

TODO ???

If there is an object containing all sets, it must not be a set itself. The notion of class seems inevitable. Either directly the ways for example the Bernays–Gödel set theory, we will also discuss later in this paper, does in, or on a meta-level like the Zermelo–Fraenkel set theory, that doesn't refer to them in the axioms but often works with the notion of a universal class. Another obstacle of constructing a set of all sets comes from Georg Cantor, who proved that the set of all subsets of a set (let  $A$  be the set and  $\mathcal{P}(A)$  its powerset) is strictly larger than  $A$ . That would turn every aspiration to finally establish an universal set into a contradictory infinite regression.<sup>3</sup> We

<sup>3</sup>An intuitive analogy of this *reductio ad infinitum* is the status of  $\omega$ , which was originally thought to be an unreachable absolute, only to become starting point of Cantor's hierarchy of sets growing beyond all boundaries around the end of the 19<sup>th</sup> century

<sup>154</sup> will use  $V$  to denote the class of all sets.



From previous thoughts we can easily argue, that it is impossible to construct a property that holds for  $V$  and no set and is neither paradoxical like  $\{x|x = x\}$  nor trivial. Previous observation can be transposed to a rather naive formulation of the reflection principle:

(Refl) Any property which holds in  $V$  already holds in some initial segment of  $V$ .

To avoid vagueness of the term "property", we could informally reformulate the above statement into a schema:

For every first-order formula<sup>4</sup>  $\varphi$  holds in  $V \leftrightarrow \varphi$  holds in some initial segment of  $V$ .

Interested reader should note that this is a theorem scheme rather than a single theorem.<sup>5</sup>

## 1.2 A few historical remarks on reflection

Reflection made it's first in set-theoretical appearance in Gödel's proof of GCH in  $L$  (citace Kanamori ? Lévy and set theory), but it was around even earlier as a concept. Gödel himself regarded it as very close to Russel's reducibility axiom (an earlier equivalent of the axiom schema of Zermelo's separation). Richard Montague then studied reflection properties as a tool for verifying that Replacement is not finitely axiomatizable (citace?). A few years later Lévy proved (citace? 1960a) equivalence of reflection with Axiom of infinity together with Replacement in proof we shall examine closely in chapter 2.

TODO co dal? recent results?

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<sup>4</sup>this also works for finite sets of formulas [3, p. 168]

<sup>5</sup>If there were a single theorem stating "for any formula  $\varphi$  that holds in  $V$  there is an initial segment of  $V$  where  $\varphi$  also holds", we would obtain the following contradiction with the second Gödel's theorem: In ZFC, any finite group of axioms of ZFC holds in some initial segment of the universe. If we take the largest of those initial segments it is still strictly smaller than the universe and thus we have, via compactness, constructed a model of ZFC within ZFC. That is, of course a harsh contradiction. This also leads to an elegant way to prove that ZFC is not finitely axiomatizable.

## 2 Lévy's Original Proof

### 2.1 Introduction

This section will try to present Lévy's proof of a general reflection principle being equivalent to Replacement and Infinity under ZF minus Replacement and Infinity. We will first introduce a few axioms and definitions that were a different in Lévy's paper[2], but are equivalent to today's terms. We will write them in contemporary notation, our aim is the result, not history of set theory notation.

Please note that Lévy's paper was written in a period when Set theory was oriented towards semantics, which means that everything was done in a model. All proofs were theodel that of ZFC was  $V_\alpha$  (notated as  $R(\alpha)$  at the time) for some cardinal  $\alpha$ , which means that  $\alpha$  is a inaccessible cardinal. Please bear in mind that this is vastly different from saying that there is an inaccessible  $\alpha$  inside the model. This  $V_\alpha$  is also referred to as  $Scm^Q(u)$ , which means that  $u$  ( $u = V_\alpha$  in our case) is a standard complete model of an undisclosed axiomatic set theory  $Q$  formulated in the "non-simple applied first order functional calculus", which is second-order theory is today's terminology, we are allowed to quantify over functions and thus get rid of axiom schemes. (Note that Lévy always speaks of "the axiom of replacement"). Besides placeholder set theory  $Q$  and ZF, which the reader should be familiar with, theories  $Z$ ,  $S$ , and  $SF$  are used in the text.  $Z$  is ZF minus replacement,  $S$  is ZF minus replacement and infinity, and finally  $SF$  is ZF minus infinity. "The axiom of subsets" is an older name for the axiom scheme of specification (and it's not a scheme since we are now working in second order logic). Also note that universal quantifier does not appear,  $\forall x\varphi(x)$  would be written as  $(x)\varphi(x)$ , the symbol for negation is " $\sim$ ".

Lévy then mentions Mahlo's arithmetic construction of cardinals, noting, that he will use similar strategy to build higher levels of strong axioms of infinity.

TODO porovnani Mahlovy a Lévyho konstrukce

TODO asi doplnit jak to souvisi se soucasnou definici slabe Mahlovych kardinalu pres stacionarni mnoziny?

**Definition 2.1**  $N_0(\varphi)$

$$\exists u(Scm^S(u) \& x_1, \dots, x_n \in u \rightarrow \varphi \leftrightarrow \varphi^u) \quad (2.1)$$

where  $\varphi$  is a formula which does not contain free variables except  $x_1, \dots, x_n$ .

TODO muzu vyhodit

213 **Theorem 2.2** *In  $\mathbf{S}$ , the schema  $N_0$  implies the Axiom of Infinity.*

214 *Proof.* For any  $\varphi$ ,  $N_0$  gives us  $\exists u Scm^S(u)$ , which means that there is a set  $u$   
 215 that is identical to  $V_\alpha$  for some alpha, so  $\exists \alpha Scm^S(V_\alpha)$ . We don't know the  
 216 exact size of this  $\alpha$ , but we know that  $\alpha \geq \omega$ , otherwise  $\alpha$  would be finite,  
 217 therefore not closed under the powerset operation, which would contradict  
 218 the axiom of powersets. In order to prove that it is a model of  $\mathbf{S}$ , we would  
 219 need to verify all axioms of  $\mathbf{S}$ . We have already shown that  $\omega$  is closed under  
 220 the powerset operation. Foundation, extensionality and comprehension are  
 221 clear from the fact that we work in  $ZF^6$ , pairing is clear from the fact, that  
 222 given two sets  $A, B$ , they have ranks  $a, b$ , without loss of generality we can  
 223 assume that  $a \leq b$ , which means that  $A \in V_a \in V_b$ , therefore  $V_b$  is a set that  
 224 satisfies the paring axiom: it contains both  $A$  and  $B$ .

225 TODO vyhodit axiomy, staci vyrobit  $\omega$

226 We now want to prove that  $V_\alpha$  leads to existence of an inductive set,  
 227 which is a set that satisfies  $\exists A(\emptyset \in A \& \forall x \in A((x \cup \{x\}) \in A)$ . If we can  
 228 find a way to construct  $V_\omega$  from any  $V_\alpha$  satisfying  $\alpha \geq \omega$ , we are done. Since  
 229  $\omega$  is the least limit ordinal, all we need is the following

$$\bigcap \{V_\kappa \mid \forall \lambda(\lambda < \kappa \rightarrow \exists \mu(\lambda < \mu < \kappa))\} \quad (2.2)$$

230 because  $V_\kappa$  is a transitive set for every  $\kappa$ , thus the intersection is non-empty  
 231 unless empty set satisfies the property or the set of  $V_\kappa$ s is itself empty.  $\square$

232 **Theorem 2.3** *In  $\mathbf{S}$ , the schema  $N_0$  implies Replacement schema.*

233 *Proof.* TODO vysvetlit!

234 Let  $\varphi(v, w)$  be a formula wth no free variables except  $v, w, x_1, \dots, x_n$   
 235 where  $n$  is any natural number. Let  $\chi$  be an instance of replacement schema  
 236 for this  $\varphi$ :

$$\chi = \forall r, s, t(\varphi(r, s) \& \varphi(r, t) \rightarrow s = t) \rightarrow \forall x \exists y \forall w(w \in y \leftrightarrow \exists v(v \in x \& \varphi(v, w))) \quad (2.3)$$

237 We can deduce the following from  $N_0$ :

- 238 (i)  $x_1, \dots, x_n, v, w \in u \rightarrow (\varphi \leftrightarrow \varphi^u)$
- 239 (ii)  $x_1, \dots, x_n, v \in u \rightarrow (\exists w \varphi \leftrightarrow (\exists w \varphi)^u)$
- 240 (iii)  $x_1, \dots, x_n, x \in u \rightarrow (\chi \leftrightarrow \chi^u)$
- 241 (iv)  $\forall x_1, \dots, x_n \forall x(\chi \leftrightarrow (\forall x_1, \dots, x_n \forall x \chi)^u)$

---

<sup>6</sup>We only need to verify axioms that provide means of constructing larger sets from smaller to make sure they don't exceed  $\omega$ . Since  $\omega$  is an initial segment of  $ZF$ , the axiom scheme of specification can't be broken, the same holds for foundation and extensionality.

242 Note that (i), (ii), (iii) are obtained from instances of  $N_0$  for  $\varphi$ ,  $\exists w\varphi$  and  
 243  $\chi$  respectively. From relativization we also know that  $(\exists w\varphi)^u$  is equivalent to  
 244  $\exists w(w \in u \& \varphi^u)$ . Therefore (ii) is equivalent to  $x_1, \dots, x_n, v \in u \rightarrow (\exists w(w \in$   
 245  $u \& \varphi^u))$ .

246 If  $\varphi$  is a function  $(\forall r, s, t(\varphi(r, s) \& \varphi(r, t) \rightarrow r = t))$ , then for every  $x \in u$ ,  
 247 which is also  $x \subset u$  by  $Scm^S(u)$ , it maps elements of  $x$  onto  $u$ . From the  
 248 axiom scheme of comprehension<sup>7</sup>, we can find a set of all images of elements  
 249 of  $x$ . Let's call it  $y$ . That gives us  $x_1, \dots, x_n, x \in u \rightarrow \chi$ . By (iii) we get  
 250  $x_1, \dots, x_n, x \in u \rightarrow \chi^u$ , closure of this formula is  $(\forall x_1, \dots, x_n \forall x \chi)^u$ , which  
 251 together with (iv) yields  $\forall x_1, \dots, x_n \forall x \chi$ . By the means of specification we  
 252 end up with  $\chi$ , which is all we need for now.

253 TODO btw co je x? nemela by tam tam byt nejaka volna promenna?  $\square$

## 254 2.2 Contemporary restatement

255 TODO nejaký uvod. Levy rika ze existuje  $Scm^S(u)$  reflektujici varphi, coz  
 256 uz nepotrebuje. atd.

257 The following lemma is usually done in parts, the first being with one  
 258 formula and the other with  $n$ . Will will only state and prove the generalised  
 259 version for  $n$  formulas.

260 **Lemma 2.4** *Lemma Let  $\varphi_1, \dots, \varphi_n$  be any formulas with  $m$  parameters<sup>8</sup> .*  
 261 *For each set  $M_0$  there is such  $M$  that  $M_0 \subset M$  and the following holds for*  
 262 *every  $i \leq n$*

$$\exists x \varphi_i(u_1, \dots, u_{m-1}, x) \rightarrow (\exists x \in M) \varphi_i(u_1, \dots, u_{m-1}, x) \quad (2.4)$$

263 *Proof.* Let us first define operation  $H(u_1, \dots, u_{m-1})$  that gives us the set of  
 264  $x$ 's with minimal rank, satisfying  $\varphi_i(u_1, \dots, u_{m-1}, x)$  for given parameteres  
 265  $u_1, \dots, u_{m-1}$  for every  $i \leq n$ .

$$H_i(u_1, \dots, u_n) = \{x \in C_i : (\forall z \in C)(rank(x) \leq rank(z))\} \text{ for } i \leq n \quad (2.5)$$

266 where

$$C_i = \{x : \varphi_i(u_1, \dots, u_{m-1}, x)\} \text{ for } i \leq n \quad (2.6)$$

<sup>7</sup>axiom of subsets in Levy's version

<sup>8</sup>For formulas with different number of parameters take for  $m$  the highest number of parameters among given formulas. Add spare parameters to the other formulas so that  $x$  remains the last parameter. That can be done in a following manner: Let  $\varphi'_i$  be the a formula with  $k$  parameters,  $k < m$ . Let us set  $\varphi_i(u_1, \dots, u_{m-1}, x) = \varphi'_i(u_1, \dots, u_{k-1}, u_k, \dots, u_{m-1}, x)$ , notice that  $u_k, \dots, u_{m-1}$  are spare variables added just for formal simplicity.

267 Next, let's construct  $M$  from given  $M_0$  by induction.

$$M_{i+1} = M_i \cup \bigcup_{j=0}^n \{ \bigcup \{ H_j(u_1, \dots, u_{m-1}) : u_1, \dots, u_{m-1} \in M_i \} \} \quad (2.7)$$

268 In other words, in each step we add the elements satisfying  $\varphi(u_1, \dots, u_{m-1}, x)$   
 269 for those parameters that were either available earlier or were added in the  
 270 previous step. The final  $M$  is obtained by joining all incremental steps to-  
 271 gether.

$$M = \bigcup_{i=0}^{\infty} M_i \quad (2.8)$$

272 Let's try to construct a set  $M'$  that satisfies the same conditions like  
 273  $M$  but is kept as small as possible. Assuming the Axiom of Choice, we can  
 274 modify the process so that cardinality of  $M'$  is at most  $|M_0| \cdot \aleph_0$ . Note that the  
 275 size of  $M'$  is determined by the size of  $M_0$  and, most importantly, by the size of  
 276  $H_i(u_1, \dots, u_{m-1})$  for any  $i \leq n$  in individual levels of the construction. Since  
 277 the lemma only states existence of some  $x$  that satisfies  $\varphi_i(u_1, \dots, u_{m-1}, x)$   
 278 for any  $i \leq n$ , we only need to add one  $x$  for every set of parameters but  
 279  $H_i(u_1, \dots, u_{m-1})$  can be arbitrarily large. Since Axiom of Choice ensures  
 280 that there is a choice function, let  $F$  be a choice function on  $\mathcal{P}((\cup)M')$ . Also  
 281 let  $h_i(u_1, \dots, u_{m-1}) = F(H_i(u_1, \dots, u_{m-1}))$  for  $i \leq n$ , which means that  $h$  is  
 282 a function that outputs an  $x$  that satisfies  $\varphi_i(u_1, \dots, u_{m-1}, x)$  for  $i \leq n$  and  
 283 has minimal rank among all such witnesses. The induction step needs to be  
 284 redefined to

$$M'_{i+1} = M'_i \cup \bigcup_j = 0^n \{ h_j(u_1, \dots, u_{m-1}) : u_1, \dots, u_{m-1} \in M'_i \} \quad (2.9)$$

285 In every step, the amount of elements added in  $M'_{i+1}$  is equivalent to the  
 286 amount of sets of parameters the yielded elements not included in  $M'_i$ . So  
 287 the cardinality of  $M'_{i+1}$  exceeds the cardinality of  $M'_i$  only for finite  $M'_i$ . It  
 288 is easy to see that if  $M_0$  is finite,  $M'$  is countable because it was built from  
 289 countable union of finite sets. If  $M_0$  is countable or larger, cardinality of  $M'$   
 290 is equal to the cardinality of  $M_0$ .<sup>9</sup> Therefore  $|M'| \leq |M_0| \cdot \aleph_0$

291 □

292 TODO proc  $\leq$  a ne =?

293 TODO nemel bych neco rict o  $u_1, \dots, u_n$ ? urcite se dostanou do nektereho  
 294  $M_i$ ?

---

<sup>9</sup>It can not be smaller because  $|M'_{i+1}| \geq |M'_i|$  for every  $i$ . It may not be significantly larger because the maximum of elements added is the number of  $n$ -tuples in  $M'_i$ , which is of the same cardinality is  $M'_i$ . ((proc? Ramsey?))

295 **Theorem 2.5** *First-order Reflection*

296 *Proof.*

297

□

298 **Theorem 2.6** *(Refl) is equivalent to (Infinity) & (Replacement) under ZFC*  
 299 *minus (Infinity) & (Replacement)*

300 *Proof.* Since (Refl) is a sound theorem in ZFC, we are only interested in  
 301 showing the converse: (Refl)  $\rightarrow$  (Infinity)

302 This is the easy part since Infinity says that *there is an infinite set* and  
 303 (Refl) is just a stronger version that says "there is an inaccessible cardinal"  
 304 which is all we need. ((PREFORMULOVAT))

305 (Refl)  $\rightarrow$  (Replacement)

306

□

307 TODO Plagiat – prepsat a vysvetlit

308 **Definition 2.7** *Let  $\varphi(R)$  be a  $\Pi_m^n$ -formula which contains only one free vari-*  
 309 *able  $R$  which is second-order. Given  $R \subseteq V_\kappa$ , we say that  $\varphi(R)$  reflects in  $V_\kappa$*   
 310 *if there is some  $\alpha < \kappa$  such that:*

$$\text{If } (V_\kappa, \in, R) \models \varphi(R), \text{ then } (V_\alpha, \in, R \cap V_\alpha) \models \varphi(R \cap V_\alpha). \quad (2.10)$$

## 3 Large Cardinals

TODO pozor na opsane definice, prefurmulovvat!!!

### 3.1 Preliminaries

To avoid confusion<sup>10</sup>, let's first define some basic terms.

**Definition 3.1** (*weak limit cardinal*)  $\kappa$  is a weak limit cardinal if it is  $\aleph_\alpha$  for some limit  $\alpha$ .

**Definition 3.2** (*strong limit cardinal*)  $\kappa$  is a strong limit cardinal if for every  $\lambda < \kappa$ ,  $2^\lambda < \kappa$

### 3.2 Inaccessibility

**Definition 3.3** (*weak inaccessibility*)  $\kappa$  is weakly inaccessible  $\leftrightarrow$  it is regular and weakly limit.

**Definition 3.4** (*inaccessibility*)  $\kappa$  is inaccessible  $\leftrightarrow$  it is regular and strongly limit.

**Theorem 3.5** [Lévy] The following are equivalent:

- (i)  $\kappa$  is inaccessible.
- (ii) For every  $R \subseteq V_\kappa$  and every first-order formula  $\varphi(R)$ ,  $\varphi(R)$  reflects in  $V_\kappa$ .
- (iii) For every  $R \subseteq V_\kappa$ , the set  $C = \{\alpha < \kappa \mid \langle V_\alpha, \in, R \cap V_\alpha \rangle \prec \langle V_\kappa, \in, R \rangle\}$  is closed unbounded.

*Proof.* Let's start with (i)  $\rightarrow$  (iii) in a way similar to [1].

The set  $\{\alpha < \kappa \mid \langle V_\alpha, \in, R \cap V_\alpha \rangle \prec \langle V_\kappa, \in, R \rangle\}$  is clearly closed, it remains to show that it is also unbounded. To do so, let  $\alpha < \kappa$  be arbitrary. Define  $\alpha_n < \kappa$  for  $n \in \omega$  by recursion as follows:

Set  $\alpha_0 = \alpha$ . Given  $\alpha_n < \kappa$  define  $\alpha_{n+1}$  to be the least  $\beta \geq \alpha_n$  such as whenever  $y_1, \dots, y_k \in V_{\alpha_n}$  and  $\langle V_\kappa, \in, R \rangle \models \exists v_0 \varphi[v_0, y_1, \dots, y_k]$  for some formula  $\varphi$ , there is an  $x \in V_\beta$  such that  $\langle V_\kappa, \in, R \rangle \models \varphi[x, y_1, \dots, y_k]$ .

Since  $\kappa$  is inaccessible,  $|V_{\alpha_n}| < \kappa$  and so  $\alpha_{n+1} < \kappa$ .

Finally, set  $\alpha = \sup(\alpha_n \mid n \in \omega)$ . Then  $\langle V_\alpha, \in, R \cap V_\alpha \rangle \prec \langle V_\kappa, \in, R \rangle$  by the usual (Tarski) criterion for elementary substructure.

<sup>10</sup>While in most sources refer to *weak limit cardinal* as a *limit cardinal* and to *strong limit cardinal*, in some cases the distinction is *weak limit cardinal* and *limit cardinal* respectively. That's why I have decided to explicitly define those otherwise elementary terms.

The next part, proving  $(iii) \rightarrow (ii)$ , should be elementary since  $C$  is closed unbounded, which means that it contains at least countably many elements but we need only one such  $\alpha$  to satisfy (2.7).

Finally, we shall prove that  $(ii) \rightarrow (i)$ . Since it obviously holds that  $\kappa > \omega$ , we have yet to prove that  $\kappa$  is regular and a strong limit. Let's argue by contradiction that it is regular. If it wasn't, there would be a  $\beta < \kappa$  and a function  $F : \beta \Rightarrow \kappa$  with range unbounded in  $\kappa$ . Set  $R = \{\beta\} \cup F$ . By hypothesis there is an  $\alpha < \kappa$  such that  $\langle V_\alpha, \in, R \cap V_\alpha \rangle \prec \langle V_\kappa, \in, R \rangle$ . Since  $\beta$  is the single ordinal in  $R$ ,  $\beta \in V_\alpha$  by elementarity. This yields the desired contradiction since the domain of  $F \cap V_\alpha$  cannot be all of  $\beta$ .

Next, let's see whether  $\kappa$  is indeed a strong limit, again by contradiction. If not, there would be a  $\lambda < \kappa$  such that  $2^\lambda \geq \kappa$ . Let  $G : \mathcal{P}(\lambda) \Rightarrow \kappa$  be surjective and set  $R = \{\lambda + 1\} \cup G$ . By hypothesis, there is an  $\alpha < \kappa$  such that  $\langle V_\alpha, \in, R \cap V_\alpha \rangle \prec \langle V_\kappa, \in, R \rangle$ .  $\lambda + 1 \in V_\alpha$  and so  $\mathcal{P}(\lambda) \in V_\alpha$ , but this is again a contradiction.  $\square$

### 3.3 Mahlo cardinals

TODO reflektuji nedosazitelnost? TODO zminit Mahlovu konstrukci?

**Definition 3.6** *Weakly Mahlo Cardinals*  $\kappa$  is weakly Mahlo  $\leftrightarrow$  it is a limit ordinal and the set of all regular ordinals less than  $\kappa$  is stationary in  $\kappa$

**Definition 3.7** *Mahlo cardinals* The following definitions are equivalent:

- (i)  $\kappa$  is Mahlo
- (ii)  $\kappa$  is weakly Mahlo and strong limit
- (iii)  $\kappa$  is inaccessible and the regular cardinals below  $\kappa$  form a stationary subset of  $\kappa$ .
- (iv)  $\kappa$  is regular and the stationary sets below  $\kappa$  form a stationary subset of  $\kappa$ .

**Theorem 3.8**  $\kappa$  is Mahlo  $\leftrightarrow$  for any  $R \subset V_\kappa$  there is an inaccessible cardinal  $\alpha < \kappa$  such that  $\langle V_\alpha, \in, R \cap V_\alpha \rangle \prec \langle V_\kappa, \in, R \rangle$ .

*Proof.* Start with the proof of (3.5) and add the following:

$\kappa$  is Mahlo by the following contradiction. If not, there would be a  $C$  closed unbounded in  $\kappa$  containing no inaccessible cardinals. By the hypothesis there is an inaccessible  $\alpha < \kappa$  such that  $\langle V_\alpha, \in, C \cap V_\alpha \rangle \prec \langle V_\kappa, \in, C \rangle$ . By elementarity  $C \cap \alpha$  is unbounded in  $\alpha$ . But then,  $\alpha \in C$ , which is the contradiction we need.  $\square$



376 Note that Mahlo cardinals were first described in 1911, almost 50 years  
 377 before Lévy's reflection, which was heavily inspired by those.

### 378 3.4 Weakly Compact Cardinals

379 TODO souvislost s reflexi!

380 In this section, we will introduce various well-known large cardinals and  
 381 establish them via reflection.

382 **Definition 3.9** *A cardinal  $\kappa$  is weakly compact if it is uncountable and*  
 383 *satisfies the partition property  $\kappa \rightarrow (\kappa)^2$*

384 **Lemma 3.10** *Every weakly compact cardinal is inaccessible*

385 *Proof.* Let  $\kappa$  be a weakly compact cardinal. To show that  $\kappa$  is regular, let  
 386 us assume that  $\kappa$  is the disjoint union  $\bigcup\{A_\gamma : \gamma < \lambda\}$  such that  $\lambda < \kappa$  and  
 387  $|A_\gamma| < \kappa$  for each  $\gamma < \lambda$ . We define a partition  $F : [\kappa]^2 \rightarrow \{0, 1\}$  as follows:  
 388  $F(\{\alpha, \beta\}) = 0$  just in case  $\alpha$  and  $\beta$  are the same size  $A_\gamma$ . Obviously, this  
 389 partition does not have a homogenous set  $H \subset \kappa$  of size  $\kappa$ . That  $\kappa$  is a  
 390 strong limit cardinal follows from Lemma 9.4: (?? doplnit z jecha): If  $\kappa \geq 2^\lambda$   
 391 for some  $\lambda < \kappa$ , then because  $2^\lambda \leq (\lambda^+)^2$ , we have  $\kappa \leq (\lambda^+)^2$  and hence  
 392  $\kappa \leq (\kappa)^2$ .  $\square$

393 **Theorem 3.11** *Let  $\kappa$  be a weakly compact cardinal. Then for every station-*  
 394 *ary set  $S \subset \kappa$  there is an uncountable regular cardinal  $\lambda < \kappa$  such that the*  
 395 *set  $S \cap \lambda$  is stationary in  $\lambda$ .*

396 *Proof.* TODO  $\square$

### 397 3.5 Indescribable Cardinals

398 **Definition 3.12 (Indescribability)** *For  $Q$  either  $\Pi_n^m$  or  $\Sigma_n^m$*   
 399 *A cardinal  $\kappa$  is  $Q$ -indescribable if whenever  $U \subseteq V_\kappa$  and  $\varphi$  is a  $Q$  sentence*  
 400 *such that  $\langle V_\kappa, \in, U \rangle \models \varphi$ , then for some  $\alpha < \kappa$ ,  $\langle V_\alpha, \in, U \cap V_\alpha \rangle \models \varphi$ .*

### 3.6 Bernays–Gödel Set Theory

TODO Plagiat – prepsat a vysvetlit

Gödel–Bernays set theory, also known as Von Neumann–Bernays–Gödel set theory is an axiomatic set theory that explicitly talks about proper classes as well as sets, which allows it to be finitely axiomatizable, albeit our version stated below contains one schema. It is a conservative extension of Zermalo–Fraenkel set theory. Using forcing, one can prove equiconsistency of BGC and ZFC.

Bernays–Gödel set theory contains two types of objects: proper classes and sets. The notion of set, usually denoted by a lower case letter, is identical to set in ZF, whereas proper classes are usually denoted by upper case letters. The difference between the two is in a fact, that proper classes are not members of other classes, sets, on the other hand, have to be members of classes.

**Definition 3.13** (*Gödel–Bernay set theory*)

(i) extensionality for sets

$$\forall a \forall b [\forall x (x \in a \leftrightarrow x \in b) \rightarrow a = b] \quad (3.11)$$

(ii) pairing for sets

$$\forall x \forall y \exists z \forall w [w \in z \leftrightarrow (w = x \vee w = y)] \quad (3.12)$$

(iii) union for sets

$$\forall a \exists b \forall c [c \in b \leftrightarrow \exists d (c \in d \wedge d \in a)] \quad (3.13)$$

(iv) powers for sets

$$\forall a \exists p \forall b [b \in p \leftrightarrow (c \in b \rightarrow c \in a)] \quad (3.14)$$

(v) infinity for sets

$$\text{There is an inductive set.} \quad (3.15)$$

(vi) Extensionality for classes

$$\forall x (x \in A \leftrightarrow x \in B) \rightarrow A = B \quad (3.16)$$

(vii) Foundation for classes

$$\text{Each nonempty class is disjoint from each of its elements.} \quad (3.17)$$

424 (viii) Limitation of size for sets

For any class  $C$  a set  $x$  such that  $x=C$  exists iff (3.18)

425

there is no bijection between  $C$  and the class  $V$  of all sets (3.19)

426 (ix) Comprehension schema for classes

For any formula  $\varphi$  with no quantifiers over classes, there is a class  $A$  such that  $\forall x(x \in A \leftrightarrow \varphi(x))$  (3.20)

427 The first five axioms are identical to axioms in ZF.

428 Comprehension schema tells us, that proper classes are basically first-order  
429 predicates. ... TODO Plagiat – prepsat a vysvetlit

430 **Definition 3.14** We say that  $\varphi(R)$  with a class parameter  $R$  reflects if there  
431 is  $\alpha$  such that

$$\varphi(R) \rightarrow (V_\alpha, V_{\alpha+1}) \models \varphi(R \cap V_\alpha). \quad (3.21)$$

432 **Theorem 3.15** There is a second-order sentence  $\varphi$  which is provable in GB  
433 such that if  $\varphi$  reflects at  $\alpha$ , i.e. if

$$\varphi \rightarrow (V_\alpha, V_{\alpha+1}) \models \varphi, \quad (3.22)$$

434 then  $\alpha$  is an inaccessible cardinal.

435 *Proof.* Take  $\varphi$  to say “there is no function from  $\gamma \in \text{ORD}$  cofinal in ORD  
436 and for every  $\gamma \in \text{ORD}$ ,  $2^\gamma \in \text{ORD}$ ”. Clearly, if  $\varphi$  reflects at some  $\alpha$ ,  
437 then  $\alpha$  is inaccessible (here we use that the second-order variable range over  
438  $\mathcal{P}(V_\alpha) = V_{\alpha+1}$ ).  $\square$

439 As a corollary we obtain:

440 **Corollary 3.16** Second-order reflection in GB implies the existence of an  
441 inaccessible cardinal.

### 442 3.7 Morse–Kelley Set Theory

443 Axioms not

444 (i) *Extensionality*

$$\forall X \forall Y (\forall z (z \in X \leftrightarrow z \in Y) \rightarrow X = Y). \quad (3.23)$$

445 (ii) *Pairing*

$$asdfg \quad (3.24)$$

446 (iii) *Foundation For Classes*

$$asdf \quad (3.25)$$

447 (iv) *Class Comprehension*

$$\forall W_1, \dots, W_n \exists Y \forall x (x \in Y \leftrightarrow (\phi(x, W_1, \dots, W_n) \& set(x))). \quad (3.26)$$

448 Where  $set(x)$  is monadic predicate stating that class  $x$  is a set.

449 (v) *Limitation Of Size For Classes*

$$asdf \quad (3.27)$$

450 (vi) *Pairing*

$$asdf \quad (3.28)$$

451 (vii) *Pairing*

$$asdf \quad (3.29)$$

452 TODO

### 3.8 Reflection and the constructible universe

TODO reflektovat muzeme jenom kardinaly konzistentni s  $V=L$ , proc?

TODO Plagiat – prepsat a vysvetlit

$L$  was introduced by Kurt Gödel in 1938 in his paper *The Consistency of the Axiom of Choice and of the Generalised Continuum Hypothesis* and denotes a class of sets built recursively in terms of simpler sets, somewhat similar to Von Neumann universe  $V$ . Assertion of their equality,  $V = L$ , is called the *axiom of constructibility*. The axiom implies GCH and therefore also AC and contradicts the existence of some of the large cardinals, our goal is to decide whether those introduced earlier are among them.

On order to formally establish this class, we need to formalize the notion of definability first:

#### Definition 3.17 (Definable sets)

$$Def(X) := \{\{y|x \in X \wedge \langle X, \in \rangle \models \varphi(y, z_1, \dots, z_n)\} \mid \varphi \text{ is a first-order formula, } z_1, \dots, z_n \in X\} \quad (3.30)$$

Now we can recursively build  $L$ .

#### Definition 3.18 (The Constructible universe) (i)

$$L_0 := \emptyset \quad (3.31)$$

(ii)

$$L_{\alpha+1} := Def(L_\alpha) \quad (3.32)$$

(iii)

$$L_\lambda = \bigcup_{\alpha < \lambda} L_\alpha \text{ If } \lambda \text{ is a limit ordinal} \quad (3.33)$$

(iv)

$$L = \bigcup_{\alpha \in Ord} L_\alpha \quad (3.34)$$

TODO Plagiat – prepsat a vysvetlit

**Fact 3.19** *The reflection – constructed as explained in the previous paragraph (!!! preformulovat !!!) – with second-order parameters for higher-order formulas (even of transfinite type) does not yield transcendence over  $L$ .*

TODO zduvodneni

TODO kratka diskuse jestli refl implikuje transcendenci na  $L$  - polemika, nazor -  $V=L$  a slaba kompaktnost a dalsi

## 473 **4 Higher-order reflection**

474 TODO rict ze to je zobecneni a nejaky dalsi uvodni veci

### 475 **4.1 Sharp**

476 TODO je tohle higher-order vec?

### 477 **4.2 Welek: Global Reflection Principles**

478 TODO

## 479 **5 Conclusion**

480 TODO asi tady neco bude, ne?

## References

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